Abstract

Financial Market is very attracting topic in finance and mathematics world. Recently we have heard lot about Gold Prices inflations. It is the the hot topic in today's finance market. So how will be combine mathematics with today's asset changes? How can we determine the tale of asset's volatility for future? These are the questions which we will consider in this thesis. We will study non parametric estimator Floren Zmirou in local real time on compact domain with stochastic differential equation which has unknown drift and diffusion coeificents. Once we will have volatility from floren zmirou then we will able to use RKHS to estimates function which will extrapolate the tale of function.

1 Introduction

Today's economy, financial asset bubbles are exciting and hot topic. In most recent market news, we have read or seen big changes in Gold prices. Everyone is interested to know what will happen in the future. How we can able to detect or estimate the future changes of any asset (stock, gold, housing, commodity)? How quickly asset price will jump? These are the question which we will consider in this study. We will study how to determine whether any asset is experiencing a price bubble in real time. How we will detect asset bubbles in real time?

First of all we will introduce Stochastic Differential Equations. SDE are being used in various fields for example biology, physics, mathematics and of course finance. In finance, SDE is used to model asset price included with Brownian motion. Here we will use constant parameters drift and diffusion coeficients. With these constants , we will use euler muruyama method to model asset price.

Second we will introduce Floren Zmirou's nonparametric estimator which is based on local time of the diffusion process on compact domain. we only can estimate $\sigma(x)$ on points visited by the process. So we cant determine the tail of the volatility function to check whether the we have bubble or not.

Third, now we want to check the tail, so we will use RKHS (Reproducing Kernal Hilbert Space) method which will allow us to construct an interpolationg function that extends the nonparametric estimator from the observation interval to the entire real line.

With the help of all these steps, we can able to estimate the fuction which will detect whether asset price has bubble or not.

2 Definitions

Finance Definations

2.1 Asset Price Bubbles

The difference between the market and fundamental price, if any, is a price bubble.

2.1.1 Strike Price

The strike price or exercise price of an option is the fixed price at which the owner of the option can buy(in the case of call) or sell (in the case of a put) the underlying security or commodity.

2.1.2 Volatility

Rate at which the price of security moves up and down.

Mathematical Definations

2.2 Introduction to Stochastic Differential Equations

We treat the asset price as a stochastic process:

2.2.1 Stochastic Process

Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process with state space X is a collection of X-valued random variables, S_t, on Ω indexed by a set T (e.g. time).

$$S = \{S_t : t \in T\} \tag{1}$$

One can think of S_t as a asset price at time t.

2.2.2 Stochastic Differential Equation

A differential equation with one or more terms is a stochastic process.

2.3 The Price Asset Model using an SDE

Consider the linear SDE with a Brownian Motion $\{S_t : 0 \leq t \leq T\}$:

$$\begin{aligned} dS_t &= \sigma(S_t)dW_t + \mu(S_t)dt \\ S_0 &= 0 \end{aligned} \tag{2}$$

- W_t denotes the standard Brownian Motion.
- $\mu(S_t)$ called the drift coefficient.
- $\sigma(S_t)$ called the volatility coefficient.

2.4 The Price Asset Model using an SDE

2.5 Brownian Motion

A continuous-time stochastic process $\{S_t : 0 \leq t \leq T\}$ is called a *Standard Brownian Motion* on $[0, T]$ if it has the following four properties:

(i) $S_0 = 0$

(ii) The increment of S_t are independent; given

$$0 \leq t_1 < t_2 < t_3 < \cdots < t_n \leq T$$

the random variables $(S_{t_2} - S_{t_1})$, $(S_{t_3} - S_{t_2})$, $\cdots$, $\left(S_{t_n} - S_{t_{n-1}}\right)$ are independent.

(iii) $(S_t - S_s)$, $0 \leq s \leq t \leq T$ has the Gaussian distribution with mean zero and variance $(t - s)$

(iv) $S_t(W)$ is a continuous function of t, where $W \in \Omega$.

2.6 Martingales

(a) $E[\mid S_n \mid] < +\infty$, for all n.

(b) S_n is said to be *adapted* if and only if S_n is $\mathcal{F}_n$-measurable.

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *martingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, for all n, almost surely and:

- S satifies (a) and (b).

2.7 Supermartingale

The stochastic process $S = \{S_n\}_{n=0}^{\infty}$ is a *supermartingale* with respect to $(\{\mathcal{F}_n\}, P)$ if $E[S_{n+1} \mid \mathcal{F}_n] \leq S_n$, for all n, almost surely and:

- S satifies (a) and (b).

2.8 Local Martingale

If $\{S_n\}$ is adapted to the filtration $\{\mathcal{F}_n\}$, for all $0 \leq t \leq \infty$, then $\{S_n : 0 \leq t \leq \infty\}$ is called a *local martingle* provided that there is nondecreasing sequence $\{\tau_k\}$ of stopping times with the property that $\tau_k \to \infty$ with probability one as $k \to \infty$ and such that for each k, the process defined by

$$S_t^{(k)} = S_{t \wedge \tau_k} - S_0$$

for $t \in [0, \infty)$ is a martingale with respect to the filtration

$$\{\mathcal{F}_n : 0 \leq t < \infty\}$$

2.9 Remark

A strict local martingale is a non-negative local martingale.

2.10 Theorem

If for any strict local martingale

$$\{S_t : 0 \leq t \leq T\}$$

with $E\left[|S_0|\right] < \infty$ is also a supermartingale and $E\left[S_T\right] = E\left[S_0\right]$, then $\{S_t : 0 \leq t \leq T\}$ is in fact a martingale.

2.11 Remark

- $\{S_t : 0 \leq t \leq T\}$ is a supermartingale and a martingale if and only if it has constant expectation.
- For a strict local martingale its expectation decreases with time.

2.12 Relating Martingales and Bubbles

2.13 theorem

$\{S_t : 0 \leq t \leq T\}$ is a strict local martingale if and only if

$$\int_{\alpha}^{\infty} \frac{x}{\sigma^2(x)} dx < \infty \tag{3}$$

for all $\alpha > 0$.

- A bubble exists if and only if (3) is finite.
- We shall call (3) the volatility of asset return.
- In this scope, the difference between a martingale and a strict local martingale is whether the volatility of asset return, (3), is finite or not finite.

2.14 Methods for Determining Price Bubbles

- *Florens-Zimirou Estimator*
- Smooth Kernel Estimator
- Unbounded Volatility Function Estimator
- Parametric Estimation
- Reproducing Kernel Hilbert Space Methods

2.15 What is the Florens-Zimirou Estimator?

This estimator is a non-parametric estimator based on the local time of the diffusion process. The local time of a diffusion is given by:

2.16 Random Variable

Random Variable is a variable whose value is subject to variations due to chance. Random variable conceptually does not have a single, fixed value rather, it can take on a set of possible different values, each with an associated probability.

Given a probability space $(\Omega, \mathcal{F}, P)$ the function X: $(\Omega, \mathcal{R})$s a real-valued random variable if
$w : X(w) \leq r \in \mathcal{F} \forall r \in \mathcal{R}$

2.17 Reproducing Kernal Hilbert Space Definations

A inner product $\langle u, v \rangle$can be 1. a usual dot product: $\langle u, v \rangle = v'w = \sum_i v_i w_i$ 2. a kernal product : $\langle u, v \rangle = k(v, w) = \varphi(v)'\varphi(w)$ (where $\varphi(u)$ may have infinite dimensions)

2.18 Hilbert Space

A hilbert space is an inner product that is complete and sepatable with respect to the norm defined by inner product.

2.19 Reproducing Kernal Hilbert Space

$k()$ is a reproducing kernal of hilbert space
$\mathcal{H}$ if $\forall f \in \mathcal{H}, f(x) = \langle k(x,.), f(.)\rangle$
A Reproducing Kernal Hilbert Space (RKHS) is a hilbert space H with a reproducing kernal whose span is danse in H.

2.20 Kernal

$k : \mathcal{X} X \mathcal{X} \to \mathcal{R}$ is a kernal if

1. k is symmetric: $k(x, y) = k(y, x)$.

2. k is positive semi-definite, i.e., $\forall x_1, x_2......., x_n \in \mathcal{X}$ the "Gram Matrix" K defined by $K_{ij} = k(x_i, xj$

A RKHS possesses many useful properties for data interpolation and function approximation problems.

2.21 1. Reproducing Property

:
There exists a kernal function $Q(x, x')$ the reproducing kernal in $H(D)$ such tht the following properties hold
$f(x) = \langle f(x'), Q(x, x')\rangle'$

$Q(x, y) = \langle Q(x, x'), Q(y, x')\rangle'$ where $\langle, \rangle$ is inner product over x'

2.22 2. Uniqueness

The RKHS $H(D)$ has one and only one reproducing kernal $Q(x, x')$

2.23 3. Symmetry and positivity

$Q(x, x')$ is symmetric means $Q(x, x') = Q(x', x)$
Positive definite means $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i Q(x_i, x_j) c_j \geq 0$ where c_i any set of real numbers and x_i any countable set of points

2.24 Well Conditined solution

Small changed in input to the expression will make small changes in output.

2.25 Ill-Conditioned solution

Small changes in input to the expression will make large changes in putput.

2.26 Beta Function

$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$
for $Re(x), Re(y)$ 0

2.27 Gaussian Hypergeometric function

for $|z| < 1$
$F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$

it is undefined (or infinite) if c equals a non=positive integer. Here $(q)_n$ is the (rising) Pochhammer symbol which is defined by :

BACKUP DEFINATIIONS:

2.28 Miscellaneous Financial Terms

- Market Price - The current price of an asset.
- Fundamental Price - The actual value of an asset based on an underlying perception of its *true value.*
- Risk - Variance of return on an asset
- Portfolio - Set of Assets.

2.29 Miscellaneous Mathematical Terms

- Probability Space $(\Omega, \mathcal{F}, P)$ where Ω is a set (sample space), $\mathcal{F}$ is a sigma algebra of subsets (events) of Ω, and P is a Probability Measure.

- Random Variable - Measurable functions of real analysis. $X : \Omega \to \mathcal{R}$ map $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{R}, \mathcal{B})$ and X is random variable if $X^{-1}(A)\epsilon\mathcal{F}, \forall A\epsilon\mathcal{B})$,
  where $X^{-1}(A) := \omega\epsilon\Omega \mid X(\omega)\epsilon A$

2.30 Types of Martingales

Martingale - Fair Game

- S_n is Total winning per dollar stock up to time n.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $E[S_{n+1} \mid \mathcal{F}_n] = S_n$, $\forall n$.

Super Martingale - Unfavorable Game

- S_n is Total winning per dollar stock up to time n.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $E[S_{n+1} \mid \mathcal{F}_n] < S_n$, $\forall n$

Sub Martingale - Unfavorable Game

- S_n is Total winning per dollar stock up to time n.
- $(S_{n+1} - S_n)$ is net winning in game $n + 1$.
- $S[M_{n+1} \mid \mathcal{F}_n] > S_n$, $\forall n$

2.31 Chapter 1

Chapter 1

In this chapter, we will focus on numerical solution of stochastic differential equations (SDE). It will give us better understanding toward the theory behind SDE. SDE are used in various areas like biology, chemistry, economics and of course finance. We will also study Brownian motion and compute Brownian paths with different methods. Euler -Maruyama method, strong and weak convergence , milstein method are being used to show solutions of SDE.
Now let's start with finance knowledge, Suppose the market price of an asset increases significantly. How can one determine if the market price is inflated above the actual price of an asset? This price behavior is know as a bubble.
To model price bubbles, we want to consider the following:

- What is an asset price bubble?

- How does one determine if an asset price is experiencing a bubble?
- Can one detect an asset price bubble in *real-time*?

Lets's consider finance definations :

- Market Price - The current price of an asset.
- Fundamental Price - The actual value of an asset based on an underlying perception of its *true value*.
- Risk - Variance of return on an asset
- Portfolio - Set of Assets.
- Asset Bubble- The difference between the market and fundamental price, if any, is a price bubble.
- Strike price -The strike price or exercise price of an option is the fixed price at which the owner of the option can buy(in the case of call) or sell (in the case of a put) the underlying security or commodity.
- Volatility-Rate at which the price of security moves up and down.
- Probability Space $(\Omega, \mathcal{F}, P)$ where Ω is a set (sample space), $\mathcal{F}$ is a sigma algebra of subsets (events) of Ω, and P is a Probability Measure.
- Random Variable - Measurable functions of real analysis. $X : \Omega \to \mathcal{R}$ map $X : (\Omega, \mathcal{F}) \mapsto (\mathcal{R}, \mathcal{B})$ and X is random variable if $X^{-1}(A)\epsilon\mathcal{F}, \forall A\epsilon\mathcal{B}),$
  where $X^{-1}(A) := \omega\epsilon\Omega \mid X(\omega)\epsilon A$

2.32 Introduction to Stochastic Differential Equations

We treat the asset price as a stochastic process:

2.33 Stochastic Process

Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process with state space X is a collection of X-valued random variables, S_t, on Ω indexed by a set T (e.g. time).

$$S = \{S_t : t \in T\} \tag{4}$$

One can think of S_t as a asset price at time t.

2.34 Stochastic Differential Equation

A differential equation with one or more terms is a stochastic process.

The Price Asset Model using an SDE

Consider the linear SDE with a Brownian Motion $\{S_t : 0 \leq t \leq T\}$:

$$\begin{aligned} dS_t &= \sigma(S_t)dW_t + \mu(S_t)dt \\ S_0 &= 0 \end{aligned} \tag{5}$$

- W_t denotes the standard Brownian Motion.
- $\mu(S_t)$ called the drift coefficient.
- $\sigma(S_t)$ called the volatility coefficient.

2.35 The Price Asset Model using an SDE

Brownian Motion A continuous-time stochastic process $\{S_t : 0 \leq t \leq T\}$ is called a *Standard Brownian Motion* on $[0, T]$ if it has the following four properties:

(i) $S_0 = 0$

(ii) The increment of S_t are independent; given

$$0 \leq t_1 < t_2 < t_3 < \cdots < t_n \leq T$$

the random variables $(S_{t_2} - S_{t_1})$, $(S_{t_3} - S_{t_2})$, $\cdots$, $\left(S_{t_n} - S_{t_{n-1}}\right)$ are independent.

(iii) $(S_t - S_s)$, $0 \leq s \leq t \leq T$ has the Gaussian distribution with mean zero and variance $(t - s)$

(iv) $S_t(W)$ is a continuous function of t, where $W \in \Omega$.

2.36 The Price Asset Model using an SDE

For $t \in [0, T]$, (5) can be represented in an integral form in the following way:

$$\begin{aligned} dS_t &= \sigma(t)dW_t + \mu(t)dt \\ \int_0^t dS_t &= \int_0^t \sigma\left(S_t\right) dW_t + \int_0^t \underbrace{\mu\left(S_t\right)}_{\in \mathcal{R}^+} dt \\ S_t - S_0 &= \int_0^t \sigma\left(S_t\right) dW_t + \left(\underbrace{\mu\left(S_t\right) \cdot t}_{x_0} - \mu\left(S_t\right) \cdot 0 \right) \\ S_t &= x_0 + \int_0^t \sigma\left(S_t\right) dW_t \end{aligned}$$

2.37 What is $S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t$?

The price model is

$$S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t \tag{6}$$

2.38 What is $S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t$?

$$\begin{aligned} dS_t &= \mu(S_t)dt + \sigma(S_t)dW_t \\ S_0 \in \mathcal{R} & \end{aligned} \tag{7}$$

2.39 The Euler-Maruyama Method

Equation (4) can be written into integral form as:

$$S_t = S_0 + \int_0^t f\left(S_s\right) ds + \int_0^t g\left(S_s\right) dW(s), t \in [0, T] \tag{8}$$

f, g are scaler function with $S_0 = x_0$ random variable

$$\{ \, d \; S_t = \mu(S_t)dt + \sigma(S_t)dW_t S(0) = S_0$$

Using Euler Maruyama method:
$w_0 = S_0$

$w_{i+1} = w_i + a(t_i, w_i) \triangle t_{i+1} + b(t_i, w_i) \triangle W_{i+1}$

$w_{i+1} = w_i + \mu w_i \triangle t_i + \sigma w_i \triangle W_i$

$\triangle t_{i+1} = t_{i+1} - t_i$

$\triangle W_{i+1} = W(t_{i+1} - W(t_i)$
Now drift coefficient μ and diffusion coefficient σ are constants, the SDE has an exact solution:

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \tag{9}$$

2.40 What is $S_t = x_0 + \int_0^t \sigma\left(S_t\right) dW_t$?

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right) \tag{10}$$

For an example, we use the Euler-Maruyama Approximation Method on the SDE where the constants $\mu = 2$, $\sigma = 1$, and $S_0 = 1$ are given.

2.41 What is $S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t$?

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{11}$$

Figure 1: Euler-Maruyama Approximation

2.42 What is $S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t$?

$$S(t) = S_0 \cdot Exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{12}$$

Earlier we stated the price model is

$$S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t \tag{13}$$

There are other methods such as Strong and weak convergence of the Euler Muruyama method, Milstein's Higher Order Method, Linear Stability and Stochastic Chain Rule are also used for numerical solutions for SDE.

- From this, we will focus on real time stock data. We will have couple estimators to determine volatility function. For instance,
- We will assume that σ is not constant. We will approximate σ with non parametric estimator method on local time.

2.43 Chapter 2

Chapter 2

Lets consider following equation:

$$S_t = x_0 + \int_0^t \sigma(S_t)\, dW_t \tag{14}$$

Stock price is strict local martingale if and only if

$$\int_\alpha^\infty \frac{x}{\sigma(x)} dx < \infty for all \alpha > 0 \tag{15}$$

According to Jarrow, Protter show that there are three types of asset price bubbles, from those three, two types of bubbles exist only in infinite horizon economies and the third which is called type 3 bubbles which exits in finite horizon economy. Type 3 bubbles are realted to martingales. When we have price process under a risk neutral measure strict local martingale, then there will be bubble in asset price. It is also depend on asymptotic behavior of the asset's price volatility. If the asset's price volatility is large enough, then a bubble exits. Type 3 bubbles exist if and only if integral is finite. Before we can evaluate the integrals, we would need volatility function (sigma).

There are many estimators used to estimate volatility function. We will study floren Zmirou in this chapter and Smooth kernal estimators. Floren Zmirou propose a nonparametric estimator based on the local time of the diffusion process. This method will only can estimate volatility on visited points by the process. We can not know the tail of volatility function. Therefore we can not check if the integral is finite or infinite.