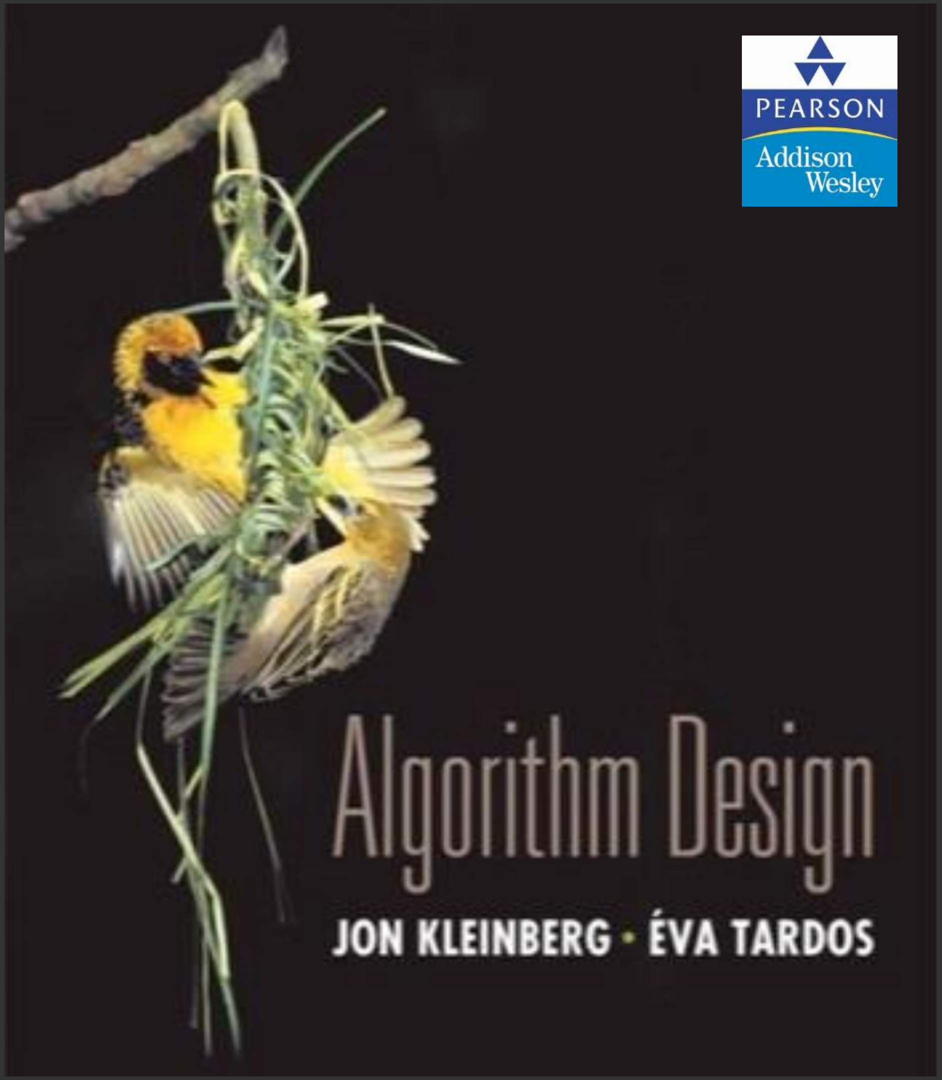


5. DIVIDE AND CONQUER I

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Divide-and-conquer paradigm

Divide-and-conquer.

- Divide up problem into several subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems into overall solution.

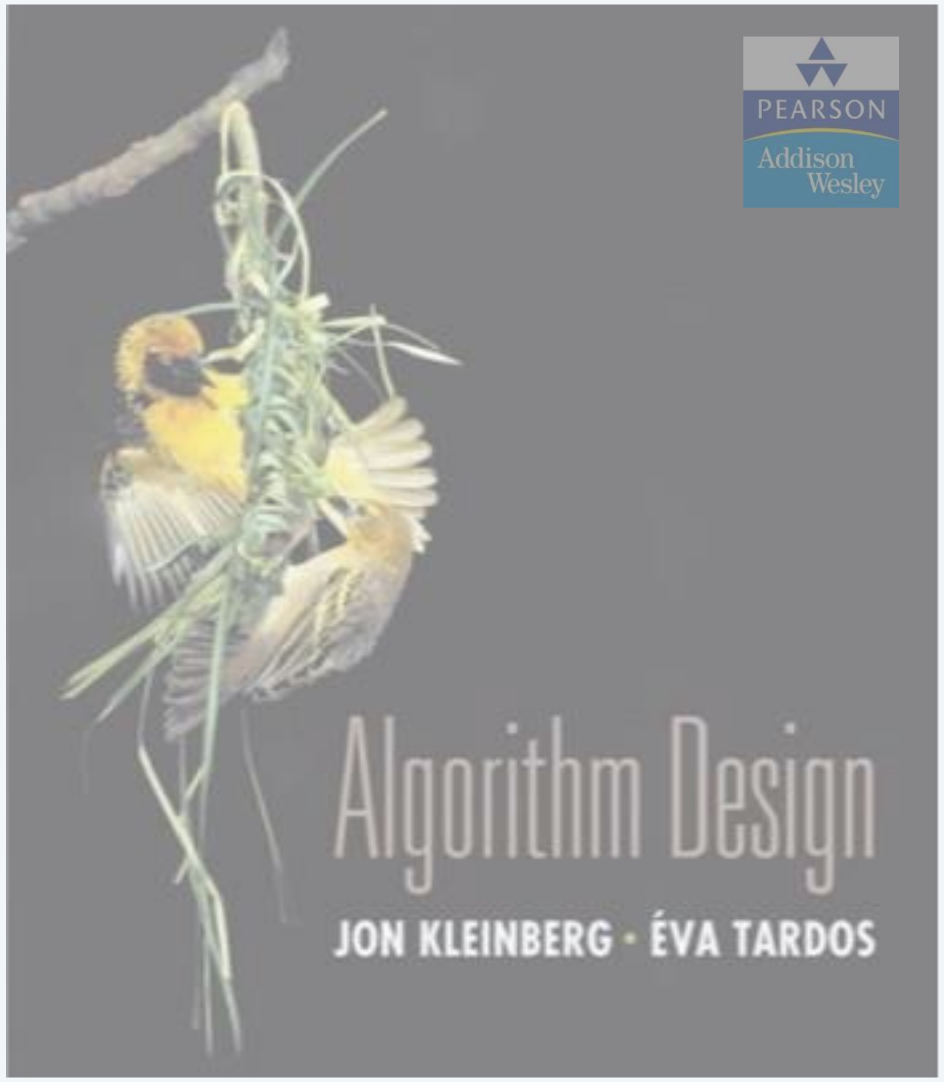
Most common usage.

- Divide problem of size n into two subproblems of size $n/2$ in linear time.
- Solve two subproblems recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $\Theta(n \log n)$.

attributed to Julius Caesar

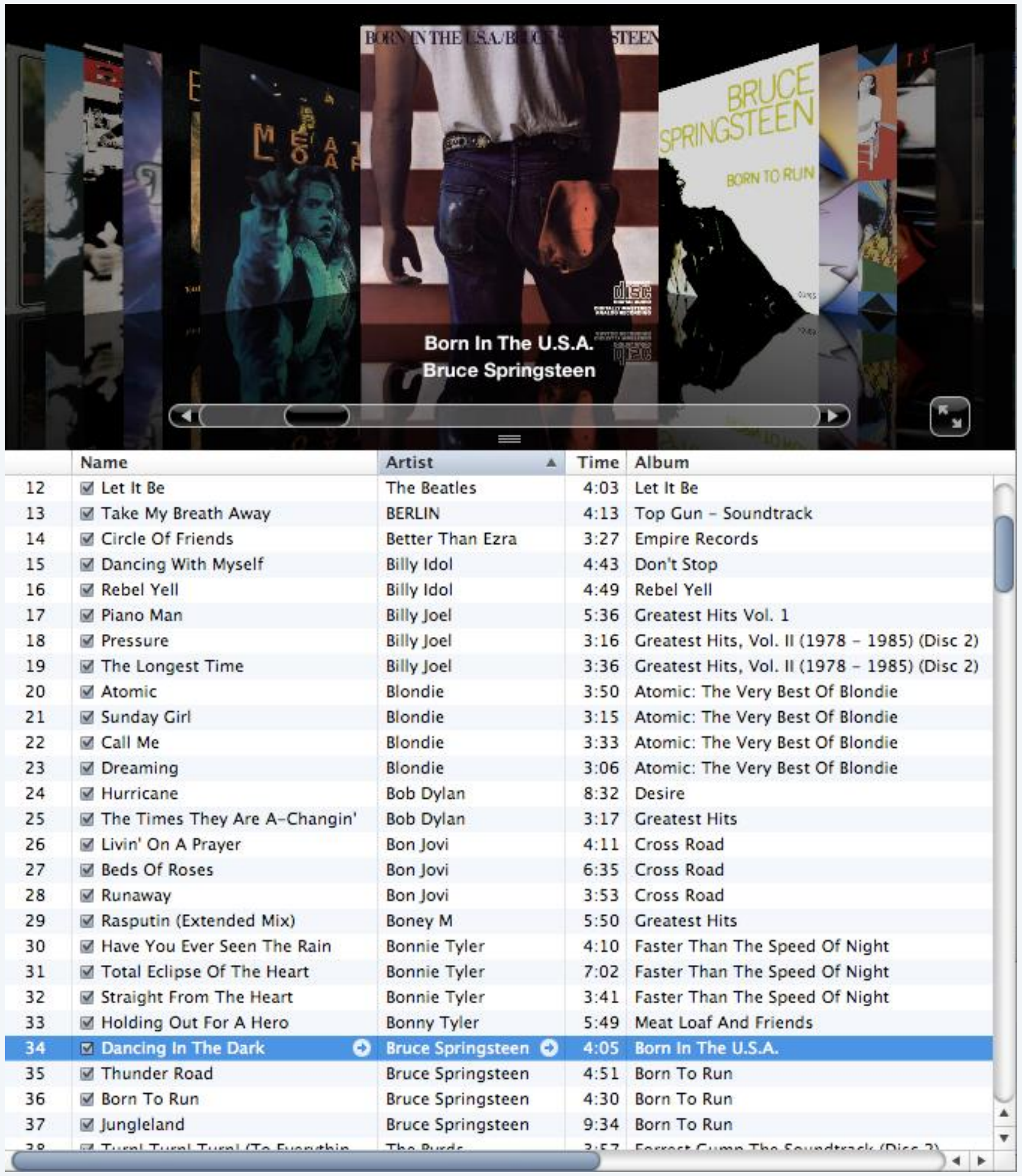


5. Divide and Conquer

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Sorting problem

Problem. Given a list of n elements from a totally-ordered universe, rearrange them in ascending order.



	Name	Artist	Time	Album
12	Let It Be	The Beatles	4:03	Let It Be
13	Take My Breath Away	BERLIN	4:13	Top Gun – Soundtrack
14	Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	Dancing With Myself	Billy Idol	4:43	Don't Stop
16	Rebel Yell	Billy Idol	4:49	Rebel Yell
17	Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
19	The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 – 1985) (Disc 2)
20	Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	Hurricane	Bob Dylan	8:32	Desire
25	The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	Beds Of Roses	Bon Jovi	6:35	Cross Road
28	Runaway	Bon Jovi	3:53	Cross Road
29	Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	Thunder Road	Bruce Springsteen	4:51	Born To Run
36	Born To Run	Bruce Springsteen	4:30	Born To Run
37	Jungleland	Bruce Springsteen	9:34	Born To Run
38	Turn! Turn! Turn! (To Everythin	The Byrds	3:57	Forrest Gump The Soundtrack (Disc 2)

Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

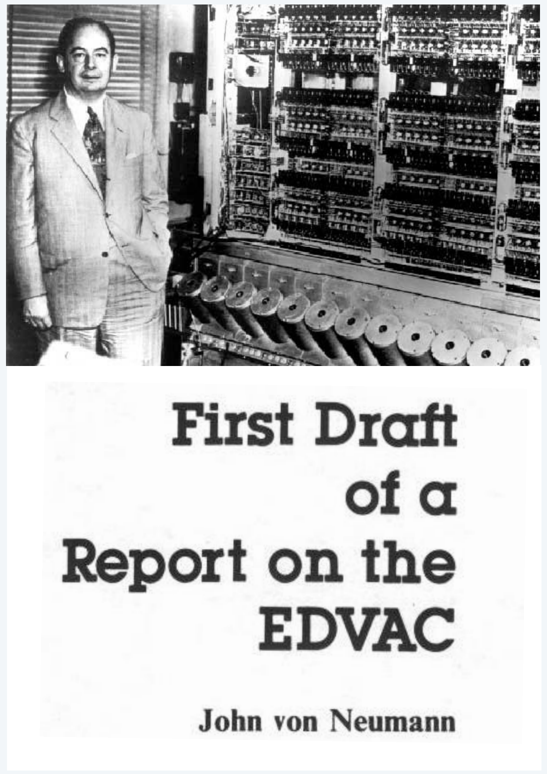
- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.



input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

sort right half

A	G	L	O	R	H	I	M	S	T

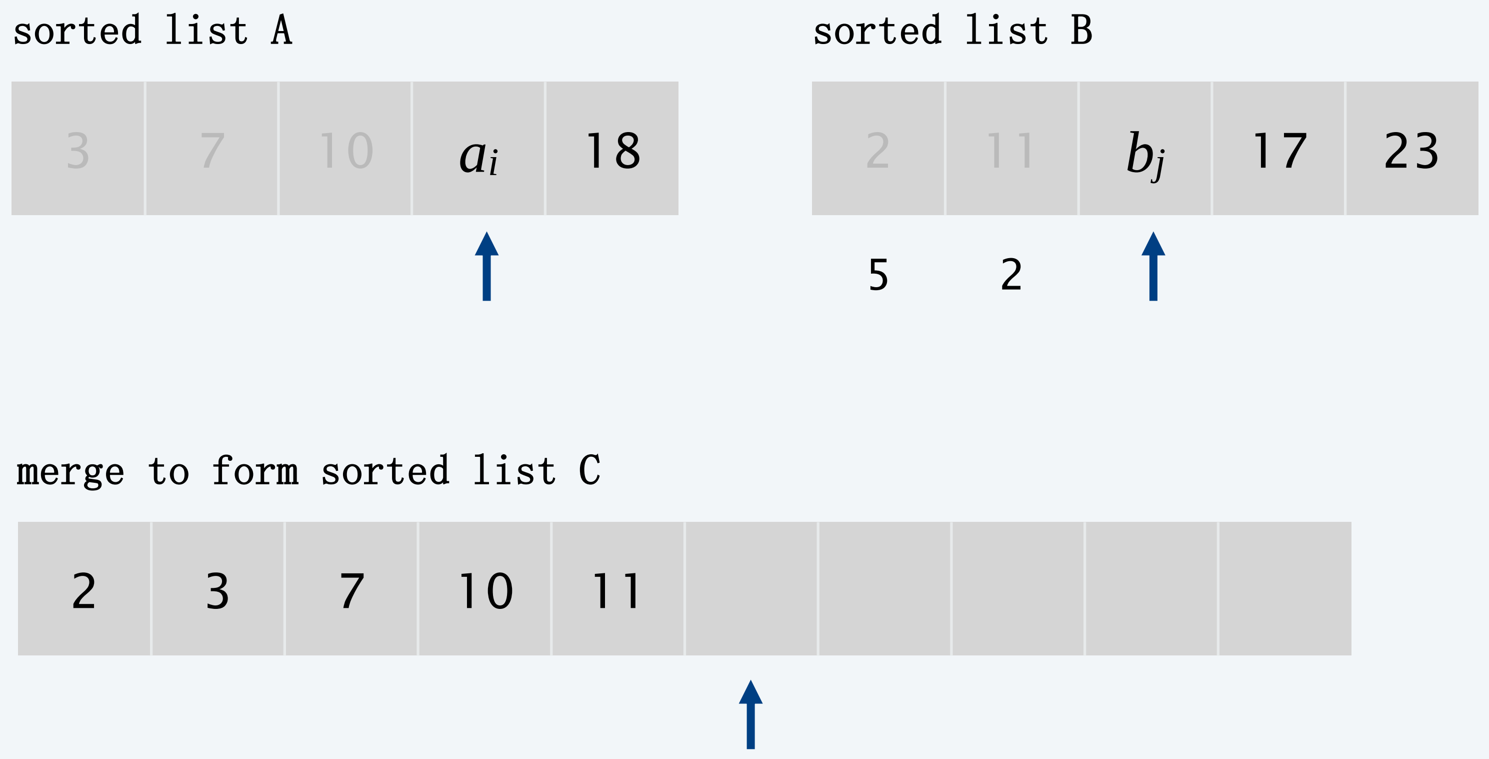
merge results

A	G	H	I	L	M	O	R	S	T

Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).



A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.

Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence.
Initially we assume n is a power of 2 and replace $\leq$ with $=$.

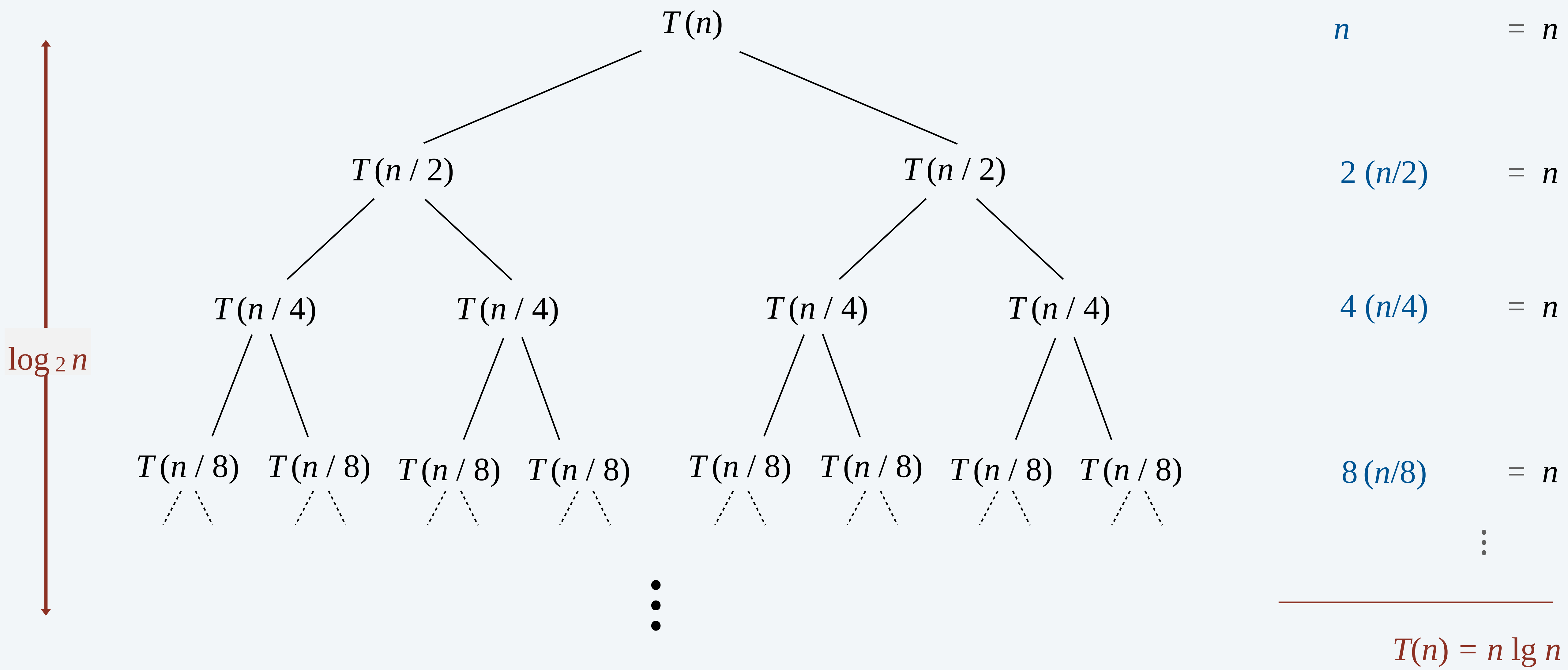
Divide-and-conquer recurrence: proof by recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 1.



Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad ▪ \end{aligned}$$

Analysis of mergesort recurrence

Claim. If $T(n)$ satisfies the following recurrence, then $T(n) \le n \lceil \log_2 n \rceil$.

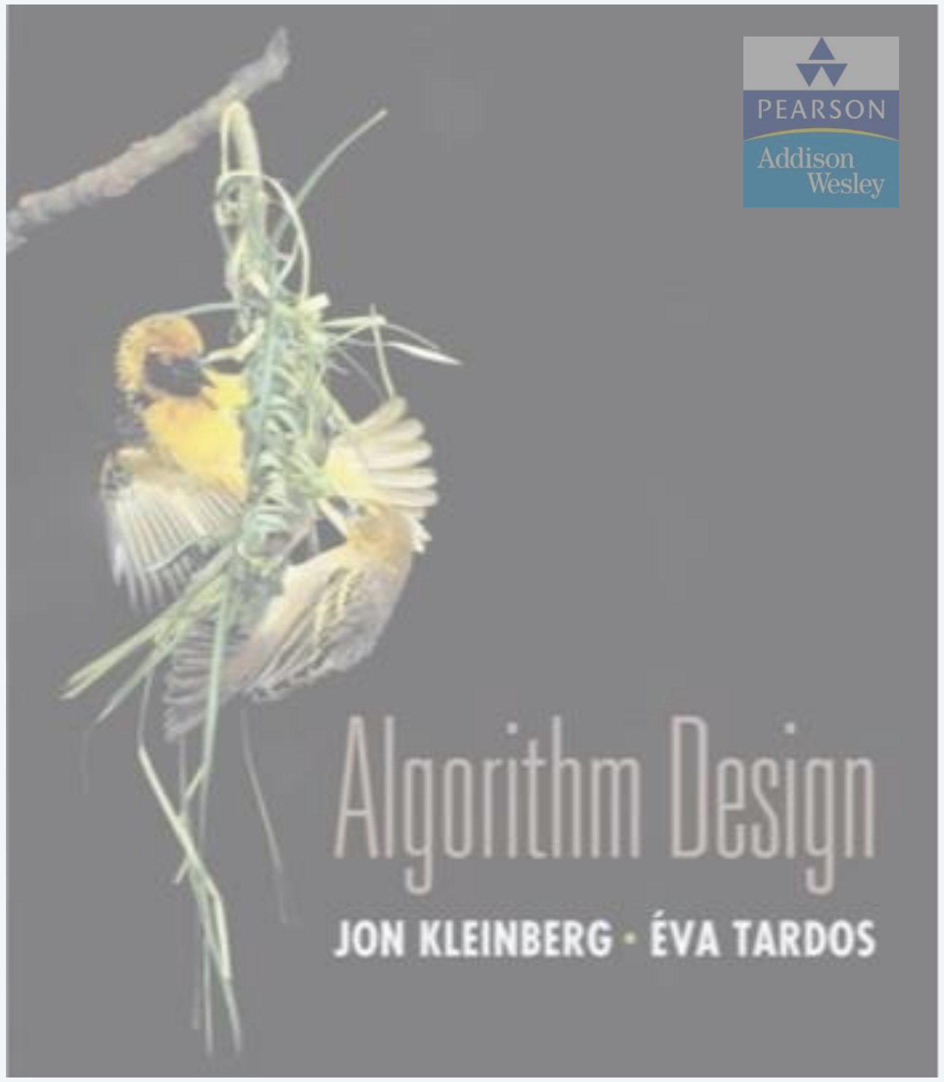
$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.
- Induction step: assume true for $1, 2, \ldots, n-1$.

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \end{aligned}$$

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\le n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \longleftarrow \log_2 n_2 \le \lceil \log_2 n \rceil - 1 \\ &\le n (\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil. \quad \blacksquare \end{aligned}$$



5. Divide and Conquer

- *mergesort*
- *counting inversions*
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- *median and selection*

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, \ldots, n$.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

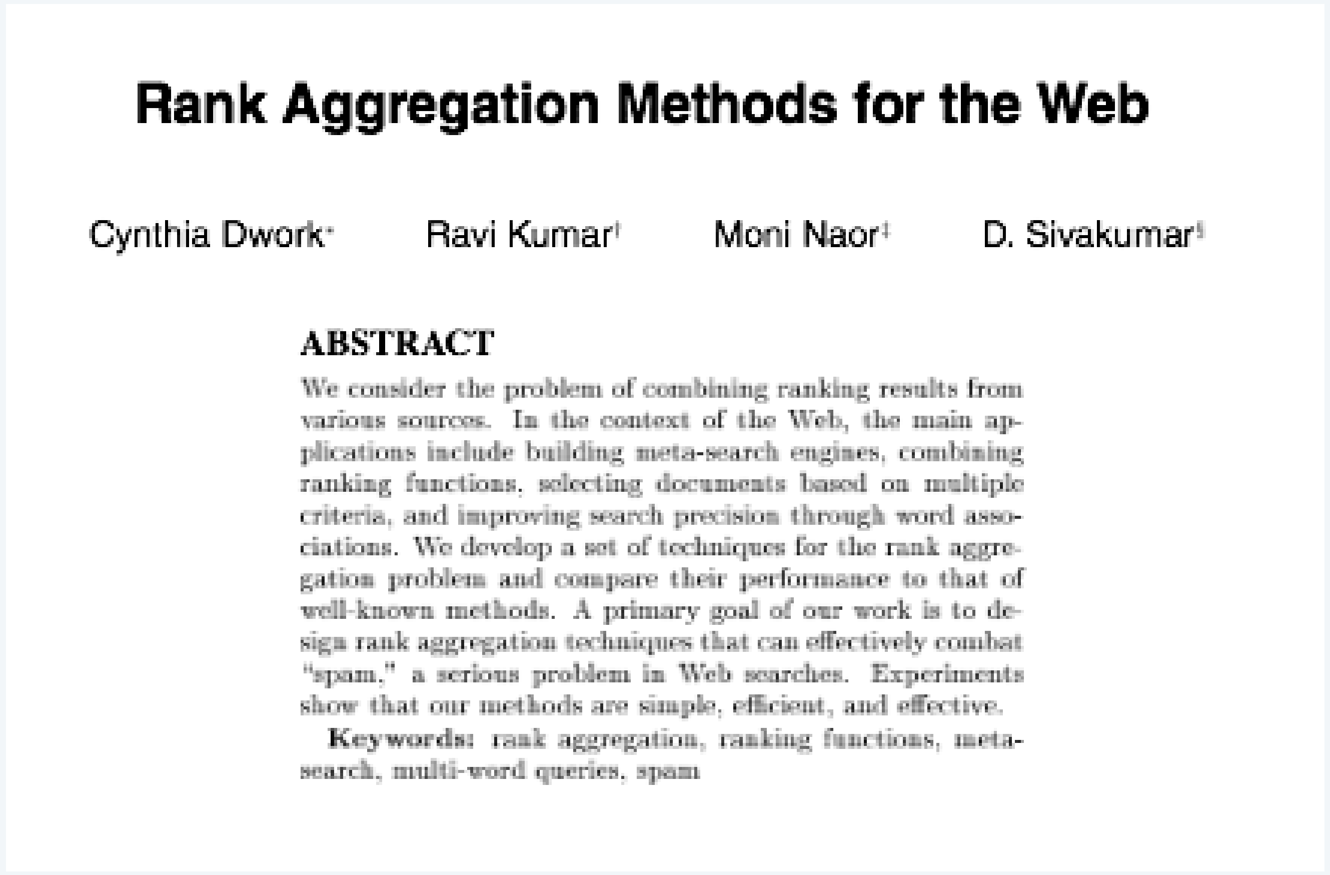
	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's tau distance).



Rank Aggregation Methods for the Web

Cynthia Dwork[*] Ravi Kumar[†] Moni Naor[‡] D. Sivakumar[§]

ABSTRACT

We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat "spam," a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10

5-4

count inversions in right half B

2	6	9	3	7

6-3 9-3 9-7

count inversions (a, b) with a ∈ A and b ∈ B

1	5	4	8	10

2	6	9	3	7

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10-3 10-6 10-7 10-9

output 1 + 3 + 13 = 17

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element $b \in B$,
  - binary search in A to find how elements in A are greater than b.

list A

7	10	18	3	14

list B

17	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	17	23

binary search to count inversions (a, b) with a ∈ A and b ∈ B

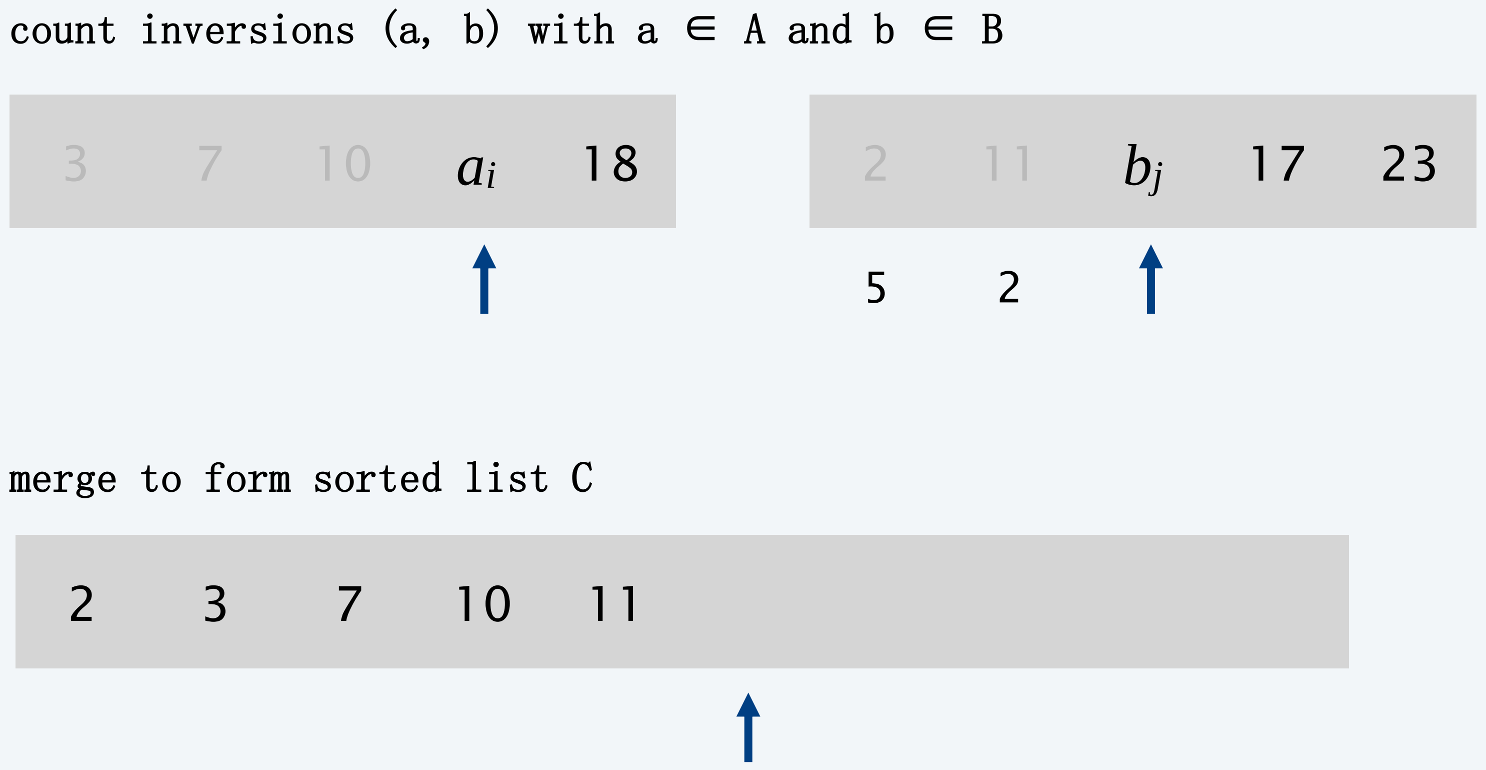
3	7	10	14	18

2	11	16	17	23
5	2	1	1	0

Counting inversions: how to combine two subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.


count inversions (a, b) with a ∈ A and b ∈ B
3 7 10 a_i 18
2 11 b_j 17 23
5 2
merge to form sorted list C
2 3 7 10 11

Counting inversions: divide-and-conquer algorithm implementation

Input. List L.

Output. Number of inversions in L and sorted list of elements L'.

```
SORT-AND-COUNT ($L$)

IF list $L$ has one element
    RETURN $(0, L)$.

DIVIDE the list into two halves $A$ and $B$.
$(r_A, A) \leftarrow$ SORT-AND-COUNT($A$).
$(r_B, B) \leftarrow$ SORT-AND-COUNT($B$).
$(r_{AB}, L') \leftarrow$ MERGE-AND-COUNT($A$, $B$).

RETURN $(r_A + r_B + r_{AB}, L')$.
```

Counting inversions: divide-and-conquer algorithm analysis

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

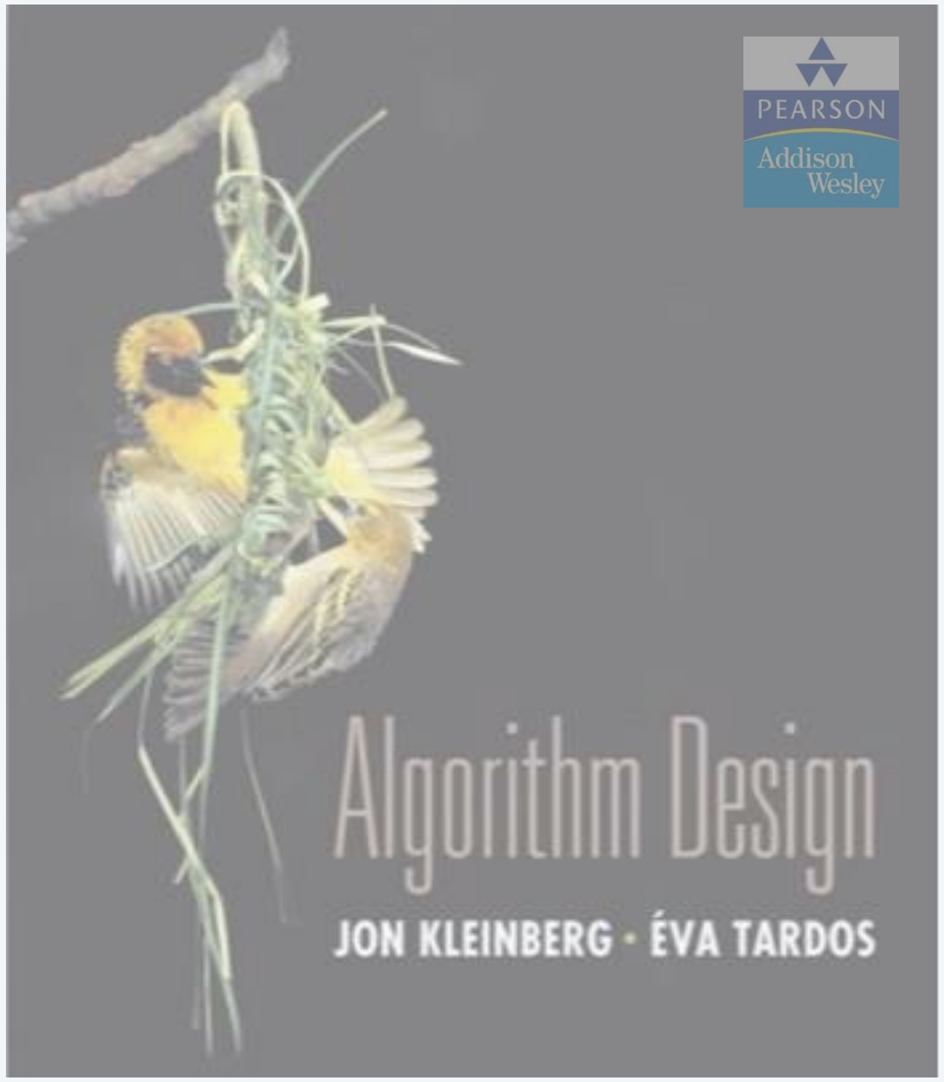
Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Divide the list into two halves

A contains the first $\lceil n/2 \rceil$ elements

B contains the remaining $\lfloor n/2 \rfloor$ elements



5. Divide and Conquer

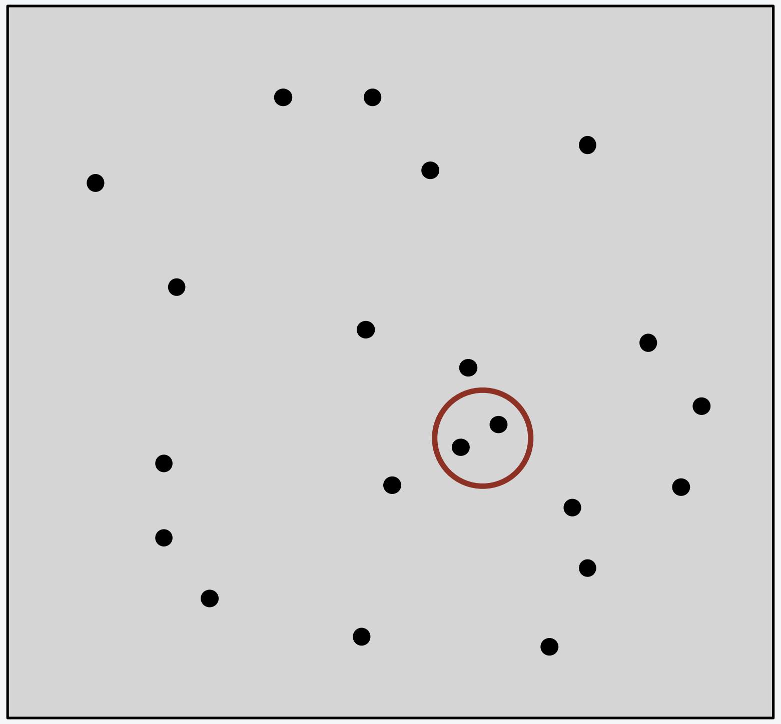
- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- $\underbrace{\text{Special case of nearest neighbor, Euclidean MST, Voronoi.}}_{\text{fast closest pair inspired fast algorithms for these problems}}$



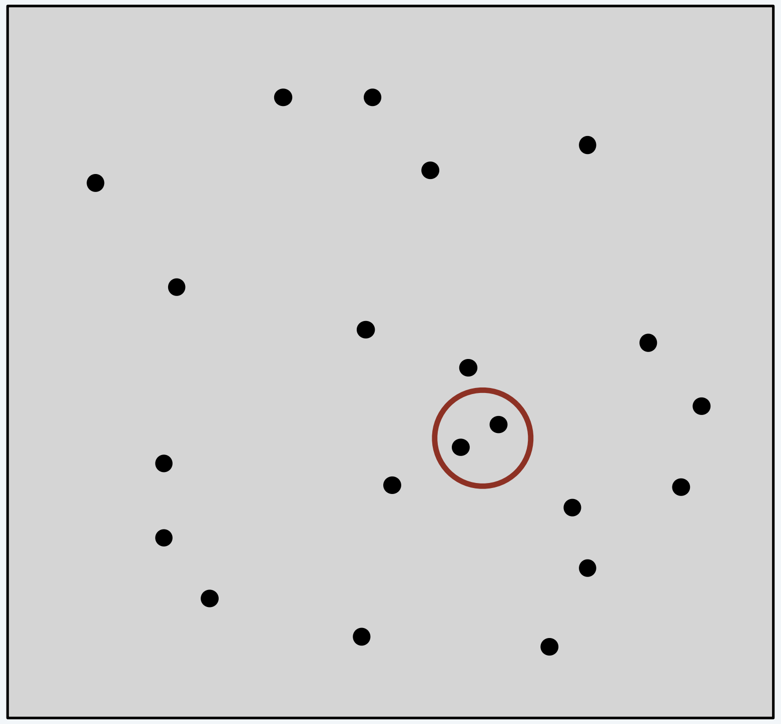
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1d version. Easy $O(n \log n)$ algorithm if points are on a line.

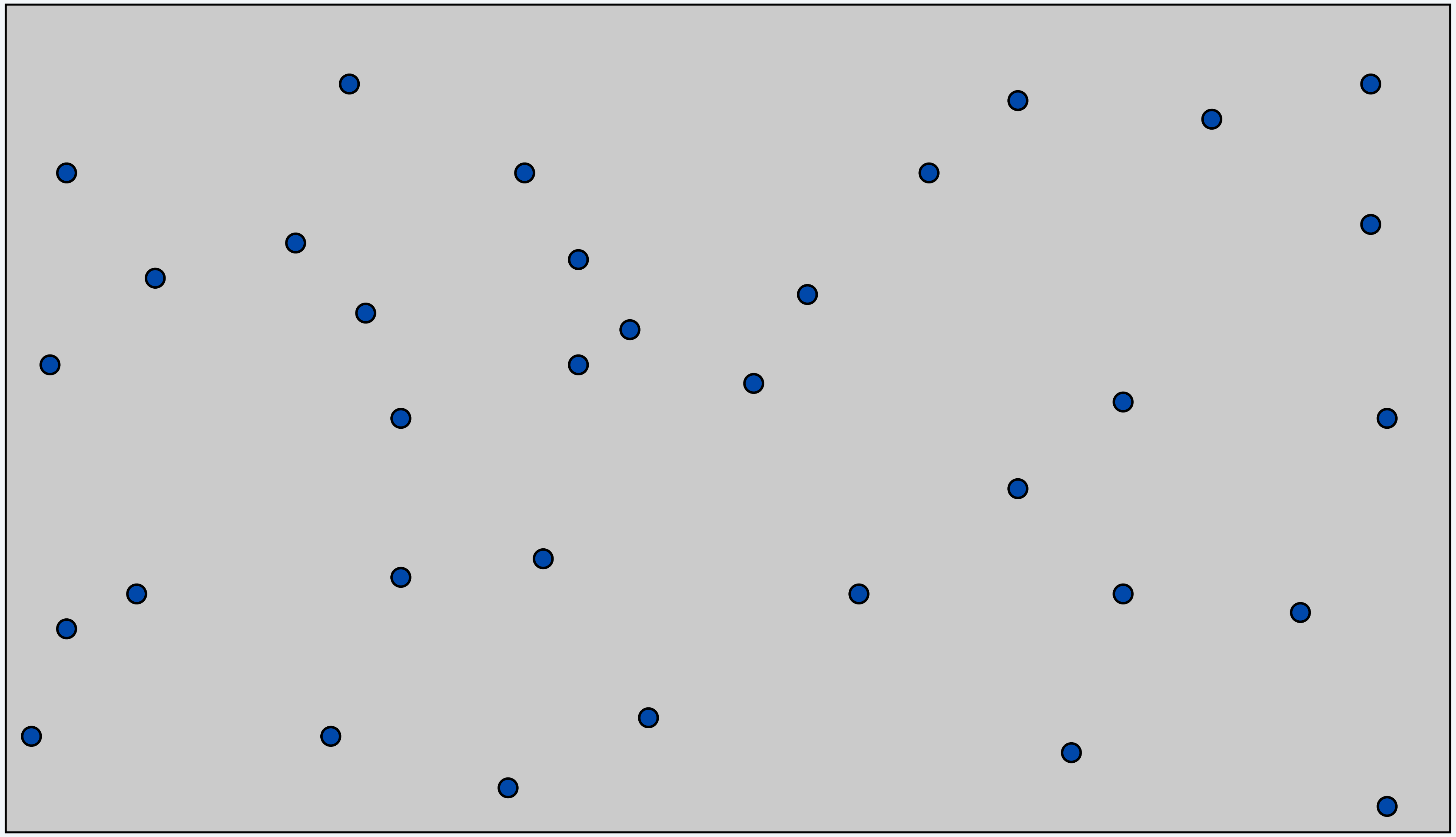
Nondegeneracy assumption. No two points have the same x-coordinate.



Closest pair of points: first attempt

Sorting solution.

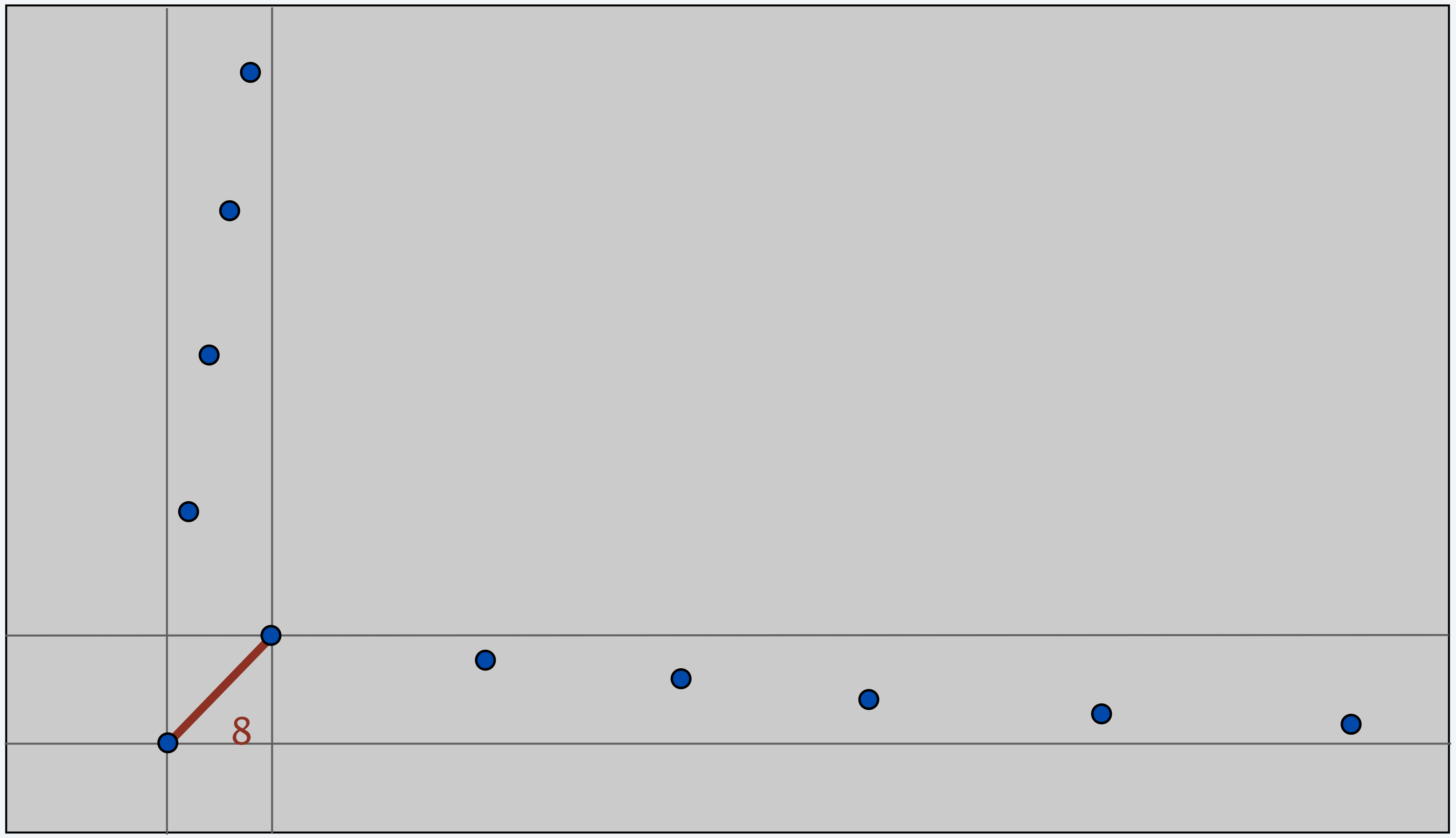
- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: first attempt

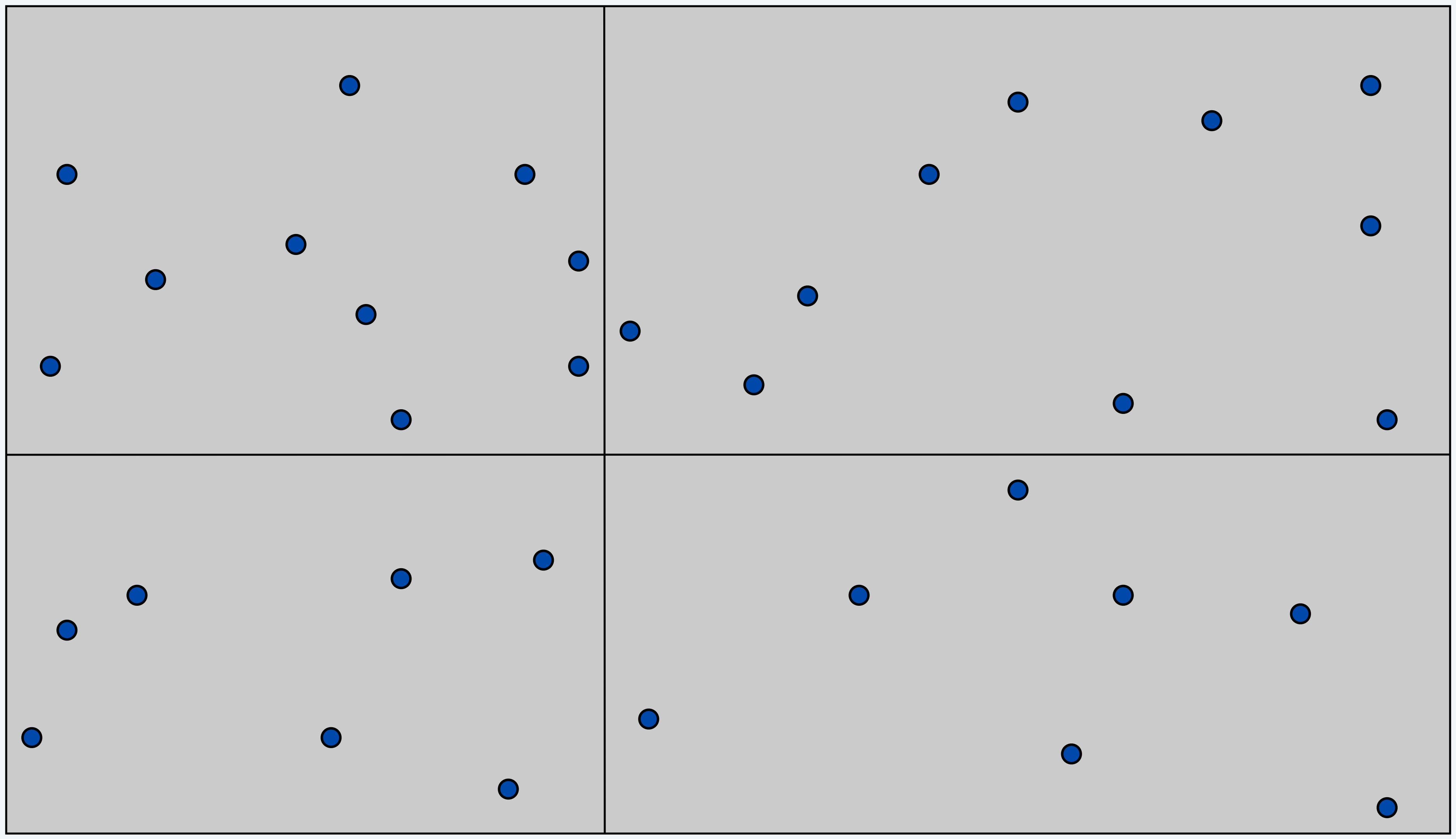
Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: second attempt

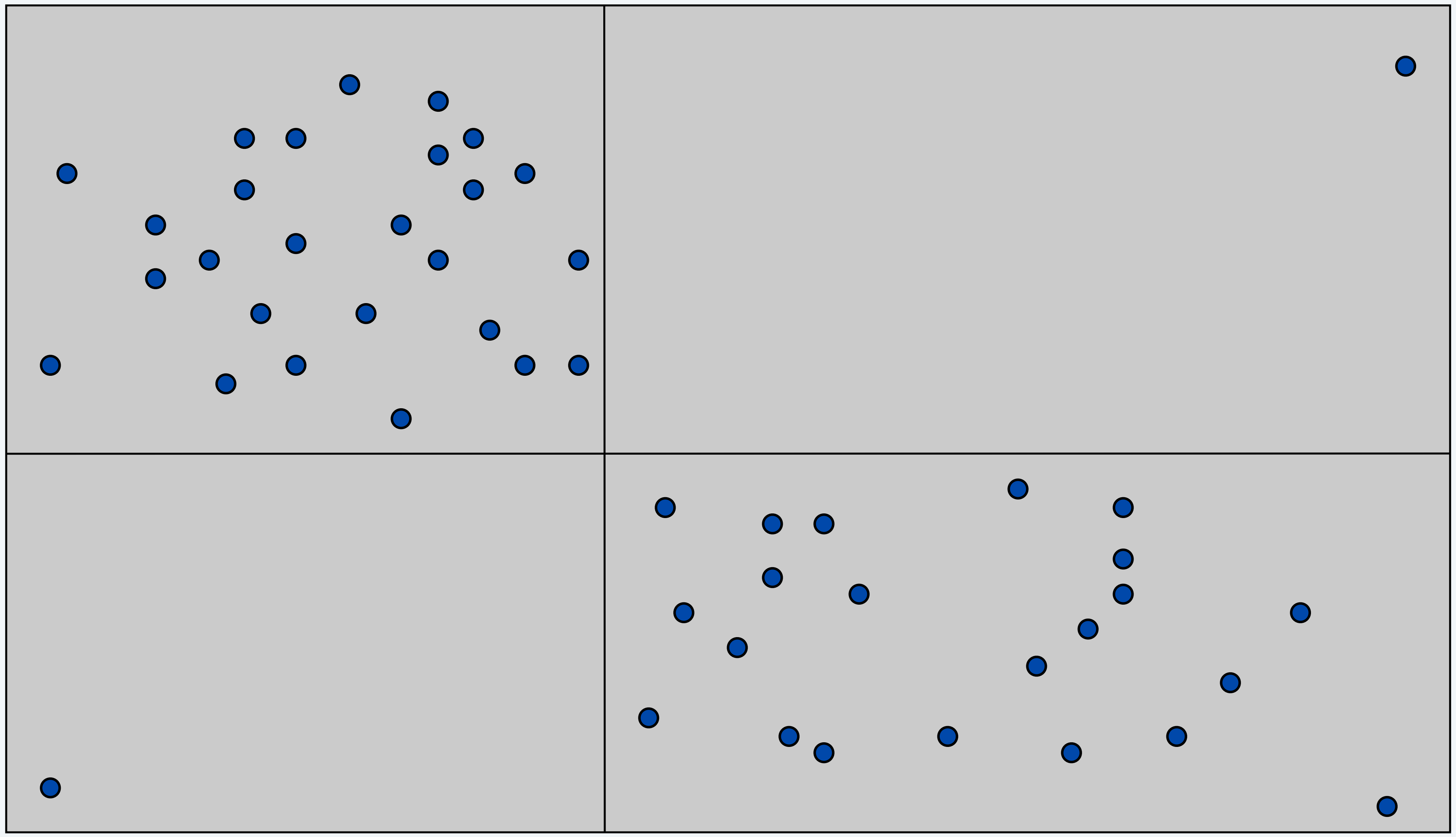
Divide. Subdivide region into 4 quadrants.



Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

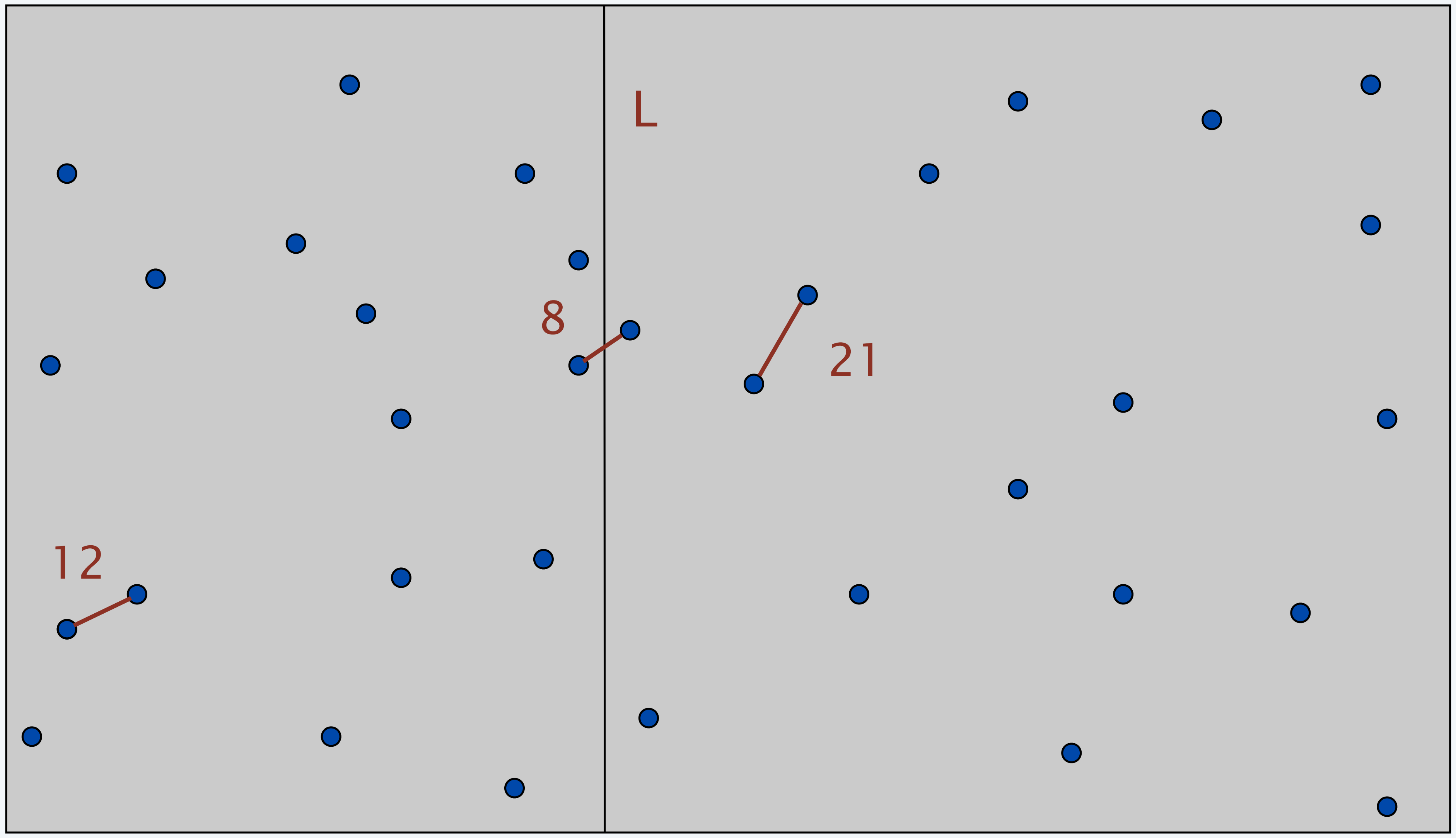
Obstacle. Impossible to ensure $n/4$ points in each piece.



Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

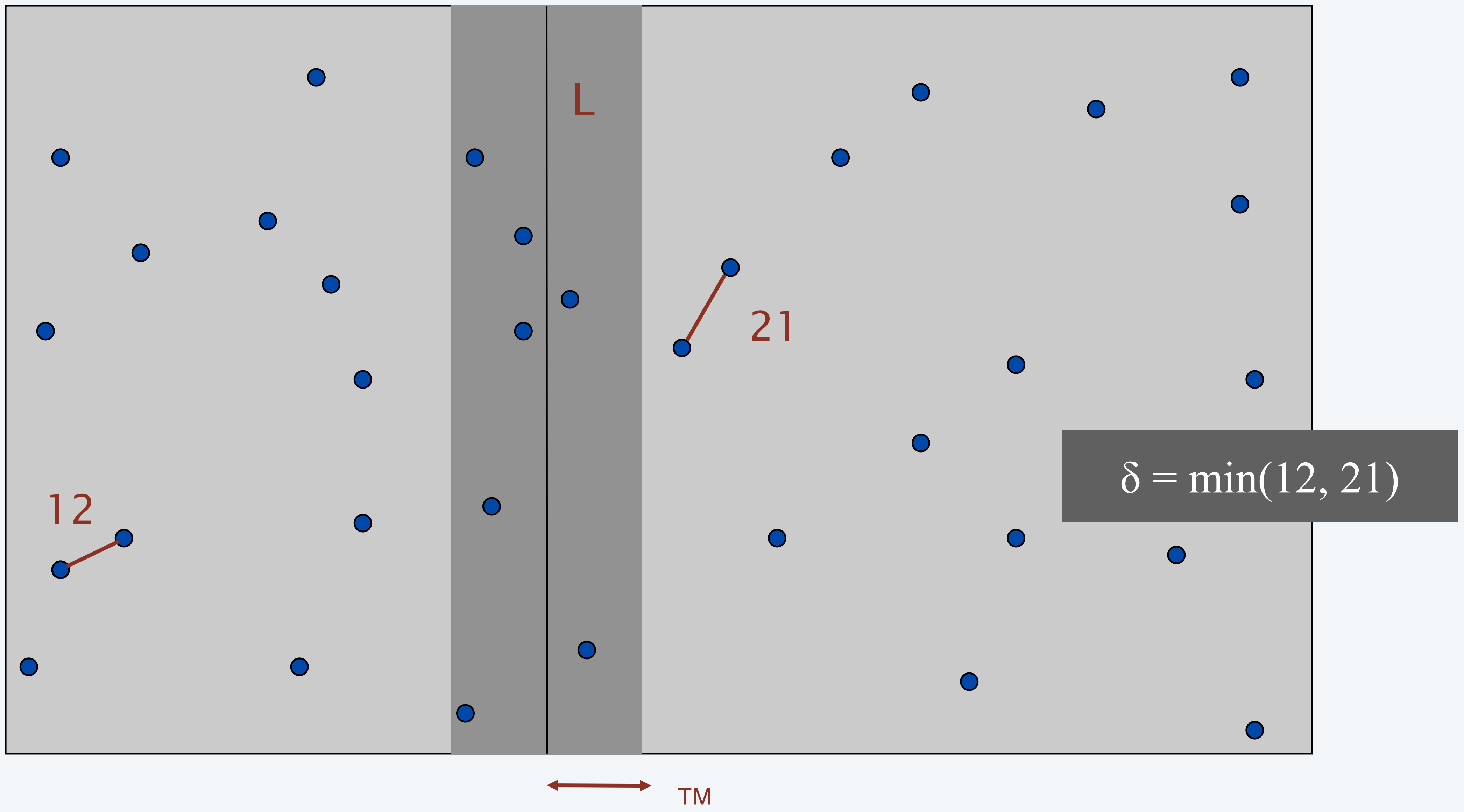
seems like $\Theta(N^2)$



How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.

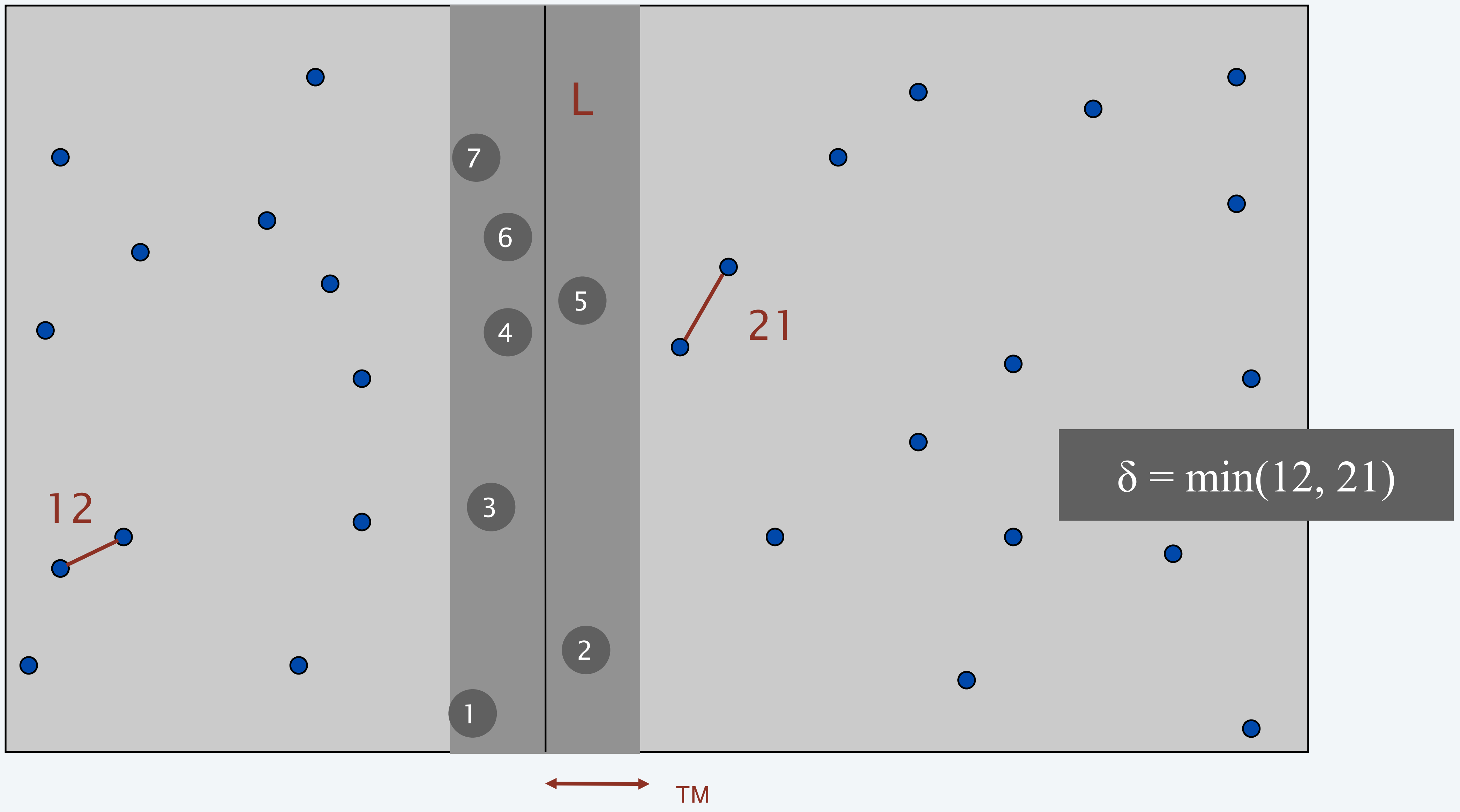


How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < ™.

- Observation: only need to consider points within ™ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Only check distances of those within 11 positions in sorted list!

why 11?



How to find closest pair with one point in each side?

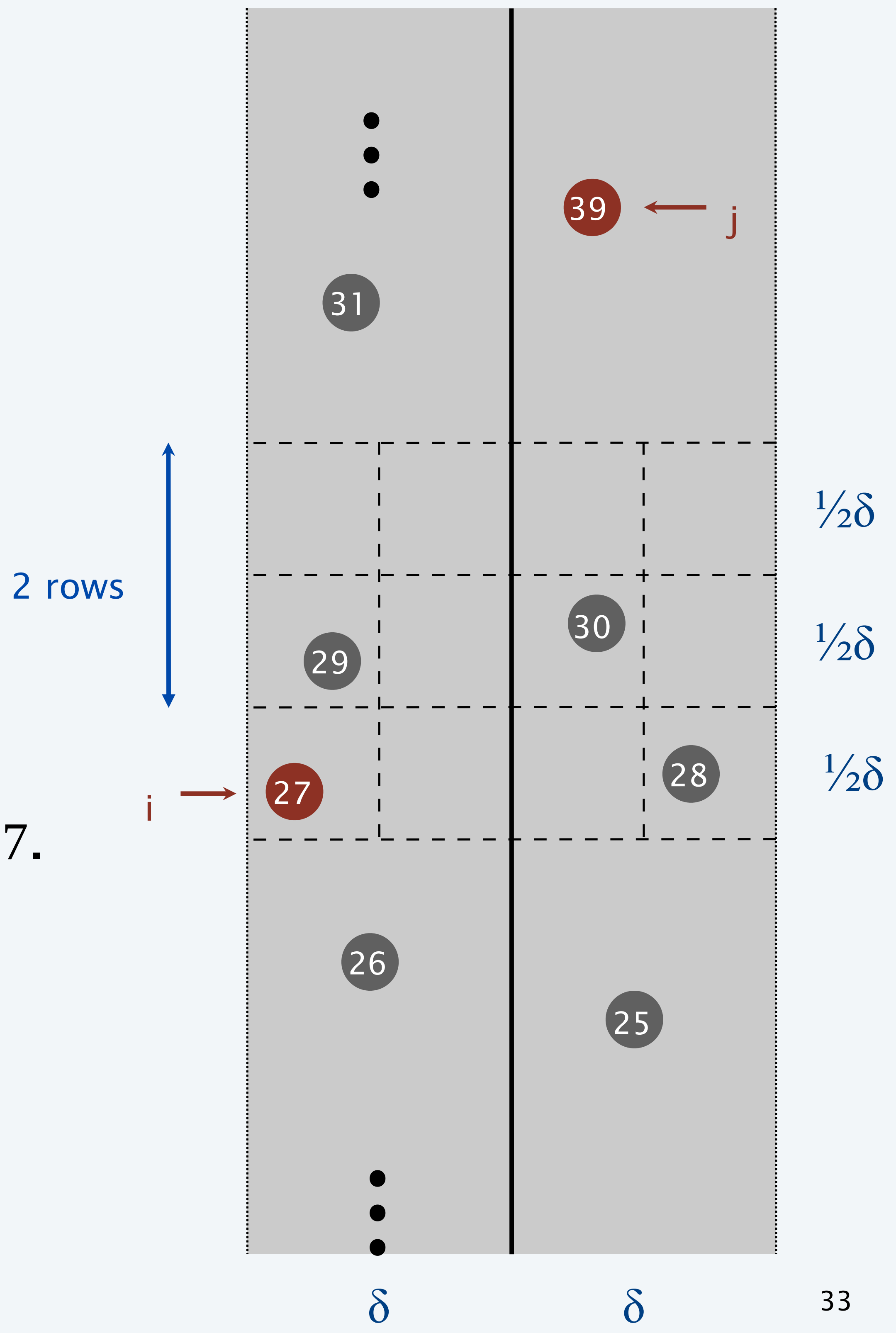
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Claim remains true if we replace 12 with 7.



Closest pair of points: divide-and-conquer algorithm

```
CLOSEST-PAIR $(p_1, p_2, \ldots, p_n)$

Compute separation line $L$ such that half the points        ← $O(n \log n)$
are on each side of the line.
$\delta_1 \leftarrow$ CLOSEST-PAIR (points in left half).
$\delta_2 \leftarrow$ CLOSEST-PAIR (points in right half).         ← $2\ T(n / 2)$
$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$.
Delete all points further than $\delta$ from line $L$.             ← $O(n)$
Sort remaining points by $y$-coordinate.                      ← $O(n \log n)$
Scan points in $y$-order and compare distance between
each point and next 11 neighbors. If any of these             ← $O(n)$
distances is less than $\delta$, update $\delta$.
RETURN $\delta$.
```

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n \log n) & \text{otherwise} \end{cases}$$

$(x_1 - x_2)^2 + (y_1 - y_2)^2$

Lower bound. In quadratic decision tree model, any algorithm for closest pair (even in 1D) requires $\Omega(n \log n)$ quadratic tests.

Improved closest pair algorithm

Q. How to improve to $O(n \log n)$?

A. Yes. Don't sort points in strip from scratch each time.

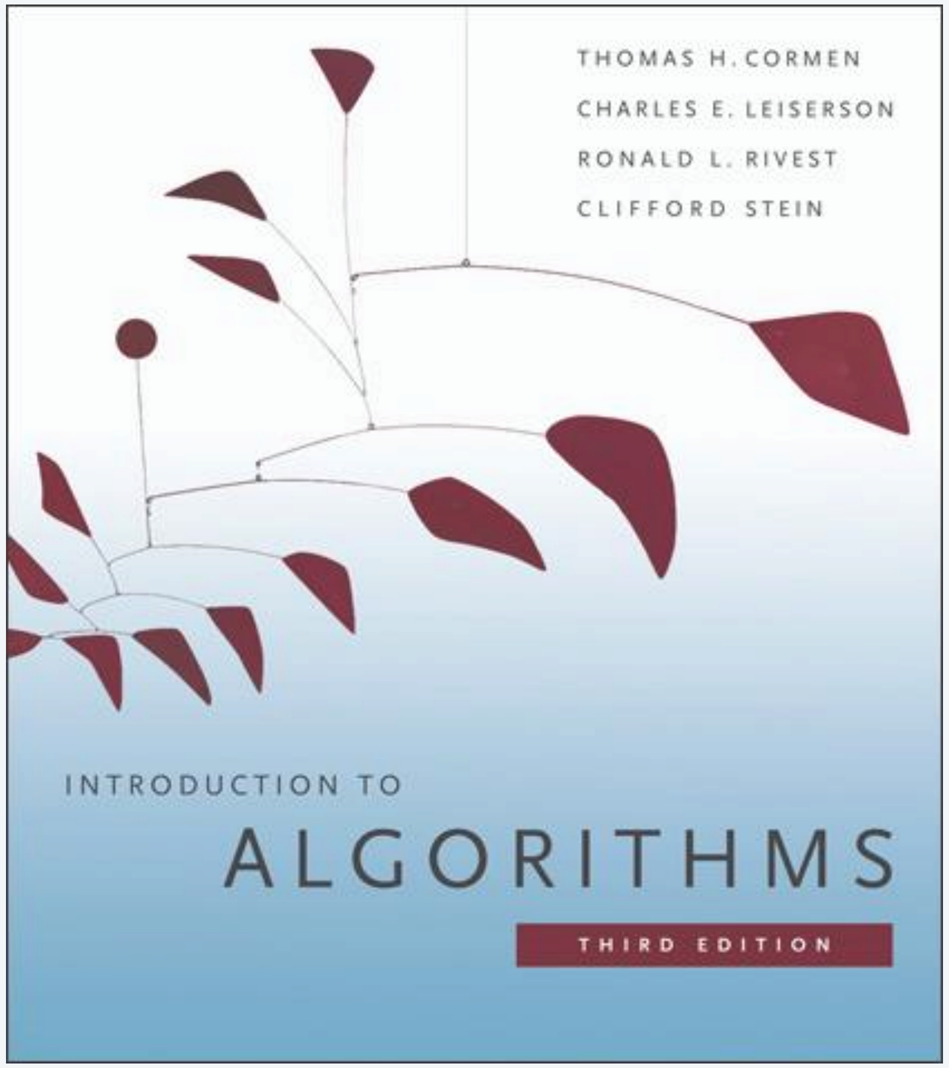
- Each recursive returns two lists: all points sorted by x-coordinate, and all points sorted by y-coordinate.
- Sort by merging two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Note. See SECTION 13.7 for a randomized $O(n)$ time algorithm.

not subject to lower bound
since it uses the floor function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CLIFFORD STEIN
INTRODUCTION TO
ALGORITHMS
THIRD EDITION

CHAPTER 7

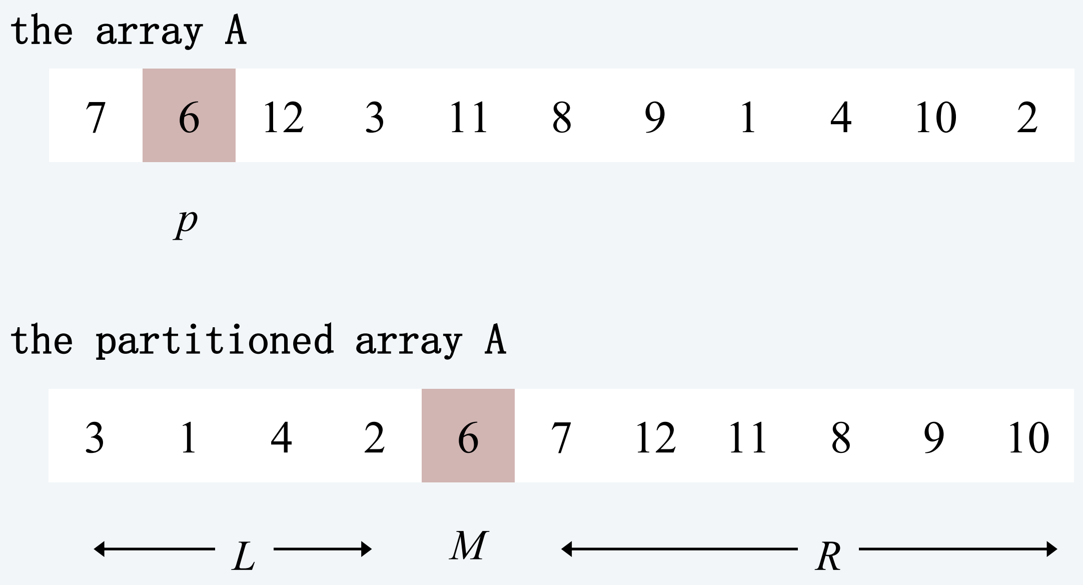
5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- *median and selection*

Randomized quicksort

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.



Recur in both left and right subarrays.

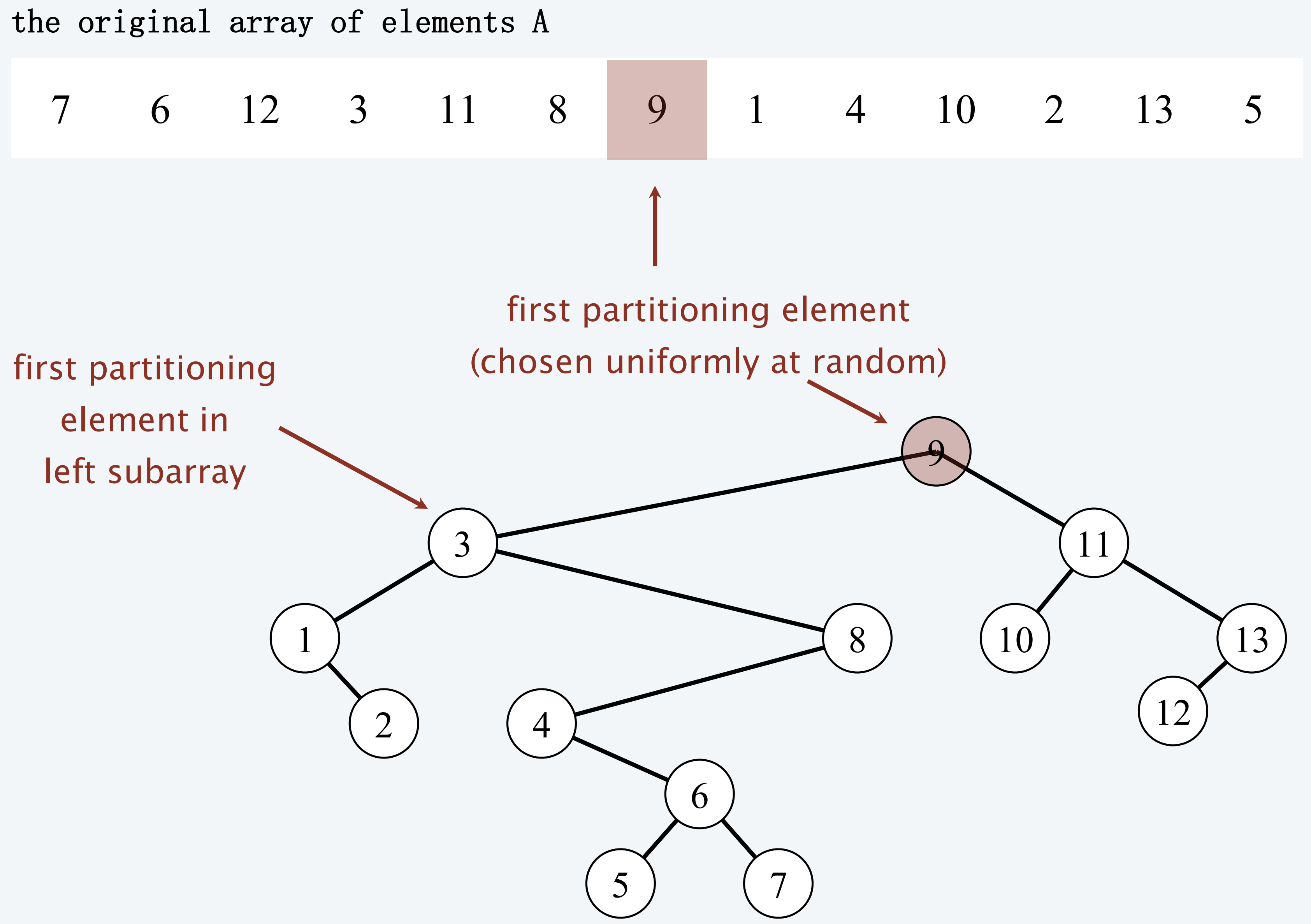
```
RANDOMIZED-QUICKSORT (A)

IF list A has zero or one element
    RETURN.
Pick pivot $p \in A$ uniformly at random.
$(L, M, R) \leftarrow$ PARTITION-3-WAY $(A, a_i)$.
RANDOMIZED-QUICKSORT($L$).
RANDOMIZED-QUICKSORT($R$).
```

3-way partitioning can be done in-place (using n–1 compares)

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

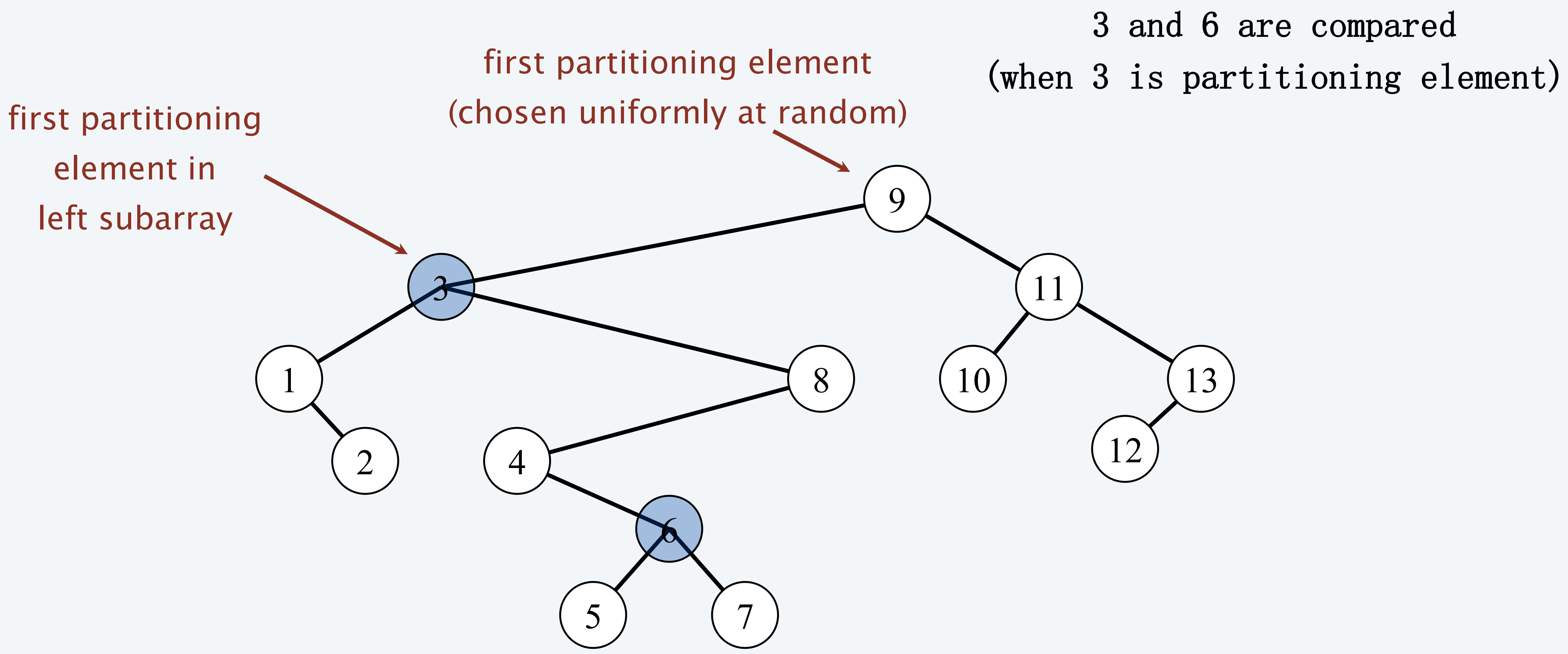

the original array of elements A
7 6 12 3 11 8 9 1 4 10 2 13 5
first partitioning element (chosen uniformly at random)
first partitioning element in left subarray
9
3
11
1
8
10
13
2
4
12
6
5
7

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.

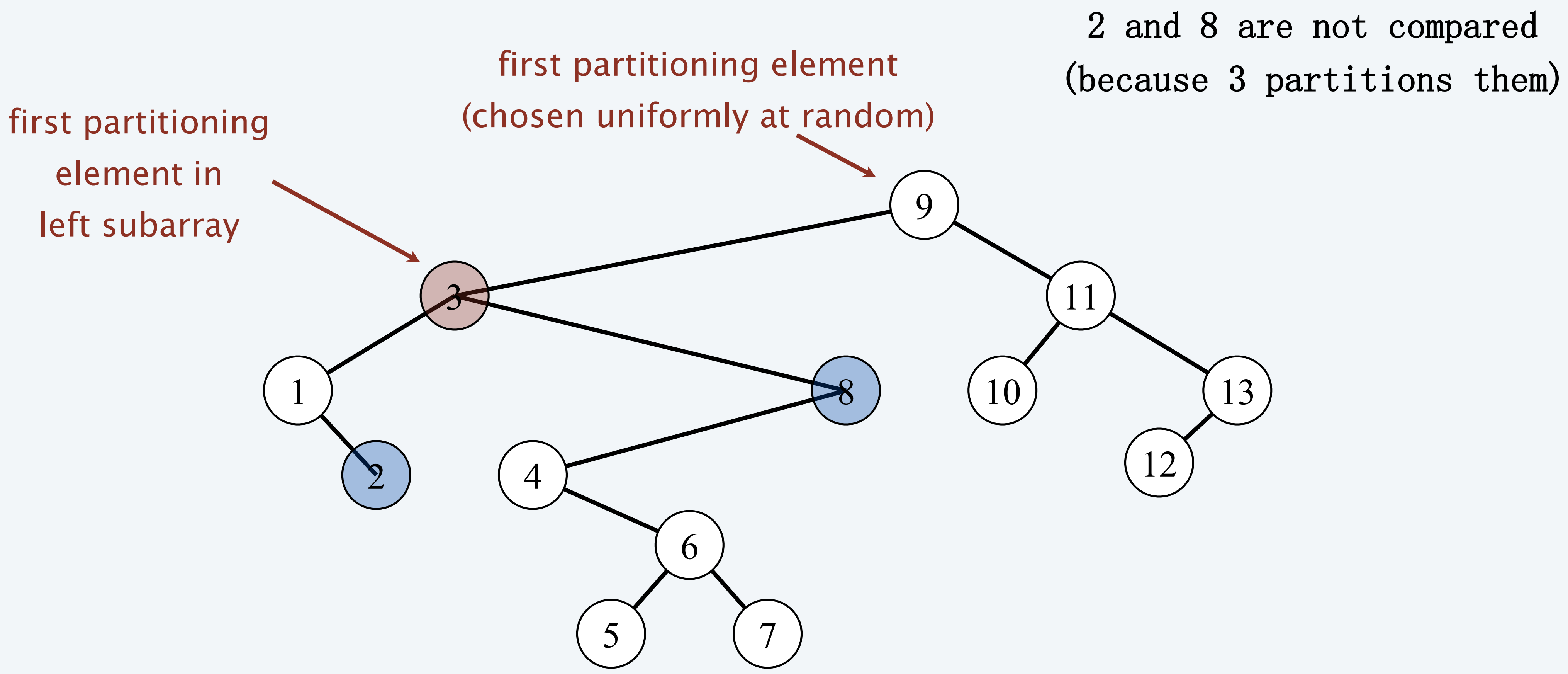

3 and 6 are compared
(when 3 is partitioning element)
first partitioning element
(chosen uniformly at random)
first partitioning
element in
left subarray
9
3
11
1
8
10
13
2
4
12
6
5
7

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.

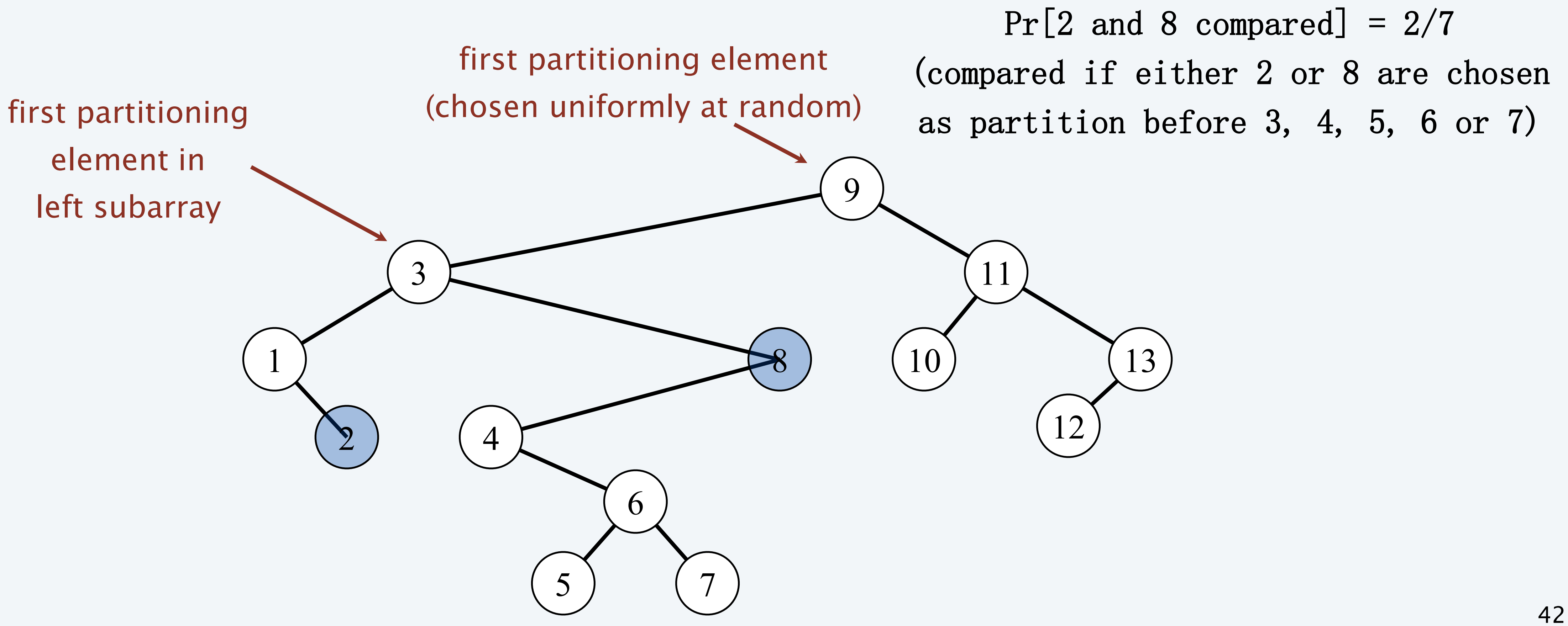


Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.
- Pr [a_i and a_j are compared] $= 2 \;/\; |j - i + 1|$.


first partitioning element in left subarray
first partitioning element (chosen uniformly at random)
Pr[2 and 8 compared] = 2/7 (compared if either 2 or 8 are chosen as partition before 3, 4, 5, 6 or 7)
9
3
11
1
8
10
13
2
4
12
6
5
7

Analysis of randomized quicksort

Proposition. The expected number of compares to quicksort an array of n distinct elements is $O(n \log n)$.

Pf. Consider BST representation of partitioning elements.

- An element is compared with only its ancestors and descendants.
- Pr [a_i and a_j are compared] $= 2 \;/\; |j - i + 1|$.
- Expected number of compares $= \displaystyle\sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{2}{j-i+1}$

all pairs i and j

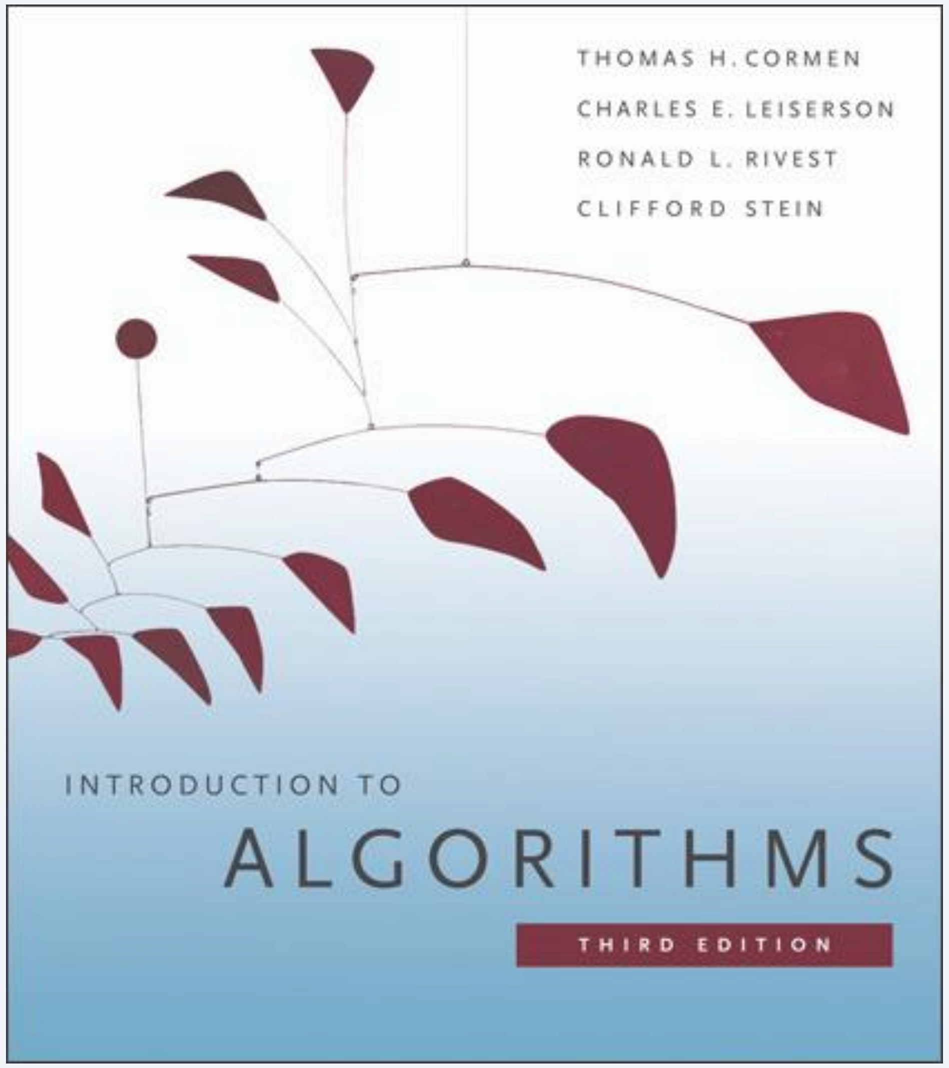
$$= 2\sum_{i=1}^{N} \sum_{j=2}^{N-i+1} \frac{1}{j}$$

$$\leq 2N \sum_{j=1}^{N} \frac{1}{j}$$

$$\sim 2N \int_{x=1}^{N} \frac{1}{x}\, dx$$

$$= 2N \ln N$$

Remark. Number of compares only decreases if equal elements.


THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN
INTRODUCTION TO
ALGORITHMS
THIRD EDITION

CHAPTER 9

5. DIVIDE AND CONQUER

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *randomized quicksort*
- ~~*median and selection*~~

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n + 1) / 2 \rfloor$.
- $O(n)$ compares for min or max.
- $O(n \log n)$ compares by sorting.
- $O(n \log k)$ compares with a binary heap.

Applications. Order statistics; find the "top k"; bottleneck paths, …

Q. Can we do it with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

Quickselect

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Recur in one subarray—the one containing the k^{th} smallest element.

```
QUICK-SELECT (A, k)
________________________________________________
Pick pivot p ∈ A uniformly at random.
(L, M, R) ← PARTITION-3-WAY (A, p).
IF       k ≤ |L|            RETURN QUICK-SELECT (L, k).
ELSE IF  k > |L| + |M|      RETURN QUICK-SELECT (R, k – |L| – |M|)
ELSE                        RETURN p.
________________________________________________
```

3-way partitioning can be done in-place (using n–1 compares)

Quickselect analysis

Intuition. Split candy bar uniformly ⇒ expected size of larger piece is ¾.

$$T(n) \le T(\tfrac{3}{4} n) + n \quad \Rightarrow \quad T(n) \le 4n$$

Def. $T(n, k)$ = expected # compares to select kth smallest in an array of size ≤ n.

Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \le 4n$.

Pf. [by strong induction on n]

- Assume true for $1, 2, \ldots, n - 1$.
- $T(n)$ satisfies the following recurrence:

can assume we always recur on largest subarray since T(n) is monotonic and we are trying to get an upper bound

$$\begin{aligned} T(n) &\le n + 2/n\,[\; T(n/2) + \ldots + T(n-3) + T(n-2) + T(n-1)\;] \\ &\le n + 2/n\,[\; 4\,n/2 + \ldots + 4(n-3) + 4(n-2) + 4(n-1)\;] \\ &= n + 4\,(3/4\;n) \\ &= 4\,n. \quad ▪ \end{aligned}$$

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst case linear time

Goal. Find pivot element p that divides list of n elements into two pieces so that each piece is guaranteed to have $\leq 7/10\, n$ elements.

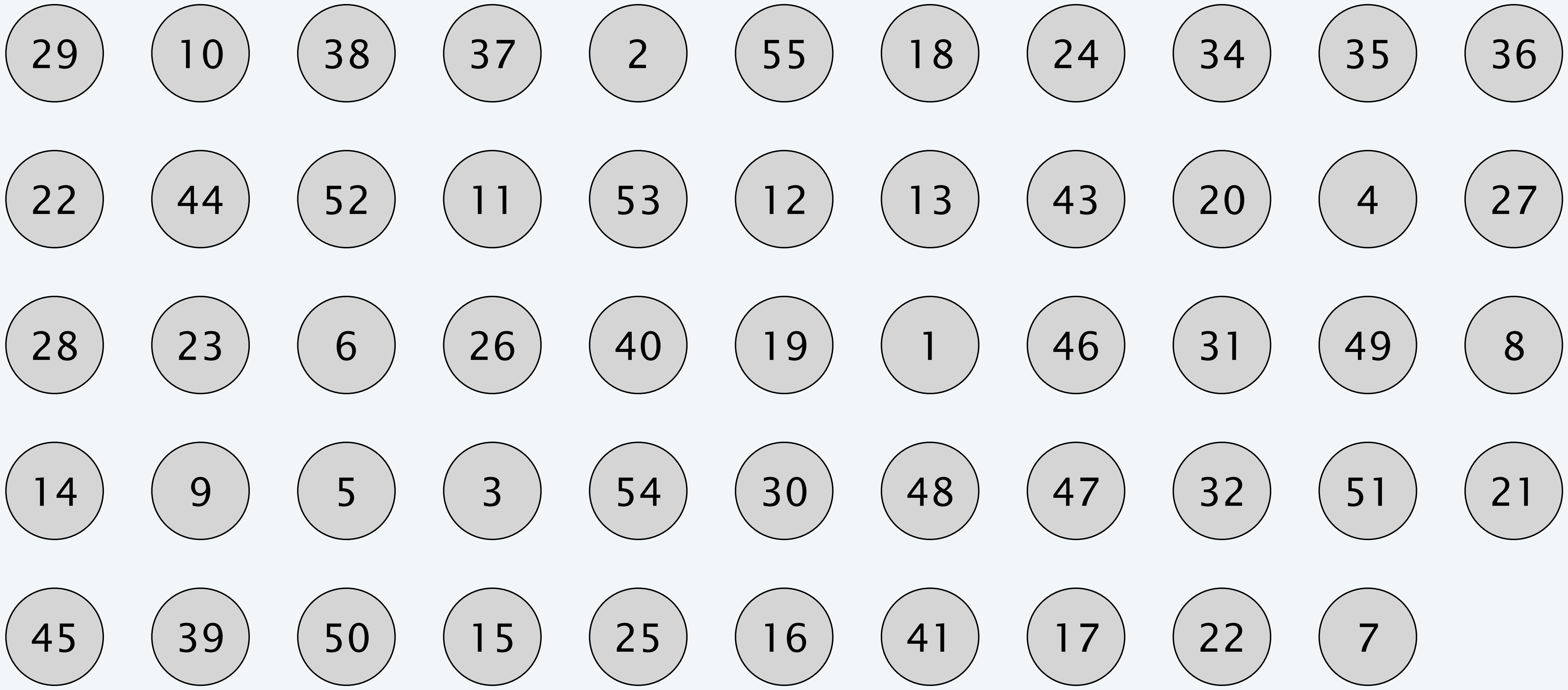
Q. How to find approximate median in linear time?

A. Recursively compute median of sample of $\leq 2/10\, n$ elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ \underbrace{T(7/10\, n)} + \underbrace{T(2/10\, n)}_{\text{two subproblems of different sizes!}} + \Theta(n) & \text{otherwise} \end{cases}$$

Choosing the pivot element

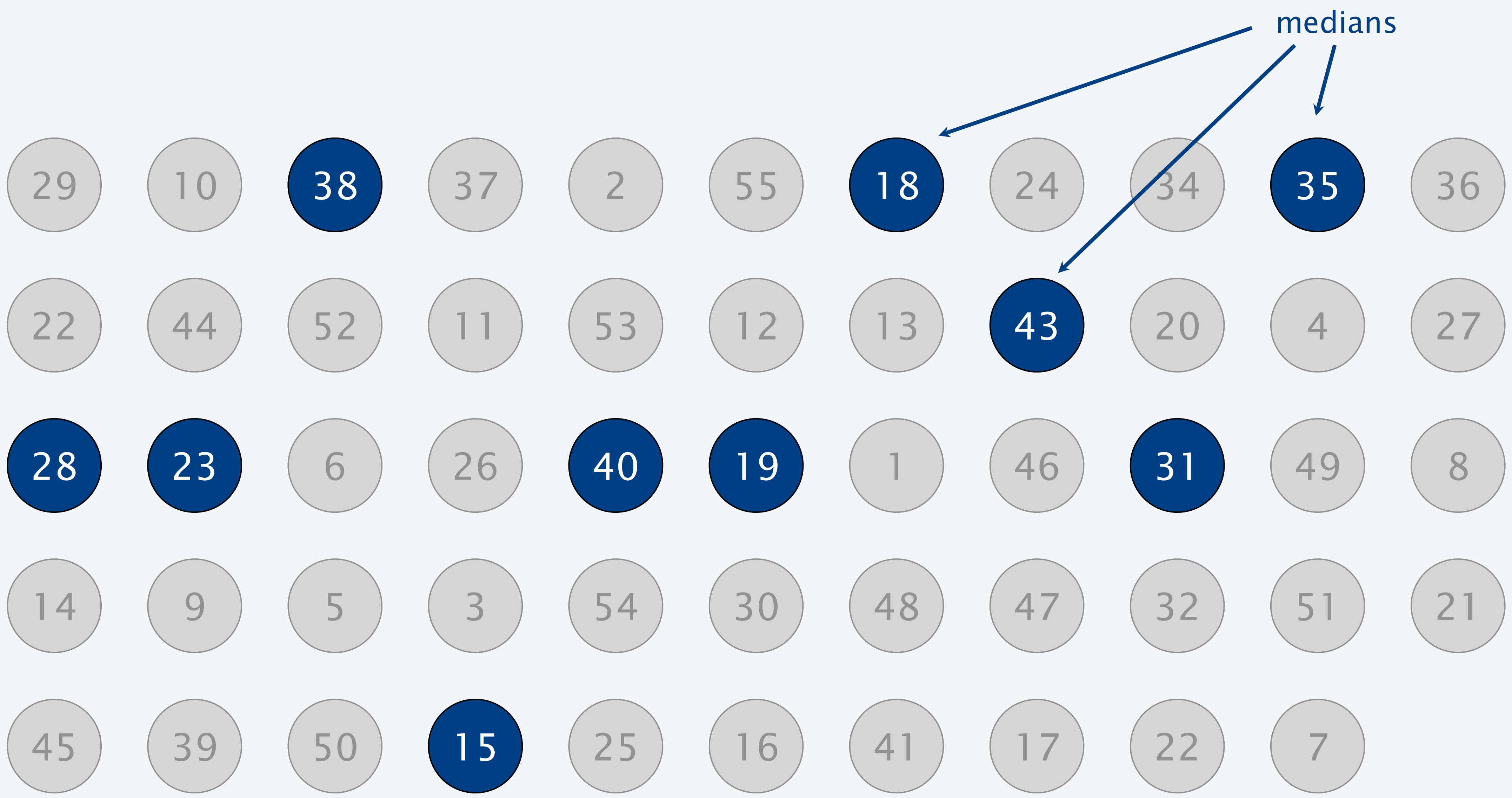
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).



N = 54

Choosing the pivot element

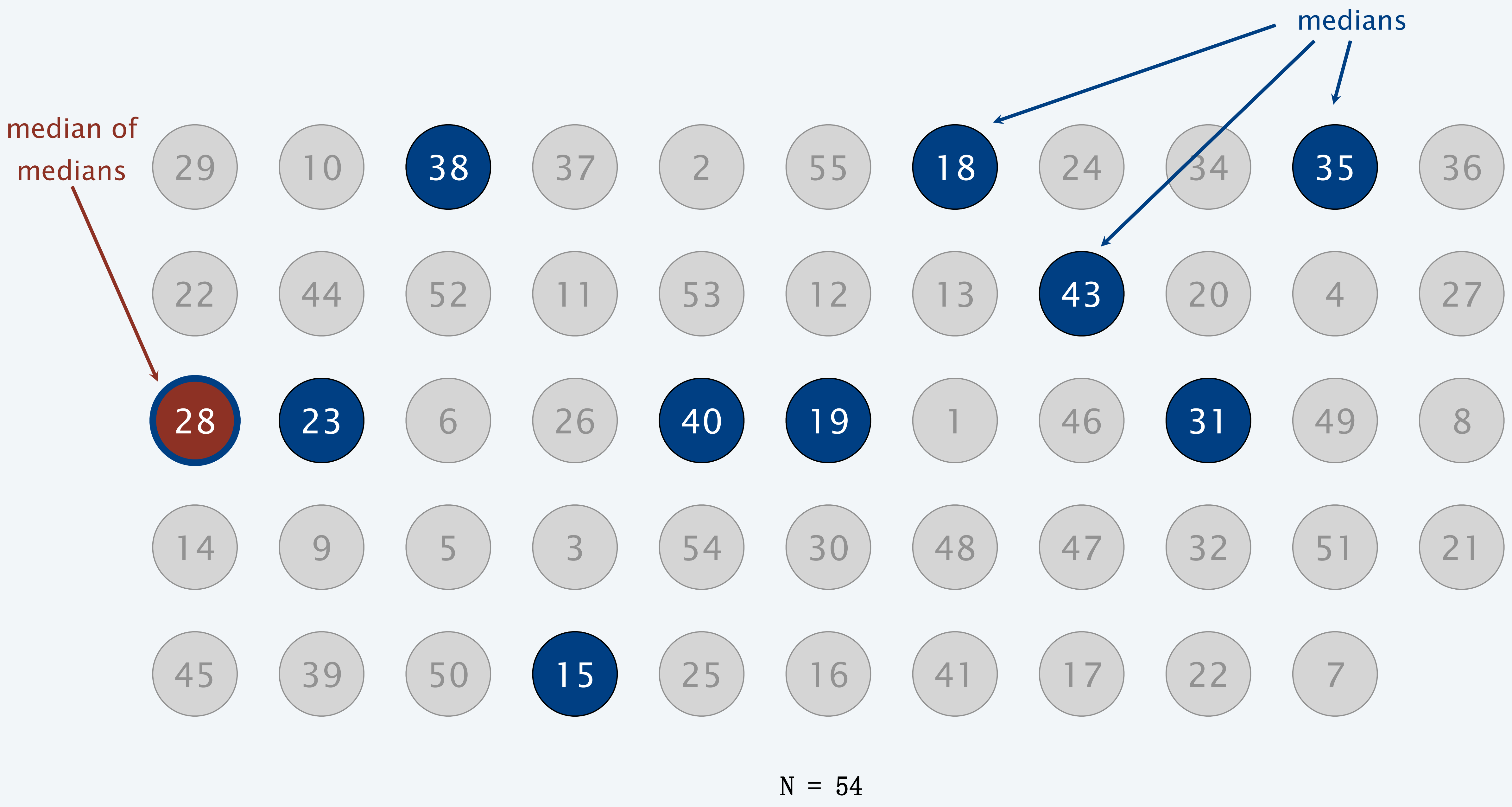
- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).



N = 54

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n / 5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.

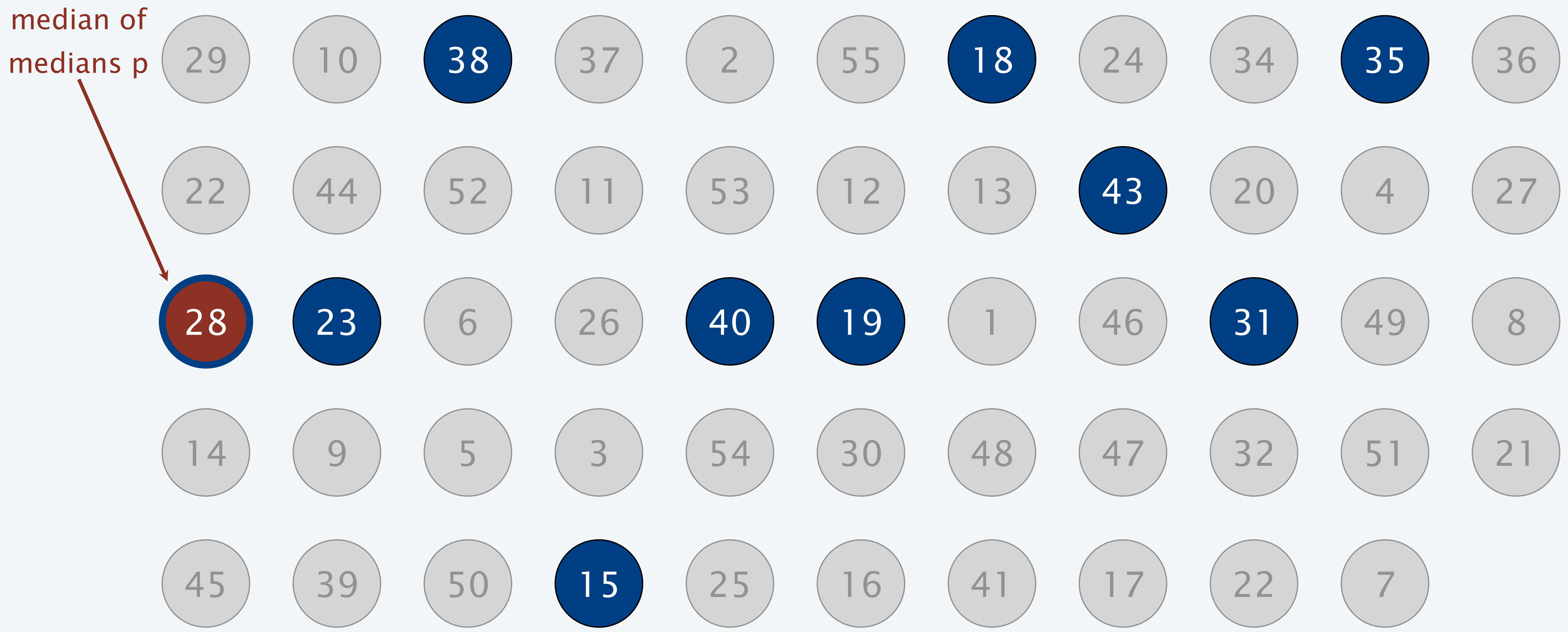


Median-of-medians selection algorithm

```
MOM-SELECT (A, k)
---------------------------------------------------------------
$n \leftarrow |A|$.
IF  $n < 50$  RETURN  $k^{th}$ smallest of element of $A$ via mergesort.

Group $A$ into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
$B \leftarrow$ median of each group of 5.
$p \leftarrow$ MOM-SELECT($B$, $\lfloor n / 10 \rfloor$)  ←—— median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY $(A, p)$.
IF        $k \le |L|$              RETURN MOM-SELECT $(L, k)$.
ELSE IF   $k > |L| + |M|$          RETURN MOM-SELECT $(R, k - |L| - |M|)$
ELSE                               RETURN $p$.
---------------------------------------------------------------
```

Analysis of median-of-medians selection algorithm

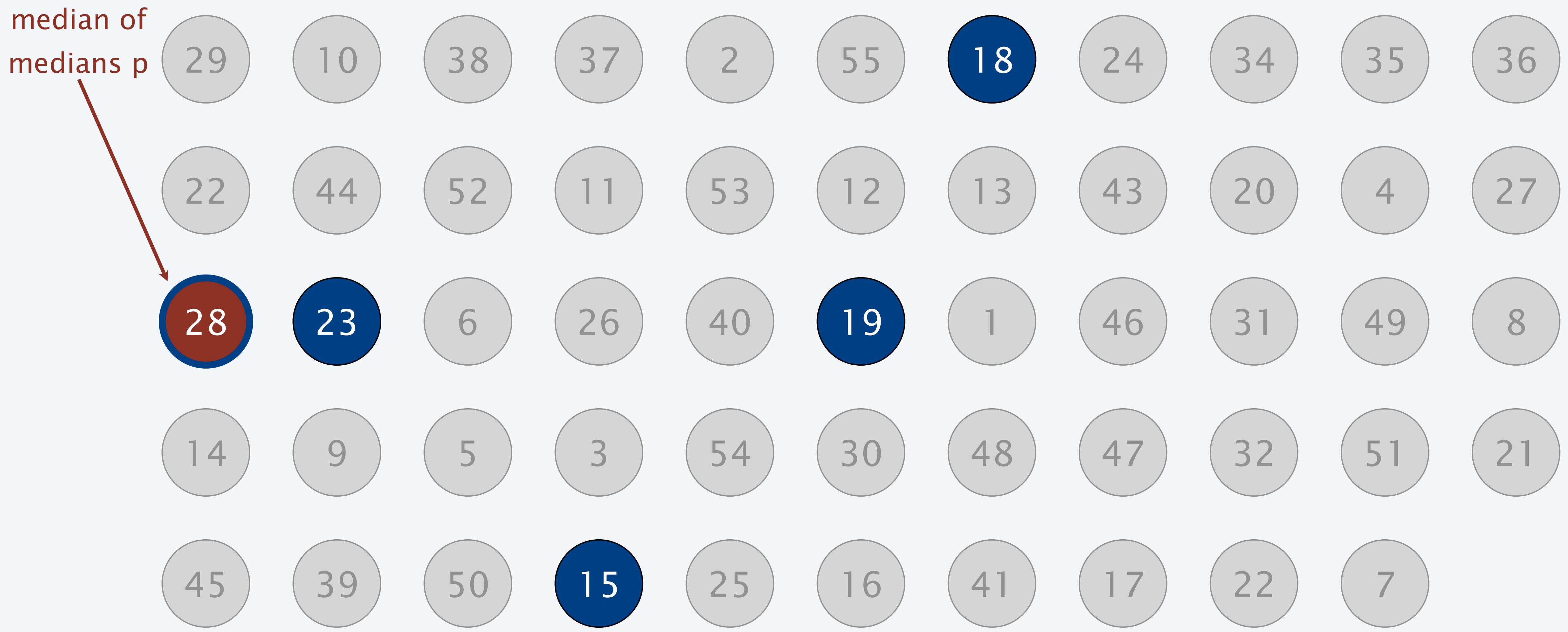
- At least half of 5-element medians ≤ p.



N = 54

Analysis of median-of-medians selection algorithm

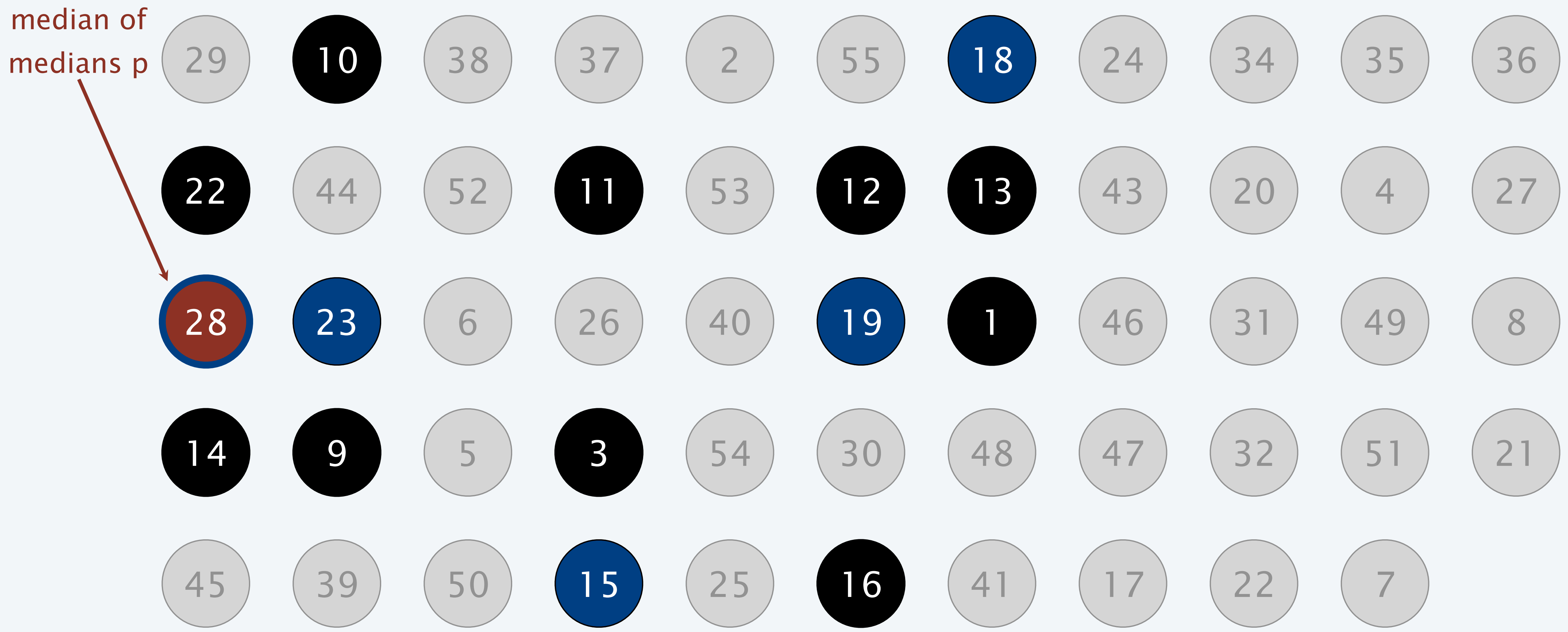
- At least half of 5-element medians ≤ p.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians ≤ p.



N = 54

Analysis of median-of-medians selection algorithm

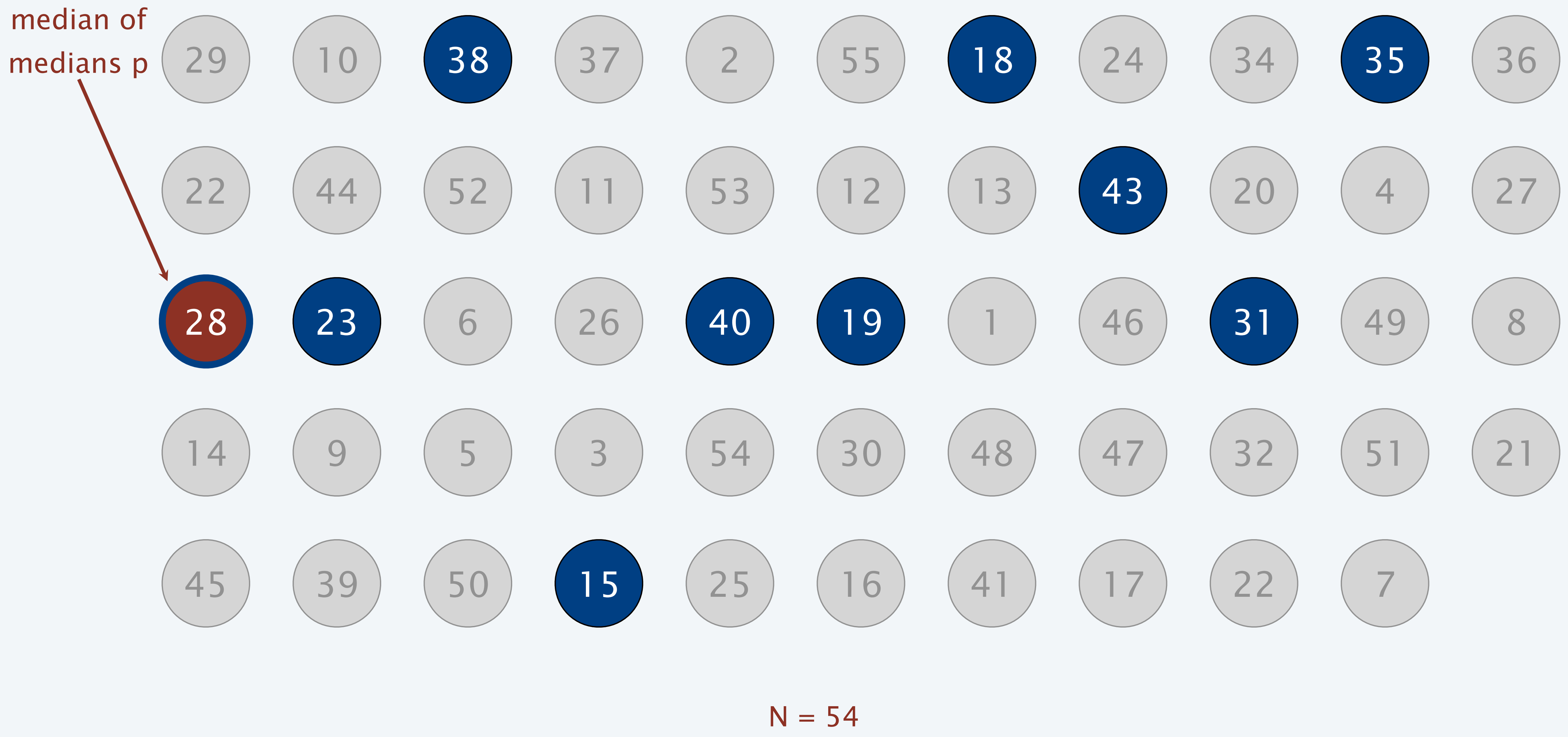
- At least half of 5-element medians ≤ p.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians ≤ p.
- At least $3 \lfloor n / 10 \rfloor$ elements ≤ p.



N = 54

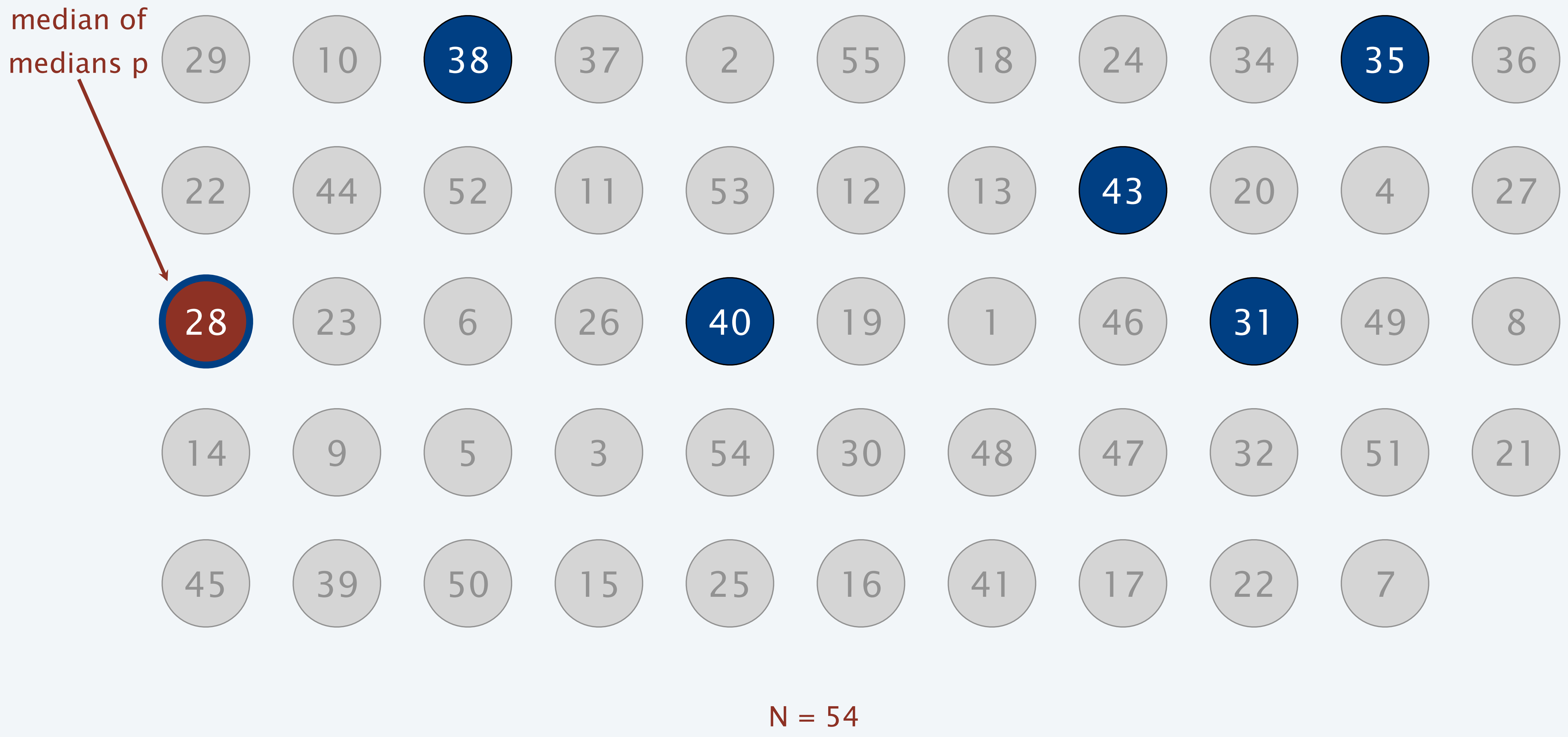
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.



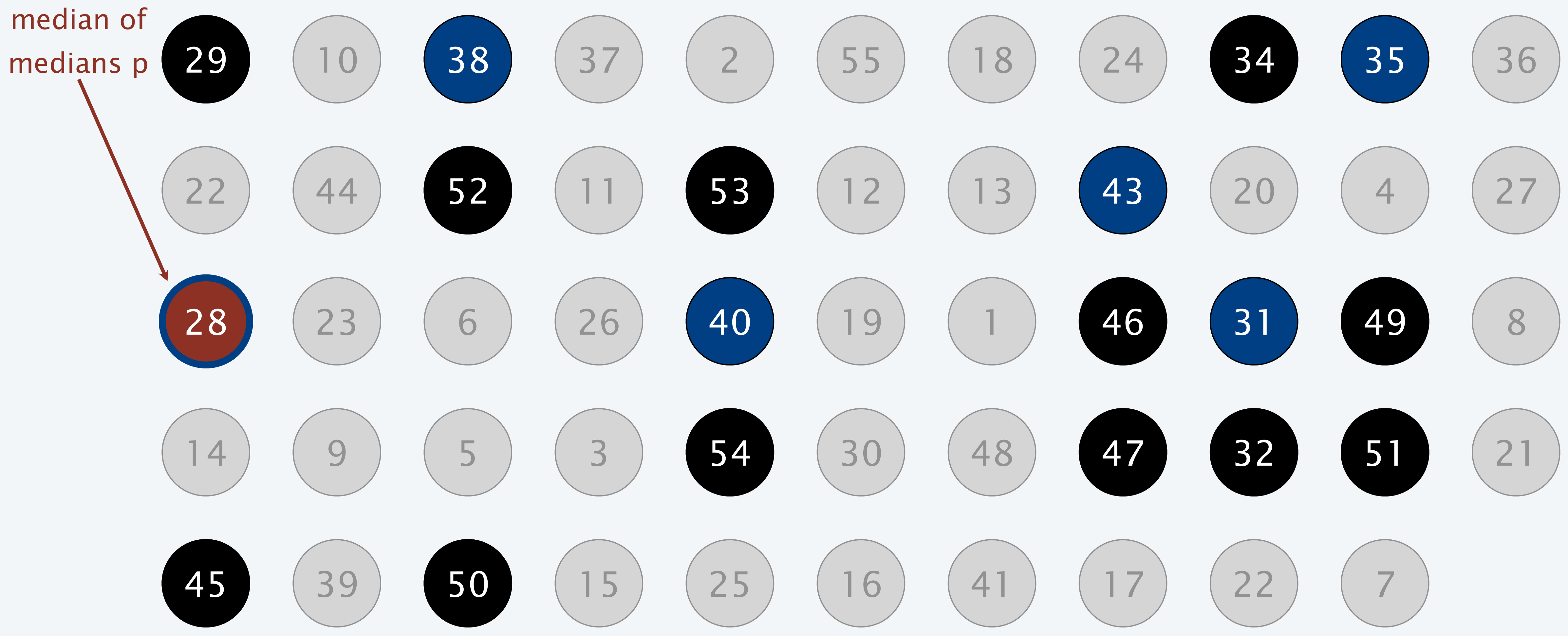
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.
- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians ≥ p.



Analysis of median-of-medians selection algorithm

- At least half of 5-element medians ≥ p.
- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians ≥ p.
- At least $3 \lfloor n / 10 \rfloor$ elements ≥ p.



N = 54

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p.
- At least $3 \lfloor n / 10 \rfloor$ elements $\le p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\ge p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on an array of n elements.

$$C(n) \le C(\lfloor n/5 \rfloor) + C(n - 3\lfloor n/10 \rfloor) + \tfrac{11}{5}\, n$$

median of medians

recursive select

computing median of 5 (6 compares per group)

partitioning (n compares)

Now, solve recurrence.

- Assume n is both a power of 5 and a power of 10?
- Assume $C(n)$ is monotone nondecreasing?

Median-of-medians selection algorithm recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # compares on an array of $\le n$ elements.
- $T(n)$ is monotone, but $C(n)$ is not!

$$T(n) \le \begin{cases} 6n & \text{if } n < 50 \\ T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \tfrac{11}{5}\, n & \text{otherwise} \end{cases}$$

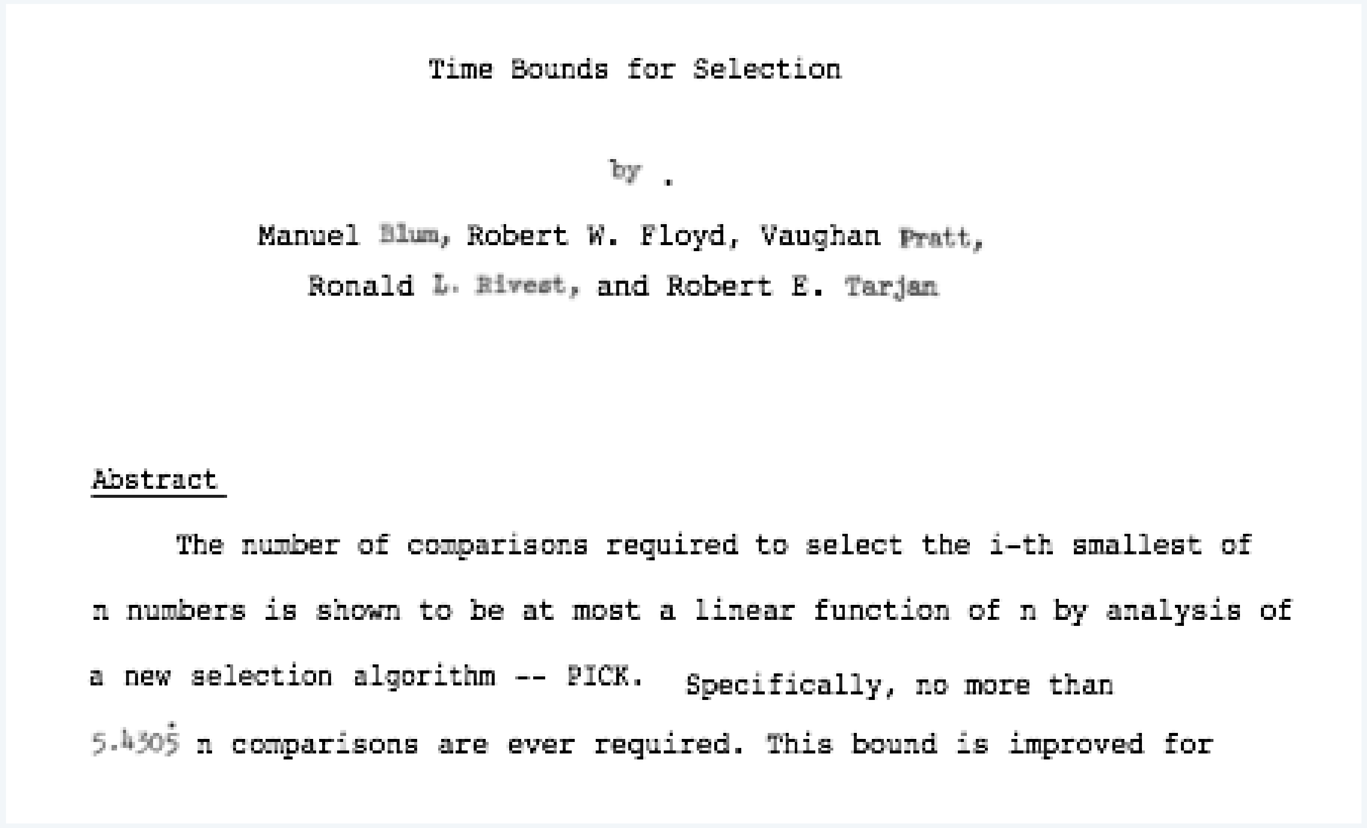
Claim. $T(n) \le 44\,n$.

- Base case: $T(n) \le 6\,n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- Induction step: for $n \ge 50$, we have:

$$\begin{aligned} T(n) &\le T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5\; n \\ &\le 44\,(\lfloor n/5 \rfloor) + 44\,(n - 3\lfloor n/10 \rfloor) + 11/5\; n \\ &\le 44\,(n/5) + 44\,n - 44\,(n/4) + 11/5\; n \quad \longleftarrow \quad \text{for } n \ge 50,\ 3\lfloor n/10 \rfloor \ge n/4 \\ &= 44\,n. \quad \blacksquare \end{aligned}$$

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.



Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Theory.

- Optimized version of BFPRT: $\leq 5.4305\, n$ compares.
- Best known upper bound [Dor-Zwick 1995]: $\leq 2.95\, n$ compares.
- Best known lower bound [Dor-Zwick 1999]: $\geq (2 + \varepsilon)\, n$ compares.

Linear-time selection postmortem

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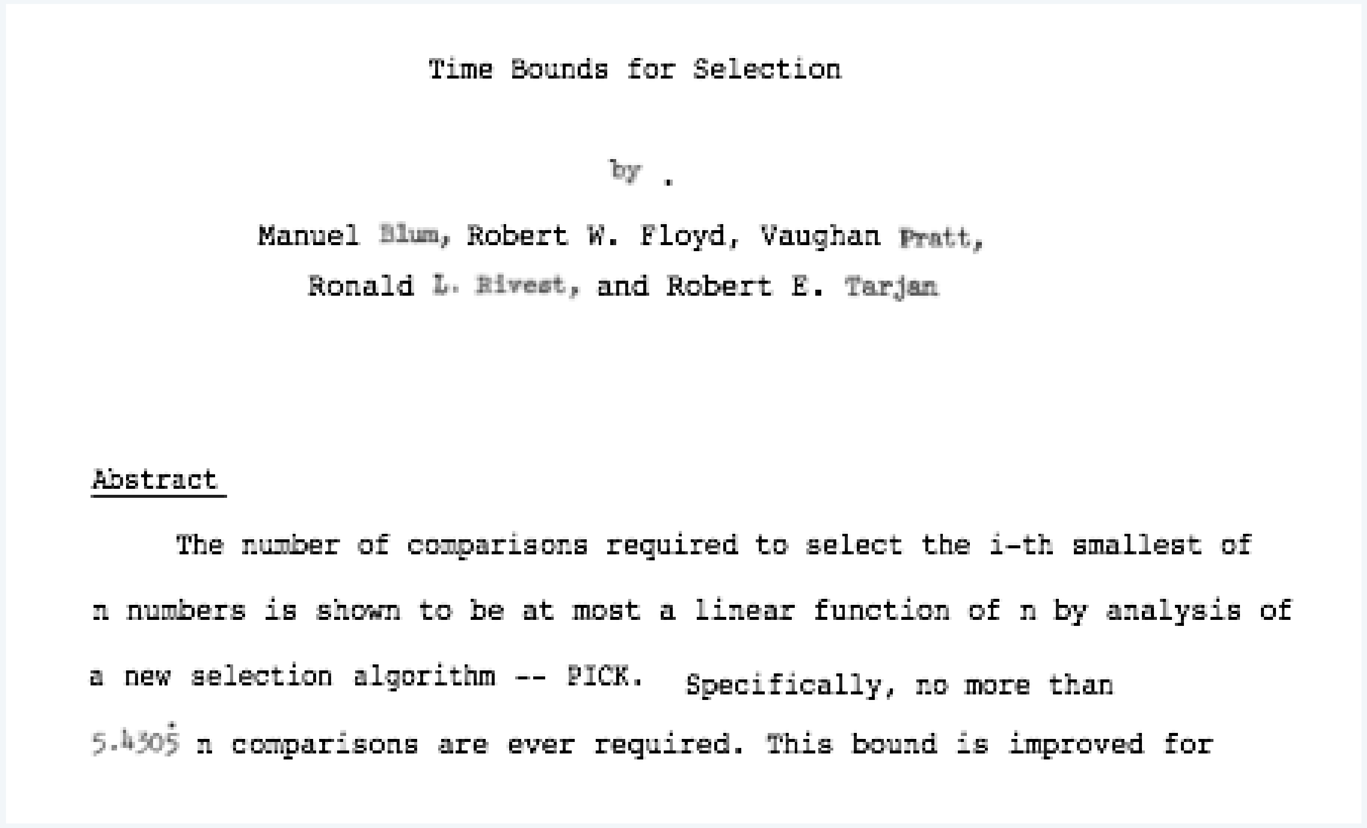

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The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Practice. Constant and overhead (currently) too large to be useful.

Open. Practical selection algorithm whose worst-case running time is $O(n)$.