## 0.1 Outline

As for the outline, i think you should have four chapters:

(1) Introduction:

(1.1) What problem are you trying to solve? Why is it important? What is your goal?

(1.2) Challenges

(1.3) Summary of how youve addressed those challenges, what have you found/built?

(1.4) Overview of following chapters

(2) Theoretical Background The Math:

(2.1) Where does your problem come from? (youll have to talk about stochastic PDEs, volatility, assets price modeling, martingales, etc.)

(2.2) Describe the math problem: parametric estimation, interpolation, extrapolation, minimization

(2.3) What is your solution? Based on all this theory connecting math and finance and assets and asset bubbles, how can you get an answer to your (math) problem as you described in chapter (1).

(3) Algorithm and Implementation The Software Tool:

We can discuss later, but you may want to consider different format options: do you want to have a section for each sub problem (acquiring data, fitting data, extrapolation, etc.) or you may want to group some of those and have a separate section talking about software tools that you have downloaded/modified, and tools you have developed

Also a section describing the Algorithm step by step.

(4) Numerical Examples Once your code works, run some experiments on real data, some that you know the answer, some that you dont know

You may want to have a final chapter with conclusions. future work, etc

**Abstract**

Financial Market is very attracting topic in finance and mathematics world. Recently we have heard lot about Gold Prices inflations. It is the the hot topic in today's finance market. So how will be combine mathematics with today's asset changes? How can we determine the tale of asset's volatility for future? These are the questions which we will consider in this thesis. We will study non parametric estimator Floren Zmirou in local real time on compact domain with stochastic differential equation which has unknown drift and diffusion coeificents. Once we will have volatility from floren zmirou then we will able to use RKHS to estimates function which will extrapolate the tale of function.

# 1 Introduction

Chapter 1

Today's economy, financial asset bubbles are exciting and hot topic. In most recent market news, we have read or seen big changes in Gold prices. Everyone is interested to know what will happen in the future. How we can able to detect or estimate the future changes of any asset ( stock, gold, housing, commodity)? How quickly asset price will jump? These are the question which we will consider in this study. We will study how to determine whether any asset is experiencing a price bubble in real time. How we will detect asset bubbles in real time?

Our problem will be deciding if an asset price is experiencing a price bubble in finite and infinite time period.

We will able to determine the volatility of asset price.Using some helpful techniques to determine asset bubble in real time will help finanial corporations, banks,and money marrkets.

They can lower their money damaging risks by using our methodology. According to " There is a bubble" paper paragraph 2, "indeed 2009 the federal reserve chairman Ben bernanke said in congress testimony[1]

"It is extraordinary difficult in real time to know if an asset price is appropriate or not".

Our goal is to estimate stock price volatility by Floren Zmirou estimator and then we will extrapolate the volatility tale in order to check the integral. wheather the integral is finite or infinite. The process for bubble detection depends on a mathematical analysis that determines when an asset is undergoing speculative pricing

i.e its market price is greater than its fundamental price. The difference between market and fundamential price, is a price bubble

As stated above, we will use a nonparametric estimator Floren -Zmirou which is based on local time of the diffusion process. The biggest challenge we have forced that using non parametric estimator, we can only estimate $\sigma(x)$ volatility function on the points which are visisted by the process. Only finite number of data points are used which is a compact subset of R+.Therefore we can not able to estimate the tail of the volatility. But by determining the tail of volatility, we can see if the integral if finite or infinite. We don't know the asympotice behavior of the volatility.In order to check the tail of the volatility, we need to use extrapolation method.

After estimation of volatility function $\sigma(x)$, we will interpolate the function using cubic splines and Reproducing Kernal Hilbert Spaces. . Once we have interpolated function then we will focus on extending the function to infinity which is our extrapolation method. Using Reproducing Kernal Hilbert Spaces combined with optimization we can get best possible extension of the interpolation function.

Our work is orginized as follow: in chapter 2 we present an overview of previos work, background of the problem, how the problem is connected to finance and mathematics, the methods to solve the problem, and our best possible solution to the problem. In chapter 3 we will discuss the details of our algorithm and it's implementation and in chapter 4 we present several numberical examples, conclusion and future work.

# 2 Definitions

# Finance Definations

## 2.1 Asset Price Bubbles

The difference between the market and fundamental price, if any, is a price bubble.

### 2.1.1 Strike Price

The strike price or exercise price of an option is the fixed price at which the owner of the option can buy( in the case of call) or sell (in the case of a put) the underlying security or commodity.

### 2.1.2 Volatility

Rate at which the price of security moves up and down.

# Mathematical Definations

## 2.2 Introduction to Stochastic Differential Equations

We treat the asset price as a stochastic process:

### 2.2.1 Stochastic Process

Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process with state space X is a collection of X-valued random variables, $S_t$, on $\Omega$ indexed by a set $T$ (e.g. time).

$$S = \{S_t : t \in T\} \tag{1}$$

One can think of $S_t$ as a asset price at time $t$.

### 2.2.2 Stochastic Differential Equation

A differential equation with one or more terms is a stochastic process.

## 2.3 The Price Asset Model using an SDE

Consider the linear SDE with a Brownian Motion $\{S_t : 0 \le t \le T\}$:

$$\begin{aligned} dS_t &= \sigma(S_t)dW_t + \mu(S_t)dt \\ S_0 &= 0 \end{aligned} \tag{2}$$

- $W_t$ denotes the standard Brownian Motion.
- $\mu(S_t)$ called the drift coefficient.
- $\sigma(S_t)$ called the volatility coefficient.

## 2.4 The Price Asset Model using an SDE

## 2.5 Brownian Motion

A continuous-time stochastic process $\{S_t : 0 \le t \le T\}$ is called a *Standard Brownian Motion* on $[0, T]$ if it has the following four properties:

(i) $S_0 = 0$

(ii) The increment of $S_t$ are independent; given

$$0 \le t_1 < t_2 < t_3 < \cdots < t_n \le T$$

the random variables $(S_{t_2} - S_{t_1})$, $(S_{t_3} - S_{t_2})$, $\cdots$, $\left(S_{t_n} - S_{t_{n-1}}\right)$ are independent.

(iii) $(S_t - S_s)$, $0 \le s \le t \le T$ has the Gaussian distribution with mean zero and variance $(t - s)$

(iv) $S_t(W)$ is a continuous function of $t$, where $W \in \Omega$.

## 2.6 Martingales

(a) $E[\mid S_n \mid] < +\infty$, for all n.

(b) $S_n$ is said to be *adapted* if and only if $S_n$ is $\mathcal{F}_n$-measurable.

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *martingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, for all n, almost surely and:

- S satifies (a) and (b).

## 2.7 Supermartingale

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *supermartingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] \le S_n$, for all n, almost surely and:

- S satifies (a) and (b).

## 2.8 Local Martingale

If $\{S_n\}$ is adapted to the filtration $\{\mathcal{F}_n\}$, for all $0 \leq t \leq \infty$, then $\{S_n : 0 \leq t \leq \infty\}$ is called a *local martingle* provided that there is nondecreasing sequence $\{\tau_k\}$ of stopping times with the property that $\tau_k \to \infty$ with probability one as $k \to \infty$ and such that for each $k$, the process defined by

$$S_t^{(k)} = S_{t \wedge \tau_k} - S_0$$

for $t \in [0, \infty)$ is a martingale with respect to the filtration

$$\{\mathcal{F}_n : 0 \leq t < \infty\}$$

## 2.9 Remark

A strict local martingale is a non-negative local martingale.

## 2.10 Theorem

If for any strict local martingale

$$\{S_t : 0 \leq t \leq T\}$$

with $E[|S_0|] < \infty$ is also a supermartingale and $E[S_T] = E[S_0]$, then $\{S_t : 0 \leq t \leq T\}$ is in fact a martingale.

## 2.11 Remark

- $\{S_t : 0 \leq t \leq T\}$ is a supermartingale and a martingale if and only if it has constant expectation.
- For a strict local martingale its expectation decreases with time.

## 2.12 Relating Martingales and Bubbles

## 2.13 theorem

$\{S_t : 0 \leq t \leq T\}$ is a strict local martingale if and only if

$$\int_\alpha^\infty \frac{x}{\sigma^2(x)} dx < \infty \tag{3}$$

for all $\alpha > 0$.

- A bubble exists if and only if (6) is finite.
- We shall call (6) the volatility of asset return.
- In this scope, the difference between a martingale and a strict local martingale is whether the volatility of asset return, (6), is finite or not finite.

## 2.14 Methods for Determining Price Bubbles

- *Florens-Zimirou Estimator*
- Smooth Kernel Estimator
- Unbounded Volatility Function Estimator
- Parametric Estimation
- Reproducing Kernel Hilbert Space Methods

## 2.15 What is the Florens-Zimirou Estimator?

This estimator is a non-parametric estimator based on the local time of the diffusion process. The local time of a diffusion is given by:

## 2.16 Random Variable

Random Variable is a variable whose value is subject to variations due to chance. Random variable conceptually does not have a single, fixed value rather, it can take on a set of possible different values, each with an associated probability.

Given a probability space $(\Omega, \mathcal{F}, P)$ the function $X$: $(\Omega, \mathcal{R})$s a real-valued random variable if
$w : X(w) \leq r \in \mathcal{F} \forall r \in \mathcal{R}$

## 2.17 Reproducing Kernal Hilbert Space Definations

A inner product $\langle u, v\rangle$can be 1. a usual dot product: $\langle u, v\rangle = v'w = \sum_i v_i w_i$ 2. a kernal product : $\langle u, v\rangle = k(v, w) = \varphi(v)'\varphi(w)$ (where $\varphi(u)$ may have infinite dimensions)

## 2.18 Hilbert Space

A hilbert space is an inner product that is complete and sepatable with respect to the norm defined by inner product.

## 2.19 Reproducing Kernal Hilbert Space

$k()$ is a reproducing kernal of hilbert space
$\mathcal{H}$ if $\forall f \in \mathcal{H}, f(x) = \langle k(x,.), f(.)\rangle$
A Reproducing Kernal Hilbert Space (RKHS) is a hilbert space H with a reproducing kernal whose span is danse in H.

## 2.20 Kernal

$k : \mathcal{X} X \mathcal{X} \rightarrow \mathcal{R}$ is a kernal if

1. $k$ is symmetric: $k(x, y) = k(y, x)$.

2. $k$ is positive semi-definite, i.e., $\forall x_1, x_2 ......., x_n \in \mathcal{X}$ the "Gram Matrix" K defined by $K_{ij} = k(x_i, xj$

A RKHS possesses many useful properties for data interpolation and function approximation problems.

## 2.21 1. Reproducing Property

:
There exists a kernal function $Q(x, x')$ the reproducing kernal in $H(D)$ such tht the following properties hold
$f(x) = \langle f(x'), Q(x, x')\rangle'$

$Q(x, y) = \langle Q(x, x'), Q(y, x')\rangle'$ where $\langle, \rangle$ is inner product over $x'$

## 2.22 2. Uniqueness

The RKHS $H(D)$ has one and only one reproducing kernal $Q(x, x')$

## 2.23 3. Symmetry and positivity

$Q(x, x')$ is symmetric means $Q(x, x') = Q(x', x)$
Positive definite means $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i Q(x_i, x_j) c_j \geq 0$ where $c_i$ any set of real numbers and $x_i$ any countable set of points

## 2.24 Well Conditined solution

Small changed in input to the expression will make small changes in output.

## 2.25 Ill-Conditioned solution

Small changes in input to the expression will make large changes in putput.

## 2.26 Beta Function

$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$
for $Re(x), Re(y)\ 0$

## 2.27 Gaussian Hypergeometric function

for $|z| < 1$
$F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$

it is undefined (or infinite) if $c$ equals a non=positive integer. Here $(q)_n$ is the (rising) Pochhammer symbol which is defined by :

BACKUP DEFINATIIONS:

## 2.28 Miscellaneous Financial Terms

- Market Price - The current price of an asset.
- Fundamental Price - The actual value of an asset based on an underlying perception of its *true value.*
- Risk - Variance of return on an asset
- Portfolio - Set of Assets.

## 2.29 Miscellaneous Mathematical Terms

- Probability Space $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set ( sample space), $\mathcal{F}$ is a sigma algebra of subsets (events) of $\Omega$, and $P$ is a Probability Measure.

- Random Variable - Measurable functions of real analysis. $X : \Omega \to \mathcal{R}$ map $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{R}, \mathcal{B})$ and $X$ is random variable if $X^{-1}(A)\epsilon\mathcal{F}, \forall A\epsilon\mathcal{B})$,
  where $X^{-1}(A) := \omega\epsilon\Omega \mid X(\omega)\epsilon A$

### 2.30 Types of Martingales

Martingale - Fair Game

- $S_n$ is Total winning per dollar stock up to time $n$.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, $\forall n$.

Super Martingale - Unfavorable Game

- $S_n$ is Total winning per dollar stock up to time $n$.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $E[S_{n+1} \mid \mathcal{F}_n] < S_n$, $\forall n$

Sub Martingale - Unfavorable Game

- $S_n$ is Total winning per dollar stock up to time $n$.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $S[M_{n+1} \mid \mathcal{F}_n] > S_n$, $\forall n$

# 3 Chapter 2

## 3.1 Theoretical Background

First of all we will introduce Stochastic Differential Equations. SDE are being used in various fields for example biology, physics, mathematics and of course finance. In finance, SDE is used to model asset price included with Brownian motion. Here we will use constant parameters drift and diffusion coeficients. With these constants , we will use euler muruyama method to model asset price.

In this chapter, we will focus on numerical solution of stochastic differential equations (SDE). It will give us better understanding toward the theory behind SDE. SDE are used in various areas like biology, chemistry, economics and of course finance. We will also study Brownian motion and compute Brownian paths with different methods. Euler -Maruyama method, strong and weak convergence , milstein method are being used to show solutions of SDE.

Now let's start with finance knowledge, Suppose the market price of an asset increases significantly. How can one determine if the market price is inflated above the actual price of an asset? This price behavior is know as a bubble.
To model price bubbles, we want to consider the following:

- What is an asset price bubble?
- How does one determine if an asset price is experiencing a bubble?
- Can one detect an asset price bubble in *real-time*?

Lets's consider finance definations :

- Market Price - The current price of an asset.
- Fundamental Price - The actual value of an asset based on an underlying perception of its *true value*.
- Risk - Variance of return on an asset
- Portfolio - Set of Assets.
- Asset Bubble- The difference between the market and fundamental price, if any, is a price bubble.
- Strike price -The strike price or exercise price of an option is the fixed price at which the owner of the option can buy( in the case of call) or sell (in the case of a put) the underlying security or commodity.
- Volatility-Rate at which the price of security moves up and down.

## 3.2 Probability Measure

### 3.2.1 Probability Space

$(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set ( sample space), $\mathcal{F}$ is a sigma algebra of subsets (events) of $\Omega$, and $P$ is a Probability Measure.

### 3.2.2 Random Variable

- Measurable functions of real analysis. $X : \Omega \to \mathcal{R}$ map $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{R}, \mathcal{B})$ and $X$ is random variable if
$X^{-1}(A) \epsilon \mathcal{F}, \forall A \epsilon \mathcal{B})$,
where $X^{-1}(A) := \omega \epsilon \Omega \mid X(\omega) \epsilon A$

## 3.3 Stochastic Differential Equations

We treat the asset price as a stochastic process:

### 3.3.1 Stochastic Process

Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process with state space X is a collection of X-valued random variables, $S_t$, on $\Omega$ indexed by a set $T$ (e.g. time).

$$S = \{S_t : t \in T\} \tag{4}$$

One can think of $S_t$ as a asset price at time $t$.

### 3.3.2 Stochastic Differential Equation

A differential equation with one or more terms is a stochastic process.

## 3.4 Brownian Motion

$\{S_t : 0 \leq t \leq T\}$:

$$\begin{aligned} dS_t &= \sigma(S_t)dW_t + \mu(S_t)dt \\ S_0 &= 0 \end{aligned} \tag{5}$$

- $W_t$ denotes the standard Brownian Motion.
- $\mu(S_t)$ called the drift coefficient.
- $\sigma(S_t)$ called the volatility coefficient.

### 3.4.1 Defination of Brownian motion

A continuous-time stochastic process $\{S_t : 0 \leq t \leq T\}$ is called a *Standard Brownian Motion* on $[0, T]$ if it has the following four properties:

(i) $S_0 = 0$

(ii) The increment of $S_t$ are independent; given

$$0 \leq t_1 < t_2 < t_3 < \cdots < t_n \leq T$$

the random variables $(S_{t_2} - S_{t_1})$, $(S_{t_3} - S_{t_2})$, $\cdots$, $\left(S_{t_n} - S_{t_{n-1}}\right)$ are independent.

(iii) $(S_t - S_s)$, $0 \leq s \leq t \leq T$ has the Gaussian distribution with mean zero and variance $(t - s)$

(iv) $S_t(W)$ is a continuous function of $t$, where $W \in \Omega$.

## 3.5 Martingales

(a) $E[| S_n |] < +\infty$, for all n.

(b) $S_n$ is said to be *adapted* if and only if $S_n$ is $\mathcal{F}_n$-measurable.

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *martingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, for all n, almost surely and:

- S satifies (a) and (b).

## 3.6 Supermartingale

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *supermartingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] \leq S_n$, for all n, almost surely and:

- S satifies (a) and (b).

## 3.7 Local Martingale

If $\{S_n\}$ is adapted to the filtration $\{\mathcal{F}_n\}$, for all $0 \leq t \leq \infty$, then $\{S_n : 0 \leq t \leq \infty\}$ is called a *local martingle* provided that there is nondecreasing sequence $\{\tau_k\}$ of stopping times with the property that $\tau_k \to \infty$ with probability one as $k \to \infty$ and such that for each $k$, the process defined by

$$S_t^{(k)} = S_{t \wedge \tau_k} - S_0$$

for $t \in [0, \infty)$ is a martingale with respect to the filtration

$$\{\mathcal{F}_n : 0 \leq t < \infty\}$$

## 3.8 Remark

A strict local martingale is a non-negative local martingale.

## 3.9 Theorem

If for any strict local martingale

$$\{S_t : 0 \leq t \leq T\}$$

with $E\left[|S_0|\right] < \infty$ is also a supermartingale and $E\left[S_T\right] = E\left[S_0\right]$, then $\{S_t : 0 \leq t \leq T\}$ is in fact a martingale.

## 3.10 Remark

- $\{S_t : 0 \leq t \leq T\}$ is a supermartingale and a martingale if and only if it has constant expectation.
- For a strict local martingale its expectation decreases with time.

## 3.11 Relating Martingales and Bubbles

## 3.12 theorem

$\{S_t : 0 \leq t \leq T\}$ is a strict local martingale if and only if

$$\int_{\alpha}^{\infty} \frac{x}{\sigma^2(x)} dx < \infty \tag{6}$$

for all $\alpha > 0$.

- A bubble exists if and only if (6) is finite.
- We shall call (6) the volatility of asset return.
- In this scope, the difference between a martingale and a strict local martingale is whether the volatility of asset return, (6), is finite or not finite.

## 3.13 Numerical Methods of SDE

For $t \in [0, T]$, (5) can be represented in an integral form in the following way:

$$\begin{aligned}
dS_t &= \sigma(t) dW_t + \mu(t) dt \\
\int_0^t dS_t &= \int_0^t \sigma(S_t)\, dW_t + \int_0^t \underbrace{\mu(S_t)}_{\in \mathcal{R}^+} dt \\
S_t - S_0 &= \int_0^t \sigma(S_t)\, dW_t + \left( \underbrace{\mu(S_t) \cdot t}_{x_0} - \mu(S_t) \cdot 0 \right) \\
S_t &= x_0 + \int_0^t \sigma(S_t)\, dW_t
\end{aligned}$$

### 3.13.1 What is $S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t$ ?

The price model is

$$S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t \tag{7}$$

### 3.13.2 What is $S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t$ ?

$$\begin{aligned}
dS_t &= \mu(S_t) dt + \sigma(S_t) dW_t \\
S_0 &\in \mathcal{R}
\end{aligned} \tag{8}$$

### 3.13.3 The Euler-Maruyama Method

Equation (4) can be written into integral form as:

$$S_t = S_0 + \int_0^t f\left(S_s\right) ds + \int_0^t g\left(S_s\right) dW(s), t \in [0, T] \tag{9}$$

f, g are scaler function with $S_0 = x_0$ random variable

$$\{\, d\ S_t = \mu(S_t)dt + \sigma(S_t)dW_t S(0) = S_0$$

Using Euler Maruyama method:
$w_0 = S_0$

$w_{i+1} = w_i + a(t_i, w_i) \triangle t_{i+1} + b(t_i, w_i) \triangle W_{i+1}$

$w_{i+1} = w_i + \mu w_i \triangle t_i + \sigma w_i \triangle W_i$

$\triangle t_{i+1} = t_{i+1} - t_i$

$\triangle W_{i+1} = W(t_{i+1} - W(t_i)$
Now drift coefficient $\mu$ and diffusion coefficient $\sigma$ are constants, the SDE has an exact solution:

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \tag{10}$$

## 3.14 What is $S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t$ ?

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \tag{11}$$

For an example, we use the Euler-Maruyama Approximation Method on the SDE where the constants $\mu = 2$, $\sigma = 1$, and $S_0 = 1$ are given.

There are other methods such as Strong and weak convergence of the Euler Muruyama method, Milstein's Higher Order Method, Linear Stability and Stochastic Chain Rule are also used for numerical solutions for SDE.

- From this, we will focus on real time stock data. We will have couple estimators to determine volatility function. For instance,
- We will assume that $\sigma$ is not constant. We will approximate $\sigma$ with non parametric estimator method on local time.

## 3.15 Methods used to solve the problem

In this classical setting, Jarrow, Protter, and Shimbo [19], [20] show that there are three types of asset price bubbles possible. Two of these price bubbles exist only in infinite horizon economies, the thirdcalled type 3 bubblesexist in finite horizon settings. Consequently, type 3 bubbles are those most rele- vant to actual market experiences. For this type of bubble, saying whether or not a bubble exists amounts to determining whether the price process under a risk neutral measure is a martingale or a strict local martingale: if it is a strict local martingale, there is a bubble.

Stock price is strict local martingale if and only if

$$\int_{\alpha}^{\infty} \frac{x}{\sigma(x)} dx < \infty forall \alpha > 0 \tag{12}$$

Floren Zmirou's non parametric estimator is based on the local time of the Diffusion Process.

### 3.15.1 Diffusion Process

In probability theory, a branch of mathematics, a diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths.

## 3.16 Floren Zmirou

Lets consider following equation:

$$S_t = S_0 + \int_0^t \sigma(S_t)\, dW_t \tag{13}$$

In Floren Zmirou, the drift coefficient $\mu(S_t)$ is null which is ignored without loss of generality. It is not involved in our problem. $\sigma(S_t)$ the volatility coefficient is unknown. We will follow steps for Floren Zmirou method:

- $(S_{t1} \ldots S_{tn})$ are the stock prices in the interval $t_1 \ldots t_n \in [0, T]$
- Without loss of generality, we assume T = 1, therefore $t_i = i/n$

Estimator as follows:
Local time = $l_T(x) = lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^T 1_{\{|S_s - x| < \epsilon\}} d\langle S, S \rangle_s$

where $d\langle S, S \rangle_s = \sigma^2(S_s)$

$L_T(x) = lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^T 1_{\{|S_s - x| < \epsilon\}} dS$

$\Longrightarrow l_T(x) = \sigma^2(S_s) L_T(x)$

$\Longrightarrow l_T(x) Ł_T(x) = \sigma^2(S_s)$

Now we will define local time of $S_s$ in $x$ during [0,t].

Let's assume that $nh_n \to \infty$ and $h_n \to 0$

$\implies L_T^n(x) = \frac{T}{2nh_n} \Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}}$

estimator of $\sigma^2(S_x)$ as follows:

$S_n(x) = \frac{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}} n(S_{t_i+1} - S_{t_i})^2}{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}}}$

## 3.17 Theorem

If $Sigma$is bounded above and below from zero, has three continuous and bounded derivatives, and if $(h_n)_n$1satisfies $nh_n \to \infty$ and $nh_4n \to 0$, then $S_n(x)$ is a consistent estimator of $\sigma^2(x)$.

Now we have $\sigma^2(x)$ and we will use interpolation methods to see function's behaviour.

## 3.18 Interpolation

Interpolation is a method of constructing new data points within the range of a discrete set of known data points. There are many methods to do interpolation. For example Linear Interpolation,polynomial interpolation, piecewise constant interpolation, spline interpolation. Here we will interpolate an estimate of $\sigma^2(x_i)$ where $i \in [1.M]$ within the bounded finite interval $D$ where we have observations. We used cubic spline interpolation and Reproducing Kernal Hilbert Spaces to get interpolation function.

- Interpolation is seen as inverse problem.
- We will have two types of solutions for inverse problem.

### 3.18.1 Normal Solution

It will allows an exact interpolation with minimal squared norm.

### 3.18.2 Regularized Solution

it will yields quasi interpolative results, accompained by an error bound analysis with Tikhonov Regularization produces an approximate solution $f_\alpha$ which belongs to $H(D)$and that can be obtained via the minimization of the regularization functional. Regularized solution with Kernal function For interpolation, we denote kernal function $K_{n,\tau}^{a,b}$ for n =1 and n = 2. of $H^n(a,d)$ where $D = (a,b)$

Now one we have interpolation function and extended form of the estimated

$\sigma^2(x)$, we can decide if there is need of extrapolation.

- if the volatility $\sigma^2(x_i)$ doesn not diverge to $\infty$ when $x \to \infty$ and it remains bounded on $\mathcal{R}^+$.

  - No extrapolation is required.

  - $\int_\epsilon^\infty \frac{x}{\sigma^2(x)}$ is infinite.

  - The process is true martingale.

- If the volatility diverges to $\infty$ when $x \to \infty$ then we will extrapolate.

## 3.19 Chapter 2

Chapter 2