The C*-envelope

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Chapter 1

Introduction

Consider the disk algebra, $A(\mathbb{D})$, defined to be the collection of all bounded analytic function on the open unit disk $\mathbb{D}$ which can be extended continuously to the boundary. This is a Banach algebra when equipped with the supremum norm. We can identify $A(\mathbb{D})$ as a subalgebra of $C(\overline{\mathbb{D}})$ isometrically by continuously extending each $f \in A(\mathbb{D})$ to $\overline{\mathbb{D}}$, where $C(\overline{\mathbb{D}})$ is a C*-algebra with the supremum norm. Moreover, by the maximum modulus principal (see [Lan93, page 91]), the restriction map $r : A(\mathbb{D}) \to C(\mathbb{T})$ given via

$$f \mapsto f\mid_{\mathbb{T}}$$

is isometric, where $C(\mathbb{T})$ also has the supremum norm. Hence, $\mathbb{T} \subseteq \overline{\mathbb{D}}$ is a closed subset where for every $f \in A(\mathbb{D})$, $|f|$ attains its maximum value. Now $A(\mathbb{D})$ also separates points of $\overline{\mathbb{D}}$ and so by the Stone-Weierstrass Theorem we get that the C*-algebra generated by $A(\mathbb{D})$ is $C(\overline{\mathbb{D}})$. We can ask if there is a smallest closed subset of $\overline{\mathbb{D}}$ where for every $f \in A(\mathbb{D})$, $|f|$ attains its maximum.

Definition 1.1. Let X be a compact Hausdorff space and $A \subseteq C(X)$ a closed subalgebra which separates points of X. A subset $Y \subseteq X$ is said to be a *Shilov boundary* of A if it is the smallest closed subset such that every $f \in A$ attains its maximum.

It is a result of Shilov (see [Pal94, Chapter 3]) that the Shilov boundary Y always exists for such $A \subseteq C(X)$ and is unique. Moreover for any isometric isomorphism $A \to C(Z)$ where A separates the points of Z we must have that Y is homeomorphic to the Shilov boundary of A inside $C(Z)$. In Corollary 3.29 we present a proof given by Hamana (see [Ham79b]) of the existence and uniqueness of the Shilov boundary using methods of injective envelopes.

For another example consider the collection of all bounded analytic functions on $\mathbb{D}$ with the supremum norm, denoted $\mathcal{A}$. By Fatou's theorem (see [Hof62, Chapter 3]) for every $f \in \mathcal{A}$ there exists $F \in L^\infty(\mathbb{T})$ such that

$$F(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

with $\|F\| = \|f\|$. So we may isometrically identify $\mathcal{A}$ with a closed subalgebra of $L^\infty(\mathbb{T})$, denoted $H^\infty(\mathbb{T})$. Now by Gelfand duality we get that $L^\infty(\mathbb{T}) \simeq C(X)$ via $\phi \mapsto \widehat{\phi}$ for some compact Hausdorff space X. We can identify X as the collection of all nonzero $\mathbb{C}$-valued homomorphisms of $L^\infty(\mathbb{T})$ with the weak*-topology inherited from $L^\infty(\mathbb{T})^*$. By Uryshon's lemma we get that a base for the topology on X is given by open sets of the form

$$U = \{x \in X : |f(x)| < \varepsilon\}$$

for $f \in C(X)$ and $\varepsilon > 0$. As simple functions are dense in $L^\infty(\mathbb{T})$, it is enough to consider open sets of the form

$$U = \{x \in X : |\widehat{\chi_E}(x)| < \varepsilon\}$$

where χ_E is the characteristic function for some measurable $E \subseteq \mathbb{T}$ and $\varepsilon > 0$. As χ_E is a projection in $L^\infty(\mathbb{T})$, we get that for any $0 < \varepsilon < 1$

$$U = \{x \in X : |\widehat{\chi_E}(x)| < \varepsilon\} = \{x \in X : \widehat{\chi_E}(x) = 0\}.$$

It follows that X has a basis of *clopen* (closed and open) sets. Given such an open set U define $f \in \mathcal{A}$ via

$$f(\lambda) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+\lambda}{e^{i\theta}-\lambda}(1-\chi_E(e^{i\theta}))d\theta\right\}$$

for every $\lambda \in \mathbb{D}$. Then the corresponding $\widehat{F} \in C(X)$ satisfies

$$|\widehat{F}(x)| = \begin{cases} e & x \in U, \\ 1 & x \in X \setminus U. \end{cases}$$

In particular the collection $\widehat{F}$ for $F \in H^\infty(\mathbb{T})$ separate the points of X. Moreover, the above argument shows that X itself is the smallest closed subset itself where all functions in $H^\infty(\mathbb{T})$ achieve their maximum. Hence, X is the Shilov boundary of $\mathcal{A}$. For more information on this example see [Hof62, Chapter 10] and [Sch61].

The classic Shilov boundary corresponds to subalgebras of unital commutative C*-algebras which generate the C*-algebra. A natural question to ask is if an analogous result holds for subalgebras of arbitrary C*-algebras which generate the C*-algebra. Our main goal is to present a proof of the existence of the non-commutative analogue of the Shilov boundary. We have the following definition made by Arveson in [Arv69].

Definition 1.2. Let $\mathcal{S} \subseteq \mathscr{A}$ be an operator system inside a C*-algebra $\mathscr{A}$ such that $C^*(\mathcal{S}) = \mathscr{A}$. A closed ideal $J \triangleleft \mathscr{A}$ is said to be a *boundary ideal* for $\mathcal{S}$ if the quotient map $q : \mathscr{A} \to \mathscr{A}/J$ is a complete isometry on $\mathcal{S}$. A boundary ideal is called the *Shilov boundary* for $\mathcal{S}$ if it contains every other boundary ideal.

In that case when $\mathscr{A} = C(X)$ we have by Gelfand duality that every ideal J inside $\mathscr{A}$ is exactly of the form $J = \{f \in C(X) : f\,|_K = 0\}$ for some closed set $K \subseteq X$. It follows that the quotient norm on $\mathscr{A}/J$ is given via

$$\|f\| = \sup_{x \in K} |f(x)|.$$

Therefore, if $\mathcal{S} \subseteq \mathscr{A}$ is an operator system such that $C^*(\mathcal{S}) = \mathscr{A}$, then $J \triangleleft \mathscr{A}$ is a Shilov boundary for $\mathcal{S}$ if and only if the associated closed subset $K \subseteq X$ is the Shilov boundary for $\mathcal{S}$.

When Arveson introduced the non-commutative Shilov boundary it was an open question whether such a boundary always existed for any operator system. It was first shown by Hamana in [Ham79b] the existence of such a non-commutative Shilov boundary for operator algebras. This was done using the theory of injective envelopes developed in [Ham79a]. This approach avoided the use of "boundary representations", a theory Arveson originally developed in [Arv69] and [Arv72] in hopes of proving the existence of the non-commutative Shilov boundary. Building on the works of [Arv69], [Arv72], [MS98], and [DM05] it was shown by Arveson in [Arv08] the existence of the non-commutative Shilov boundary for any separable operator system. The general case for arbitrary operator systems was then proved by Davidson and Kennedy in [DK15].

We present the proof by Hamana in Section 2 and the proof by Davidson and Kennedy in Section 3.

Chapter 2

Preliminaries

Throughout H and K will always denote Hilbert spaces unless otherwise stated. Given a subset $F \subseteq H$ let $[F]$ denote the closed linear span of F. Let $\mathcal{B}(H)$ denote the collection of all bounded linear operators on the Hilbert space H. Given a subset $A \subseteq \mathcal{B}(H)$ let $C^*(A)$ denote the C*-algebra generated by A.

Given a compact Hausdorff space X, let $C(X)$ denote the collection of continuous $\mathbb{C}$-valued functions on X. Then $C(X)$ is a commutative unital C*-algebra with the supremum norm defined via

$$\|f\| := \sup_{x \in X} |f(x)|$$

for all $f \in C(X)$. By Gelfand duality every unital commutative C*-algebra is of this form. Indeed, let $\mathscr{A}$ be a unital commutative C*-algebra and let $\Gamma(\mathscr{A})$ be the collection of all nonzero homomorphisms of $\mathscr{A}$ into $\mathbb{C}$. We see that $\Gamma(\mathscr{A})$ is a compact Hausdorff space with the weak*-topology induced from $\mathscr{A}^*$. Lastly, the map $\widehat{\cdot} : \mathscr{A} \to C(\Gamma(\mathscr{A}))$ via

$$\widehat{f}(\gamma) = \gamma(f)$$

for every $f \in \mathscr{A}$, $\gamma \in \Gamma(\mathscr{A})$ is a *-isomorphism.

The following results can be found in the excellent treatise [Pau02].

Definition 2.1. Let $\mathscr{A}$ be a unital C*-algebra. A subspace $V \subseteq \mathscr{A}$ is said to be an *operator space*. A unital self-adjoint subspace $\mathcal{S} \subseteq \mathscr{A}$ is said to be an *operator system*.

For example the collection of all n by n upper triangular matrices is an operator space. It is a subspace of the C*-algebra $M_n(\mathbb{C})$. As the upper triangular matrices are not self-adjoint, they are not an operator system.

Let $z : \mathbb{T} \to \mathbb{T}$ be the identity map. Notice that $\mathcal{S} := \mathrm{span}\{1, \mathrm{z}, \mathrm{z}^*\}$ is an operator system inside the C*-algebra, $C(\mathbb{T})$, consisting of all continuous functions defined on the unit circle with the supremum norm. Every operator system is also seen to be an operator space.

By the GNS construction we can always view operator spaces and operator systems as subspaces of $\mathcal{B}(H)$. Now given an operator space $V \subseteq \mathcal{B}(H)$ and $n \in \mathbb{N}$ define $M_n(V)$ to be the collection of all n by n matrices with entries from V. This is said to be the n-matrix amplification of V and is easily identified as a subspace of $\mathcal{B}(H^n)$ where $H^n := H \oplus \cdots \oplus H$, n-times. In particular, if $\mathcal{S} \subseteq \mathcal{B}(H)$ is an operator system, then $M_n(\mathcal{S})$ is also an operator system inside $\mathcal{B}(H^n)$. For example the n-matrix amplification of the complex numbers is the usual n by n matrices.

Let $\phi : V \to \mathscr{A}$ be a linear map from an operator space to a C*-algebra and $n \in \mathbb{N}$. We can define the linear map $\phi_n : M_n(V) \to M_n(\mathscr{A})$ given via

$$\phi_n((a_{ij})_{i,j=1}^n) := (\phi(a_{ij}))_{i,j=1}^n.$$

The map ϕ is said to be *completely bounded* if and only if

$$\sup_{n\in\mathbb{N}} \|\phi_n\| < \infty.$$

Given a completely bounded map ϕ define

$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\|.$$

The map ϕ is said to be *completely contractive* if each ϕ_n is a contractive map or equivalently $\|\phi\|_{cb} \leq 1$. Moreover, ϕ is said to be *complete isometric* if each ϕ_n is an isometric map.

We claim that any operator system $\mathcal{S}$ has plenty of positive elements, i.e., $\mathcal{S}$ is the span of its positive elements. Indeed, given any self-adjoint $a \in \mathcal{S}$ it can be shown that a is the difference of the following positive elements

$$a = \frac{1}{2}(\|a\|1 + a) - \frac{1}{2}(\|a\|1 - a).$$

As the self-adjoint elements of $\mathcal{S}$ span $\mathcal{S}$ we obtain our desired result. So we can consider positive maps defined on operator systems. A linear map $\phi : \mathcal{S} \to \mathscr{A}$ from an operator system to a C*-algebra is said to be *completely positive* if for every $n \in \mathbb{N}$, each ϕ_n is a positive map. In particular for all $n \in \mathbb{N}$

$$(a_{ij}) \in M_n(\mathcal{S})^+ \implies (\phi(a_{ij})_{ij}) \geq 0$$

where $M_n(\mathcal{S})^+$ denotes the set of all positive elements in $M_n(\mathcal{S})$. Furthermore it is a *unital completely positive* (UCP) map if ϕ is completely positive and unital.

For example given an operator system $\mathcal{S}$ and any *-homomorphism $\pi : \mathcal{S} \to \mathcal{B}(H)$ we get that $\pi_n : M_n(\mathcal{S}) \to \mathcal{B}(H^n)$ is also a *-homomorphism for every $n \in \mathbb{N}$. As each *-homomorphism is a positive map we obtain that π is a completely positive map. Furthermore, if π is a unital *-homomorphism, then π is seen to be a UCP map. Moreover, we claim that if $\phi : \mathcal{S} \to \mathcal{B}(K)$ a completely positive map and $V : H \to K$ is a bounded operator, then $V^*\pi(\cdot)V : \mathcal{S} \to \mathcal{B}(H)$ is also a completely positive map. Indeed for $n \in \mathbb{N}$, $(a_{ij}) \in M_n(\mathcal{S})^+$, $\xi_1, \ldots, \xi_n \in H$ we get

$$\begin{aligned}\left\langle (V^*\phi(a_{ij})V) \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\rangle &= \sum_{i,j} \langle \phi(a_{ij})V\xi_j, V\xi_i \rangle \\ &= \left\langle (\phi(a_{ij})) \begin{bmatrix} V\xi_1 \\ \vdots \\ V\xi_n \end{bmatrix}, \begin{bmatrix} V\xi_1 \\ \vdots \\ V\xi_n \end{bmatrix} \right\rangle \geq 0.\end{aligned}$$

In particular, adjoints of *-representations are also completely positive maps. In turns out by a result of Stinespring that every completely positive map is of this form.

Theorem 2.2 (Stinespring's dilation, [Pau02, page 43])**.** *Let $\mathscr{A}$ be a C^*-algebra and ϕ a completely positive map from $\mathscr{A}$ to $\mathcal{B}(H)$. Then there exists a Hilbert space K containing H, a *-representation $\pi : \mathscr{A} \to \mathcal{B}(K)$, and a linear map $V : H \to K$ such that*

$$\phi(s) = V^*\pi(s)V$$

for every $s \in \mathscr{A}$. Moreover, $\|\phi\|_{cb} = \|V\|^2$.

Definition 2.3. Let V be an operator space and $\phi : V \to \mathcal{B}(H)$ a linear map. We say that the linear map $\psi : V \to \mathcal{B}(K)$ is a *dilation* of ϕ if $H \subseteq K$ and

$$\phi(\cdot) = P_H\psi(\cdot)\mid_H$$

where P_H denotes the projection onto H.

If in the above theorem ϕ is also unital, then V can be assumed to be an isometry. Hence, we can identify $H \subseteq K$ and $\phi(\cdot) = P_H \pi(\cdot) \mid_H$ where P_H denotes the projection onto H. In particular, π is a dilation of ϕ. Moreover, we can always restrict π to the closed linear span of $\pi(\mathscr{A})VH$ to obtain the *minimal Stinespring dilation.*

Definition 2.4. Let $\mathscr{A}$ be a C*-algebra and $\phi : \mathscr{A} \to \mathcal{B}(H)$ a completely positive map. We say that (π, V, K) is a *minimal Stinespring dilation* of ϕ if $\pi : \mathscr{A} \to \mathcal{B}(K)$ is a *-representation, $V : H \to K$ a linear map, $K = [\pi(A)VH]$, and

$$\phi(\cdot) = V^* \pi(\cdot) V.$$

Lemma 2.5 ([Pau02, page 46])**.** *Let $\mathscr{A}$ be a C*-algebra, $\phi : \mathscr{A} \to \mathcal{B}(H)$ a completely positive map, and (π_i, V_i, K_i) minimal Stinespring dilations for $i = 1, 2$. Then there exists unitary operator $U : K_1 \to K_2$ such that $UV_1 = V_2$ and $U\pi_1(\cdot)U^* = \pi_2(\cdot)$.*

In particular, the minimal Stinespring dilation for a completely positive map is unique up to unitary equivalence.

We shall require a number of results concerning completely positive and completely contractive maps. We state them without proof.

Lemma 2.6 ([Arv08, page 29])**.** *Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a completely positive map. Then $\|\phi\|_{cb} = \|\phi\| = \|\phi(1)\|$.*

From the above lemma we see that every UCP map will also be completely contractive. The converse also holds if we assume the completely contractive map is unital.

Theorem 2.7 ([Pau02, page 28])**.** *If $\mathcal{S}$ is an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a unital completely contractive map, then ϕ is a UCP map.*

Lemma 2.8 ([Pau02, page 33])**.** *Let V be an operator space and X a compact Hausdorff space. If $\phi : V \to C(X)$ is a bounded linear map, then ϕ is completely bounded with $\|\phi\|_{cb} = \|\phi\|$.*

Lemma 2.9 (Cauchy-Schwarz for UCP maps, [Pau02, page 27])**.** *Let $\mathcal{S}$ be an operator system. If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a unital completely positive map, then $\phi(x)^*\phi(x) \leq \phi(x^*x)$ for every $x \in \mathcal{S}$.*

Theorem 2.10 (Multiplicative domain, [Pau02, page 38]). *If $\mathscr{A}$ is a unital C^*-algebra and $\phi : \mathscr{A} \to \mathcal{B}(H)$ is a UCP map, then*

$$\begin{aligned}&\{x \in \mathscr{A} : \phi(x^*x) = \phi(x)^*\phi(x) \text{ and } \phi(xx^*) = \phi(x)\phi(x)^*\}\\ &= \{x \in \mathscr{A} : \phi(xa) = \phi(x)\phi(a) \text{ and } \phi(ax) = \phi(a)\phi(x) \text{ for every } a \in \mathscr{A}\}\end{aligned}$$

is a C^*-subalgebra of $\mathscr{A}$ and ϕ is a $*$-homomorphism when restricted to it. The above C^*-subalgebra is said to be the multiplicative domain of ϕ.

Given Banach spaces X, Y we can define the *projective tensor norm* on $X \otimes Y$ as follows. For any $u \in X \otimes Y$ let

$$\|u\|_\gamma := \inf\left\{\sum_{k=1}^n \|x_k\|\|y_k\| : u = \sum_{k=1}^n x_k \otimes y_k\right\}.$$

Let $X \otimes^\gamma Y$ denote the complete of $X \otimes Y$ with respect to the projective tensor norm.

Given any operator space V we can consider the space, $\mathcal{B}(V, \mathcal{B}(H))$, of all bounded operators from V to $\mathcal{B}(H)$. We can endow this with the *bounded weak (BW) topology*. This is a weak*-topology given by the identification

$$\mathcal{B}(V, \mathcal{B}(H)) \simeq (V \otimes^\gamma \mathcal{T}(H))^*,$$

where $\mathcal{T}(H)$ is the collection of trace class operators. A bounded net $(\phi_\lambda)_\lambda$ in $\mathcal{B}(V, \mathcal{B}(H))$ converges in the BW-topology if and only if for every $x, y \in H$ and $s \in V$

$$\langle \phi_\lambda(s)x, y\rangle \to \langle \phi(s)x, y\rangle.$$

As the BW-topology is a weak*-topology we obtain the closed unit ball of $\mathcal{B}(V, \mathcal{B}(H))$, $\overline{b_1}(\mathcal{B}(V, \mathcal{B}(H)))$, is compact in the BW-topology. An important observation is that the collection of all completely contractive maps, $\mathrm{CC}(\mathrm{V}, \mathcal{B}(\mathrm{H}))$, from an operator space V is a closed subset of $\overline{b_1}(\mathcal{B}(V, \mathcal{B}(H)))$ in the BW-topology. Indeed, this follows from the convergence characterization given for bounded nets in the BW-topology given above. Hence, it is compact in the BW-topology as it is a closed subset of a compact set. Similarly, the collection of UCP maps, $\mathrm{UCP}(\mathcal{S}, \mathcal{B}(H))$, from an operator system $\mathcal{S}$ to $\mathcal{B}(H)$ is a compact subset of $\overline{b_1}(\mathcal{B}(\mathcal{S}, \mathcal{B}(H)))$ in the BW-topology.

We have the following fundamental extension theorems.

Theorem 2.11 (Arveson's extension theorem, [Pau02, page 86]). *Let $\mathcal{S} \subseteq \mathscr{A}$ be an operator system contained in the C^*-algebra $\mathscr{A}$. If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a* *completely positive map, then there exists completely positive map* $\psi : \mathscr{A} \to \mathcal{B}(H)$ *extending* ϕ.

Theorem 2.12 (Wittstock's extension theorem, [Pau02, page 99])**.** *Let* $V \subseteq \mathscr{A}$ *be an operator space contained in the* C^**-algebra* $\mathscr{A}$. *If* $\phi : V \to \mathcal{B}(H)$ *is a completely contractive map, then there exists completely contractive map* $\psi : \mathscr{A} \to \mathcal{B}(H)$ *that extends* ϕ.

Chapter 3

Injectivity

We first present the theory of injective envelopes developed by Hamana (see [Ham79a] and [Ham79b]) towards showing the existence of the non-commutative Shilov boundary for operator algebras. We mostly follow Chapter 15 from [Pau02].

Definition 3.1. We say an operator space V is *injective* if, given any other operator space $W \subseteq \mathcal{B}(H)$ and any completely contractive map $\phi : W \to V$, there exists a completely contractive map $\psi : \mathcal{B}(H) \to V$ that extends ϕ.

In particular, we get the following commutative diagram:

$$\begin{tikzcd}
\mathcal{B}(H) \arrow[dr, dashed, "\exists\psi"] & \\
W \arrow[u, hook] \arrow[r, "\phi"'] & V.
\end{tikzcd}$$

Notice Wittstock's extension theorem, Theorem 2.12, implies that $\mathcal{B}(H)$ is always injective. Using this we obtain the following characterization of injectivity.

Definition 3.2. We say the linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a *completely contractive projection* if ϕ is a completely contractive map such that $\phi \circ \phi = \phi$. We similarly define a *completely positive projection*.

Proposition 3.3. *Let $V \subseteq \mathcal{B}(H)$ be an operator space. The space V is injective if and only if there exists a completely contractive projection $\phi : \mathcal{B}(H) \to V$ onto V.*

Proof. Suppose V is injective. Notice, the identity map $\mathrm{id} : \mathrm{V} \to \mathrm{V}$ is completely contractive. Hence, we can extend it to a completely contractive map

$\phi : \mathcal{B}(V) \to V$. We see that ϕ is also idempotent since for any $v \in V$, $\phi(v) = \mathrm{id(v)} = \mathrm{v}$. Hence, ϕ is a completely contractive projection onto V.

Conversely suppose $\phi : \mathcal{B}(H) \to V$ is a completely contractive projection. Given an operator space $W \subseteq \mathcal{B}(K)$ and a completely contractive map $\gamma : W \to V$, consider the completely contractive map $\iota \circ \gamma : W \to \mathcal{B}(H)$, where $\iota : V \to \mathcal{B}(H)$ is the inclusion map. Since $\mathcal{B}(H)$ is injective, we obtain an extension of $\iota \circ \gamma$ to a completely contractive map $\psi : \mathcal{B}(K) \to \mathcal{B}(H)$. Notice, $\phi \circ \psi : \mathcal{B}(K) \to V$ gives the required extension of γ, showing that V is injective. $\square$

Definition 3.4. Let $\mathcal{S}$ be an operator system. We say that $\mathcal{S}$ is *injective with respect to UCP maps* if for every operator system $W \subseteq \mathcal{B}(H)$ and UCP map $\phi : W \to \mathcal{S}$, there exists a UCP map $\psi : \mathcal{B}(H) \to \mathcal{S}$ that extends ϕ.

More generally injectivity can be defined in any category as follows. Let $\mathcal{C}$ be a category and $\mathrm{Mor}(\mathcal{C})$ denote the collection of morphisms in $\mathcal{C}$. Fix a subset $\mathrm{Enb}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C})$. A morphism $\phi \in \mathrm{End}(\mathcal{C})$ is said to be an embedding. We say an object I in $\mathcal{C}$ is *injective* if, for all objects A, B in $\mathcal{C}$, morphisms $\phi \in \mathrm{Mor}(\mathcal{C})$ with $\phi : A \to I$, embeddings $\iota \in \mathrm{End}(\mathcal{C})$ with $\iota : A \to B$, there exists a morphism $\psi \in \mathrm{Mor}(\mathcal{C})$ such that $\psi : B \to I$ and $\psi \circ \iota = \phi$.

Hence, Definition 3.4 refers to injectivity in the category where objects are operator systems, morphisms are UCP maps, and the embeddings are the inclusions of operator systems.

By Arveson's extension theorem, Theorem 2.11, we obtain that $\mathcal{B}(H)$ is also injective with respect to UCP maps. By following the exact same proof as in Proposition 3.3 we obtain the following.

Lemma 3.5. *Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an operator system. Then $\mathcal{S}$ is injective with respect to UCP maps if and only if there exists a UCP projection $\phi : \mathcal{B}(H) \to \mathcal{S}$.*

Now any completely contractive projection $\phi : \mathcal{B}(H) \to \mathcal{S}$ onto $\mathcal{S}$ will be unital and hence completely positive by Theorem 2.7. Conversely any UCP projection $\phi : \mathcal{B}(H) \to \mathcal{S}$ is also a completely contractive projection. Hence, using the above lemma we immediately obtain the following.

Corollary 3.6. *Let $\mathcal{S}$ be an operator space. The following are equivalent.*

1. *$\mathcal{S}$ is injective.*
2. *$\mathcal{S}$ is injective with respect to UCP maps.*

Using the above characterization, for any injective operator system we can define a multiplication that induces a C*-algebra structure. This is know as the *Choi-Effros product.*

Theorem 3.7 (Choi-Effros, [CE77])**.** *Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an injective operator system and $\phi : \mathcal{B}(H) \to \mathcal{S}$ a unital completely positive projection onto $\mathcal{S}$. Given any $x, y \in \mathcal{S}$, defining $x \circ y := \phi(xy)$ equips $\mathcal{S}$ with a multiplication that induces a C*-algebra structure.*

Proof. Notice that $\mathcal{S}$ is norm-closed, since it is the range of a completely positive projection $\phi : \mathcal{B}(H) \to \mathcal{S}$. We first show the multiplication $x \circ y := \phi(xy), x, y \in \mathcal{S}$ is associative. It is enough to show that for every $x \in \mathcal{S}$, $y \in \mathcal{B}(H)$ we get $\phi(x\phi(y)) = \phi(xy)$ and $\phi(\phi(y)x) = \phi(xy)$. From this it follows that

$$a \circ (b \circ c) = \phi(a\phi(bc)) = \phi(abc) = \phi(\phi(ab)c) = (a \circ b) \circ c$$

for every $a, b, c \in \mathcal{S}$. Towards this, let $x \in \mathcal{S}$, $y \in \mathcal{B}(H)$ and notice that by the Cauchy-Schwarz Inequality (see lemma 2.9) we get that

$$\phi_2\left(\begin{bmatrix} x & 0 \\ y^* & 0 \end{bmatrix}\right)\phi_2\left(\begin{bmatrix} x^* & y \\ 0 & 0 \end{bmatrix}\right) \leq \phi_2\left(\begin{bmatrix} x & 0 \\ y^* & 0 \end{bmatrix}\begin{bmatrix} x^* & y \\ 0 & 0 \end{bmatrix}\right).$$

Expanding and taking the difference we obtain

$$0 \leq \begin{bmatrix} \phi(xx^*) - xx^* & \phi(xy) - x\phi(y) \\ \phi(y^*x^*) - \phi(y^*)x^* & \phi(y^*)\phi(y) - \phi(y^*y) \end{bmatrix}.$$

Applying ϕ_2 once again yields

$$0 \leq \begin{bmatrix} 0 & \phi(xy) - \phi(x\phi(y)) \\ \phi(y^*x^*) - \phi(\phi(y^*)x^*) & \phi(\phi(y^*)\phi(y)) - \phi(y^*y) \end{bmatrix}.$$

It is not hard to check that any positive two by two operator matrix with a zero entry in its diagonal must also have its off-diagonal entries to be zero. Indeed if

$$\begin{bmatrix} 0 & a \\ a^* & x \end{bmatrix} \in \mathcal{B}(H^2)$$

is a positive matrix, then for every $\xi, \eta \in H$ we must have

$$0 \leq \left\langle \begin{bmatrix} 0 & a \\ a^* & x \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}, \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right\rangle = 2\mathrm{Re}\langle a\eta, \xi\rangle + \langle x\eta, \eta\rangle.$$

It follows that we must have $a = 0$. Hence, $\phi(xy) = \phi(x\phi(y))$ by looking at the (1,2)-entry. Now looking at the (2,1)-entry and taking adjoints we obtain

$\phi(yx) = \phi(\phi(y)x)$. So associativity of the product follows. Lastly, we have to show the C*-identity holds. Let $x \in \mathcal{S}$ and notice that

$$\|x \circ x^*\| = \|\phi(xx^*)\| \leq \|xx^*\| = \|x\|^2.$$

Moreover, by the Cauchy-Schwarz Inequality we get

$$\|x \circ x^*\| = \|\phi(xx^*)\| \geq \|\phi(x)\phi(x)^*\| = \|xx^*\| = \|x\|^2$$

for every $x \in \mathcal{S}$. So $\|x \circ x^*\| = \|x\|^2$ for every $x \in \mathcal{S}$ and our desired result follows. □

As an example consider the row Hilbert space defined as $H_r := \mathcal{B}(H, \mathbb{C})$. We can view $H_r \subseteq \mathcal{B}(H \oplus \mathbb{C})$ as an operator space. We claim it is an injective operator space. Indeed, consider the map $V : \mathcal{B}(H \oplus \mathbb{C}) \to H_r$ defined via

$$Vx = P_{\mathbb{C}} x P_H$$

for every $x \in \mathcal{B}(H \oplus \mathbb{C})$ where $P_{\mathbb{C}}, P_H$ denote the projections onto $\mathbb{C}$ and H respectively. This can be easily checked to be a completely contractive projection onto H_r. Therefore, H_r is an injective operator space. The above argument shows in general, operator spaces of the form $\mathcal{B}(H_1, H_2) \subseteq \mathcal{B}(H)$ for closed subspaces $H_1, H_2 \subseteq H$ are injective.

Definition 3.8. Let V be an operator space. We say that (E, κ) is an *injective envelope* of V if E is an injective operator space, $r : V \to E$ is a complete isometry, and for any other injective operator space W with $r(V) \subseteq W \subseteq E$, we have $E = W$.

The existence of such injective envelopes was shown by Hamana in [Ham79a] and [Ham79b]. One of the main ideas in Hamana's proof is the existence of certain minimal completely contractive projections.

Let $V \subseteq \mathcal{B}(H)$ be an operator space. Define a *V-map* to be a completely contractive map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ such that $\phi(x) = x$ for every $x \in V$. To each V-map we can associate a seminorm on $\mathcal{B}(H)$ given via $\rho_\phi(x) := \|\phi(x)\|$. A seminorm induced by a V-map is said to be a *V-seminorm*. Define a partial order on the collection of V-seminorms by $\rho_\phi \leq \rho_\psi$ if and only if $\rho_\phi(x) \leq \rho_\psi(x)$ for every $x \in \mathcal{B}(H)$. The following lemma establishes the existence of minimal V-seminorms.

Lemma 3.9. *Let $V \subseteq \mathcal{B}(H)$ be an operator space. Then there exists a minimal V-seminorm on $\mathcal{B}(H)$.*

Proof. Let $(\rho_{\phi_\lambda})_\lambda$ be a decreasing net of V-seminorms. Consider the net of associated V-maps $(\phi_\lambda)_\lambda$. As the collection of completely contractive maps, $\mathrm{CC}(\mathcal{B}(\mathrm{H}), \mathcal{B}(\mathrm{H}))$, is compact in the BW-topology, there exists a subnet $(\phi_{\lambda_i})_{\lambda_i}$ which converges to a completely contractive map ϕ in the BW-topology. It follows that ϕ is also a V-map since for every $x \in V$, $\xi, \eta \in H$

$$\langle x\xi, \eta\rangle = \langle \phi_\lambda(x)\xi, \eta\rangle \to \langle \phi(x)\xi, \eta\rangle.$$

Hence $\phi(x) = x$ for every $x \in V$. We claim that ρ_ϕ is a lower bound for the net. Indeed for every $x \in V, \xi, \eta \in H$

$$|\langle \phi(x)\xi, \eta\rangle| = \lim_i |\langle \phi_{\lambda_i}(x)\xi, \eta\rangle| \leq \liminf_i \|\phi_{\lambda_i}(x)\| \|\xi\| \|\eta\|.$$

Hence, $\|\phi(x)\| \leq \liminf_i \|\phi(x)\|$ for every $x \in V$. As the net is decreasing, we obtain that ρ_ϕ is a lower bound for our net. Therefore, every decreasing chain has a minimal element and so by Zorn's lemma there exists a minimal V-seminorm. $\square$

Using the existence of minimal V-seminorms we can now proof the existence of an injective envelope.

Theorem 3.10 (Hamana, [Ham79a])**.** *Let $V \subseteq \mathcal{B}(H)$ be an operator system. If ϕ is a V-map corresponding to a minimal V-seminorm, then $E := \phi(\mathcal{B}(H))$ is an injective envelope of V.*

Proof. We first show that ϕ is a projection using an averaging trick. As ϕ is completely contractive, $\|\phi(\phi(x))\| \leq \|\phi(x)\|$ for every $x \in \mathcal{B}(H)$. It is clear that if ϕ is a V-map, then so is $\phi^2 := \phi \circ \phi$. Hence, by minimality of ρ_ϕ, we get that $\|\phi^2(x)\| = \|\phi(x)\|$ for every $x \in \mathcal{B}(H)$. For $k \in \mathbb{N}$, define $\phi^{k+1} = \phi^k \circ \phi$. Similarly, by minimality, we obtain $\|\phi^k(x)\| = \|\phi(x)\|$ for all $k \in \mathbb{N}$ and $x \in \mathcal{B}(H)$. Furthermore for every $k \in \mathbb{N}$, define

$$\psi_k := \frac{1}{k}(\phi + \cdots + \phi^k).$$

Once again we get for every $x \in V$, $k \in \mathbb{N}$

$$\|\psi_k(x)\| \leq \frac{1}{k}(\|\phi(x)\| + \cdots + \left\|\phi^k(x)\right\|) = \frac{k}{k}\|\phi(x)\| = \|\phi(x)\|.$$

It follows that, $\|\psi_k(x)\| = \|\phi(x)\|$ for every $k \in \mathbb{N}$ and $x \in \mathcal{B}(H)$ because ρ_ϕ is a minimal V-seminorm. Hence, for every $x \in \mathcal{B}(H)$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\|\phi(x) - \phi^2(x)\| &= \|\phi(x - \phi(x))\| = \|\psi_n(x - \phi(x))\| \\
&= \left\| \frac{\phi(x) + \cdots + \phi^n(x)}{n} - \frac{\phi^2(x) + \cdots + \phi^{n+1}(x)}{n} \right\| \\
&= \left\| \frac{\phi(x) - \phi^{n+1}(x)}{n} \right\| \leq 2\frac{\|\phi(x)\|}{n}.
\end{aligned}$$

Since this holds for every $n \in \mathbb{N}$, we get that $\phi = \phi^2$. Therefore, ϕ is a completely contractive projection whose range contains V. It follows that $E = \phi(\mathcal{B}(H))$ is an injective operator space. Next, suppose there exists an injective operator system R such that $V \subseteq R \subseteq E$. Then there exists a completely contractive projection $\gamma : \mathcal{B}(H) \to R$ onto R. Now, by minimality, we must have $\|\gamma(\phi(x))\| = \|\phi(x)\|$ for every $x \in \mathcal{B}(H)$. In particular, γ is an isometry on E. Since

$$\gamma(\phi(x) - \gamma(\phi(x))) = 0$$

we obtain that $\phi(x) = \gamma(\phi(x))$ for every $x \in \mathcal{B}(H)$. Therefore, $R = E$ and it follows that E is an injective envelope of V. $\square$

Let V, ϕ, E be as above. Suppose we are given a completely contractive map $\gamma : E \to E$ that fixes V. By minimality of the seminorm ρ_ϕ, we obtain that

$$\|\gamma(\phi(x))\| = \|\phi(x)\|$$

for every $x \in \mathcal{B}(H)$. Hence, γ is an isometry on E. It is clear that $\gamma \circ \phi$ is also associated to a minimal V-seminorm. So by the above proof it is a projection. Since, $\phi \circ (\gamma \circ \phi) = \gamma \circ \phi$ we get

$$\gamma \circ \phi = \gamma \circ \phi \circ (\gamma \circ \phi) = \gamma \circ \gamma \circ \phi.$$

It follows that $\gamma \circ (\phi - \gamma \circ \phi) = 0$ and so $\phi = \gamma \circ \phi$ as γ is an isometry. In particular, if a completely contractive map from E to itself fixes V it must be the identity. We can use this to prove uniqueness of the injective envelope.

Theorem 3.11. *Let (E_1, κ_1) and (E_2, κ_2) be two injective envelopes of the operator space V. The map $\gamma : \kappa_1(V) \to \kappa_2(V)$ given via, $\gamma(\kappa_1(x)) := \kappa_2(x)$ extends uniquely to a completely isometric isomorphism of E_1 onto E_2.*

Proof. Assume $V \subseteq \mathcal{B}(H)$ and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be the completely contractive projection corresponding to a minimal V-seminorm as in the proof of Theorem 3.10. Let $E = \phi(\mathcal{B}(H))$ also be an injective envelope. It is enough to show that for $i = 1, 2$ the map $\kappa_i : V \to E_i$ extends uniquely to a completely isometric isomorphism $\gamma_i : E \to E_i$. Then $\gamma_2 \circ \gamma_1^{-1} : E_1 \to E_2$ will given the required complete isometric isomorphism.

By injectivity κ_1 extends to a completely contractive map $\gamma_1 : E \to E_1$. Moreover, by injectivity of E we can extend the map

$$\begin{aligned} \Phi : \kappa_1(V) &\to E_1 \\ \kappa_1(x) &\mapsto x \end{aligned}$$

to a complete contraction $\beta : E_1 \to E$. It is clear that $\beta \circ \gamma_1(x) = x$ for all $x \in V$. Hence, by the discussion preceding this theorem we obtain that $\beta \circ \gamma_1$ is the identity on E. It follows that γ_1 must be a complete isometry. As $\gamma_1(E)$ is an injective operator space such that $\kappa_1(V) \subseteq \gamma_1(E) \subseteq E_1$, we must have $\gamma_1(E) = E_1$. Therefore, $\gamma_1 : E \to E_1$ is the required complete isometric isomorphism. Uniqueness once again follows from the preceding discussion. A similar argument shows the same for κ_2. □

Now given an operator space V, let $I(V)$ denote the unique injective envelope of V as given in Theorem 3.10.

An important concept in proving uniqueness was the rigidity of the injective envelope. In particular, the only possible extension of the identity map on a operator space to its injective envelope is the identity. We formulate this more precisely and prove another characterization of the injective envelope using rigidity.

Definition 3.12. Let V, E be operator spaces. We say $r : V \to E$ is a *rigid extension* if r is a complete isometry and for every complete contraction $\phi : E \to E$ such that $\phi \circ r = r$ we get $\phi = \mathrm{id_E}$.

Lemma 3.13. *Let V, E be operator spaces and $r : V \to E$ a complete isometry. The following are equivalent.*

1. *(E, r) is the injective envelope of V.*
2. *E is an injective operator space and $r : V \to E$ is a rigid extension.*

Proof. We have already seen that the injective envelope is a rigid extension. Conversely let E be an injective operator space and $r : V \to E$ a rigid extension. Suppose there exists an injective operator space W such that $r(V) \subseteq W \subseteq E$. By injectivity we can extend the inclusion $r(V) \subseteq W$ to a completely contractive map $\gamma : E \to W$. As, $\gamma \circ r = r$ it follows by rigidity that $\gamma = \mathrm{id_E}$. Therefore, $W = E$ and so E is the injective envelope of V. □

We also present another proof of the existence of the injective envelope in the appendix. It is due to Sinclair (see [Sin15]) and uses compact semigroup techniques.

We digress into studying injectivity for unital commutative C*-algebras. Our main goal will be to present a result of Gleason, Hadwin, and Paulsen (see [Gle58], [HP07]) which shows that injectivity for a unital commutative C*-algebra is completely characterized by an intrinsic topological property.

In the case when $\mathscr{A}$ is a unital commutative C*-algebra, we know by Gelfand duality that $\mathscr{A} \simeq C(\Gamma(\mathscr{A}))$ where $\Gamma(\mathscr{A})$ is a compact Hausdorff space defined to be the collection of all nonzero homomorphisms of $\mathscr{A}$ into $\mathbb{C}$. This *-isomorphism is given by $\widehat{\cdot} : \mathscr{A} \to C(\Gamma(\mathscr{A}))$ such that $\widehat{a}$ is point evaluation at a. Given another unital commutative C*-algebra $\mathscr{B}$ and a *-homomorphism $\pi : \mathscr{A} \to \mathscr{B}$, there exists a continuous map $\phi_\pi : \Gamma(\mathcal{B}) \to \Gamma(\mathscr{A})$ defined by

$$\phi_\pi(\gamma)(a) = \gamma(\pi(a))$$

for every $\gamma \in \Gamma(\mathscr{B})$ and $a \in \mathscr{A}$. If π is injective, then ϕ_π will be surjective. Also if π is surjective, then ϕ_π will be injective. Conversely, given a continuous map $\phi : \Gamma(\mathscr{B}) \to \Gamma(\mathscr{A})$, there exists a *-homomorphism $\pi_\phi : \mathscr{A} \to \mathscr{B}$ defined by

$$\pi_\phi(\widehat{a})(\gamma) = \widehat{a}(\phi(\gamma))$$

for every $a \in \mathscr{A}$, $\gamma \in \Gamma(\mathscr{B})$ where we are identifying $\mathscr{A}$ and $\mathscr{B}$ with $C(\Gamma(\mathscr{A}))$ and $C(\Gamma(\mathscr{B}))$ respectively. It can also be shown that the correspondence given above of *-homomorphisms from $\mathscr{A}$ to $\mathscr{B}$ and continuous maps from $\Gamma(\mathscr{B})$ to $\Gamma(\mathscr{A})$ is a bijective correspondence.

Definition 3.14. Let X be a compact Hausdorff space. We say X is *extremally disconnected* if for every open $U \subseteq X$, the closure of U, denoted $\overline{U}$, is clopen.

Definition 3.15. A compact Hausdorff space X is said to be *projective* if the following holds. Let Y, Z be compact Hausdorff spaces. If there exists continuous maps $\phi : X \to Y$ and $\pi : Z \to Y$ where π is surjective, then there exists a continuous map $\psi : X \to Z$ with $\pi \circ \psi = \phi$.

For example, the Stone-Čech compactification of $\mathbb{N}$, $\beta\mathbb{N}$, is projective. Indeed, consider a continuous map $\phi : \beta\mathbb{N} \to Y$ and a surjective continuous map $\pi : Z \to Y$. Then $\phi|_{\mathbb{N}} : \mathbb{N} \to Y$ can extended to a map $\widehat{\psi} : \mathbb{N} \to Z$ by defining $\widehat{\psi}(x)$ be be some element in $\pi^{-1}(\phi(x))$. By the universal property of the Stone-Čech compactification, $\widehat{\psi}$ extends to a continuous map $\psi : \beta\mathbb{N} \to Z$. By continuity, we obtain $\pi \circ \psi = \phi$ showing that $\beta\mathbb{N}$ is projective. The same argument applies to show that βX is projective for any discrete space X.

We now present a classic result of Gleason (see [Gle58]) showing that a compact Hausdorff space X is projective if and only if X is extremally disconnected. We first present the easier forward direction.

Theorem 3.16. *Let X be a compact Hausdorff space. If X is projective, then X is extremally disconnected.*

Proof. Let $U \subseteq X$ be an open subset. Consider the space

$$A := ((X \setminus U) \times \{1\}) \cup (\overline{U} \times \{2\}).$$

Observe that A is a closed subset of the compact Hausdorff space $X \times \{1,2\}$. Notice that $\overline{U} \times \{2\}$ is clopen in A as it is closed and

$$\overline{U} \times \{2\} = A \cap ((X \times \{1,2\}) \setminus (X \times \{1\})).$$

Let $\pi_1 : A \to X$ be the projection onto the first component. By construction of A, we get that π is a surjective map. As X is projective, there exists a continuous map $\phi : X \to A$ such that $\pi_1 \circ \phi = \mathrm{id}_X$. Since, $\pi_1^{-1}(U) \subseteq \overline{U} \times \{2\}$ we obtain that

$$U = \mathrm{id}_X^{-1}(\mathrm{U}) = \phi^{-1}(\pi_1^{-1}(\mathrm{U})) \subseteq \phi^{-1}(\overline{\mathrm{U}} \times \{2\}).$$

Taking closures we get $\overline{U} \subseteq \phi^{-1}(\overline{U} \times \{2\})$. If $x \in \phi^{-1}(\overline{U} \times \{2\})$, then $\phi(x) \in \overline{U} \times \{2\}$ and so

$$x = \mathrm{id}_X(\mathrm{x}) = \pi_1(\phi(\mathrm{x})) \in \overline{\mathrm{U}}.$$

Therefore, $\overline{U} = \phi^{-1}(\overline{U} \times \{2\})$ and it follows that $\overline{U}$ is clopen. □

From this we see that $\beta\mathbb{N}$ is an extremally disconnected space since we had already shown it was projective. The following technical lemmas are needed to show the converse of the above theorem.

Lemma 3.17. *Let A, E be compact Hausdorff spaces and $\rho : E \to A$ a continuous surjection such that for any proper closed $E_0 \subseteq E$ we have $\rho(E_0) \neq A$. If $G \subseteq E$ is an open subset, then*

$$\rho(G) \subseteq \overline{A \setminus \rho(E \setminus G)}.$$

Proof. We can without loss of generality take $G \neq \emptyset$ or else the result holds trivially. If $a \in \rho(G)$, then it is enough to show that for every open neighborhood N of a we have

$$N \cap (A \setminus \rho(E \setminus G)) \neq \emptyset.$$

Let N be an open neighborhood of a. As $a \in \rho(G)$, it follows that $G \cap \rho^{-1}(N) \neq \emptyset$. Hence, $E \setminus (G \cap \rho^{-1}(N))$ is a proper closed subset of E and so

$$\rho(E \setminus (G \cap \rho^{-1}(N))) \neq A.$$

Choose an element

$$x \in A \setminus \rho(E \setminus (G \cap \rho^{-1}(N))) \subseteq A \setminus \rho(E \setminus G).$$

Since ρ is surjection, there exists $y \in G \cap \rho^{-1}(N)$ such that $x = \rho(y)$. Therefore, we obtain that

$$x \in N \cap (A \setminus \rho(E \setminus G)).$$

Therefore, $a \in \overline{A \setminus \rho(E \setminus G)}$. $\square$

Lemma 3.18. *Let X be an extremally disconnected compact Hausdorff space. If $U_1, U_2 \subseteq X$ are disjoint open sets, then $\overline{U_1}$ and $\overline{U_2}$ are also disjoint.*

Proof. Since $U_2 \subseteq X \setminus U_1$ and $X \setminus U_1$ is closed we get

$$\overline{U_2} \subseteq X \setminus U_1.$$

Therefore, $\overline{U_2}$ and U_1 are disjoint. As $\overline{U_2}$ is also open, repeating the same argument yields the desired result. $\square$

Lemma 3.19. *Let A, E be compact Hausdorff spaces where A is extremally disconnected. If $\rho : E \to A$ is a continuous surjection such that for any proper closed subset $E_0 \subseteq E$ we have $\rho(E_0) \neq A$, then ρ is a homeomorphism.*

Proof. It is enough to show that ρ is injective as any bijective continuous function between compact Hausdorff spaces is a homeomorphism. Suppose $x_1, x_2 \in E$ are distinct points. As E is Hausdorff there exists disjoint open neighborhoods of x_1 and x_2 denoted U_1 and U_2 respectively. By Lemma 3.17 we have

$$\rho(U_i) \subseteq \overline{A \setminus \rho(E \setminus U_i)}$$

for $i = 1, 2$. Notice that

$$\begin{aligned}
(A \setminus \rho(E \setminus U_1)) \cap (A \setminus \rho(E \setminus U_2)) &= A \setminus (\rho(E \setminus U_1) \cup \rho(E \setminus U_2)) \\
&= A \setminus \rho(E \setminus (U_1 \cap U_2)) \\
&= A \setminus \rho(E) \\
&= \emptyset
\end{aligned}$$

showing the open sets $A \setminus \rho(E \setminus U_1)$ and $A \setminus \rho(E \setminus U_2)$ are disjoint. Therefore, as A is extremally disconnected the closures are still disjoint by Lemma 3.18, i.e.,

$$\overline{A \setminus \rho(E \setminus U_1)} \cap \overline{A \setminus \rho(E \setminus U_2)} = \emptyset.$$

In particular, $\rho(U_1) \cap \rho(U_2) = \emptyset$ and so $\rho(x_1) \neq \rho(x_2)$. $\square$

Lemma 3.20. *Let E, A be compact Hausdorff spaces. If $\rho : E \to A$ is a continuous surjection, then there exists a compact subset $D \subseteq E$ such that $\rho(D) = A$ and for every proper closed subset $C \subseteq D$ we get $\rho(C) \neq A$.*

Proof. Assume $\{D_\alpha\}_{\alpha \in I}$ is a decreasing chain of closed subsets of E such that $\rho(D_\alpha) = A$ for every $\alpha \in I$. Let $x \in A$ and consider the family of subsets $\left\{D_\alpha \cap \rho^{-1}(x)\right\}_{\alpha \in I}$. It is clear this family has the finite intersection property; meaning that for any finite $\alpha_1, \ldots, \alpha_n \in I$ we get

$$\bigcap_{i=1}^{n} D_{\alpha_i} \cap \rho^{-1}(x) \neq \emptyset.$$

By compactness of E, there exists $x \in \cap_{\alpha \in I} D_\alpha$ with $\rho(x) = y$. It follows that

$$\rho(\cap_{\alpha \in I} D_\alpha) = A.$$

An application of Zorn's lemma gives us our desired result. $\square$

The follow theorem, due to Gleason, gives us a converse to Theorem 3.16.

Theorem 3.21 (Gleason, [Gle58])**.** *If X is an extremally disconnected compact Hausdorff space, then X is projective.*

Proof. Let Y, Z be compact Hausdorff spaces. Assume $\phi : X \to Y$ and $\pi : Z \to Y$ are continuous maps where π is a surjection. Define

$$W := \{(x, z) : \phi(x) = \pi(z)\} \subseteq X \times Z.$$

It is clear that W is closed and hence compact. As π is surjective, we get that for every $x \in X$ there exists $z \in Z$ such that $(x, z) \in W$. Let $\pi_1 : W \to X$ denote the projection onto the first component. Notice, π_1 is continuous and surjective. We know that there exists a compact subset $D \subseteq W$ such that $\pi_1(D) = X$ and for every proper closed $C \subseteq D$ we have $\pi_1(C) \neq X$. Define ψ to be the restriction of π_1 to D. As X is extremally disconnected, we obtain by Lemma 3.19 that ψ is a homeomorphism from D to X. Let $\pi_2 : W \to Z$ denote the projection onto the second component. Defining

$$\gamma := \pi_2 \circ \psi^{-1} : X \to Z$$

yields the required continuous map such that $\phi = \pi \circ \gamma$. $\square$

Using Gleason's characterization we obtain the following theorem as it appears in [HP07].

Theorem 3.22. *Let X be a compact Hausdorff space and $C(X)$ the associated unital commutative C*-algebra. The following are equivalent.*

1. *$C(X)$ is injective as an operator space.*
2. *$C(X)$ is injective in the category of unital commutative C*-algebras with *-homomorphisms.*
3. *X is projective.*
4. *Let $\mathcal{E}, \mathcal{F}$ be nonempty subsets of $C(X)^{sa}$; the collection of all self-adjoint elements in $C(X)$, which in this case is the collection of $\mathbb{R}$-valued continuous functions on X. If $\mathcal{E} \leq \mathcal{F}$ (meaning $a \leq b$ for all $a \in \mathcal{E}$ and $b \in \mathcal{F}$), then there exists $a \in C(X)^{sa}$ with $\mathcal{E} \leq a \leq \mathcal{L}$.*
5. *$C(X)$ is injective in the category of Banach spaces with contractive linear maps.*

Proof. We claim $3 \implies 2$ follows by Gelfand duality. Let $\mathscr{A}, \mathscr{B}$ be unital commutative C*-algebra, $\iota : \mathscr{A} \to \mathscr{B}$ the inclusion map, and $\phi : \mathscr{A} \to C(X)$ a *-homomorphism. It is enough to show that ϕ extends to a *-homomorphism on $\mathscr{B}$. By Gelfand duality, there exists compact Hausdorff spaces Y, Z such that $\mathscr{A} \simeq C(Y)$ and $\mathscr{B} \simeq C(Z)$. Also, there exists continuous surjection $\pi : Z \to Y$ such that

$$\iota(f)(z) = f(\pi(z))$$

for every $f \in C(Y)$ and $z \in Z$. Lastly, there exists continuous map $\gamma : X \to Y$ such that

$$\phi(f)(x) = f(\gamma(x))$$

for every $f \in C(Y)$ and $x \in X$. By projectivity of X, there exists a continuous map $\alpha : X \to Z$ such that $\pi \circ \alpha = \gamma$. It follows by Gelfand duality that there exists a *-homomorphism $\psi : C(Z) \to C(X)$ such that

$$\psi(f)(x) = f(\alpha(x))$$

for every $f \in C(Z)$ and $x \in X$. It follows by a routine calculation that ψ is the required *-homomorphism satisfying $\psi \circ \iota = \phi$.
Conversely, $2 \implies 3$ is shown by a similar Gelfand duality argument. Hence, $2 \iff 3$ holds.

We now show the equivalences of conditions $1, 3, 4$, and 5.

We first prove that $1 \implies 4$. Notice that, $C(X) \subseteq \ell^\infty(X)$ isometrically. Moreover, as $C(X)$ is injective we can extend the identity map on $C(X)$ to a unital completely contractive projection $\phi : \ell^\infty(X) \to C(X)$. By Theorem 2.7 we have that ϕ is also a UCP map. Assume there exists $\mathcal{E}, \mathcal{F} \in C(X)^{sa}$ such that $\mathcal{E} \leq \mathcal{F}$. Viewing $\mathcal{E}, \mathcal{F} \subseteq \ell^\infty(X)$ we can define $a = (a_i)_{i \in X}$ via

$$a_k = \sup\{e_k : (e_i)_{i \in X} \in \mathcal{E}\}.$$

It follows that $a \in \ell^\infty(X)$ since, by assumption $\mathcal{E} \leq \mathcal{F}$ and so every element in $\mathcal{E}$ is uniformly bounded above. Since ϕ is completely positive, it preserves order and so

$$\mathcal{E} \leq \phi(a) \leq \mathcal{F}.$$

where $\phi(a) \in C(X)^{sa}$.

To show $4 \implies 3$ it is enough to prove X is extremally disconnected. Let $U \subseteq X$ be open. Consider the following subsets

$$\mathcal{E} := \{f \in C(X) : 0 \leq f \leq \chi_U\} \subseteq C(X)^{sa}$$

and

$$\mathcal{F} := \{f \in C(X) : \chi_{\overline{U}} \leq f \leq 1\} \subseteq C(X)^{sa}.$$

It is clear that $\mathcal{E} \leq \mathcal{F}$. By assumption, there exists $g \in C(X)^{sa}$ such that $\mathcal{E} \leq g \leq \mathcal{F}$. We claim that $g = \chi_{\overline{U}}$, which implies that $\overline{U}$ is open. For $x \in U$, by Urysohn's lemma, there exists $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$, and $f\mid_{X \setminus U} = 0$. Therefore, $f \in \mathcal{E}$, consequently showing that $f \leq g$ and $g(x) = 1$. Hence, $g\mid_U = 1$ and by continuity $g\mid_{\overline{U}} = 1$. Given $x \notin \overline{U}$, once again by Urysohn's lemma, there exists $f \in C(X)$ with $0 \leq f \leq 1$, $f(x) = 0$, and $f\mid_{\overline{U}} = 1$. It follows that $f \in \mathcal{F}$, and so $g \leq f$. In particular, $g(x) = 0$. As this holds for every $x \notin \overline{U}$, we get $g \leq \chi_{\overline{U}}$. This forces $g = \chi_{\overline{U}}$. Since g is continuous we obtain that

$$\overline{U} = g^{-1}(\{1\}) = g^{-1}((\frac{1}{2}, \frac{3}{2}))$$

is a clopen set. Hence, X is extremally disconnected.

Now we show that $3 \implies 5$. Let $B \subseteq C$ be Banach spaces such that $\phi : B \to C(X)$ is a contractive map. We can define the continuous map $\tilde{\phi} : X \to \overline{b_1}(B^*)$ given via $\tilde{\phi}(x)(b) := \phi(b)(x)$ for every $b \in B$ and $x \in X$ where $\overline{b_1}(B^*)$ has the weak*-topology inherited from B^*. As $B \subseteq C$, by the Hahn-

Banach Theorem there exists a continuous surjective map $f : \overline{b_1}(C^*) \to \overline{b_1}(B^*)$ such that $f(\gamma)(b) = \gamma(b)$ for every $b \in B$ and $\gamma \in \overline{b_1}(C^*)$. Since X is projective, there exists a continuous map $\psi : X \to \overline{b_1}(C^*)$ such that

$$f \circ \psi = \tilde{\phi}.$$

Define the contractive map $\Phi : C \to C(X)$ via $\Phi(c)(x) := \psi(x)(c)$ for all $x \in X$. Given $b \in B \subseteq C$ and $x \in X$ we obtain

$$\begin{aligned}
\Phi(b)(x) &= \psi(x)(b) \\
&= f(\psi(x))(b) \\
&= \tilde{\phi}(x)(b) \\
&= \phi(b)(x).
\end{aligned}$$

Therefore we get $\Phi \mid_B = \phi$, showing $C(X)$ is injective as a Banach space with contractive maps.

Lastly, we show that $5 \implies 1$. Let $V \subseteq W$ be operator spaces and $\phi : V \to C(X)$ a completely contractive map. Without loss of generality take V and W to be norm-closed operators spaces as we can always uniquely extend ϕ to the norm-closure. In particular, V and W are Banach spaces. Hence, ϕ extends to a contractive map $\psi : W \to C(X)$. Then by Lemma 2.8, it follows that ψ is a completely contractive map; thus showing $C(X)$ is injective.

□

By a result of Dixmier (see [Dix51]) the injective envelope of any unital commutative C*-algebra $C(X)$ is the *Dixmier algebra* $\mathcal{D}(X)$; defined as the algebra of bounded Borel functions, $B(X)$, modulo the ideal of bounded Borel functions whose support is a meagre set[1], $\mathcal{M}(X)$, i.e.

$$I(C(X)) = \mathcal{D}(X) := B(X)/\mathcal{M}(X).$$

For another example we look at the finite rank operators. Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be an operator system that contains the finite rank operators. We claim that $I(\mathcal{S}) = \mathcal{B}(H)$. Indeed, by Lemma 3.13 it is enough to show the inclusion $\mathcal{S} \subseteq \mathcal{B}(H)$ is a rigid extension. Suppose $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a completely contractive map which fixes $\mathcal{S}$. In particular, ϕ is a unital completely positive map which fixes the finite rank operators. Hence, by Theorem 2.10, the finite rank operators are contained in the multiplicative domain of ϕ. Moreover, Theorem 2.10 tells us

[1] Recall a meagre set in a topological space is defined to be a countable union of nowhere dense sets, where a set X is said to be nowhere dense if and only if $\mathrm{int}(\overline{\mathrm{X}}) = \emptyset$.

that ϕ is a $\mathcal{F}(H)$-bimodule map, meaning that for every $x \in \mathcal{B}(H)$, $F \in \mathcal{F}(H)$ we have

$$\phi(xF) = \phi(x)\phi(F)$$

and

$$\phi(Fx) = \phi(F)\phi(x).$$

In particular, as ϕ fixes the finite rank operators, we obtain that for every finite rank projection $F \in \mathcal{F}(H)$ and $x \in \mathcal{B}(H)$ we get

$$\phi(FxF) = F\phi(x)F.$$

Furthermore, as $FxF \in \mathcal{F}(H)$, we obtain that $\phi(FxF) = FxF$ for every $F \in \mathcal{F}(H)$ and $x \in \mathcal{B}(H)$. In particular, we get that

$$FxF = F\phi(x)F$$

for every $x \in \mathcal{B}(H)$ and $F \in \mathcal{F}(H)$. This forces $\phi(x) = x$ for every $x \in \mathcal{B}(H)$, thus showing rigidity of the inclusion $\mathcal{S} \subseteq \mathcal{B}(H)$. Therefore, $I(\mathcal{S}) = \mathcal{B}(H)$.

Lemma 3.23. *Let $V \subseteq \mathcal{B}(H)$ be a unital operator space. Then $I(V) = I(V+V^*)$.*

Proof. Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a V-map. In particular, ϕ is a unital completely contractive map. Hence, by Theorem 2.7, we get that ϕ is a UCP map. Therefore, for every $x, y \in V$

$$\phi(x + y^*) = \phi(x) + \phi(y)^* = x + y^*.$$

Hence, ϕ is a $V+V^*$-map. Conversely, it is clear that every $V+V^*$-map is also a V-map. Therefore, the V-maps coincide with the $V+V^*$-maps. In particular, a minimal V-seminorm will coincide with a minimal $V+V^*$-seminorm. It follows by Theorem 3.10, that $I(V) = I(V+V^*)$. □

Corollary 3.24. *Let V be a unital operator space. Then $I(V)$ is a C*-algebra.*

Proof. By Lemma 3.23 we see that $I(V) = I(V+V^*)$. Since $V+V^*$ is an operator system, by Theorem 3.7, $I(V+V^*)$ is a C*-algebra when equipped with the Choi-Effros product. □

Corollary 3.25. *If $\mathscr{A} \subseteq \mathcal{B}(H)$ be a unital operator algebra (meaning a unital subalgebra), then $I(\mathscr{A})$ is a C*-algebra and there exists a complete isometric homomorphism $\psi : \mathscr{A} \to I(\mathscr{A})$.*

Proof. By Lemma 3.23, we get that $I(\mathscr{A}) = I(\mathscr{A}+\mathscr{A}^*)$. As $\mathscr{A}+\mathscr{A}^*$ is an operator system, $I(\mathscr{A})$ will be a C*-algebra with the Choi-Effros product. Therefore, the inclusion of $\mathscr{A}$ into $I(\mathscr{A})$ will be a completely isometric homomorphism. □

For another example consider the collection of upper triangular operators $T(\ell^2(\mathbb{N}))$ inside $\mathcal{B}(\ell^2(\mathbb{N}))$ with respect to the usual basis $(e_i)_{i\in\mathbb{N}}$. By Lemma 3.23, as $T(\ell^2(\mathbb{N}))$ is a unital operator space, we get that

$$I(T(\ell^2(\mathbb{N}))) = I(T(\ell^2(\mathbb{N})) + T(\ell^2(\mathbb{N}))^*).$$

Let $\mathcal{S}$ denote the operator system $T(\ell^2(\mathbb{N}))+T(\ell^2(\mathbb{N}))^*$. For every $i, j \in \mathbb{N}$ define

$$T_{ij} := \langle \cdot, e_i \rangle e_j.$$

It is seen that for every $i, j \in \mathbb{N}$, $T_{ij} \in \mathcal{S}$. Let $\mathcal{C}$ be the collection of all finite linear combinations of the operators T_{ij} for $i, j \in \mathbb{N}$. Hence, $\mathcal{C} \subseteq \mathcal{S}$. For every $n \in \mathbb{N}$ define

$$P_n := \sum_{k=1}^{n} \langle \cdot, e_k \rangle e_k \in \mathcal{S}.$$

It is computed that for every $x \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $n \in \mathbb{N}$, $P_n x P_n \in \mathcal{S}$. Similar to our example dealing with the finite rank operators, let $\phi : \mathcal{B}(\ell^2(\mathbb{N})) \to \mathcal{B}(\ell^2(\mathbb{N}))$ be a completely contractive map which fixes $\mathcal{S}$. By Theorem 2.7, we get that ϕ is a UCP map which fixes $\mathcal{S}$. Hence, by Theorem 2.10, we obtain that $\mathcal{S}$ is contained within the multiplicative domain of ϕ and that ϕ is a $\mathcal{S}$-bimodule map. It follows that for every $x \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $n \in \mathbb{N}$

$$P_n x P_n = \phi(P_n x P_n) = P_n \phi(x) P_n.$$

This implies that $\phi(x) = x$ for every $x \in \mathcal{B}(\ell^2(\mathbb{N}))$. Therefore, the inclusion $\mathcal{S} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ is rigid and hence

$$I(T(\ell^2(\mathbb{N}))) = I(\mathcal{S}) = \mathcal{B}(\ell^2(\mathbb{N})).$$

Definition 3.26. Let $\mathscr{A}$ be a unital operator algebra. Define the *C^*-envelope* $C_e^*(\mathscr{A})$ to be the C*-algebra generated by $\mathscr{A}$ inside $I(\mathscr{A})$.

Theorem 3.27 (Hamana)**.** *Let $\mathscr{A}$ be a unital operator algebra and $\rho : \mathscr{A} \to \mathcal{B}(H)$ a completely isometric homomorphism. Then there exists a unique surjective *-homomorphism from $C^*(\rho(\mathscr{A}))$ to $C_e^*(\mathscr{A})$ extending the map $\rho(a) \mapsto a$.*

Proof. We first show the existence of such a map. We may view $\mathscr{A} \subseteq I(\mathscr{A})$. By injectivity of $I(\mathscr{A})$ we can extend the completely isometric map $\rho(a) \mapsto a$ to a completely contractive map $\phi : \mathcal{B}(H) \to I(\mathscr{A})$. As ρ is unital, we obtain that ϕ is unital and consequently, by Theorem 2.7, a UCP map. Similarly, by injectivity of $\mathcal{B}(H)$ we get that ρ extends to a UCP map $\psi : I(\mathscr{A}) \to \mathcal{B}(H)$.
We now show that ϕ is multiplicative on $C^*(\rho(\mathscr{A}))$ and consequently is the re-

quired *-homomorphism.

Consider the completely contractive map, $\phi \circ \psi : I(\mathscr{A}) \to I(\mathscr{A})$. For every $a \in \mathscr{A}$, we have $\phi(\rho(a)) = a$. Therefore, by rigidity, $\phi \circ \psi$ must be the identity map.
It is enough to show for every $a \in \mathscr{A}$ that

$$\phi(\rho(a)^*\rho(a)) = a^*a$$

and

$$\phi(\rho(a)\rho(a)^*) = aa^*.$$

As then $\rho(\mathscr{A})$ will be a subset of the multiplicative domain of ϕ. Since the multiplicative domain is a C*-algebra, $C^*(\rho(\mathscr{A}))$ will also be a subset. It then follows that ϕ restricted to $C^*(\rho(\mathscr{A}))$ will be a *-homomorphism.
By the Cauchy-Schwarz inequality we get that

$$\phi(\rho(a)^*\rho(a)) \geq \phi(\rho(a))^*\phi(\rho(a)) = a^*a.$$

Similarly, $\psi(a^*a) \geq \psi(a)^*\psi(a) = \rho(a)^*\rho(a)$. Applying ϕ we see that

$$a^*a = \phi \circ \psi(a^*a) \geq \phi(\rho(a)^*\rho(a)).$$

Therefore, $\phi(\rho(a)^*\rho(a)) = a^*a$. Similarly, $\phi(\rho(a)\rho(a)^*) = aa^*$ follows. By Theorem 2.10 we get our desired result that $\rho(\mathscr{A})$ is a subset of the multiplicative domain. Surjectivity and uniqueness is clear as any other completely isometric *-homomorphism would have to act the same on $\rho(\mathscr{A})$ and consequently on the C*-algebra generated by $\rho(\mathscr{A})$. □

Let $\mathscr{A} \subseteq C^*(\mathscr{A}) \subseteq \mathcal{B}(H)$ be a unital operator algebra and I be the kernel of the surjective *-homomorphism $C^*(\mathscr{A}) \to C^*_e(\mathscr{A})$ given by Hamana's theorem. We claim that I is a Shilov boundary of $\mathscr{A}$ as defined in the introduction.
Indeed, it is clear that I itself is a boundary ideal as the map $C^*(\mathscr{A}) \to C^*_e(\mathscr{A}) \simeq C^*(\mathscr{A})/I$ is a completely isometric homomorphism when restricted to $\mathscr{A}$. If $J \lhd C^*(\mathscr{A})$ is another boundary ideal, then the quotient map $q : C^*(\mathscr{A}) \to C^*(\mathscr{A})/J$ is a completely isometric homomorphism when restricted to $\mathscr{A}$. By Hamana's theorem there exists a surjective *-homomorphism $C^*(\mathscr{A})/J \to C^*_e(\mathscr{A}) \simeq C^*(\mathscr{A})/I$. It follows that $J \subseteq I$ and hence I is a Shilov boundary for $\mathscr{A}$.

Corollary 3.28. *Let $\mathscr{A} \subseteq \mathcal{B}(H)$ be a unital operator algebra. If $C^*(\mathscr{A})$ is a simple C*-algebra, then $C^*(\mathscr{A}) \simeq C^*_e(\mathscr{A})$.*

Proof. Let $\pi : C^*(\mathscr{A}) \to C^*_e(\mathscr{A})$ be the given surjective *-homomorphism. As $C^*(\mathscr{A})$ is simple π must also be injective and our desired result follows. □

Corollary 3.29 (Shilov, [Pau02]). *Let X be a compact Hausdorff space and $\mathscr{A} \subseteq C(X)$ a subalgebra that separates points. There exists unique closed subset $Y \subseteq X$ such that the following holds.*

1. *The restriction map $r : C(X) \to C(Y)$ via $f \mapsto f|_Y$ is isometric on $\mathscr{A}$.*
2. *If $W \subseteq X$ is another closed subset such that 1 holds, then $Y \subseteq W$.*

Moreover given any isometric isomorphism $\rho : \mathscr{A} \to C(X_1)$ such that $\rho(\mathscr{A})$ is a closed subalgebra which separates points and $Y_1 \subseteq X_1$ is the unique closed set given above, there exists a homeomorphism $h : Y_1 \to Y$ such that $\rho(f)|_{Y_1} = f \circ h$.

Proof. As $\mathscr{A}$ separates points, by the Stone-Weierstrass Theorem, we get $C^*(\mathscr{A}) = C(X)$. We can apply Hanama's theorem to get that there exists a surjective *-homomorphism $\pi : C(X) \to C_e^*(\mathscr{A})$ which is isometric on $\mathscr{A}$. By Gelfand duality, $C_e^*(\mathscr{A})$ is homeomorphic to $C(Z)$ for some compact Hausdorff space Z. Hence, there is no loss of generality replacing $C_e^*(\mathscr{A})$ by $C(Z)$. Then there exists an injective continuous function $h : Z \to X$ such that $\pi(f) = f \circ h$. Defining $Y := h(Z)$, we get condition 1. Uniqueness of Y will follow once we show condition 2.

Now let $W \subseteq X$ be a closed subset that satisfies condition 1. As the restriction map

$$\begin{aligned} r : C(X) &\to C(W) \\ f &\mapsto f|_W \end{aligned}$$

is isometric on $\mathscr{A}$ we may view $\mathscr{A} \subseteq C(W)$. By Hanama's theorem there exists a surjective *-homomorphism $\pi_1 : C(W) \to C(Z)$ which is isometric on $\mathscr{A}$ and consequently an injective continuous function $h_1 : Z \to W$ such that $\pi_1(f) = f \circ h_1$ for every $f \in C(W)$. Let $i : W \to X$ the inclusion map. In particular, $r(f) = f \circ i$ for every $f \in C(X)$. Then, by the uniqueness given in Hanama's theorem, we see that $\pi = \pi_1 \circ r$. Therefore, we obtain

$$\pi(f) = \pi_1(r(f)) = \pi_1(f \circ i) = f \circ i \circ h_1.$$

Hence, $h = i \circ h_1$ and it follows that $Y = h(Z) = i(h_1(Z)) \subseteq W$.

Uniqueness regardless of the representation of $\mathscr{A}$ follows by the uniqueness of the C*-envelope. $\square$

Chapter 4

Boundary Representations

The study of boundary representations was started by Arveson in [Arv69] and [Arv72] in hopes of proving the existence of the noncommmutative analogue of the Shilov boundary. The existence of the noncommutative Shilov boundary was shown in the case of a separable operator system by Arveson in [Arv08]. The proof heavily relies on disintegration theory for Hilbert spaces. We instead present the proof of existence in full generality as shown by Davidson and Kennedy in [DK15], which avoids the use of disintegration theory.

We first discuss the properties and structure given by a partial ordering on the collection of all unital completely positive maps from an operator system as seen in [Arv69].

Let $\mathcal{S}$ be an operator system. There exists natural partial ordering on $\mathrm{CP}(\mathcal{S}, \mathcal{B}(\mathrm{H}))$ given by $\phi_1 \leq \phi_2$ if and only if $\phi_2 - \phi_1$ is completely positive. Given a completely positive map ϕ, define $[0, \phi] := \{\psi \in \mathrm{CP}(\mathcal{S}, \mathcal{B}(\mathrm{H})) : \psi \leq \phi\}$.

Definition 4.1. We say $\phi \in \mathrm{CP}(\mathcal{S}, \mathcal{B}(\mathrm{H}))$ is *pure* if for all $\psi \in [0, \phi]$ we have $\psi = \lambda\phi$ for some $0 \leq \lambda \leq 1$.

Recall the minimal Stinespring dilation for a completely positive map (see Theorem 2.2 and Lemma 2.5) on a C*-algebra.

Let $\mathscr{A}$ be a C*-algebra and $\phi : \mathscr{A} \to \mathcal{B}(H)$ a completely positive map. Then there exists *-representation $\pi : \mathscr{A} \to \mathcal{B}(K)$ and bounded linear operator $V : H \to K$ such that $K = [\pi(A)VH]$ and

$$\phi(\cdot) = V^*\pi(\cdot)V.$$

The pair (π, V, K) is called the minimal Stinespring dilation of ϕ and is unique up to unitary equivalence.

Lemma 4.2. *Let $\mathcal{A}$ be a C^*-algebra, $\phi_1, \phi_2 \in \mathrm{CP}(\mathscr{A}, \mathcal{B}(\mathrm{H}))$, and $\phi_i(\cdot) = V_i^* \pi_i(\cdot) V_i$ be the minimal Stinespring dilations where $\pi_i : \mathscr{A} \to \mathcal{B}(H_i)$ are *-representations for $i = 1, 2$. If $\phi_1 \leq \phi_2$, then there exists unique contraction $T \in \mathcal{B}(H_2, H_1)$ such that $TV_2 = V_1$ and $T\pi_2(\cdot) = \pi_1(\cdot)T$.*

Proof. Let $\xi_1, \ldots, \xi_n \in H, x_1, \ldots, x_n \in \mathscr{A}$. Notice

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \pi_1(x_k) V_1 \xi_k \right\|^2 &= \sum_{i,j} \langle \phi_1(x_i^* x_j) \xi_j, \xi_i \rangle \\
&\leq \sum_{i,j} \langle \phi_2(x_i^* x_j) \xi_j, \xi_i \rangle \\
&= \left\| \sum_{k=1}^{n} \pi_2(x_k) V_1 \xi_k \right\|^2 .
\end{aligned}$$

Therefore, there exists unique contraction T on $[\pi_2(\mathscr{A})V_2 H] = H_2$ such that $T\pi_2(x)V_2\xi = \pi_1(x)V_1\xi$ for every $x \in \mathscr{A}$. It is clear the contraction T satisfies the desired properties and is unique. $\square$

Theorem 4.3. *Let $\mathscr{A}$ be a C^*-algebra, $\phi \in \mathrm{CP}(\mathscr{A}, \mathcal{B}(\mathrm{H}))$, and let $\phi(\cdot) = V^*\pi(\cdot)V$ be the minimal Stinespring dilation. There exists affine order isomorphism[1] from $\{T \in \pi(\mathscr{A})' : 0 \leq T \leq I\}$ onto $[0, \phi]$ via $T \mapsto \phi_T$, where $\phi_T(\cdot) := V^*T\pi(\cdot)V$.*

Proof. Let $T \in \pi(\mathscr{A})'$ with $0 \leq T \leq I$ and $\xi_1, \ldots, \xi_n \in H$, $(x_{ij}) \in M_n(\mathscr{A})^+$. As $\sqrt{T} \in \pi(\mathscr{A})'$, we get

$$\sum_{i,j} \langle \phi_T(x_{ij}) \xi_j, \xi_i \rangle = \sum_{i,j} \left\langle \pi(x_{ij}) \sqrt{T} V \xi_j, \sqrt{T} V \xi_i \right\rangle \geq 0.$$

So ϕ_T is completely positive. Since $I - T \in \pi(\mathscr{A})'$ and $0 \leq I - T \leq I$, by repeating the same argument we get $0 \leq \phi_{I-T} = \phi - \phi_T$ and consequently $\phi_T \leq \phi$. By similar argument, we can show if $T_1, T_2 \in \pi(\mathscr{A})'$ such that $0 \leq T_1 \leq T_2 \leq I$, then $\phi_{T_1} \leq \phi_{T_2}$. So the map $T \mapsto \phi_T$ is order preserving. It is clear the map is also linear. We now show that it is an order isomorphism onto its range.

[1]Meaning that a bijective map such that $T_1 \leq T_2 \iff \phi_{T_1} \leq \phi_{T_2}$ and $\phi_{tT_1+(1-t)T_2} = t\phi_{T_1} + (1-t)\phi_{T_2}$ for every $T_1, T_2 \in \{T \in \pi(\mathscr{A})' : 0 \leq T \leq I\}, 0 \leq t \leq 1$.

First we claim if $T \in \pi(\mathscr{A})'$ such that ϕ_T is completely positive, then $T \geq 0$. Consider a vector of the form $\xi = \pi(x_1)V\xi_1 + \cdots + \pi(x_n)V\xi_n$ and notice that

$$\langle T\xi, \xi \rangle = \sum_{i,j} \langle \phi_T(x_i^* x_j)\xi_j, \xi_i \rangle \geq 0.$$

As T acts on $[\pi(\mathscr{A})VH]$, we obtain $T \geq 0$. Moreover, if we assume $0 \leq \phi_T \leq \phi$, then $\phi - \phi_T = \phi_{I-T}$ is completely positive. Therefore $0 \leq I - T$ and so $T \leq I$. Similarly, if $T_1, T_2 \in \pi(\mathscr{A})'$ with $0 \leq \phi_{T_1} \leq \phi_{T_2} \leq \phi$, then $0 \leq T_1 \leq T_2 \leq I$. Lastly, we show the map $T \mapsto \phi_T$ is surjective.

Assume $\psi \in [0, \phi]$ and let $\psi(\cdot) = W^*\sigma(\cdot)W$ be its minimal Stinespring dilation. By the previous lemma we know there exists contraction X from $[\pi(\mathscr{A})VH]$ to $[\sigma(\mathscr{A})WH]$ such that $X\pi(\cdot) = \sigma(\cdot)X$ and $XV = W$. Defining $T = X^*X$, it is easily checked that $T \in \pi(\mathscr{A})'$, $0 \leq T \leq I$, and $\phi_T = \psi$. □

Corollary 4.4. *Let $\mathscr{A}$ be a unital C^*-algebra, $\phi : \mathscr{A} \to \mathcal{B}(H)$ a unital completely positive map, and $\phi(\cdot) = V^*\pi(\cdot)V$ its minimal Stinespring dilation. We have that ϕ is pure if and only if π is an irreducible representation of $\mathscr{A}$.*

Proof. Let ϕ be pure. If P is a nonzero projection onto a $\pi(\mathscr{A})$ invariant subset, then $P \in \pi(\mathscr{A})'$. The above theorem tells us that the set $\{T \in \pi(\mathscr{A})' : 0 \leq T \leq I\}$ is in a bijective correspondence with the set $[0, \phi]$. Since ϕ is pure, $[0, \phi]$ only consists of scalar multiples of ϕ. Hence, $\{T \in \pi(\mathscr{A})' : 0 \leq T \leq I\}$ must only consist of scalar multiples of the identity. In particular, we get that $P = I$. Therefore, π is irreducible.

Conversely if π is irreducible, then every projection in $\pi(\mathscr{A})'$ is a scalar multiple of the identity. Moreover, $\pi(\mathscr{A})'$ is a von Neumann algebra and hence generated by its projections. It follows that $\pi(\mathscr{A})'$ only consists of scalar multiples of the identity. An application of the above theorem show us that ϕ is pure. □

Let ϕ be a state on a unital C*-algebra $\mathscr{A}$. If ϕ is an extreme point of the state space of $\mathscr{A}$ it is said to be a pure state. It is well-known that ϕ is a pure state if and only if the representation given in the minimal Stinespring dilation of ϕ is irreducible (see [Dav96, page 30]). Hence by the above corollary, there is no difference in the definition of ϕ being pure as a unital completely positive map and ϕ being a pure state in the usual sense.

Definition 4.5. Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a unital completely positive map. We say ϕ has the *unique extension property* (UEP) if ϕ has a unique completely positive extension to $C^*(\mathcal{S})$ and the extension is a

*-representation. Furthermore, if the extension is an irreducible representation we say ϕ is a *boundary representation.*

Unique extensions seems likely to be a difficult condition to satisfy in general. Regardless, it was shown by Colin Krisko and myself in an undergrad research project supervised by Raphaël Clouâtre that any pure state defined on an operator system $\mathcal{S} \subseteq M_2(\mathbb{C})$ extends uniquely to a state on $M_2(\mathbb{C})$. This was done by a rather laborious calculation, the proof of which and more can be found in [Clo23].

We now present an important characterization of the unique extension property due to [MS98].

Define a partial ordering on the collection of all unital completely positive maps on $\mathcal{S}$ given as follows. If $\phi_i : \mathcal{S} \to \mathcal{B}(H_i)$ are unital completely positive maps for $i = 1, 2$ we say that $\phi_1 \prec \phi_2$ if $H_1 \subseteq H_2$ and

$$\phi_1(\cdot) := P_{H_1}\phi_2(\cdot) \mid_{H_1} .$$

In this case, we say that ϕ_2 is a dilation of ϕ_1.

Definition 4.6. Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a unital completely positive map. We say ϕ is *maximal* if for any $\phi \prec \psi$ we have $\psi = \phi \oplus \lambda$ for some unital completely positive map λ. We can interpret this as saying ϕ only has "trivial" dilations.

Let $\phi : \mathcal{S} \to \mathcal{B}(H)$ be a unital completely positive map and ψ a dilation of ϕ. We can then define $\widehat{\psi}$ to be the restriction of ψ to $[C^*(\psi(\mathcal{S}))H]$. It is clear that $\phi \prec \widehat{\psi} \prec \psi$ and $\psi = \widehat{\psi} \oplus \lambda$ for some unital completely positive map λ. So to show ϕ is maximal, it is enough to show that dilations $\psi : \mathcal{S} \to \mathcal{B}(K)$ such that $K = [C^*(\psi(\mathcal{S}))H]$ must satisfy $\phi = \psi$.

Lemma 4.7. *Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a UCP map. Then ϕ has the UEP if and only if ϕ is maximal.*

Proof. Assume ϕ has the UEP and take $\psi : \mathcal{S} \to \mathcal{B}(K)$ to be a dilation of ϕ where $K = [C^*(\psi(\mathcal{S}))H]$. Let $\Phi : C^*(\mathcal{S}) \to \mathcal{B}(K)$ be a completely positive extension of ψ. As ϕ has the UEP, we get that $P_H\Phi(\cdot) \mid_H$ is the unique (multiplicative) completely positive extension of ϕ. Hence, for any $x \in \mathcal{S}$ we get by the Cauchy-Schwarz inequality that

$$P_H\Phi(x)^*P_H\Phi(x)P_H = P_H\Phi(x^*x)P_H \geq P_H\Phi(x)^*\Phi(x)P_H.$$

Taking the difference we obtain

$$P_H\Phi(x)^*(I - P_H)\Phi(x)P_H \leq 0.$$

In particular

$$0 \leq ((I - P_H)\Phi(x)P_H)^*((I - P_H)\Phi(x)P_H) = P_H\Phi(x)^*(I - P_H)\Phi(x)P_H \leq 0.$$

Hence, $(I-P_H)\Phi(x)P_H = 0$ and it follows that $\psi(S)H \subseteq H$. Therefore, $\psi = \phi \oplus \lambda$ for some completely positive map λ. So we obtain the desired result that ϕ is maximal.

Conversely suppose ϕ is maximal. Let ψ be a completely positive extension of ϕ to $C^*(\mathcal{S})$. Taking the minimal Stinespring dilation we get that $\psi(\cdot) = P_H\pi(\cdot) \mid_H$ for some *-representation $\pi : C^*(\mathcal{S}) \to \mathcal{B}(K)$ such that $K = [C^*(\pi(\mathcal{S}))H]$. Now by maximality of ϕ, we obtain that $\phi = \pi \mid_{\mathcal{S}}$. This implies that $H = K$. Hence it follows that $\psi = \pi$. As every completely positive extension to $C^*(\mathcal{S})$ is multiplicative we obtain that every completely extension acts the same on polynomials in $\mathcal{S}$. Hence, every completely positive extension acts the same on $C^*(\mathcal{S})$ and consequently ϕ has the UEP. $\square$

We motivate why boundary representations are important in the construction of the noncommutative Shilov boundary.

Theorem 4.8 ([Arv03])**.** *For $i = 1, 2$ let $\mathcal{S}_i \subseteq C^*(\mathcal{S}_i)$ be operator systems and $\theta : S_1 \to S_2$ a unital complete isometry onto S_2. If $\pi_1 : \mathcal{S}_1 \to \mathcal{B}(H)$ is a unital completely positive map with the UEP, then there exists unital completely positive map $\pi_2 : \mathcal{S}_2 \to \mathcal{B}(H)$ with the UEP such that $\pi_2 \circ \theta = \pi_1$.*

Proof. Define $\pi_2 : \mathcal{S}_2 \to \mathcal{B}(H)$ via $\pi_2 := \pi_1 \circ \theta^{-1}$. As θ^{-1} is a unital completely contractive map we obtain by Theorem 2.7 that θ^{-1} is a UCP map. Hence, π_2 is also a UCP map. It is enough to show that π_2 is maximal. Let $\phi : \mathcal{S}_2 \to \mathcal{B}(K)$ be a dilation of π_2 satisfying $K = [C^*(\pi_2(\mathcal{S}_2))H]$. It follows that $\phi \circ \theta$ is a dilation of π_1 satisfying $K = [C^*(\pi_1(\mathcal{S}))H]$. Therefore, by maximality of π_1 we obtain $\phi \circ \theta = \pi_1$ and so $\phi = \pi_2$. $\square$

Recall the following definition as seen in the introduction.

Definition 4.9. Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be an operator system. A closed ideal $J \triangleleft C^*(\mathcal{S})$ is said to be a *boundary ideal* for $\mathcal{S}$ if the quotient map $q : C^*(\mathcal{S}) \to C^*(\mathcal{S})/J$ is a complete isometry on $\mathcal{S}$. A boundary ideal is called the *Shilov boundary* for $\mathcal{S}$ if it contains every other boundary ideal.

Definition 4.10. Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be an operator system. We say $\mathcal{S}$ is an *admissible* operator system if the intersection of the kernels of the boundary representations for $\mathcal{S}$ is a boundary ideal for $\mathcal{S}$.

Let bd $\mathcal{S}$ denote the collection of all boundary representation of $\mathcal{S}$, up to unitary equivalence. In particular two boundary representations π_1, π_2 are unitarily equivalence if and only if there exists a unitary operator U such that $U\pi_1 U^* = \pi_2$. If $\mathcal{S}$ is admissible, then the quotient map

$$q : C^*(\mathcal{S}) \to C^*(\mathcal{S}) / \bigcap_{\omega \in \text{bd } \mathcal{S}} \ker \omega$$

is a complete isometry on $\mathcal{S}$. From this it is easily seen that $\mathcal{S}$ is admissible if and only if for every $n \in \mathbb{N}$ and $s \in M_n(\mathcal{S})$ we have

$$\|s\| = \sup_{\omega \in \text{bd } \mathcal{S}} \|\omega_n(s)\|.$$

Lemma 4.11. *Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be an operator system. If $J \triangleleft C^*(\mathcal{S})$ is a boundary ideal for $\mathcal{S}$ and $\omega \in$ bd $\mathcal{S}$, then $J \subseteq \ker \omega$.*

Proof. Let $q : C^*(\mathcal{S}) \to C^*(\mathcal{S})/J$ be the quotient map. By assumption we have $q\mid_{\mathcal{S}}: \mathcal{S} \to q(\mathcal{S})$ is a unital complete isometry onto the operator system $q(\mathcal{S})$. Moreover, $C^*(q(\mathcal{S})) = q(C^*(\mathcal{S})) = C^*(\mathcal{S})/J$. Since ω is a boundary representation, in particular, it is a UCP map with the UEP. Hence, by Theorem 4.8 there exists a UCP map ω_1 of $q(\mathcal{S})$ with the UEP such that $\omega_1 \circ q = \omega$. If $x \in J = \ker q$, then $\omega(x) = \omega_1(q(x)) = 0$. Therefore, we obtain $J \subseteq \ker \omega$. □

Theorem 4.12. *Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be an admissible operator system. If K denotes the intersection of the kernels of boundary representations, then K is the Shilov boundary of $\mathcal{S}$.*

Proof. By assumption K is a boundary ideal for $\mathcal{S}$. An application of the previous lemma yields any other boundary ideal must lie inside K. Therefore, K is the required Shilov boundary. □

By the above theorem to show that every operator system $\mathcal{S} \subseteq C^*(\mathcal{S})$ has a Shilov boundary, it is enough to show $\mathcal{S}$ is admissible. The remainder of this section is devoted to proving exactly this.

Lemma 4.13. *Let $\mathcal{S}$ be an operator system. If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure maximal UCP map, then ϕ is a boundary representation.*

Proof. We know ϕ is maximal and hence has the UEP. So let ϕ extends uniquely to a *-representation $\pi : C^*(\mathcal{S}) \to \mathcal{B}(H)$. Suppose π is a reducible representation and $H_0 \subseteq H$ a non-trivial invariant subspace for $\pi(C^*(\mathcal{S}))$. If P is the projection onto H_0, then $P \in \pi(C^*(\mathcal{S}))'$ and $\psi(\cdot) := P\phi(\cdot) = P\phi(\cdot)P$ is a completely positive map such that $\psi \leq \phi$. As ϕ is pure we obtain that $\psi = \lambda\phi$ for some $0 \leq \lambda \leq 1$. Therefore

$$P = \psi(I) = \lambda\phi(I) = \lambda I$$

and so $P = I$. This contradicts our assumption that H_0 is a non-trivial invariant subspace. Hence, π is irreducible, showing that ϕ is a boundary representation. $\square$

Definition 4.14. Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a UCP map. We say that ϕ is *maximal* at $(s_0, x_0) \in \mathcal{S} \times H$ if for every UCP dilation ψ of ϕ we have

$$\psi(s_0)x_0 = \phi(s_0)x_0.$$

It can be easily seen that a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is maximal if and only if ϕ is maximal at every element in $\mathcal{S} \times H$.
We show it is possible to dilate any UCP map to a UCP map maximal at some point in $\mathcal{S} \times H$.

Lemma 4.15. *Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathcal{B}(H)$ a UCP map. If $(s_0, x_0) \in \mathcal{S} \times H$, then there exists UCP dilation of ϕ to $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ which is maximal on (s_0, x_0).*

Proof. It is enough to show that there exists a UCP map dilation of ϕ to $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ such that $\|\psi(s_0)x_0\| = \sup\{\|\rho(s_0)x_0\| : \phi \prec \rho\}$, where the supremum is taken over all UCP dilations ρ of ϕ. Indeed, if $\psi \prec \gamma$, then in particular $\phi \prec \gamma$, and so $\|\psi(s_0)x_0\| = \|\gamma(s_0)x_0\|$. From which it follows that $\psi(s_0)x_0 = \gamma(s_0)x_0$.

Consider an arbitrary UCP map ρ such that $\phi \prec \rho$. Let ρ' be the compression of ρ to the closed linear span of $\{H, \rho(s_0)x_0\}$, meaning if P is the projection onto the closed linear span of $\{H, \rho(s_0)x_0\}$, then

$$\rho'(\cdot) = P\rho(\cdot)P.$$

This yields a dilation $\rho' : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ of ϕ such that

$$\|\rho'(s_0)x_0\| = \|\rho(s_0)x_0\|.$$

So the value $\sup\{\|\rho(s_0)x_0\| : \phi \prec \rho\}$ is achieved by only looking at dilations into $\mathcal{B}(H \oplus \mathbb{C})$. It is easily checked that the collection of such dilation of ϕ is a closed subset of $\mathrm{UCP}(\mathcal{S}, \mathcal{B}(\mathrm{H} \oplus \mathbb{C}))$ in the BW-topology. Hence, it is compact and so the superemum is achieved by some dilation $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$. $\square$

We now prove the following key lemma which allows us to find a maximal dilation at some point of $\mathcal{S} \times H$ while preserving purity of the maps.

Lemma 4.16. *Let $\mathcal{S}$ be an operator system, $\phi : \mathcal{S} \to \mathcal{B}(H)$ a pure UCP map, and $(s_0, x_0) \in \mathcal{S} \times H$. If ϕ is not maximal at (s_0, x_0), then there exists a pure unital completely positive dilation $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ which is maximal on (s_0, x_0).*

Proof. Step 1: A candidate maximal dilation.
Define $L := \sup\{\|\rho(s_0)x_0\| : \phi \prec \rho\}$, where the superemum is taken over all UCP dilations ρ of ϕ. Furthermore, define $\eta := (L^2 - \|\phi(s_0)x_0\|^2)^{1/2}$ and

$$X := \{\psi \in \mathrm{UCP}(\mathcal{S}, \mathcal{B}(H \oplus \mathbb{C})) : \phi \prec \psi \text{ and } \psi(s_0)x_0 = \phi(s_0)x_0)x_0 \oplus \eta\}.$$

Notice the maximal dilation $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ given in the last lemma satisfies $\psi(s_0)x_0 = \phi(s_0)x_0 \oplus \eta$ and so X is nonempty. It is not hard to check that X is also a compact convex set in then BW-topology. By the Krien-Milman theorem, X is the convex hull of its extreme points. Thus we may choose an extreme point $\psi_0 \in X$. We claim that ψ_0 is pure and hence the required dilation.
Step 2: Dealing with invertibility issues.
Suppose $\psi_1 \in [0, \psi_0]$ and define $\psi_2 := \psi_0 - \psi_1$. Let $\varepsilon > 0$ be small enough such that $1 - 2\varepsilon > 0$. To ensure that $\psi_1(I), \psi_2(I)$ are invertible we define

$$\psi_i' := (1 - 2\varepsilon)\psi_i + \varepsilon\psi_0$$

for $i = 1, 2$. In order to prove that ψ_1 is a scalar multiple of ψ_0, it is enough to show ψ_1' is a scalar multiple of ψ_0. Indeed, if $\psi_1' = \lambda\psi_0$, then

$$\psi_1 = \frac{\lambda - \varepsilon}{1 - 2\varepsilon}\psi_0$$

where we can choose ε small enough so that $\lambda - \varepsilon > 0$.
Step 3: Properties of ψ_i'.
An easy calculation shows that $\psi_1' + \psi_2' = \psi_0$. Compressing each ψ_i' to H we get $P_H\psi_i'(\cdot) \mid_H \leq \phi$ for $i = 1, 2$. As ϕ is pure, there exists $\lambda_1, \lambda_2 > 0$ such that

$$P_H\psi_i'(\cdot) \mid_H = \lambda_i\phi(\cdot)$$

for $i = 1, 2$. Moreover as $P_H(\psi_1' + \psi_2') \mid_H = P_H\psi_0 \mid_H = \phi$, we get $\lambda_1 + \lambda_2 = 1$. Our aim will be to prove $\psi_1' = \lambda_1\psi_0$.
Define $Q_i := \psi_i'(I)$ and notice

$$Q_i = (1 - 2\varepsilon)\psi_i(I) + \varepsilon I \geq \varepsilon I$$

for $i = 1, 2$.
Step 4: Matrix forms of Q_i.
Writing Q_i as a matrix with respect to $H \oplus \mathbb{C}$ we can assume there exist $x_i \in H, \alpha_i \in \mathbb{C}$ such that

$$Q_i = \begin{bmatrix} \lambda_i & \lambda_i^{1/2}x_i \\ \lambda_i^{1/2}x_i^* & \alpha_i \end{bmatrix}$$

where $x_i^* : H \to \mathbb{C}$ defined via $x_i^*(\xi) = \langle \xi, x_i \rangle$ for $i = 1, 2$. Since $\psi_1' + \psi_2' = \psi_0$, we get $Q_1 + Q_2 = I$ and so $\alpha_1 + \alpha_2 = 1$. Moreover as Q_i is invertible, $\alpha_i > 0$ for $i = 1, 2$. Furthermore, $\lambda_1^{1/2} x_1 + \lambda_2^{1/2} x_2 = 0$. We can now decompose Q_i as

$$Q_i = \begin{bmatrix} \lambda_i^{1/2} & 0 \\ x_i^* & \beta_i \end{bmatrix} \begin{bmatrix} \lambda_i^{1/2} & x_i \\ 0 & \beta_i \end{bmatrix}$$

where $\beta_i = (\alpha_i - x_i^*(x_i))^{1/2}$ for $i = 1, 2$. Again by invertibility of Q_i, we obtain $\beta_i > 0$ for $i = 1, 2$. Define

$$\gamma_i = \begin{bmatrix} \lambda_i^{1/2} & x_i \\ 0 & \beta_i \end{bmatrix}$$

for $i = 1, 2$. So, $I = Q_1 + Q_2 = \gamma_1^* \gamma_1 + \gamma_2^* \gamma_2$ and it can be computed that

$$\gamma_i^{-1} = \begin{bmatrix} \lambda_i^{-1/2} & -\lambda_i^{-1/2} \beta_i^{-1} x_i \\ 0 & \beta_i^{-1} \end{bmatrix}$$

for $i = 1, 2$.

Step 5: Matrix forms of ψ_i'.

Given $s \in \mathcal{S}$ and $i = 1, 2$ define the map $\tau_i(s) = (\gamma_i^{-1})^* \psi_i'(s) \gamma_i^{-1}$. Writing this in matrix form we get

$$\tau_i(s) = \begin{bmatrix} \lambda_i^{-1/2} & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \lambda_i \phi(s) & * \\ * & * \end{bmatrix} \begin{bmatrix} \lambda_i^{-1/2} & * \\ 0 & * \end{bmatrix} = \begin{bmatrix} \phi(s) & S_i(s) \\ T_i(s) & f_i(s) \end{bmatrix}$$

where $S_i \in \mathcal{B}(\mathcal{S}, H), T_i \in \mathcal{B}(\mathcal{S}, H^*), f_i \in \mathcal{B}(\mathcal{S}, \mathbb{C})$ for $i = 1, 2$. As each τ_i is a completely positive dilation of ϕ we must have $\|\tau_i(s_0)x_0\|^2 \leq L^2$ for $i = 1, 2$. Therefore, $|T_i(s_0)x_0| \leq (L^2 - \|\phi(s_0)x_0\|^2)^{1/2} = \eta$ for $i = 1, 2$. Since for every $s \in \mathcal{S}$, $\psi_i'(s) = \gamma_i^* \tau_i(s) \gamma_i$ we obtain

$$\psi_i'(s) = \begin{bmatrix} \lambda_i^{1/2} & 0 \\ x_i^* & \beta_i \end{bmatrix} \begin{bmatrix} \phi(s) & S_i(s) \\ T_i(s) & f_i(s) \end{bmatrix} \begin{bmatrix} \lambda_i^{1/2} & x_i \\ 0 & \beta_i \end{bmatrix} = \begin{bmatrix} \lambda_i \phi(s) & * \\ \lambda_i^{1/2}(x_i^* \phi(s) + \beta_i T_i(s)) & * \end{bmatrix}$$

for $i = 1, 2$.

Step 6: Looking at the (2,1) entry of the matrices of ψ_i'.

By the choice of ψ_0 we obtain

$$\begin{aligned} \eta &= P_{\mathbb{C}} \psi_0(s_0) x_0 \\ &= P_{\mathbb{C}}(\psi_1'(s_0)x_0 + \psi_2'(s_0)x_0) \\ &= \lambda_1^{1/2}(x_1^* \phi(s) + \beta_1 T_1(s))x_0 + \lambda_2^{1/2}(x_2^* \phi(s) + \beta_2 T_2(s))x_0 \\ &= (\lambda_1^{1/2} x_1 + \lambda_2^{1/2} x_2)^* \phi(s_0)x_0 + \lambda_1^{1/2} \beta_1 T_1(s_0)x_0 + \lambda_2^{1/2} \beta_2 T_2(s_0)x_0. \end{aligned}$$

Since $\lambda_1^{1/2}x_1 + \lambda_2^{1/2}x_2 = 0$, we get

$$\eta = \lambda_1^{1/2}\beta_1 T_1(s_0)x_0 + \lambda_2^{1/2}\beta_2 T_2(s_0)x_0.$$

As $|T_i(s_0)x_0| \leq \eta$ and $\beta_i = (\alpha_i - x_i^*(x_i))^{1/2} \leq \alpha_i^{1/2}$ for $i = 1, 2$, we obtain that

$$\eta \leq (\lambda_1^{1/2}\alpha_1^{1/2} + \lambda_2^{1/2}\alpha_2^{1/2})\eta.$$

An application of the Cauchy-Schwarz Inequality and the fact that $\lambda_1 + \lambda_2 = \alpha_1 + \alpha_2 = 1$ gives us

$$\eta \leq (\lambda_1 + \lambda_2)^{1/2}(\alpha_1 + \alpha_2)^{1/2}\eta = \eta.$$

Hence, we must have $\beta_i = \alpha_i^{1/2}$, $x_i = 0$, and $|T_i(s_0)x_0| = \eta$ for $i = 1, 2$. Moreover, as the Cauchy-Schwarz Inequality was also an equality, the vectors $(\lambda_1^{1/2}, \lambda_2^{1/2})$ and $(\alpha_1^{1/2}, \alpha_2^{1/2})$ must be linearly dependent. As they both have positive entries with the same norm, we get equality of the two vectors. In particular, $\beta_i = \alpha_i^{1/2} = \lambda_i^{1/2}$ and $\gamma_i = \lambda_i^{1/2} I$ for $i = 1, 2$. Hence

$$\psi_i'(\cdot) = \gamma_i^* \tau_i(\cdot)\gamma_i = \lambda_i \tau_i(\cdot)$$

for $i = 1, 2$. If follows that $\psi_0 = \psi_1' + \psi_2' = \lambda_1\tau_1 + \lambda_2\tau_2$. Since $|T_i(s_0)x_0| = \eta$, it follows that $\tau_i \in X$ for $i = 1, 2$. Since ψ_0 was chosen to be an extreme point of X, we obtain that $\psi_0 = \tau_1 = \tau_2$. In particular, $\psi_1' = \lambda_1\tau_1 = \lambda_1\psi_0$. $\square$

We now use transfinite induction to show that we can always obtain a pure maximal dilation. We briefly recall the meaning of transfinite induction.
Successor ordinals are defined to be ordinals α such that $\alpha = \lambda + 1$ for some ordinal λ. Any ordinal which is not a successor ordinal is said to be a *limit ordinal*. Let $P(\lambda)$ be a property defined for any ordinal λ. Suppose the following three conditions holds.

1. $P(0)$ holds.
2. For any ordinal λ, $P(\lambda + 1)$ holds whenever $P(\lambda)$ does.
3. For any limit ordinal λ, if $P(\alpha)$ holds for all ordinal $\alpha < \lambda$, then $P(\lambda)$ also holds.

Then $P(\lambda)$ holds for all ordinals λ. The process of showing the above three conditions are true is called transfinite induction.

We shall need the following technical lemma.

Lemma 4.17. *Let Λ be an ordinal and $\mathcal{S}$ an operator system such that for every $\lambda < \Lambda$ there exists Hilbert space H_λ and unital completely positive map $\phi_\lambda : \mathcal{S} \to \mathcal{B}(H_\lambda)$. If for every $\lambda < \beta < \Lambda$ we have $\phi_\lambda \prec \phi_\beta$, then there exists unique completely positive map $\phi_\Lambda : \mathcal{S} \to \mathcal{B}(H)$ where H is the completion of $\cup_{\lambda<\Lambda} H_\lambda$ such that $\phi_\lambda \prec \phi_\Lambda$ for every $\lambda < \Lambda$. Furthermore, if we assume each ϕ_λ for $\lambda < \Lambda$ is pure, then ϕ_Λ is also pure.*

Proof. Notice for every $s \in \mathcal{S}$ define the operator $\phi_\Lambda(s) : \cup_{\lambda<\Lambda} H_\lambda \to H$ via

$$\phi_\Lambda(s)\xi := \lim_{\lambda<\Lambda} \phi_\lambda(s)\xi.$$

This is a well-defined operator as $\sup_{\lambda<\Lambda} \|\phi_\lambda(s)\xi\| \leq \|s\|\|\xi\|$. It is easily seen that ϕ_Λ is also a completely positive map. Since $\cup_{\lambda<\Lambda} H_\lambda$ is dense in H, we get that ϕ_Λ uniquely extends to a UCP map on H. By construction we get $\phi_\lambda \prec \phi_\Lambda$ for every $\lambda < \Lambda$. Uniqueness follows from the observation that any other UCP map $\psi : \mathcal{S} \to \mathcal{B}(H)$ satisfying $\phi_\lambda \prec \psi$ for every $\lambda < \Lambda$, would have to act the same on the dense subspace $\cup_{\lambda<\Lambda} H_\lambda$.

Suppose ϕ_λ is pure for every $\lambda < \Lambda$. Towards showing ϕ_Λ is pure, choose a completely positive map $\psi : \mathcal{S} \to \mathcal{B}(H)$ such that $\psi \leq \phi_\Lambda$. Compressing to each H_λ we see that

$$P_{H_\lambda} \phi(\cdot) \mid_{H_\lambda} \leq \phi_\lambda$$

for every $\lambda < \Lambda$. So there exists a scalar t_λ such that $P_{H_\lambda} \psi(\cdot) \mid_{H_\lambda} = t_\lambda \phi_\lambda$ for each $\lambda < \Lambda$. Therefore, $t_\lambda I = P_{H_\lambda} \psi(I) \mid_{H_\lambda}$ and so t_λ does not depend on $\lambda < \Lambda$. This forces $\psi = t\phi_\Lambda$ where $t = t_\lambda$ does not depend on $\lambda < \Lambda$. It follows that ϕ_Λ is pure. □

Theorem 4.18. *Let $\mathcal{S}$ be an operator system. If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure unital completely positive map, then there exists a pure maximal dilation ψ of ϕ.*

Proof. Consider, $b_1(\mathcal{S}) \times b_1(H)$, the product of the unit balls of $\mathcal{S}$ and H respectively. Enumerate a dense subset of $b_1(\mathcal{S}) \times b_1(H)$ via

$$\{(s_\lambda, x_\lambda) : \lambda < \Lambda\},$$

for some appropriate ordinal Λ. We claim the following holds. For every $\lambda < \Lambda$ there exists a pure unital completely positive map $\phi_\lambda : \mathcal{S} \to \mathcal{B}(H_\lambda)$ that is maximal on (s_λ, x_λ). Moreover, if $\gamma < \lambda < \Lambda$ we have $\phi_\gamma \prec \phi_\lambda$. We proceed via transfinite induction.

If ϕ is maximal at (s_0, x_0) define $\phi_0 := \phi$ or else let ϕ_0 be a pure unital completely positive 1-dimensional dilation of ϕ which is maximal at (s_0, x_0).

Suppose the claim holds for all $\gamma \le \lambda$ for some $\lambda + 1 < \Lambda$. Now if ϕ_λ is maximal at $(s_{\lambda+1}, x_{\lambda+1})$ define $\phi_{\lambda+1} := \phi_\lambda$; otherwise let $\phi_{\lambda+1}$ be a pure unital completely positive 1-dimensional dilation of ϕ_λ that is maximal at $(s_{\lambda+1}, x_{\lambda+1})$.

Suppose the claim holds for all $\gamma < \lambda$ where $\lambda < \Lambda$ is a limit ordinal. Applying the previous lemma we obtain a pure unital completely positive map $\widehat{\phi}_\lambda : \mathcal{S} \to \mathcal{B}(H'_\lambda)$ such that H'_λ is the completion of $\cup_{\alpha<\lambda} H_\alpha$ and $\phi_\alpha \prec \widehat{\phi}_\lambda$ for every $\alpha < \lambda$. If $\widehat{\phi}_\lambda$ is maximal at (s_λ, x_λ) let $\phi_\lambda := \widehat{\phi}_\lambda$; otherwise let ϕ_λ be a 1-dimension pure dilation which is maximal on (s_λ, x_λ).

Hence our claim holds and by another application of the previous lemma we obtain a pure maximal unital completely map $\phi_1 : \mathcal{S} \to \mathcal{B}(K_1)$ such that $\phi \prec \phi_1$ and ϕ_1 is maximal on $\{(s_\lambda, x_\lambda) : \lambda < \Lambda\}$. By continuity of ϕ_1, we obtain that ϕ_1 is maximal on $\mathcal{S} \times H$.

Repeating the above argument we obtain a sequence of pure unital completely positive maps $\phi_n : \mathcal{S} \to \mathcal{B}(K_n)$ such that $\phi_n \prec \phi_m$ for $n \le m$ and ϕ_{n+1} is maximal on $S \times K_n$. Once again applying the previous lemma we obtain a pure unital completely positive map $\phi_\infty : \mathcal{S} \to \mathcal{B}(K_\infty)$ such that $\phi_n \prec \phi_\infty$ for every $n \in \mathbb{N}$ and is maximal on $\cup_{n\in\mathbb{N}} K_n$. Lastly, by continuity of ϕ_∞ we obtain that ϕ_∞ is maximal on $S \times K_\infty$ and hence is the desired pure maximal dilation of ϕ. $\square$

Before proceeding to the main theorem, we introduce a result which allows us to produce a boundary representation on $\mathcal{S}$ whenever we are given one on $M_n(\mathcal{S})$. These results and a converse to the above statement can be found in [Hop73].

Let $\mathscr{A}$ be a unital C*-algebra, $n \in \mathbb{N}$, and $\rho : M_n(\mathscr{A}) \to \mathcal{B}(H)$ a representation. Define for each $a \in \mathscr{A}$, $E_{ij}(a)$ to be the matrix in $M_n(\mathscr{A})$ with a in the (i,j)-entry and zeroes elsewhere. Notice, for $i = 1, \dots, n$ we get that $P_i := \rho(E_{ii}(I))$ are orthogonal projections in $\mathcal{B}(H)$ such that $\sum_{i=1}^n P_i = I$.
Let $H_i := P_i H$ and it can be checked that $H_i \simeq H_j$ via $E_{ji}(I) : H_i \to H_j$ for all $i, j = 1, \dots, n$. In particular, we get $H \simeq H_1 \oplus \cdots \oplus H_n \simeq H_1^n$.
Consider a representation $\pi : \mathscr{A} \to \mathcal{B}(H_1)$ defined to be $\rho(E_{11}(\cdot))$ compressed via the projection P_1. It can now be checked that $\pi_n : M_n(\mathscr{A}) \to \mathcal{B}(H_1^n)$ is unitarily equivalent to ρ.
Now suppose $\mathcal{S} \subseteq \mathscr{A}$ is an operator system such that $C^*(\mathcal{S}) = \mathscr{A}$. Hence we also obtain $C^*(M_n(\mathcal{S})) = M_n(C^*(\mathcal{S})) = M_n(\mathscr{A})$. If ρ is a boundary representation for $M_n(\mathcal{S})$, we claim that π is a boundary representation for $\mathcal{S}$. Indeed, let ψ be a unital completely positive extension of $\pi(\cdot) \mid_{\mathcal{S}}$ to $\mathscr{A}$. It follows that ϕ_n is unital completely positive dilation of $\pi_n(\cdot) \mid_{M_n(\mathcal{S})}$. Since π_n is unitary equivalent to ρ, it must also be a boundary representation. Therefore, $\pi_n = \phi_n$ and hence $\pi = \phi$;

showing that π is a boundary representation.

Theorem 4.19. *Let $\mathcal{S}$ be an operator system and $n \in \mathbb{N}$. For every $s \in M_n(\mathcal{S})$ there exists a boundary representation π of $\mathcal{S}$ such that $\|\pi_n(s)\| = \|s\|$.*

Proof. It is enough to show that $\|\pi_n(x)\| = \|x\|$ for $x := ss^* \in M_n(\mathcal{S})$. Indeed, if $\|\pi_n(x)\| = \|x\|$, then

$$\|\pi_n(s)\|^2 = \|\pi_n(ss^*)\| = \|ss^*\| = \|s\|^2.$$

It is not hard, using functional calculus, to show there exists a multiplicative linear functional ϕ on $C^*(x)$ such that $\phi(x) = \|x\|$ (see [Dav96, page 32]). This functional extends to a state on $M_n(C^*(\mathcal{S}))$. It follows that

$$X := \{\phi \in S(M_n(C^*(\mathcal{S})) : \phi(x) = \|x\|\}$$

is a non-empty weak*-compact convex set. By the Krein-Milman theorem, X is the convex hull of its extreme points. Thus we may choose an extreme point $f \in X$. It can be easily checked that X is a face of the state space on $M_n(C^*(\mathcal{S}))$; meaning that for all states f_1, f_2 on $M_n(C^*(\mathcal{S}))$ if $\frac{1}{2}(f_1+f_2) \in X$, then $f_1, f_2 \in X$. It follows that f is a pure state. Therefore, by Theorem 4.18, there exists a boundary representation σ on $M_n(\mathcal{S})$ which is a dilation of f. Notice $\|\sigma(x)\| = \|x\|$ still holds. By the discussion preceding this theorem, we get that there exists a boundary representation π of $\mathcal{S}$ such that π_n is unitary equivalent to σ. It follows that $\|\pi_n(x)\| = \|x\|$. ☐

Corollary 4.20. *Let $\mathcal{S} \subseteq C^*(\mathcal{S})$ be an operator system. Then the Shilov boundary for $\mathcal{S}$ exists.*

Proof. By Theorem 4.19 it follows that $\mathcal{S}$ is an admissible operator system. Hence, by Theorem 4.12, we obtain that K, the intersection of the kernels of boundary representations for $\mathcal{S}$, is the Shilov boundary. ☐

Appendix

Another proof of the existence of the injective envelope

The following proof is due to Sinclair [Sin15]. We now introduce the required compact semigroup concepts required for the proof (see [FK89]).

Definition 4.21. A (left) *compact semigroup* is a semigroup S endowed with a compact Hausdorff topology such that for every $s \in S$, the map

$$S \ni x \mapsto x \cdot s$$

is continuous.

Theorem 4.22. *If S is a compact semigroup, then S contains an idempotent.*

Proof. By an application of compactness and Zorn's lemma there exists a minimal nonempty compact sub-semigroup $M \subseteq S$. Let $e \in M$ and consider $Me \subseteq M$. Due to continuity of the action Me is a compact semigroup and so by minimality $Me = M$. In particular

$$\emptyset \neq P = \{x \in M : xe = e\}.$$

Once again P is seen to be a compact semigroup and so $P = M$. Hence $e^2 = e$. □

We say a nonempty subset $J \subseteq S$ is a *left ideal* if J is closed and $SJ \subseteq J$. By Zorn's lemma, it follows that there always exists minimal left ideals.

We define a partial ordering on the set of idempotents of S as follows. Let x, y be idempotent in S. We say $x \prec y$ if and only if $xy = yx = x$.

Theorem 4.23. *Let S be a compact semigroup. If e is an idempotent of S, then there exists minimal idempotent $z \in S$ such that $z \prec e$.*

Proof. Notice Se is a left ideal. There exists minimal left ideal $J \subseteq Se$. Since J is also a compact semigroup there exists idempotent $\theta \in J$ where $\theta = se$ for some $s \in S$. It follows that $\theta e = se^2 = \theta$. Defining $z := e\theta$ it is clear that $z^2 = z$ and $z \prec e$.

Lastly we show z is a minimal idempotent. If $\gamma \prec z$, then $\gamma z = \gamma \in J$. By minimality of J we get $J\gamma = J$. Hence $z = j\gamma$ for some $j \in J$. In particular, $z\gamma = j\gamma^2 = j\gamma = z$. Also by assumption $z\gamma = \gamma$ and so $\gamma = z$. □

Theorem 4.24. *Let $V \subseteq \mathcal{B}(H)$ be an operator space. Then there exists an injective envelope of V.*

Proof. Let S be the collection of all completely contractive map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ such that $\phi(x) = x$ for every $x \in V$. Clearly S is a semigroup and we now show it is closed in the BW-topology. Let $(\phi_\lambda)_\lambda \subseteq S$ such that $\phi_\lambda \to \phi$ in the BW-topology. So for every $x \in V$, $\xi, \eta \in H$

$$\langle x\xi, \eta\rangle = \langle \phi_\lambda(x)\xi, \eta\rangle \to \langle \phi(x)\xi, \eta\rangle.$$

Hence $\phi(x) = x$ for every $x \in V$ and so $\phi \in S$. This also shows S is compact in the BW-topology as it is contained in the closed unit ball of $\mathcal{B}(\mathcal{B}(H), \mathcal{B}(H))$. Lastly for a fixed $\psi \in S$ consider the map

$$S \ni \phi \mapsto \phi \circ \psi.$$

To show continuity to the map consider $(\phi_\lambda)_\lambda$ in S converging in the BW-topology to ϕ. Now for every $x \in \mathcal{B}(H), \xi, \eta \in H$

$$\langle \phi_\lambda(\psi(x))\xi, \eta\rangle \to \langle \phi(\psi(x))\xi, \eta\rangle.$$

As the net $(\phi_\lambda \circ \psi)_\lambda$ is bounded this shows $\phi_\lambda \circ \psi \to \phi \circ \psi$ in the BW-topology. Hence S is a compact semigroup and so there exists minimal idempotent $\phi \in S$. Defining $E := \phi(\mathcal{B}(H))$, we see that E is the injective envelope of V. Indeed, consider an injective operator system K with $V \subseteq K \subseteq E$. Then there exists a completely contractive projection $\psi : \mathcal{B}(H) \to K$ onto K. Therefore $\psi \circ \phi = \phi \circ \psi = \psi$ and by minimality $\psi = \phi$, showing that $K = E$. □

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