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CSCI 3104, Algorithms **Due January 29, 2021**
Problem Set 2 (50 points) **Spring 2021, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.
Advice 2: Verbal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solution:

- The solutions **should be typed** and we cannot accept hand-written solutions. Here's a short intro to Latex.
- You should submit your work through **Gradescope** only.
- The easiest way to access Gradescope is through our Canvas page. There is a Gradescope button in the left menu.
- Gradescope will only accept **.pdf** files.
- It is vital that you match each problem part with your work. Skip to 1:40 to just see the matching info.

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1. The following problems are a review of logarithm and exponent topics.

   (a) Solve for x.

      i. $3^{2x} = 81$
      ii. $3(5^{x-1}) = 375$
      iii. $\log_3 x^2 = 4$

   (b) Solve for x.

      i. $x^2 - x = \log_5 25$
      ii. $\log_{10}(x+3) - \log_{10} x = 1$

   (c) Answer each of the following with a TRUE or FALSE.

      i. $a^{\log_a x} = x$
      ii. $a^{\log_b x} = x$
      iii. $a = b^{\log_b a}$
      iv. $\log_a x = \frac{\log_b x}{\log_b a}$
      v. $\log b^m = m \log b$

Solution:

(a) i. $3x^2 = 81$
$\log_3(81) = 2x$ (Apply logarithm rules)
$4 = 2x$ (Simplify)
$\boxed{\mathbf{2 = x}}$ (Divide both sides by 2)

ii. $3(5^{x-1}) = 375$
$5^{x-1} = 125$ (Divide both sides by 3)
$\log_5(125) = x - 1$ (Apply logarithm rules)
$3 = x - 1$ (Simplify)
$\boxed{\mathbf{4 = x}}$ (Add 1 to both sides)

iii. $\log_3 x^2 = 4$
$3^{\log_3(x^2)} = 3^4$ (Raise each side as the exponent to 3)
$x^2 = 3^4$ (Simplify)
$x^2 = 81$ (Simplify)
$\sqrt{x^2} = \sqrt{81}$ (Take the square root of both sides)
$\boxed{\mathbf{x = \pm 9}}$ (Simplify)

(b) i. $x^2 - x = \log_5 25$
$x^2 - x = 2$ (Simplify)
$x(x-1) = 2$ (Factor)
$\boxed{\mathbf{x = 2, -1}}$ (Solve for roots)

ii. $\log_{10}(x+3) - \log_{10} x = 1$
$\log_{10}\left(\frac{x+3}{x}\right) = 1$ (Apply log rules (subtraction))
$10^{\log_{10}\left(\frac{x+3}{x}\right)} = 10^1$ (Raise each side as the exponent to 10)
$\frac{x+3}{x} = 10$ (Simplify)
$x + 3 = 10x$ (Multiply both sides by x)
$3 = 10x - x$ (Subtract x from both sides)
$3 = 9x$ (Simplify)
$\boxed{\mathbf{\frac{1}{3} = x}}$ (Divide both sides by 9)

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(c) i. $a^{\log_a x} = x$ - $\boxed{\textbf{TRUE}}$

ii. $a^{\log_b x} = x$ - $\boxed{\textbf{FALSE}}$

iii. $a = b^{\log_b a}$ - $\boxed{\textbf{TRUE}}$

iv. $\log_a x = \frac{\log_b x}{\log_b a}$ - $\boxed{\textbf{TRUE}}$

v. $\log b^m = m \log b$ - $\boxed{\textbf{TRUE}}$

2. Compute the following limits at infinity. Show all work and justify your answer.

   (a) $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$

   (b) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$

   (c) $\lim_{x\to\infty} \frac{\ln x^4}{x^3}$

Solution:

(a) $\lim_{x\to\infty} \frac{3x^3+2}{9x^3-2x^2+7}$

$\lim_{x\to\infty} \frac{3+\frac{2}{x^3}}{9-\frac{2}{x}+\frac{7}{x^3}}$ (Divide by x^3)

$\frac{3+0}{9-0+0} = \frac{3}{9} = \boxed{\mathbf{\frac{1}{3}}}$ (Evaluate at infinity and simplify)

(b) $\lim_{x\to\infty} \frac{x^3}{e^{x/2}}$

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{\frac{e^{x/2}}{2}} = \lim_{x\to\infty} \frac{6x^2}{e^{x/2}}$ (Apply L'Hôpital's and simplify)

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{12x}{\frac{e^{x/2}}{2}} = \lim_{x\to\infty} \frac{24x}{e^{x/2}}$ (Apply L'Hôpital's and simplify)

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{24}{\frac{e^{x/2}}{2}} = \lim_{x\to\infty} \frac{48}{e^{x/2}}$ (Apply L'Hôpital's and simplify)

$\frac{48}{\infty} = \boxed{\mathbf{0}}$ (Evaluate at infinity and simplify)

(c) $\lim_{x\to\infty} \frac{\ln x^4}{x^3}$

$\lim_{x\to\infty} \frac{4\ln x}{x^3}$ (Apply logarithm rules)

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{4\frac{1}{x}}{3x^2}$ (Apply L'Hôpital's)

$\stackrel{L'H}{=} \lim_{x\to\infty} \frac{4}{3x^3}$ (Simplify)

$\frac{4}{\infty} = \boxed{\mathbf{0}}$ (Evaluate at infinity and simplify)

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3. Compute the following limits at infinity. Show all work and justify your answer.

   (a) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

   (b) What does this tell us about the rate at which e^{nx} approaches infinity relative to x^m? A brief explanation is fine for this part.

   (c) For real numbers $m, n > 0$ compute $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

   (d) What does this tell us about the rate at which $(\ln x)^n$ approaches infinity relative to x^m? A brief explanation is fine for this part.

Solution:

(a) $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$ (Apply L'Hôpital's and simplify)

$\overset{L'H}{=} \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$ (Apply L'Hôpital's and simplify)

We see a pattern forming. If we repeat the pattern, we will eventually get $\boxed{\lim_{\mathbf{x}\to\infty} \frac{\mathbf{m!}}{\mathbf{n^m e^{nx}}} = \mathbf{0}}$ for real numbers $m, n > 0$

(b) This shows that x^m approaches infinity slower than e^{nx}.

(c) $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$

$\lim_{x\to\infty} \frac{\ln^n x}{x^m}$ (Apply logarithm rules)

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{n\ln^{n-1} x}{x}}{mx^{m-1}} = \lim_{x\to\infty} \frac{n\ln^{n-1} x}{mx^m}$ (Apply L'Hôpital's and simplify)

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{n(n-1)\ln^{n-2} x}{x}}{m^2x^{m-1}} = \lim_{x\to\infty} \frac{n(n-1)\ln^{n-2} x}{m^2x^m}$ (Apply L'Hôpital's and simplify)

We see a pattern forming. If we repeat the pattern, we will eventually get $\boxed{\lim_{\mathbf{x}\to\infty} \frac{\mathbf{n!}}{\mathbf{m^n x^m}} = \mathbf{0}}$ for real numbers $m, n > 0$

(d) This shows that $(\ln x)^n$ approaches infinity slowerr than x^m.

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4. The problems in this question deal with the Root and the Ratio Tests. Determine the convergence or divergence of the following series. State which test you used.

   (a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

   (b) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   (c) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

   (d) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

Solution:

(a) **Root Test**

$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n} \to \lim_{n\to\infty} \left|\frac{e^{2n}}{n^n}\right|^{\frac{1}{n}}$ (Apply root test)

$\lim_{n\to\infty} \left|\left(\frac{e^2}{n}\right)^n\right|^{\frac{1}{n}}$ (Factor)

$\lim_{n\to\infty} \left|\frac{e^2}{n}\right|$ (Simplify)

$\left|\frac{e^2}{\infty}\right| = 0$ (Evaluate at infinity and simplify)

$0 < 1$ Therefore by the root test, the series is $\boxed{\textbf{Convergent}}$

(b) **Ratio Test**

$\sum_{n=0}^{\infty} \frac{2^n}{n!} \to \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$ (Apply ratio test).

$\lim_{n\to\infty} \frac{n! \cdot 2^{n+1}}{2^n \cdot (n+1)!}$ (Simplify)

$\lim_{n\to\infty} \frac{n! \cdot 2^n \cdot 2}{2^n \cdot n! \cdot (n+1)}$ (Expand to allow for cancellation)

$\lim_{n\to\infty} \frac{2}{(n+1)}$ (Cancel like terms)

$\frac{2}{\infty} = 0$ (Evaluate at infinity and simplify)

$0 < 1$ Therefore by the ratio test, the series is $\boxed{\textbf{Convergent}}$

(c) **Ratio Test**

$$\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n} \rightarrow \frac{\frac{(n+1)^2 \cdot 2^{n+1}}{3^{n+1}}}{\frac{n^2 \cdot 2^n}{3^n}}$$ (Apply ratio test).

$$\lim_{n\to\infty} \frac{3^n \cdot (n+1)^2 \cdot 2^{n+1}}{3^{n+1} \cdot n^2 \cdot 2^n}$$ (Simplify)

$$\lim_{n\to\infty} \frac{3^n \cdot (n^2+2n+1) \cdot 2^n \cdot 2}{3^n \cdot 3 \cdot n^2 \cdot 2^n}$$ (Expand to allow for cancellation)

$$\lim_{n\to\infty} \frac{2(n^2+2n+1)}{3n^2}$$ (Cancel like terms)

$$\lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2}$$ (Distribute)

$$\lim_{n\to\infty} \frac{2+\frac{4}{n}+\frac{2}{n^2}}{3}$$ (Divide by n^2)

$$\frac{2+0+0}{3} = \frac{2}{3}$$ (Evaluate at infinity and simplify)

$\frac{2}{3} < 1$ Therefore by the ratio test, the series is $\boxed{\textbf{Convergent}}$

(d) **Root Test**

$$\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n \rightarrow \lim_{n\to\infty} \left|\left(\frac{\ln n}{n}\right)^n\right|^{\frac{1}{n}}$$ (Apply root test)

$$\lim_{n\to\infty} \left|\frac{\ln n}{n}\right|$$ (Simplify)

$$\stackrel{L'H}{=} \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{1}\right| = \lim_{n\to\infty} \left|\frac{1}{n}\right|$$ (Apply L'Hôpital's and simplify)

$$\left|\frac{1}{\infty}\right| = 0$$ (Evaluate at infinity and simplify)

$0 < 1$ Therefore by the root test, the series is $\boxed{\textbf{Convergent}}$