ENGINEERING

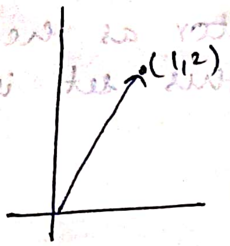
MATHS

LINEAR ALGEBRA

(1-2 questions)
(2-3 marks)

- → system of linear equations
- → eigen values & eigen vectors
- → determinants.

Linear algebra helps in data representation



$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$

Linear Combination of vectors

Multiplying vectors by scalers and adding them together.

- $x_1 A_1 + x_2 A_2 + x_3 A_3 =$ ____ linear combination of A_1 A_2 A_3

Linearly Dependent vectors –

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linear dependent [one is scaler multiple of other

- property is for set of vectors

- 3 vectors are linearly dependent if one of them is linear combination of the other two.

$\rightarrow \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right)$ are linearly dependent

$$\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$\rightarrow \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right)$ linearly dependent

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$\{u, v, w, p, q, r\} \longrightarrow$ set of vectors

If we can represent atleast one vector as the linear combination of other vectors, then, this set is linearly dependent.

⊙ A set containing zero vector is always linearly dependent set.

$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{w = 0.u + 0 \cdot v} \longrightarrow \therefore$ linearly dependent.

⊙ $v_1 = c_2 v_2 + c_3 v_3 + c_4 v_4 + \ldots\ldots c_n v_n$

Can v_2 be represented as linear combination of other vectors?

Ans - ~~Yes only if~~ if $c_2 = 0$, then, no

$c_2 \neq 0$ then yes

Let u_i's be the vectors in R^n for $i = 1, 2, 3, 4$

Which of the following is/are correct?

A. If $\{u_1, u_2, u_3\}$ is linearly independent, so is $\{u_1\ u_2\}$

B. If u_4 is not linear combination of $\{u_1\ u_2\ u_3\}$ then $\{u_1\ u_2\ u_3\ u_4\}$ is linearly independent

C. Any set containing zero vector is linealy dependent

D. If $\{u_1\ u_2\ u_3\}$ is linearly dependent, so is $\{u_1 u_2 u_3 u_4\}$

* If a subset is linearly dependent, then, its superset is ~~linea~~ linearly dependent.

Linear independence

a set of vectors is linearly independent iff they are not linearly dependent.

If $c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots c_n v_n = 0$

If any one of $c_1\ c_2\ c_3 \dots c_n$ is non, zero, then, it is linearly dependent set

supp. $c_1 \neq 0$

then, $v_1 = -\frac{1}{c_1}(c_2 v_2 + c_3 v_3 + \dots c_n v_n)$

else may or may not be linearly independent

Convenient vectors in R^2

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are convenient vectors

$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If there are 2 linearly independent vectors, then, any other vector in the vector space can be derived from it

* If 2 vectors in R^2 are linearly independent, then, any set containing those 2 vectors ~~are~~ is linearly dependent.

Can we have more than i independent vectors in R^i → No

* If there are more than n vectors $\in R^n$ in the set, then, the set is definitely linearly dependent

Multiplying a matrix by a vector —

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$= x \begin{bmatrix} a \\ b \\ c \end{bmatrix} + y \begin{bmatrix} d \\ e \\ f \end{bmatrix} + z \begin{bmatrix} g \\ h \\ i \end{bmatrix}$$

· ~~linear~~ result is linear combination of column vectors ~~combinations~~ of the matrix where coefficients are from vector x

$$\underset{\text{matrix}}{A}\ \underset{\text{vector}}{X} = \underset{\text{vector}}{B}$$

★ If $AX = 0$ has some non trivial solution then, columns of A are linearly ~~dependendes~~ dependent.

System of linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots \dots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots \dots a_{nn}x_n &= b_n \end{aligned}\right\} \text{system of linear equations.}$$

linear combination of columns

$$x - 2y = 1$$
$$3x + 2y = 11$$

coefficient matrix

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{AX = B} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$Ax = 0$ has some non trivial solution then columns of A are linearly dependent? True/False

Ans → True

$$\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \text{linear combination of column vector}$$

$$c_1a_1 + c_2a_2 + c_3a_3 = 0$$

∵ non trivial,
atleast one of c_1, c_2 & c_3 is non zero

$$\therefore a_1 = -\frac{1}{c_1}(c_2a_2 + c_3a_3)$$

∴ not linearly independent

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\, a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c\begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad b\begin{bmatrix} 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 2 \\ 4 \end{bmatrix} \,\right]$$

$$= \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$$\underbrace{\overset{A}{\begin{bmatrix} () & () & \cdots & () \end{bmatrix}}}_{\text{2 LI vectors}} \overset{B}{\begin{bmatrix} \cdot \end{bmatrix}} = \underbrace{\overset{C}{\begin{bmatrix} \end{bmatrix}}}_{\text{atmost 2 linear independent vectors}}$$

★ B is not a linear combination of columns of A, then, we can never get the solution of $AX = B$.

Solutions of $AX = B$

$A_{m \times n}$ has m linearly independent columns?

- Yes → always solution exists
- No → may / may not have solutions

Solutions of $AX = B$

B is a linear combination of columns of A

- Yes → solution always exists
- No → solution never exists

Solutions of $AX = B$

B is a linear combination of column of A

Yes → Solution always exists

No → never a solution

Solution always exists → columns are LI

Yes → Unique soln

No → Infinite soln

If a vector (B) can be represented as linear combination of a few vectors and those vectors are linearly independent, then there is a unique soln

Step by step solution of $AX = B$ (Quassian elimination)

Quassian elimination –
An algorithm to solve the systems of linear equations

$$\boxed{\text{Matrix} \xrightarrow[\text{elimination}]{\text{Quassian}} \text{Echleon form of matrix}}$$

Terms
- Echleon form of matrix
- Pivot and free variable
- Elementary row operations

Echleon form –

$$\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

leading entries

- All non zero rows are above any rows of all zeros
- All entries in a column below leading entry are zero
- leading entry of any row occurs to the right of the leading entry of the row above it.

leading entry → first non zero entry in each row.

Priyanshu (GATE 2024 AIR 379, GATE 2023 AIR 384) @ https://x.com/Priyanshu__2109

$$\begin{bmatrix} 5 & 1 & -6 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -6 \end{bmatrix} \qquad \begin{bmatrix} 0 & -3 & 3 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot and free variables —

- Leading entries ~~a~~ (~~no~~ first non zero entry in each row) is called pivot variable.

  Column containing pivot element is pivot column other columns ~~a~~ are called free columns.

Identify basic and free variables in given augmented matrices —

$$\begin{bmatrix} 1 & 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since augmented matrix is given
∴ last column is neither free nor basic.

pivot column

pivot colum

free column

neither free nor basic

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

pivot column

neither free nor pivot

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

✗

pivot colm

free colum

* Free column is always linearly dependent on pivot columns.

Elementary Row operations

- Swap positions of 2 rows $R_i \leftrightarrow R_j$
- Multiply a row by non zero scaler $R_i \to cR_i$
- Add to one row scaler multiple of other $R_i \to R_i + cR_j$

Guassian elimination

$$\begin{bmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row echleon form

$$\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & 1 & 1 \\ 9 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{3}{2} R_2} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

row echleon form.

Rank of a matrix

- linearly independent rows/ columns
- pivot elements in row echleon form of matrix
- non zero rows of an echleon form of matrix

$\boxed{\text{Rank of zero matrix} = 0}$

* Rank :- no. of pivot columns (r)

Nullity :- no. of free columns ($n-r$)

dimension of null space

n = total no. of columns in ~~augmented~~ coefficient matrix

$\boxed{\text{Rank + Nullity = Total no. of columns}}$

System of linear equations

- Homogeneous: $Ax=0$
- Heterogeneous (non homogeneous): $Ax=b$

Find echleon form of augmented matrix

- if [0000|non zero] exists → no solution, inconsistent eqns
- if [00000|non zero] doesn't exist → Free variables?
  - Yes → infinite solutions
  - No → unique soln

No. of linearly independent solutions = no of free variables

= Nullity

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

$\text{Rank}(A) = \text{Rank}(A|b) \iff$ b is Linear combination of columns of A.

$\therefore$ solution exists.

$\text{Rank}(A) \neq \text{Rank}(A|b) \iff$ b is not linear combinati[on] of columns of A.

$\therefore$ soln does not exist

$\text{Rank}(A|b) = \text{Rank}(A) + 1$

Find Rank(A) & Rank(A|b)

- $\text{Rank}(A) \neq \text{Rank}(A|b)$ → Inconsistent equations, no solution
- $\text{Rank}(A) = \text{Rank}(A|b)$ → Consistent eqns. $\text{Rank}(A) = n$?
  - Yes → Unique solution
  - No → Infinite solution

<u>Given:-</u> $A \to m \times n$ matrix

① $\text{rank}(A) = m = n$

$$\left[\begin{array}{cccc|c} \blacksquare & x & x & x & \uparrow \\ 0 & \blacksquare & x & x & b \\ 0 & 0 & \blacksquare & x & \\ \underbrace{0 \quad 0 \quad 0}_{} & & & \blacksquare & \downarrow \end{array}\right]$$

every column has pivot

$\therefore \text{rank}(A) = \text{rank}(A|b) = n$

$\therefore$ <u>unique solution.</u>

② $\text{rank}(A) = m$ (m≠n)

$$\left[\begin{array}{cccccc|c} \blacksquare & x & x & x & x & x & \uparrow \\ & \blacksquare & x & x & x & x & \\ & & \blacksquare & x & x & x & b \\ & & & \blacksquare & x & x & \downarrow \end{array}\right]$$

$\text{rank}(A) = \text{rank}(A|b) = m$

There exist free variables

<u>infinite solutions</u> (∵ m≠n ∴ can't be unique so… free variable will always be pre…

③ $\text{rank}(A) = n$

$$\left[\begin{array}{cccc|c} \blacksquare & x & x & x & \uparrow \\ 0 & \blacksquare & x & x & \\ 0 & 0 & \blacksquare & x & b \\ 0 & 0 & 0 & \blacksquare & \downarrow \\ 0 & 0 & 0 & 0 & \end{array}\right]$$

$\text{rank}(A) = \text{rank}(A|b)$ if $0\ 0\ 0\ 0 \,|\, 0$ exists

$\therefore$ unique solution (no free var…

$\therefore$ 0 or 1 <u>solutions</u>

$\text{rank}(A) \neq \text{rank}(A|b)$ if $0\ 0\ 0\ 0 \,|$ non zero entr…

no solution

(8) (4) rank(A) < m & rank(A) < n

$$\left[\begin{array}{ccccc|c} & & & & 0 & \\ & & & & 0 & \\ & & & & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

If 0 then, infinite soln (∵ free variable is present)

if not zero, no solution.

0 or ∞ solution.

* If all columns are linearly independent, then only trivial solution exists for $AX = 0$.

$A_{m \times n}$, rank(A) = n.

$AX = 0 \Rightarrow X = 0$

Chapter 2: Determinants, Eigenvalues & eigen vector.

<all matrices are square matrices>

Det(A) ← scaler

Properties -

- Det(Identity matrix) = 1

- Determinant changes sign when 2 rows are exchanged.

1 mark question in Gate 2023

$$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \qquad \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

- $$\begin{vmatrix} a+p & b+q \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} p & q \\ c & d \end{vmatrix}$$

only 1 row can be split at a time.

Linearity of one row at a time

$$\begin{vmatrix} a+p & b+q \\ c+r & d+w \end{vmatrix} = \begin{vmatrix} a & b \\ c+r & d+w \end{vmatrix} + \begin{vmatrix} p & q \\ c+r & d+w \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ r & w \end{vmatrix} + \begin{vmatrix} p & q \\ c & d \end{vmatrix} + \begin{vmatrix} p & q \\ r & w \end{vmatrix}$$

→ $$\begin{vmatrix} ta & tb \\ tc & td \end{vmatrix} = t\begin{vmatrix} a & b \\ tc & td \end{vmatrix} = t^2\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

$$2\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = \begin{vmatrix} 2a & b \\ 2c & d \end{vmatrix}$$

- $|A+B| \neq |A| + |B|$

- Determinant of diagonal matrix /upper triangular/ lower triangular is product of diagonal elements.

- Number of terms in the determinant of $n \times n$ matrix $= n!$

  For 2×2 mat. $\longrightarrow 2! = 2$ terms ($ad-bc$)

  For 3×3 mat $\longrightarrow 3! = 6$ terms.

Imp. properties of determinants-

- $\det(AB) = \det(A) * \det(B)$

- $\det(A^{-1}) = \dfrac{1}{\det(A)}$

- $\det(A^T) = \det(A)$

Sign of cofactor of a_{ij} : $(-1)^{i+j}$

without sign $\longrightarrow$ minor $\rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

with sign $\longrightarrow$ cofactor $\rightarrow (-1)^{i+j}\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

★ ⊙ If elements of a row / column are multiplied with cofactors of any other row / column, then, their sum is $\underline{0}$.

$$A^{-1} = \frac{1}{|A|} \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T}_{(\text{coefficient matrix})^T}$$

$$\text{adjoint of } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$$

crammer's rule

→ mathematics of millionaires.

· very expensive method of finding solutions of $AX = b$.

more resources / computing power required.

→ only for theory purpose.

$$AX = b \quad \Rightarrow \quad X = A^{-1}.b = \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} b_1 c_{11} + b_2 c_{21} + b_3 c_{31} \\ b_1 c_{12} + b_2 c_{22} + b_3 c_{32} \\ b_1 c_{13} + b_2 c_{23} + b_3 c_{33} \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} \det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{23} & b_3 & a_{33} \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}\right) \end{bmatrix}$$

अमीरों का शौक !!

$$X = \frac{1}{\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)}\begin{bmatrix} \det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & b_{22} & b_{23} \\ b_3 & b_{32} & b_{33} \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}\right) \end{bmatrix}$$

very expensive.)

Imp ques

Give conditions when

$$2x_1 + hx_2 = k$$
$$x_1 - x_2 = 2$$

have (i) Infinite soln ii) No soln (iii) Unique soln

$$\begin{bmatrix} 2 & h \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 2 \end{bmatrix}$$

i) For infinite solutions —

↳ b should be LC of columns of A

↳ Columns of A should be L.D

$\therefore$ $h = -2$ and $k = 4$

ii) For no solution —

↳ Columns of A should be L.D.

↳ b should not be LC of columns of A.

$\therefore$ $h = -2$ and $k \neq 4$

iii) for unique solution,

↳ b should be L.C. of columns of A

↳ columns of A should be L.I.

R^2 है तो सिर्फ 2nd cond. ही satisfy करे तो क्योंकि 2 LI vectors in R^2 can generate anything.

$\therefore$ $h \neq -2$ & $k \in R$

Eigenvalues and eigen vectors

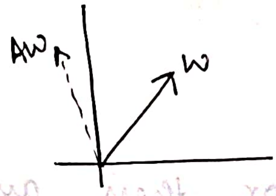
(only square matrix)

$Ax = \lambda x$ — vector gets scaled.

$A x$: eigen vector of A

λ: scaler

$\lambda = 1$	no change
$\lambda > 1$	stretched
$\lambda < 1$	shrunk



vector w is not eigen vector of vector A

vector w is eigen vector of A.

$\underbrace{\lambda = 0.5}_{\text{eigen value}}$

○ several properties of matrices can be analysed based on their eigen values.

Application ⟶ core concept of Regularization is eigen value & eigen vectors

machine learning cannot survive without this.

⟶ Principal Component Analysis

By definition, eigen vectors are non-zero.

eigen values ~~zero~~ ~~cannot be~~ can be zero

$$Av = \lambda v$$

A: matrix

v: eigen vector

λ: eigen value

⊙ There are infinitely many eigen vectors for every λ identity matrix I.

No. of LI eigen vectors = n for I_n.

For $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, No. of LI eigen vectors = 3.

⊙ If X is eigenvector, then, any non zero multiple of X is also an eigen vector.

$$\boxed{\text{Characteristic equation :- } |A - \lambda I| = 0}$$

Eigen vectors from different eigen values are linearly independent.

λ_1 & λ_2 are eigenvalues.

if $\lambda_1 \neq \lambda_2$ → corresponding eigen vectors are linearly independent.

if $\lambda_1 = \lambda_2$ → if λ. is repeating 2 times, then, we can have ~~eg~~ either one or 2 LI eigen vectors.

No. of LI eigen vectors for following characteristic equation -

$$(\lambda-1)^2 (\lambda-5)(\lambda-7)^3 = 0$$

(a) atleast 6 (c) exactly 6

(b) atmost 6 (d) atleast 3

(e) exactly 3.

$$(\lambda - x_1)^{m_1} (\lambda - x_2)^{m_2} (\lambda - x_3)^{m_3} \cdots\cdots (\lambda - x_k)^{m_k}$$

↳ atleast k LI eigen vectors

↳ atmost $m_1 + m_2 + m_3 + \dots m_k = n$ LI eigen vectors

$(\lambda - \lambda_1)^{m}$ ⟷ arithemetic multiplicity (AM) ⟹ atmost m LI eigen vectors

⊙ Geometric multiplicity (GM) :- no. of LI eigenvectors corresponding to $\lambda = \lambda_1$.

$$\boxed{A.M. \geq G.M.}$$

Condition when AM = GM.

i.e. if $(\lambda-\lambda_1)^m$, then, what should be the value of λ_1 such that there are m LI eigenvectors for $\lambda = \lambda_1$.

Geometric multiplicity :- Number of linearly independent eigenvectors corresponding to λ.

~~Arithem~~

Algebraic multiplicity :- Number of times λ is repeating

$$GM \le A.M.$$

→ ~~Sq~~ Real symmetric matrices have G.M. = A.M.

↳ n real eigen values.

↳ n orthogonal eigen vectors.

This concept is needed in diagonalization in machine learning!!

In real symmetric matrices, all eigen vectors are LI even if eigen values are repeating.

For real symmetric matrix

$$(\lambda-1)^3 (\lambda-5)^7 (\lambda-11) = 0$$

11 LI eigen vectors

Corresponding to $\lambda = 1$, 3 LI eigen vectors

" " $\lambda = 5$, 7 LI " "

" " $\lambda = 11$, 1 LI " "

Imp.

Two Important properties of eigen values —

ये properties
भूलने की
इजाज़त नहीं है
समझे ?
2023 में तुम
भूलने की खता
कर चुके हो !!

☹

- Determinant of any matrix is product of eigen values.

- Trace (sum of elements of main diagonal) of matrix is sum of eigen values.

$$|A| = \lambda_1 \lambda_2 \lambda_3 \dots \lambda_n.$$

$$\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 + \dots \lambda_n$$

- If 2 rows of A are same, one of the eigen values is 0.

$$\begin{bmatrix} 3 & 1 & 4 \\ c & 1 & 5 \\ c & 1 & 5 \end{bmatrix}$$

R_2 & R_3 are same

∴ one of the eigen values is 0.

If columns of a matrix are linearly dependent

⇒ determinant of the matrix is 0.

⇒ product of eigen values is 0.

⇒ one of the eigen values is 0.

Relationship between rank and eigen values.

If one of the λ eigen values of $A \neq 0 \implies \text{rank}(A) = n$

If one of the λ eigen values of $A = 0 \implies \text{rank}(A) < n$

$\boxed{A}$

eigenvalue of $A \neq 0$ → $\boxed{\text{rank}(A) = n}$

eigenvalue of $A = 0$ → $\boxed{\text{rank}(A) < n}$

Number of LI eigenvectors corresponding to zero $= k$.

Free variables $= k$

Rank $= n-k$.

when $\lambda = 0$,

$$[A - \lambda I]X = 0 \longrightarrow AX = 0$$

i.e. it is similar to solving $AX=0$.

$\therefore$ rank $= n -$ no. of LI vectors.

No. of LI eigenvectors for $\lambda = 0$ are 2. What is the rank of matrix.

$A_{10 \times 10}$

$$(A - \lambda I)x = 0 \quad \because \lambda = 0 \quad \therefore \quad Ax = 0$$

soln has 2 LI eigen vectors

i.e. it is of the form

$$s[\;\;] + t[\;\;]$$

i.e. 2 free variables.

nullity $= 2$

rank $= n - 2 = 10 - 2 = 8$.

$\boxed{(A - \lambda I)x = 0 \text{ and } Ax = 0 \text{ can be connected by } \lambda = 0}$

No. of LI eigen vectors corresponding to $\lambda = 0$ → Nullity.

Cayley - Hamilton Theorem

- Every square matrix satisfies its own characteristic equation.

$\boxed{\text{Application}}$

→ To calculate positive integral powers of A.
→ To calculate inverse of a square matrix A.

Imp.
- If λ is the eigenvalue of AB, then, it is also the eigenvalue of B.A.

$$AB.x = \lambda x$$

$$\Rightarrow B \cdot ABx = B\lambda x \quad \Rightarrow \quad (BA)(Bx) = \lambda(Bx)$$

inferred.

Bx is the eigenvector of BA
λ is eigen value of BA.

i.e. AB and BA share non zero eigen values.

just make sure $Bx \neq 0$

$A_{2\times10}$ $B_{10\times2}$

$AB_{2\times2} \rightarrow$ eigen values = 1, 5

$BA_{10\times10} \rightarrow$ eigen values = 1, 5, 0, 0, 0, 0, 0, 0, 0, 0

coz only non zero eigen values are shared.
remaining should be 0.

$A_{4\times3}$ $B_{3\times4}$

$AB_{4\times4}$ cannot have 4 non zero eigen values.
one of the eigen values should be 0.
coz. $BA_{3\times3}$ can have only 3 eigen values
& all non zero e.v. should be shared
∴ not possible if 4 non zero e.v. are there for AB.

Imp question

If $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}_{6\times1}$ $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}_{6\times1}$

~~larges~~ largest eigen value of uv^T

non 0 eigen values of uv^T = non zero eigen value of $v^T u$

$v^T u = \begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = [17]_{1\times1}$

∴ Ans = 17

Eigen values of powers of A –

- If λ_1, λ_2 are eigen value of A, eigen values of A^k are $\lambda_1^k, \lambda_2^k \ldots$

- eigen value of $f(A)$ is $f(\lambda)$

  if λ is eigen value of A

  eigen value of $A^k + 3A^{k-1} + I$ is $\lambda^k + 3\lambda^{k-1} + 1$

Ques Suppose x is an eigenvector of A such that $Ax = \lambda_1 x$ and x is also eigenvector of B such that $Bx = \lambda_2 x$. What is the eigen value of $\left(A + \frac{1}{2}B\right)^{-1}$?

A) $\dfrac{\lambda_1}{2\lambda_1 + \lambda_2}$ B) $\dfrac{\lambda_2}{2\lambda_1 + \lambda_2}$ C) $\dfrac{2}{2\lambda_1 + \lambda_2}$ ✓ D) $\dfrac{\lambda_1}{2\lambda_2 + \lambda_1}$

$$\left(A + \tfrac{1}{2}B\right)x = Ax + \tfrac{1}{2}Bx = \lambda_1 x + \tfrac{1}{2}\cdot\lambda_2 x$$

$$= \left(\lambda_1 + \frac{\lambda_2}{2}\right)x$$

$\therefore$ eigen value of $A + \frac{1}{2}B$ is $\lambda_1 + \frac{\lambda_2}{2}$

$\therefore$ eigen value of $\left(A + \frac{1}{2}B\right)^{-1}$ is $\left(\lambda_1 + \frac{\lambda_2}{2}\right)^{-1}$

$$= \frac{2}{2\lambda_1 + \lambda_2}$$

- If eigen values are imaginary — then, its conjugate will also be eigen value.

  i.e. if $2+i$ is e.v. of A then, $2-i$ is also eigen value of A

LU Decomposition

Decomposing a matrix into 2 parts — L and U
where L = lower triangular matrix with 1 at diagonal
U = upper triangular matrix.

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}}_{U}$$

Method

- Convert A into row echelon form without swapping the rows.
- The A in row echelon form gives upper triangular matrix.
- The ~~corresponding~~ negative of coefficients applied to rows in operations gives lower triangular matrix.

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 3 & 4 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & -8 & 16 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 15.5 \end{bmatrix}$$

$\hookrightarrow U$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix}$$

Types of matrices

1. **Identity matrix** – 1s at principal diagonal 0's everywhere else.

2. **Inverse of a matrix** –

A and B are inverse of each other iff

$$AB = BA = I_n$$

$(A^{-1})^{-1} = A$

$(kA)^{-1} = k^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

For square matrices, if $AB = I_n$ then $BA = I_n$.

3. **Transpose of a matrix** –

Transpose of a matrix is an operator that flips a matrix over its diagonal.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(cA)^T = cA^T$
5. dot product of 2 column vectors can be calculated by

$$a \cdot b = a^T b$$

6. $(A^T)^{-1} = (A^{-1})^T$

4. Triangular matrix.

(i) lower triangular if all the elements above its main diagonal are zero. $a_{ij} = 0 \quad i > j$

(ii) upper triangular if all the elements below its main diagonal are zero. $a_{ij} = 0 \quad i < j$

→ Product of 2 lower triangular matrices is lower triangular

→ Product of 2 upper triangular matrices is upper triangular

5. Diagonal matrix

all the entries outside the main diagonal are all zero.

a_{ij} if $i \neq j$, $a_{ij} = 0$

6. Symmetric matrix

matrix equal to its own transpose

$$A = A^T$$

7. Skew symmetric matrix

$$A^T = -A$$

elements of principle diagonal are zero.

$$\begin{bmatrix} 0 & 2 & -45 \\ -2 & 0 & -4 \\ 45 & 4 & 0 \end{bmatrix}$$

- For any matrix A, both AA^T and A^TA are symmetric.
- For any matrix A with real no. entries,

$A + A^T$ is symmetric
$A - A^T$ is skew symmetric.

- Every matrix can be represented as a sum of symmetric & skew symmetric matrix.

$$A = \frac{1}{2}\Big[\underbrace{(A + A^T)}_{\text{symmetric}} + \underbrace{(A - A^T)}_{\text{skew symmetric}}\Big]$$

Ques:- If A and B are symmetric, then, ABA is

A. Symmetric
B. Skew symmetric
C. Diagonal
D. Triangular.

Given:- $A^T = A$ $B^T = B$

$(ABA)^T = A^T B^T A^T$
$= ABA$

$\therefore$ symmetric

8. Orthogonal matrix

Orthogonal matrix is a real square matrix whose columns and rows are orthogonal vectors

i.e. $Q^T \cdot Q = Q \cdot Q^T = I$

If a matrix is orthogonal, then, its transpose is equal to its inverse.

$$\boxed{Q^T = Q^{-1}}$$

$A_{n \times n}$ matrix is diagonalizable only if it has n LI eigen vectors

~~Ques~~

Consider the matrix $A_{n \times n}$ having the following characteristic equation

$$\lambda^2 (\lambda-3)(\lambda+2)^3 (\lambda-4)^3$$

What could be rank of A?

A. 6
~~B.~~ 7
~~C.~~ 8
D. 9

$\lambda = 0$ — Yes → $R < n$ → This is the case

No → Corresponding to $\lambda = 0$, 1 LI eigen vector; 2 LI eigen vector

Nullity = 1 or 2

$\therefore$ Rank = 9-1 or 9-2
8 or 7

Solutions is not unique when they exist $\Rightarrow$ column of A are L.D.

The eigen vectors corresponding to eigen values of real symmetric matrix are orthogonal.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 = \text{trace}(AA^T)$$

PROBABILITY

- Conditional Probability
- Probability distribution
  - Random variable
  - Expectation

Inclusion Exclusion Principle

For any 2 events E and F,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

For any 3 events E, F & G,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G)$$

De Morgan's Rule for Probability

$$\cancel{P((E \cap F)^c)} \quad P((E \cap F)^c) = P(E^c \cup F^c)$$

&

$$P((E \cup F)^c) = P(E^c \cap F^c)$$

Conditional Probability

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$P(A \cap B) = P(B)\, P(A|B) = P(A)\, P(B|A).$$

$$\begin{aligned} P(A \cap B \cap C) = P(A, B, C) &= P(C)\, P(B|C)\, P(A|B, C) \\ &= P(A)\, P(B|A)\, P(C|A, B) \\ &= P(B)\, P(A|B)\, P(C|A, B) \\ &= P(B)\, P(C|B)\, P(A|C, B) \end{aligned}$$

A and B are mutually exhaustive $\Rightarrow$ $\cancel{P(A)}$ $P(A \cup B) = 1$

A and B are mutually exclusive $\Rightarrow$ $P(A \cap B) = 0$.

Marginalization (also called Total Probability)

Let A_1, A_2 and A_3 partition the sample space then, for any event B,

$$P(B) = \cancel{P(B)\ A}\ P(B|A_1) + P(B \cap A_2) + P(B \cap A_3)$$

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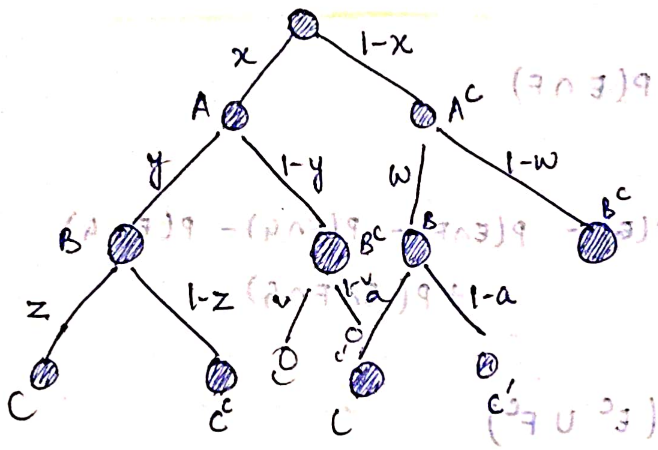


$$P(A\cap B\cap C) = xyz$$

$$P(C \mid A,B) = z$$

$$P(C \mid B) = \frac{P(B\cap C)}{P(B)}$$

$$P(B\cap C) = P(A\cap B\cap C) + P(A^c\cap B\cap C)$$
$$= x\cdot y\cdot z + (1-x)\cdot w\cdot a$$

$$P(B) = P(A\cap B) + P(A^c\cap B)$$
$$= xy + (1-x)\cdot w$$

★

$$\therefore \boxed{P(C \mid B) = \frac{xyz + (1-x)wa}{xy + (1-x)\cdot w}}$$

$$P(C \mid A) = \frac{P(C\cap A)}{P(A)}$$

$$P(C\cap A) = P(A\cap B\cap C) + P(A\cap B^c\cap C)$$
$$= xyz + x(1-y)v$$

$$P(A) = x$$

$$\therefore P(C \mid A) = \frac{xyz + x(1-y)v}{x} = \underline{\underline{yz + (1-y)v}}$$

Independent Events

$A \perp\!\!\!\perp B \rightarrow$ A and B are independent events.

$$P(B \mid A) = P(B)$$

$$P(A\cap B) = P(A)\,P(B)$$

$$P(A\cap B\cap C) = P(A)\,P(B)\,P(C)$$

3 events A, B & C are independent if all following conditions are satisfied -

$$P(A\cap B) = P(A)\,P(B)$$
$$P(B\cap C) = P(B)\,P(C)$$
$$P(A\cap C) = P(A)\,P(C)$$
$$P(A\cap B\cap C) = P(A)\,P(B)\,P(C)$$

Conditional independence

2 events A & B are independent given C —

$$P(A \cap B \mid C) = P(A|C)\,P(B|C)$$

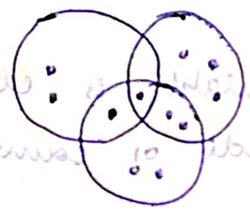
$$= P(A \mid B, C)\,P(B|C)$$

$$= P(B \mid A, C)\,P(A|C)$$

★ It is possible that earlier A & B were independent but after C, they got dependent

i.e. $P(A \cap B) = P(A)\,P(B) \not\Rightarrow P(A \cap B \mid C) = P(A|C)\,P(B|C)$

Independence does not imply conditional independence and vice versa.



$P(A \cap B|C) = P(A|C)\,P(B|C)$

but

$P(A \cap B) \neq P(A)\,P(B)$

Two events A and B are independent —

$P(A \cap B) = P(A)\,P(B)$

$P(A|B) = P(A)$

$P(B|A) = P(B)$

If A & B are independent,

A & B^c are independent

A^c & B are independent

A^c & B^c are independent.

Two events A & B are conditionally independent —

$P(A \cap B|C) = P(A|C)\,P(B|C)$

$P(A|B,C) = P(A|C)$

$P(B|A,C) = P(B|C)$

If A & B are conditionally independent given C, then,

A & B^c are ~~condi~~ independent given C

A^c & B are independent given C

A^c & B^c are independent given C.

Random variables

Function that maps an outcome to the real number

$x < 10 \equiv \{\omega \mid X(\omega) < 10\}$ $\rightarrow$ set of some item = event.

⊙ Any condition on random variables is an event.

$\rightarrow$ X maps any outcomes to either a, b, or c.
Then, $x = a$, $x = b$ and $x = c$ there are 3 events are mutually exhaustive

Types of Random variables

1. Discrete - Random variable is called discrete if it takes either finite or countably infinite no of values.

   Ex $\rightarrow$ no. of sixes in 2 rolls.

2. Continuous - Random variable takes an uncountably infinite no. of values.

   Ex $\rightarrow$ choosing a point from $[-1, 1]$

Probability Mass Function

Listing down probability of each value for a discrete random variable is called 'Probability Mass Function'

Expectation

- single number that summarises PMF.
- weighted average (in proportion to probabilities) of the possible values of X.

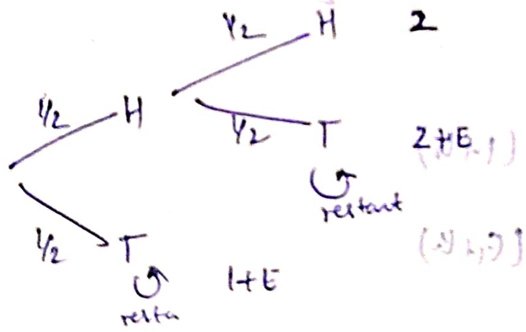
$$\boxed{E(X) = \sum_k k\, P(X=k)}$$

Expectation is just average on enough data.

As amount of data increases, Average ⟿ Expectation.

Gate 2015

Let the random variable X represent the no of times a fair coin needs to be tossed till 2 consecutive heads appear for the first time. The expectation of X



$\therefore$ Expectation =

$$E = 2 \times \frac{1}{4} + \frac{1}{4} \times (2+E) + \frac{1}{2} \times (1+E)$$

$$\Rightarrow E = \frac{1}{2} + \frac{2+E+2+2E}{4} = \frac{1}{2} + \frac{4+3E}{4}$$

$$\Rightarrow \frac{4E-4-3E}{4} = \frac{1}{2} \Rightarrow \frac{E-4}{4} = \frac{1}{2}$$

$$\Rightarrow E-4 = 2 \Rightarrow \underline{\underline{E = 6}}$$

Gate 2005

An unbiased coin is tossed repeatedly until the outcome of 2 successive tosses is same. Assuming the trials are independent, the expected number of tosses is

- A. 3 ✓
- B. 4
- C. 5
- D. 6

$\xrightarrow{1/2}$ H $(1+E_1)$

$\xrightarrow{1/2}$ T $(1+E_2)$

Let E_1 = no. of expected tossess to get HH TT when we have H in hand.

E_2 = no of expected tossesses to get HH TT when we have T in hand

$$E = \frac{1}{2}(1+E_1) + \frac{1}{2}(1+E_2) \quad —③$$

H $\xrightarrow{1/2}$ H 1

H $\xrightarrow{1/2}$ T $(1+E_2)$

$$\therefore E_1 = \frac{1}{2} + \frac{1}{2}(1+E_2) \quad —①$$

T $\xrightarrow{1/2}$ H $(1+E_1)$

T $\xrightarrow{1/2}$ T 1

$$\therefore E_2 = \frac{1}{2} + \frac{1}{2}(1+E_1) \quad —②$$

from ① & ②,

$$E_1 + E_2 = 1 + \frac{1}{2}(2 + E_1 + E_2) = 2 + \frac{E_1+E_2}{2}$$

$$\Rightarrow \frac{E_1+E_2}{2} = 2$$

Now,

$$E = \frac{1}{2}(1+E_1) + \frac{1}{2}(1+E_2)$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}(E_1+E_2)$$

$$= 1 + \frac{E_1+E_2}{2} = 1 + 2 = 3$$

Variance

- Variance is average squared distance from mean.
- always non negative

  - If variance is low, then, for any random data point we can expect value of that datapoint to be very close to what we expect!

	Calculate using data	Calculate using probability distribution
Expectation	$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots x_n}{n}$	$\cancel{\sum_k k\,P =}$ $\sum_k k P(X=k)$
Variance	squared distance from mean $= \frac{(x_i - \bar{x})^2}{n}$	$E\left((X - E[X])^2\right)$

variance of any cost $= 0$

$$Var(X) = E\left[(X - E(X))^2\right]$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$Var[X_1 + 2X_2 + 3X_3 + 4X_4]$$

only valid if X_1, X_2, X_3, X_4 are independent

$$= Var[X_1] + Var[2X_2] + Var[3X_3] + Var[4X_4]$$

$$= Var[X_1] + 2^2 Var[X_2] + 3^2 Var[X_3] + 4^2 Var[X_4]$$

Properties of expectations and variance:

Let $Y = aX + b$

① $E[Y] = E[ax+b] = E[ax] + E[b] = aE[x] + \cancel{E[b]}$

$= aE[x] + b$

② $Var[Y] = Var[ax+b] = var[ax] + \cancel{var[b]}^{\to 0}$

$= a^2 var[x]$

③ $E[x_1 + x_2] = E[x_1] + E[x_2]$

$Var[x_1 + x_2] = Var[x_1] + var[x_2]$

[if x_1 & x_2 are independent]

$Var[x_1 - x_2] = var[x_1] + var[x_2]$

$$\boxed{\text{Standard deviation (SD)} = \sqrt{\text{variance}}}$$

Covariance $(X, Y) = E[(X - E[X])(Y - E[Y])]$

$cov(X, Y)^2 \leq var[X] var(Y)$

Discrete Random variable

- Bernoulli (only 2 possible outcomes — 0 or 1)
- Binomial
- Poisson
- Uniform

1. Bernoulli Random variable —

If we have some experiment where we can classify every outcome as success or failure then, we can say that we have bernoulli distribution

To Be Remembered

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$var[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

$P(X=1) = p$

$$PMF = X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

2. Binomial Random variable

Repeated independent trials of Bernoulli

Performing experiment n times

No of success $= k$ No. of failures $= n-k$

$$P(X=k) = {}^nC_k \cdot p^k (1-p)^{n-k}$$

$$E[X] = np$$

$$var[X] = np(1-p) = npq$$

3. Poisson distribution

$n \to$ very large
$p \to$ very small
binomial distribution changes to poisson d...

parameter λ

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2 \ldots \infty$$

[k successes)

$$E[X] = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k^{\lambda}} = \lambda$$

$$\text{var}[X] = E[X^2] - (E[X])^2 = \lambda$$

$$\boxed{E[X] = \text{var}[X] = \lambda}$$

Ques :- On an average there are 2 accidents per day. What is the probability of 4 accidents on given d...

$\lambda = 2$

$$P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!} = \frac{e^{-2}2^4}{4!} = \frac{16}{24e^2} = \frac{2}{3e^2}$$

Some patterns where poisson distribution is to be used

- No. of misprints on a page
- No of people in a community surviving to 100.
- No. of wrong telephone nos dialed in a day.
- No. of ----

Rate

4. Discrete Uniform Random variable

A random variable X has a discrete uniform distribution if each of the n values in its range, say, $x_1, x_2, \ldots, x_n$ has equal probability.

i.e. $\boxed{P(x_i) = 1/n}$

where $P(x)$ represents the probability mass function (PMF)

Ques If X is uniformly distributed over $\{a, a+1, \ldots, b\}$ then find out

$P(X=x)$

$E(x)$

$var(x)$

No. of elements, $n = b-a+1$

$\therefore P(X=x) = \frac{1}{b-a+1}$

$$E(x) = \frac{1}{b-a+1}\left[a + a+1 + \ldots\ldots b-1+b\right]$$

$$= \frac{1}{b-a+1}\left[\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right] = \frac{1}{b-a+1}\left[\frac{b^2-a^2+b+a}{2}\right]$$

$$= \frac{1}{b-a+1}\left[\frac{(b-a)(b+a)+(b+a)}{2}\right] = \frac{b-a+1}{b-a+1}\cdot\frac{b+a}{2} = \frac{b+a}{2}$$

variance does not change on shifting R.V. to left or right by a constant

$\therefore var(x+a) = var(x)$

$\therefore$ required variance $=$ variance of $\{1, 2, \ldots\, b-a+1\}$

$$E[X^2] = \frac{1}{n}(1+4+9+\ldots n^2) = \frac{n(n+1)(2n+1)}{6\cdot n} = \frac{(n+1)(2n+1)}{6}$$

$$E[X] = \frac{1}{n}(1+2+3+\ldots n) = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2}$$

$\therefore var(x) = E[X^2] - (E(X))^2 = \frac{n^2-1}{12} = \frac{(b-a+1)^2-1}{12}$

Continuous Random variable

- Uniform
- Exponential
- Normal.

Probability Density Function -

Random variable X has a PDF $f(x)$ if ~~$P(a \le X \le b)$~~

$$P(a \le X \le b) = \int_a^b f(x)\,dx \quad \text{for all } a, b.$$

$$\left[a = b \Rightarrow \int_a^a f(x)\,dx = 0\right]$$

For valid PDF,

$$P(X \le \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$P(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$

1. Uniform Continuous Random variable

Probability $\propto$ length of interval

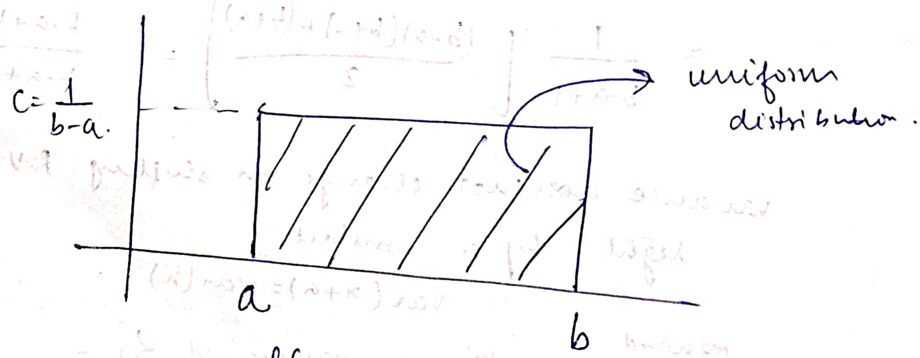
$$f(x) = \begin{cases} c & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

~~Prob~~
P(interval) = c (length of interval)

$$c = \frac{1}{b-a}$$

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$



$$\int_3^6 f(x)\,dx = \int_6^9 f(x)\,dx \quad \text{coz interval is same.}$$

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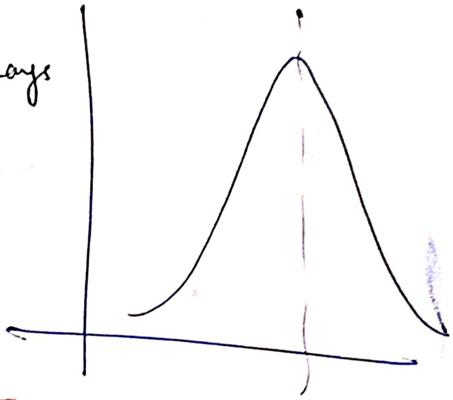
2. Normal Distribution / Guassian Distribution

symmetrical distribution (about mean)

$E[X] = \mu$

$Var[X] = \sigma^2$

σ & μ are always given in question



Standard normal distribution $\rightarrow$ $\sigma = 1, \mu = 0$

The given distribution (x) is converted into standard normal distribution (z)

* $Z = \dfrac{x-\mu}{\sigma}$

$E[X] = \mu$ $\quad$ $E[z] = 0$

$Var[X] = \sigma^2$ $\quad$ $Var[z] = 1$

3. Exponential distribution

continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$E[X] = \dfrac{1}{\lambda}$

$Var(X) = \dfrac{1}{\lambda^2}$

Calculus

$$\lim_{x\to a} f(x) = f(a)$$ if $f(a)$ does not exist, $\lim_{x\to a}$ exists only if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

LHL = $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

RHL = $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$

1. $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$

2. $\lim_{x\to a} (f(x) \times g(x)) = \boxed{\lim_{x\to a} f(x) \times \lim_{x\to a} g(x) =} \; l \times m$

3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l}{m}$

4. $\lim_{x\to a} (f(x))^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} = l^m$ if $\lim_{x\to a} f(x) = l > 0$

5. $\lim_{x\to a} f \circ g(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$ provided $f(x)$ is continuous at $g(x) = m$

Frequently used limits –

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ $\quad \lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$ $\quad \lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

* $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \log_e a$

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1^∞ form

1. $l = \lim_{x\to a} (1+f(x))^{g(x)}$ $\quad \lim_{x\to a} g(x) = \infty \quad \lim_{x\to a} f(x) = 0 \quad [1^\infty$ form$]$

$\therefore \log l = \lim_{x\to a} g(x) \log(1+f(x))$

$= \lim_{x\to a} g(x) \dfrac{\log(1+f(x))}{f(x)} \cdot f(x)$

$= \lim_{x\to a} g(x) \cdot \cancel{\lim_{x\to a} \dfrac{\log(1+f(x))}{f(x)}} \xrightarrow{} 1 \cdot \lim_{x\to a} f(x)$

$= \lim_{x\to a} g(x) f(x)$

$\therefore \boxed{l = e^{\lim_{x\to a} g(x) f(x)}}$

2. $l = \lim_{x\to a} (f(x))^{g(x)}$ $\quad \lim_{x\to a} f(x) = 1 \quad \lim_{x\to a} g(x) = \infty \quad [1^\infty$ form$]$

$\log l = \lim_{x\to a} g(x) \log f(x) = \lim_{x\to a} g(x) \log(1+f(x)-1)$

$= \lim_{x\to a} g(x) \cdot \dfrac{\log(1+(f(x)-1))}{f(x)-1} \cdot (f(x)-1)$

$= \lim_{x\to a} g(x) \cancel{\lim_{x\to a} \dfrac{\log(1+(f(x)-1))}{f(x)-1}} \xrightarrow{} 1 \; f(x)-1$

$= \lim_{x\to a} g(x) \cdot (f(x)-1)$

$\therefore \boxed{l = e^{\lim_{x\to a} g(x)(f(x)-1)}}$

Imp

- If $f(x)$ and $g(x)$ are continuous at $x=a$, then,
  1. $f(x) \pm g(x)$ is also continuous at $x=a$
  2. $f(x) \cdot g(x)$ & $f(x)/g(x)$ are also continuous at $x=a$. $[g(a) \neq 0]$

$f(g(x))$ is continuous at $x=a$.
if $g(x)$ is continuous at $x=a$
then $f(x)$ is continuous at $g(a)$.

Differentiability

A function $f(x)$ is differentiable at $x=a$ iff
Left hand derivative = Right hand derivative
LHD = RHD

i.e $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}$

Basic properties and formulae –

let $f(x)$ & $g(x)$ be differentiable functions c & n real nos.

1. $(cf)' = cf'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + g'f$
4. $\left(\frac{f}{g}\right)' = \frac{f'g + fg'}{g^2}$
5. $\frac{d}{dx} c = 0$
6. $\frac{d}{dx} x^n = nx^{n-1}$
7. $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

Common derivatives

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(\sin x) = \cos x$
3. $\frac{d}{dx}(\cos x) = -\sin x$
4. $\frac{d}{dx}(\tan x) = \sec^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(a^x) = a^x \ln a$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
9. $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

Intermediate value Theorem

If f is continuous function in the closed interval $[a, b]$, and if d is b/w $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ with $f(c) = d$

Rolle's Theorem

If a function f is
- continuous in $[a, b]$
- differentiable in (a, b)
- $f(a) = f(b)$

then, there exists c in (a, b) such that $f'(c) = 0$

Indeterminate forms in limit –

$\frac{\infty}{\infty}$	$\frac{0}{0}$	$\infty - \infty$	0^0	$0 \cdot \infty$	∞^0	1^∞
1	2	3	4	5	6	7

1, 2 → L Hospital rule

4, 6 → taking log

7 → $L = \lim_{x \to a} f(x)^{g(x)}$ form

$L = e^{\lim_{x \to a} g(x)(f(x) - 1)}$ [taking log]

Integration

① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

② $\int \frac{1}{x}\,dx = \ln|x| + C$

③ $\int e^x\,dx = e^x + C$

④ $\int a^x\,dx = \frac{a^x}{\ln a} + C$

⑤ $\int \sin x\,dx = -\cos x + C$

⑥ $\int \cos x\,dx = \sin x + C$

Properties of definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd function i.e. $f(-x) = -f(x)$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even function i.e. $f(-x) = f(x)$

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

integration by parts