Control Systems PPT

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1 State Space Analysis

1. **Select States of System**

$$X = \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \text{Angle between pendulum and vertical} \\ \text{Angular velocity of pendulum} \\ \text{Angle made by reaction wheel w.r.t pendulum} \\ \text{Angular velocity of reaction wheel} \end{bmatrix}$$

2. **State Space Model**

$$\dot{X} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

$$\therefore \dot{X} = A \cdot X$$

$$\therefore \frac{dX}{dt} = A \cdot X$$

$$\therefore X(t) = e^{At} \cdot X(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + ... \tag{1}$$

But this expansion will be tough to calculate and hence the equations become unnecessarily complex.
So instead we can make use of diagonalization to linearly transform our system to eigenvalue co-ordinates.

$$\text{i.e } X = P \cdot Z \text{ (P = Modal Matrix)}$$

$$\therefore \dot{X} = P \cdot \dot{Z} = A \cdot X$$

$$\therefore P \cdot \dot{Z} = A \cdot P \cdot Z$$

$$\therefore \dot{Z} = P^{-1} \cdot A \cdot P \cdot Z$$

Considering P is orthogonal Matrix

$$\dot{Z} = D \cdot Z$$

D = Diagonal Matrix

$$\therefore \dot{Z} = \begin{bmatrix} \lambda 1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda n \end{bmatrix} \cdot Z$$

Equations become decoupled and thus have simple to calculate solutions.

$$i.e\ \dot{z1} = \lambda 1.z1$$
$$\dot{z2} = \lambda 2.z2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

Also,

$$Z(t) = e^{Dt} \cdot Z(0)$$

$$\therefore Z(t) = \begin{bmatrix} e^{\lambda 1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & e^{\lambda n} \end{bmatrix} \cdot Z(0)$$

Now,
Converting back to original state since it is important to preserve our required system state variables.
Thus, Using Power of Matrix Theorem in equation (1),

$$e^{At} = P^{-1} \cdot P + P^{-1} \cdot D \cdot Pt + \frac{P^{-1} \cdot D^2 \cdot Pt^2}{2!} + ...$$

$$\therefore e^{At} = P\left[I + D + \frac{D^2 t^2}{2!} + \frac{D^3 t^3}{3!} + ...\right]P^{-1}$$

$$\therefore e^{At} = P^{-1} \cdot e^{Dt} \cdot P$$

$$\therefore \boxed{X(t) = P^{-1} \cdot e^{Dt} \cdot P \cdot X(0)} \tag{2}$$

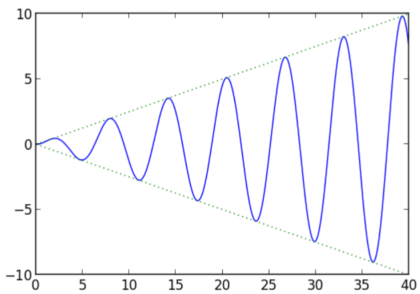
3. **Studying the Effect of Eigenvalues on Stability of System**
From equation (2), we can infer that X(t) will be some linear combination of $e^{\lambda t}$ terms.
Now,

$$\lambda \in \Im$$

$$\therefore \lambda = a + ib$$

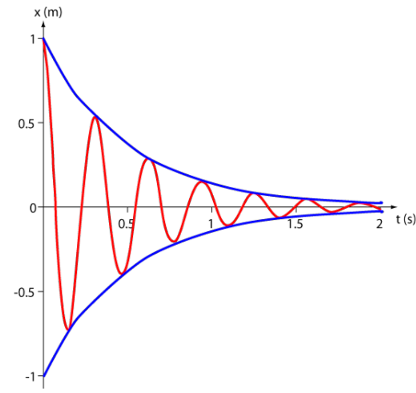
$$\therefore e^{\lambda t} = e^{at} \underbrace{(cos(bt) + i.sin(bt))}_{\text{always 1}}$$

if $a > 0$



System oscillations grow overtime so unstable.

if $a < 0$

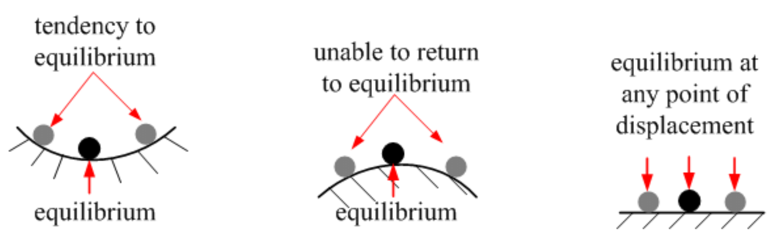


System oscillations increase overtime so reaches stability.

However, this modelling only works if the system is linear. But in most practical applications, the systems are non-linear. So, in order to linearize them we make use of Jacobians.

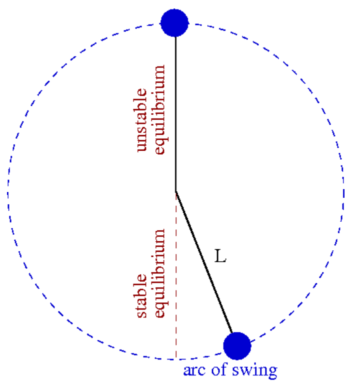
4. **Linearize around a fixed point.**

   (a) Find the stable equilibrium points for the system.



(a) stable equilibrium (b) unstable equilibrium (c) neutral equilibrium

For our inverted pendulum system, the equilibrium are at:
$\theta = \pi \ \ and \ \ \theta = 0$



(b) Linearize around these equilibrium points:

i.e Non-Linear system acts linear when we look rally close to equilibrium points.

$$\text{Let } \bar{X} => \text{Equilibrium point}$$

For Non-Linear System,

$$\dot{X} = f(x)$$

where f(x) is a non-linear combination of state variables.
Around Equilibrium point,

$$\therefore \dot{X} = f(\bar{X}) + \left.\frac{Df}{Dx}\right|_{\bar{X}} (X - \bar{X}) + \left.\frac{D^2 f}{DX^2}\right|_{\bar{X}} (X - \bar{X})^2 + \dots.$$

This is the Taylor-Series expansion around $\bar{X}$
where, $\left.\frac{Df}{DX}\right|_{\bar{X}}$ is the Jacobian of $\dot{X} = f(X) at \bar{X}$

$$\left.\frac{Df}{DX}\right|_{\bar{X}} = \begin{bmatrix} \frac{\delta f1}{\delta x1} & \frac{\delta f1}{\delta x2} & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\delta fn}{\delta x1} & . & . & \frac{\delta fn}{\delta xn} \end{bmatrix} at \bar{X}$$

Taking only the linear component of this expansion,

$$\dot{X} = \left.\frac{Df}{DX}\right|_{\bar{X}} (X - \bar{X})$$

$$\therefore \dot{X} = A \cdot X$$

where,

$$A = \begin{bmatrix} \frac{\delta f1}{\delta x1} & \frac{\delta f1}{\delta x2} & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\delta fn}{\delta x1} & . & . & \frac{\delta fn}{\delta xn} \end{bmatrix} at\bar{X}$$

Thus, we come to realize that for a Non-Linear system, the matrix which defines our state space model linearly is basically just the Jacobian of the system taken at the equilibrium point.

2 Controllability

Now that we have out System Modelled, we can work towards making a controller for the same.

$$\boxed{\dot{X} = A \cdot X + B \cdot u} \tag{3}$$

where,

$$X \in \Re^n;$$

$$A \in \Re^{nxn};$$

$$B \in \Re^{nxq};$$

$$u \in \Re^q;$$

Controllability Matrix C

$$C = \begin{bmatrix} B & A.B & A^2.B & . & . & . & A^{n-1}.B \end{bmatrix}$$

For system to be controllable,

$$\boxed{rank(C) = n}$$