Physics 704 Group Project
Determining the Potential of a Cylindrical System Using the Modified Bessel Functions

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Abstract

For the system of a point charge q located at the point (ρ, ϕ, z) inside a grounded cylindrical box bounded by surfaces z=0, z=L, $\rho = $ a, we have derived the Green's function and thereby the potential inside the box using the modified Bessel functions of the first and second kinds: $I_n(x)$ and $K_n(x)$, respectively. Using this result, we find the potential inside a cylindrical box held at zero potential at all surfaces except for a disc in the upper end of radius b, where $b < a$, held at potential V. In the case that $\rho = 0$, z=L/2, and b=L/4=a/2, we find the ratio Φ/V to 10 significant figures to be $\Phi/V = 0.0715293729$.

1 Jackson Problem 3.23

Following the discussion of Jackson[2], section 3.7, we solve for our potential first by a separation of variables. Per assignment, we are interested in the modified Bessel functions of the first and second kinds, the I_m's and K_m's, and we take our separation constant k^2 to $-k^2$. Having done so, we divide our cylindrical box in ρ. Then our associated orthogonal functions are $\sin kz$ and $\cos kz$. Our boundary condition in z is

$$G(z=0) = G(z=L) = 0$$

so we set our separation constant k equal to $k = \frac{n\pi}{L}$ and we choose to use only $sinkz = sin\frac{nz\pi}{L}$ to describe the boundary condition in z. Therefore, in Region I, defined by $\rho > b > \rho'$

$$G_I(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \tfrac{n\pi z}{L} I_m(\tfrac{n\pi\rho}{L}) e^{im\phi}$$

In Region II, $a > \rho > \rho'$, so we may include both modified Bessel functions:

$$G_{II}(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \tfrac{n\pi z}{L} \Big(I_m(\tfrac{n\pi\rho}{L}) + C_{mn} K_m(\tfrac{n\pi\rho}{L}) \Big) e^{im\phi}$$

Now, $G_{II} = 0$ at $\rho = a$, so

$$0 = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \tfrac{n\pi z}{L} \Big(I_m(\tfrac{n\pi a}{L}) + C_{mn} K_m(\tfrac{n\pi a}{L}) \Big) e^{im\phi}$$

By the orthogonality of the $e^{\pm ikz} e^{im\phi}$ in z and ϕ, we can equate each term separately to zero. We then find

$$C_{mn} = -\frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})}$$

Therefore,

$$G_{II}(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi z}{L} \Big(I_m(\frac{n\pi\rho}{L}) - \frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})} K_m(\frac{n\pi\rho}{L}) \Big) e^{im\phi} \tag{1}$$

Now, the potential, and therefore the Green's function G, is continuous across the boundary at $\rho = \rho'$, so

$$G_I\Big|_{\rho=\rho'} = G_{II}\Big|_{\rho=\rho'}$$

so

$$\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi z}{L} I_m(\frac{n\pi\rho'}{L}) e^{im\phi} = \tag{2}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi z}{L} \Big(I_m(\frac{n\pi\rho'}{L}) - \frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})} K_m(\frac{n\pi\rho'}{L}) \Big) e^{im\phi}. \tag{3}$$

By the orthogonality of the functions in z and ϕ,

$$\int_0^{2\pi} e^{im\phi} e^{-im'\phi} \, d\phi = 2\pi\delta_{mm'} \tag{4}$$

$$\int_0^{L} \sin \frac{n\pi z}{L} \sin \frac{n'\pi z}{L} \, dz = \frac{L}{2}\delta_{nn'} \tag{5}$$

So we find that

$$A_{mn} = B_{mn}\Big(1 - \tfrac{I_m(\frac{n\pi a}{L})K_m(\frac{n\pi\rho'}{L})}{I_m(\frac{n\pi\rho'}{L})K_m(\frac{n\pi a}{L})}\Big)$$

Thus,

$$G_I(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi z}{L}\Big(1 - \frac{I_m(\frac{n\pi a}{L})K_m(\frac{n\pi\rho'}{L})}{I_m(\frac{n\pi\rho'}{L})K_m(\frac{n\pi a}{L})}\Big)I_m(\frac{n\pi\rho}{L})e^{im\phi} \tag{6}$$

We may combine results (1) and (2) to get

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi z}{L}g(\rho,\rho')e^{im\phi} \tag{7}$$

Where

$$g_I = \Big(1 - \frac{I_m(\frac{n\pi a}{L})K_m(\frac{n\pi\rho'}{L})}{I_m(\frac{n\pi\rho'}{L})K_m(\frac{n\pi a}{L})}\Big)I_m(\frac{n\pi\rho}{L}) \tag{8}$$

and

$$g_{II} = \Big(I_m(\frac{n\pi\rho}{L}) - \frac{I_m(\frac{n\pi a}{L})}{K_m(\frac{n\pi a}{L})}K_m(\frac{n\pi\rho}{L})\Big) \tag{9}$$

Now we may make use of the definition of G in terms of the differential equation

$$\nabla^2 G = -4\pi\delta(\vec{x}-\vec{x}') \tag{10}$$

Expressing the delta function on the right in cylindrical coordinates, we have

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial\rho G}{\partial\phi} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial^2\phi} + \frac{\partial^2 G}{\partial^2 z} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \tag{11}$$

so

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} B_{mn}(\frac{n\pi}{L})\sin\frac{n\pi z}{L}e^{im\phi}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g}{\partial\rho} - \frac{m^2}{\rho^2}g - (\frac{n\pi}{L})^2 g\right\} = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \tag{12}$$

Now we multiply both sides by $e^{-im'\phi}$ and integrate from 0 to 2π in ϕ. Similarly we multiply both sides by $\sin\frac{n'\pi z}{L}$ and integrate over all z. Using the sifting property on the RHS, and the orthogonality of the $e^{im\phi}$ and $\sin\frac{n\pi z}{L}$ on the LHS, we find

$$B_{mn'}L\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g}{\partial\rho} - \frac{m^2}{\rho^2}g - (\frac{n'\pi}{L})^2 g\right\} = -\frac{4}{\rho}\delta(\rho-\rho')\sin(\frac{n'\pi z'}{L})e^{im\phi'} \tag{13}$$

Dropping the primes on n', and remembering that z' and ϕ' are fixed for the moment, we relabel:

$$B_{mn} \equiv \beta_{mn}\sin(\frac{n'\pi z'}{L})e^{im\phi'} \tag{14}$$

Now we multiply both sides by ρ and integrate across the boundary at $\rho = \rho'$

$$\int_{\rho'-\epsilon}^{\rho'-\epsilon}\beta_{mn}L\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g}{\partial\rho} - \frac{m^2}{\rho^2}g - (\frac{n\pi}{L})^2 g\right\}d\rho = -\int_{\rho'-\epsilon}^{\rho'-\epsilon}\delta(\rho-\rho')\,d\rho \tag{15}$$

Making use of the continuity of g, we get

$$\beta_{mn}\rho\frac{\partial g}{\partial\rho}\bigg|_{\rho'-\epsilon}^{\rho'+\epsilon} = -\frac{4}{L} \tag{16}$$

At the upper limit we are in region II where $g = g_{II}$, and at the lower limit we are in region I where $g = g_I$. Evaluating at these limits, we find that β_{mn} is a fraction whose denominator is proportional to the Wronskian $W(\frac{n\pi\rho'}{L})$ of the modified Bessel differential equation. Since $W(x)$ is the same function for all x, we may use the large argument form of the modified Bessel functions to evaluate it. Using 3.147,

$$W(x) = -\frac{1}{x} \tag{17}$$

We have

$$\beta_{mn} = -\frac{4}{L}\frac{K_m(\frac{n\pi a}{L})I_m(\frac{n\pi\rho'}{L})}{I_m(\frac{n\pi a}{L})} \tag{18}$$

Thus we arrive at new forms for G_I and G_{II} in terms of the modified Bessel functions I_m and K_m. Combining the two forms using the notation

$$\rho_< = min(\rho, \rho') \tag{19}$$

$$\rho_> = max(\rho, \rho') \tag{20}$$

we arrive at our final form of G:

$$G = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}\frac{I_m(\frac{n\pi\rho_<}{L})}{I_m(\frac{n\pi a}{L})}\left\{I_m(\frac{n\pi a}{L})K_m(\frac{n\pi\rho_>}{L}) - K_m(\frac{n\pi a}{L})I_m(\frac{n\pi\rho_>}{L})\right\} \tag{21}$$

2 Problem 3.24

The problem states: "The walls of the conducting cylindrical box of Problem 3.23 are all at zero potential, except for a disc in the upper end, defined by $\rho = b < a$, at potential V." Then we are to: a) find an expansion for the potential inside the cylinder using the Greens function found in Problem 3.23, and b) "calculate numerically the ratio of the potential at $\rho = 0$, z = L/2 to the potential of the disc, Φ/V, assuming b=L/4=a/2.

Given that our potential is defined on the surface, our solution for the Green's function obeys Dirichlet boundary conditions. This then gives us the following equation for the potential in terms of the Dirichlet Green's function:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V G_D(\vec{x},\vec{x}')\rho(\vec{x}')\,dV - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\vec{\nabla}G_D(\vec{x},\vec{x}')\cdot\hat{n}'\,dS' \tag{22}$$

Note here that ρ refers to the charge density inside the volume. Because for our present case there is no charge inside the cylinder, $\rho = 0$, and thus the first integral identically goes to zero. This leaves our potential as being determined only by the surface integral at the bounding surface $z' = L$.

$$\Phi(\rho,\phi,z) = -\frac{1}{4\pi}\oint_S \Phi_S\vec{\nabla}G_D(\vec{x},\vec{x}')\cdot\hat{n}'\,dS' \tag{23}$$

Where $\Phi_S = V$ on the upper surface at $z' = L$ and zero everywhere else. Then our surface integral can be expressed in cylindrical coordinates, using $\rho'\,d\phi' d\rho'$ as our area element, where ρ' varies from 0 to b and ϕ' from 0 to 2π. Because our potential on the surface is held at constant V, it can be taken outside the integral.

$$\Phi(\rho,\phi,z) = -\frac{V}{4\pi}\int_0^{2\pi}\int_0^b \left.\frac{\partial G}{\partial z'}\right|_{z'=L}\rho'\,d\phi' d\rho' \tag{24}$$

Given that our Green's function from the previous problem contains the term $e^{im\phi'}$, integrating $d\phi'$ from 0 to 2π will identically yield zero for integer m unless we set $m = 0$. That this is the case corresponds to the system's azimuthal symmetry. Then integrating over $d\phi'$ simply gives us a factor of 2π, and our potential reduces to a function of ρ and z:

$$\Phi(\rho,z) = -\frac{V}{2}\int_0^b \left.\frac{\partial G}{\partial z'}\right|_{z'=L}\rho'\,d\rho' \tag{25}$$

Then using the form of G from Problem 3.23, we arrive at

$$\left.\frac{\partial G}{\partial z'}\right|_{z'=L} = \frac{4}{L}\sum_{n=1}^{\infty}(-1)^n\frac{n\pi}{L}\sin\frac{n\pi z}{L}\frac{I_0(\frac{n\pi\rho_<}{L})}{I_0(\frac{n\pi a}{L})}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho_>}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho_>}{L})\right\} \tag{26}$$

Where the $(-1)^n$ term is equivalent to $cos(n\pi)$, having taken the derivative of $sin(\frac{n\pi z'}{L})$ and evaluated at $z' = L$. We then integrate over ρ', dividing our range of integration into two regions: Region II is defined by $b < \rho < a$ and Region I by $\rho < b$. In Region II, $\rho' = \rho_<$ and $\rho = \rho_>$ everywhere, so

$$\Phi_{II}(\rho,z) = -\frac{V}{2}\left\{\int_0^b \left.\frac{\partial G}{\partial z'}\right|_{z'=L,\rho_>=\rho}\rho'\,d\rho'\right\} \tag{27}$$

In Region I, we must break up our range of integration into two parts, depending on whether ρ is greater than or less than ρ':

$$\Phi_I(\rho, z) = -\frac{V}{2}\left\{\int_0^\rho \frac{\partial G}{\partial z'}\bigg|_{z'=L,\rho_>=\rho} \rho'\, d\rho' + \int_\rho^b \frac{\partial G}{\partial z'}\bigg|_{z'=L,\rho_>=\rho'} \rho'\, d\rho'\right\} \tag{28}$$

Now, to evaluate Φ_I, we write

$$\Phi_I = \frac{2V\pi}{L^2}\sum_{n=1}^{\infty}(-1)^{n+1} n \sin\frac{n\pi z}{L}\frac{1}{I_0(\frac{n\pi a}{L})}F \tag{29}$$

Where

$$F = \int_0^\rho \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho'}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho' \tag{30}$$

$$+\int_\rho^b \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho'}{L})I_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho' \tag{31}$$

$$\tag{32}$$

$$= \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^\rho I_0(\frac{n\pi\rho'}{L})\rho' d\rho' \tag{33}$$

$$+ I_0(\frac{n\pi\rho}{L})\int_\rho^b \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho' -}{L})K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho' \tag{34}$$

We may make use of the recursion relations given in 8.107 of Lea:

$$\frac{d}{dx}(x^m I_m) = x^m I_{m-1} \tag{35}$$

$$\frac{d}{dx}(x^m K_m) = -x^m K_{m-1} \tag{36}$$

With m=1 we have

$$\frac{d}{dx}(xI_1) = xI_0 \tag{37}$$

$$\frac{d}{dx}(xK_1) = -xK_0 \tag{38}$$

Now, letting $x' = \frac{n\pi\rho'}{L}$, so that $dx' = \frac{n\pi}{L}(d\rho')$, we have

$$\int_0^\rho I_0(\frac{n\pi\rho'}{L})\rho' d\rho' = \left(\frac{L}{n\pi}\right)^2 \int_0^{\frac{L}{n\pi}x} x' I_0(x) dx' \tag{39}$$

$$= \left(\frac{L}{n\pi}\right)^2 \int_0^{\frac{L}{n\pi}x} \frac{d}{dx}(x' I_1(x')) dx' \tag{40}$$

$$= \left(\frac{L}{n\pi}\right)^2 \left[x' I_1(x')\right]\Bigg|_0^{\frac{L}{n\pi}x} \tag{41}$$

$$= \left(\frac{L}{n\pi}\right)\rho I_1(\frac{n\pi\rho}{L}) \tag{42}$$

Similarly, we find

$$\int_{\rho}^{b}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho'}{L})K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho' \tag{43}$$

$$= I_0(\frac{n\pi a}{L})\int_{\rho}^{b}K_0(\frac{n\pi\rho'}{L})\rho' d\rho' - K_0(\frac{n\pi a}{L})\int_{\rho}^{b}I_0(\frac{n\pi\rho'}{L})\rho' d\rho' \tag{44}$$

where the second integral has already been evaluated with different limits; in this case we find:

$$\int_{\rho}^{b}I_0(\frac{n\pi\rho'}{L})\rho' d\rho' = \left(\frac{L}{n\pi}\right)^2\left[x' I_1(x')\right]\Bigg|_{\frac{L}{n\pi}\rho}^{\frac{L}{n\pi}b} \tag{45}$$

$$= \left(\frac{L}{n\pi}\right)\left\{bI_1(\frac{n\pi b}{L}) - \rho I_1(\frac{n\pi\rho}{L})\right\} \tag{46}$$

The first integral is

$$\int_{0}^{\rho}K_0(\frac{n\pi\rho'}{L})\rho' d\rho' = \left(\frac{L}{n\pi}\right)^2\int_{\frac{L}{n\pi}\rho}^{\frac{L}{n\pi}b}x' K_0(x)dx' \tag{47}$$

$$= -\left(\frac{L}{n\pi}\right)^2\int_{\frac{L}{n\pi}\rho}^{\frac{L}{n\pi}b}\frac{d}{dx}(x' K_1(x'))dx' \tag{48}$$

$$= -\left(\frac{L}{n\pi}\right)^2\left[x' K_1(x')\right]\Bigg|_{\frac{L}{n\pi}\rho}^{\frac{L}{n\pi}b} \tag{49}$$

$$= \left(\frac{L}{n\pi}\right)\left\{\rho K_1(\frac{n\pi\rho}{L}) - bK_1(\frac{n\pi b}{L})\right\} \tag{50}$$

We have thus solved for F. Accounting for terms that cancel, we may substitute back into our equation for Φ to arrive at

$$\Phi_I = \frac{2V}{L}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi z}{L}\frac{1}{I_0(\frac{n\pi a}{L})}\times \tag{51}$$

$$\left(\rho I_0(\frac{n\pi a}{L})\left(K_0(\frac{n\pi\rho}{L})I_1(\frac{n\pi\rho}{L}) + I_0(\frac{n\pi\rho}{L})K_1(\frac{n\pi\rho}{L})\right)- \right. \tag{52}$$

$$\left. bI_0(\frac{n\pi\rho}{L})\left(I_0(\frac{n\pi a}{L})K_1(\frac{n\pi b}{L}) + K_0(\frac{n\pi a}{L})I_1(\frac{n\pi b}{L})\right)\right) \tag{53}$$

Now, using Lea[1] 8.106 and 8.110,

$$\frac{d}{dx}\frac{I_m(x)}{x^m} = \frac{I_{m+1}(x)}{x^m} \tag{54}$$

$$\frac{d}{dx}\frac{K_m(x)}{x^m} = -\frac{K_{m+1}(x)}{x^m} \tag{55}$$

we see that for m=0,

$$I_0' = I_1 \tag{56}$$
$$K_0' = -K_1 \tag{57}$$

Substituting these into Φ, we may write

$$\Phi_I = \frac{2V}{L}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi z}{L}\frac{1}{I_0(\frac{n\pi a}{L})}\times \tag{58}$$
$$\left(\rho I_0(\frac{n\pi a}{L})\left(K_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho}{L}) - I_0(\frac{n\pi\rho}{L})K_0(\frac{n\pi\rho}{L})\right)- \tag{59}$$
$$bI_0(\frac{n\pi\rho}{L})\left(I_0(\frac{n\pi a}{L})K_1(\frac{n\pi b}{L}) + K_0(\frac{n\pi a}{L})I_1(\frac{n\pi b}{L})\right)\right) \tag{60}$$

Observe that the first term in the parenthesis above is the Wronskian of I_0 and K_0. According to , section 3.11, the Wronskian is proportional to $\frac{1}{x}$ for all values of x, so we may evaluate at large x. In this limit, according to 3.147,

$$W[I_m(x), K_m(x)] = -\frac{1}{x} \tag{61}$$

Or, since the Wronskian is anti-symmetric,

$$W[K_m(x), I_m(x)] = \frac{1}{x} \tag{62}$$

Which is exactly the form that appears in Φ, with $x = \frac{n\pi\rho}{L}$. We thus arrive at a general form for Φ_I:

$$\Phi_I(\rho, z) = \frac{2V}{L}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi z}{L}\left\{\frac{L}{n\pi} - bI_o(\frac{n\pi\rho}{L})\left(K_1(\frac{n\pi b}{L}) + \frac{K_0(\frac{n\pi a}{L})}{I_0(\frac{n\pi a}{L})}I_1(\frac{n\pi b}{L})\right)\right\} \tag{63}$$

For Φ_{II}, we evaluate

$$\Phi_{II}(\rho, z) = -\frac{V}{2}\left\{\int_0^b \frac{\partial G}{\partial z'}\bigg|_{z'=L,\rho_>=\rho}\rho'\,d\rho'\right\} \tag{64}$$

We may re-write this as

$$\Phi_{II} = \frac{2V\pi}{L^2}\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\frac{n\pi z}{L}\frac{1}{I_0(\frac{n\pi a}{L})}U \tag{65}$$

where

$$U = \int_0^b\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho'}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})I_0(\frac{n\pi\rho'}{L})\right\}\rho' d\rho' \tag{66}$$
$$= \left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\}\int_0^b I_0(\frac{n\pi\rho'}{L})\rho' d\rho' \tag{67}$$
$$\tag{68}$$

Using a previous result to evaluate this integral, we find

$$\int_0^b I_0(\frac{n\pi\rho'}{L})\rho' d\rho' = \left(\frac{L}{n\pi}\right)^2 \left[x' I_1(x')\right]\Bigg|_0^{\frac{L}{n\pi}b} \tag{69}$$

$$= \left(\frac{bL}{n\pi}\right) I_1(\frac{n\pi b}{L}) \tag{70}$$

So we have

$$U = \left(\frac{bL}{n\pi}\right)\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\} I_1(\frac{n\pi b}{L}) \tag{71}$$

And thus

$$\Phi_{II} = \frac{2Vb}{L}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi z}{L}\frac{1}{I_0(\frac{n\pi a}{L})}\left\{I_0(\frac{n\pi a}{L})K_0(\frac{n\pi\rho}{L}) - K_0(\frac{n\pi a}{L})I_0(\frac{n\pi\rho}{L})\right\} I_1(\frac{n\pi b}{L}) \tag{72}$$

Therefore, we have a piecewise solution for Φ, where

$$\Phi = \Phi_I \quad 0 \leq \rho \leq b \tag{73}$$

$$= \Phi_{II} \quad b < \rho < a \tag{74}$$

As expected, Φ is continuous at the boundary where $\rho = b$, as can be seen by substituting b for ρ into Φ_I and Φ_{II}. Furthermore, at $\rho = a$, we see that the terms in the curly brackets in Φ_{II} cancel identically; $\Phi(\rho = a) = 0$, as expected. At $\rho = 0$, Φ is well behaved since the dependence on K_m's vanishes there. Thus, as required, Φ is well-behaved at the origin and adheres to the boundary conditions of the configuration.

Now, suppose that the radius of the cylinder is $a = L/2$ and that the radius of the disk on which the potential is equal to V is $b = L/4$. If we want to find the potential along the axis of the cylinder at a height of $z = \frac{1}{2}$, we need only to plug the above values into Φ_I and evaluate the solution numerically to desired precision. Dividing by V, we have the ratio

$$\frac{\Phi(\rho=0, z=L/2)}{V} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi}{2}\left\{\frac{1}{n\pi} - \frac{1}{4}I_0(0)\left(K_1(\frac{n\pi}{4}) + \frac{K_0(\frac{n\pi}{2})}{I_0(\frac{n\pi}{2})}I_1(\frac{n\pi}{4})\right)\right\} \tag{75}$$

Notice however that $\sin\frac{n\pi}{2} = 0$ for n=even, and alternating +1, -1, +1 for n = 1, 3, 5, etc. Thus we can pick out only the odd terms in this series. To do this, we write (2n-1) in every function in place of n. Notice also that $I_0(0) = 1$. Combining these simplifications, we are left with:

$$\frac{\Phi(\rho=0, z=L/2)}{V} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{1}{(2n-1)\pi} - \frac{1}{4}\left(K_1(\frac{(2n-1)\pi}{4}) + \frac{K_0(\frac{(2n-1)\pi}{2})}{I_0(\frac{(2n-1)\pi}{2})}I_1(\frac{(2n-1)\pi}{4})\right)\right\} \tag{76}$$

Here we have an infinite sum consisting of two alternating terms. In general we know that for convergent sequences a_n and b_n, we can split an infinite sum of two terms into a sum of two infinite sums (Stewart[4], page 719):

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}a_n + \sum_{n=1}^{\infty}b_n \tag{77}$$

Then for the sake of clarity, let us further distinguish our sums by introducing different summation indices, albeit over the same bounds, such that the first sum is the "m-Series" and the second is the "n-Series":

$$\begin{aligned}\frac{\Phi(\rho=0, z=L/2)}{V} &= 2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{(2m-1)\pi} \\ &\quad - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2}\left(K_1(\frac{(2n-1)\pi}{4}) + \frac{K_0(\frac{(2n-1)\pi}{2})}{I_0(\frac{(2n-1)\pi}{2})}I_1(\frac{(2n-1)\pi}{4})\right)\end{aligned} \tag{78}$$

Because these are both alternating series, we should apply the Alternating Series test to each, which states that an alternating series converges if $|a_n| \to 0$ as $n \to \infty$ and $|a_{n+1}| < |a_n|$. Testing the convergence of the m-Series first, we find using Mathematica that the limit of the terms indeed approaches zero. Further, each term is indeed greater than its subsequent term. Both of these follow from the 1/n character of the series. Thus we can say that the m-Series converges. That the n-Series converges is less obvious, but is nonetheless readily shown by plotting terms in Mathematica. For arbitrary n, we can see that $|a_{n+1}| < |a_n|$, and by taking the limit, we see that the terms do go to zero. Thus the n-Series also converges.

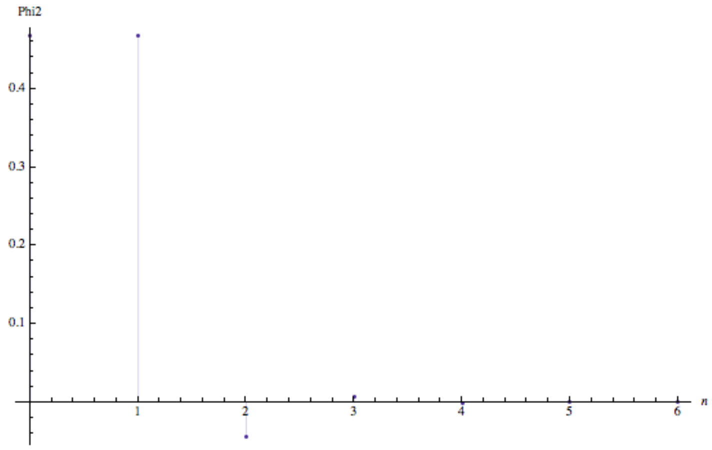

Figure 1: The first six terms in the n-Series

Using Mathematica's built-in NSum function, we can take the infinite of these two series independently of one another. In so doing, the m-Series converges to an exact result. Indeed, this series is known as the Leibniz series and has historically been used to determine the value of π, where

$$\sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{(2m-1)}=\frac{\pi}{4} \tag{79}$$

and thus our m-Series converges to the exact result

$$2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{(2m-1)\pi}=\frac{1}{2}. \tag{80}$$

By requiring that Mathematica use NSum with a WorkingPrecision of 20 decimals, we can take the infinite sum of the n-Series to great accuracy. In so doing, we get a highly precise value for our n-Series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2}\left(K_1(\frac{(2n-1)\pi}{4})+\frac{K_0(\frac{(2n-1)\pi}{2})}{I_0(\frac{(2n-1)\pi}{2})}I_1(\frac{(2n-1)\pi}{4})\right)=0.42847062711242885491 \tag{81}$$

Combining these results, we find that the overall potential Φ/V inside the cylinder is given by the equation:

$$\frac{\Phi}{V}=\frac{1}{2}-0.42847062711242885491=0.07152937288757114509. \tag{82}$$

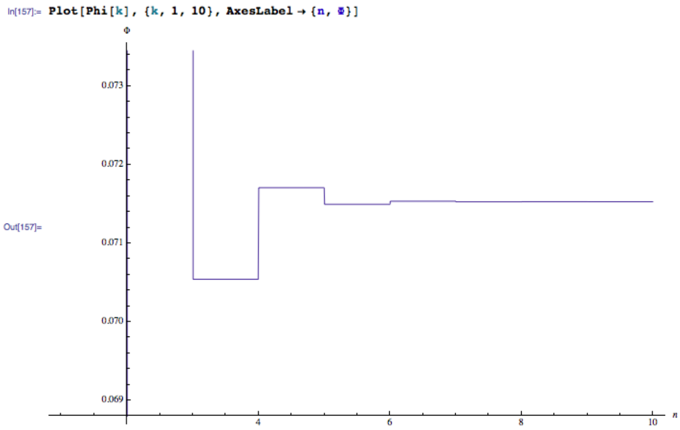

Figure 2: Φ *vs. Number of Terms in Series*

3 Error Analysis

Although Mathematica employs floating-point algorithms that allow it to achieve almost arbitrary accuracy, it is worthwhile to consider possible sources of error in our numerical solution above. First, let us consider the rates of convergence of our two series. We find in Dahlquist[3] (p. 622) the defintion of linear convergence:

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|} = C \tag{83}$$

Where α is the value that the sequence converges to, and C must be between 0 and 1, inclusive. For both of our series, $\alpha = 0$, as has already been shown. The value of C determines the rate of convergence, with C=1 representating sublinear convergence, C=0 representing superlinear convergence, and intermediate values giving linear convergence. We find then for our two series:

$$\lim_{m\to\infty} \frac{\frac{1}{2m+1}}{\frac{1}{2m-1}} = 1 \tag{84}$$

and

$$\lim_{n\to\infty} \frac{\frac{1}{2}\left(K_1(\frac{(2n+1)\pi}{4}) + \frac{K_0(\frac{(2n+1)\pi}{2})}{I_0(\frac{(2n+1)\pi}{2})} I_1(\frac{(2n+1)\pi}{4})\right)}{\frac{1}{2}\left(K_1(\frac{(2n-1)\pi}{4}) + \frac{K_0(\frac{(2n-1)\pi}{2})}{I_0(\frac{(2n-1)\pi}{2})} I_1(\frac{(2n-1)\pi}{4})\right)} = e^{-\pi/2} \tag{85}$$

Thus for the m-Series C = 1, which corresponds to slow sublinear convergence. The n-Series has C = 0.20788, indicating relatively fast linear convergence. From these rates we can conclude that if we were to truncate our series, we would need many more terms for the m-Series than the n-Series to achieve comparable accuracy between them. Indeed, the main sources of error in a computer experiment such as ours are rounding errors and truncation errors. As defined by Dahlquist (p.88), rounding errors may occur in the cases that i) only s digits can be stored in the computer, so multiplication by two terms with s digits with require that the product must be rounded off, or ii) small terms are "shifted out" from a floating-point sum. Truncation occurs when, for example, an infinite series is broken off after a finite number of terms. In Mathematica, the MachinePrecision calculates numbers to 16 decimal places by default. Inherent in that design is a precision up to 2.22045×10^{-16}–found by the "MachineEpsilon" command–which in essence gives the default uncertainty per calculation. Then for j sums, we would expect an uncertainty on the order of $j \times 2.22045 \times 10^{-16}$.

However, it is possible to demand higher precision than MachinePrecision in Mathematica. Indeed, a precision of hundreds of decimal places can be specified. So at least for our Mathematica solution, the foregoing is not likely to introduce appreciable error. A more likely source of error then would be truncation. Although Mathematica claims to take an infinite sum, this is of course impossible. Thus a truncation necessarily takes place, whereby the software extrapolates an approximate value for those terms remaining past a cutoff point. To test for possible effects of truncation, we compare NSum values for 10, 100, 1000, and "Infinite" terms out to 20 decimals for our n-Series. Although our m-Series converges more slowly and therefore would be more susceptible to truncation error, because it is an exact result, Mathematica treats it with infinite precision–i.e., there is no possibility of rounding loss–and so we will treat it as a baseline from which to investigate the n-Series sum. Because the n-Series converges relatively quickly, we find that for as few as 10 terms, our potential Φ/V agrees with the results for higher terms up to seven decimal places. Upon careful investigation the potential is accurate to 20 decimals places for a mere 30 terms, as compared to all higher sums. The speed of convergence is evident also in the fractional error of the first several terms:

$$\Phi_{error} = |\Phi_{infinite} - \Phi_n|/\Phi_{infinite} \tag{86}$$

Which gives $\Phi_{error} = 0.54$ for $n = 1; 0.00046$ for $n = 5$; and 1.3×10^{-7} for $n = 10$.

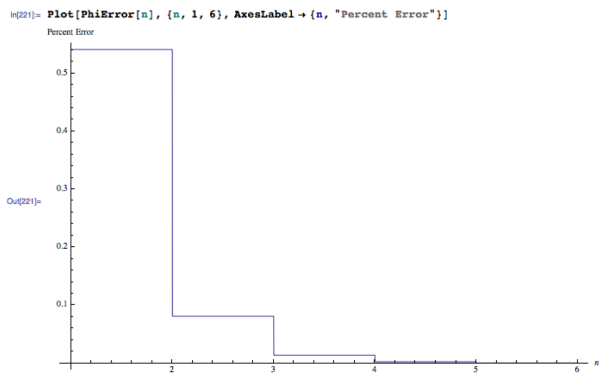

Figure 3: Percent Error vs. Number of Terms in Series

If instead of Mathematica, we use Microsoft Excel to perform our numerical calculations, we encounter more sources of error. Because Excel follows the IEEE 754 specification for floating-point numbers, the program can only store 15 significant digits of precision. As such, in doing our n-Series sum, the constituent Bessel I and Bessel K terms grow very large and small, respectively, as n increases. As these terms are added and multiplied, the number of decimals required to account for all significant digits grows as n increases. And yet because of the fundamental cutoff of 15 digits of precision, a rounding necessarily takes place. In effect, the presence of many terms to the right of the decimal is neglected. This lack of precision is reflected in our results for Excel. Our series converges rapidly to approximately the correct answer (as compared to Mathematica), agreeing to eight decimal places after a mere 11 terms. And yet for higher terms, there is no improvement in agreement. The Excel series terminates at eight decimals of agreement for any n, and for $n > 25$ terms, there is no change in the sum at all. Comparing the fractional error of the Excel solution to the infinite Mathematica sum gives for the first several terms:

$\Phi_{error} = 0.54$ for $n = 1$; 0.00046 for $n = 5$; and 1.0×10^{-7} for $n = 10$.

Thus our Excel solution agrees with our Mathematica solution and converges almost equivalently for the first 10 terms. Comparing the converging Excel value with the converging Mathematica value gives:

$$|\Phi_{Mathematica} - \Phi_{Excel}| = |0.07152937288757 - 0.07152937103909| = 1.8 \times 10^{-9} \quad (87)$$

Seeing that these two differ in barely one part in a billion, we are confident that these values are correct out to eight decimal places. And thus we can conclude that despite Excel's inherent precision limitations, the only error on the following number is on the rounding of the last digit:

$$\Phi/V = 0.07152937$$

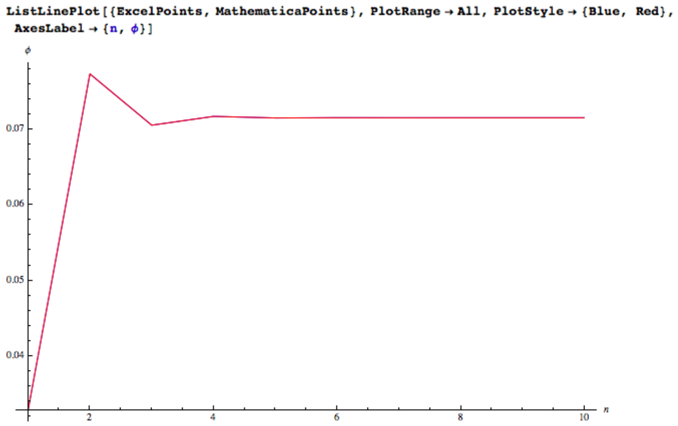

Figure 4: Comparison of Φ vs. n for the Mathematica and Excel Solutions

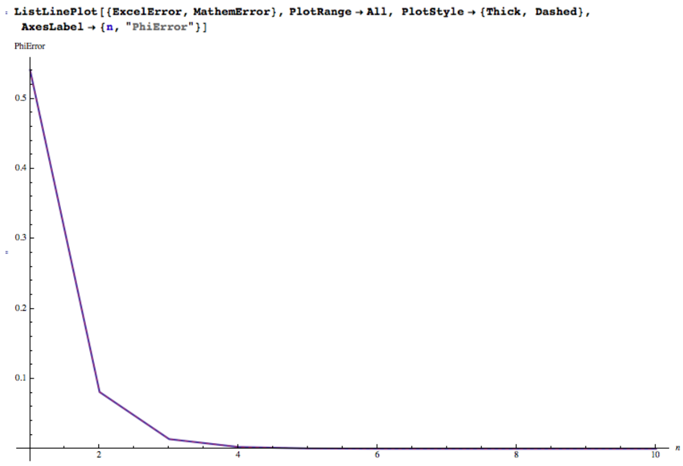

Figure 5: Comparison of Percent Error vs. n for the Mathematica and Excel Solutions

References

[1] Dr. Susan Lea, *Mathematics for Physicists*. Thomson,California, 1st Edition, 2004.

[2] John David , *Classical Electrodynamics*. John Wiley and Sons,New Jersey, 3rd Edition, 1999.

[3] Germund Dahlquist, *Numerical Methods in Scientific Computing Vol.1*. Siam,Philadelphia, 1st Edition, 2008.

[4] James Stewart, *Calculus*. Thomson,California, 5th Edition, 2003.