# Linearization, with offsets

Usually, we consider a framework with linear dynamics, so:

$$\dot{x} = Ax + Bu$$

with state feedback:

$$u = -Kx$$

which yields:

$$\dot{x} = (A - BK)x. \tag{1}$$

If the eigenvalues of (A-BK) are stable poles, the states are all driven asymptotically to zero.

Now, assume we want to go to some other operating point, $x_{eq}$ and that our equations of motion have an "offset" (so they are *affine* rather than strictly *linear*). We can write the dynamics as:

$$\dot{x} = A(x - x_{eq}) + Bu + \Phi_{off}, \tag{2}$$

Our new control law will be:

$$u = -K(x - x_{eq}) + u_{off}. \tag{3}$$

Let's define a new variable, $p = x - x_{eq}$, and combine the two equations above. We get[1]:

$$\dot{p} = Ap - BKp + Bu_{off} + \Phi_{off}.$$

If we can ensure $Bu_{off} = -\Phi_{off}$, then:

$$\dot{p} = (A - BK)p,$$

which matches Eq. 1.

This is the new framework we will adopt. A few issues deserve discussion at this point:

1. Linearization: In general, our dynamics are neither linear nor affine, so we will need to *linearize* them to create an approximation that matches the form of Eq. 2. Text

2. Picking $x_{eq}$: We cannot necessarily ensure $Bu_{off} = -\phi_{off}$ for just any, arbitrary configuration of the system. In particular, for the two-link systems we will study, we have only one input torque in $u$, acting at only one of the two joints. We need to be sure that our desired operating point can be made into an equilibrium configuration (via feedback control), even with zero torque applied at the unactuated joint! For these partiular systems, that is equivalent to ensuring that "equilibrium" velocites are zero (obviously) and the configuration defined by the angle states within $x_{eq}$ results in **no net torque** about the unactuated joint.

3. $F = ma$ form of equations: Typically, we start with equations of motion of the form $F = ma = m\ddot{x}$. Our state space equations require the form $\ddot{x} = \frac{1}{m}F$. Also, we typically have multiple degrees of freedom, so $F = ma$ is represented in a matrix form, e.g., $M\ddot{q} = F$. After linearization, $F$ should look something like $F = -C\dot{q} - Kq + f(u) + F_{off}$. To get various terms needed to fill the $A$ matrix for Eq. 2, we will need to pre-multiply $C$ and $K$ by the inverse of M, so we have a vector of accelerations, $\ddot{q}$, alone on the lefthand side of our matrix equations.

(Don't worry if the last bullet item is somewhat confusing; we'll go over this in more detail later.)

[1]Since $x_{eq}$ is a constant, its derivative is zero, so $\dot{p} = \dot{x}$.

Simple Pendulum

Let's go over the full steps required using a simple (single-link) pendulum, first. We can linearize this system about one of its two natural equilibriums (straight up, or straight down), in which case $u_{off}$ will be zero. Or, we can pick some other, arbitrary angle as an operating point, in which case $u_{off}$ will need to be some non-zero value.

We will define the angle of pendulum, $\theta$, as being measured from the horizontal axis, going counter-clockwise. The corresponding equation of motion for the system is:

$$mL^2\ddot{\theta} = -mgL\cos\theta + u, \tag{4}$$

where $u$ is our input torque (to be set via some feedback law).

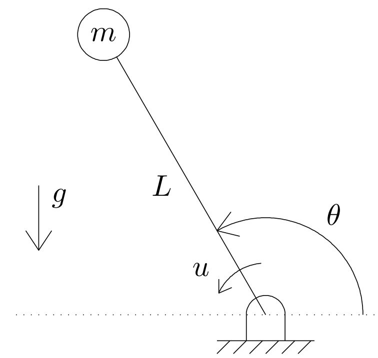


Figure 1: Simple Pendulum.

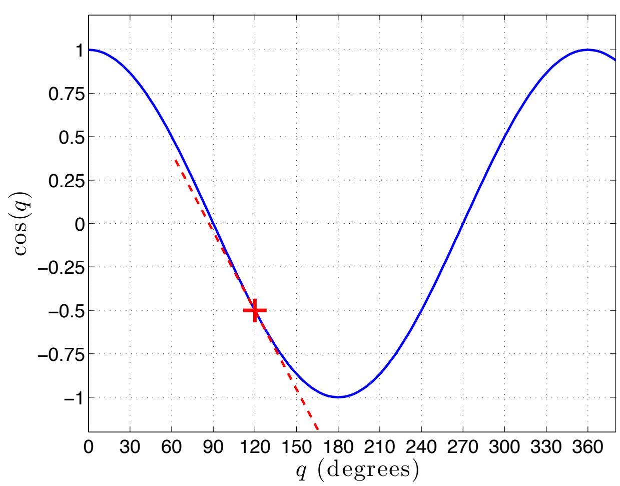


Figure 2: Cosine function and operating point.

Say we wish to control the system so that it is stabilized at the orientation shown in Fig. 1, with $\theta_{eq} = \frac{4\pi}{3} = 120°$ and $\dot{\theta}_{eq} = 0$. Our first step is to linearize the system about this operation point, $x_{eq}$, so that we represent the equation of motion in Eq. 4 as an *affine approximation*, matching the form shown in Eq. 2. To do so, we need to represent the function $y = cos(x)$ in the form $y \approx mx + b$, for $\theta$ close to $\theta_{eq} = \frac{4\pi}{3}$.

For the general case, $y = f(x)$, the affine approximation near $x = x_{eq}$ would be:

$$y = f(x) \approx f(x_{eq}) + \left.\frac{df}{dx}\right|_{x_{eq}} (x - x_{eq}). \tag{5}$$

If $x$ is a vector of states, $x_i$ through $x_n$, then the linearized approximation for $y = f(x)$ would involve *partial* derivatives, i.e.,

$$y = f(x) \approx f(x_{eq}) + \sum_{i=1}^{n} \left( \left.\frac{\partial f}{\partial x}\right|_{x_{eq}} (x_i - x_{i,eq}) \right). \tag{6}$$

In words, this is akin to $y \approx mx + b$, with details below:

- Here, $b = f(x_{eq})$ is what the function equals at *exactly* the operating point, $x_{eq}$. For our example, $b = cos\left(\frac{4\pi}{3}\right) = -0.5$, which is the highlighted operating point (+ symbol) in Fig. 2.

- The slope of $f(x)$ evaluated at $x_{eq}$ gives us $m$, i.e., $m = f'(x_{eq}) = \frac{df}{dx}|_{x_{eq}}$. For our pendulum example, $m = \frac{d}{dx}\cos(x)|_{x_{eq}} = -sin(x_{eq})$. Therefore, $m = -sin\left(\frac{4\pi}{3}\right) = \frac{-1}{2}\sqrt{3} \approx -0.866$. The slope is illustrated by the dashed line in Fig. 2.

- Finally, although we wrote "$y = mx + b$" as a familiar example of an affine equation, the term $x$ here really corresponds to some $\Delta x$, with respect to $x_{eq}$. (As is common, we abuse poor variable name $x$ throughout, since it has so many pre-defined uses! Pretend $x_{eq} = \theta_{eq}$ is a scalar for the moment...) What we really have is: $y \approx m\Delta x + b = m(x - x_{eq}) + b$.

Next, let's use $m$, $\Delta x$ and $b$ to write a linearized version of Eq. 4:

$$\begin{aligned} mL^2\ddot{\theta} &= -mgL\left(cos\left(\frac{4\pi}{3}\right) - sin\left(\frac{4\pi}{3}\right)\left(\theta - \frac{4\pi}{3}\right)\right) + u \\ &= -mgL\left(-0.5 - 0.866\left(\theta - \frac{4\pi}{3}\right)\right) + u \\ &= mgL\left(0.5 + 0.866\left(\theta - \frac{4\pi}{3}\right)\right) + u \\ &= 0.866 \cdot mgL\left(\theta - \frac{4\pi}{3}\right) + u + 0.5 \cdot mgL. \end{aligned} \tag{7}$$

We are getting closer to the form in Eq. 2. Note we can rewrite the last equation as:

$$\ddot{\theta} = \tfrac{1}{mL^2}\left(0.866 \cdot mgL\left(\theta - \tfrac{4\pi}{3}\right) + u + 0.5 \cdot mgL\right)$$

To put this in the state space format of Eq. 2, we first define a state <u>vector</u>, $x$:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix},$$

and the equilibrium state (i.e., our desired operating point) is:

$$x_{eq} = \begin{bmatrix} x_{1,eq} \\ x_{2,eq} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4\pi}{3} \end{bmatrix}$$

Our linearized equation of motion is our first state equation:

$$\begin{aligned} mL^2\dot{x}_1 = mL^2\ddot{\theta} &= 0.866 \cdot mgL\left(\theta - \frac{4\pi}{3}\right) + u + 0.5 \cdot mgL \\ &= 0.866 \cdot mgL\left(x_2 - x_{2,eq}\right) + u + 0.5 \cdot mgL, \end{aligned} \tag{8}$$

and we relate $x_1$ and $x_2$ with a second state equation:

$$\dot{x}_2 = x_1. \tag{9}$$

Putting both state equations (Eq. 8 and 9) together in matrix form, we have:

$$\begin{bmatrix} mL^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.866(mgL) \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_{1,eq} \\ x_{2,eq} \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5(mgL) \\ 0 \end{bmatrix} \tag{10}$$

Let's write that as:

$$M\dot{x} = \tilde{A}\left(x - x_{eq}\right) + \tilde{B}u + \tilde{\Phi}_{off}. \tag{11}$$

Then, pre-multiplying each term by the inverse of $M$, we get:

$$\begin{aligned} \dot{x} &= M^{-1}\tilde{A}\left(x - x_{eq}\right) + M^{-1}\tilde{B}u + M^{-1}\tilde{\Phi}_{off} \\ &= A\left(x - x_{eq}\right) + Bu + \Phi_{off}. \end{aligned}, \tag{12}$$

which matches Eq. 2. From the above equations, $A = M^{-1}\tilde{A}$, $B = M^{-1}\tilde{B}$, and $\Phi_{off} = M^{-1}\tilde{\Phi}_{off}$. Let us define $m = 0.5$ (kg), $L = 1.2$ (m), and $g = 9.81$ (m/s$^2$). Then, Eq. 10 becomes:

$$\begin{bmatrix} 0.720 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 5.097 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_{1,eq} \\ x_{2,eq} \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 2.943 \\ 0 \end{bmatrix}, \tag{13}$$

and Eq. 12 becomes:

$$\begin{aligned} \dot{x} &= A\left(x - x_{eq}\right) + Bu + \Phi_{off} \\ &= \begin{bmatrix} 0 & 7.0798 \\ 1.0000 & 0 \end{bmatrix} \left(x - x_{eq}\right) + \begin{bmatrix} 1.3889 \\ 0 \end{bmatrix} u + \begin{bmatrix} 4.0875 \\ 0 \end{bmatrix}. \end{aligned} \tag{14}$$

It is trivial to find an appropriate $u_{off}$ to satisfy Eq. 3. Recall: $u = -K(x - x_{eq}) + u_{off}$ in our framework. To cancel $\Phi_{off}$, we require (as previously stated):

$$\begin{aligned} Bu_{off} &= -\Phi_{off} \\ \begin{bmatrix} 1.3889 \\ 0 \end{bmatrix} u_{off} &= -\begin{bmatrix} 4.0875 \\ 0 \end{bmatrix} \\ u_{off} &= \frac{-4.0875}{1.3889} = -2.9430. \end{aligned} \tag{15}$$

This should make sense intuitively, because $u_{off}$ is simply equal and opposite to the torque due to gravity, which (at 120°) is $\frac{1}{2}mgL \approx 2.9430$, since the moment arm in Fig. 1 is $L\cos\left(\theta_{eq}\right) = \frac{1}{2}L$. Using this $u_{off}$, the linearized dynamics become:

$$\dot{x} = (A - BK)\left(x - x_{eq}\right) = \left( \begin{bmatrix} 0 & 7.0798 \\ 1.0000 & 0 \end{bmatrix} - \begin{bmatrix} 1.3889 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right) \left(x - x_{eq}\right). \tag{16}$$

To place the closed-loop poles at (for example) $s = -3 \pm 3j$ requires:

$$K = \begin{bmatrix} 4.3200 & 18.0574 \end{bmatrix},$$

which results in

$$(A - BK) = \begin{bmatrix} -6 & -18 \\ 1 & 0 \end{bmatrix}.$$

Two-Link Pendulum Systems

For this system, there are now two links, but only one is actuated. The angles of the two links will be called $q_1$ and $q_2$ (instead of using "$\theta$"). Angle $q_1$ is measured as shown in Fig. 1, as an *absolute* angle, going CCW from the $x$ axis, and $q_2$ is the *relative* angle of the second joint (so that the absolute angle of the second link would be $q_1 + q_2$).

The same basic steps can be used here and for more complex systems. Here are guidelines:

1. Find a valid operating point: $x_{eq}$. This should contain two angles, $q_1$ and $q_2$, and two angular velocities, $\dot{q}_1$ and $\dot{q}_2$. For any unactuated joint, there must be no net torque when in this configuration. Gravity is the key consideration here. The center of mass of all links "going outward" from this joint must be along a vertical line passing through the joint, i.e., either directly below or directly above the joint, to avoid producing net torque in a static (i.e., zero velocity) configuration.
2. Find the linearized matrices to describe the equations of motion, matching the form in Eq. 11. That is, find $M$, $\tilde{A}$, $\tilde{B}$, and $\tilde{\Phi}_{off}$.
   - $M$ will be configuration-dependent, meaning you need to plug in the states corresponding to $x_{eq}$ to find it. (Inertia values may depend on values of the angles, $q_1$ and $q_2$, and some rows in $M$ will simply have zeros and ones, as was the case in the second row of $M$, in Eq. 13.)
   - In $\tilde{A}$, some rows will simply have zeros and ones, again analogous to Eq. 13. The other rows will contain partial derivatives. Thinking in terms of some nonlinear matrix equations analogous to "$F = ma$", the matrix $M$ includes effective mass terms (inertia terms, to be more precise) that are multiplied by accelerations ($\ddot{q}$ terms). Within the "$F$" part, $\tilde{A}$ includes slope values, similar to the illustration from Fig. 2. Once you have $M\ddot{x}$ on one side of your matrix equations, take partial derivatives of all the stuff on the other side of the equation (the "$F$" part), and evaluate these partial derivatives at the operating point, $x_{eq}$. Plug in the results as appropriate within $\tilde{A}$, just as we did for Eqs. 10 and 13.
   - $\tilde{B}$ is simple. For your example, $u$ is a single torque input, and you will have 4 states in $x$ (two angles and two angular velocities). Therefore, $\tilde{B}$ is a $4x1$ matrix. It will contain 3 zeros and 1 one. The one goes in the row corresponding to the "$F = ma$" type equation for the actuated joint. This is analogous to $\tilde{B}$ for our simple pendulum, from Eqs. 10 and 13.
   - $\tilde{\Phi}_{off}$ is whatever the "$F$" part of the equation evaluates to when you plug in $x = x_{eq}$, once again directly analogous to what was done in Eqs. 10 and 13.

The MATLAB code provided includes the full equations of motion for the acrobot system, in which only the second (elbow) joint is actuated. The other system is the pendubot, which has the same equations of motion, except that now only the first joint is actuated (so only $\tilde{B}$ and $B$ are different here). Also, note that MATLAB can be used to take derivatives (or partial derivatives), via the command "diff", if you wish.

Linearization results in angle-dependent forces that are analogous to stable or unstable "spring" forces, but terms analogous to "damping" are all exactly zero. This was the case for the inverted pendulum system in Lab 3 and for our simple pendulum, because we have no passive damping at the joints, so do not be surprised to see many zeros appearing in $\tilde{A}$.