Discrete Mathematics

GO Classes

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$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \sum \frac{x^i}{i!}$$

Imp formulae

$$1+2+3+4+\ldots\ldots n = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+4^2+\ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+4^3+\ldots\ldots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Sequence and series

1. Arithematic Series

n^{th} term, $a_n = a+(n-1)d$

Sum of n terms, $S_n = \frac{n}{2}[2a+(n-1)d]$

2. Geometric Series

n^{th} term, $a_n = ar^{n-1}$

Sum of n terms, $S_n = a\left(\frac{1-r^n}{1-r}\right)$

$S_\infty = \frac{a}{1-r}$

3. Harmonic series.

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} + \ldots \frac{1}{a+(n-1)d}$

n^{th} term, $a_n = \frac{1}{a+(n-1)d}$

Sum of n terms, $S_n = \frac{1}{d} \ln \left\{\frac{2a+(2n-1)d}{2a-d}\right\}$

ARITHEMATIC - GEOMETRIC PROGRESSION

A.P. :- $1, 2, 3, \ldots\ldots n$

G.P. :- $x, x^2, x^3, \ldots\ldots n$

$1x + 2x^2 + 3x^3 + \ldots\ldots nx^n$ A.G.P.

series obtained by multiplying corresponding terms of A.P. & G.P.

Ques:- $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

$$S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots nx^{n-1}$$

$$-\quad xS_n = x + 2x^2 + 3x^3 + 4x^4 \ldots\ldots\ldots nx^n$$

$$(1-x)S_n = 1 + x + x^2 + x^3 + \ldots\ldots x^{n-1} - nx^n$$

$$= 1\left(\frac{1-x^n}{1-x}\right) - nx^n$$

For infinite terms -

$$\therefore (1-x)S_\infty = \frac{1}{1-x} \qquad \therefore S_\infty = \frac{1}{(1-x)^2}$$

Ques:- $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots$

A.P. :- $1, 2, 3, 4 \ldots$

G.P. :- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

$$S_\infty = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots$$

$$\frac{1}{2}S_\infty = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots$$

$$S_\infty - \frac{1}{2}S_\infty = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$\left(1-\frac{1}{2}\right)S_\infty = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}}$$

$$\therefore S_\infty = \cancel{\frac{1}{2}} \times \frac{1}{\cancel{1/2}} \times \frac{1}{1/2} = 2$$

Ques:- $2 + 5x + 8x^2 + 11x^3 + \dots \infty$

$$S_\infty = 2 + 5x + 8x^2 + 11x^3 + \dots \infty$$

$$xS_\infty = 2x + 5x^2 + 8x^3 + 11x^4 + \dots \infty$$

$$S_\infty - xS_\infty = 2 + 3x + 3x^2 + 3x^3 + \dots \infty$$

$$(1-x)S_\infty = \cancel{2+3(x+} \; 2 + 3x(1 + x + x^2 + \dots \infty)$$

$$(1-x)S_\infty = 2 + 3x \cdot 1 \cdot \frac{1}{1-x} \Rightarrow (1-x)S_\infty = 2 + \frac{3x}{1-x} = \frac{2+x}{1-x}$$

$$\therefore S_\infty = \frac{2+x}{(1-x)^2}$$

Ques:- $1 - 3x + 5x^2 - 7x^3 + \dots \infty$

$$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \dots \infty$$

$$-xS_\infty = -x + 3x^2 - 5x^3 + 7x^4 + \dots \infty$$

$$(1+x)S_\infty = 1 - 2x + 2x^2 - 2x^3 + \dots \infty$$

$$(1+x)S_\infty = 1 - 2x(1 - x + x^2 + \dots \infty)$$

$$(1+x)S_\infty = 1 - 2x \cdot 1 \cdot \frac{1}{1+x} = \frac{1+x-2x}{1+x} = \frac{1-x}{1+x}$$

$$\Rightarrow S_\infty = \frac{1-x}{(1+x)^2}$$

Modular Arithematic

Divisibility –

If a and b are integers, a divides b if there exists an integer c such that

$$ac = b$$

$a | b \longrightarrow$ a divides b?

Division algorithm.

Let a be an integer and d be a positive integer. Then, there are unique integers q and r with $0 \le r < d$ such that

$$a = dq + r.$$

$$r = a \bmod d$$

★ Remainder should be greater than 0

$-11 \div 3 \rightarrow q = -4, \; r = 2$

★ If 2 numbers a and b give the same remainder when divided by m, then, we say a is congruent to b modulo m

$$a \equiv b \pmod{m}$$

$$a \% m = b \bmod m.$$

$$10 \equiv 15 \pmod{5}$$

$$\boxed{\begin{array}{l} a \bmod n = r \\ a \pm nk \bmod n = r \end{array}}$$

$a \equiv b \pmod{n}$ $\quad$ $a = nk_1 + r$

$b = nk_2 + r$

$\Rightarrow a - b = n(k_1 - k_2)$

$\boxed{\Rightarrow a-b \text{ is multiple of } n.}$ $\quad$ $n \mid a-b \rightarrow T$

$a \equiv b \pmod{n} \iff n \mid (a-b) \iff a \bmod n = b \bmod n$

$a \equiv b \pmod{n}$

$\Rightarrow ((((a \bmod n) \bmod n) \bmod n) \bmod n) \equiv b \pmod{n}$

Important theorems

If $a \equiv b \pmod{m}$ and
$c \equiv d \pmod{m}$ then,

$a + c \equiv b + d \pmod{m}$

$a * c \equiv b * d \pmod{m}$

$a - c \equiv b - d \pmod{m}$

★ Division does not hold !!

Ques

$994 \cdot 996 \cdot 997 \cdot 998 \pmod{1000}$

$= (994 \bmod 1000)(996 \bmod 1000)(997 \bmod 1000)(998 \bmod 1000)$

$= (-6 \bmod 1000)(-4 \bmod 1000)(-3 \bmod 1000)(-2 \bmod 1000)$

$= (-6)(-4)(-3)(-2) \bmod 1000$

$= 144 \bmod 1000$

$= 144$

Ques

$17^{753} \bmod 9$

$= 8^{753} \bmod 9 = (8 \bmod 9)^{753} \bmod 9$

$= (-1 \bmod 9)^{753} \bmod 9 = (-1)^{753} \bmod 9$

$= -1 \bmod 9 = \underline{\underline{8}}$

PROOF TECHNIQUES

- Disproof by counterexample.
- Exhaustive proof
- Direct proof
- Proof by ~~count~~ contraposition
- Proof by contradiction

1. Direct Proof

To proove :- If P then Q

Start with P (Assume)

↓ Apply facts that are already known.

Derive Q

If n is even, n^2 is even.

2. Proof by ~~count~~ contraposition

To proove :- If P then Q,

$P \Rightarrow Q$

$\overline{Q} \rightarrow \overline{P}$

Start with $\overline{Q}$.

↓ Apply facts

Derive $\overline{P}$

Ex → If n^2 is even, n is even

3. Proof by contradiction

If $P \Rightarrow Q$ ⟶ We need to proove that $\overline{P}$ is impossible

Proove $\sqrt{2}$ is irrational

Let $\sqrt{2} = \frac{a}{b}$ — a, b = no common factor other than 1

$2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is even $\Rightarrow a$ is even

Let $a = 2k$

$2b^2 = (2k)^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$

b^2 is even $\Rightarrow b$ is even

$\therefore a, b$ have 2 as factor

i.e. $GCD(a, b) \neq 1$

$\therefore$ assumption is wrong!

$\therefore \sqrt{2}$ is not rational.

$\therefore \sqrt{2}$ is irrational.

4. Disproof by counter example

Find an example where the assumption is false.

Logarithm

$$\log_b a^m = m\log_b a$$

$$\log_b mn = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$a = b^{\log_b a}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^{\log_b c} = c^{\log_b a}$$

Discrete Mathematics

- Mathematical Logic (37) + (39)
- Set Theory (62)
- Combinatorics (27+)
- Graph Theory (20+)
- Group Theory (40+)

⊙ field of mathematics where we study discrete objects.

1. Mathematical Logic

⊙ Language for Computers / AI / Automated Reasoning

⊙ Mathematical logic gives precise, unambiguous meaning to mathematical statements / theorems, etc.

- Propositional Logic
- First Order Logic.

→ Propositional logic

⊙ simplest logic

⊙ variable can be either $T_{(1)}$ or $F_{(0)}$

Proposition :-

declarative sentence that can be either true or false (cannot be both)

commands, questions, paradoxes, free variable } not propositions

Propositional variable :-

- each proposition is represented by a propositional variable.
- denoted by lower case letters
- each variable can be either T or F (not both)

Atomic Proposition :- proposition whose truth or falsity does not depend on the truth or falsity of any other proposition

⊙ atomic proposition is represented by proposition variable.

Compound Proposition :- propositions formed by combination of one or more atomic propositions using logical connectives

Standard Logical Connectives

- NOT ($\neg P \equiv \sim p \equiv \bar{p} \equiv p'$)
- AND ($\cdot$, $\wedge$)
- OR ($+$, $\vee$)
- Exclusive OR ($\oplus$)
  - OR :- but not both
- Implication ($\rightarrow$)
- Biimplication ($\leftrightarrow$)
- NAND ($\uparrow$)
- NOR ($\downarrow$)

(And = But = Although = however = moreover = yet
still = even though = though = nevertheless)

→ translate to conjunction

$p \rightarrow q$

- p → Condition / Antecedent / Hypothesis / Premise
- q → Conclusion / Consequence

P is necessary for Q $=$ $Q \rightarrow P$

P is sufficient for Q $=$ $P \rightarrow Q$

P is sufficient for Q = Q is necessary for P

$A \leftrightarrow B = A \rightarrow B$ AND $B \rightarrow A$

★ $P \leftrightarrow Q = \overline{P \oplus Q}$

① Implication = Property
② Biimplication = Definition

$P \Leftrightarrow Q$
- P biimplication Q
- P implies Q & Q implies P
- P implies Q & vice versa
- Q if and only if P.
- P is sufficient and necessary for Q

Propositional Logic

collection of ~~p~~ all propositional formulae.

Precedence -

$\neg > \wedge > \vee > \rightarrow > \leftrightarrow$

$T \rightarrow Q = Q$

$F \rightarrow Q = T$

$P \rightarrow T = T$

$P \rightarrow F = \neg P$

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Propositional formula –
- Symbols T and F are propositional formulae.
- Every propositional variable is propositional formula.
- Propositional formulae connected by logical connectives are propositional formula.

Truth Table

Truth Table tells about the truth value of a compound proposition for each combination of truth value of atomic propositions.

Tautology :- compound proposition which is always TRUE for all truth values of atomic propositions.

Contradiction :- compound proposition which is always (Fallacy) FALSE for all truth values of atomic proposition

Unsatisfiable

Contingency :- compound proposition which is true for some truth values and false for some.

Satisfiable :- compound statement can be made true for atleast one combination of truth values.

~~Unsatisfiable :- compound statement can be made false for atleast one combination of truth value.~~

Valid = Tautology = always true

Invalid :- sometimes false = contradiction / contingency.

(consistent)
Satisfiable = sometimes true = tautology / contingency

Unsatisfiable = contradiction = always false.
(inconsistent)

Falsifiable = sometimes false

not falsifiable = always true = tautology = valid.

Imp. –

1. always True
   - → Tautology
   - → Valid
   - → Logically true
   - → Unfalsifiable.

   satisfiable

2. sometimes true, sometimes false
   - → contingency
   - → satisfiable
   - → falsifiable
   - → invalid.

3. always false
   - → unsatisfiable
   - → logically false

By case method –

Create 2 cases –

(I) P = True (II) P = False

Other variables can be either T or F.

simplify the expression for each case.

Ques

Show that $p \wedge \neg(q \vee p)$ is contradiction

Case I :- P is True

$T \wedge \neg(q \vee T)$

$T \wedge \neg T$

$T \wedge F$

F

Case II :- P is False

$F \wedge \neg(q \vee F)$

F

$\therefore$ The proposition is false contradiction

$(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

Case I :- P is True

$(\neg(T \wedge q)) \leftrightarrow (\neg T \vee \neg q)$

$\neg q \leftrightarrow \neg q$

True

Case II :- P is False

$(\neg(F \wedge q)) \leftrightarrow (\neg F \vee \neg q)$

$(\neg F) \leftrightarrow (T \vee \neg q)$

$T \leftrightarrow T$

$= T$

$\therefore$ tautology

$(p \vee q) \leftrightarrow (q \vee p)$

Case I :- p is T

$(T \vee q) \leftrightarrow (q \vee T)$

$T \leftrightarrow T$

$\underline{\underline{T}}$

Case II :- p is F

$(F \vee q) \leftrightarrow (q \vee F)$

$q \leftrightarrow q$

$\underline{\underline{T}}$

tautology

Propositional expressions : α, β

$\alpha \equiv \beta$ (α is equivalent to β) iff

- They have same truth table
- always equal truth values
- $\alpha \leftrightarrow \beta$ is tautology
- $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ are tautologies

$P \rightarrow Q$
Converse :- $Q \rightarrow P$
Contrapositive :- $\neg Q \rightarrow \neg P$
Inverse :- $\neg P \rightarrow \neg Q$

AND = But = Although = Though = Even Though = However = Yet = Still = Moreover = Nevertheless = Nonetheless = comma (12)

$P \to Q$

- If P then Q
- Q if P
- Q <u>whenever</u> P
- Q is necessary for P
- Q <u>provided that</u> P
- Whenever P then also Q
- P is sufficient for Q
- P only if Q. / Q only when

$P \leftrightarrow Q$

- p is necessary and sufficient for q.
- If p then q and conversely
- p iff q.

★ P <u>unless</u> Q = If not Q, then, P

$\neg Q \to P$

$\neg P \to Q$

unless = if not

provided that = if

P unless Q = $\neg Q \to P$

= $Q + P$ = OR

∴ unless = OR

If p then q unless r

$\equiv (p \to q)$ unless r

$\equiv \neg r \to (p \to q)$

Logical Laws / Logical Identities

$\neg(\neg p) = p$

Domination ~~True~~ Laws

$p \vee T = T$

$p \wedge F = F$

Negation Laws

$A \vee \bar{A} = T$

$A \wedge \bar{A} = F$

Identity Law

$p \wedge T = p$

$p \vee F = p$

Idempotent law

$p \vee p = p$

$p \wedge p = p$

(only $\vee, \wedge$)

De Morgan's law

$\overline{p \vee q} = \bar{p} \wedge \bar{q}$

$\overline{p \wedge q} = \bar{p} \vee \bar{q}$

Absorption law

$p \vee pq = \cancel{p \vee q}\ p$

$p \wedge (p \vee q) = p$

Implication laws

$p \to q = \bar{p} \vee q$

$p \to q = \bar{q} \to \bar{p}$

Commutative law

$p \vee q = q \vee p$

$p \wedge q = q \wedge p$

[$\wedge, \vee, \leftrightarrow, \oplus, \uparrow, \downarrow$ commutative operators]

Associative law

$(p \vee q) \vee r = p \vee (q \vee r)$

$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

[$\wedge, \vee, \leftrightarrow, \oplus$ associative] [$\to, \uparrow, \downarrow$ not associative]

Distributive law

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

[$\vee$ is distributive over $\wedge$
$\wedge$ is distributive over $\vee$]

$p \to (q \wedge r) = (p \to q) \wedge (p \to r)$

[$\to$ is distributive over $\wedge$]

$p \wedge (q \to r) = (p \wedge q) \to (p \wedge r)$

[$\wedge$ is distributive over $\to$]

$p \to (q \vee r) = (p \to q) \vee (p \to r)$

$p \vee (q \to r) = (p \vee q) \to (p \vee r)$

[$\to$ is distributive over $\vee$]

[$\vee$ is distributive over $\to$]

Formal definition of valid argument

Argument $\left\{ \begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline C \end{array} \right.$ $\left. \begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \end{array} \right\}$ Premises, C } Conclusion

The argument is true iff $P_1 \wedge P_2 \wedge P_3 \wedge \dots P_n \rightarrow C$ is tautology

To check if argument is valid or not –

↦ Make conclusion false and try to make all the premises true.

If possible, argument is invalid

If not, argument is valid.

$$\begin{array}{c} A \rightarrow B \\ A \\ \hline B \end{array}$$

argument is very popular known as <u>modus ponens</u>

(1) $\begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array}$ modus ponens

(2) $\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \neg P \end{array}$ modus Tollens

(3) $\begin{array}{c} p \\ q \\ \hline p \end{array}$ Conjunctive simplification.

(4) $\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$ Disjunctive syllogism.

(5) $\begin{array}{c} A \rightarrow B \\ B \rightarrow C \\ \hline A \rightarrow C \end{array}$ Transitivity of implications Hypothetical syllogism.

(6) $\begin{array}{c} p \\ \hline p \vee q \end{array}$

(7) $\begin{array}{c} p \wedge q \\ \hline p \end{array}$

(8) $\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$ Resolution

Inference symbol

KB = Knowledge Base (set of premises)

$KB \models Y$ means Knowledge base infers Y

$\models$ → logically infers

means $\frac{K.B.}{Y}$

$P_1, P_2, P_3 \models C$ i.e. $P_1 \wedge P_2 \wedge P_3 \rightarrow C$

$\not\models$ → does not infer $\qquad$ $\models$ infers

infers = entails = implies

C is consequence of K.B. $\qquad KB \models C$

First Order Logic / Predicate calculus

All men are mortals. Socrate is a man
∴ Socrate is mortal.

No way to express the above argument with propositional logic.

Propositional logic ~~have~~ has ~~very~~ limited expressive power.

∴ first order logic is needed

First Order Logic –

A world of objects, their properties, their relationships, their transformation (function)

Properties

- → Each variable refers to some objects in a set called the domain of discourse
  - set of all possible values.

Properties
- Domain
- Quantifier
- Predicate

- → FOL is also concerned with properties of these objects.
- → In FOL, we also have relations over/between/among objects (called Predicates)

* unary relations/predicate are called properties.

$P(x, y) : x < y$.

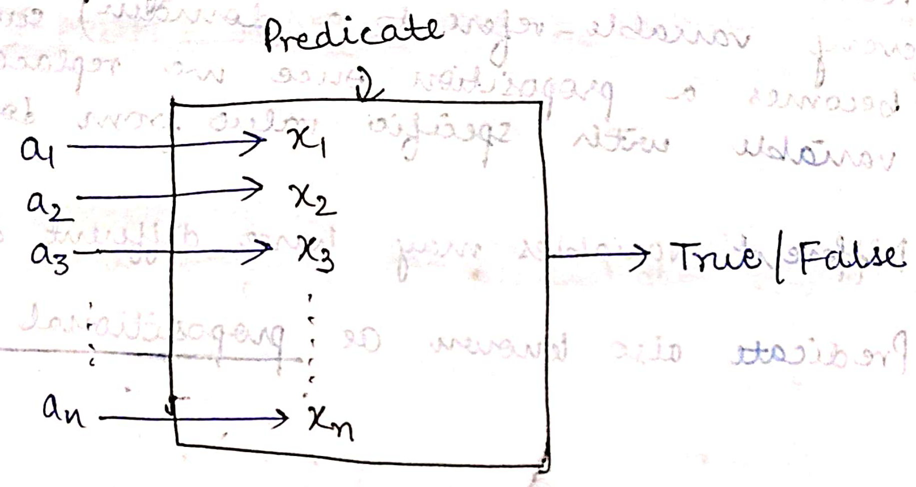
* → Predicate tells about properties of object / relation between objects.

* Once we replace every variable with some value from their domain, then, it becomes <u>proposition</u>.

* Predicate with 0 variable = 0-ary predicate = proposition

Predicate with 1 variable = unary predicate

Predicate with 2 variables = binary predicate



n-ary predicate.

→ Quantification words can be represented in FOL.

all, some

for all → $\forall$

for all x, $P(x)$ is true $= \forall_x P(x)$

$\forall_x P(x) = P(a) \wedge P(b) \wedge P(c) \wedge P(d)$ $\quad D(x) = \{a, b, c, d\}$

there exists → $\exists$

$\exists_x P(x)$ → There is atleast one/some x for which $P(x)$ is true.

Predicate -

Predicate tells us about the properties of objects and relationship among objects.

$S(x)$: x is a student (unary predicate)

$F(x, y)$: x and y are friends (binary predicate)

- Predicate is a sentence containing variables (where every variable refers to a domain) such that it becomes a proposition once we replace each variable with specific value from domain.

- Different variables may have different domains.

- Predicate also known as propositional function.

★→ Domain in FOL is ALWAYS non empty; unless explicitly stated.

★→ By default, domain of every variable is same

★ When predicate is quantified, it becomes proposition.

$E(x)$ = x is even → predicate

There exists x in the domain, $E(x)$ is true → proposition.

(Quantification words → there exists, some, forall)

Quantification words in english —

few, all, many, some, any ...

Quantification words in FOL —

All ≡ every

some ≡ atleast one

↳ 2 quantifiers —

① for all ≡ every ≡ universal quantifier

② there exists ≡ atleast one ≡ existential quantifier.

* Quantifier is a way of creating proposition from ~~som~~ predicate.

Quantification of a property :-

Quantification of a property is talking about if a property is satisfied by multiple objects.

Universal quantification

- Universal quantification means saying that a property P is satisfied by all elements in the domain.
- i.e. all elements in the domain satisfy property P.

$\forall$: universal quantifier symbol
(read as 'for all')

- For all elements x in the domain, P(x) is true

$$\forall_{x \in D} (P(x)) \equiv \forall_x (P(x))$$ [$x \in D$ is implicit]

- For finite set with domain, $D = \{a, b, c, d\}$

$$\forall_x P(x) = P(a) \wedge P(b) \wedge P(c) \wedge P(d)$$

universal quantifier is conjunction of all elements.

Existential quantification

- Existential quantification means saying that there exists atleast one element in domain for which the property P is true.

- some element in the domain satisfies the property.

$\exists$: existential quantifier symbol
(read as 'there exists')

There exists atleast one element for which P(x) is true

$$\exists_x P(x) \equiv \exists_{x \in D} P(x)$$

Example - Domain : N

$$\exists_x \left(even(x) \wedge prime(x) \right) \rightarrow \text{True} \quad (x=2)$$ IMP

⊙ Finite set with domain $D = \{a, b, c, d\}$

$$\exists_x P(x) = P(a) \vee P(b) \vee P(c) \vee P(d)$$

existential quantification is a disjunction of all elements.

1. When domain is empty,

→ ⊙ universal quantifier statement is always true. (bcoz no counterexample)

$$\forall_x \alpha(x) = T$$

→ ⊙ existential quantifier statement is always false (bcoz no witness)

$$\exists_x \beta(x) = F.$$

2. When there is no free variable.

→ ⊙ $\forall_x P(x) = P(x)$

[bcoz ∵ free variable is not present, ∴ P(x) is proposition]

Ex. $\boxed{\begin{array}{l} P: 2+2=6 \\ \forall_x P = \text{false} \end{array}}$

no free variable

→ ⊙ $\exists_x P(x) = P$

3. Predicate with no variable is proposition

IMP ★ Domain : set of all animals

1. Every rabit is cute

$$\forall_x (\text{Rabbit}(x) \rightarrow \text{cute}(x))$$ ✓

$$\forall_x (\text{Rabbit}(x) \wedge \text{cute}(x))$$ ✗

↳ (bcoz it means ~~for~~ ~~at~~ every element is rabbit & cute)

2. Some rabbit is cute.

$$\exists_x (\text{Rabbit}(x) \wedge \text{cute}(x))$$ ✓

$$\exists_x (\text{Rabbit}(x) \rightarrow \text{cute}(x))$$ ✗

↳ (bcoz it is true for those animals which are not rabbit)

Useful intuition

All P's are Q's $\equiv$ $\forall_x (P(x) \rightarrow Q(x))$

Some P's are Q's $\equiv$ $\exists_x (P(x) \wedge Q(x))$

[with universal quantifier, we use implication. with existential quantifier we use conjunction]

$$\exists_x (R(x) \rightarrow H(x)) = \exists_x (\overline{R(x)} \vee H(x))$$

There is some animal which is either not a rabbit or it hops.

1. All P's are Q's

$\forall_x \left(P(x) \rightarrow Q(x) \right) \equiv$

P=Rabbit
Q= Cute.

2. Some Ps are Qs

$\exists_x \left(P(x) \wedge Q(x) \right)$

3. No Ps are Qs

$\equiv$ All p's are not Qs

$\equiv$ It is not the case that some Ps are Qs

$\forall_x \left(P(x) \rightarrow \neg Q(x) \right)$

$\neg \exists_x \left(P(x) \wedge Q(x) \right)$

4. Some Ps are not Qs

$\exists_x \left(P(x) \wedge \neg Q(x) \right)$

5. Only A's are B's / Every A is B

$\forall_x \left(\neg A(x) \rightarrow \neg B(x) \right)$

$\forall_x \left(B(x) \rightarrow A(x) \right)$

7. Not all As are B's

$\exists_x \left(A(x) \wedge \neg B(x) \right)$

6. All and only A's are B's

$\forall_x \left(A(x) \Leftrightarrow B(x) \right)$

7. $\forall_{P(x)} Q(x) \equiv \forall_x \left(P(x) \rightarrow Q(x) \right)$

8. $\exists_{P(x)} \left(Q(x) \right) \equiv \exists_x \left(P(x) \wedge Q(x) \right)$

IMPORTANT

No A is B

Not all A's are B's

* $\neg \forall_x (A(x) \to B(x)) \equiv \exists_x (A(x) \land \neg B(x))$

* Every student loves someone

Love(x, y) : x loves y.

$\forall_x (\text{student}(x) \to \exists_y (\cancel{\text{stu}}\ \text{Love}(x,y)))$

Bounded Variable

$\forall_x$ Even(x) [x is bounded by ~~domain~~ quantifier

proposition ← $\forall_x P(x)$

↳ Bounded variable / Quantified variable / Dummy variable

We cannot replace x by any value from domain. $\forall_3 P(3)$ doesn't make sense

Free Variable

$E(x) = x$ is even — not a proposition

↳ free variable

x is free to take any value from domain.

* also known as real variable

$\forall_x P(x)$ — proposition

Bounded

$\exists_x Q(x)$ — proposition

Bounded

$E(x)$ — not proposition

free.

$N(x) \longrightarrow$ free variable

$\forall_x N(x) \longrightarrow$ bounded variable

Free variable
- Not bounded by any quantifier
- free to take any value from domain

If any expression does not contain free variable, it is proposition

Domain : N

S: $\forall_x (x > y)$ $\quad S(y) = \forall_x (x>y)$
- $x \to$ Bounded variable
- $y \to$ Free variable

[Not a proposition]

$\forall_y \left(\underbrace{\forall_x (x > y)}_{\text{every natural no. is greater than } y} \right)$ } proposition.

false proposition.

$\exists_y \left(\underbrace{\forall_x (x > y)}_{S(y)} \right) = \underbrace{S(1)}_{F} \vee \underbrace{S(2)}_{F} \vee \underbrace{S(3)}_{F} \vee \underbrace{S(4)}_{F} \ldots$

False proposition.

Convert predicate to proposition —
- Replace free variable with some value from domain
- quantify free variable.

Bounded v/s Free variable

↦ A bounded variable is a variable that is subject to a quantifier. A variable that is not bound is called free variable.

↦ A proposition can only contain bounded variables, no free variables.

Important NOTE

If A doesn't have any free variable x, then,

$\forall_x A \equiv A$

$\exists_x A \equiv A$

$\exists_x P(x) : A$ → no free variable → proposition

$\forall_x A \equiv A$

$\exists_x A \equiv A$

i.e. applying quantifier on a proposition gives same value as that of the proposition

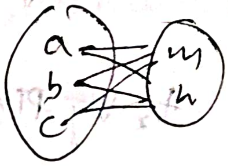
Scope of quantifier

The part of a logical expression to which a quantifier is applicable (applied) is called scope of the quantifier

$\forall_x [\, . \; x \; \ldots \; x \ldots]$

Types of nested quantifiers

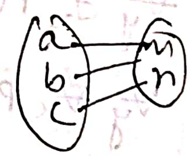
① $\forall_x \forall_y \; P(x, y)$



$x \in \{a, b, c\} \qquad y \in \{p, q, r\}$

$$\forall_x \forall_y P(x,y) = P(a,p) \wedge P(a,q) \wedge P(a,r) \wedge P(b,p) \wedge P(b,q) \wedge P(b,r) \wedge P(c,p) \wedge P(c,q) \wedge P(c,r)$$

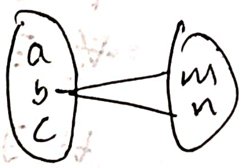
② $\forall_x \; \exists_y \; P(x,y)$



$x \in \{a, b, c\} \qquad y \in (m, n)$

$$(P(a,m) \vee P(a,n)) \wedge (P(b,m) \vee P(b,n)) \wedge (P(c,m) \vee P(c,n))$$

③ $\exists_x \; \forall_y \; (P(x,y))$



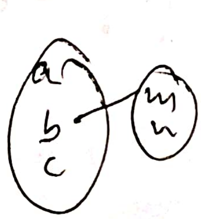
$x \in \{a, b, c\} \qquad y \in (m, n)$

?

$\cancel{P(a,m) \wedge P(b,m) \wedge P(c,m}$

$$(P(a,m) \wedge P(a,n)) \vee (P(b,m) \wedge P(b,n)) \vee (P(c,m) \wedge P(c,n))$$

④ $\exists_x \exists_y \; P(x,y)$



$x \in \{a, b, c\} \qquad y = \{m, n\}$

$$P(a,m) \vee P(a,n) \vee P(b,m) \vee P(b,n) \vee P(c,m) \vee P(c,n)$$

$x \rightarrow$ set of students
$y \rightarrow$ set of courses

$P(x,y) \rightarrow x$ has taken course y

$\forall_x \forall_y P(x,y) \rightarrow$ Every student has taken every course

$\forall_x \exists_y P(x,y) \rightarrow$ Every student has taken atleast one cou

$\exists_x \forall_y P(x,y) \rightarrow$ Some student has taken all the courses

$\exists_x \exists_y P(x,y) \rightarrow$ Some student has taken some course.

$\forall_y \exists_x P(x,y) \rightarrow$ Every course is taken by atleast one stud

$\forall_y \forall_x P(x,y) \rightarrow$ Every course is taken by every student

$\exists_y \forall_x P(x,y) \rightarrow$ Some course is taken by ~~eve~~ every stud

$\exists_x \exists_x P(x,y) \rightarrow$ Some course is taken by some student.

* Order of quantifiers matters for different quantifiers

$$\forall_x \exists_y \alpha \neq \exists_x \forall_y \alpha$$
$$\forall_x \exists_y \alpha \neq \exists_y \forall_x \alpha$$

* In case of same quantifiers, order does not matter

$$\forall_x \forall_y \alpha = \forall_y \forall_x \alpha$$
$$\exists_x \exists_y \alpha = \exists_y \exists_x \alpha$$

English to FOL translations –

① ~~There~~ All large cubes are nice $\quad \forall_x (\text{large}(x) \wedge \text{cube}(x) \to \text{nice}(x)$

② There is atleast one cube $\quad \exists_x (\text{cube}(x))$

③ There are atleast 2 cubes $\quad \exists_x \exists_y (\text{cube}(x) \wedge \text{cube}(y) \wedge (x \neq y$

④ There are atleast 3 cubes

$$\exists_x \exists_y \exists_z (\text{cube}(x) \wedge \text{cube}(y) \wedge \text{cube}(z) \wedge x \neq y \wedge y \neq z \wedge x \neq z)$$

⑤ There is atmost one cube

~~$\exists_x \exists_y (\text{cube}(x)$~~ $\quad$ [no cube / 1 cube]

$$\forall_x \forall_y ((\text{cube}(x) \wedge \text{cube}(y)) \to x = y)$$

⑥ There is exactly one cube

atleast one cube $\wedge$ atmost one cube

① $\exists_x (\text{cube}(x)) \wedge \forall_x \forall_y ((\text{cube}(x) \wedge \text{cube}(y)) \to x = y)$

OR

② $\exists_x (\text{cube}(x) \wedge \forall_y (\text{cube}(y) \to x = y))$

$\exists_x \forall_y (\text{cube}(x) \wedge (\text{cube}(y) \to x = y))$

③ $\exists_x \forall_y (\text{cube}(y) \longleftrightarrow x = y)$

⑦ There are atmost ~~3~~ 2 cubes

$$\forall_x \forall_y \forall_z \left(cube(x) \wedge cube(y) \wedge cube(z) \rightarrow (x=y) \vee (y=z) \vee (x=z) \right)$$

⑧ There are exactly 2 cubes

atmost 2 cubes $\wedge$ atleast 2 cubes

$$\Big(\forall_x \forall_y \forall_z \left(cube(x) \wedge cube(y) \wedge cube(z) \rightarrow x=y \vee y=z \vee x=z \right) \wedge$$
$$\exists_x \exists_y \left(cube(x) \wedge cube(y) \wedge x \neq y \right) \Big)$$

OR

$$\exists_x \exists_y \Big(cube(x) \wedge cube(y) \wedge x \neq y \wedge$$
$$\forall_z \left(cube(z) \rightarrow (x=z) \vee (y=z) \right) \Big)$$

x is prime —

$$x>1 \wedge (\forall_y (y \mid x \rightarrow (y=1) \vee (y=x)))$$

Negation of quantifier

$S = \forall_x P(x)$ $\qquad \neg S = \neg \forall_x P(x) = \exists_x(\neg P(x))$

$S = \exists_x P(x)$ $\qquad \neg S = \neg \exists_x P(x) = \forall_x(\neg P(x))$

Ex 1.

$$\neg \forall_x \exists_y (P(x) \rightarrow Q(y))$$

$$= \exists_x (\neg \exists_y (P(x) \rightarrow Q(y)))$$

$$= \exists_x \forall_y (\neg (P(x) \rightarrow Q(y)))$$

$$= \exists_x \forall_y (P(x) \wedge \neg Q(y))$$

$$\left[\neg (P \rightarrow Q) = \overline{\overline{P} \vee Q} = \overline{\overline{P}} \wedge \overline{Q} = P \wedge \overline{Q}\right]$$

Ex.2.

$$\neg \exists_x \forall_y (P(x) \wedge Q(y))$$

$$= \forall_x (\neg \forall_y (P(x) \wedge Q(y)))$$

$$= \forall_x \exists_y (\neg (P(x) \wedge Q(y)))$$

$$= \forall_x \exists_y (\neg P(x) \vee \neg Q(y))$$

Valid FOL Expression

$\forall_x P(x) \longrightarrow \exists_x P(x)$

$\forall_x (P(x) \wedge Q(x)) \longrightarrow \forall_x P(x) \wedge \forall_x Q(x)$

$\forall_x P(x) \wedge \forall_x Q(x) \longrightarrow \forall_x (P(x) \wedge Q(x))$

$\forall_x (P(x) \wedge Q(x)) \longrightarrow \forall_y P(y)$

$\forall_x (P(x)) \vee \forall_x Q(x) \longrightarrow \forall_x (P(x) \vee Q(x))$

Distribution of quantifiers over logical connectives

$\underbrace{\Pi_x (P(x) \# Q(x))}_{\text{Compact expression}} \underset{\text{valid?}}{\overset{\text{valid?}}{\rightleftarrows}} \underbrace{\Pi_x (P(x)) \# \Pi_x Q(x)}_{\text{Expanded expression}}$

Π: quantifier $(\exists, \forall, \exists!)$

$\exists!$: uniqueness quantifier (there exists one & only one)

$\#$: logical connective

	$\forall$	$\exists$
$\wedge$	$\leftrightarrow$	$\rightarrow$
$\vee$	$\leftarrow$	$\leftrightarrow$
$\rightarrow$	$\rightarrow$	$\leftarrow$
$\leftrightarrow$	$\rightarrow$	X
$\oplus$	X	$\leftarrow$
$\uparrow$	$\rightarrow$	$\leftarrow$
$\downarrow$	$\rightarrow$	

$\rightarrow$: Compact to expanded

$\leftarrow$: Expanded to compact

X : both aren't valid

$\leftrightarrow$: both sides are valid.

Null quantification rule

Distribution of quantifiers over logical connectives, when some expression isn't affected by quantifiers.

$\equiv$ Some part has no free variable.

A: has no free variable

$\forall_x A = A$, $\exists_x A = A$

[By case method]

① $\forall_x (P(x) \vee A) \equiv \forall_x P(x) \vee A$

② $\exists_x (P(x) \vee A) \equiv \exists_x P(x) \vee A$

③ $\forall_x (P(x) \wedge A) \equiv \forall_x P(x) \wedge A$

④ $\exists_x (P(x) \wedge A) \equiv \exists_x P(x) \wedge A$

⑤ $\forall_x (A \rightarrow P(x)) \equiv A \rightarrow \forall_x P(x)$

⑥ $\exists_x (A \rightarrow P(x)) \equiv A \rightarrow \exists_x P(x)$

* ⑦ $\forall_x (P(x) \rightarrow A) \equiv \exists_x P(x) \rightarrow A$

* ⑧ $\exists_x (P(x) \rightarrow A) \equiv \forall_x P(x) \rightarrow A$

Interpretation $\neq \phi$

The values of variables (true/false) is called interpretation.

For n variables,

no of interpretations $= 2^N$

Model

Interpretations for which the formula evaluates to true are called models.

Co model

Interpretations for which formula evaluates to false

$$\frac{\forall_x P(x)}{P(b)}$$ universal instantiation

$$\frac{\text{for arbitrary } n \in D, P(n)}{\forall_x P(x)}$$ universal generalization

$$\frac{P(a)}{\exists_x P(x)}$$ Existential generalization

$$\frac{\exists_x P(x)}{\therefore P(c) \text{ for some element } c}$$ Existential instantiation

Set Theory

Set – collection of objects is called set.
Set is represented by $\{\ \}$.

Ex:- $\{a, b, c, d, e\}$ = English vowels

⊙ order of ~~colle~~ elements does not matter
⊙ no duplicate elements

→ unordered collection of distinct elements.

$S = \{\{a\}, \{a, b\}, c, d\}$ 4 elements in S.

P: ϕ = empty set $|P| = 0$
Q: $\{\phi\}$ = set containing 1 element $|Q| = 1$
$\phi \neq \{\phi\}$

$|S| = \infty$
if S is infinite set
(set of all integers)

Set Representations –
- → Verbal representation
- → Roster (list) representation
- → Venn diagram representation
- → Set Builder Representation

Subset

ϕ is the subset of every set
Every set is subset of itself
$\phi \subseteq \phi$

If n elements, no. of subsets $= 2^n$

If n elements, no. of proper subsets $= 2^n - 1$.

Power set –

The set of all subsets of a set S is called power set of

$S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,$

$|P(S)| = 2^n = 2^{|S|}$

$|P(|P(S)|)| = 2^{|P(S)|} = 2^{2^{|S|}}$

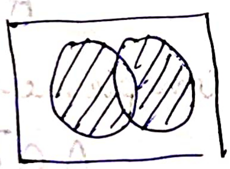
$U =$ Universal set (contains everything)

$\phi =$ Empty set (contains nothing)

Operations on set

Union, Intersection, Difference, Complement

$\Delta =$ symmetric difference (exclusive OR)



Disjoint sets :- Intersection of sets is ϕ.

$A \,\Delta\, B = (A - B) \cup (B - A)$

$A \,\Delta\, B = (A \cup B) - (A \cap B)$

$A - B = A \cap \overline{B}$

$\overline{A} \cup B = \overline{A - B}$

$\overline{A} - B = \overline{A \cup B}$

Equal sets –

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

Set Identities

① Identity law –

$A \cup \phi = A$
$A \cap U = A$

② Domination law –

$A \cup U = U$

$A \cap \phi = \phi$

③ Idempotent law

$A \cup A = A$
$A \cap A = A$

④ Complementation law

$\overline{\overline{A}} = A$

⑤ Complement law

$A \cap \overline{A} = \phi \qquad A \cup \overline{A} = U$

If $S \subseteq A$ and $S \subseteq B$ then,

$S \subseteq A \cap B.$

Ordered pair –

pairs of elements in which order ~~does~~ matters

$(a, b) \neq (b, a)$

$(a, b) = (c, d)$
iff $a = c$ & $b = d$.

$A \times B = \{(x,y) \mid x \in A \text{ \& } y \in B\}$

Cross product with empty set $= \phi$

$$A \times \phi = \phi \times A = \phi \times \phi = \phi$$

$A \times B = B \times A$ iff $A = B$ or $A = \phi$ or $B = \phi$

Cross product
- → Not associative
- → Not commutative

Relation

Subset of $A \times B$ is relation

$A = m$ elements

$B = n$ elements

No. of relations from set $A \to B$ =

$R \subseteq A \times B$

No. of elements in $A \times B = m \times n$

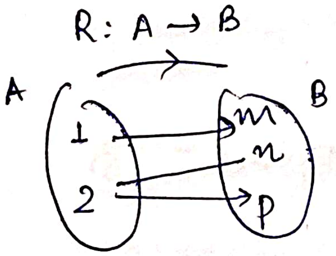
No. of subsets in $A \times B = 2^{m \times n}$

$\therefore$ No. of relations possible from $A \to B = \underline{\underline{2^{mn}}}$

$R: A \to A$ = Relation R is on set A

$R \subseteq N \times N$ = Relation R is on set $(N\times N) \times (N \times N)$

Representations of relations



Arrow diagram representation

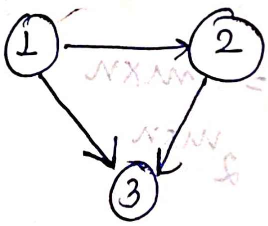
$R = \{(1,m)\ (2,n)\ (2,p)\}$

Set Representation

$A = \{1, 2, 3\}$
↳ Base set

xRy iff $x<y$

$R = \{(1,2)\ (1,3)\ (2,3)\}$



Graph Representation

↳ arrow if $a \to b$
node for each element of base set.

	1	2	3
1	x	✓	✓
2	x	x	✓
3	x	x	x

Matrix Representation

Types of Relations

valid if relation exists on a single set $A \to A$

1. Reflexive Relation

every element is related to itself.

$A = \{a, b, c\}$ R is defined on set A

R contains $\{(a,a)\ (b,b)\ (c,c),\ (a,c)\}$

↳ other elements can be present

Relation R is reflexive iff

$\forall_{x \in A} (xRx)$

Not Reflexive $\to \exists_{x \in A} (x \not R x)$ main diagonal contains atleast one 1 & 1 zero

Irreflexive $\to \forall_{x \in A} (x \not R x)$ main diagonal contains all 0s

Reflexive $\to \forall_{x \in A} (xRx)$ main diagonal contains all 1s

2. Symmetric Relation

$\forall_{a,b \in A} (aRb \to bRa)$

bidirectional edges

$M = M^T$

no unidirectional edges in graph

Antisymmetric –

unidirectional edges

if x & y are different,

$xRy \longrightarrow y \not R x$

$\forall_{x,y} \left[(x \neq y \wedge xRy) \to y \not R x \right]$ $\forall_{x,y \in A} \left[x \neq y \to (xRy \wedge y \not R x) \right]$

$$\forall_{a,b \in A} \left[(aRb \wedge bRa) \rightarrow a = b \right]$$

Asymmetric :- antisymmetric and irreflexive

unidirectional edges

no self loop

$$\forall_{a,b \in A} \left(aRb \rightarrow bR\!\!\!/\,a \right)$$

3. Transitive Relation

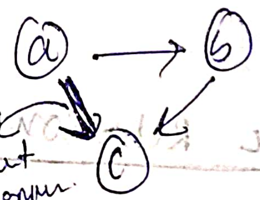
If aRb and bRc then aRc

$$\forall_{a,b,c \in A} \left((aRb \wedge bRc) \rightarrow aRc \right)$$

Violation of transitivity

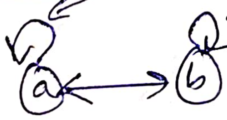
$$\exists_{a,b,c \in A} \left(aRb \wedge bRc \wedge aR\!\!\!/\,c \right)$$

★ solve ques. using graph representation



must happen

self loops should be there



parity of integer → no. is even or odd

parity of 2 = even

3 = odd

0 = even

Equivalence Relation

A relation R is equivalence relation iff it is Reflexive, Transitive & symmetric

Partition of set

A set S is partitioned into k non empty subsets $A_1 \ A_2 \ A_3 \ldots \ A_k$ if

- every pair of subsets is disjoint i.e. $A_i \cap A_j = \phi \quad i \neq j$
- $A_1 \cup A_2 \cup A_3 \cup \ldots \ A_k = S$
- $A_i \neq \phi$
- $A_i \subseteq S$

① Every part is a set
② Partition is a set.

$S = \{1, 2\}$

equivalence relation always creates partition of some set.

Partitions of $S = \{\{1\}, \{2\}\}, \{\{1, 2\}\}$

$[x]_R = \{y \mid xRy\} = \{y \mid yRx\}$

equivalence relation

equivalence class containing zero

If there are 4 equivalence classes of size n. Then,

$|R| = |E_1|^2 + |E_2|^2 + |E_3|^2 + \ldots$

$= n^2 + n^2 + n^2 + \ldots$

equivalence relation has unique equivalence class

$R = (E_1 \times E_1) \cup (E_2 \times E_2)$

fiction

Base set = A

Largest equivalence relation that can be created = R = A × A

No. of equivalence classes = 1

Cardinality = $|A|^2$

(RST)

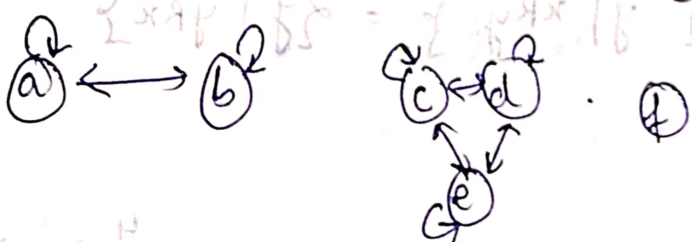
Smallest equivalence relation = identity relation

No of equivalence classes = $|A|$

Cardinality = $|A|$

Graph of equivalence relation –

$\{a, b\}$ $\{c, d, e\}$ $\{f\}$ are equivalence classes



Complete directed graph for each equivalence ~~relat~~ class.

$[a]_R$ → equivalence class of element a

{set of all elements to which a is related}

No. of equivalence relations on a set of 3 elements.

No. of equivalence relations = No. of partitions possible

for 3 elements, partitions possible are

$\{abc\}$ $\{a, bc\}$ $\{ab, c\}$ $\{ac, b\}$ $\{a, b, c\}$.

5 partitions

$\therefore$ 5 equivalence relations

Partial Order Relations

atleast 2 elements b/w which no order is present } partial order

order exists b/w every pair of elements in the set } total order

Partial Order Relations :-

- A relation R on a set A is a partial order for A if R is reflexive, antisymmetric and transitive
- Set A with a partial order is called partially ordered set (Poset).

(Base set, POR) $\longrightarrow$ Poset

(A, R)

Standard ~~POR~~ Poset —

↦ $(N, \leq)$ is Poset
bcoz reflexive, antisymmetric & transitive

↦ $(N, \geq)$

↦ $(P(A), \subseteq)$ subset relation on powerset of set.

↦ $(P(A), \supseteq)$ superset relation...

↦ divisibility relation
aRb → $a|b$

Partial Order Relations

① Relation on non empty base set

Ref ✗

Symmetric ✓

Antisymmetric ✓

Transitive ✓

Irreflexive ✓

ϕ relation on empty base set.

Ref ✓

Symmetric ✓

Antisymmetric ✓

Transitive ✓

Irreflexive ✓

② Divides relation on integers is not antisymmetric

$2|-2$ $-2|2$

③ Divides relation on natural numbers is antisymmetric placeholder for POR

★ "$\leq$" → used for partial order relation

"~" → used for equivalence relation

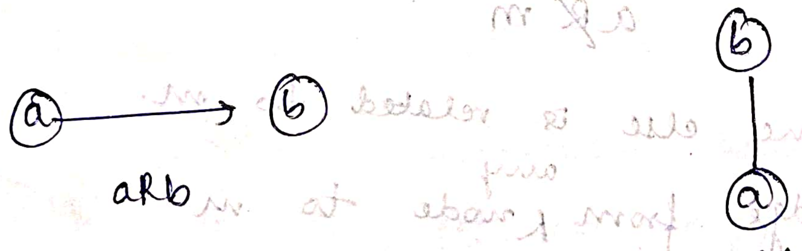
Total order → partial order + every pair of elements should be comparable.

Hasse Diagram (HD)

used for representation of partial order relations (property & definition of partial order relation)

- self loops are not shown (coz POR is reflexive, so, no need to explicitly mention)
- transitive edges are not shown. (remove arrow heads)

- Graph like representation ~~of~~ POR.
- all arrows are in upward direction.

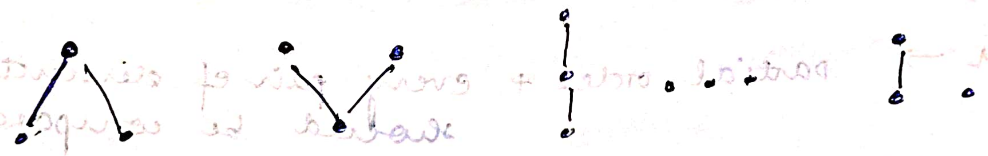


aRb   aRb.

aRb iff there is some upward path (not edge) from a to b.

- No horizontal edges
- no concept of levels   3 . 4 . 8 $\cong$ 3, 4, 8

⊚ For 3 elements (unlabelled), following structures are possible for HD —



3+1+1 = 5 | 3+1+1 5 | 3+1+1+1 6 | 3 3 | 3+1 4 | cardinality of relations

Maximal element 'M'

$\forall_{a \neq M}$; $M \not R a$

M is not related to anyone else.

no edge from M to other node.

Minimal element 'm'

$\forall_{a \neq m}$ $a \not R m$

no one else is related to m.

no edge from any node to m.

Greatest / Maximum element (G)

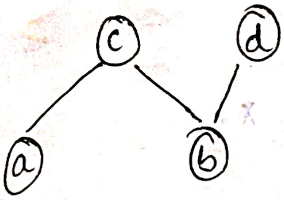
$\forall_a$ aRG

every element is related to G

Least / Minimum element (L)

$\forall_a$ LRa.

related to every element

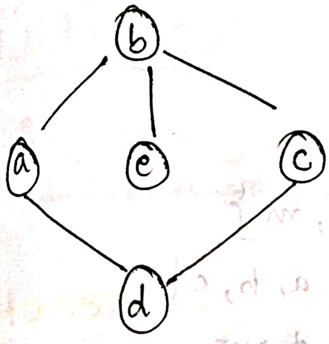


Maximal element = c, d

Minimal element = a, b

Maximum element = DNE (Does not exist)

Minimum element = DNE



Maximal element = b

Minimal element = d, e

Maximum element = b

Minimum element = DNE

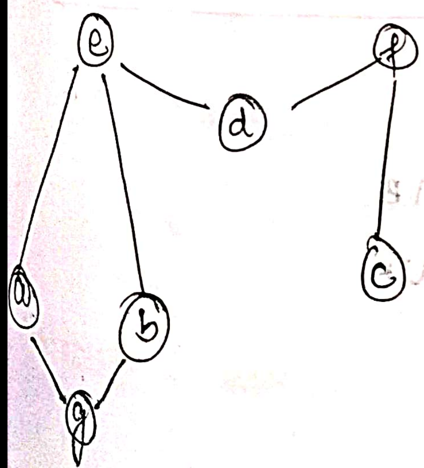
Upper bound (U.B.)

upper bound ~~of a set of~~ ~~x of~~ of a set S is

$\forall_{x \in S} (x R (U.B.))$

lower bound (LB)

$\forall_{x \in S} (L.B.\ R\ x)$



Maximal = {e, f}

Minimal = {g, d, c}

UB {a, g, b} = {e}

UB {a, g} = {a, e}

LB {a, g} = {g}

LB {e, f} = {d}

UB {e, f} = ϕ

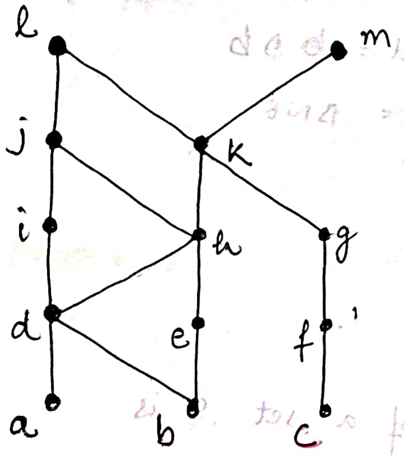
UB {a} = {a, e}

Least upper bound of a set X (LBU):

The least element in the upper bound of X.

Greatest lower bound of a set X (GLB):

The greatest element in the lower bound of X.



Maximal element = {l, m}

Minimal element = {a, b, c}

Greatest element = ϕ DNE

Least element = ϕ DNE

L.B{d, k, f} = ϕ

GLB{d, k, f} = DNE

U.B. {d, k, f} = {k, l, m}

LUB {d, k, f} = k

LB{l, m} = {k, h, d, a, e, b, g, f, c}

G.L.B.{l, m} = k

U.B.{l, m} = ϕ

L.U.B.{l, m} = DNE

UB{d, e} = {h, j, k, l, m}

LUB {d, e} = h

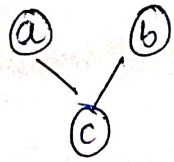
If a poset has more than one maximal elements, greatest element does not exist

* GLB{a, b} = $a \wedge b$ or $a \cap B$

LUB{a, b} = $a \vee b$ or $a \cup B$

LUB{a, b} = $a \vee b$ = $a \cup B$ = Join {a, b}

GLB{a, b} = $a \wedge b$ = $a \cap B$ = meet {a, b}

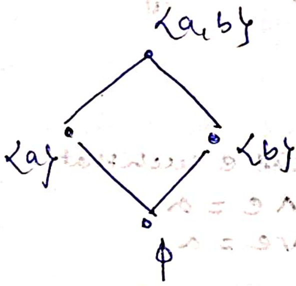


$GLB(a,b,c) = a \wedge b \wedge c = c$

$LUB(a,b,c) = a \vee b \vee c = LUB(a,b)$

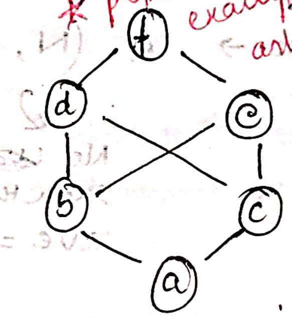
Lattices

A lattice is a poset $(A, \leq)$ in which any 2 elements a, b have an LUB (a,b) & a GLB(a,b)



$(P(A), \subseteq)$

* popular example asked in Gate

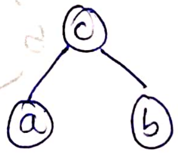


$b \vee c$ = DNE (not unique join point)

$d \wedge e$ = DNE

∴ not lattice

Cardinality of partial order relation



No. of relations of type (a, x) = 2 (aRa, aRc)

" " " " type (b, x) = 2 (bRb, bRc)

" " " " " (c, x) = 1 (cRc)

∴ Cardinality = 2+2+1 = 5

★ On 'n' elements, the largest possible POR is a chain (Total order relation)

Cardinality = 3+2+1 = 6

aRx bRx cRx

Cardinality $\frac{n(n+1)}{2}$

smallest por → a b c Cardinality = n

★ AXA is POR if and only if $|A| \leq 1$

$(N, |)$ is a lattice

becoz for every pair of elements a, b

$LUB(a,b) = a \vee b = LCM(a,b)$

$GLB(a,b) = a \wedge b = GCD(a,b)$

★ Every total order is a lattice

★ Identity property is not satisfied by all Lattices

ex-→ $(N, \leq)$ $(R, \leq)$

No ~~LUB~~ e such that $a \vee e = a$

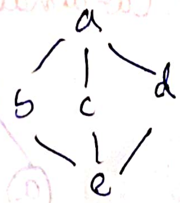
No ~~GLB~~ e such that $a \wedge e = a$, $a \vee e = a$

★ Distributive property is not satisfied by some lattice

$b \wedge (c \vee d) \neq (b \wedge c) \vee (b \wedge d)$

$(b \wedge a)$ e $\wedge$ e

b $\neq$ e



→ If an element is deleted from base set, of POSET then, the resulting relation is also POSET

If (A, R) is a POSET, then, (B, R) is also a POSET $\forall B \subseteq A$.

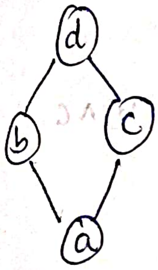
If (A, R) is equivalence relation, then, (B, R) is also an equivalence relation $\forall B \subseteq A$.

Sublattice

Given a lattice L

S is sublattice of L if

- S is subset of L
- S is a lattice
- ~~GLB~~, LUB, GLB of L, S should be same (a, b ∈ S)



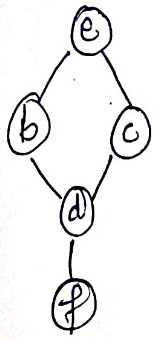
Delete c →

Subset of L ✓

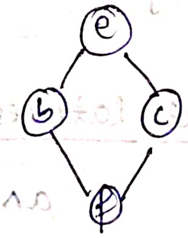
lattice ✓ ∴ sublattice ✓

	L	S
a∨b	b	b
b∨d	d	d
a∨d	d	d
a∧b	a	a
⋮		

✓



delete d →

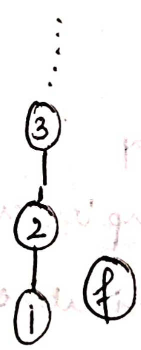


Subset of L ✓

lattice ✓ ∴ not sublattice ✓

	L	S
b∧c	d	f ✗

★ In the subset of L, i.e. base set of S, if a, b are present, their GLB and LUB should also be present.



maximal element = f

greatest element = DNE

★ In finite poset,

unique maximal → greatest

unique minimal → least

In infinite poset,

no guarantee.

Properties satisfied by all lattices.

- Commutative $a \vee b = b \vee a$ $\quad a \wedge b = b \wedge a$
- Associative $(a \vee b) \vee c = a \vee (b \vee c)$ $\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- Idempotence $a \wedge a = a$ $\quad a \vee a = a$
- Absorption $a \vee (a \wedge b) = a$ $\quad a \wedge (a \vee b) = a$
- $\forall_{x,y \in L}$ $\{ x R (x \vee y)$, $(x \wedge y) R x$, $(x \wedge y) R (x \vee y) \}$

Properties do not satisfied by all lattices

- Identity $a \vee 0 = a$ $\quad a \wedge 1 = a$
- Distributive $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- Complement $a \vee \neg a = 1$ $\quad a \wedge \neg a = 0$

Types of lattices

1. Bounded lattice — identity
2. Complemented Lattice — complement
3. Distributive lattice — distributive
4. Boolean lattice — all

1. Bounded Lattice

- A lattice is bounded if it has minimum & maximum element.
- These are denoted by 0 & 1.

→ Infinite lattice that is bounded — $([0,1], \leq)$, $(P(N), \subseteq)$ power set of natural nos

$(W, |)$ — least = 1, Greatest = 0
whole nos

→ Every finite lattice is bounded.

Minimum element → 0
Maximum element → 1

$\forall_x$ $x \vee 1 = 1$
$x \wedge 1 = x$

$\forall_x$ $x \vee 0 = x$
$x \wedge 0 = 0$

- Bounded lattice is a lattice with identity element for both GLB and LUB.

Identity element for GLB = Greatest element (1)
Identity element for LUB = Least element (0)

Domination law = $S \cup U = U$ U = universal set.
$S \cap \phi = \phi$ ϕ = empty set

U is dominator for union
ϕ is ~~donim~~ dominator for intersection

Dominator for ~~GLB~~ LUB = $a \vee G = G$ (greatest element)
Dominator for ~~LUB~~ GLB = $a \wedge L = L$ (least element)
G' 0

- Bounded lattice is a lattice with dominator for both GLB and LUB.

Greatest element — Identity element for GLB
— Dominator for LUB

Least element — Identity element for LUB
— Dominator for GLB.

2. Complemented Lattice

⊙ lattice which follows complementary property.

Complement of an element 'a':

'b' is complement of a iff

$a \vee b = 1$ (greatest element)

$a \wedge b = 0$ (least element)

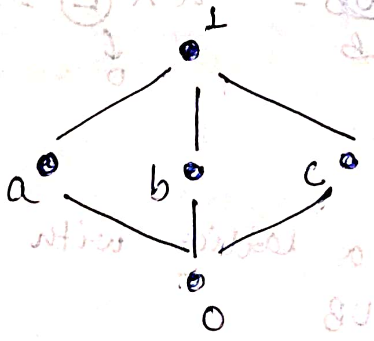
⊙ If lattice is not bounded, then, complement cannot be defined.

Complement of an element 'a' $= a^{-1}$

$(\text{least})^{-1} = \text{greatest}$

$(\text{greatest})^{-1} = \text{least}$

⊙ If every element in the lattice has atleast one complement, then, it is called complemented lattice.



$a^{-1} = b, c$

$b^{-1} = a, c$

$c^{-1} = a, b$

$0^{-1} = 1$

$1^{-1} = 0$

⊙ A TOSET can be complemented lattice iff it has ≤ 2 elements.

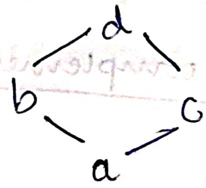
3. Distributive ~~Pro~~ Lattice

Distributive property :

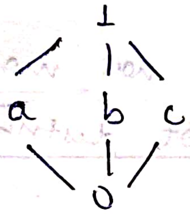
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

⊙ Lattices which satisfy distributive property are called distributive lattice. $\forall_{x,y,z}$



distributive lattice ✓

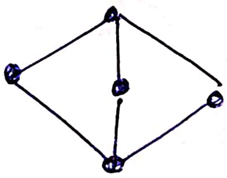


$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

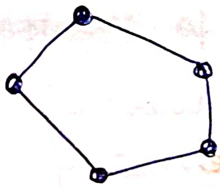
$= a \vee 0 \qquad = 1 \wedge 1$

$= a \qquad = 1$

∴ distributive property not satisfied



Kite lattice
Diamond lattice
M_3



Pentagon lattice
N_5

Popular non distributive lattice

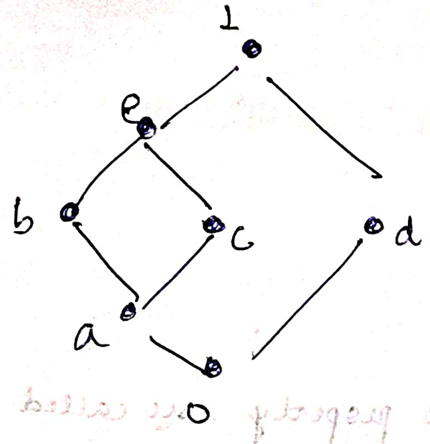
Theorem :-

⊙ A lattice L is distributive iff there is no sublattice of L which is (kite) or pentagon.

⊙ If a lattice has ≤ 4 elements, it is definitely distributive.

Theorem:-

⊙ A distributive lattice can have atmost one complement



$b^{-1} = d$

$c^{-1} = d$

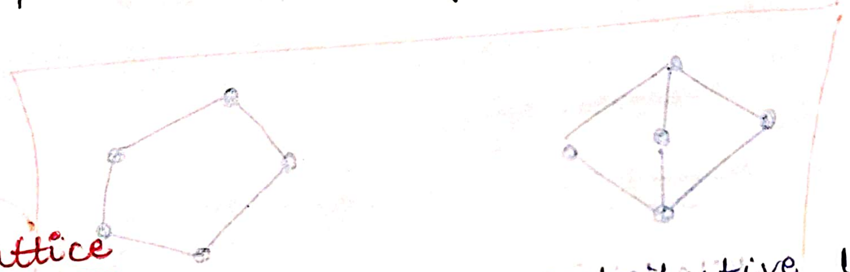
$d^{-1} = (b, c)$

$\therefore$ not distributive

[distributive lattices can have atmost one complement)

★ If some element has more than one complement, then the lattice is not distributive

If ≤ 1 complements, lattice may / may not be distributive.



4. Boolean Lattice

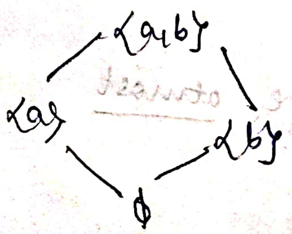
Boolean lattice is complemented distributive lattice.

- → complemented lattice (hence bounded)
- → distributive lattice

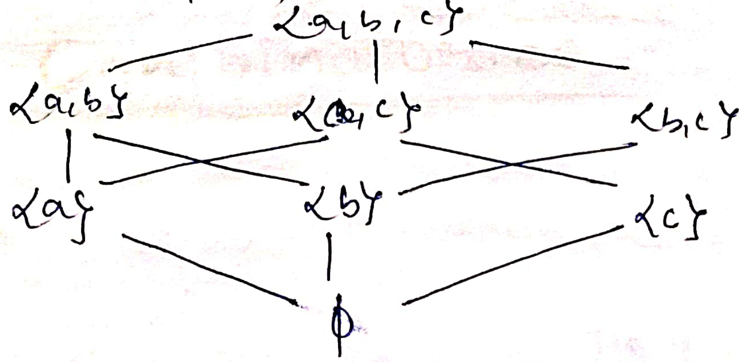
→ Every boolean algebra has the same structure as $(P(A), \subseteq)$ structure.

→ 2^n elements in boolean algebra

$n=2$: $(\{a,b\}, \subseteq)$ $\qquad$ $n=0$: $(\phi, \subseteq)$ $\qquad$ $n=1$: $(\{a\}, \subseteq)$

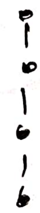


$(\{a, b, c\}, \phi)$



→ Not every lattice with 2^n elements is boolean lattice.

ex →



⊙ Boolean lattice is also called boolean algebra because it satisfies all properties of boolean algebra

- Commutative
- Idempotent
- Associative
- Absorption
- Consistency
- Identity
- Complement
- Distributive

Combinatorics

Sum Rule

Task T

Task1 (n1 ways) OR Task2 (n2 ways) OR Task3 (n3 ways)

mutually disjoint

No. of ways of completing task $= (n_1 + n_2 + n_3)$ ways.

2 imp. points –

- subtasks must be mutually disjoint
- OR

If there are $n(A)$ ways to do task A and distinct from them, $n(B)$ ways to do B, then, the no. of ways to do A or B is $n(A) + n(B)$.

How many k long palindromes can be formed from an n-set?

The first $\lceil k/2 \rceil$ elements are to be selected arbitrarily from n ~~lm~~ set.

$\therefore$ Ans $\rightarrow n^{\lceil k/2 \rceil}$

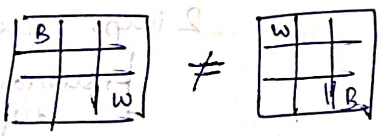
Ques :- In how many ways can we place 2 identical rooks on a 8X8 chessboard so that they occupy different rows & different columns ?

1st rook $\longrightarrow$ 64 choices

2nd rook $\longrightarrow$ $64 - (2\times8 - 1)$ choices $= 64 - 15 = 49$ choices

Since the 2 rooks are identical,

$\therefore$ No of ways $= \dfrac{64 \times 49}{2} = \underline{32 \times 49}$



different rooks

identical rooks

Ques How many subsets of exactly 2 elements are there for a set of n elements ?

1st element $\longrightarrow$ n choices

2nd element $\longrightarrow$ $(n-1)$ choices

$n(n-1)$ choices of ordered pairs

For a set, $\{a, b\} = \{b, a\}$ (2-1 correspondence)

$\therefore$ No. of ways $= \dfrac{n(n-1)}{2}$

Ques :- How many linear orders of 6 elements a, b, c, d, e, f are there such that 'a' comes before 'b' (not necessarily immediately)

No. of ~~ways~~ orders of 6 elements $= 6!$

Half of them contain a before b

$\therefore$ No. of order $= \dfrac{6!}{2}$

Ques – How many linear orders of 6 elements a, b, c, d, e, f are there such that 'a' comes before 'b' and 'b' comes before 'c' (not necessarily immediately) ('abc' subsequence is there.)

No. of possible orders = $6! = 720$

No. of possible orders of 'abc' : $3! = 6$

$\therefore$ Ans = $\frac{6!}{3!} = \frac{720}{6} = \underline{\underline{120}}$

$\boxed{e}$ a b $\boxed{d}$ $\boxed{e}$ c
$\boxed{e}$ b a $\boxed{d}$ $\boxed{e}$ c
$\boxed{e}$ c b $\boxed{d}$ $\boxed{e}$ a
$\boxed{e}$ b c $\boxed{d}$ $\boxed{e}$ a
$\boxed{e}$ a c $\boxed{d}$ $\boxed{e}$ b
$\boxed{e}$ c a $\boxed{d}$ $\boxed{e}$ b

e a b d e c

for every six possible arrangements, 1 is chosen

...mutation ...ically ...duct rule.

$${}^nP_r = \frac{\lfloor n}{\lfloor n-r} = \frac{n!}{(n-r)!}$$

$${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{n!}{r!\,(n-r)!}$$

$${}^nP_r = (r!)({}^nC_r)$$

$${}^nC_r = {}^nC_{r-1}$$

Ques – No. of orderings / permutations / arrangements of r distinct elements from a set of n elements?

${}^nP_r$

n	n-1	n-2	...	n-r+1

$= n(n-1)(n-2)(n-3) \cdots (n-r+1)$

$= \frac{n!}{(n-r)!} = {}^nP_r$

Ques - No. of ways to <u>select</u> r people from a set of n people.

^nC_r

(apply division rule to permutation)

Why 0! is 1?

Ans → By convention / definition

↳ no. of subsets of size r of a set of size n $= {}^nC_r$

not proof
one of the reasons behind this convention

no. of subsets of size 0 of a set of size $n = \overset{1}{\phi}$ (ϕ)

$\therefore {}^nC_0 = 1 \Rightarrow \frac{n!}{n!\,0!} = 1$

$\Rightarrow \underline{0! = 1}$

$^nC_r \longrightarrow$ binomial coefficient

$^nC_r = \frac{n!}{(n-r)!\,r!}$ algebraic meaning

^nC_r = selecting r people from n people   combinatorial meaning

If repeatation is allowed ⇒ ^nP_r, ^nC_r cannot <u>be used.</u>

Poker hand → (poker hand consists of 5 randomly chosen cards out of 52 cards.

Ques How many bit strings of length n contain exactly r 1s?

^nC_r

Templates

1. Some elements are never together -
First arrange remaining elements and then in gaps put these elements.

Ques - 8 men 5 women; stand in line; no 2 women stand next to each other

8 men can stand in 8! ways

| M_1 | M_2 | M_3 | M_4 | M_5 | M_6 | M_7 | M_8 |

9 gaps are there.
5 to be occupied by women

$\therefore$ Ans $\rightarrow$ $8! \times {}^9C_5 \times 5!$

Ques - 7 women & 9 men are on the faculty.

1) ways to select a committee of five members if atleast one woman must be in the committee?

$^7C_1 \times {}^{15}C_4$ ✗ wrong (overcounting)

Ans - 1 - no woman
$= {}^{16}C_5 - {}^9C_5$

2) ways to select a committee of five members if atleast 1 man & atleast one woman must be in the committee.

$^7C_1 \times {}^9C_1 \times {}^{14}C_3$ ✗ wrong

Ans - $^{16}C_5$ - no men - no women $= {}^{16}C_5 - {}^7C_5 - {}^9C_5$

Pascal's identity

can be used in dp of c program

Recursive definition of binomial coefficient.

(1) $$\boxed{{}^{n+1}C_k = \underbrace{{}^{n-1}C_{k-1}} + \underbrace{{}^{n-1}C_k}}$$

from (n+1) Students, selects k ~~students~~

1 particular student is taken
Select k-1 students from n students

1 particular student is not taken.
select k students from n student

take 2 students x, y (2)

$$\boxed{{}^{n}C_k = \underbrace{{}^{n-2}C_{k-2}} + \underbrace{{}^{n-2}C_k} + \underbrace{2 \cdot {}^{n-2}C_{k-1}}}$$

Select k students from n students

~~take 2 stude~~
both are selected
∴ select k-2 from n-2 student

both rejected.
∴ select k from n-2

one select one rejected
∴ select ~~n-2~~ k-1 from n-2

(3) $$\boxed{{}^{n+m}C_r = {}^{n}C_0 \, {}^{m}C_r + {}^{n}C_1 \, {}^{m}C_{r-1} + \dots - {}^{n}C_r \cdot {}^{m}C_0}$$

$$\sum_{k=0}^{r} {}^{n}C_k \, {}^{m}C_{r-k}$$

$$\cancel{\binom{n+m}{r}} \binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

Vandermonde's identity ← $[{}^{n}C_r = {}^{n}C_{n-r}]$

$$= \sum_{k=0}^{r} \binom{n}{n-k}\binom{m}{m-r+k} \dots$$

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$$

$$\boxed{\binom{2n}{2} = 2\binom{n}{1} + n^2}$$

n boys n girls select 2 = both boys + both girls + 1 boy 1 girl

$$\binom{n}{2} \quad \binom{n}{2} \quad \binom{n}{1}\cdot\binom{n}{1}$$

$$= 2\binom{n}{2} + n^2$$

n^2

(4)

$$\sum_{r=0}^{n} {}^{n}C_r = \underbrace{2^n}_{\text{total subsets}}$$

i.e. ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \dots {}^{n}C_n = 2^n$

${}^{n}C_0$: subsets of length 0

${}^{n}C_1$: subsets of length 1.

$\binom{n}{r} = \left(\frac{n}{r}\right)\binom{n-1}{r-1}$

(5)

$$\boxed{\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}}$$

$k \cdot {}^{n}C_k = n^{\,n-1}C_{k-1}$

select k subset from n people, choose 1 president among them.

choose one person as president. Select subset of (n-1) people.

Hockey Stick Identity

(6)

$$\boxed{{}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots {}^{r}C_r} = \sum_{k=r}^{n} {}^{k}C_r$$

select max number

${}^{n+1}C_{r+1}$: from n+1 numbers, select r+1 numbers

${}^{n}C_r$: max. no is (n+1) selected from remaining n elements, select r elements

${}^{n-1}C_r$: max no. n is selected from remaining (n-1) elements select r elements.

${}^{r}C_r$: max. no. selected = r+1 remaining r elements (smaller than r) select r elements.

Binomial Theorem

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 (x)^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots {}^nC_n x^0 y^n$$

$$\Rightarrow (1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \ldots {}^nC_n x^n$$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots \binom{n}{n}x^n$$

$$\Rightarrow (1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Putting $x=-1$,

$$\boxed{\begin{aligned}(1-1)^n = 0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \binom{n}{n} \\ \Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots &= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\end{aligned}}$$

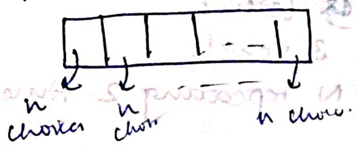
4

i.e. sum of even binomial coefficients = sum of odd binomial coefficients

Permutations with ~~Repeat~~ Repetitions

Repetition allowed $\therefore$ ${}^nP_r$ cannot be used.

Q. n distinct objects
permute r of them with repetition allowed.



$= (n)^r$

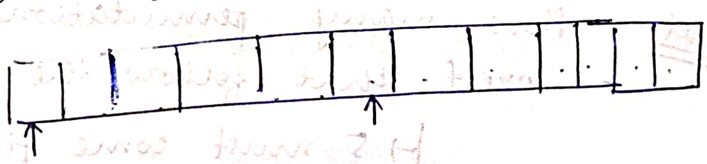
Q Total n elements
n_1 of type 1, n_2 of type 2, ... n_k of type k

No. of permutations $= \dfrac{n!}{n_1!\,(n_2)!\ldots(n_k)!}$ $\boxed{n_1+n_2+n_3\ldots n_k = n}$

No. of permutations of word 'MISSISSIPPI' $= \dfrac{11!}{4!\,4!\,2!\,1!}$

$= \underbrace{{}^{11}C_4}_{S} \cdot \underbrace{{}^{7}C_4}_{I} \cdot \underbrace{{}^{3}C_2}_{P} \cdot \underbrace{{}^{1}C_1}_{M}$

Ques No. of ways in which 'PERMUTATIONS' can be arranged if there are always 4 letters b/w P & S.



7 choices for P & S.

$7 \times 2! \times \dfrac{10!}{2!}$

Ques In how many ways can the letters in 'WONDERING' be arranged with ~~ex~~ exactly 2 consecutive vowels?

WONDERING — ~~8~~ 9 letters
3 vowels
N repeating 2 times

$$\underbrace{{}^3C_2 \cdot 2!}_{\text{no. of ways of arranging 2 vowels from 3}} \cdot \underbrace{\frac{6!}{2!}}_{\text{arranging all the consonants}} \cdot \underbrace{{}^7C_2 \cdot 2!}_{\text{group of 2 vowels \& 3rd vowel cannot be together coz exactly 2 should be together}}$$

7 gaps

Ques No. of permutations of the word 'ABCDEF' in which A comes before B and C comes before D. (not necessarily immediately)

$$\underbrace{{}^6C_2 \times 1}_{\text{A comes before B}} \cdot \underbrace{{}^4C_2 \times 1}_{\text{C comes before D}} \cdot \underbrace{2!}_{\text{permute remaining}}$$

Ques How many permutations of [5 2 1 8 9 7] exist that follow the rule

- 5 must come first
- 8 must come before both 9 & 7
- 2 must come before 1.

$$\underbrace{1}_{\text{1st cond}} \times \underbrace{{}^5C_3 \times 2!}_{\text{2nd cond}} \; \underbrace{{}^2C_2 \cdot 1}_{\text{3rd cond}}$$

Ques How many sequences of A B C D E F G H contain the subsequences of <C, A, B> or <B, E, D>

$= \cancel{8!} \quad {}^8C_3 \cdot 5! + {}^8C_3 \cdot 5! - {}^8C_5 \cdot 3!$

$= \underbrace{\frac{8!}{3!}}_{CAB} + \underbrace{\frac{8!}{3!}}_{BED} - \underbrace{\frac{8!}{5!}}_{CABED}$

Distributing Objects Into Boxes

DODB → Different objects different Boxes.

① order of elements does not matter in distributing objects into boxes.

1. DODB Template

Distinguishable objects into distinguishable boxes.

$Obj_1 \quad Obj_2 \quad Obj_3$ into $B_1, B_2 \ldots B_n$

[Product rule with simple combination]

Ques:- distribute hands of 5 cards to each of 4 players from deck of 52 cards

${}^{52}C_5 \cdot {}^{47}C_5 \cdot {}^{42}C_5 \cdot {}^{37}C_5 \cdot$ ways

$= \frac{52!}{5!\,5!\,5!\,5!\cdot 32!}$

Ques- n distinct objects
k different people.

A_1 A_2 A_3 - - - A_k

n_1 obj n_2 obj n_3 obj n_k obj

$(n_1+n_2+n_3+\dots n_k) = n$

~~No. of ways = ${}^{n}C_{n_1} * {}^{n}C_{n_2} \times {}^{n}C_{n_3} \dots {}^{n}C_{n_k}$~~

$\boxed{n_1+n_2+\dots n_k = n}$

No. of ways $= {}^{n}C_{n_1} \cdot {}^{n-n_1}C_{n_2} \cdot {}^{n-(n_1+n_2)}C_{n_3} \dots {}^{n-\dots-n_{k-1}}C_{n_k}$

$$= \frac{n!}{n_1!\, n_2! \dots n_k!}$$

Ques- 8 distinct objects; 3 boys;
everyone gets atleast 2 object.

2 cases < 2+2 2 2 ; 2 2+1 2+1

$${}^{3}C_1 \cdot {}^{8}C_4 \cdot {}^{4}C_2 \cdot {}^{2}C_2 + {}^{3}C_1 \cdot {}^{8}C_2 \cdot {}^{6}C_3 \cdot {}^{3}C_3.$$

Ques- 52 cards to 4 people
everyone gets 13 cards.
oldest player gets queen of spades.

$${}^{51}C_{13} \cdot {}^{39}C_{13} \cdot {}^{26}C_{13} \cdot {}^{13}C_{13}$$

$$\binom{n}{n_1\ n_2\ n_3 \ldots n_k} = \frac{n!}{n_1!\ n_2!\ n_3! \ldots n_k!}$$ $n_1+n_2+\ldots n_k = n$

← notation

Ques – 15 objects 5 boxes

~~one box~~ boxes have 5, 4, 3, 2, 1 objects

$$\frac{15!}{1!\,2!\,3!\,4!\,5!}\ 5! \rightarrow$$ because it is not given which box contain how many items

Ques – ~~m~~ n distinguishable balls into n boxes

no. of balls in each box can vary

Ball$_1$ Ball$_2$ - - - Ball$_n$

n choice n . ——— n

Ans → $\underline{\underline{n^n}}$

n balls m boxes. Then?

$m \times m \times \ldots$ n times $= \underline{\underline{m^n}}$

Analogy — Santa has identical chocolates to give to children.

2. IODB Template

Identical/Indistinguishible objects into distinguishible boxes.

IODB problem is same as Star - Bar problem

where star → no. of chocolates to be distributed

bar → no. of children − 1

xx | xx | x

ways of ~~distributing~~ permuting n stars and ~~n~~ to r bars $= \frac{(n+r)!}{n!\,r!}$

[n identical objects, r identical object]

Ques distributing 30 identical objects into 3 distinct boxes each box must have ~~0~~ atleast 5 items

5+x | 5+y | 5+z $\qquad x+y+z = \cancel{0}\,30-15=15$

problem narrows down to permutation 15 stars and 2 bars

$\frac{\cancel{15}}{\cancel{2!}} \quad \frac{17!}{15!\,2!} = {}^{17}C_2 = {}^{17}C_{15}$

→ Combination with repetition

== Starbars

Selection from items with repetition.

3 fruits — mango, ~~org~~ orange, apples

ways of selecting 4 fruits.

≡ 4 stars 2 bar problem : $^{6}C_2$

In how many ways can we choose r objects from n objects → $^{n}C_r$

In how many ways can we choose r objects from n kinds of objects → $^{n+r-1}C_r$

→ Non negative Integer solutions of a equations:

① $x_1 + x_2 + x_3 = 11$
where $x_1\ x_2\ x_3$ are non negative integers
How many solutions?

$\equiv$ 11 stars 2 bar $= \underline{\underline{^{13}C_2}}$

② $a+b+c = 10$

$a \geq 1,\ b \geq 2,\ c \geq 3$

Convert to standard temp. by putting min value of a, b, c

$x+y+z = 4 \qquad [x = a-1 \quad y = b-2 \quad z = c-3]$

$x, y, z \geq 0$

2 bars 4 stars $= \underline{\underline{^{6}C_2}}$

IODB Template is equivalent to—

- no. of combinations of (n) objects taken r at a time with repetition.
- no. of ways r identical objects can be distributed among n distinct container.
- non negative integer solns of eqn

$$\boxed{x_1 + x_2 + \dots + x_n = r}$$

- a k element multiset
- non decreasing subsequence.

→ Multiset problem—

k element multiset from n element set

$$\binom{n+k-1}{k} = {}^{n+k-1}C_k$$

$(n-1)$ bars
k stars.

Buy 3 hats and there are 5 colors.

4 bars. 3 stars

$^7C_3$

Buy 5 hats and there are 3 colors

2 bars 5 star

$^7C_2$

→ Non decreasing integer sequence problem.

[GATE 2015] No. of non decreasing subsequences of length n from the set $\{1, 2, 3 \dots m\}$

n stars $(m-1)$ bars.

$$^{n+m-1}C_n$$

Ques - No. of n digit natural numbers in which digits are in non decreasing order

0 will not be a digit

$\therefore \{1, 2, 3 \dots 9\} \Rightarrow$ 8 bars
n stars.

$\therefore$ Ans — ${}^{n+8}C_8$

Ques - No. of n digit natural numbers in which digits are in non increasing order

0 can be there

$\therefore \{0, 1, 2 \dots 9\} \Rightarrow$ 9 bars
n stars

Ans — ${}^{n+9}C_9 - 1$ → for all zeros

Ques - No. of n digit natural numbers in which digits are ~~always~~ strictly increasing / ~~decreasing~~

${}^{10}C_n$ $n \leq 10$
coz if $n > 10$ repetition will occur.

for strictly decreasing —
0 cannot be selected

$\therefore$ ${}^{9}C_n$

(order matters)

→ Integer composition of a number

composition of an integer n is a way of writing n as the sum of a sequence of positive integer.

$4 = 4+0 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1$

Ques- set of compositions of 6 into 3 parts

$a+b+c = 6$ $\quad a \geq 1,\ b \geq 1,\ c \geq 1$

$\therefore x+y+z = 3$

$\therefore$ Ans - ${}^{3+2}C_2$

Compositions of n into k parts

$x_1 + x_2 + \dots + x_k = n-k$ (cos positive can't be zero)

$\therefore$ Ans $\rightarrow$ ${}^{n-k+k-1}C_{k-1} = {}^{n-1}C_{k-1}$

Total no. of composition $= \sum_{r=1}^{n} \binom{n-1}{r-1}$

let $x = r-1$

$r=1 \Rightarrow x=0$

$r=n \Rightarrow x=n-1$

$= \sum_{x=0}^{n-1} \binom{n-1}{x} = 2^{n-1}$

(No. of equivalence relations on a set = No. of partition)

No. of partitions / equivalence relations of set $\{1,2,3,4\}$

GATE 1997

1 part partition = 4 $\rightarrow$ 1 way

2 part partition = (1,3) (2,2) $\rightarrow$ $^4C_1 \cdot {}^3C_3 + \dfrac{^4C_2 \cdot {}^2C_2}{2!}$

$= 4 + \dfrac{4 \times 3}{2 \times 3} = 7$

3 part partition $\rightarrow$ (1, 1, 2) $\rightarrow$ $\dfrac{^4C_1 \cdot {}^3C_1 \cdot {}^2C_2}{2!} = 6$

4 part partition $\rightarrow$ (1, 1, 1, 1) $\rightarrow$ 1 way

Total no of ways $= 1 + 7 + 6 + 1$

$= \boxed{\underline{15} \text{ partitions}}$

Partition of a set $\rightarrow$ DOIB template (order matters)
Different object identical boxes.

Partition of integer (number) $\rightarrow$ IOIB template

Composition of number $\rightarrow$ IODB template
$\sum {}^{n-1}C_{k-1}$

DOIB
- $\rightarrow$ Partition of a set
- $\rightarrow$ No. of equivalence relations on a set.

IOIB
- $\rightarrow$ partition of integer
  (partition no of n)

Generating Functions

The generating function on a sequence $a_0, a_1, a_2 \ldots a_k$ of real numbers is the infinite series

$$\boxed{G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k}$$

Generating functions are defined only for infinite sequences.

AGP function / sequence

$$S = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots$$

$$rS = ar + (a+d)r^2 + (a+2d)r^3 + \ldots$$

$$(S - rS) = a + dr + dr^2 + dr^3 + \ldots$$

$$(S - rS) = a + dr(1 + r + r^2 + \ldots)$$

$$S(1-r) = a + dr \cdot \frac{1}{1-r}$$

$$\Rightarrow \boxed{S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$$ imp. remember the formula.

(AGP series) $1 + 2x + 3x^2 + 4x^3 + \ldots$

$$GF(s) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \frac{1}{1-x} + \frac{1 \cdot x}{(1-x)^2} = \frac{1}{1-x} + \frac{x}{(1-x)^2}$$

$$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

imp If generating function for the sequence $a_0\ a_1\ a_2 \ldots \Longleftrightarrow G(x)$

Then,

GF for seq. $a_1, 2a_2, 3a_3 + 4a_4 + \ldots \Longleftrightarrow G'(x)$

Interesting obs —

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$

$\boxed{a_0 = G(0)}$ but $G(1) = a_0 + a_1 + a_2 + \ldots$

$\therefore$ To find a_1 —

$G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

$\boxed{G'(0) = a_1}$

$\boxed{G''(0) = 2a_2}$ $\boxed{G'''(0) = 3!\, a_3}$

imp $\boxed{G^n(0) = n!\, a_n}$

Extended Binomial Theorem —

$^{1/2}C_2 = \frac{(1/2)(1/2 - 1)}{2!} = \frac{-1}{4\times 2} = -1/8$

$^{-3}C_2 = \frac{(-3)(-3-1)}{2!} = \frac{(-3)(-4)}{2} = 6$

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots \binom{n}{n}x^n + \underbrace{\binom{n}{n+1}x^{n+1} + \ldots}$

There terms are 0 if $n \in N$. else non zero

$\therefore$ In the expansion of $(1+x)^{-2}$, coeff of $x^4 = \binom{-2}{4}$

$= {}^{-2}C_4 = \frac{(-2)(-3)(-4)(-5)}{4!}$

$= 5$

coeff of $x^{-4} = 0$ coz x^{negative} is not possible

$(2+x)^{-2}$ coeff of $x^4 =$ $\cancel{\binom{-2}{4} x^4\, 2^{-3}}$

$= 2^{-2}\left[1+\frac{x}{2}\right]^{-2}$ $\therefore$ coff of $x^4 = 2^{-2}\cdot\binom{-2}{4}\left(\frac{1}{2}\right)^4$

$= \frac{1}{4}\cdot {}^{-2}C_4\left(\frac{1}{2}\right)^4$

$= \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4\cdot\frac{(-2)\cancel{(-3)(-4)}(-5)}{\cancel{4!}}$

$= \frac{5}{2^6} = 5/64$

~~${}^{-n}C_r = (-1)^r \binom{n+r-1}{r}$~~

negative term

$$\boxed{{}^{-n}C_r = (-1)^r \binom{n+r-1}{r}}$$

Imp

Recurrence Relations

Recurrence relations occur when some term in the sequence depends upon the previous terms of the sequence.

Recurrence relation is the equation that expresses a_n in terms of one or more of the previous terms of the sequence.
Initial conditions are required to be specified to find first term of seq.

Recursive function – Function that is represented by itself in terms of smaller elements.

$fib(n) = fib(n-1) + fib(n-2)$ ← R. Function

$a_n = a_{n-1} + a_{n-2}$ ← R. Relation

Ques :- No. of bit strings that do not have 2 consecutive 0's

$\underbrace{[\quad\quad\quad]}_{\text{(n-1) length string no consecutive 0's}}[1] + \underbrace{[\quad\quad\quad]}_{\text{(n-2) length string no consecutive 0's}}[1|0]$

$$\boxed{f(n) = f(n-1) + f(n-2)}$$

$f(1) = 2$
$f(2) = 3$

Ques :- No. of permutations of a set with n elements

$$P(n) = n\,P(n-1)$$ (factorial ka)

Ques :- No. of bit strings containing a pair of consecutive 0's

$\underbrace{[\quad\quad\quad]}_{n-1}[1] \rightarrow T(n-1)$

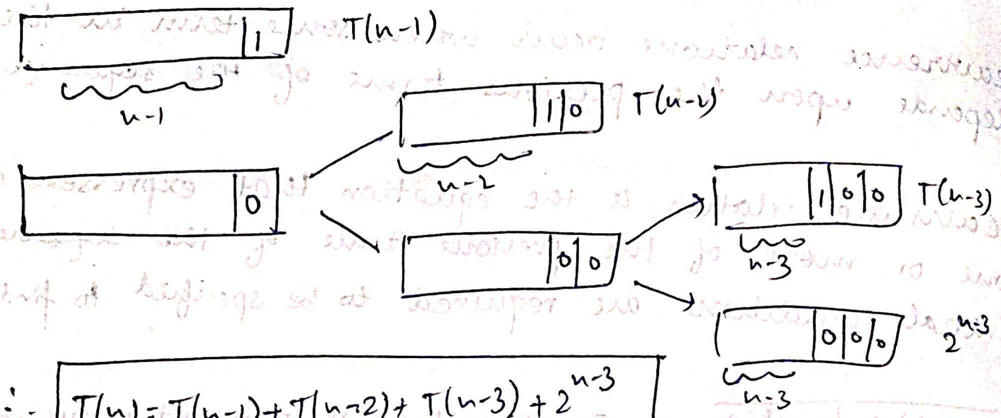
$[\quad\quad\quad][0] \rightarrow \underbrace{[\quad\quad\quad]}_{n-2}[0|0]$ all strings of n-2 length are counted $= 2^{n-2}$

$[\quad\quad\quad][0] \rightarrow \underbrace{[\quad\quad\quad]}_{n-2}[1|0]$ $T(n-2)$

$T(1) = 0$
$T(2) = 1$

$$\therefore \boxed{T(n) = T(n-1) + T(n-2) + 2^{n-2}}$$

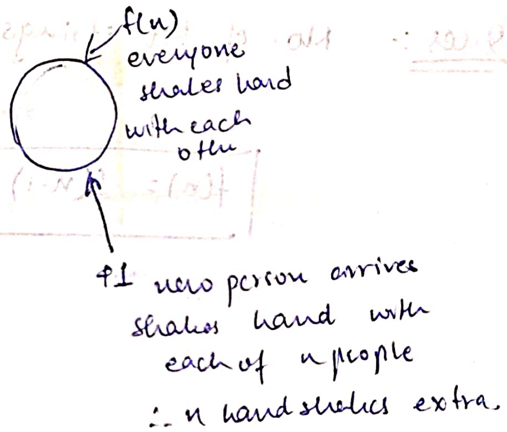
Ques- No. of bit strings of length n that contain three consecutive 0's



$$\therefore \boxed{T(n) = T(n-1) + T(n-2) + T(n-3) + 2^{n-3}}$$

$\cancel{a}$ $\cancel{f(0)}$ $f(1) = 0$ $f(2) = 0$ $f(3) = 1$

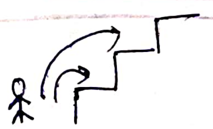
Ques :- Recurrence relation for number of handshakes among n people.

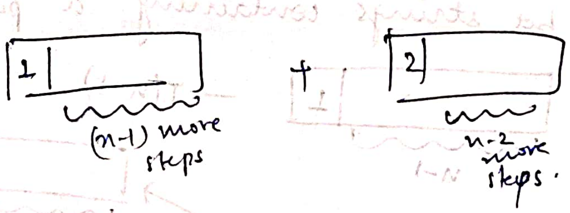


$$\boxed{\therefore T(n+1) = T(n) + n}$$

$\frac{n(n+1)}{2}$ $\quad {}^{n}C_2$

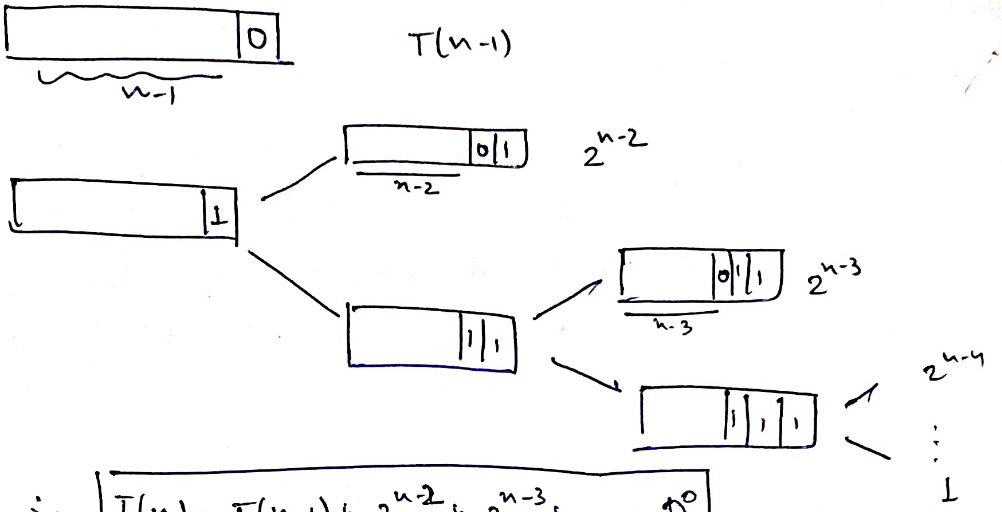
Ques :- No. of ways of climbing n stairs if one or 2 steps can be taken





$$\therefore S(n) = S(n-1) + S(n-2)$$

Ques - Recurrence relation for the number of ~~bits~~ bit strings containing the string '01'



$$\therefore \boxed{T(n) = T(n-1) + 2^{n-2} + 2^{n-3} + \cdots 2^{0}}$$

$$\therefore T(n) = T(n-1) + 2^{n-1} - 1$$

Graph Type	Edges	Multiple edges?	self loop?
Simple graph	undirected	✗	✗
Multigraph	undirected	✓	✗
Pseudograph	undirected	✓	✓
Simple directed graph.	directed	✗	✗
Directed multigraph.	directed	✓	✓
Mixed graph	Directed/undirected	✓	✓

GRAPH THEORY

(~~Enginee~~ Discrete Mathematics)

Types of graph –

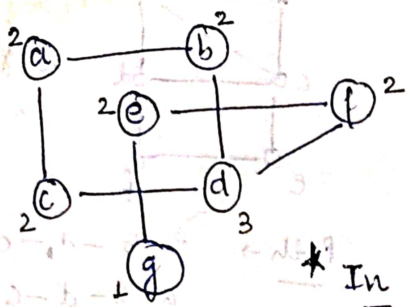
- → ① undirected graph (~~[illegible]~~)
- → ② directed graph
- → ③ Multigraph
  - ↳ undirected graph
  - ↳ no self loop

  } multiedges / parallel ~~pos~~ edges allowed.

- → ④ Pseudo graph
  - ↳ any undirected graph parallel edges, self loop allowed

Degree of a vertex — Number of edges incident on the vertex.

not defined for directed graph

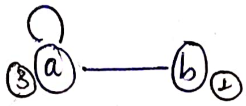


Even degree vertices = a b c e f

odd degree vertices = d g

* In pseudograph —

self loop gives degree of 2.



self loop contributes to twice in the degree of a vertex.

1. Degree 1 vertex
   ↳ pendant vertex
2. Degree 0 vertex
   ↳ isolated vertex

Σ Degree = 2* size of graph

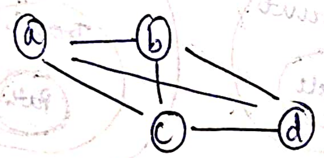
Σ Degree = 2* no of edges

Order of graph — cardinality of vertex set.

Size of graph — cardinality of edge set.

$\delta(G) \rightarrow$ minimum degree in G.

$\Delta(G) \rightarrow$ maximum degree in G.



Order of graph = 4

Size of graph = 6

$\delta(G) \le$ Avg. degree $\le \Delta(G)$

$\Rightarrow n\delta(G) \le n *$ Avg. degree $\le n\cdot\Delta(G)$

$\Rightarrow n\delta(G) \le$ Total degree $\le n\Delta(G)$

$\rightarrow n\delta(G) \le$ No. of edges $\le n\Delta(G)$

Handshaking Theorem —

G(V, E) is any undirected graph with m edges. Then,

$$2m = \sum_{v \in V} \deg(v)$$

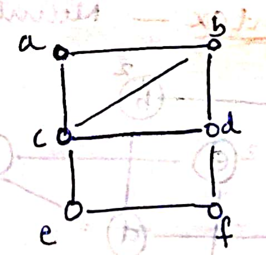
(applies even if self loop & multiple edges are present).

For directed graph,

Total indegree = Total outdegree = No. of edges

* In any graph, the no. of odd degree vertices is always even.

Walk - vertex repetition is allowed
edge repetition is allowed



length of walk = no. of edges in the walk

Trail - vertex repetition is ~~not~~ allowed
edge repetition is not allowed.

Path - vertex repetition is not allowed
edge repetition is not allowed.
$v_1 \neq v_n$

Path → a − b − d − c − b − d − f
a − b − d − c
a − c − e − f − d − c − a − b.

Trail → a − b − d − c − b

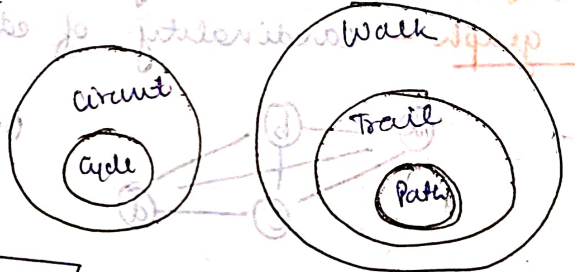
Path → a − b − d − c
a − b − d − f
a − c − e − f.

* Closed trail is called circuit.

a — b — d — c — a

* Path in which first & last vertices are same is called cycle
min. 3 vertices in a cycle.

* If u−v walk exists then, u−v path also exist.



$\boxed{n\delta \leq 2e}$ ← used often in min

Connected Graph

A graph G is connected iff ~~there~~ is a path ~~from~~ b/w every pair of vertices.

Distance b/w a & b is length of shortest path from a to b.

*

$$\boxed{\text{Diameter of a connected graph} = \max_{\forall a,b} \{dist(a,b)\}}$$

Regular graph – Every vertex has same degree.

2 reg. 2 reg. 0 regular

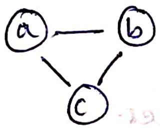
Analysis of regular graph –

d-regular graph.

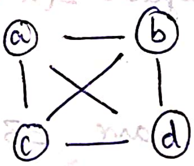
No. of vertices = n.

1. Degree sequence = $d, d, d \dots d$ (n times)
2. $\Delta(G) = d$ $\delta(G) = d$ Avg deg. = d Total deg. = nd
3. No. of edges = $\frac{nd}{2}$
4. Diameter = infinite (disconnected) or finite.

Complete graph – edge exists b/w every pair of vertices.



K_3



n vertices

$\frac{n(n-1)}{2}$ edges.

K_n: $|V| = n$

↳ degree of every vertex = $n-1$

↳ $\Delta(G) = \delta(G) = $ Avg deg = $n-1$

↳ $(n-1)$ regular

↳ Deg. seq. → $(n-1), (n-1), \dots (n-1)$ [n times]

↳ Diameter = 1

↳ connected graph.

Empty graph / Null graph / Edgeless graph.

graph without edges (E_n) → n vertices.

E_n

↳ Degree of each vertex = 0

↳ 0 - Regular

↳ $\delta = \Delta = $ Avg. = 0

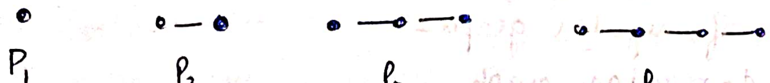
↳ No. of edges = 0

↳ Diameter = 0 if $n=1$

∞ if $n>1$

Path graph – looks like a straight line

$a_1 - a_2 - a_3 - \cdots - a_n$



P_n

↳ Deg. seq. = $\begin{cases} 1 & P_1 \\ 1,1 & P_2 \\ \underbrace{2,2,\ldots 2}_{n-2 \text{ times}},1,1 & P_{n \geq 3} \end{cases}$

↳ Diameter = $n-1$

↳ No of edges = $n-1$

↳ Total degree = $2(n-1)$.

Cycle Graph – must have ≥ 3 vertices.

n vertices n edges.

C_n

↳ Degree of each vertex = 2

↳ 2- regular

↳ $\delta = \Delta =$ avg = 2

↳ No. of edges = n

↳ Diameter = $\left\lfloor \frac{n}{2} \right\rfloor$ $\begin{cases} n/2 & n \text{ is even} \\ \frac{n-1}{2} & n \text{ is odd} \end{cases}$

↳ deg. seq. → $\underbrace{2,2,2\ldots}_{n \text{ times}}$

Wheel graph

add a node to C_n and connect it with every other node

W_n → wheel graph on $n+1$ vertices.

only W_3 is regular

Diameter = 2

W_n

↳ $(n+1)$ vertices

↳ Deg. seq. → $\ldots n, \underbrace{3, \ldots 3}_{n \text{ times}}$

↳ ~~n regular~~, not regular

↳ edges = $\frac{nd}{2} = \frac{n^2}{2}$

$\delta = 3$ $\quad \Delta = n$ $\quad$ avg. deg = $\frac{n+3n}{n+1} = 4n/n+1$

edges = ~~3n~~ 2n

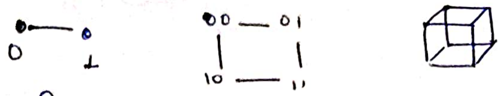
Hypercube graph

Q_n.

- ↳ no. of vertices $= 2^n$
- ↳ no. of edges $= 2^{n-1} * n$
- ↳ Diameter $= n$
- ↳ Degree of each vertex $= n$.

$Q_n \rightarrow$ n bit sequence. each seq. is a node $\therefore 2^n$ nodes.

edge exists b/w u & v if u & v have 1 hamming distance i.e. differ by one bit position.



Q_1 Q_2 Q_3

Boolean lattice is hypercube graph.

Ques - How many edges in Hasse diagram of $(P(A), \subseteq)$ where $|A| = n$

$|P(A)| = 2^n$ Q_n $\therefore$ no. of edges $= n \times 2^n$

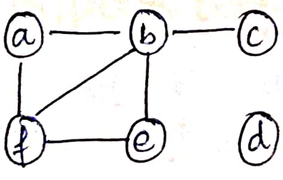
Graph H is subgraph of graph G.

iff $V(H) \subseteq V(G)$ &&
$E(H) \subseteq E(G)$

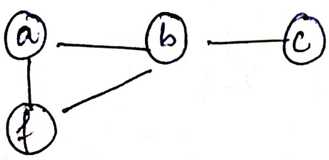
Graph G $\xrightarrow[\text{more edges/ vertices or both}]{\text{delete one or}}$ Graph H

Subgraph
- Spanning subgraph: vertex deletion is not allowed
- Induced subgraph: edge deletion is not allowed (vertex deletion allowed — edge deleted due to vertex deletion is allowed, no extra edge should be deleted)

No. of spanning & induced subgraphs of graph G $= 1$



Subgraph induced by the vertices a, b, c, and f $\cong$ subgraph obtained after deleting d, e vertices.



Imp -
- ↦ Every induced subgraph of a complete graph is also a complete graph.
- ↦ Every graph of n vertices is subgraph of n vertices complete graph.
- ↦ There is exactly 1 subgraph of G (G itself) which is both induced and spanning subgraph.

Ques - n vertices simple graph

Max. no. of edges = ${}^nC_2 = \frac{n(n-1)}{2}$

$\therefore$ No. of simple graphs = $2^{\frac{n(n-1)}{2}}$

Graph isomorphism (very hard problem NP Intermediate)

exp. time complexity.

graphs with same structure

⊙ 2 graphs are isomorphic if there exists bijection in the ~~no. of~~ vertices of the graph which preserves edges.

i.e. $a,b \in E_1$ iff $f(a), f(b) \in E_2$

$|A| = 3 \quad |B| = 2$

No. of bijections from $A \to B$ $= 0$

coz $|A| \neq |B|$

$|A| = 3 \quad |B| = 3$

No. of bijections from $A \to B$ $= 3! = 6$

Graph invariates are necessary for isomorphism but not sufficient → order, degree, size, degree seq., n-length cycle etc.

- No of vertices
- $|E|$
- connectedness
- Degree sequence.

- No. of 3 length cycles
- No. of 4 " "
- No. of 5 " "

* Graph isomorphism is equivalence relation.

Complement of a graph

complement of a graph with n vertices = Complete graph with n vertices − edges in G.

$G(V, E)$

$\bar{G}(V, \bar{E})$ ← comp.

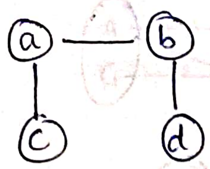
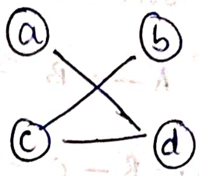
$$\boxed{\bar{E} = E(K_n) - E}$$

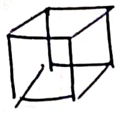
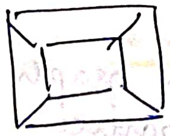
$$|\bar{E}| = {}^nC_2 - |E|$$

i.e. $|E| + |\bar{E}| = {}^nC_2$

Self complementary graph –

If $G = \overline{G}$ i.e. G & $\overline{G}$ are isomorphic, then, G is self complementary

G:  $\equiv$ $\overline{G}$: 

 $\equiv$ isomorphic to 

★ complement of disconnected graph is always connected

If G is disconnected, then, $\overline{G}$ is connected

For every simple graph G,

either G or $\overline{G}$ is definitely connected

★ If a graph has exactly 2 vertices of odd degree, then, they are connected by a path.

★ In any simple graph there is a simple path from any vertex of odd degree to some other vertex of odd degree (not every)

For a simple graph with n vertices, what is max. no of edges?

↳ if G is undirected → ${}^nC_2 = \frac{n(n-1)}{2}$

↳ if G is directed → $2 \times {}^nC_2 = n(n-1)$

Bipartite graph

$G(V, E)$ is bipartite iff

$\exists$ a ~~bijection~~ bipartition X, Y of V such that

① $X \cap Y = \phi$

② $\forall a, b \in X \quad (a, b) \notin E(G)$

$\forall a, b \in Y \quad (a, b) \notin E(G)$

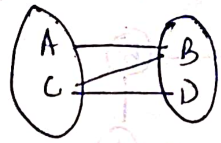
③ $X \cup Y = V$

④ X, Y can be empty.

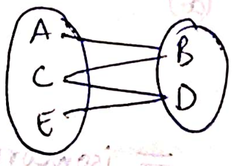
Ques – P_n is bipartite, $n = ?$

~~should be even~~

A — B — C — D

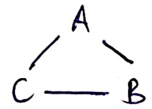


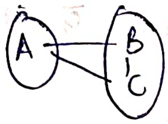
A — B — C — D — E

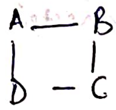


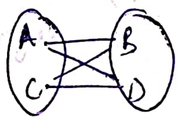
∴ every P_n graph is bipartite..

Ques – C_n is bipartite?

C_3 = 

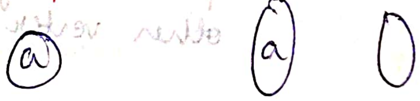
 not bipartite

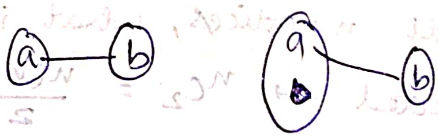
C_4 = 

 bipartite

∴ C_n is bipartite when n is even

Ques – K_n is bipartite?

K_1 =  bipartite

K_2 =  bipartite

K_3 =  not bipartite

K_4 :  not bipartite

∴ only K_1 & K_2 are bipartite

K_n $n \geq 3$ is not bipartite

Theorem

↳ A graph is bipartite if and only if it ~~has even length cycle~~ doesn't have odd length cycle.

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$$

Total degree of vertices in X = Total degree in Y.

For k regular bipartite graph,
all the vertices have degree k. $k \geq 1$

$$\sum_{v \in A} \deg(v) = k|A|$$

$$\sum_{v \in B} \deg(v) = k|B|$$

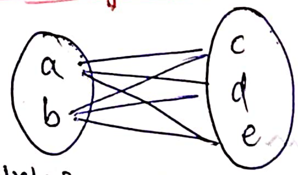
$$\because \sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$$

$$\therefore k(|A|) = k|B| \Rightarrow |A| = |B|$$

- Every subgraph of a bipartite graph is bipartite.

Complete Bipartite graph

$K_{2,3}$



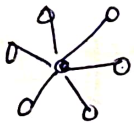
$K_{m,n} \rightarrow m \geq 1 \quad n \geq 1$

$|V| = m+n$

$|E| = m \times n$

deg. seq. = $\underbrace{n, n, n, \ldots n}_{m \text{ times}}, \underbrace{m, m \ldots m}_{n \text{ times}}$

- Star graph is always complete bipartite graph.



$K_{1,6}$ $\quad K_{1,n}$

$\Delta = \max(m,n)$

$\delta = \min(m,n)$

diameter = 1 for $K_{1,1}$

2 for $K_{m,n}$ otherwise

- E_n is complete bipartite graph.
  - edgeless graph

Maximize the number of ~~vertices~~ edges in bipartite graph of n vertices

let $|A| = m$

$|B| = n-m$.

No. of edges in complete b.g. $= |A||B| = m(n-m) = mn - m^2$

e

$$\frac{de}{dm} = n - 2m = 0 \Rightarrow 2m = n \Rightarrow m = n/2$$

$\therefore$ $\lfloor \frac{n}{2} \rfloor$ elements should be in A
$\lceil \frac{n}{2} \rceil$ elements should be in B
} for maximum no. of edges.

Max $\rightarrow \left(\frac{n}{2}\right)^2$ edges.

* If a graph on n vertices has more than $\frac{n^2}{4}$ edges, then, the graph is not bipartite.

Which graphs are bipartite?

(complete graph)	K_n	only for $n=1,2$
(Path graph)	P_n	✓
(Cyclic graph)	C_n	only if n is even
(wheel graph)	W_n	never
(edgeless graph)	E_n	✓
(Hypercube graph)	Q_n	✓
(Complete bipartite graph)	$K_{m,n}$	✓

Cyclic graph

graph containing atleast one cycle.

- There are atleast 2 vertices which have more than one path between them.

Tree
- connected
- undirected
- acyclic

Forest → collection of trees

In a tree, vertex with degree 1 is called leaf.

Complete analysis of tree –

- Tree on n vertices
- ~~n-1~~ (n-1) edges should be there (minimally connected)
- connected, acyclic

Connected graph on n vertices (n-1) edges. Is this tree? ⇒ Yes absolutely YES!

- Tree is maximally acyclic
  - adding one more edge leads to cycle

→ Every tree T with atleast 2 vertices has minimum 2 vertices of degree 1

Height of a node in rooted tree = no. of edges from that node to the farthest leaf.

~~Deapth~~ Depth of a node in rooted tree = no. of edges from the root to that node

* In any cycle of cyclic graph, every vertex in it has a degree greater than or equal to 2.

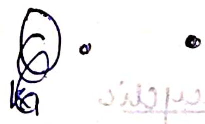
* Tree + 1 edge $\longrightarrow$ graph with exactly one cycle.

No. of edges in a graph with n vertices to guarantee that it is connected — $\underbrace{{}^{n-1}C_2}_{\text{complete graph with } n-1 \text{ vertices}} + \underbrace{1}_{\text{to add } n^{th} \text{ vertex}}$

* Every simple graph with more than $(n-1)(n-2)/2$ edges is connected.

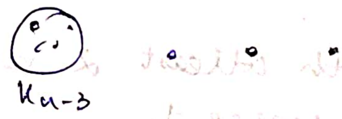
* Maximum number of possible edges in an undirected graph with n vertices and k components.

$k=2$



$k=3$

$k=4$



$\therefore$ for k components, $(k-1)$ components will have 0 edge

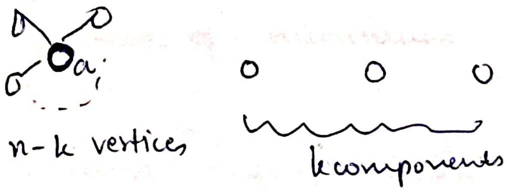
The k^{th} component will be $K_{n-(k-1)} = K_{n-k+1}$

$\therefore$ Max. no. of edges: $\underline{\underline{{}^{n-k+1}C_2}} = \dfrac{(n-k+1)(n-k)}{2}$

* Let G be an arbitrary graph with n vertices and k components. If a vertex is removed from G the number of components in the resultant graph must lie down b/w

A. k & n B. k-1 & k+1 C. k-1 & n-1 ✓ D. k+1 & n-k

For max comp. —

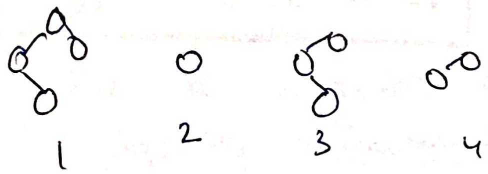


On deleting a_1 no. of components generated from 1st component

$= n-k-1$

$\therefore$ Total no. of components $= n-\cancel{k}-1+\cancel{k} = n-1$

For min comp. —



On deleting ~~com~~ 2, no. of remaining components $= k-1$

Clique

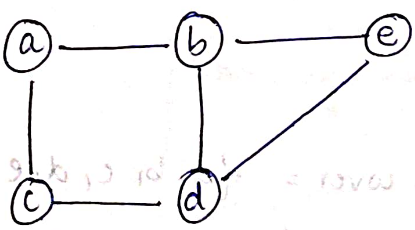
Clique of a graph is the subgraph which is complete.

Size of maximum clique = clique no. = $\omega(G)$

Independent set

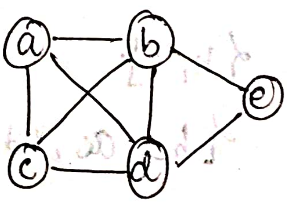
subgraph with vertices not connected to each other.

size of maximum independent set = independence number $\alpha(G)$ ↳ stability no.



Clique → {a, b} {b, d, e} ~~{a, d, e}~~

Independent set → {a, e} {a} ~~{a, d, e}~~ {c, e}



$\omega(G) = 4$ $\alpha(G) = 2$

{a, b, c, d} ← maximum clique   {a, e} ← maximum independent set

$$C_n \begin{cases} \alpha(G) = \left\lfloor \frac{n}{2} \right\rfloor \\ \omega(G) = \begin{cases} 3 & n=3 \\ 2 & n>3 \end{cases} \end{cases}$$

$$K_n \begin{cases} \alpha(G) = 1 \\ \omega(G) = n \end{cases}$$

$$P_n \begin{cases} \alpha(G) = \left\lceil \frac{n}{2} \right\rceil \\ \omega(G) = \begin{cases} 1 & n=1 \\ 2 & n\geq 2 \end{cases} \end{cases}$$

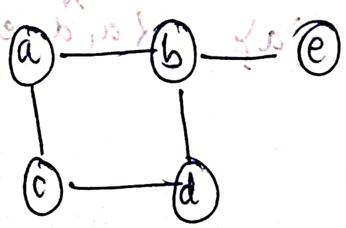
Clique in G $\rightleftarrows$ Independent set in $\bar{G}$

Vertex Cover

set of vertices that cover all the edges.

Edge Cover

edges covering all the vertices

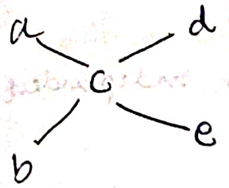


Maximum vertex cover = {a, b, c, d, e}

Maximum edge cover = {ab, ac, cd, bd, be}

Minimum ~~edge~~ vertex cover = {b, c}

Minimum edge cover = {be, ac, bd}



Vertex cover → {c}   {a, b, e, d}

Edge cover → {ac, bc, ec, dc}

★ Edge cover exists only if no isolated vertex in the graph

Size of minimum vertex cover = $\beta(G)$
size of minimum edge cover = $\beta'(G)$

Maximum degree vertex should be in minimum vertex cover ⟶ False

Analysis of edge cover

For n vertices graph,
size of minimum edge cover, $\beta' \geq \lceil \frac{n}{2} \rceil$

K_n — β(vertex cover) = $n-1$

P_n — $\beta = \lfloor \frac{n}{2} \rfloor$, $\beta' = \lceil \frac{n}{2} \rceil$

C_n — $\beta = \lceil \frac{n}{2} \rceil$, $\beta' = \lceil \frac{n}{2} \rceil$

W_n — $\beta = \lceil \frac{n}{2} \rceil + 1$, $\beta' = \lceil \frac{n}{2} \rceil$

Covering number of graph = vertex cover.

Relation between vertex cover and a independent set.

If $S \subseteq V$ is vertex cover, then $\overline{S}$ or $V-S$ is independent set & vice versa.

$\alpha \rightarrow$ size of largest independent set.

$\beta \rightarrow$ size of minimum vertex cover.

$\boxed{\alpha + \beta = n}$ → no. of vertices.

Matching

set of edges that are independent (non adjacent)

- A set of pairwise non adjacent edges in a graph is called a matching.
- The maximum number of edges in a matching in a graph G is called matching number of G and is denoted by ~~μ(G)~~. $\alpha'(G) = \mu(G)$

Perfect matching means that all the vertices are saturated → Matching is perfect if it covers all vertices of the graph.

Cycle graph $C_n \longrightarrow \alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

(perfect matching exists when n = even)

Complete graph. $K_n \longrightarrow \alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

Complete bipartite graph $K_{m,n} \longrightarrow \alpha'(G) = \mu(G) = \min(m,n)$

perfect matching exists only if $m = n$

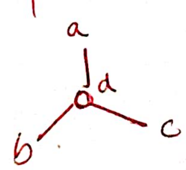
For every graph,

$\mu(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$

size of ~~perfect~~ largest matching

Prefect matching exists means no. of vertices in graph is even.

Vice versa is not true



* If there exists a matching of size k, then, every vertex cover has size $\geq k$

$\mu \leq \beta(G)$

μ → max matching

$\beta(G)$ → min vertex cover

⊙ The cardinality of any matching is less than or equal to the cardinality of any vertex cover.

> **NOTE:** If S is the maximum matching then every edge of the graph is incident on some vertex covered by S.

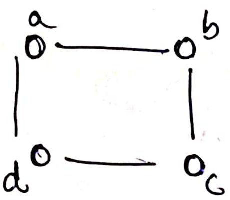
So, vertices covered by S will cover all the edges.

$$\therefore \quad \mu(G) \leq \beta(G) \leq 2\mu(G)$$

K_n (even): $\beta = n-1$, $\mu = \lfloor \frac{n}{2} \rfloor$

$K_{m,n}$: $\mu = \min(m,n)$, $\beta = \min(m,n)$

$\mu = \beta$



$\mu(G) = |\{ab, cd\}| = 2$

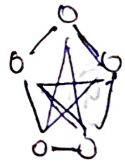
$\beta(G) = |\{a, c\}| = 2$

Cycle graph.

$\mu = 2\beta$

K_{odd}: $\beta = n-1$, $\mu = \lfloor \frac{n}{2} \rfloor$

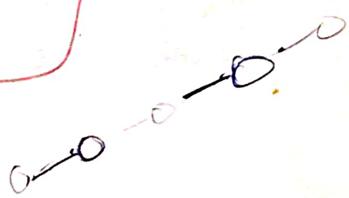
$\beta = 2\mu$

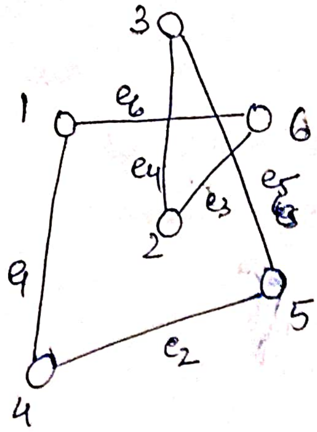


$\beta = 4$

$\mu = 2$

for bipartite graph, $\mu = \beta$





$= C_6$

α = Independent set = 3

β = min vertex cover = 6 - 3 = 3

β' = min edge cover = $\lceil 6/2 \rceil$ = 3

μ = max matching = 3

⊙ For any graph, the size of maximum independent set is less than or equal to edge cover.

$$\boxed{\alpha \le \beta'}$$

Conclusion

α = Maximum independent set

$\alpha' = \mu$ = Maximum matching

β = Minimum vertex cover.

β' = Minimum edge cover.

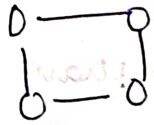
1. $\alpha + \beta = n = |V|$ $\quad \alpha' + \beta' = n$
2. $\mu \le \beta \le 2\mu$
3. $\beta' \ge \left\lceil \frac{n}{2} \right\rceil$
4. $\alpha' = \mu \le \left\lfloor \frac{n}{2} \right\rfloor$
5. $\alpha \le \beta'$
6. Bipartite graph $\rightarrow \mu = \beta$

Graph coloring

assignment of labels / colors to vertices of a graph such that no two adjacent vertices share the same color.

Graph is K-colorable?- graph can be colored using atmost k colors?

Minimum number of colors needed — chromatic number



1 colourable ?	No
2 colorable ?	Yes
3 colorable ?	Yes

↳ denoted by $\chi(G)$ ← geta

K_n complete graph $\quad \chi = n$

E_n edgeless graph $\quad \chi = 1$

P_n path graph $\quad \chi = 2, \; n \geq 2$ else 1.

~~C_n cycle graph~~ $\quad \chi = \begin{cases} \lceil \frac{n}{2} \rceil \\ \frac{n}{2} \end{cases}$ ~~odd length cycle~~ ~~even length cycle~~

$n - 2\lfloor \frac{n}{2} \rfloor + 2$ $\quad C_n$ cycle graph $\quad \chi = \begin{cases} 3 & n \text{ is odd} \\ 2 & n \text{ is even} \end{cases}$

simple connected graph which does not contain odd length cycle

Bipartite graph $\quad \chi = 2$ if edge exists else $\chi = 1$

$K_{m,n}$ complete bipartite graph. $\quad \chi = \begin{cases} 1 & \text{if } m=0 \text{ or } n=0 \\ 2 & \text{otherwise.} \end{cases}$

$\chi(G) \leq \Delta(G)+1$

$Q_n \rightarrow$ Hypercube graph (bipartite) $= \chi = \begin{cases} 2 & n \geq 1 \\ 1 & n = 0 \end{cases}$

$W_3 = K_4$
4 colors needed

$W_n \rightarrow$ wheel graph on $n+1$ vertices $= \chi = \begin{cases} 4 & n \text{ is odd} \\ 3 & n \text{ is even} \end{cases}$

Chromatic number is always greater than or equal to clique number

$$\chi(G) \geq \omega(G)$$

For any graph,

$$\boxed{\omega(G) \leq \chi(G) \leq \Delta(G)+1}$$

$$|E(G)| \geq \frac{\chi(G)\,(\chi(G)-1)}{2} \quad \text{for every graph}$$

χC_2

b/w every color class min 1 edge should be there.

$$|V(G)| \leq \chi(G)\,\alpha(G)$$

$\frac{n}{\chi} \rightarrow$ avg. size of color class.

$\chi = \Delta+1$ for only 2 graphs
- $\rightarrow$ odd length cycle graph
- $\rightarrow$ complete graph (K_n)

Brooke's Theorem.

Vizing Theorem

For any simple graph,
edge chromatic number can be ~~eig~~ either $\Delta(G)$ or $\Delta(G)+1$.

$$\boxed{\Delta(G) \le \chi'(G) \le \Delta(G)+1}$$

* Bipartite graph $\rightarrow$ $\chi'(G) = \Delta(G)$

* Every graph having $\chi'(G) = \Delta(G)+1$ must have atleast three vertices of maximum degree

$$\left.\begin{array}{l} C_{odd} \\ K_{odd} \\ \text{regular}_{odd} \end{array}\right\} \chi'(G) = \Delta(G)+1$$

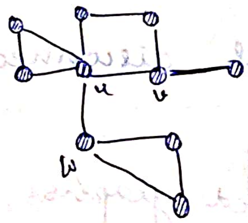
Cut vertex (Articulation point).

A vertex v of a graph G is called cut vertex of G if ~~ck~~ ~~no~~ on removal of v from G, the no. of components increases.

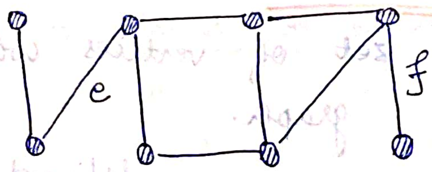
i.e. $k(G-v) > k(G)$ $k(G) \rightarrow$ no. of components of G

Cut edge (Bridges)

An edge e of a graph is said to be a cut edge if $k(G-e) > k(G)$



$u, v, w \rightarrow$ cut vertices.



$e, f \rightarrow$ cut edges.

Theorem If e is the bridge incident on vertex v, then, v is cut vertex of G and only if $\deg(v) \geq 2$.

$$\underbrace{k(G)-1}_{\text{when } v \text{ is isolated vertex}} \leq \underbrace{k(G-v)}_{\text{no. of components in graph after removing a vertex}} \leq \underbrace{n-1}_{\text{star graph}}$$
} Graph with isolated vertex

$\delta \geq 0$

$$k(G) \leq k(G-v) \leq n-1$$
} Graph without isolated vertex

$\delta \geq 1$

On deleting an edge,

no. of components = $k(G)$ or $k(G)+1$

Every path b/w 2 vertices u and v in a graph contains edge e $\Rightarrow$ e is a cut edge / bridge.

~~every graph is~~
~~every edge~~

all edges in all graphs are cut edges $\longrightarrow$ graph is forest

all edges in connected graph are cut edges $\rightarrow$ graph is tree

Vertex cut

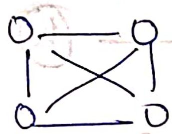
set of vertices whose removal disconnects the graph.

defined for only connected graphs.

Size of smallest vertex cut = Connectivity number
$K(G)$ ← kappa

If cut vertex is present in the graph,
then $K(G) = 1$

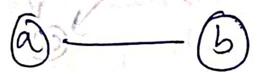
Full definition of vertex cut -
In a connected graph, a vertex cut is a subset of vertices whose removal either disconnects the graph or a single vertex remains.

K_4 :  → $K = 3$ (single vertex remain)

$\boxed{K_n = n-1}$

$K(\text{disconnected graph}) = 0$

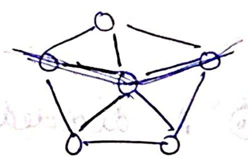
$K(C_n) = 2$ $n \geq 4$
$\quad\quad\quad\ \ 2$ $n = 3$ } 2 for $\forall n$



No cut vertex (articulation point)
{b}; {a} is vertex cut

k connected graph ← $K(G) \geq k$

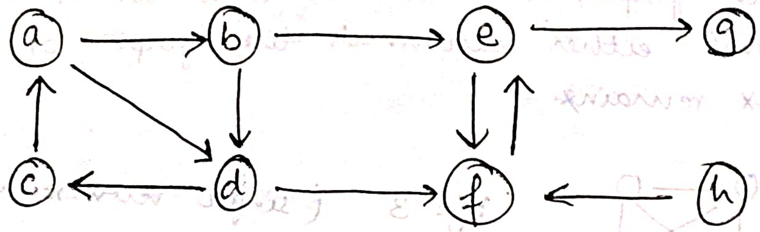
$K_{m,n} \longrightarrow K(K_{m,n}) = \min(m, n)$

$W_n \longrightarrow K(W_n) = 3$ 

Imp -
The connectivity of a graph is atmost its minimum degree (removing all vertices around vertex with minimum degree gives only 1 vertex disconnected graph)
$K \leq \delta$

Strongly Connected Component

Strongly connected component of a directed graph G is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u & v, there is a directed path from u to v and a directed path from v to u.

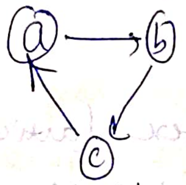


4 strongly connected components

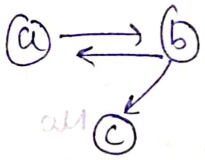
$[a]_R = \{a, b, c, d\} = [b]_R = [c]_R = [d]_R$

$[e]_R = \{e, f\} = [f]_R$

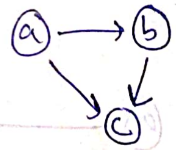
$[g]_R = \{g\}$ $\quad [h]_R = \{h\}$



No. of strongly connected components = 1



#SCC = 2



#SCC = 3

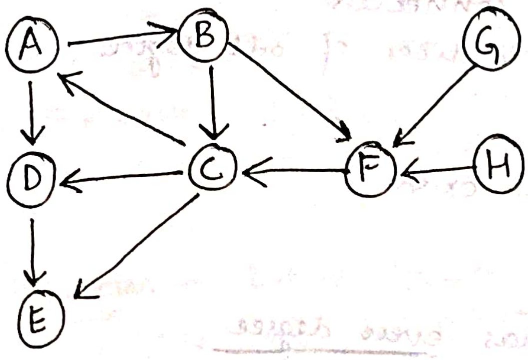
Graph is strongly connected if no. of components in the graph = 1

all vertices are reachable from every other vertex

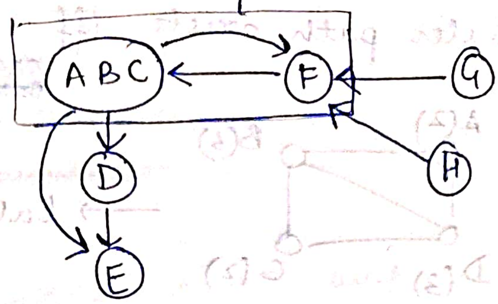
⊙ A directed graph is weakly connected if there is a path b/w every 2 vertices in the underlying undirected graph, which is the undirected graph obtained by ignoring the directions of the edges of the graph.

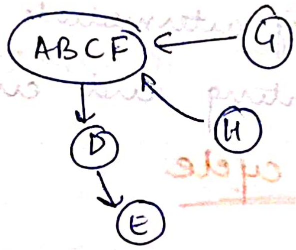
Every strongly connected graph is also weakly connected.

* Every directed graph is DAG of strongly connected compon



cycle :: can be merged.





Euler graph

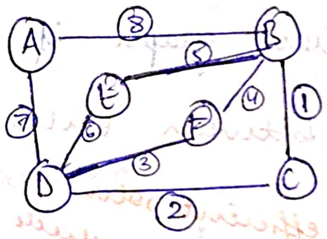
Connected graph in which it is possible to have a walk that crosses each edge exactly once and returns to the starting point.

Circuit → closed trail

closed: starting = ending

trail: edge must not repeat.

circuit containing every edge



B C D F B E D A B

Euler circuit

Euler graph.
Connected graph with existence of Euler circuit

connected even degree graph.

An undirected graph G is Eulerian if and only if it is connected and every node has even degree.

Graph with all vertices of even degree but not euler graph → disconnected graph

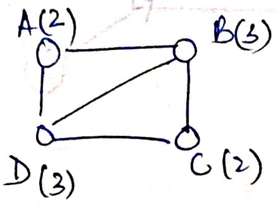
Connected graph	+	even degree vertices	=	Euler graph

graph → isolated vertex are allowed.

Euler Path — starting & ending vertices ~~can~~ must be different.

open trail →

Euler path exists iff G is connected
~~only~~ exactly 2 vertices of odd degree



⟶ Euler path exists.

→ Every intermediate vertex has even degree

→ Starting and ending ~~degree~~ vertex have odd degree

Hamiltonian cycle

Euler circuit	v/s	Hamiltonian ~~circuit~~ cycle
visit every edge exactly once and return to the starting vertex		visit every vertex exactly once and return to the starting point

G is Hamiltonian graph if there exists hamiltonian cycle

★ No relationship between Euler circuit and hamiltonian cycle.

↳ no efficient soln
↳ manually check करना पड़ेगा

★ Clay Institute will give $1,000,000 to the person who will give polynomial time solution

Ready, Priyanshu?

Hamiltonian ~~circuit~~ cycle exists ⟶ Hamiltonian path also exists

Imp.

- Every Hamiltonian graph must be connected.
- No tree is hamiltonian (coz acyclic)
- For each $n \geq 3$, C_n is hamiltonian
- For each $n \geq 3$, K_n is hamiltonian
- for each $n \geq 2$, $K_{n,n}$ is hamiltonian

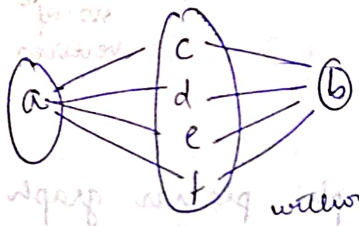
Dirac theorem — Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a hamiltonian cycle

$\delta(G) \geq n/2 \longrightarrow$ hamiltonian graph exists.

Planar graph

A graph is called a planar graph if there is some way to draw it in a 2D plane without any of the edges crossing.

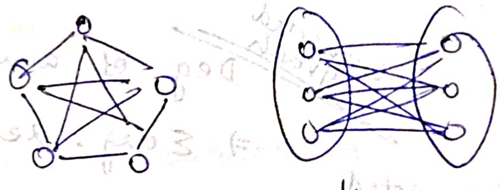
every $K_{2,n}$ is planar



without edge crossing.

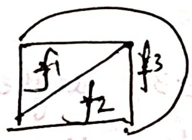
K_5 and $K_{3,3}$ are non planar

→ most common examples.



K_5 $K_{3,3}$

Planar representation divides the plane into ~~regions~~ faces



K_4

- Any tree → always planar with 1 face.
- forest → always planar with 1 face

→ All planar ~~gr~~ representations have same number of faces.

Degree of face —
Number of sides of edges the face touches is called degree of face.

For any planar graph,

$$\sum_{\forall i} Deg(f_i) = \text{Degree sum of faces} = 2|E|$$

even no. of faces have odd degree

Total degree of faces = Total degree of vertices = $2|E|$

→ Degree of face can be zero (edgeless graph)
→ Degree of face cannot be 1.
→ ⓐ—ⓑ K_2 is only this graph has face with degree 2

Euler's formula—

No. of faces (F) = $E - V + C + 1$

very important for GATE

or $\boxed{V + F = E + C + 1}$

V → no. of vertices, F → no. of faces, E → no. of edges, C → no. of components

⊙ In a simple planar graph with ≥ 3 vertices, degree of every face is >= 3.

connected graph

Deg of any face ≥ 3

⇒ $\sum deg = 2e$ $\quad 3 \cdot F \le \sum deg$

connected planar graph → ⇒ $3F \le 2e$ ⇒ $3(e - v + 1 + 1) \le 2e$ ⇒ $3e - 3v + 6 \le 2e$

⇒ $\boxed{e \le 3v - 6}$

⊙ In any connected planar graph with $|V| \ge 3$, degree of every face ≥ 3 and if there is no triangle, then, degree of every face ≥ 4. △ → only △ leads to degree 3.

Connected No triangle in simple planar graph.

$deg(f) \ge 4$

⇒ $\sum deg(f) = 2e$ ⇒ $4F \le 2e$

⇒ $4(e - v + 2) \le 2e$ ⇒ $4e - 4v + 8 \le 2e$ ⇒ $\boxed{e \le 2v - 4}$

⊙ Every planer graph is 4 colorable.

i.e. $\chi(G) \le 4$.

↳ Four colorable theorem has not been proved by any human! It was proved by COMPUTER

Adjacency Matrix

If M is the adjacency matrix of a graph.

$M[i][j] \longrightarrow$ denotes no. of walks from vertex i to j of length 1.   i — j

$M^2[i][j] \longrightarrow$ no. of walks of length 2 from vertex i to j   ~~i — □ — □ — j~~

i — □ — j

⋮

$M^k \longrightarrow$ walks of length exactly k b/w every 2 vertices.

Applications

↳ diagonal entries in M^2 give the degree of that vertex in simple undirected graph.

$$\text{Trace}(M^2) = \Sigma \deg(v) = 2e$$

If self loops are present, then,

$$\text{Trace}(M) + \text{Trace}(\overset{\text{self loop counted once}}{M^2}) = 2e$$

$$\Rightarrow \boxed{\text{Trace}(M + M^2) = 2e}$$

after putting all self loops

$(M + I_n)^{(n-1)}$ ← all non zero entries ⇒ graph is connected

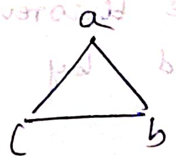
$(n-1) \to n$ for directed graph

$(M^0 + M^1 + M^2 + \dots M^{n-1})$ ← all non zero entries ⟺ graph is connected

$M^{n-1} \to +M^n$ for directed graph

Number of three length cycles in undirected simple graph

One triangle contribution in $trace(M^3) = 6$



a — b — c — a
a — c — b — a
b — a — c — b
b — c — a — b
c — a — b — c
c — b — a — c

$\therefore$ No. of triangles = $\frac{1}{6}$ trace(M^3)

trace (M^3) = 6 (No of triangles)

In directed simple graph,

trace(A^3) = ~~No. of~~ 3 (No. of triangles)

Syllabus
- magma
- semigroup
- monoid
- group
- quasi group

Algebraic structure (AS)
↳ defining operations on base set
(Base set, *, #, - - - -)

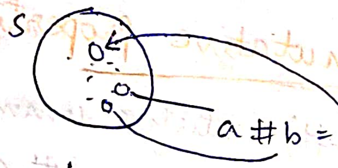
Algebraic structure with single binary operation → GATE Syllabus

1. Closure Property

Applying operation on any 2 elements of a set, the result also belongs to the set.

अगर अंदर की बात अंदर ही रहे तो closure property satisfy हो जाएगी।

(S, #) → AS



$\forall_{a,b}\ a \# b = c$

$a \# b = c$

Set S is closed under # operation

↳ (S, #) follows closure property under # operation

({0}, +) → closed

({1}, +) → not closed

* Binary operation ⟺ closure property satisfied

(G, *) * is binary operation iff $\forall_{a,b} \in G;\ a * b \in G$

2. Associative property

(S, #) is associative iff

$(a \# b) \# c = a \# (b \# c)$

3. Identity Property

Identity element does not affect operation

(S, #) $a \# e = a$ & $e \# a = a$

e is fixed for all elements e ∈ Base set

(N, +) ⟶ No identity element

(W, +) ⟶ e = 0.

(Z, −) ⟶ no identity element.

rational ↷ (Q, ×) ⟶ e = 1

4. Inverse property

$(S,\#)$ satisfies inverse property iff $\forall_{a \in S}, \exists b \in S$ such that $a\#b = e$ and $b\#a = e$

$\underline{b = a^{-1} \text{ and } a = b^{-1}}$

① an element can have multiple inverses.

inverse of element is not unique.

$(R, \times)$ does not satisfy inverse property coz 0^{-1} doesn't exist.

$(N, +)$ " " " " " coz e doesn't exist

$\boxed{e^{-1} = e}$

5. Commutative property

$(S,\#)$ satisfies commutative property iff

$\forall_{a,b \in S} \quad a\#b = b\#a$

Classification of algebraic structures based on properties —

Abelian = commutative

∴ Abe

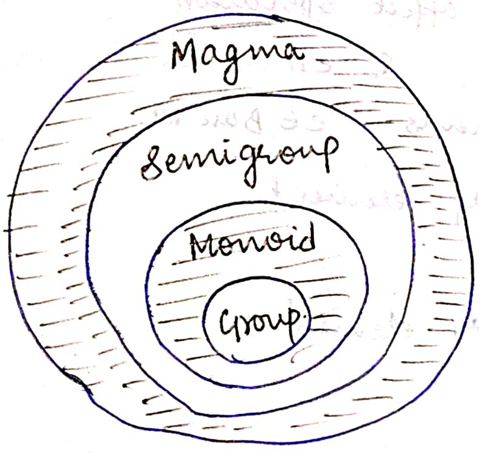
Magma (groupoid) → closure

Semigroup → closure + Associative

Monoid → closure + Associative + Identity

Group → closure + Associative + Identity + Inverse.

Abelian group / monoid / semigroup / magma → group / monid / semigroup / mangma + commutative property.



★ Order of any algebraic structure = Cardinality of Base set

★ No. of binary operations $*$ on a set of size n $= n^{n^2}$ [No of functions from $S \times S \to S$]

No. of commutative binary operations $= n^n \cdot n^{\frac{n^2-n}{2}}$

$= n^{n+\frac{n^2-n}{2}} = n^{\frac{n^2+n}{2}}$

Properties of ~~mono~~ monoid.

↦ Identity element is unique (true for all algebraic structures)

↦ left and right cancellation property does not hold.

do not hold on monoid

$a \# b = a \# c \rightarrow b = c$ (left cancellation)

$b \# a = c \# a \rightarrow b = c$ (right cancellation)

$(\underbrace{\{1, \omega, \omega^2\}}, \times) \longrightarrow$ Abelian group

cube roots of unity

$1+\omega+\omega^2=0$

$\omega^3 = 1$

closure ✓ $\omega^2 \times \omega^2 = \omega^4 = \omega \in S$

Asso ✓ coz multiplication

Identity ele $\rightarrow 1$

Inverse ✓ $1^{-1} = 1$

$\omega^{-1} = \omega^2$

$(\omega^2)^{-1} = \omega$.

Commutative ✓

∴ abelian group

Note → for all $n \geq 1$, n^{th} roots of unity are abelian groups under ~~us~~ multiplication.

$(\{1, -1, i, -i\}, \times)$

↳ 4^{th} root of unity.

closure $(-1)\times(-i) = -1$ ✓

$i \times i = 1$

Associative ✓ coz multiplication

Identity ele $= 1$

Inverse → $1^{-1} = 1$

$-1^{-1} = -1$

$i^{-1} = -i$

$-i^{-1} = i$

Commutative ✓

∴ abelian group.

Addition modulo n is abelian group ✓

$\{ Z_n , \oplus_n \}$ $\quad a \oplus_n b = \cancel{a \bmod b}\ (a+b) \bmod n$

set of 0 to n-1

Closure property ✓

Associative? ✓ $\longrightarrow$ $a \oplus_n (b \oplus_n c)$ $\qquad (a \oplus_n b) \oplus_n c$

Identity ele = 0 $\qquad a \oplus_n (b+c) \bmod n \qquad ((a+b) \bmod n) \oplus_n c$

Inverse $\rightarrow n-a$ $\qquad [a + ((b+c) \bmod n)] \bmod n \qquad (((a+b) \bmod n)+c) \bmod n$

Commutative ✓ $\qquad (a+b+c) \bmod n = (a+b+c) \bmod n$

$a^{-1} = n-a \quad \forall a \neq 0$

$a^{-1} = 0 \quad$ for $a=0$.

① In a group, every element has unique inverse ✓

let a be 2 inverses b and c

$\therefore a^{-1} = b \ \& \ a^{-1} = c \quad \Rightarrow a * b = e$ and $a * c = e$

$\Rightarrow a*b = a*c \quad \Rightarrow a^{-1} * (a*b) = a^{-1} * (a*c)$

$\Rightarrow (a^{-1}*a)*b = (a^{-1}*a)*c \quad \Rightarrow e*b = e*c \Rightarrow \underline{b=c}$

$\therefore$ unique inverse.

Proof that group has left & right cancellation property.

$a*b = b*c$ does not imply $a=c$

common sense है भाई

<u>In a group,</u>

$(a*b)^{-1} = b^{-1} a^{-1}$

Proof— $(a*b)*(a*b)^{-1} = e$

$a^{-1} * a * b * (a*b)^{-1} = a^{-1}$

$e*b*(a*b)^{-1} = a^{-1}$

$b*(a*b)^{-1} = a^{-1}$

$b^{-1}*b*(a*b)^{-1} = b^{-1}a^{-1}$

$e*(a*b)^{-1} = b^{-1}a^{-1}$

$(a*b)^{-1} = b^{-1}\underline{a^{-1}}$

Property of numbers that allows us to remove parenthesis from expressions $\longrightarrow$ Associativity.

Ques $a\#b\#c$

For which of the following $a\#b\#c$ makes sense (is unambiguous)

1. Groupoid
2. Semigroup ✓
3. Monoid ✓
4. Group. ✓

} bcoz associative property holds.

If operation # is ~~unambiguous~~, ~~then~~ associative, then, $a\#b\#c\#d$ is unambiguous expression.

Cayley Table / Multiplication Table / Operation Table.

$(\{a, b, c\}, \#)$

	a	b	c
a	a#a	a#b	a#c
b	b#a	b#b	b#c
c	c#a	c#b	c#c

multiplication operation यहीं है पर ये multiplication table नाम है HaHaHaHa

a, b, x are elements of a group.

Soln of $ax = b \longrightarrow x = a^{-1}b$

Soln of $xa = b \longrightarrow x = ba^{-1}$

	....	y	z
a		b	b

← not possible becoz soln is always unique

If it were possible,

$a*y = b \quad a*z = b$

$\Rightarrow a*y = a*z$

$\rightarrow y = z$ (left cancellation)

∴ no 2 entries in same row can be same

In a group,

if $a*b = e$ then, $b*a = e$

coz if $a = b^{-1}$ then $b = a^{-1}$ unique

If $a*a = a$, then, $a = e$

Properties of Cayley Table for groups -

- Every element $g \in G$ appears exactly once in each row and each column.
- Every row/column is permutation of all elements i.e every element appears once.
- Row and column of identity element is same as header.

$(e\ a\ b\ c)$ $a^{-1} = a$ $b^{-1} = b$ $c^{-1} = c$

$(e\ a\ b\ c)$ $a^{-1} = a$ $b^{-1} = c$ $c^{-1} = c$

GATE 2007

No. of isomorphic groups of order 4 = 2 (both abelian)

No. of isomorphic groups of order 3 = 1

No. of isomorphic groups of order 2 = 1

No. of isomorphic groups of order 1 = 1.

Cayley Table	Multiplication Table	Operation Table

For $\forall a \in G$, $(G, *)$ is group.

$a^0 = e$ $\quad a^2 = a * a$ $\quad a^{-n} = (a^{-1})^n$

$a^1 = a$ $\quad a^{-1} = a^{-1}$ $\quad a^{-2} = (a^{-1})^2 = (a^{-1}) * (a^{-1})$

'a' is an element of group.

$$\left.\begin{aligned} a^n \cdot a^{-n} &= e \\ a^m \cdot a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \end{aligned}\right\} \text{becoz of associativity.}$$

Subgroups

Subset is a group under same operation

$(H, *)$ is subgroup of $(G, *)$ iff

$(G, *)$ is a group, $H \subseteq G$ and $(H, *)$ is a group.

$G = \{1, -1, i, -i\}$ under multiplication

Subgroups of G

1. $(\{1\}, \times)$
2. $(\{1, -1\}, \times)$
3. $(\{1, -1, i, -i\}, \times)$

Not subgroups

$(\{-1\}, \times) \rightarrow$ not closed

$(\{i, -1\}, \times)$ $-i$

$(\{1, i, -1\}, \times)$ $-i$

- Identity element of G should be present in H.
- If $a_1 \in H$, then $a_1^{-1}, a_1^2, a_1^3 \ldots \in H$

* $(Z, +)$ is a group.

$\forall n \in Z$, $(nZ, +)$ is a subgroup of $(Z, +)$

multiples of n.

$G = \{e, a, b, c \ldots\}$

Assume, $a^{-1} = d$.

subgroup of G that contains 'a'

$H = \{e, a, a^2, a^3, a^4 \ldots, d, d^2, d^3 \ldots\}$

$(G, *)$ is a group let $a \in G$

$\langle a \rangle$ = smallest subgroup generated by a.

$\langle a \rangle = \{a^n \mid n \in Z\}$ $\quad a^0 = e \quad a^{-1} \ldots$

$a^1 = \ldots \quad a^{-2} \ldots$

$a^2 \ldots$

Generator:- Group $(G, *)$ $g \in G$ is generator of G iff $\langle g \rangle = G$.

1. $(Z, +) \rightarrow$ Group G

Generator of G = 1.

2. $(\{0,1,2,3\}, \oplus_4) \rightarrow$ Group G

0 1 2 3

Generator of G = 1, 3

$$\boxed{\langle a \rangle = \langle a^{-1} \rangle}$$

Multiplication modulo n

$(Z_4 = \{0,1,2,3\}, \otimes_4)$

- Closure ✓
- Associative ✓
- Identity element = 1 ✓
- 0^{-1} = DNE 2^{-1} = DNE
  $1^{-1} = 1$ $3^{-1} = 3$
- commutative ✓

∴ Multiplication modulo n is abelian monoid.

NOT A GROUP.

$(U_4 = \{1,3\}, \otimes_4)$

Only those elements from Z_4 are taken which have inverse.

abelian group.

$U_n = \{m \in Z_n \mid m$ is coprime with n or $GCD(m,n)=1\}$

$U_n \rightarrow$ contains only those elements which are coprime to n.

$(Z_8 = \{0,1,2,3,4,5,6,7\}, \otimes_8)$

$U_8 = \{1,3,5,7\}, \otimes_8$

abelian group

Ques.

Let G be a group which can be generated by an element 'a' which is inverse of itself. Then, what is the order of G?

Given:- $a = a^{-1} \Rightarrow a^2 = e \quad a^3 = a \cdot a^2 = a \cdot e = e$

∴ The set is $\{a, e\}$ or $\{e\}$

∴ Cardinality / Order of G = 1 or 2

→ since order is different, 2 non isomorphic groups are possible.

each is abelian.

Order of an element / Period length / Period = ~~Number~~ Order of the subgroup generated by that element.

→ least positive integer n such that $a^n = e$.

Cyclic group – Group that can be generated by a single element

$(\{1, 3, 5, 7\}, \otimes_8) \rightarrow$ not cyclic group.

$(\{1, 2, 3, 4\}, \otimes_5) \rightarrow$ cyclic group

2 is the generator.

$(Z, +)$ is cyclic generated by 1

$(Z_n, \oplus_n)$ is cyclic generated by 1

when $|G| = n$, then for finite group G
$\forall a,\ a^n = e$

- Every cyclic group is abelian.

- Order of smallest group that is not cyclic = 4.

Every group of order 1, 2, 3 is cyclic.

- Order of identity element = 1.

- For any finite group,
$|a| \le |G| \quad a \in G$

- $\langle g \rangle$ is finite cyclic group of order n
Then $\{1, g, g^2 \dots g^{n-1}\}$ are distinct.

Imp Theorem
- Subgroup of cyclic group is cyclic.

Theorem —
Any cyclic group is isomorphic to either $(Z, +)$ or $(Z_n, \oplus_n)$

→ infinite cyclic group isomorphic to $(Z, +)$
→ finite cyclic group isomorphic to $(Z_n, \oplus_n)$

Lagrange's Theorem –

Order of subgroup divides order of group.

- If $|G| = \text{prime}$, then, G is cyclic
- $|G| = \text{prime} \Rightarrow G$ is abelian
  $|G| = (\text{prime})^2 \Rightarrow G$ is abelian
- $|G| = \text{prime} \Rightarrow G$ is cyclic
  $|G| = (\text{prime})^2 \not\Rightarrow G$ is cyclic

$U_8 \rightarrow$ not cyclic group

$\{1, 3, 5, 7\}$

हर non cyclic grp के example में चिपका दो

Abelian group

① If every element of a group is its own inverse, or $a^2 = e$, then, the group is abelian.

vice versa not true

For every group,

$$(ab)^{-1} = b^{-1}a^{-1} \Rightarrow (ba)^{-1} = b^{-1}a^{-1}$$

$$\Rightarrow a^{-1}b^{-1} = b^{-1}a^{-1} \Rightarrow \underline{ab = ba}$$

② If G is a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$ then show that G is abelian

$$(ab)^2 = a^2b^2 \Rightarrow \cancel{a}ba\cancel{b} = \cancel{a}ab\cancel{b}$$

$$\Rightarrow ba = ab \quad \therefore \text{commutative}$$

$\therefore$ abelian group.

③ Every subgroup of an abelian group have to be abelian

④ $xy = zx$ implies $y = z$

Then, G is abelian

vice versa also true

⑤ $xyz = ayz$ implies $xz = az$

Then, G is abelian

vice versa also true

Subgroup

S is subgroup of G iff S is non empty and whenever $a, b \in S$ then $ab^{-1} \in S$.

If S is subset of finite group G, then, S is a subgroup of G if and only if S is non empty and closure property holds.

GRAPH THEORY

Degree of vertex = No. of edges incident on the vertex

Pseudograph – graph in which parallel edges and self loop are allowed.

Multigraph – undirected graph in which parallel edges are allowed, no self loops.

Self loop contributes to twice the degree of vertex in pseudograph.

Σ Degree = 2 * size of graph
= 2 * no. of edges.

Order of graph – Cardinality of vertex set

Size of graph – Cardinality of edge set.

$\delta(G) \rightarrow$ Minimum degree in G

$\Delta(G) \rightarrow$ Maximum degree in G

$$\delta(G) \leq \text{Avg. degree} \leq \Delta(G)$$

$$n\,\delta(G) \leq n * \text{Avg. degree} \leq n\,\Delta(G)$$

$$n \cdot \delta(G) \leq \text{Total degree} \leq n\,\Delta(G)$$

$$\frac{n\,\delta(G)}{2} \leq \text{No. of edges} \leq \frac{n\,\Delta(G)}{2}.$$

Handshaking Theorem

G(V, E) any undirected graph with m edges. Then,

$$2m = \sum_{v \in V} \deg(v)$$

No. of odd degree vertices is always even.

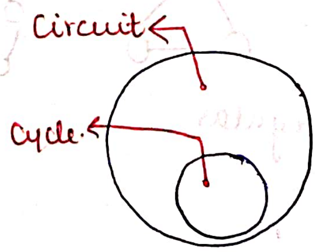
Walk – vertex repetition & edge repetition allowed.

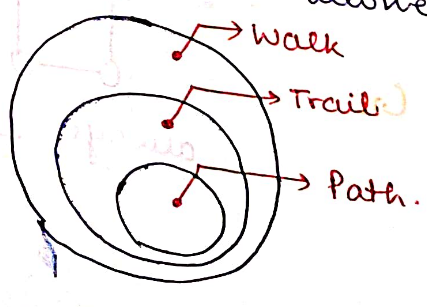
Trail – vertex repetition is allowed & edge repetition not allowed

Path – niether vertex repetition nor edge repetition is allowed.

Closed trail is called circuit.

Closed path is called cycle

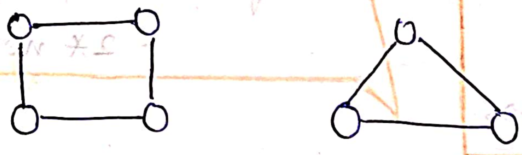




$$\text{Diameter of a connected graph} = \max_{\forall a,b} \{ \text{dist}(a,b) \}$$

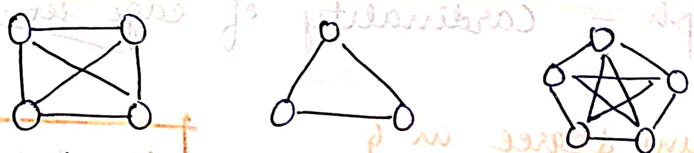
Regular graph – every vertex has ~~even~~ ~~d~~ same degree.

n-reg.



Complete graph – edge exists b/w every pair of vertices.

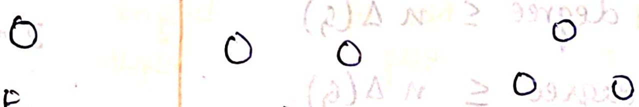
K_n



K_4 K_3 K_5

Empty graph – graph without edge
Degree of each vertex = 0.

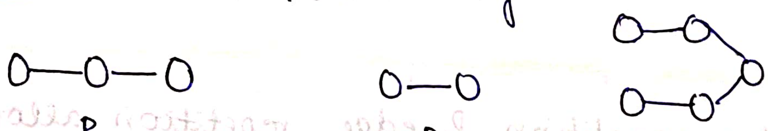
E_n



E_1 E_2 E_3

Path graph – ~~look~~ vertices arranged in straight line.

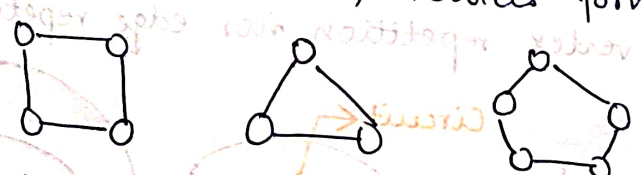
P_n



P_3 P_2 P_5

Cycle graph – ≥ 3 vertices, vertices form a cycle.

C_n

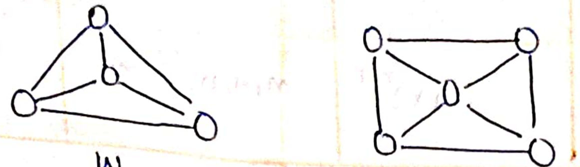


always 2 regular.

Wheel graph — add a node to C_n and connect it with every other node.

W_n

W_n contains $(n+1)$ nodes.



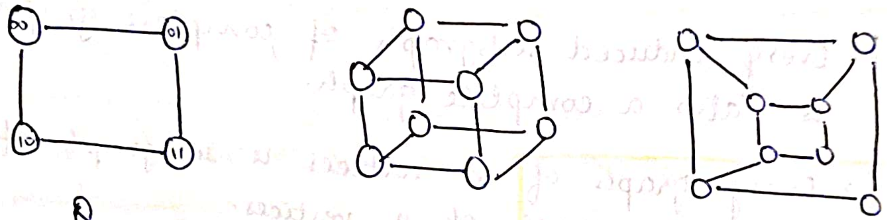
W_3 W_4.

Q_n

Hypercube graph

edge exists b/w u and v iff u and v differ by only one bit

$Q_n \rightarrow n$ bit seq. no. $\rightarrow 2^n$ nodes.



Q_2 Q_3 Q_3. (planar rep.).

Type of graph	No. of vertices	No. of edges.	Degree sequence	Diameter
d reg.	n	$\frac{nd}{2}$	$\underbrace{d, d, \ldots\ldots d}_{n \text{ times}}$	infinite (if disconnected) finite (if connected)
K_n	n	$\frac{n(n-1)}{2}$	$\underbrace{(n-1)(n-1)\ldots\ldots(n-1)}_{n \text{ times}}$. $(n-1)$ regular	1 (always connected)
P_n	n	$n-1$	$\underbrace{2,2,\ldots 2}_{n-2 \text{ times}},1,1$ $n \geq 3$	$n-1$
C_n	n	n	$\underbrace{2,2,2 \ldots - 2}_{n \text{ times}}$	$\frac{n}{2}$ if n is even $\frac{n-1}{2}$ if n is odd

	No. of vertices	No. edges	Degree seq.	Diameter	
W_n	$n+1$	$2n$	$n, \underbrace{3, 3, \ldots 3}_{n \text{ times}}$	2	only W_3 is regular
Q_n	2^n	$n \times 2^{n-1}$	$n, n, n \ldots n$	n	

Subgraph

- Spanning subgraph
  vertex deletion is not allowed
- Induced subgraph
  edge deletion is not allowed

↦ Every induced subgraph of complete graph is also a complete graph.

↦ Every graph of n vertices is subgraph of complete graph of n vertices.

↦ There is exactly one subgraph of G (G itself) which is both induced and spanning subgraph.

> n vertices simple graph
>
> Max. no of edges $= {}^nC_2 = \frac{n(n-1)}{2}$
>
> No. of simple graph $= 2^{\frac{n(n-1)}{2}}$

Complement of a graph $= \bar{G} =$

Complete graph with n vertices − Edges in G.

$G(V, E)$ given graph

$\bar{G}(V, \bar{E})$ complement of G.

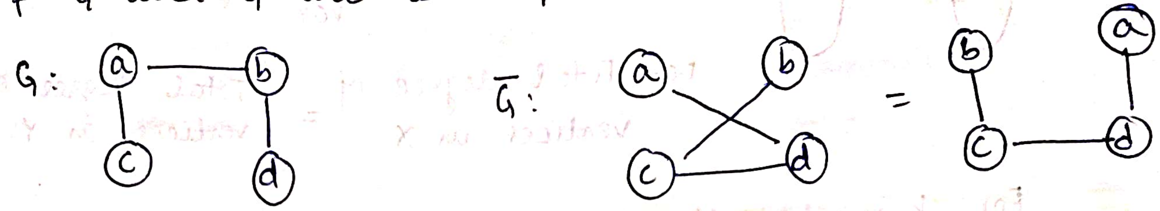
$\bar{E} = E(K_n) - E.$

$|\bar{E}| = {}^nC_2 - |E|$

i.e. $|\bar{E}| + |E| = n.$

Self Complementary graph –

If G and $\bar{G}$ are isomorphic, then, G is self complementary



⊙ Complement of disconnected graph is always connected.

⊙ For every simple graph G,
either G or $\bar{G}$ is definitely connected.

Imp.

If a graph has exactly 2 vertices of odd degree, they are connected by a path.

In any graph there is a simple path from any vertex of odd degree to <u>some</u> other vertex of odd degree.

Bipartite graph

$G(V,E)$ is bipartite iff $\exists$ a bijection X, Y of V such that

① $X \cap Y = \phi$

② $\forall a,b \in X,\ (a,b) \notin E(G)$
$\forall a,b \in Y,\ (a,b) \notin E(G)$

③ $X \cup Y = V$

④ X, Y can be empty

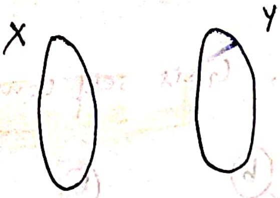
★ A graph is bipartite iff it does not contain odd length cycle.

Imp.

↦ Every P_n is bipartite graph

↦ C_n is bipartite only when n is even.

★ ↦ ~~K_n~~ only K_1 and K_2 are bipartite. $K_{n \geq 3}$ is not bipartite.



$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$$

i.e. Total degree of vertices in X = Total degree of vertices in Y —①

For k regular bipartite graph,
all vertices ~~of~~ have degree k.

$\Rightarrow k|A| = k|B|$ [From ①]

$\Rightarrow |A| = |B|$

* i.e. equal elements in A and B if bipartite graph is regular.

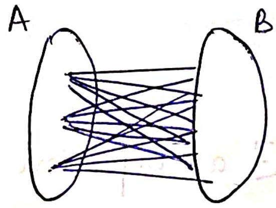
Imp.
Every subgraph of bipartite graph is bipartite.

Interesting result
Important

मस्त। वाह! वाह!
ये बढ़िया था गुरु!!

— Mahatma Gandhi

Complete bipartite graph.
No. of vertices in graph = n



$|A| = m$
$|B| = n-m$

No. of edges, $e = m(n-m)$
$= mn - m^2$

* **Theorem**

If a graph on n vertices has more than $n^2/4$ edges, then, graph is not complete bipartite.

For max no. of edges,

$\frac{de}{dm} = n - 2m = 0 \Rightarrow n = 2m$

$\frac{d^2e}{dm^2} = -2 < 0 \therefore n = 2m$ is point of maxima.

$\Rightarrow m = n/2$

$\therefore$ Maximum no. of edges $= mn - m^2$

$= \cancel{m(2m) - m^2 = 2m^2 - m^2}$

$= \frac{n}{2} \cdot n - \left(\frac{n}{2}\right)^2 = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}$.

Tree

- connected, undirected, acyclic graph.
- n vertices
  - ↦ (n-1) edges
  - ↦ connected, acyclic
- Connected graph on n vertices & (n-1) edges is tree.
- Tree is <u>maximally acyclic</u>

Height of a node in rooted tree	=	No. of ~~nodes~~ edges from that node to the farthest leaf.
Depth of a node in rooted tree	=	No. of edges from the root to that node.

Tree + 1 edge → graph with exactly one cycle.

Imp.

Minimum number of edges in a graph with n vertices to guarantee that it is connected $= {}^{n-1}C_2 + 1$

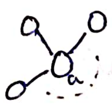
Imp

Every simple graph with more than $\frac{(n-1)(n-2)}{2}$ vertices is connected.

Good question.

Let G be an arbitrary graph with n vertices and k components. If a vertex is removed from G, the no. of components in the resultant graph must lie between

A) k & n B) k-1 & k+1 C) k-1 & n-1 ✓ D) k+1 & n-k



n-k+1 vertices.

k-1 component

On deleting a,
no. of components $= n - \cancel{k} + \cancel{k} - 1$
$= n-1$

k components

On deleting a,
no. of components $= k-1$.

Clique :- clique of a ~~subgraph~~ graph is the subgraph which is complete.

Size of maximum clique = Clique no. = $\omega(G)$

Independent set :-

subgraph with vertices not connected to each other.

Size of maximum independent set = Independence number = $\alpha(G)$.

Vertex cover

set of vertices that cover all the edges.

Size of minimum vertex cover = $\beta(G)$

Edge cover

~~edge~~ set of edges covering all the vertices.

Size of minimum edge cover = $\beta'(G)$

Matching

- set of edges that are independent (non adjacent)
- A set of pairwise non adjacent edges in a graph is called matching.

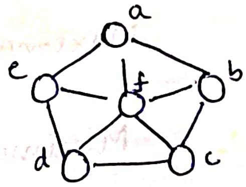
Maximum no. of edges in a matching of graph = $\alpha'(G) = \mu(G)$

→ Matching is perfect if it covers all the vertices of graph.

Imp.

- Clique in G ⟷ Independent set in $\overline{G}$

- Maximum degree vertex need not be included in minimum vertex cover.



Min. VC = {a, d}

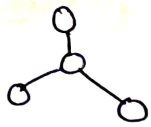
- Edge cover exists only if no isolated vertex in the graph.

- If $S \subseteq V$ is the vertex cover, then, $\overline{S}$ or $V-S$ is independent set & vice versa.

  $\alpha \rightarrow$ Size of maximum independent set.
  $\beta \rightarrow$ Size of minimum vertex cover.

$$\boxed{\alpha + \beta = n}$$

- If perfect matching exists, then, no. of vertices in the graph are even. (vice versa not true)



$\{$ odd no. of vertices $\Rightarrow$ no perfect matching exists $\}$

- For every graph,

$$\boxed{\mu(G) = \underbrace{\alpha'(G)}_{\text{size of largest matching}} \leq \left\lfloor \frac{n}{2} \right\rfloor}$$

1. Complete graph (K_n)

Clique number $= \omega(G) = n$

Maximal independent set $= \alpha(G) = 1$

Minimum vertex cover, $\beta(G) = n-1$

Minimum edge cover

Matching number of $G = \lfloor \frac{n}{2} \rfloor$

2. Cycle graph (C_n)

Clique number $= \begin{cases} 3 & \text{if } n=3 \\ 2 & \text{if } n>3 \end{cases}$

Maximum independent set, $\alpha(G) = \lfloor \frac{n}{2} \rfloor$

Minimum vertex cover, $\beta(G) = \lceil \frac{n}{2} \rceil$

Minimum edge cover, $\beta'(G) = \lceil \frac{n}{2} \rceil$

Matching number $= \lfloor \frac{n}{2} \rfloor$

3. **Path graph (P_n)**

Clique number $= \omega(G) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n \geq 2 \end{cases}$

Maximum independent set, $\alpha(G) = \left\lceil \frac{n}{2} \right\rceil$

Minimum vertex cover, $\beta(G) = \left\lfloor \frac{n}{2} \right\rfloor$

Minimum edge cover, $\beta'(G) = \left\lceil \frac{n}{2} \right\rceil$

Matching number, $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

4. **Complete bipartite graph ($K_{m,n}$)**

Clique number $= \omega(G) = 2$

Maximum independent set, $\alpha(G) = \max(m, n)$

Minimum vertex cover, $\beta(G) = \min(m, n)$

Minimum edge cover, $\beta'(G) = \max(m, n)$

Matching number, $\alpha'(G) = \min(m, n)$

Imp.

- If there exists a matching of size k, then every vertex cover has size $\geq k$.

$$\boxed{\alpha'(G) = \mu \leq \beta(G)}$$

- For bipartite graph, $\alpha'(G) = \beta(G)$.

- $\underbrace{\alpha'(G) \leq \beta(G)}$ $\leq 2\alpha'(G)$

$C_4 \rightarrow \alpha'(G) = \beta(G)$

$K_{odd} \rightarrow 2\alpha'(G) = \beta(G)$

- For any graph, size of maximum independent set is less than or equal to edge cover.

$$\boxed{\alpha(G) \leq \beta'(G)}$$

- For any graph, size of maximum matching is less than or equal to vertex cover.

$$\boxed{\alpha'(G) \leq \beta(G)}$$

1. $\alpha(G) + \beta(G) = n$
2. $\alpha'(G) + \beta'(G) = n$
3. $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$
4. $\beta'(G) \geq \left\lceil \frac{n}{2} \right\rceil$
5. $\alpha'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$
6. $\alpha \leq \beta'(G)$
7. $\alpha'(G) \leq \beta(G)$
8. For bipartite graph, $\mu = \beta$.

Important conclusions.

याद ही कर लो

Graph coloring -

assignment of labels or colors to vertices of a graph.

Chromatic number

Minimum number of colors required to color all the vertices of the graph such that no 2 adjacent vertices share the same color.

Denoted by $\chi(G)$

① $\chi(K_n) = n$

② $\chi(E_n) = 1$

③ $\chi(P_n) = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n \geq 2 \end{cases}$

④ $\chi(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$ ← $n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2$

⑤ $\chi(K_{m,n}) = \begin{cases} 1 & \text{if } m=0 \text{ or } n=0 \\ 2 & \text{otherwise.} \end{cases}$

⑥ $\chi(Q_n) = \begin{cases} 2 & n \geq 1 \\ 1 & n=0 \end{cases}$

⑦ $\chi(W_n) = \begin{cases} 4 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$

Imp.

- Chromatic number is always greater than or equal to clique number.

$$\boxed{\chi(G) \geq \omega(G)}$$

- For any graph,

$$\boxed{\omega(G) \leq \chi(G) \leq \Delta(G) + 1}$$

- Between every colour class, atleast one edge should be there.

$\therefore \quad |E(G)| \geq {}^{\chi}C_2$

i.e. $\boxed{|E(G)| \geq \frac{\chi(G)(\chi(G)-1)}{2}}$

$\boxed{|V(G)| \leq \underbrace{\chi(G)}_{\text{chromatic number}} \underbrace{\alpha(G)}_{\text{Maximum independent set}}}$

Brooke's Theorem –

$\chi(G) = \Delta(G) + 1$ for only 2 graphs –

↳ odd length cycle graph: C_{odd}

↳ complete graph K_n

Vizeing Theorem –

For edge chromatic number.

- For any simple graph, edge chromatic number can be either $\Delta(G)$ or $\Delta(G)+1$.

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

Bipartite graph $\rightarrow \chi'(G) = \Delta(G)$

- Every graph having $\chi'(G) = \Delta(G)+1$ must have atleast 3 vertices of maximum degree.

$$\left.\begin{array}{l} C_{odd} \\ K_{odd} \\ \text{Regular}_{odd} \end{array}\right\} \chi'(G) = \Delta(G)+1.$$

Cut vertex and cut edge

- If e is any ~~edge~~ bridge incident on a vertex v, then, v is cut vertex if and only if $\deg(v) \geq 2$.

$$\boxed{k(G) - 1 \leq k(G-v) \leq n-1}$$

Graph with isolated vertex. $\delta(G) \geq 0$

$$\boxed{k(G) \leq k(G-v) \leq n-1}$$

Graph without isolated vertex $\delta(G) \geq 1$.

- On deleting an edge, no. of components can be either $k(G)$ or $k(G)+1$.

- Every path b/w 2 vertices u and v contains edge e $\Rightarrow$ e is cut edge / bridge.

- All edges in graph are cut edges $\rightarrow$ G is forest.
  All edges in connected graph are cut edges $\rightarrow$ G is tree.

Vertex cut

set of vertices whose removal disconnects the graph.

Smallest vertex cut = Connectivity number. $K(G)$

If cut vertex is present in the graph, $K(G) = 1$.

$K_n \longrightarrow K(G) = n-1$

$C_n \longrightarrow K(G) = 2$

$K_{m,n} \longrightarrow K(G) = \min(m,n)$

$W_n \longrightarrow K(G) = 3$

Imp.

Connectivity of a graph is atmost its minimum degree.

$$\boxed{K \le \delta(G)}$$

Euler graph

connected graph with existance of euler circuit is euler graph.

euler circuit → closed trail

closed → Starting = ending

trail → edge should not repeat.

Imp.

An undirected graph is euler if and only if

↦ connected

↦ every node has even degree.

Connected graph	+	Even degree vertices	=	Euler graph

Euler Path -

starting and ending vertices must be different.

* Euler path exists iff

⊙ G is connected

⊙ exactly 2 vertices have odd degree.

* In euler path

↳ every intermediate vertex has even degree

↳ ⊙ starting and ending vertex have odd degree.

Hamiltonian cycle

- Visit every node exactly once and return to the starting point.

Hamiltonian cycle exists $\longrightarrow$ Hamiltonian Path also exists.

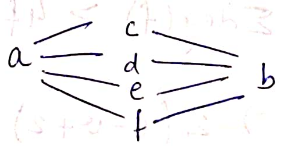
↦ every hamiltonian graph must be connected.
↦ no tree is hamiltonian (coz acyclic)
↦ for each

No. of hamiltonian cycles in complete graph, K_n $= \dfrac{(n-1)!}{2}$

No. of hamiltonian cycles in complete bipartite graph $K_{m,n} = \dfrac{n!\,(n-1)!}{2}$
↳ graph is hamiltonian iff $m=n \geq 2$

Planar graph

Every $K_{2,n}$ is planar



K_5 and $K_{3,3}$ are non planar
↳ most common example.

For any planar graph,

$$\boxed{\sum_{\forall i} Deg(f_i) = \text{Degree sum of faces} = 2|E|}$$

1. Degree of face can be zero (edge less graph)
2. Degree of face cannot be 1. ③ only K_2 has degree of face 2.

Euler's formula –

$$V + F = E + K + 1$$

V → no. of vertices
F → no. of faces
E → no. of edges
K → no. of components.

⊙ In a simple graph with ≥ 3 vertices.
Degree of every face ≥ 3

<u>Connected graph</u> –

Degree of any face ≥ 3

$\Sigma deg = 2e$

$3f \leq \Sigma deg \Rightarrow 3f \leq 2e$

$\Rightarrow 3(e - v + 2) \leq 2e \Rightarrow 3e - 3v + 6 \leq 2e$

$\Rightarrow \boxed{e \leq 3v - 6}$

If no 3 length cycle is present,
Degree of every face ≥ 4

$\Sigma deg(f) \geq 4f \Rightarrow 4f \leq 2e \Rightarrow 2f \leq e$

$\Rightarrow 2(e - v + 2) \leq e \Rightarrow 2e - 2v + 4 \leq e$

$\Rightarrow \boxed{e \leq 2v - 4}$

<u>Imp.</u>

4 colorable theorem

Every planar graph is 4 colorable.

i.e. $K(G) \leq 4$.