Trigonometry from Ground Up

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1 Trigonometry is a Blend of Ideas[1]

Trigonometry is often taught in schools as a separate entity from the rest of mathematics while it is actually a complicated blend of different mathematical ideas from different domains. This article aims to make explicit this blend by building the concepts of trigonometry from basic principles and thus elucidate the confusion that students can potentially face while dealing with the subject.[1]

1.1 Blend 1: The Cartesian Plane

The Cartesian Plane is often taken for granted, but it is actually a mixture of two different ideas—namely, two number-lines and the Euclidean Plane with two perpendicular lines in it. When Euclid first introduced his namesake plane in his book *Elements*(c.300 BC) as a tool to prove results about geometry, he never intended to use numbers to describe lengths, angles, or areas. It was not until more than 1600 years later would René Descartes create what we might call the *x-y plane* today, and he did so by combining a simple concept with another one.

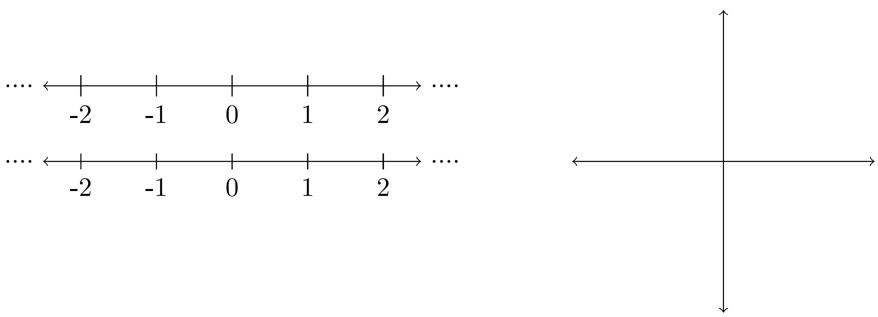

Figure 1: A pair of number-lines and a pair of perpendicular lines in the Euclidean Plane

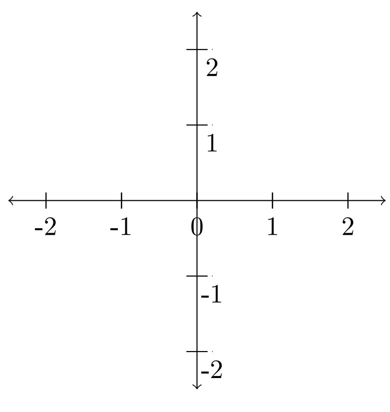

Figure 2: The Cartesian Plane

[1] A large part of information in doing so was in reference to (Lakoff and Núñez, 2000). This section is simply a shorter and more accessible form of the first part of the last chapter from their book.

1.2 Blend 2: The Unit Circle

The Unit Circle is often taken for granted as well, but it is the most important and complex blend in Trigonometry. This blend uses the concepts of a Circle in the Euclidean Plane and the concept of the Cartesian Plane obtained in the previous subsection. The first thing to notice is that, as mentioned, the notion of length is not quantifiable in the Euclidean Plane, so the introduction of quantified length in the Cartesian Plane is crucial for the concept of a *Unit* Circle. Certainly, the word "*Unit*" entails quantity. By superimposing the circle's center with the Cartesian Plane's origin and setting the length of the radius of the circle r to "1" as defined in the Cartesian Plane, we get a framework for the concept.

To add the missing details, we use the concept of angle the size θ of which is, again, not quantifiable in Euclidean Plane. By superimposing the angle's vertex with Cartesian Plane's origin, one of the legs onto the horizontal axis (often called the "X-axis") and the other leg to a line that connects the origin and some arbitrary point P on the circle, we can now quantify the concept of "angle". The size of an angle is now defined as the length of the arc of the circle of radius 1 that the angle subtends—this is the origin of the "Radian" measure of angle.

Finally, we add the last missing piece from the concept by taking the construct of a Right Triangle in the Euclidean Plane and superimposing it on the angle which is now nicely incorporated into the Unit Circle framework. Because we can now quantify lengths, the Right Triangle's legs can now be represented as numbers a and b and its hypotenuse c. We can further define a right angle in the Cartesian Plane as an angle of measure $\frac{\pi}{2}$, or one-fourth of the circumference of a circle.[2] Now, we have our complete concept of the Unit Circle.

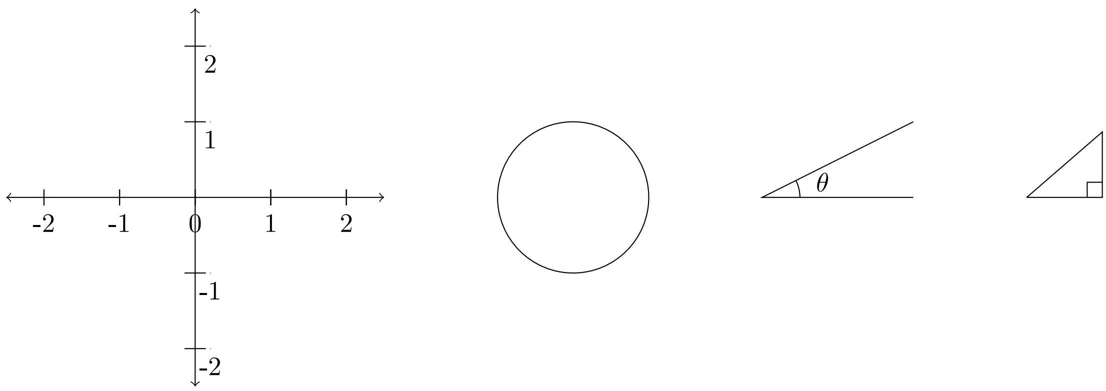

Figure 3: The Cartesian Plane, the Euclidean Circle, Angle, and Right Triangle

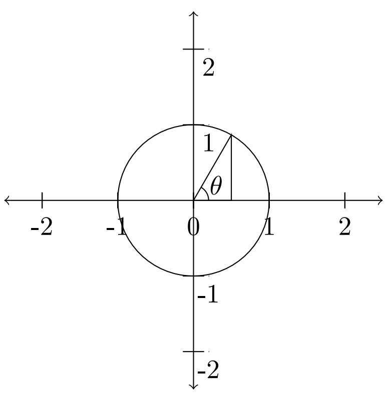

Figure 4: The Unit Circle

[2]Note that the concept of π had long been around before Euclid's time, and its introduction into the very first framework in this subsection defines the circumference of the Unit Circle to be 2π.

1.3 Blend 3: The Trigonometry Blend

As noted, we have prescribed lengths to the legs of the right triangle. Extending this fact, we note that the lengths of these legs change as we change the angle between the hypotenuse and the x axis while maintaining the blended structure of the Unit Circle. Moreover, it is now easy to notice that the right triangle returns to its original look after a rotation of 2π. Indeed, this intimate connection to circularity is why periodicity underlies Trigonometry. The definitions given based on this framework of blended concepts dictated that trigonometric functions, something that some students acquaint via right triangles, is periodic in nature. When a number (length) change in response to another (angle), we can conceptualize the correspondence as a function. Therefore we define the lengths of the vertical and horizontal legs as a functions of θ: $sin(\theta)$ and $cos(\theta)$ respectively. We have now made sense of the basis of trigonometry from ground up.

$$y \leftrightarrow \sin(\theta) \tag{1}$$

$$x \leftrightarrow \cos(\theta) \tag{2}$$

Figure 5: Basis of Trigonometry

2 Extending the Basis

A key fact entails the two equations in Figure 5 in conjunction with the Pythagorean Theorem. For the right triangle in Figure 4 with hypotenuse length 1:

$$x^2 + y^2 \equiv \boxed{\cos^2(\theta) + \sin^2(\theta) = 1} \tag{3}$$

From this important and fundamental equation, it is possible to obtain all other definitions of trigonometric functions via division.

$$1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \rightarrow \boxed{1 + \cot^2(\theta) = \csc^2(\theta)} \tag{4}$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta} + 1 = \frac{1}{\cos^2(\theta)} \rightarrow \boxed{\tan^2(\theta) + 1 = \sec^2(\theta)} \tag{5}$$

Therefore it is clear that even though there are 6 basic trigonometric functions, one only has to know about sine and cosine, which are at the basis of trigonometry, to have access to all other trigonometric functions.

With the help of Analytic Geometry that is inherited by the Unit Circle from the Cartesian Plane, we can also calculate the values for sine and cosine exactly at certain values of θ. This process is fairly trivial and is left to readers as an exercise.

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Figure 6: Table of Selected Values

For all other values of θ between 0 and 2π, instead of looking at the value for θ in itself, you can make an angle between the hypotenuse that corresponds to that θ and the x-axis. If the magnitude of the resulting angle is in the leftmost column of the table above, then it is possible to use simple geometric arguments (again, the benefits of Analytic Geometry) to compute trigonometric functions at their values when needed.

Keep in mind that 2π is a full traversal around the Unit Circle. However, if we are looking at the function $g(\theta) = \sin(2\theta)$ as opposed to $f(\theta) = \sin(\theta)$, then $g(\theta)$ will complete a full rotation at twice the rate of $f(x)$. This is true because 2θ increases twice as fast as θ. We can now define a quantity T, called the *period*, to be the how "long" it takes for a trigonometric function $F(\theta)$ to complete one full traversal around the Unit Circle. The period of $f(\theta)$ is evidently 2π, which follows from the Unit Circle's circumference. So if $g(\theta)$ completes a full traversal twice as fast, then its period bust be half of that of $f(\theta)$'s. Therefore $T_g = \frac{T_f}{2} = \frac{2\pi}{2} = pi$.

Take further notice that the function $\tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$ repeats itself every for every increment of π as opposed to 2π for all other trigonometric functions. A more detailed explanation will not be given here.

In general, if a basic trigonometric function has the form $f(\theta)$ and has period T, then $f(\omega\theta)$ has period $\frac{T}{\omega}$. The quantity ω is called the *angular frequency* of the function, in correspondence to the rotation of a line segment linking the origin and an arbitrary point on the Unit Circle going through angular displacements.

The periodic nature of the trigonometric functions inherited from the Unit Circle implies that any angle greater than $\theta = nT + \phi$ is equivalent to just the extraneous angle ϕ when computing values for an arbitrary trigonometric function with period T (n is any integer). This is the algebraic view of the account immediately following the table of Figure 6. Concisely stated, this is the defining property of a periodic function f with period T:

$$f(nT + \phi) = f(\phi) \tag{6}$$

Note that all simple trigonometric functions are periodic, but not all periodic functions are trigonometric. This is a general definition and is not constrained to trigonometric functions. Also note that this property is a statement about the algebraic *values* of periodic functions only, since n, T, and ϕ all indicate constants, as opposed to variables.

Of course, with the help of the Cartesian Plane, it is also possible to represent algebraic functions as geometric curves. A point in the Cartesian Plane simply is a geometric conceptualization of a pair of numbers. A function f, roughly speaking, provides all possible pairs of numbers, and thus points, that satisfy it. When you need to recall information about the graph of some trigonometric function, whether you are being asked to graph it, or just need it as a smaller piece in a larger problem, it is often helpful to figure out ω or T first. This is the case because the periodicity, which is encoded in ω and T, of trigonometric functions gives a subtle but holistic structure to the underlying Unit Circle. As a general rule of thumb, when you are informed about the bigger structure, thinking about specific cases under it becomes easier.

These information should build a foundation to understand more advanced trigonometric concepts and results. The blend between the complex plane and trigonometry, which is a further layer built upon trigonometry, is left out because it reaches beyond the purpose of this particular article, but will be covered in a future one.

As a matter of fact, mathematics is made up of layers and layers of metaphors and concepts. The way that math is taught to us in school is "efficient" in that we only learn about the top level abstractions of complicated blends of ideas—but they do not provide sufficient insight into the concepts. As a general rule of thumb, it is important to know which lower-level concepts and ideas you are dealing with—you don't need to be actively thinking about them all the time, but you should acknowledge and be able to recognize them when needed.

References

[1] M. Lakoff and R. Núñez. *Where Mathematics Comes From.* 2000. ISBN: 9780465037711.