THE LANE-EMDEN EQUATION

Theoretical Physics

Chapter 3, Problem 28

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THE LANE-EMDEN EQUATION

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Abstract. In this paper we solve the Lane-Emden equation with polytropic index n. We show two analytical solutions in which one is a more general expression satisfying any n polytropic index up to the fourth term. We also solve the second order differential equation by means of the Runge-Kutta numerical method for $n = 2$. Initial conditions had to be taken into consideration for the numerical approach in which the system described is a star with equation of state $P = K\rho^{(n+1)/n}$ and density $\rho = \lambda\phi^n$ in hydrostatic equilibrium.

1. Introduction

In astrophysics, the Lane-Emden equation, named after Jonathan Homer Lane and Robert Emden, describes the structure of a star with a polytropic equation of state $P = K\rho^{(n+1)/n}$. This equation is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid and therefore, it can be applied to different types of stars by varying n: the polytropic index.

In Section 2.1, we will show the case when $n = 1$, and the relevant physical solution applicape to the system. Section 2.2 will consider some general n from where the $n = 1$ case can be retrieved. Lastly, in Section 2.3, we will demonstrate different polytropic indices that do not have an analytic solution to them, but instead we direct ourselves the numerical approach. Graphs are presented at the end of the section. In section 3 we present conclusions from the aftermath. Finally, two codes are presented in the appendix, we show the numerical output of one of them and give references.

2. Solving the Lane-Emden equation

With the definition $\rho = \lambda\phi^n$, hydrostatic equailibrium gives the Lane-Emden equation:

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\phi}{dx}\right) + \phi^n = 0 \tag{1}$$

where x is a dimensionless distance variable. The center of the star corresponds to $x = 0$. The Lane-Emden equation can be put in standard form as follows

$$\frac{d^2\phi}{dx^2} + \frac{2}{x}\frac{d\phi}{dx} + \phi^n = 0 \tag{2}$$

which has a singular point at $x = 0$. One of the methods to find a solution for a differential equation about a singular point is Frobenius method, in which the solution takes the shape of a series given by

$$\phi(x) = (x - x_0)^p \sum_{m=0}^{\infty} a_m (x - x_0)^m \tag{3}$$

where x_0 is the singular point. In our case

$$\phi(x) = \sum_{m=0}^{\infty} a_m x^{m+p}$$

where p may be any number, positive or negative, integer or noninteger, real or complex. The first and second derivatives follow[1]

$$\phi'(x) = \sum_{m=0}^{\infty} a_m (m+p) x^{m+p-1}$$

$$\phi''(x) = \sum_{m=0}^{\infty} a_m (m+p)(m+p-1) x^{m+p-2}$$

2.1. **Finding a series solution for ϕ in the case $n = 1$.** Substituting the first and second derivatives of ϕ into our equation (2) with $n = 1$ we get

$$\sum_{m=0}^{\infty} a_m (m+p)(m+p-1) x^{m+p-2} + 2 \sum_{m=0}^{\infty} a_m (m+p) x^{m+p-2} + \sum_{m=0}^{\infty} a_m x^{m+p} = 0$$

and combining like powers of x:

$$\sum_{m=0}^{\infty} a_m (m+p)(m+p+1) x^{m+p-2} + \sum_{m=0}^{\infty} a_m x^{m+p} = 0 \tag{4}$$

The two lowest powers of x are x^{p-2} and x^{p-1}, its coefficients are

$\underline{x^{p-2}}$:

$$a_0 p(p+1) = 0$$

$\underline{x^{p-1}}$:

$$a_1 (p+1)(p+2) = 0$$

We cannot have $a_0 = 0$ and $a_1 = 0$ since we want a nontrivial solution. So,

a. If we set $a_0 \neq 0$ and $a_1 = 0$, we are left with the indicial equation

[1]To see why we can differentiate as usual please see Lea, Mathematics for Physicists, p.188.

$$p(p+1) = 0$$

which solutions are

$$p = 0$$
$$p = -1$$

where $p = 0$ will give an even solution and $p = -1$ an odd solution.

b. If we set $a_0 = 0$ and $a_1 \neq 0$, we are left with the indicial equation

$$(p+2)(p+1) = 0$$

which solutions are

$$p = -1$$
$$p = -2$$

where $p = -1$ will give an even solution and $p = -2$ an odd solution.

For the following powers of x, all the sums in the algebraic equation contribute. So we can draw a general conclusion valid for all higher powers of x. Let's look at the coefficient of x^{m+p} in order to obtain a recursion relation. We need $m \to m+2$ in the first term of equation (4) and $m \to m$ in the second term:

$$a_{m+2}(m+p+2)(m+p+3) + a_m = 0$$

If we go about solving for a_{m+2}, our recursion relation is

$$a_{m+2} = -\frac{a_m}{(m+p+2)(m+p+3)} \tag{5}$$

Equation (5) is the general result that can give us all coefficients of the series solution.

Since equation (2) is a second order differential equation, it will have two independent solutions. If Frobenius method gives us two independent solutions[2] for case **a** and two independent solutions for case **b**, solutions in case **a** will have to be the same as the solutions in case **b**. Let's analyze case **a** first for simplicity and see what we get.

[2]See Lea, Mathematics for Physicists, p.190.

2.1.1. *The p=0 condition.* When $p=0$, our recursion relation takes the form

$$a_{m+2} = -\frac{a_m}{(m+2)(m+3)}$$

But we don't know what a_m is explcitly, so we make $m \to m-2$ from our recursion relation directly to retrieve it

$$a_m = -\frac{a_{m-2}}{(m+1)(m)}$$

If we iterate for a_{m-2} as well ($m \to m-4$) we get

$$a_{m-2} = -\frac{a_{m-4}}{(m-2)(m-1)}$$

Therefore,

$$a_{m+2} = -\frac{a_{m-4}}{(m+3)(m+2)(m+1)(m)(m-1)(m-2)}$$

Since $p=0$ corresponds to an even solution, m is an even number, $m=2k$, and we can write the general a_{2k} in terms of a_0 as follows

$$a_{2k} = a_0\frac{(-1)^k}{(2k+1)!}$$

and our even solution is

$$\phi_1(x) = \sum_{k=0}^{\infty} a_0\frac{(-1)^k}{(2k+1)!}x^{2k}$$

Finally, if we multiply and divide by x, we realize that we have the sine series divided by x, so the even solution becomes

$$\boxed{\phi_1(x) = a_0\frac{\sin(x)}{x}} \tag{6}$$

When we look at this equation, we would think that it has a pole at $x=0$, but we have to be careful because if we take the limit when $x \to 0$, by L'Hopital's rule we realize that $x=0$ is actually a removable singularity. Thus, equation (6) is valid everywhere and is well defined at $x=0$, the center of the star.

2.1.2. *The p=-1 condition.* Plugging in $p = -1$ in recursion relation (5) we get

$$a_{m+2} = -\frac{a_m}{(m+1)(m+2)}$$

Here we take the same approach as in subsection 2.1.1 to find the coefficients a_{m+2}, a_m, a_{m-2},... and so forth

$$a_{m+2} = -\frac{a_{m-4}}{(m+2)(m+1)(m)(m-1)(m-2)(m-3)}$$

The first coefficients in terms of a_0 are

$$a_2 = -\frac{a_0}{1 \cdot 2}$$
$$a_4 = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4}$$

and so on. In general,

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$$

This leads to the following solution

$$\phi_2(x) = \frac{1}{x}\left[\sum_{k=0}^{\infty} a_{2k} x^{2k}\right]$$

$$\phi_2(x) = \frac{1}{x}\left[a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}\right]$$

$$\boxed{\phi_2(x) = a_0 \frac{\cos(x)}{x}} \tag{7}$$

Equation (6) and (7) are two independent solutions because the Wronskian

$$W(\phi_1, \phi_2) = \phi_1 \phi_2^{'} - \phi_2 \phi_1^{'} = a_0 \frac{\cos(2x) + x\sin(2x)}{x^3}$$

is different from zero in general.

The above equation (7) goes to infinity when x approaches zero. ϕ is directly proportional to the density of the star (for the $n = 1$ case) and so this solution implies that the density is infinite at the center of star, $x = 0$. This is not realistic and hence we can discard this solution. And so a valid series solution for the $n = 1$ case is

$$\boxed{\phi(x) = a_0 \frac{\sin(x)}{x}} \tag{8}$$

2.2. **The n case.** For the arbitrary n case, we can apply the conditions that we found in Section 2.1. This includes setting $p = 0$ and using only even powers of x for the series solution. We still find the same constraints on p since the term ϕ^n does not contrinute to find the lowest two powers of x: x^{p-2} and x^{p-1}. Thus we can let $\phi(x)$ take the form of a series with only even powers of x.

Additionally, differential equation (2) has the same form when we do the switch $x \to -x$ since

$$\frac{d}{dx} = \frac{d(-x)}{dx}\frac{d}{d(-x)} = -\frac{d}{d(-x)}$$
$$\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx} = \frac{d}{d(-x)}\frac{d}{d(-x)} = \frac{d^2}{d(-x)^2}$$

The even solution can be written as

$$\phi(x) = \sum_{m=0}^{\infty} a_{2m}x^{2m}$$
$$\phi'(x) = \sum_{m=0}^{\infty} (2m)a_{2m}x^{2m-1}$$
$$\phi''(x) = \sum_{m=0}^{\infty} (2m)(2m-1)a_{2m}x^{2m-2}$$

With this approach our Lane-Emden equation (2) becomes

$$\sum_{m=0}^{\infty} a_{2m}(2m)(2m-1)x^{2m-2} + 2\sum_{m=0}^{\infty} a_{2m}(2m)x^{2m-2} + \left(\sum_{m=0}^{\infty} a_{2m}x^{2m}\right)^n = 0$$

Once again, we can simplify the equation by combining the first two series. For the series elevated to the power of n, we only consider the first three terms

$$\sum_{m=0}^{\infty} a_{2m}(2m)(2m+1)x^{2m-2} + \left(\sum_{m=0}^{\infty} a_{2m}x^{2m}\right)^n = 0 \tag{9}$$

$$(2\cdot 3)a_2x^0 + (4\cdot 5)a_4x^2 + (6\cdot 7)a_6x^4 + \cdots + \left(a_0x^0 + a_2x^2 + a_4x^4 + \cdots\right)^n = 0$$

or

$$(2\cdot 3)a_2 + (4\cdot 5)a_4x^2 + (6\cdot 7)a_6x^4 + \cdots + \left(a_0 + a_2x^2 + a_4x^4 + \cdots\right)^n = 0 \tag{10}$$

To evaluate the last term, we will use the binomial expansion

$$\left[a_0 + (a_2x^2 + a_4x^4)\right]^n = \sum_{k=0}^{n} \binom{n}{k} (a_0)^{n-k}(a_2x^2 + a_4x^4)^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient.

Using it again for the term $(a_2x^2 + a_4x^4)^k$:

$$\sum_{k=0}^{n} \binom{n}{k} (a_0)^{n-k}(a_2x^2 + a_4x^4)^k = \sum_{k=0}^{n} \binom{n}{k} (a_0)^{n-k} \left[\sum_{\ell=0}^{k} \binom{k}{l} (a_2x^2)^{k-\ell}(a_4x^4)^{\ell}\right]$$

Since we are only interested in the first three nonzero terns, we are only interested in the terms x^0, x^2 and x^4 in the sum over l: $\sum_{\ell=0}^{k} \binom{k}{l}(a_2x^2)^{k-\ell}(a_4x^4)^{\ell}$. Thus we want terms where $k = 0, 1, 2$. The terms under these conditions are:

$k = 0$, $\ell = 0$:

$$\binom{n}{0}(a_0)^{n-0}(a_2x^2)^{0-0}(a_4x^4)^0 = \binom{n}{0}(a_0)^n = a_0^n$$

$k = 1$, $\ell = 0$:

$$\begin{aligned}\binom{n}{1}(a_0)^{n-1}(a_2x^2)^{1-0}(a_4x^4)^0 &= \binom{n}{1}(a_0)^{n-1}a_2x^2 \\ &= na_0^{n-1}a_2x^2\end{aligned}$$

$k = 1$, $\ell = 1$:

$$\begin{aligned}\binom{n}{1}(a_0)^{n-1}(a_2x^2)^{1-1}(a_4x^4)^1 &= \binom{n}{1}(a_0)^{n-1}a_4x^4 \\ &= na_0^{n-1}a_4x^4\end{aligned}$$

$k = 2$, $\ell = 0$:

$$\begin{aligned}\binom{n}{2}(a_0)^{n-2}(a_2x^2)^{2-0}(a_4x^4)^0 &= \binom{n}{2}(a_0)^{n-2}a_2^2x^4 \\ &= \frac{n(n-1)}{2}a_0^{n-2}a_2^2x^4\end{aligned}$$

Thus:

$$(a_0+a_2x^2+a_4x^4+\cdots)^n = (a_0)^n+n(a_0)^{n-1}a_2x^2+\left[n(a_0)^{n-1}a_4 + \frac{n(n-1)}{2}(a_0)^{n-2}a_2^2\right]x^4+\cdots$$

Now we can substitute this into equation (10) to get:

$$\left[(2\cdot 3)a_2 + a_0^n\right]x^0+\left[(4\cdot 5)a_4 + na_0^{n-1}a_2\right]x^2+\left[(7\cdot 6)a_6 + na_0^{n-1}a_4 + \frac{n(n-1)}{2}a_0^{n-2}a_2^2\right]x^4 = 0$$

Since x^n are linearly independent, their coefficients must equal zero. Thus we obtain the following relationship:

$$(3\cdot 2)a_2 + a_0^n = 0 \quad\Rightarrow\quad a_2 = -\frac{a_0^n}{3!}$$

$$(4\cdot 5)a_4 + na_0^{n-1}a_2 = 0 \quad\Rightarrow\quad a_4 = \frac{na_0^{2n-1}}{5!}$$

$$na_0^{n-1}a_4 + \frac{n(n-1)}{2}a_2^2a_0^{n-2} + a_6(6\cdot 7) = 0 \quad\Rightarrow\quad a_6 = -a_0^{3n-2}\left[\frac{n^2}{7!} + \frac{n(n-1)}{2}\frac{1}{(3!)^2\cdot 6\cdot 7}\right]$$

The even series solution

$$\phi(x) = a_0x^0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots$$

can now be written as

(11)
$$\boxed{\phi(x) = a_0 - \frac{a_0^n}{3!}x^2 + \frac{n}{5!}a_0^{2n-1}x^4 - a_0^{3n-2}\left[\frac{n^2}{7!} + \frac{n(n-1)}{2}\frac{1}{(3!)^2\cdot 6\cdot 7}\right]x^6 + ...}$$

Lastly, to test this general expression, we make $n = 1$. And the answer is the exact same result we got in section 2.1:

$$\boxed{\phi(x) = a_0 - \frac{a_0}{3!}x^2 + \frac{a_0}{5!}x^4 - \frac{a_0}{7!}x^6 + ... = a_0\frac{\sin(x)}{x}}$$

2.3. **The Numerical Case.** Only the Lane-Emden equation with polytropic indices of $n = 0$, 1, and 5, has an analytic solution. For all other polytropic indices, the Lane-Emden equation must be numerically integrated. We used the Runge-Kutta-Method (RKM) (which help us solve a differential equation of any order N) to solve the Lane-Emden equation for $n = 2$. The RKM was coded in IDL.

Our numerical RKM can solve a differential equation of order N breaking it into a sysytem of N differential equations of order 1:

$$\frac{d^Ny}{dx^N} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^{N-1}y}{dx^{N-1}}\right)$$

with initial conditions

$$x_0, y_0, \left.\frac{dy}{dx}\right|_{x=0}, \left.\frac{d^2y}{dx^2}\right|_{x=0}, ..., \left.\frac{d^{N-1}y}{dx^{N-1}}\right|_{x=0}$$

where we define

$$y = y_1$$

$$\frac{dy}{dx} = \frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_N) = y_2$$
$$\frac{d^2y}{dx^2} = \frac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_N) = y_3$$
$$\vdots$$

and so

$$\frac{d^N y}{dx^N} = \frac{dy_N}{dx} = f_N(x, y_1, y_2, ..., y_N)$$

In our case for the Lane-Emden equation

$$\phi = y_1$$

$$\frac{d\phi}{dx} = \frac{dy_1}{dx} = f_1(x, y_1, y_2) = y_2$$
$$\frac{d^2\phi}{dx^2} = \frac{dy_2}{dx} = f_N(x, y_1, y_2)$$

where the function f_N is given by

$$f_N\left(x, y, \frac{dy}{dx}\right) = -\frac{2}{x}\frac{d\phi}{dx} - \phi^n$$

or

$$f_N(x, y_1, y_2) = -\frac{2}{x}y_2 - y_1^n$$

with initial conditions

$$x_0 = 0, \phi_0 = 1, \left.\frac{d\phi}{dx}\right|_{x=0} = 0$$

We present the following plots for different scenarios. The stepsize is $h = 0.00799$ and was determined by the number of points and the interval size, which were chosen by the user.

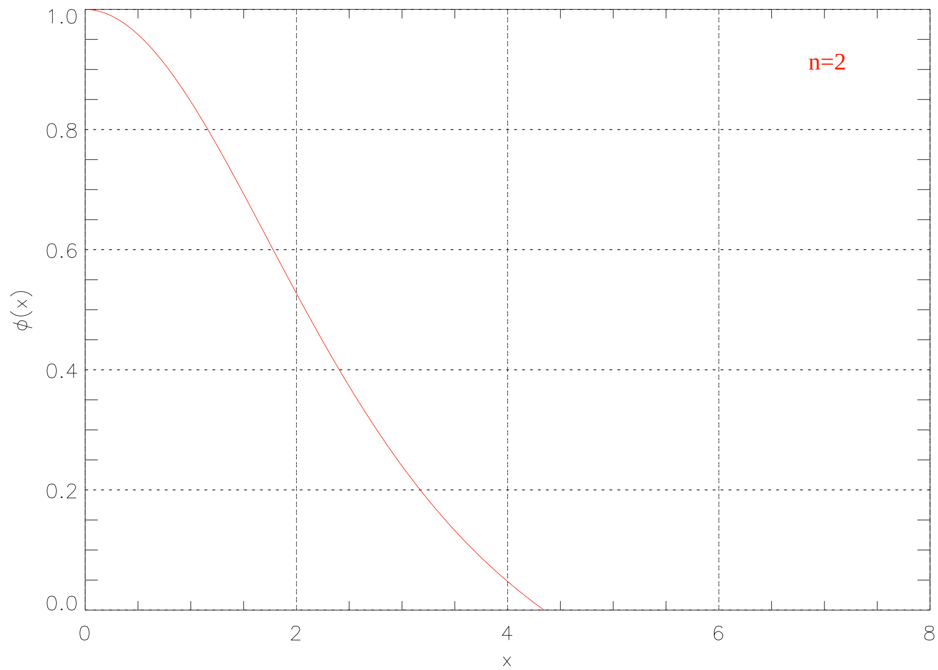

FIGURE 1. **Numerical case n=2.** This plot shows the polytrope with n=2. $\phi(x)$ decreases as a function of distance. $\phi(x)$ first equal zero at x=4.34, which corresponds to the surface of the star.

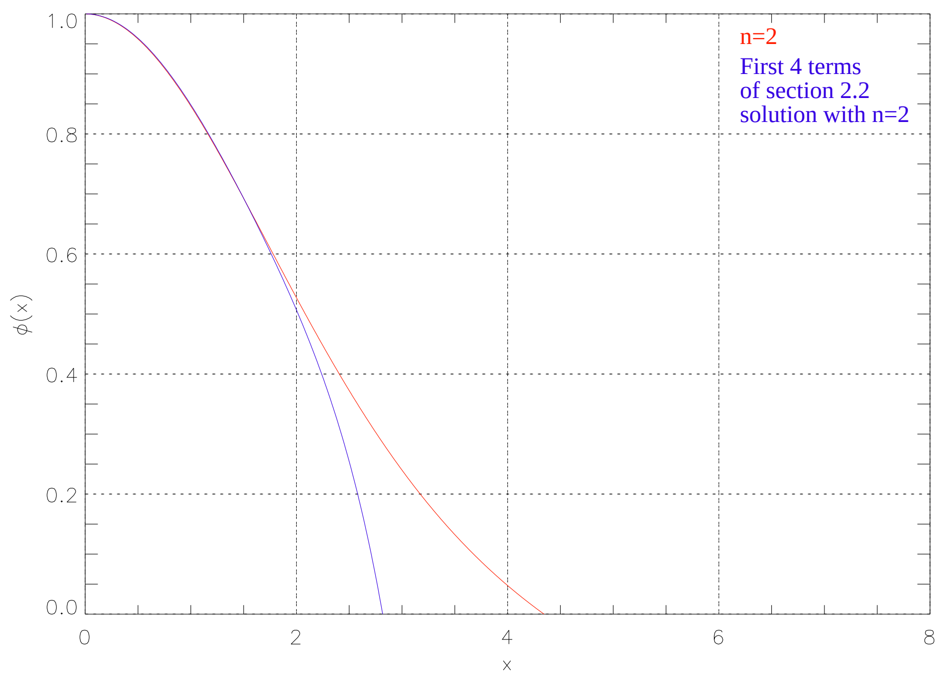

FIGURE 2. **Comparison with section 2.2** This plot shows the polytrope with n=2, red line, and the first 4 terms in the series solution obtained in section 2.2 with n=2, blue line. For $0 < x < 2$ this is a very good fit. Things start to differ for larger x, this is because we are only considering a few terms of the infinite series solution found in section 2.2.

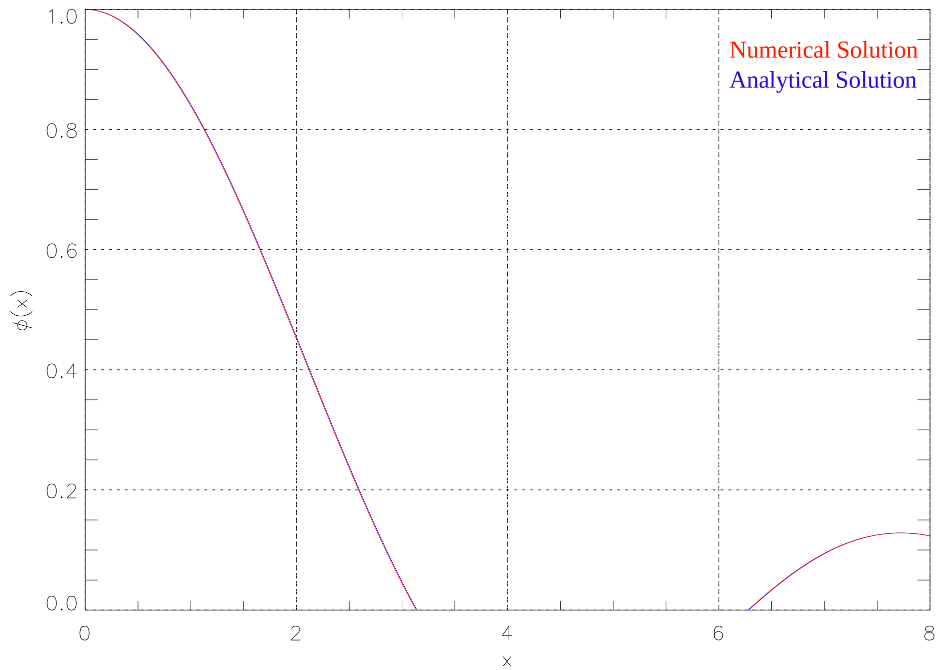

FIGURE 3. **Numerical and analytical case n=1.** This plot shows both the numerical solution, red line, and the analytical solution, blue line, for polytrope with $n = 1$. We can see that the numerical solution matches the analytical solution.

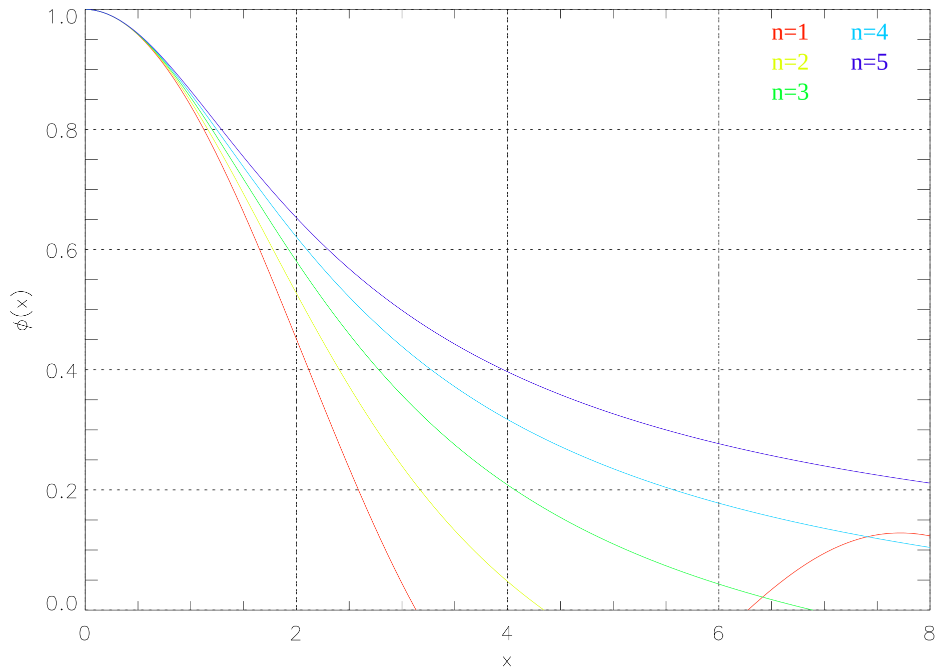

FIGURE 4. **Five polytropes.** This plot considers the five polytropes corresponding to $n = 1, 2, 3, 4$,and 5.

3. Conclusion

We were able to find two independent solutions for Lane-Emden equation (1) in the case where $n = 1$. However, we discarded one of them that was not physical for the system in consideration, a star. The solution we chose was equation (8), the physical, realistic solution which is valid everywhere and is well behaved at x=0. This assures a finite value for the central density of the star. Thus, even when Frobenius method was capable of giving us two independent solutions we had to be more conscious about their physical relevance and choose the aprropiate for our realistic system.

We also found the first four nonzero terms of a series solution for the general case n, in which the binomial expansion was very helpful in keeping the relevant powers of x. We compared the numerical solution obtained with Runge-Kutta-Method to the analytical solution given by equation (11) with n=2.

The Runge-Kutta-Method was very helpful in the task of getting numerical solutions for the polytropes of any n. To plot, there was no need to convert the Lane-Emden equation into dimensionless form, because it is already in dimensionless form since x is a dimensinless distance and ϕ is a dimensionless density.

Finally, we can see that both the physical intuition as well as mathematical and numerical techniques allowed us to correctly analyze the Lane-Emden equation in order to obtain solutions of physical significance for the star system.

APPENDIX

Runge-Kutta-Method.

```
pro RK4, fileNameRoot

;set_plot, 'ps'
;device,filename='n2.eps',/encapsulated,/times,/color,/inches,bits_per_pixel=8
;!p.font=0


;open file here to save data
get_lun,lun
openw,lun,'LaneEmden2.txt'

;Order of differential equation is 2

;Number of points
NP=1000

;Beginning and end of interval to plot
A=0.01
B=8

;Initial conditions:
y10 = 1    ;This corresponds to phi0 = 1
y20 = 0    ;This corresponds to dphi0 = 0

;Step size in x
h = (B-A)/NP
print,h

for i=0L,NP do begin
x = A + i*h
y1 = y10
y2 = y20

RK11 = F1(x,y1,y2)
RK12 = F2(x,y1,y2)

xm = x + 0.5*h
y1 = y10 + 0.5*h*RK11
y2 = y20 + 0.5*h*RK12

RK21 = F1(xm,y1,y2)
RK22 = F2(xm,y1,y2)

y1 = y10 + 0.5*h*RK21
y2 = y20 + 0.5*h*RK22

RK31 = F1(xm,y1,y2)
RK32 = F2(xm,y1,y2)

xm1 = x + h
y1 = y10 + h*RK31
y2 = y20 + h*RK32

RK41 = F1(xm1,y1,y2)
```

```
RK42 = F2(xm1,y1,y2)

y1s = y10 + h*(RK11 + 2*RK21 + 2*RK31 + RK41)/6
y2s = y20 + h*(RK12 + 2*RK22 + 2*RK32 + RK42)/6

;Here save x,y1s,y2s
;print,x,y1s,y2s
printf,lun,x,y1s,y2s

y10 = y1s
y20 = y2s
endfor
close,lun

loadct,13
readcol,'LaneEmden.txt',x,y,yy,format='f,f,f'
readcol,'LaneEmden1.txt',x1,y1,yy1,format='f,f,f'
readcol,'LaneEmden2.txt',x2,y2,yy2,format='f,f,f'
readcol,'LaneEmden3.txt',x3,y3,yy3,format='f,f,f'
readcol,'LaneEmden4.txt',x4,y4,yy4,format='f,f,f'
readcol,'LaneEmden5.txt',x5,y5,yy5,format='f,f,f'

;phi(x)
thisletter="165B
greekletter= '!4'+String(thisletter)+'!X'

plot,x1,y1,XTicklen=1.0,YTicklen=1.0,XGridStyle=1,YGridStyle=1,XTitle='x',YTitle=greekletter+'(x)',yrange=[0,1],yst
oplot,x1,y2,color=[250]

xyouts,6.85,0.90,'n=2',color=[250]

;device,/close
;set_plot,'x'
;!p.font=-1


end
```

The program above calls the functions F1 and F2 shown below

```
FUNCTION F1, x,y1,y2
RETURN, F1 = y2
END


FUNCTION F2, x,y1,y2
;Polytropic Index m
m=2
RETURN, F2 = -(2/x)*y2 - y1^m
END
```

The numerical output for the n=2 is presented below. The first column corresponds to the dimensionless distance x, the second column corresponds to the values of the dimensionless density $\phi(x)$, and the third cloumn is the first derivative of $\phi(x)$. We can see that $\phi(x) = 0$ around $x = 4.34$.

0.0100000	0.999977	-0.00492382
0.0179900	0.999924	-0.00814351
0.0259800	0.999847	-0.0110196
0.0339700	0.999748	-0.0137849
0.0419600	0.999627	-0.0165034
0.0499500	0.999485	-0.0191982
0.0579400	0.999321	-0.0218793
0.0659300	0.999135	-0.0245514
0.0739200	0.998928	-0.0272172
0.0819100	0.998700	-0.0298778
0.0899000	0.998451	-0.0325341
0.0978900	0.998180	-0.0351864
0.105880	0.997889	-0.0378350
0.113870	0.997576	-0.0404798
0.121860	0.997242	-0.0431210
0.129850	0.996887	-0.0457584
0.137840	0.996511	-0.0483919
0.145830	0.996113	-0.0510215
0.153820	0.995695	-0.0536469
0.161810	0.995256	-0.0562680
0.169800	0.994796	-0.0588847
0.177790	0.994315	-0.0614968
0.185780	0.993814	-0.0641041
0.193770	0.993291	-0.0667065
0.201760	0.992748	-0.0693037
0.209750	0.992183	-0.0718955
0.217740	0.991599	-0.0744819
0.225730	0.990993	-0.0770626
0.233720	0.990367	-0.0796374
0.241710	0.989721	-0.0822061
0.249700	0.989054	-0.0847686
0.257690	0.988366	-0.0873247
0.265680	0.987658	-0.0898742
0.273670	0.986930	-0.0924168
0.281660	0.986181	-0.0949526
0.289650	0.985412	-0.0974811
0.297640	0.984624	-0.100002
0.305630	0.983814	-0.102516
0.313620	0.982985	-0.105022
0.321610	0.982136	-0.107520
0.329600	0.981267	-0.110010
0.337590	0.980378	-0.112492
0.345580	0.979470	-0.114966
0.353570	0.978541	-0.117431
0.361560	0.977593	-0.119887
0.369550	0.976626	-0.122334
0.377540	0.975638	-0.124773
0.385530	0.974632	-0.127202
0.393520	0.973606	-0.129621
0.401510	0.972560	-0.132031
0.409500	0.971496	-0.134432
0.417490	0.970412	-0.136822
0.425480	0.969309	-0.139203
0.433470	0.968188	-0.141573
0.441460	0.967047	-0.143933
0.449450	0.965888	-0.146282
0.457440	0.964710	-0.148621
0.465430	0.963513	-0.150949

0.473420	0.962297	-0.153265
0.481410	0.961064	-0.155571
0.489400	0.959811	-0.157866
0.497390	0.958541	-0.160149
0.505380	0.957252	-0.162420
0.513370	0.955945	-0.164680
0.521360	0.954621	-0.166928
0.529350	0.953278	-0.169164
0.537340	0.951917	-0.171388
0.545330	0.950539	-0.173600
0.553320	0.949143	-0.175799
0.561310	0.947730	-0.177986
0.569300	0.946299	-0.180160
0.577290	0.944851	-0.182322
0.585280	0.943386	-0.184471
0.593270	0.941903	-0.186606
0.601260	0.940404	-0.188729
0.609250	0.938887	-0.190838
0.617240	0.937354	-0.192934
0.625230	0.935804	-0.195016
0.633220	0.934238	-0.197085
0.641210	0.932655	-0.199140
0.649200	0.931056	-0.201181
0.657190	0.929440	-0.203209
0.665180	0.927808	-0.205222
0.673170	0.926161	-0.207221
0.681160	0.924497	-0.209207
0.689150	0.922818	-0.211177
0.697140	0.921122	-0.213134
0.705130	0.919412	-0.215076
0.713120	0.917686	-0.217003
0.721110	0.915944	-0.218915
0.729100	0.914187	-0.220813
0.737090	0.912416	-0.222696
0.745080	0.910629	-0.224564
0.753070	0.908827	-0.226417
0.761060	0.907011	-0.228255
0.769050	0.905180	-0.230077
0.777040	0.903334	-0.231885
0.785030	0.901474	-0.233677
0.793020	0.899600	-0.235453
0.801010	0.897712	-0.237214
0.809000	0.895809	-0.238960
0.816990	0.893893	-0.240690
0.824980	0.891963	-0.242404
0.832970	0.890019	-0.244103
0.840960	0.888062	-0.245786
0.848950	0.886092	-0.247453
0.856940	0.884108	-0.249104
0.864930	0.882111	-0.250739
0.872920	0.880101	-0.252358
0.880910	0.878079	-0.253961
0.888900	0.876043	-0.255548
0.896890	0.873995	-0.257118
0.904880	0.871934	-0.258673
0.912870	0.869861	-0.260211
0.920860	0.867776	-0.261733
0.928850	0.865679	-0.263239

0.936840	0.863570	-0.264728
0.944830	0.861449	-0.266201
0.952820	0.859316	-0.267658
0.960810	0.857172	-0.269098
0.968800	0.855016	-0.270522
0.976790	0.852849	-0.271929
0.984780	0.850670	-0.273320
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1.03272	0.837373	-0.281316
1.04071	0.835121	-0.282591
1.04870	0.832858	-0.283849
1.05669	0.830585	-0.285090
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1.08066	0.823707	-0.288715
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1.13659	0.807335	-0.296590
1.14458	0.804962	-0.297649
1.15257	0.802579	-0.298691
1.16056	0.800188	-0.299717
1.16855	0.797790	-0.300726
1.17654	0.795383	-0.301718
1.18453	0.792968	-0.302694
1.19252	0.790546	-0.303654
1.20051	0.788116	-0.304597
1.20850	0.785678	-0.305524
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1.22448	0.780782	-0.307329
1.23247	0.778323	-0.308207
1.24046	0.775857	-0.309069
1.24845	0.773384	-0.309914
1.25644	0.770904	-0.310744
1.26443	0.768418	-0.311557
1.27242	0.765925	-0.312354
1.28041	0.763427	-0.313135
1.28840	0.760922	-0.313900
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1.30438	0.755894	-0.315381
1.31237	0.753371	-0.316099
1.32036	0.750842	-0.316800
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1.35232	0.740674	-0.319448
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1.39227	0.727852	-0.322407

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1.41624	0.720105	-0.323999
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1.44820	0.709718	-0.325911
1.45619	0.707113	-0.326352
1.46418	0.704503	-0.326778
1.47217	0.701891	-0.327189
1.48016	0.699275	-0.327585
1.48815	0.696656	-0.327967
1.49614	0.694034	-0.328335
1.50413	0.691409	-0.328688
1.51212	0.688782	-0.329027
1.52011	0.686151	-0.329352
1.52810	0.683519	-0.329662
1.53609	0.680883	-0.329959
1.54408	0.678246	-0.330241
1.55207	0.675606	-0.330509
1.56006	0.672964	-0.330764
1.56805	0.670321	-0.331005
1.57604	0.667675	-0.331232
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1.61599	0.654422	-0.332165
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1.72785	0.617193	-0.333061
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1.75182	0.609211	-0.332940
1.75981	0.606551	-0.332876
1.76780	0.603892	-0.332801
1.77579	0.601233	-0.332714
1.78378	0.598575	-0.332616
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1.79976	0.593262	-0.332386
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1.82373	0.585299	-0.331959
1.83172	0.582648	-0.331795
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1.91162	0.556219	-0.329580
1.91961	0.553587	-0.329304
1.92760	0.550957	-0.329018
1.93559	0.548329	-0.328722
1.94358	0.545704	-0.328417
1.95157	0.543081	-0.328102
1.95956	0.540461	-0.327779
1.96755	0.537843	-0.327446
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2.00750	0.524797	-0.325648
2.01549	0.522197	-0.325262
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2.25519	0.445912	-0.310183
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2.51087	0.369283	-0.288489
2.51886	0.366981	-0.287747
2.52685	0.364685	-0.287002
2.53484	0.362394	-0.286254
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2.63871	0.333175	-0.276291
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2.66268	0.326581	-0.273939
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3.73334	0.0902870	-0.170586
3.74133	0.0889266	-0.169924
3.74932	0.0875716	-0.169264
3.75731	0.0862218	-0.168607
3.76530	0.0848773	-0.167952
3.77329	0.0835379	-0.167299
3.78128	0.0822038	-0.166649
3.78927	0.0808749	-0.166002
3.79726	0.0795511	-0.165357
3.80525	0.0782325	-0.164714
3.81324	0.0769190	-0.164074
3.82123	0.0756105	-0.163436
3.82922	0.0743072	-0.162801
3.83721	0.0730090	-0.162169
3.84520	0.0717158	-0.161538
3.85319	0.0704276	-0.160911
3.86118	0.0691444	-0.160286
3.86917	0.0678662	-0.159663
3.87716	0.0665930	-0.159043
3.88515	0.0653247	-0.158426
3.89314	0.0640613	-0.157811
3.90113	0.0628029	-0.157199
3.90912	0.0615493	-0.156589
3.91711	0.0603006	-0.155981
3.92510	0.0590567	-0.155377
3.93309	0.0578177	-0.154775
3.94108	0.0565834	-0.154175
3.94907	0.0553539	-0.153578
3.95706	0.0541292	-0.152984
3.96505	0.0529092	-0.152392
3.97304	0.0516940	-0.151802
3.98103	0.0504834	-0.151216
3.98902	0.0492776	-0.150632
3.99701	0.0480763	-0.150050
4.00500	0.0468797	-0.149471
4.01299	0.0456878	-0.148895
4.02098	0.0445004	-0.148321
4.02897	0.0433176	-0.147750
4.03696	0.0421394	-0.147181
4.04495	0.0409656	-0.146615
4.05294	0.0397964	-0.146052
4.06093	0.0386317	-0.145491
4.06892	0.0374715	-0.144933
4.07691	0.0363157	-0.144378
4.08490	0.0351643	-0.143825
4.09289	0.0340174	-0.143274
4.10088	0.0328748	-0.142726
4.10887	0.0317366	-0.142181
4.11686	0.0306027	-0.141639
4.12485	0.0294732	-0.141099
4.13284	0.0283480	-0.140562
4.14083	0.0272270	-0.140027
4.14882	0.0261103	-0.139495
4.15681	0.0249979	-0.138965
4.16480	0.0238896	-0.138438
4.17279	0.0227856	-0.137914

```
4.18078    0.0216858    -0.137392
4.18877    0.0205901    -0.136873
4.19676    0.0194985    -0.136357
4.20475    0.0184111    -0.135843
4.21274    0.0173278    -0.135332
4.22073    0.0162485    -0.134823
4.22872    0.0151733    -0.134317
4.23671    0.0141021    -0.133813
4.24470    0.0130349    -0.133312
4.25269    0.0119718    -0.132814
4.26068    0.0109126    -0.132318
4.26867   0.00985731    -0.131825
4.27666   0.00880598    -0.131335
4.28465   0.00775857    -0.130847
4.29264   0.00671504    -0.130362
4.30063   0.00567538    -0.129879
4.30862   0.00463957    -0.129399
4.31661   0.00360758    -0.128921
4.32460   0.00257940    -0.128446
4.33259   0.00155501    -0.127974
4.34058  0.000534375    -0.127504
4.34857 -0.000482513    -0.127037
4.35656  -0.00149568    -0.126572
4.36455  -0.00250514    -0.126110
4.37254  -0.00351092    -0.125650
4.38053  -0.00451304    -0.125193
```

To compare the anaylitical solutions for the cases n=1, equation (8) with $a_0 = 1$, and n=2 in equation (11) to the numerical solutions we wrote the following program:

```
pro partab, fileNameRoot

set_plot,'ps'
device,filename='n1comparison.eps',/encapsulated,/times,/color,/inches,bits_per_pixel=8
!p.font=0

;open file here to save data
get_lun,lun
openw,lun,'parta.txt'

;Number of points
NP=1000

;Beginning and end of interval to plot
A=0.01
B=8

;Step size in x
h = (B-A)/NP
print,h

for i=0L,NP do begin
x = A + i*h
phia = (sin(x))/x
phib = 1 - (x^2)/6 + (x^4)/60 - (x^6)/1260 - (x^6)/1512
;Here save x,phi
print,x,phia
```

```
;print,x,phib
printf,lun,x,phia
;printf,lun,x,phib
endfor

close,lun

loadct,13
readcol,'LaneEmden1.txt',x1,y1,yy1,format='f,f,f'
readcol,'LaneEmden2.txt',x2,y2,yy2,format='f,f,f'
readcol,'parta.txt',xa,phia,format='f,f'
readcol,'partb.txt',xb,phib,format='f,f'

;phi(x)
thisletter="165B
greekletter='!4'+String(thisletter)+'!X'

plot,x1,y1,XTicklen=1.0, YTicklen=1.0,XGridStyle=1,YGridStyle=1,XTitle='x',YTitle=greekletter+'(x)',yrange=[0,1],ys
oplot,x1,phia,color=[50]
oplot,x1,y1,color=[250]
;oplot,x2,y2,color=[250]
;oplot,x2,phib,color=[50]

xyouts,6.1,0.92,'Numerical Solution',color=[250]
xyouts,6.1,0.87,'Analytical Solution',color=[50]

device,/close
set_plot,'x'
!p.font=-1

end
```

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