2D simulations with Dom & Davide

19/09/2022

$$(\partial_t + \underline{u}\cdot\underline{\nabla})\underline{\underline{Q}} = \underline{\underline{S}} + \Gamma\underline{\underline{H}} \qquad \underline{\underline{H}} = -\frac{\delta F}{\delta \underline{\underline{Q}}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\,\mathrm{Tr}\frac{\delta F}{\delta \underline{\underline{Q}}}$$

$$F = \int d\underline{r}\left[\frac{A_0}{2}\left(1-\frac{\gamma}{3}\right)\underline{\underline{Q}}^2 + \frac{A_0\gamma}{4}\left(\underline{\underline{Q}}^2\right)^2 + \frac{K}{2}\left(\underline{\nabla}\underline{\underline{Q}}\right)^2\right]$$

$$\underline{\underline{S}} = (\xi\underline{\underline{D}} + \underline{\underline{\omega}})\cdot\left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right) + \left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right)\cdot(\xi\underline{\underline{D}} - \underline{\underline{\omega}}) - 2\xi\left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right)\mathrm{Tr}\left(\underline{\underline{Q}}\cdot\underline{\nabla}\underline{u}\right)$$

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\underline{u}\right] = -\underline{\nabla}p + \eta\nabla^2\underline{u} + \underline{\nabla}\cdot\underline{\underline{\Pi}} - \zeta\underline{\nabla}\cdot\underline{\underline{Q}}$$

$$\underline{\underline{\Pi}} = 2\xi\left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right)\mathrm{Tr}\left(\underline{\underline{Q}}\cdot\underline{\underline{H}}\right) - \xi\underline{\underline{H}}\cdot\left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right) - \xi\left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{\mathbb{I}}}\right)\cdot\underline{\underline{H}} + \underline{\underline{Q}}\cdot\underline{\underline{H}} - \underline{\underline{H}}\cdot\underline{\underline{Q}}$$
$$- \left(\partial_\alpha Q_{\gamma\nu}\right)\frac{\delta F}{\delta(\partial_\beta Q_{\gamma\nu})}$$

Dimensionless units: scale length on $\ell = \sqrt{\frac{K}{A_0}}$

time on $\tau = \frac{1}{\Gamma A_0}$

velocity on $v = \ell\Gamma A_0$

pressure on $\eta\, v/\ell$

$$(\partial_t + \underline{u}\cdot\underline{\nabla})\underline{\underline{Q}} = \underline{\underline{S}} + \underline{\underline{H}}$$

$$Re\left[\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right]\underline{u} = -\underline{\nabla}p + \nabla^2\underline{u} + \kappa\,\underline{\nabla}\cdot\underline{\underline{\Pi}} - \Xi\,\underline{\nabla}\cdot\underline{\underline{Q}}$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

Here, $Re = \frac{\rho K\Gamma}{\eta} = \frac{\rho v \ell}{\eta}$ $\quad \kappa = \frac{1}{\Gamma\eta}$ $\quad \Xi = \frac{\zeta}{\eta\Gamma A_0}$ – activity

Equations in components

1. Molecular field $\underline{\underline{H}}$:

$$\mathcal{F} = \int d\underline{r} \left[\frac{1}{2}\left(1-\frac{\gamma}{3}\right) Q_{\alpha\beta} Q_{\alpha\beta} + \frac{\gamma}{4} \left(Q_{\alpha\beta} Q_{\alpha\beta}\right)^2 + \frac{1}{2} \left(\partial_\alpha Q_{\beta\gamma}\right)^2 \right.$$

$$\frac{\delta \mathcal{F}}{\delta Q_{nm}} = \left(1-\frac{\gamma}{3}\right) Q_{nm} + \gamma \left(Q_{\alpha\beta} Q_{\alpha\beta}\right) Q_{nm} - \nabla^2 Q_{nm}$$

$$\mathrm{Tr}\, \frac{\delta \mathcal{F}}{\delta Q_{nm}} = 0 \quad \Rightarrow \quad H_{nm} = -\frac{\delta \mathcal{F}}{\delta Q_{nm}} = -\left(1-\frac{\gamma}{3}\right) Q_{nm} - \gamma\left(\underline{\underline{Q}}^2\right) Q_{nm} + \nabla^2 Q_{nm}$$

▶ $\underline{\underline{Q}}^2 = Q_{\alpha\beta} Q_{\alpha\beta} = Q_{xx}^2 + 2Q_{xy}^2 + Q_{yy}^2 = 2\left(Q_{xx}^2 + Q_{xy}^2\right)$ since $Q_{yy} = -Q_{xx}$ ◀

$$\boxed{\begin{aligned} H_{xx} &= -\left(1-\frac{\gamma}{3}\right) Q_{xx} - 2\gamma\left(Q_{xx}^2 + Q_{xy}^2\right) Q_{xx} + \nabla^2 Q_{xx} \\ H_{xy} &= -\left(1-\frac{\gamma}{3}\right) Q_{xy} - 2\gamma\left(Q_{xx}^2 + Q_{xy}^2\right) Q_{xy} + \nabla^2 Q_{xy} \end{aligned}}$$

2. $\underline{\underline{S}}$ field:

$$\underline{\underline{S}} = \left(\xi \underline{\underline{D}} + \underline{\underline{\omega}}\right) \cdot \left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{I}}\right) + \left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{I}}\right) \cdot \left(\xi \underline{\underline{D}} - \underline{\underline{\omega}}\right) - 2\xi \left(\underline{\underline{Q}} + \frac{1}{2}\underline{\underline{I}}\right) \mathrm{Tr}\left(\underline{\underline{Q}} \cdot \underline{\underline{W}}\right)$$

$$\underline{\underline{D}} = \frac{\underline{\underline{W}} + \underline{\underline{W}}^T}{2} \qquad \underline{\underline{\omega}} = \frac{\underline{\underline{W}} - \underline{\underline{W}}^T}{2} \qquad W_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta}$$

$$D_{ij} = \frac{\partial_j u_i + \partial_i u_j}{2} \qquad \omega_{ij} = \frac{\partial_j u_i - \partial_i u_j}{2}$$

▶ $\frac{1}{2}\underline{\underline{I}} \cdot \left[\xi \underline{\underline{D}} + \underline{\underline{\omega}} + \xi \underline{\underline{D}} - \underline{\underline{\omega}}\right] = \xi \underline{\underline{D}}$ ◀

▶ $\mathrm{Tr}\left(\underline{\underline{Q}} \cdot \nabla \underline{u}\right) = Q_{\alpha\beta}\, \partial_\alpha u_\beta$

$$= Q_{xx}\, \partial_x u_x + Q_{xy}\left(\partial_x u_y + \partial_y u_x\right) - Q_{xx}\, \partial_y u_y$$

$$= 2 Q_{xx}\, \partial_x u_x + Q_{xy}\left(\partial_x u_y + \partial_y u_x\right)$$ ◀

$$S_{nm} = \xi D_{nm} + \left[\left(\xi \underline{\underline{D}} + \underline{\underline{\omega}}\right) \cdot \underline{\underline{Q}} + \underline{\underline{Q}} \cdot \left(\xi \underline{\underline{D}} - \underline{\underline{\omega}}\right)\right]_{nm} - 2\xi \left(Q_{nm} + \frac{1}{2}\delta_{nm}\right)\left[2 Q_{xx}\partial_x u_x + Q_{xy}\left(\partial_x u_y + \partial_y u_x\right)\right]$$

▶ $$\underline{\underline{\omega}} \cdot \underline{\underline{Q}} - \underline{\underline{Q}} \cdot \underline{\underline{\omega}} = \begin{pmatrix} 0 & \omega_{xy} \\ -\omega_{xy} & 0 \end{pmatrix} \cdot \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix} - \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix} \cdot \begin{pmatrix} 0 & \omega_{xy} \\ -\omega_{xy} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\omega_{xy} Q_{xy} & -2\omega_{xy} Q_{xx} \\ -2\omega_{xy} Q_{xx} & -2\omega_{xy} Q_{xy} \end{pmatrix} = \left(\partial_y u_x - \partial_x u_y\right) \begin{pmatrix} Q_{xy} & -Q_{xx} \\ -Q_{xx} & -Q_{xy} \end{pmatrix} = \left(\partial_x u_y - \partial_y u_x\right) \begin{pmatrix} -Q_{xy} & Q_{xx} \\ Q_{xx} & Q_{xy} \end{pmatrix}$$ ◀

▶ $\xi \underline{\underline{D}} \cdot \underline{\underline{Q}} +$ …

Thus, S_{xx} …

$S_{xx} = -(\partial_x u$ …

$\boxed{S_{xx} = \xi\, \partial \ldots}$

$S_{xy} = \xi \frac{\partial_x u_y + \ldots}{2}$

$\boxed{S_{xy} = \xi\, \partial_x u \ldots}$

$\partial_t Q_{xx} + \underline{u}$ …

$\partial_t Q_{xx} + (1-$ …

$\partial_t Q_{xy} + \underline{u} \cdot \nabla$ …

$\partial_t Q_{xy} + (1-$ …

$$\blacktriangleright\ \xi \underline{\underline{D}}\cdot\underline{\underline{Q}} + \xi \underline{\underline{Q}}\cdot\underline{\underline{D}} = \xi \left[\begin{pmatrix} d_{xx} & d_{xy} \\ d_{xy} & -d_{xx} \end{pmatrix}\begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix} + \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix}\begin{pmatrix} d_{xx} & d_{xy} \\ d_{xy} & -d_{xx} \end{pmatrix} \right]$$

$$= \xi \begin{pmatrix} 2(d_{xx} Q_{xx} + d_{xy} Q_{xy}) & \cancel{d_{xx} Q_{xy}} + d_{xy} Q_{xx} + Q_{xx} d_{xy} - \cancel{Q_{xy} d_{xx}} \\ \cancel{d_{xy} Q_{xx}} - \cancel{d_{xx} Q_{xy}} + \cancel{Q_{xy} d_{xx}} - \cancel{Q_{xx} d_{xy}} & 2(d_{xy} Q_{xy} + d_{xx} Q_{xx}) \end{pmatrix}$$

$$= \xi \begin{pmatrix} 2\partial_x u_x Q_{xx} + (\partial_x u_y + \partial_y u_x) Q_{xy} & 0 \\ 0 & -\|- \end{pmatrix}$$

Thus, $S_{xx} = \xi \partial_x u_x + 2\xi (\partial_x u_x) Q_{xx} + \xi(\partial_x u_y + \partial_y u_x) Q_{xy} - (\partial_x u_y - \partial_y u_x) Q_{xy}$

$$- 2\xi \left(Q_{xx} + \tfrac{1}{2}\right)\left[2 Q_{xx} \partial_x u_x + Q_{xy}(\partial_x u_y + \partial_y u_x)\right]$$

$$S_{xx} = -(\partial_x u_y - \partial_y u_x) Q_{xy} - 2\xi Q_{xx}\left[2 Q_{xx} \partial_x u_x + Q_{xy}(\partial_x u_y + \partial_y u_x)\right] + \xi \partial_x u_x$$

$$+ \xi\left[2\cancel{(\partial_x u_x) Q_{xx}} + \cancel{(\partial_x u_y + \partial_y u_x) Q_{xy}} - 2\cancel{Q_{xx} \partial_x u_x} - \cancel{Q_{xy}(\partial_x u_y + \partial_y u_x)}\right]$$

$$\boxed{S_{xx} = \xi \partial_x u_x - (\partial_x u_y - \partial_y u_x) Q_{xy} - 2\xi Q_{xx}\left[2 Q_{xx} \partial_x u_x + Q_{xy}(\partial_x u_y + \partial_y u_x)\right]}$$

$$S_{xy} = \xi \frac{\partial_x u_y + \partial_y u_x}{2} + Q_{xx}(\partial_x u_y - \partial_y u_x) - 2\xi Q_{xy}\left[\ \cdots\cdots\ \right]$$

$$\boxed{S_{xy} = \xi \frac{\partial_x u_y + \partial_y u_x}{2} + (\partial_x u_y - \partial_y u_x) Q_{xx} - 2\xi Q_{xy}\left[2 Q_{xx}(\partial_x u_x) + Q_{xy}(\partial_x u_y + \partial_y u_x)\right]}$$

$$\partial_t Q_{xx} + \underline{u}\cdot\underline{\nabla} Q_{xx} = \xi \partial_x u_x - (\partial_x u_y - \partial_y u_x) Q_{xy} - 2\xi Q_{xx}\left[2 Q_{xx} \partial_x u_x + Q_{xy}(\partial_x u_y + \partial_y u_x)\right]$$

$$- \left(1 - \frac{\gamma}{3}\right) Q_{xx} + \nabla^2 Q_{xx} - 2\gamma (Q_{xx}^2 + Q_{xy}^2) Q_{xx}$$

$$\partial_t Q_{xx} + \left(1 - \frac{\gamma}{3}\right) Q_{xx} - \nabla^2 Q_{xx} - \xi \partial_x u_x = -\underline{u}\cdot\underline{\nabla} Q_{xx} - (\partial_x u_y - \partial_y u_x) Q_{xy}$$
$$- 2\xi Q_{xx}\left[2 Q_{xx} \partial_x u_x + Q_{xy}(\partial_x u_y + \partial_y u_x)\right] - 2\gamma(Q_{xx}^2 + Q_{xy}^2) Q_{xx}$$

$$\partial_t Q_{xy} + \underline{u}\cdot\underline{\nabla} Q_{xy} = \xi \frac{\partial_x u_y + \partial_y u_x}{2} + (\partial_x u_y - \partial_y u_x) Q_{xx} - 2\xi Q_{xy}\left[2 Q_{xx}(\partial_x u_x) + Q_{xy}(\partial_x u_y + \partial_y u_x)\right]$$

$$- \left(1 - \frac{\gamma}{3}\right) Q_{xy} - 2\gamma (Q_{xx}^2 + Q_{xy}^2) Q_{xy} + \nabla^2 Q_{xy}$$

$$\partial_t Q_{xy} + \left(1 - \frac{\gamma}{3}\right) Q_{xy} - \nabla^2 Q_{xy} - \xi \frac{\partial_x u_y + \partial_y u_x}{2} = -\underline{u}\cdot\underline{\nabla} Q_{xy} + (\partial_x u_y - \partial_y u_x) Q_{xx}$$
$$- 2\xi Q_{xy}\left[2 Q_{xx}(\partial_x u_x) + Q_{xy}(\partial_x u_y + \partial_y u_x)\right] - 2\gamma (Q_{xx}^2 + Q_{xy}^2) Q_{xy}$$

3. Navier - Stokes equation

$$Re\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\underline{\nabla}\,\underline{u}\right] = -\underline{\nabla}p + \nabla^2\underline{u} + \kappa\,\underline{\nabla}\cdot\underline{\underline{\Pi}} - \Xi\,\underline{\nabla}\cdot\underline{\underline{Q}}$$

$$\underline{\nabla}\cdot\underline{u} = 0$$

In components:

$$Re\left[\partial_t u_\alpha + u_\beta \nabla_\beta u_\alpha\right] = -\nabla_\alpha p + \nabla^2 u_\alpha + \kappa \nabla_\beta \Pi_{\alpha\beta} - \Xi \nabla_\beta Q_{\alpha\beta}$$

① $Re\left[\partial_t u_x + (u_x\partial_x + u_y\partial_y)u_x\right] = -\nabla_x p + \nabla^2 u_x + \kappa\left(\nabla_x \Pi_{xx} + \nabla_y \Pi_{xy}\right) - \Xi\left(\partial_x Q_{xx} + \partial_y Q_{xy}\right)$

② $Re\left[\partial_t u_y + (u_x\partial_x + u_y\partial_y)u_y\right] = -\nabla_y p + \nabla^2 u_y + \kappa\left(\partial_x \Pi_{yx} + \partial_y \Pi_{yy}\right) - \Xi\left(\partial_x Q_{xy} - \partial_y Q_{xx}\right)$

Introduce the streamfunction: $u_x = \frac{\partial \psi}{\partial y}$ $\quad u_y = -\frac{\partial \psi}{\partial x}$

vorticity: $\omega_z \equiv \omega = \varepsilon_{zij}\nabla_i u_j = \partial_x u_y - \partial_y u_x$

$\partial_x ② - \partial_y ① \Rightarrow$ $\mathbf{Re}\left[\partial_t\omega + \partial_x\{u_x\partial_x u_y + u_y\partial_y u_y\} - \partial_y\{u_x\partial_x u_x + u_y\partial_y u_x\}\right]$

$$= \nabla^2\omega + \kappa\left[\partial_x^2 \Pi_{yx} + \partial_{x,y}(\Pi_{yy} - \Pi_{xx}) - \partial_y^2 \Pi_{xy}\right] - \Xi\left(\partial_x^2 Q_{xy} - 2\partial_{x,y}Q_{xx} - \partial_y^2 Q_{xy}\right.$$

▶ $(\partial_x u_x)(\partial_x u_y) + (\partial_x u_y)(\partial_y u_y) - (\partial_y u_x)(\partial_x u_x) - (\partial_y u_y)(\partial_y u_x)$

$+ u_x\partial_x^2 u_y + u_y\partial_{x,y}u_y - u_x(\partial_{x,y}u_x) - u_y\partial_y^2 u_x$

$= u_x\partial_x(\partial_x u_y - \partial_y u_x) + u_y\partial_y(\partial_x u_y - \partial_y u_x) = (u_x\partial_x + u_y\partial_y)\omega$

$$\Rightarrow Re\left[\partial_t\omega + (u_x\partial_x + u_y\partial_y)\omega\right] = \nabla^2\omega + \kappa\left[\partial_x^2\Pi_{yx} - \partial_y^2\Pi_{xy} + \partial_{x,y}(\Pi_{yy} - \Pi_{xx})\right]$$

$$- \Xi\left((\partial_x^2 - \partial_y^2)Q_{xy} - 2\partial_{x,y}Q_{xx}\right)$$

$$\boxed{\begin{aligned} Re\,\partial_t\omega - \nabla^2\omega + \Xi\left[(\partial_x^2 - \partial_y^2)Q_{xy} - 2\partial_{x,y}Q_{xx}\right] &= -Re\,(u_x\partial_x + u_y\partial_y)\omega \\ &+ \kappa\left[\partial_x^2\Pi_{yx} - \partial_y^2\Pi_{xy} + \partial_{x,y}(\Pi_{yy} - \Pi_{xx})\right]\end{aligned}}$$

N.B. Π's might contain linear terms!

$\omega = \partial_x$

$\Rightarrow$

4. $\underline{\underline{\Pi}} = 2\xi$

▶ $-\xi\underline{H}\cdot\underline{Q}$

$\begin{pmatrix} Q_{xx} & Q_x \\ Q_{xy} & -C \end{pmatrix}$

$\begin{pmatrix} H_{xx} & H \\ H_{xy} & - \end{pmatrix}$

xx: (1-

xy: (1-

yx: (1-

yy: -

▶ $-\frac{1}{2}\xi\underline{H}$

▶ $Tr(\underline{Q}\cdot\underline{H}$

▶ $\frac{\delta\mathfrak{F}}{\delta(\partial_\beta Q_{\gamma\delta}}$

xy: ∂

yx: -

yy: 2

$$\omega = \partial_x u_y - \partial_y u_x = \partial_x\left(-\frac{\partial\psi}{\partial x}\right) - \partial_y^2\psi = -\nabla^2\psi$$

$$\Rightarrow \quad \boxed{\omega + \nabla^2\psi = 0}$$

4. $$\underline{\underline{\Pi}} = 2\xi\left(\underline{\underline{Q}} + \tfrac{1}{2}\underline{\underline{I}}\right)\mathrm{Tr}\left(\underline{\underline{Q}}\cdot\underline{\underline{H}}\right) - \xi\,\underline{\underline{H}}\cdot\left(\underline{\underline{Q}} + \tfrac{1}{2}\underline{\underline{I}}\right) - \xi\left(\underline{\underline{Q}} + \tfrac{1}{2}\underline{\underline{I}}\right)\cdot\underline{\underline{H}} + \underline{\underline{Q}}\cdot\underline{\underline{H}} - \underline{\underline{H}}\cdot\underline{\underline{Q}}$$
$$- \left(\partial_\alpha Q_{\gamma\nu}\right)\frac{\delta F}{\delta(\partial_\beta Q_{\gamma\nu})}$$

▶ $-\xi\underline{\underline{H}}\cdot\underline{\underline{Q}} - \xi\underline{\underline{Q}}\cdot\underline{\underline{H}} + \underline{\underline{Q}}\cdot\underline{\underline{H}} - \underline{\underline{H}}\cdot\underline{\underline{Q}} = (1-\xi)\,\underline{\underline{Q}}\cdot\underline{\underline{H}} - (1+\xi)\,\underline{\underline{H}}\cdot\underline{\underline{Q}}$

$$\begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix}\cdot\begin{pmatrix} H_{xx} & H_{xy} \\ H_{xy} & -H_{xx} \end{pmatrix} = \begin{pmatrix} Q_{xx}H_{xx} + Q_{xy}H_{xy} & Q_{xx}H_{xy} - Q_{xy}H_{xx} \\ Q_{xy}H_{xx} - Q_{xx}H_{xy} & Q_{xx}H_{xx} + Q_{xy}H_{xy} \end{pmatrix}$$

$$\begin{pmatrix} H_{xx} & H_{xy} \\ H_{xy} & -H_{xx} \end{pmatrix}\cdot\begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{xy} & -Q_{xx} \end{pmatrix} = \begin{pmatrix} Q_{xx}H_{xx} + Q_{xy}H_{xy} & Q_{xy}H_{xx} - Q_{xx}H_{xy} \\ Q_{xx}H_{xy} - Q_{xy}H_{xx} & Q_{xy}H_{xy} + Q_{xx}H_{xx} \end{pmatrix}$$

xx: $(1-\xi-1-\xi)(Q_{xx}H_{xx} + Q_{xy}H_{xy}) = -2\xi\,(Q_{xx}H_{xx} + Q_{xy}H_{xy})$

xy: $(1-\xi)[Q_{xx}H_{xy} - Q_{xy}H_{xx}] - (1+\xi)[Q_{xy}H_{xx} - Q_{xx}H_{xy}]$
$= Q_{xx}H_{xy}(1-\xi+1+\xi) + Q_{xy}H_{xx}(-1+\xi-1-\xi) = 2(Q_{xx}H_{xy} - Q_{xy}H_{xx})$

yx: $(1-\xi)[Q_{xy}H_{xx} - Q_{xx}H_{xy}] - (1+\xi)[Q_{xx}H_{xy} - Q_{xy}H_{xx}]$
$= Q_{xx}H_{xy}(-1+\xi-1-\xi) + Q_{xy}H_{xx}(1-\xi+1+\xi) = 2(Q_{xy}H_{xx} - Q_{xx}H_{xy})$

yy: $-2\xi\,[Q_{xx}H_{xx} + Q_{xy}H_{xy}]$ ◀

▶ $-\frac{1}{2}\xi\underline{\underline{H}} - \frac{1}{2}\xi\underline{\underline{H}} = -\xi\underline{\underline{H}}$ → has linear & non-linear parts! ◀

▶ $\mathrm{Tr}(\underline{\underline{Q}}\cdot\underline{\underline{H}}) = 2[Q_{xx}H_{xx} + Q_{xy}H_{xy}]$ ◀

▶ $\dfrac{\delta F}{\delta(\partial_\beta Q_{\gamma\nu})} = \partial_\beta Q_{\gamma\nu}$; $(\partial_x Q_{\nu\beta})(\partial_x Q_{\nu\beta}) = [\partial_x Q_{xx}]^2 + 2[\partial_x Q_{xy}]^2 + [\partial_x Q_{xx}]^2$
$= 2\left[(\partial_x Q_{xx})^2 + (\partial_x Q_{xy})^2\right]$

xy: $\partial_x Q_{xx}\,\partial_y Q_{xx} + 2\,\partial_x Q_{xy}\,\partial_y Q_{xy} + \partial_x Q_{xx}\,\partial_y Q_{xx} = 2\left[(\partial_x Q_{xx})(\partial_y Q_{xx}) + (\partial_x Q_{xy})(\partial_y Q_{xy})\right]$

yx: —〃—

yy: $2\left[(\partial_y Q_{xx})^2 + (\partial_y Q_{xy})^2\right]$ ◀

[redacted]

$$\Pi_{xx} = 4\xi Q_{xx}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) - \xi H_{xx} - 2\left[(\partial_x Q_{xx})^2 + (\partial_x Q_{xy})^2\right]$$

$$\Pi_{xy} = 4\xi Q_{xy}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) + 2\left(Q_{xx}H_{xy} - Q_{xy}H_{xx}\right) - \xi H_{xy}$$
$$- 2\left[(\partial_x Q_{xx})(\partial_y Q_{xx}) + (\partial_x Q_{xy})(\partial_y Q_{xy})\right]$$

$$\Pi_{yx} = 4\xi Q_{xy}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) - 2\left(Q_{xx}H_{xy} - Q_{xy}H_{xx}\right) - \xi H_{xy}$$
$$- 2\left[(\partial_x Q_{xx})(\partial_y Q_{xx}) + (\partial_x Q_{xy})(\partial_y Q_{xy})\right]$$

$$\Pi_{yy} = -4\xi Q_{xx}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) + \xi H_{xx} - 2\left[(\partial_y Q_{xx})^2 + (\partial_y Q_{xy})^2\right]$$

The only place where linear terms can be $\rightarrow \xi H_{ij}$

Separate the linear and non-linear parts of H_{ij}:

$$\Pi_{xx}^{lin} = -\xi H_{xx}^{lin} = \xi\left[\left(1-\frac{\gamma}{3}\right)Q_{xx} - \nabla^2 Q_{xx}\right] \Rightarrow \Pi_{ij}^{lin} = \xi\left[\left(1-\frac{\gamma}{3}\right)Q_{ij} - \nabla^2 Q_{ij}\right]$$

Contribution to the N-S eqs:

$$\kappa\left[\partial_x^2\Pi_{yx} - \partial_y^2\Pi_{xy} + \partial_{x,y}\left(\Pi_{yy} - \Pi_{xx}\right)\right] = \xi\kappa\left[1 - \frac{\gamma}{3} - \nabla^2\right]\left[\partial_x^2 Q_{xy} - \partial_y^2 Q_{xy} - 2\partial_{x,y}Q_{xx}\right]$$

Thus

$$Re\,\partial_t\omega - \nabla^2\omega + \left[\Xi - \xi\kappa\left(1-\frac{\gamma}{3}\right) + \xi\kappa\nabla^2\right]\left[\left(\partial_x^2 - \partial_y^2\right)Q_{xy} - 2\partial_{x,y}Q_{xx}\right]$$
$$= -Re\left(u_x\partial_x + u_y\partial_y\right)\omega + \kappa\left[\partial_x^2\Pi_{yx} - \partial_y^2\Pi_{xy} + \partial_{x,y}\left(\Pi_{yy} - \Pi_{xx}\right)\right]$$

where: $\Pi_{xx} = 4\xi Q_{xx}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) + 2\xi\gamma\left(Q_{xx}^2 + Q_{xy}^2\right)Q_{xx} - 2\left[(\partial_x Q_{xx})^2 + (\partial_x Q_{xy})^2\right]$

$$\Pi_{xy} = 4\xi Q_{xy}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) + 2\left(Q_{xx}H_{xy} - Q_{xy}H_{xx}\right) + 2\xi\gamma\left(Q_{xx}^2 + Q_{xy}^2\right)Q_{xy}$$
$$- 2\left[(\partial_x Q_{xx})(\partial_y Q_{xx}) + (\partial_x Q_{xy})(\partial_y Q_{xy})\right]$$

$$\Pi_{yx} = 4\xi Q_{xy}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) - 2\left(Q_{xx}H_{xy} - Q_{xy}H_{xx}\right) + 2\xi\gamma\left(Q_{xx}^2 + Q_{xy}^2\right)Q_{xy}$$
$$- 2\left[(\partial_x Q_{xx})(\partial_y Q_{xx}) + (\partial_x Q_{xy})(\partial_y Q_{xy})\right]$$

$$\Pi_{yy} = -4\xi Q_{xx}\left(Q_{xx}H_{xx} + Q_{xy}H_{xy}\right) - 2\xi\gamma\left(Q_{xx}^2 + Q_{xy}^2\right)Q_{xx} - 2\left[(\partial_x Q_{xx})^2 + (\partial_x Q_{xy})^2\right]$$

$$\omega + \nabla^2\psi = 0$$