

# Stochastic Control for Optimal Market-Making

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# Trading Order Book (TOB)



# Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price  $P$  and size  $N$
- Buy LO  $(P, N)$  states willingness to buy  $N$  shares at a price  $\leq P$
- Sell LO  $(P, N)$  states willingness to sell  $N$  shares at a price  $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of (Price, Size) pairs

Bids:  $[(P_i^{(b)}, N_i^{(b)}) \mid 1 \leq i \leq m], P_i^{(b)} > P_j^{(b)} \text{ for } i < j$

Asks:  $[(P_i^{(a)}, N_i^{(a)}) \mid 1 \leq i \leq n], P_i^{(a)} < P_j^{(a)} \text{ for } i < j$

- We call  $P_1^{(b)}$  as simply *Bid*,  $P_1^{(a)}$  as *Ask*,  $\frac{P_1^{(a)} + P_1^{(b)}}{2}$  as *Mid*
- We call  $P_1^{(a)} - P_1^{(b)}$  as *Spread*,  $P_n^{(a)} - P_m^{(b)}$  as *Market Depth*
- A Market Order (MO) states intent to buy/sell  $N$  shares at the *best possible price(s)* available on the TOB at the time of MO submission

# Trading Order Book (TOB) Activity

- A new Sell LO  $(P, N)$  potentially removes best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid (i : P_i^{(b)} \geq P)]$$

- After this removal, it adds the following to the asks side of the TOB

$$(P, \max(0, N - \sum_{i: P_i^{(b)} \geq P} N_i^{(b)}))$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order  $N$  will remove the best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid 1 \leq i \leq m]$$

- A Buy Market Order  $N$  will remove the best ask prices on the TOB

$$\text{Removal: } [(P_i^{(a)}, \min(N_i^{(a)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(a)}))) \mid 1 \leq i \leq n]$$

# TOB Dynamics and Market-Making

- Modeling TOB Dynamics involves predicting arrival of MOs and LOs
- Market-makers are liquidity providers (providers of Buy and Sell LOs)
- Other market participants are typically liquidity takers (MOs)
- But there are also other market participants that trade with LOs
- Complex interplay between market-makers & other mkt participants
- Hence, TOB Dynamics tend to be quite complex
- We view the TOB from the perspective of a single market-maker who aims to gain with Buy/Sell LOs of appropriate width/size
- By anticipating TOB Dynamics & dynamically adjusting Buy/Sell LOs
- Goal is to maximize *Utility of Gains* at the end of a suitable horizon
- If Buy/Sell LOs are too narrow, more frequent but small gains
- If Buy/Sell LOs are too wide, less frequent but large gains
- Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation

# Notation for Optimal Market-Making Problem

- We simplify the setting for ease of exposition
- Assume finite time steps indexed by  $t = 0, 1, \dots, T$
- Denote  $W_t \in \mathbb{R}$  as Market-maker's trading PnL at time  $t$
- Denote  $I_t \in \mathbb{Z}$  as Market-maker's inventory of shares at time  $t$  ( $I_0 = 0$ )
- $S_t \in \mathbb{R}^+$  is the TOB Mid Price at time  $t$  (assume stochastic process)
- $P_t^{(b)} \in \mathbb{R}^+, N_t^{(b)} \in \mathbb{Z}^+$  are market maker's Bid Price, Bid Size at time  $t$
- $P_t^{(a)} \in \mathbb{R}^+, N_t^{(a)} \in \mathbb{Z}^+$  are market-maker's Ask Price, Ask Size at time  $t$
- Assume market-maker can add or remove bids/asks costlessly
- Denote  $\delta_t^{(b)} = S_t - P_t^{(b)}$  as Bid Spread,  $\delta_t^{(a)} = P_t^{(a)} - S_t$  as Ask Spread
- Random var  $X_t^{(b)} \in \mathbb{Z}_{\geq 0}$  denotes bid-shares "hit" up to time  $t$
- Random var  $X_t^{(a)} \in \mathbb{Z}_{\geq 0}$  denotes ask-shares "lifted" up to time  $t$

$$W_{t+1} = W_t + P_t^{(a)} \cdot (X_{t+1}^{(a)} - X_t^{(a)}) - P_t^{(b)} \cdot (X_{t+1}^{(b)} - X_t^{(b)}), \quad I_t = X_t^{(b)} - X_t^{(a)}$$

- Goal to maximize  $\mathbb{E}[U(W_T + I_T \cdot S_T)]$  for appropriate concave  $U(\cdot)$

# Markov Decision Process (MDP) Formulation

- Order of MDP activity in each time step  $0 \leq t \leq T - 1$ :
  - Observe  $State := (t, S_t, W_t, I_t)$
  - Perform  $Action := (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)})$
  - Experience TOB Dynamics resulting in:
    - random bid-shares hit =  $X_{t+1}^{(b)} - X_t^{(b)}$  and ask-shares lifted =  $X_{t+1}^{(a)} - X_t^{(a)}$
    - update of  $W_t$  to  $W_{t+1}$ , update of  $I_t$  to  $I_{t+1}$
    - stochastic evolution of  $S_t$  to  $S_{t+1}$
  - Receive next-step  $(t + 1)$  Reward  $R_{t+1}$

$$R_{t+1} := \begin{cases} 0 & \text{for } 1 \leq t + 1 \leq T - 1 \\ U(W_{t+1} + I_{t+1} \cdot S_{t+1}) & \text{for } t + 1 = T \end{cases}$$

- Goal is to find an *Optimal Policy*  $\pi^*$ :

$$\pi^*(t, S_t, W_t, I_t) = (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)}) \text{ that maximizes } \mathbb{E}\left[\sum_{t=1}^T R_t\right]$$

- Note: Discount Factor when aggregating Rewards in the MDP is 1



# Avellaneda-Stoikov Continuous Time Formulation

- We go over the [landmark paper by Avellaneda and Stoikov in 2006](#)
- They derive a simple, clean and intuitive solution
- We adapt our discrete-time notation to their continuous-time setting
- $X_t^{(b)}, X_t^{(a)}$  are *Poisson processes* with *hit/lift-rate* means  $\lambda_t^{(b)}, \lambda_t^{(a)}$

$$dX_t^{(b)} \sim \text{Poisson}(\lambda_t^{(b)} \cdot dt), \quad dX_t^{(a)} \sim \text{Poisson}(\lambda_t^{(a)} \cdot dt)$$

$$\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)}), \quad \lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)}) \text{ for decreasing functions } f^{(b)}, f^{(a)}$$

$$dW_t = P_t^{(a)} \cdot dX_t^{(a)} - P_t^{(b)} \cdot dX_t^{(b)}, \quad I_t = X_t^{(b)} - X_t^{(a)} \quad (\text{note: } I_0 = 0)$$

- Since infinitesimal Poisson random variables  $dX_t^{(b)}$  (shares hit in time  $dt$ ) and  $dX_t^{(a)}$  (shares lifted in time  $dt$ ) are Bernoulli (shares hit/lifted in time  $dt$  are 0 or 1),  $N_t^{(b)}$  and  $N_t^{(a)}$  can be assumed to be 1
- This simplifies the *Action* at time  $t$  to be just the pair:  $(\delta_t^{(b)}, \delta_t^{(a)})$
- TOB Mid Price Dynamics:  $dS_t = \sigma \cdot dz_t$  (scaled brownian motion)
- Utility function  $U(x) = -e^{-\gamma x}$  where  $\gamma > 0$  is coeff. of risk-aversion

# Hamilton-Jacobi-Bellman (HJB) Equation

- We denote the Optimal Value function as  $V^*(t, S_t, W_t, I_t)$

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[-e^{-\gamma \cdot (W_T + I_T \cdot S_T)}]$$

- $V^*(t, S_t, W_t, I_t)$  satisfies a recursive formulation for  $0 \leq t < t_1 < T$ :

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[V^*(t_1, S_{t_1}, W_{t_1}, I_{t_1})]$$

- Rewriting in stochastic differential form, we have the HJB Equation

$$\max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}[dV^*(t, S_t, W_t, I_t)] = 0 \text{ for } t < T$$

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma \cdot (W_T + I_T \cdot S_T)}$$

# Converting HJB to a Partial Differential Equation

- Change to  $V^*(t, S_t, W_t, I_t)$  is comprised of 3 components:
  - Due to pure movement in time  $t$
  - Due to randomness in TOB Mid-Price  $S_t$
  - Due to randomness in hitting/lifting the Bid/Ask
- With this, we can expand  $dV^*(t, S_t, W_t, I_t)$  and rewrite HJB as:

$$\begin{aligned} \max_{\delta_t^{(b)}, \delta_t^{(a)}} \{ & \frac{\partial V^*}{\partial t} dt + \mathbb{E} \left[ \sigma \frac{\partial V^*}{\partial S_t} dz_t + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} (dz_t)^2 \right] \\ & + \lambda_t^{(b)} \cdot dt \cdot V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) \\ & + \lambda_t^{(a)} \cdot dt \cdot V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) \\ & + (1 - \lambda_t^{(b)} \cdot dt - \lambda_t^{(a)} \cdot dt) \cdot V^*(t, S_t, W_t, I_t) \\ & - V^*(t, S_t, W_t, I_t) \} = 0 \end{aligned}$$

# Converting HJB to a Partial Differential Equation

We can simplify this equation with a few observations:

- $\mathbb{E}[dz_t] = 0$
- $\mathbb{E}[(dz_t)^2] = dt$
- Organize the terms involving  $\lambda_t^{(b)}$  and  $\lambda_t^{(a)}$  better with some algebra
- Divide throughout by  $dt$

$$\begin{aligned} \max_{\delta_t^{(b)}, \delta_t^{(a)}} \left\{ \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \right. \\ \left. + \lambda_t^{(b)} \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) - V^*(t, S_t, W_t, I_t)) \right. \\ \left. + \lambda_t^{(a)} \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) - V^*(t, S_t, W_t, I_t)) \right\} = 0 \end{aligned}$$

# Converting HJB to a Partial Differential Equation

Next, note that  $\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)})$  and  $\lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)})$ , and apply the max only on the relevant terms

$$\begin{aligned} & \frac{\partial V^*}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} \\ & + \max_{\delta_t^{(b)}} \{ f^{(b)}(\delta_t^{(b)}) \cdot (V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1) - V^*(t, S_t, W_t, I_t)) \} \\ & + \max_{\delta_t^{(a)}} \{ f^{(a)}(\delta_t^{(a)}) \cdot (V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1) - V^*(t, S_t, W_t, I_t)) \} = 0 \end{aligned}$$

This combines with the boundary condition:

$$V^*(T, S_T, W_T, I_T) = -e^{-\gamma \cdot (W_T + I_T \cdot S_T)}$$

# Converting HJB to a Partial Differential Equation

- We make an “educated guess” for the structure of  $V^*(t, S_t, W_t, I_t)$ :

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \quad (1)$$

and reduce the problem to a PDE in terms of  $\theta(t, S_t, I_t)$

- Substituting this into the above PDE for  $V^*(t, S_t, W_t, I_t)$  gives:

$$\begin{aligned} & \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) \\ & + \max_{\delta_t^{(b)}} \left\{ \frac{f^{(b)}(\delta_t^{(b)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(b)} - S_t + \theta(t, S_t, I_{t+1}) - \theta(t, S_t, I_t))} \right) \right\} \\ & + \max_{\delta_t^{(a)}} \left\{ \frac{f^{(a)}(\delta_t^{(a)})}{\gamma} \cdot \left( 1 - e^{-\gamma(\delta_t^{(a)} + S_t + \theta(t, S_t, I_{t-1}) - \theta(t, S_t, I_t))} \right) \right\} = 0 \end{aligned}$$

- The boundary condition is:

$$\theta(T, S_T, I_T) = I_T \cdot S_T$$

# Indifference Bid/Ask Price

- It turns out that  $\theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t)$  and  $\theta(t, S_t, I_t) - \theta(t, S_t, I_t - 1)$  are equal to financially meaningful quantities known as *Indifference Bid and Ask Prices*
- Indifference Bid Price  $Q^{(b)}(t, S_t, I_t)$  is defined as:

$$V^*(t, S_t, W_t - Q^{(b)}(t, S_t, I_t), I_t + 1) = V^*(t, S_t, W_t, I_t) \quad (2)$$

- $Q^{(b)}(t, S_t, I_t)$  is the price to buy a share with *guarantee of immediate purchase* that results in Optimum Expected Utility being unchanged
- Likewise, Indifference Ask Price  $Q^{(a)}(t, S_t, I_t)$  is defined as:

$$V^*(t, S_t, W_t + Q^{(a)}(t, S_t, I_t), I_t - 1) = V^*(t, S_t, W_t, I_t) \quad (3)$$

- $Q^{(a)}(t, S_t, I_t)$  is the price to sell a share with *guarantee of immediate sale* that results in Optimum Expected Utility being unchanged
- We abbreviate  $Q^{(b)}(t, S_t, I_t)$  as  $Q_t^{(b)}$  and  $Q^{(a)}(t, S_t, I_t)$  as  $Q_t^{(a)}$

# Indifference Bid/Ask Price in the PDE for $\theta$

- Express  $V^*(t, S_t, W_t - Q_t^{(b)}, I_t + 1) = V^*(t, S_t, W_t, I_t)$  in terms of  $\theta$ :

$$\begin{aligned} -e^{-\gamma(W_t - Q_t^{(b)} + \theta(t, S_t, I_t + 1))} &= -e^{-\gamma(W_t + \theta(t, S_t, I_t))} \\ \Rightarrow Q_t^{(b)} &= \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t) \end{aligned} \quad (4)$$

- Likewise for  $Q_t^{(a)}$ , we get:

$$Q_t^{(a)} = \theta(t, S_t, I_t) - \theta(t, S_t, I_t - 1) \quad (5)$$

- Using equations (4) and (5), bring  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in the PDE for  $\theta$

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \max_{\delta_t^{(b)}} g(\delta_t^{(b)}) + \max_{\delta_t^{(a)}} h(\delta_t^{(a)}) = 0$$

$$\text{where } g(\delta_t^{(b)}) = \frac{f^{(b)}(\delta_t^{(b)})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(b)} - S_t + Q_t^{(b)})})$$

$$\text{and } h(\delta_t^{(a)}) = \frac{f^{(a)}(\delta_t^{(a)})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(a)} + S_t - Q_t^{(a)})})$$



# Optimal Bid Spread and Optimal Ask Spread

- To maximize  $g(\delta_t^{(b)})$ , differentiate  $g$  with respect to  $\delta_t^{(b)}$  and set to 0

$$e^{-\gamma(\delta_t^{(b)*} - S_t + Q_t^{(b)})} \cdot (\gamma \cdot f^{(b)}(\delta_t^{(b)*}) - \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}(\delta_t^{(b)*})) + \frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}(\delta_t^{(b)*}) = 0$$
$$\Rightarrow \delta_t^{(b)*} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(b)}(\delta_t^{(b)*})}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}(\delta_t^{(b)*})} \right) \quad (6)$$

- To maximize  $g(\delta_t^{(a)})$ , differentiate  $h$  with respect to  $\delta_t^{(a)}$  and set to 0

$$e^{-\gamma(\delta_t^{(a)*} + S_t - Q_t^{(a)})} \cdot (\gamma \cdot f^{(a)}(\delta_t^{(a)*}) - \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}(\delta_t^{(a)*})) + \frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}(\delta_t^{(a)*}) = 0$$
$$\Rightarrow \delta_t^{(a)*} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(a)}(\delta_t^{(a)*})}{\frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}(\delta_t^{(a)*})} \right) \quad (7)$$

- (6) and (7) are implicit equations for  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  respectively

# Solving for $\theta$ and for Optimal Bid/Ask Spreads

- Let us write the PDE in terms of the Optimal Bid and Ask Spreads

$$\begin{aligned} & \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) \\ & + \frac{f^{(b)}(\delta_t^{(b)*})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(b)*} - S_t + \theta(t, S_t, I_t+1) - \theta(t, S_t, I_t))}) \\ & + \frac{f^{(a)}(\delta_t^{(a)*})}{\gamma} \cdot (1 - e^{-\gamma(\delta_t^{(a)*} + S_t + \theta(t, S_t, I_t-1) - \theta(t, S_t, I_t))}) = 0 \end{aligned} \quad (8)$$

with boundary condition  $\theta(T, S_T, I_T) = I_T \cdot S_T$

- First we solve PDE (8) for  $\theta$  in terms of  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- In general, this would be a numerical PDE solution
- Using (4) and (5), we have  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in terms of  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- Substitute above-obtained  $Q_t^{(b)}$  and  $Q_t^{(a)}$  in equations (6) and (7)
- Solve implicit equations for  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  (in general, numerically)

# Building Intuition

- Define *Indifference Mid Price*  $Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2}$
- To develop intuition for Indifference Prices, consider a simple case where the market-maker doesn't supply any bids or asks

$$V^*(t, S_t, W_t, I_t) = \mathbb{E}[-e^{-\gamma(W_t + I_t \cdot S_T)}]$$

- Combining this with the diffusion  $dS_t = \sigma \cdot dz_t$ , we get:

$$V^*(t, S_t, W_t, I_t) = -e^{-\gamma(W_t + I_t \cdot S_t - \frac{\gamma \cdot I_t^2 \cdot \sigma^2 (T-t)}{2})}$$

- Combining this with equations (2) and (3), we get:

$$Q_t^{(b)} = S_t - (2I_t + 1) \frac{\gamma \sigma^2 (T-t)}{2}, \quad Q_t^{(a)} = S_t - (2I_t - 1) \frac{\gamma \sigma^2 (T-t)}{2}$$

$$Q_t^{(m)} = S_t - I_t \gamma \sigma^2 (T-t), \quad Q_t^{(a)} - Q_t^{(b)} = \gamma \sigma^2 (T-t)$$

- These results for the simple case of no-market-making serve as approximations for our problem of optimal market-making

# Building Intuition

- Think of  $Q_t^{(m)}$  as *inventory-risk-adjusted* mid-price (adjustment to  $S_t$ )
- If market-maker is long inventory ( $I_t > 0$ ),  $Q_t^{(m)} < S_t$  indicating inclination to sell than buy, and if market-maker is short inventory,  $Q_t^{(m)} > S_t$  indicating inclination to buy than sell
- Armed with this intuition, we come back to optimal market-making, observing from eqns (6) and (7):  $P_t^{(b)*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)*}$
- Think of  $[P_t^{(b)*}, P_t^{(a)*}]$  as “centered” at  $Q_t^{(m)}$  (rather than at  $S_t$ ), i.e.,  $[P_t^{(b)*}, P_t^{(a)*}]$  will (together) move up/down in tandem with  $Q_t^{(m)}$  moving up/down (as a function of inventory position  $I_t$ )

$$Q_t^{(m)} - P_t^{(b)*} = \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(b)}(\delta_t^{(b)*})}{\frac{\partial f^{(b)}}{\partial \delta_t^{(b)}}(\delta_t^{(b)*})} \right) \quad (9)$$

$$P_t^{(a)*} - Q_t^{(m)} = \frac{Q_t^{(a)} - Q_t^{(b)}}{2} + \frac{1}{\gamma} \cdot \ln \left( 1 - \gamma \cdot \frac{f^{(a)}(\delta_t^{(a)*})}{\frac{\partial f^{(a)}}{\partial \delta_t^{(a)}}(\delta_t^{(a)*})} \right) \quad (10)$$

# Simple Functional Form for Hitting/Lifting Rate Means

- The PDE for  $\theta$  and the implicit equations for  $\delta_t^{(b)*}, \delta_t^{(a)*}$  are messy
- We make some assumptions, simplify, derive analytical approximations
- First we assume a fairly standard functional form for  $f^{(b)}$  and  $f^{(a)}$

$$f^{(b)}(\delta) = f^{(a)}(\delta) = c \cdot e^{-k \cdot \delta}$$

- This reduces equations (6) and (7) to:

$$\delta_t^{(b)*} = S_t - Q_t^{(b)} + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (11)$$

$$\delta_t^{(a)*} = Q_t^{(a)} - S_t + \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \quad (12)$$

$\Rightarrow P_t^{(b)*}$  and  $P_t^{(a)*}$  are equidistant from  $Q_t^{(m)}$

- Substituting these simplified  $\delta_t^{(b)*}, \delta_t^{(a)*}$  in (8) reduces the PDE to:

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( e^{-k \cdot \delta_t^{(b)*}} + e^{-k \cdot \delta_t^{(a)*}} \right) = 0 \quad (13)$$

with boundary condition  $\theta(T, S_T, I_T) = I_T \cdot S_T$

# Simplifying the PDE with Approximations

- Note that this PDE (13) involves  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$
- However, equations (11), (12), (4), (5) enable expressing  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  in terms of  $\theta(t, S_t, I_t - 1), \theta(t, S_t, I_t), \theta(t, S_t, I_t + 1)$
- This would give us a PDE just in terms of  $\theta$
- Solving that PDE for  $\theta$  would not only give us  $V^*(t, S_t, W_t, I_t)$  but also  $\delta_t^{(b)*}$  and  $\delta_t^{(a)*}$  (using equations (11), (12), (4), (5))
- To solve the PDE, we need to make a couple of approximations
- First we make a linear approx for  $e^{-k \cdot \delta_t^{(b)*}}$  and  $e^{-k \cdot \delta_t^{(a)*}}$  in PDE (13):

$$\frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} (1 - k \cdot \delta_t^{(b)*} + 1 - k \cdot \delta_t^{(a)*}) = 0 \quad (14)$$

- Equations (11), (12), (4), (5) tell us that:

$$\delta_t^{(b)*} + \delta_t^{(a)*} = \frac{2}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + 2\theta(t, S_t, I_t) - \theta(t, S_t, I_t + 1) - \theta(t, S_t, I_t - 1)$$

# Asymptotic Expansion of $\theta$ in $l_t$

- With this expression for  $\delta_t^{(b)*} + \delta_t^{(a)*}$ , PDE (14) takes the form:

$$\begin{aligned} \frac{\partial \theta}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 \theta}{\partial S_t^2} - \gamma \left( \frac{\partial \theta}{\partial S_t} \right)^2 \right) + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right. \\ \left. - k(2\theta(t, S_t, l_t) - \theta(t, S_t, l_t + 1) - \theta(t, S_t, l_t - 1)) \right) = 0 \end{aligned} \quad (15)$$

- To solve PDE (15), we consider this asymptotic expansion of  $\theta$  in  $l_t$ :

$$\theta(t, S_t, l_t) = \sum_{n=0}^{\infty} \frac{l_t^n}{n!} \cdot \theta^{(n)}(t, S_t)$$

- So we need to determine the functions  $\theta^{(n)}(t, S_t)$  for all  $n = 0, 1, 2, \dots$
- For tractability, we approximate this expansion to the first 3 terms:

$$\theta(t, S_t, l_t) \approx \theta^{(0)}(t, S_t) + l_t \cdot \theta^{(1)}(t, S_t) + \frac{l_t^2}{2} \cdot \theta^{(2)}(t, S_t)$$

# Approximation of the Expansion of $\theta$ in $I_t$

- We note that the Optimal Value Function  $V^*$  can depend on  $S_t$  only through the current *Value of the Inventory* (i.e., through  $I_t \cdot S_t$ ), i.e., it cannot depend on  $S_t$  in any other way
- This means  $V^*(t, S_t, W_t, 0) = -e^{-\gamma(W_t + \theta^{(0)}(t, S_t))}$  is independent of  $S_t$
- This means  $\theta^{(0)}(t, S_t)$  is independent of  $S_t$
- So, we can write it as simply  $\theta^{(0)}(t)$ , meaning  $\frac{\partial \theta^{(0)}}{\partial S_t}$  and  $\frac{\partial^2 \theta^{(0)}}{\partial S_t^2}$  are 0
- Therefore, we can write the approximate expansion for  $\theta(t, S_t, I_t)$  as:

$$\theta(t, S_t, I_t) = \theta^{(0)}(t) + I_t \cdot \theta^{(1)}(t, S_t) + \frac{I_t^2}{2} \cdot \theta^{(2)}(t, S_t) \quad (16)$$



# Solving the PDE

- Substitute this approximation (16) for  $\theta(t, S_t, I_t)$  in PDE (15)

$$\begin{aligned} & \frac{\partial \theta^{(0)}}{\partial t} + I_t \frac{\partial \theta^{(1)}}{\partial t} + \frac{I_t^2}{2} \frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \left( I_t \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} + \frac{I_t^2}{2} \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} \right) \\ & - \frac{\gamma \sigma^2}{2} \left( I_t \frac{\partial \theta^{(1)}}{\partial S_t} + \frac{I_t^2}{2} \frac{\partial \theta^{(2)}}{\partial S_t} \right)^2 + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0 \end{aligned}$$

with boundary condition:

$$\theta^{(0)}(T) + I_T \cdot \theta^{(1)}(T, S_T) + \frac{I_T^2}{2} \cdot \theta^{(2)}(T, S_T) = I_T \cdot S_T \quad (17)$$

- We will separately collect terms involving specific powers of  $I_t$ , each yielding a separate PDE:
  - Terms devoid of  $I_t$  (i.e.,  $I_t^0$ )
  - Terms involving  $I_t$  (i.e.,  $I_t^1$ )
  - Terms involving  $I_t^2$

# Solving the PDE

- We start by collecting terms involving  $I_t$

$$\frac{\partial \theta^{(1)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(1)}}{\partial S_t^2} = 0 \text{ with boundary condition } \theta^{(1)}(T, S_T) = S_T$$

- The solution to this PDE is:

$$\theta^{(1)}(t, S_t) = S_t \quad (18)$$

- Next, we collect terms involving  $I_t^2$

$$\frac{\partial \theta^{(2)}}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 \theta^{(2)}}{\partial S_t^2} - \gamma \sigma^2 \cdot \left( \frac{\partial \theta^{(1)}}{\partial S_t} \right)^2 = 0 \text{ with boundary } \theta^{(2)}(T, S_T) = 0$$

- Noting that  $\theta^{(1)}(t, S_t) = S_t$ , we solve this PDE as:

$$\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t) \quad (19)$$

# Solving the PDE

- Finally, we collect the terms devoid of  $I_t$

$$\frac{\partial \theta^{(0)}}{\partial t} + \frac{c}{k + \gamma} \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) + k \cdot \theta^{(2)} \right) = 0 \text{ with boundary } \theta^{(0)}(T) = 0$$

- Noting that  $\theta^{(2)}(t, S_t) = -\gamma \sigma^2 (T - t)$ , we solve as:

$$\theta^{(0)}(t) = \frac{c}{k + \gamma} \left( \left( 2 - \frac{2k}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right) \right) (T - t) - \frac{k \gamma \sigma^2}{2} (T - t)^2 \right) \quad (20)$$

- This completes the PDE solution for  $\theta(t, S_t, I_t)$  and hence, for  $V^*(t, S_t, W_t, I_t)$
- Lastly, we derive formulas for  $Q_t^{(b)}, Q_t^{(a)}, Q_t^{(m)}, \delta_t^{(b)*}, \delta_t^{(a)*}$

# Formulas for Prices and Spreads

- Using equations (4) and (5), we get:

$$Q_t^{(b)} = \theta^{(1)}(t, S_t) + \underbrace{(2l_t+1)}_2 \cdot \theta^{(2)}(t, S_t) = S_t - (2l_t+1) \frac{\gamma \sigma^2 (T-t)}{2} \quad (21)$$

$$Q_t^{(a)} = \theta^{(1)}(t, S_t) + \underbrace{(2l_t-1)}_2 \cdot \theta^{(2)}(t, S_t) = S_t - (2l_t-1) \frac{\gamma \sigma^2 (T-t)}{2} \quad (22)$$

- Using equations (11) and (12), we get:

$$\delta_t^{(b)*} = \frac{(2l_t+1)\gamma\sigma^2(T-t)}{2} + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right) \quad (23)$$

$$\delta_t^{(a)*} = \frac{(1-2l_t)\gamma\sigma^2(T-t)}{2} + \frac{1}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right) \quad (24)$$

$$\text{Optimal Bid-Ask Spread } \delta_t^{(b)*} + \delta_t^{(a)*} = \gamma\sigma^2(T-t) + \frac{2}{\gamma} \ln\left(1 + \frac{\gamma}{k}\right) \quad (25)$$

$$\text{Optimal "Mid" } Q_t^{(m)} = \frac{Q_t^{(b)} + Q_t^{(a)}}{2} = \frac{P_t^{(b)*} + P_t^{(a)*}}{2} = S_t - l_t \gamma \sigma^2 (T-t) \quad (26)$$

# Back to Intuition

- Think of  $Q_t^{(m)}$  as *inventory-risk-adjusted* mid-price (adjustment to  $S_t$ )
- If market-maker is long inventory ( $I_t > 0$ ),  $Q_t^{(m)} < S_t$  indicating inclination to sell than buy, and if market-maker is short inventory,  $Q_t^{(m)} > S_t$  indicating inclination to buy than sell
- Think of  $[P_t^{(b)*}, P_t^{(a)*}]$  as “centered” at  $Q_t^{(m)}$  (rather than at  $S_t$ ), i.e.,  $[P_t^{(b)*}, P_t^{(a)*}]$  will (together) move up/down in tandem with  $Q_t^{(m)}$  moving up/down (as a function of inventory position  $I_t$ )
- Note from equation (25) that the Optimal Bid-Ask Spread  $P_t^{(a)*} - P_t^{(b)*}$  is independent of inventory  $I_t$
- Useful view:  $P_t^{(b)*} < Q_t^{(b)} < Q_t^{(m)} < Q_t^{(a)} < P_t^{(a)*}$ , with these spreads:

$$\text{Outer Spreads } P_t^{(a)*} - Q_t^{(a)} = Q_t^{(b)} - P_t^{(b)*} = \frac{1}{\gamma} \ln \left( 1 + \frac{\gamma}{k} \right)$$

$$\text{Inner Spreads } Q_t^{(a)} - Q_t^{(m)} = Q_t^{(m)} - Q_t^{(b)} = \frac{\gamma \sigma^2 (T - t)}{2}$$

# Real-world Market-Making and Reinforcement Learning

- Real-world TOB dynamics are non-stationary, non-linear, complex
- Frictions: Discrete Prices/Sizes, Constraints on Prices/Sizes, Fees
- Need to capture various market factors in the *State* & TOB Dynamics
- This leads to Curse of Dimensionality and Curse of Modeling
- The practical route is to develop a simulator capturing all of the above
- Simulator is a *Market-Data-learned Sampling Model* of TOB Dynamics
- Using this simulator and neural-networks func approx, we can do RL
- References: [2018 Paper from University of Liverpool](#) and [2019 Paper from JP Morgan Research](#)
- Exciting area for Future Research as well as Engineering Design