CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

# METHODS TO SOLVE ASSET BUBBLE IN FINANCE

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by

Jaspreet Kaur

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The thesis of Jaspreet Kaur is approved:

| | |
|---|---|
| Dr. Stephen Breen | Date |
| Dr. Vladislav Panferov | Date |
| Dr. Jorge Balbás, Chair | Date |

California State University, Northridge

Dedication

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ABSTRACT

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Financial Market is very attracting topic in finance and mathematics world. Recently we have heard a lot about Gold Prices inflations. It is the the hot topic in today's finance market. So how will be combine mathematics with today's asset changes like gold? How can we determine the tale of asset's volatility for future? These are the questions which we will consider in this thesis. We will study non parametric estimator Floren Zmirou in local real time on compact domain with stochastic differential equation which has unknown drift and diffusion coeificents. Once we will have volatility from floren zmirou then we will able to use RKHS to estimates function which will extrapolate the tale of function.

## 0.1 Theoretical Background

First of all we will introduce Stochastic Differential Equations SDE. SDE are used in biology, physics, mathematics and of course finance. Our topic focus on finance, SDE is used to model asset price with Brownian motion. In this thesis work, we will use constant parameters drift $\mu$ and diffusion $\sigma$ coeficients. Let's suppose we have $\mu$ and $\sigma$ With some constant coeficients, then we will able to use euler muruyama method to model asset price.

In this chapter, we will focus on numerical solution of stochastic differential equations (SDE). By reviwing numberical solutions, we will have better understanding toward the theory behind SDE. We will also study Brownian motion with different methods for example Euler -Maruyama method, strong and weak convergence , milstein method are being used to show solutions of SDE.

Now let's start with little background of finance, Suppose the market price of an asset increases significantly. How can one determine if the market price is inflated above the actual price of an asset? This price behavior is know as a bubble.

To model price bubbles, we want to consider the following:

- What is an asset price bubble?
- How does one determine if an asset price is experiencing a bubble?

**Definition 0.1.1 (Market Price)** The current price of an asset.

**Definition 0.1.2 (Fundamental Price)** The actual value of an asset based on an underlying perception of its *true value*.

**Definition 0.1.3 (Risk)** Variance of return on an asset.

**Definition 0.1.4 (Portfolio)** Set of Assets.

**Definition 0.1.5 (Asset Bubble)** The difference between the market and fundamental price, if any, is a price bubble.

**Definition 0.1.6 (Volatility)** Rate at which the price of security moves up and down.

Lets's consider mathematical definations :

**Definition 0.1.7 (Probability Space)** $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set ( sample space), $\mathcal{F}$ is a sigma algebra of subsets (events) of $\Omega$, and $P$ is a Probability Measure.

**Definition 0.1.8 (Random Variable)** Measurable functions of real analysis $X : \Omega \mapsto \mathcal{R}$ map $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{R}, \mathcal{B})$ and $X$ is random variable if

$$X^{-1}(A) \epsilon \mathcal{F}, \forall A \epsilon \mathcal{B}$$

where $X^{-1}(A) := \omega \epsilon \Omega \mid X(\omega) \epsilon A$

We treat the asset price as a stochastic process:

**Definition 0.1.9 (Stochastic Process)** Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process with state space X is a collection of X-valued random variables, $S_t$, on $\Omega$ indexed by a set $T$ (e.g. time).

$$S = \{S_t : t \in T\} \tag{1}$$

One can think of $S_t$ as a asset price at time $t$.

**Definition 0.1.10 (Stochastic Differential Equation)** A differential equation with one or more terms is a stochastic process.

**Definition 0.1.11 (Brownian Motion)** $\{S_t : 0 \leq t \leq T\}$:

$$\begin{aligned} dS_t &= \sigma(S_t)dW_t + \mu(S_t)dt \\ S_0 &= 0 \end{aligned} \tag{2}$$

- $W_t$ denotes the standard Brownian Motion.
- $\mu(S_t)$ called the drift coefficient.
- $\sigma(S_t)$ called the volatility coefficient.

**Definition 0.1.12 (Brownian motion)** A continuous-time stochastic process $\{S_t : 0 \leq t \leq T\}$ is called a *Standard Brownian Motion* on $[0, T]$ if it has the following four properties:

(i) $S_0 = 0$

(ii) The increment of $S_t$ are independent; given

$$0 \leq t_1 < t_2 < t_3 < \cdots < t_n \leq T$$

the random variables $(S_{t_2} - S_{t_1}), (S_{t_3} - S_{t_2}), \cdots, \left(S_{t_n} - S_{t_{n-1}}\right)$ are independent.

(iii) $(S_t - S_s), 0 \leq s \leq t \leq T$ has the Gaussian distribution with mean zero and variance $(t - s)$

(iv) $S_t(W)$ is a continuous function of $t$, where $W \in \Omega$.

**Definition 0.1.13 (Martingales)**

(a) $E[\mid S_n \mid] < +\infty$, for all n.

(b) $S_n$ is said to be *adapted* if and only if $S_n$ is $\mathcal{F}_n$-measurable.

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *martingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, for all n, almost surely and:

- S satifies (a) and (b).

**Definition 0.1.14 (Supermartingale)** The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *supermartingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] \leq S_n$, for all n, almost surely and:

- S satifies (a) and (b).

**Definition 0.1.15 (Local Martingale)** If $\{S_n\}$ is adapted to the filtration $\{\mathcal{F}_n\}$, for all $0 \leq t \leq \infty$, then $\{S_n : 0 \leq t \leq \infty\}$ is called a *local martingle* provided that there is nondecreasing sequence $\{\tau_k\}$ of stopping times with the property that $\tau_k \to \infty$ with probability one as $k \to \infty$ and such that for each $k$, the process defined by

$$S_t^{(k)} = S_{t \wedge \tau_k} - S_0$$

for $t \in [0, \infty)$ is a martingale with respect to the filtration

$$\{\mathcal{F}_n : 0 \leq t < \infty\}$$

- Remark A strict local martingale is a non-negative local martingale.

**Theorem 0.1.1** *If for any strict local martingale*

$$\{S_t : 0 \leq t \leq T\}$$

*with $E[|S_0|] < \infty$ is also a supermartingale and $E[S_T] = E[S_0]$, then $\{S_t : 0 \leq t \leq T\}$ is in fact a martingale.*

Remarks

- $\{S_t : 0 \leq t \leq T\}$ is a supermartingale and a martingale if and only if it has constant expectation.
- For a strict local martingale its expectation decreases with time.

Now we will see how Martingales and Bubbles are related to each other.

**Theorem 0.1.2**
$$\{S_t : 0 \leq t \leq T\}$$
*is a strict local martingale if and only if*

$$\int_{\alpha}^{\infty} \frac{x}{\sigma^2(x)} dx < \infty \tag{3}$$

*for all* $\alpha > 0$.

- A bubble exists if and only if (3) is finite.
- We shall call (3) the volatility of asset return.

- In this scope, the difference between a martingale and a strict local martingale is whether the volatility of asset return, (3), is finite or not finite.

### 0.1.1 Numerical Methods of SDE

For $t \in [0, T]$, $(2)$ can be represented in an integral form in the following way:

$$\begin{aligned}
dS_t &= \sigma(t)dW_t + \mu(t)dt \\
\int_0^t dS_t &= \int_0^t \sigma\left(S_t\right) dW_t + \int_0^t \underbrace{\mu\left(S_t\right)}_{\in \mathcal{R}^+} dt \\
S_t - S_0 &= \int_0^t \sigma\left(S_t\right) dW_t + \left(\underbrace{\mu\left(S_t\right) \cdot t}_{x_0} - \mu\left(S_t\right) \cdot 0\right) \\
S_t &= x_0 + \int_0^t \sigma\left(S_t\right) dW_t
\end{aligned}$$

#### 0.1.1.1 What is $S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t$ ?

The price model is

$$S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t \tag{4}$$

$$\begin{aligned}
dS_t &= \mu(S_t)dt + \sigma(S_t)dW_t \\
S_0 \in \mathcal{R} &
\end{aligned} \tag{5}$$

#### 0.1.1.2 The Euler-Maruyama Method

Equation (4) can be written into integral form as:

$$S_t = S_0 + \int_0^t f\left(S_s\right) ds + \int_0^t g\left(S_s\right) dW(s) \quad t \in [0, T] \tag{6}$$

$f$, $g$ are scalar function with $S_0 = x_0$ **fix me** random variable

$$\begin{cases} dS_t = \mu(S_t)dt + \sigma(S_t)dW_t \\ S(0) = S_0 \end{cases}$$

Using Euler Maruyama method:
$w_0 = S_0$ $w_{i+1} = w_i + a(t_i, w_i) \triangle t_{i+1} + b(t_i, w_i) \triangle W_{i+1}$ $w_{i+1} = w_i + \mu w_i \triangle t_i + \sigma w_i \triangle W_i$
$\triangle t_{i+1} = t_{i+1} - t_i$ $\triangle W_{i+1} = W(t_{i+1} - W(t_i)$
Now drift coefficient $\mu$ and diffusion coefficient $\sigma$ are constants, the SDE has an exact solution:

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \tag{7}$$

For an example, we use the Euler-Maruyama Approximation Method on the SDE where the constants $\mu = 2$, $\sigma = 1$, and $S_0 = 1$ are given. There are other methods such as Strong and weak convergence of the Euler Muruyama method, Milstein's Higher Order Method, Linear Stability and Stochastic Chain Rule are also used for numerical solutions for SDE.

- What will happen if $\sigma$ is not constant. We will approximate $\sigma$ with non parametric estimator method on local real time.

### 0.1.2 Our Probelm and method used to solve it

In this classical setting, Jarrow, Protter, and Shimbo [19], [20] show that there are three types of asset price bubbles possible. Two of these price bubbles exist only in infinite horizon economies, the thirdÑcalled type 3 bubblesÑexist in finite horizon settings. Consequently, type 3 bubbles are those most rele- vant to actual market experiences. For this type of bubble, saying whether or not a bubble exists amounts to determining whether the price process under a risk neutral measure is a martingale or a strict local martingale: if it is a strict local martingale, there is a bubble. Stock price is strict local martingale if and only if

$$\int_\alpha^\infty \frac{x}{\sigma\left(x\right)} dx < \infty forall \alpha > 0 \tag{8}$$

First We will obtain data from Google Finance and Yahoo Finance. Algorithms 1, 2 and 3 shows the process how to get Stock Prices. Floren Zmirou's non parametric estimator is based on the local time of the Diffusion Process.

**Definition 0.1.16 (Diffusion Process)** In probability theory, a branch of mathematics, a diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths.

### 0.1.3 Theorem1

**Theorem 0.1.3** *If $\sigma$ is bounded above and below from zero, has three continuous and bounded derivatives, and if $(h_n)_n{}^3 1$satisfies $nh_n \to \infty$ and $nh_4 n \to 0$, then $S_n(x)$ is a consistent estimator of $\sigma^2(x)$.*

In Theorem 0.1.3 we say XYZ. . . .

### 0.1.4 Theorem2

If $nh_n \to 0$, then $\sqrt{N_x^n}((\frac{S_n(x)}{\sigma^2(x)}) - 1)$ converges in distribution to $\sqrt{2}Z$ where Z is a standard normal ransom variable and $N_x^n$ = $1_{\{|S_{ti} - x| < hn\}}$ Floren Zmirou estimator requires a grid step which is $h_n \to 0$ such that $nh_n \to \infty$ and $nh_n^4 \to 0$. We will choose step size $h_n = \frac{1}{n^{\frac{1}{3}}}$ so that all the conditions will be satisfied.

### 0.1.5 Floren Zmirou

Lets consider following equation:

$$S_t = S_0 + \int_0^t \sigma\left(S_t\right) dW_t \tag{9}$$

In Floren Zmirou, the drift coefficient $\mu(S_t)$ is null which is ignored without loss of generality. It is not involved in our problem. $\sigma(S_t)$ the volatility coefficient is unknown. We will follow steps for Floren Zmirou method:

- $(S_{t_1}, \ldots, S_{t_n})$ are the stock prices in the interval $t_1, \ldots, t_n \in [0, T]$
- Without loss of generality, we assume $T = 1$, therefore $t_i = \frac{i}{n}$

Estimator as follows:
Local time = $l_T(x) = lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^T 1_{\{|S_s - x| < \epsilon\}} d\langle S, S\rangle_s$ where $d\langle S, S\rangle_s = \sigma^2(S_s)$ $L_T(x) = lim_{\epsilon \longrightarrow 0} \frac{1}{2\epsilon} \int_0^T 1_{\{|S_s - x| < \epsilon\}} dS \Longrightarrow l_T(x) = \sigma^2(S_s) L_T(x) \Longrightarrow l_T(x)_T(x) = \sigma^2(S_s)$ Now we will define local time of $S_s$ in $x$ during [0,t]. Let's assume that $nh_n \to \infty$ and $h_n \to 0$ $\Longrightarrow L_T^n(x) = \frac{T}{2nh_n} \Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}}$ estimator of $\sigma^2(S_x)$ as follows:

$$S_n(x) = \frac{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}} n (S_{t_i+1} - S_{t_i})^2}{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}}} \tag{10}$$

$S_n(x) = \frac{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}} n (S_{t_i+1} - S_{t_i})^2}{\Sigma_{i=1}^n 1_{\{|S_{t_i} - x| < h_n\}}}$

Now we have $\sigma^2(x)$ and we will use interpolation methods to see function's behaviour.

### 0.1.6 Interpolation

Interpolation is a method of constructing new data points within the range of a discrete set of known data points. There are many methods to do interpolation. For example Linear Interpolation,polynomial interpolation, piecewise constant interpolation, spline interpolation. Here we will interpolate an estimate of $\sigma^2(x_i)$ where $i \in [1.M]$ within the bounded finite interval $D$ where we have observations. We used cubic spline interpolation and Reproducing Kernal Hilbert Spaces to get interpolation function.

#### 0.1.6.1 Cubic Spline Interpolation

- **Piecewise-polynomial approximation** : An alternative approach is to divide the interval into a collection of subintervals and construct a different approximating polynomial on each subinterval. Approximation by functions of this type is called piecewise-polynomial approximation.
- The most common piecewise-polynomial approximation uses cubic polynomial between each sucessive pair of nodes and is called **cubic spline interpolation**.
- **Cubic Spline Interpolation**
  Given a function f defined on [a,b] and a set of nodes $a = x_o < x_1 < \ldots < x_n = b$ a **cubic spline interpolant S** for f is a function that satisfies the following conditions: a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$ on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, 2, \ldots, n-1$; b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, 2, \ldots, n-1$; c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, 2, \ldots, n-2$; d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, 2, \ldots, n-2$; e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, 2, \ldots, n-2$; f) One of the following sets of boundry conditions is satisfied; i) $S''(x_0) = S''x_n) = 0$; ii) $S'(x_0) = f'x_0)$ and $S'(x_n) = f'x_n)$

- **Natural Spline** When the free boundry conditions occur, the spline is called **natural spline**.

#### 0.1.6.2 Interpolation is seen as inverse problem.

#### 0.1.6.3 We will have two types of solutions for inverse problem.

#### 0.1.6.4 Normal Solution

It will allows an exact interpolation with minimal squared norm.

#### 0.1.6.5 Regularized Solution

it will yields quasi interpolative results, accompained by an error bound analysis with Tikhonov Regularization produces an approximate solution $f_\alpha$ which belongs to $H(D)$and that can be obtained via the minimization of the regularization functional.

**Regularized solution with Kernal function** For interpolation, we denote kernal function $K^{a,b}_{n,\tau}$ for n =1 and n = 2. of $H^n(a,d)$ where $D=(a,b)$ Now one we have interpolation function and extended form of the estimated $\sigma^2(x)$ call this extended function $\sigma^b(x)$, we can decide if there is need of extrapolation.

- if the volatility $\sigma^2(x_i)$ doesn not diverge to $\infty$ when $x \to \infty$ and it remains bounded on $\mathcal{R}^+$. - No extrapolation is required. - $\int_\epsilon^\infty \frac{x}{\sigma^2(x)}$ is infinite. - The process is true martingale.
- If the volatility diverges to $\infty$ when $x \to \infty$ then we will extrapolate.

### 0.1.7 Extrapolation ,Optimization and Minimization

Now since we have extended function $\sigma^b(x)$, we will extrapolate this over $\mathcal{R}^+$.
Let's consider following optimization with minimazation problem:

$$\bar{m} = \underset{m\geq 0}{\text{argmin}} \sqrt{\int_{[a,\infty\cap D]} |\sigma_m - \sigma^b|^2 dS} \tag{11}$$

- $\sigma_m$ will interpolate the input data points when n = 2.
- $\sigma_{\bar{m}}$ has the asymptotic behavior that beast matches our function on the estimation interval. We will construct extrapolation by choosing the asymptotic weighting function parameter $m$ such that $f_m = \frac{1}{\sigma^2 m}$.
- $a$ = max S - $\frac{1}{3}$(maxS-min S)
- $H_{2,m}$ allows the best interpolation of M which is estimated points such that the extrapolation function remains as close as possible to $\sigma^b(x)$.
- We will plot function with different asymptotic weighting parameter m which is obtained from RKHS extrapolation method.
- The asymptotic weighting function's parameter $\bar{m}$ obtained by optimization and minimization is the most consistent which exactly match the input data within all the function in $H_{1,m}$