A Utility Maximization Example

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Since we couldn't finish the utility maximization problem in section, here it is solved from the beginning.

1 The ingredients

First, we start with a budget constraint: $p_x x + p_y y = M \implies 10x + 30y = 360$ (we are assuming that $p_x = 10$, $p_y = 30$, and $M = 360$).

Next, we are given a utility function: $U = U(x, y) = x^{3/4}y^{1/4}$.

An important concept captured by the utility function is that of *marginal utility*. The idea here is, if we get one more unit of a good (i.e., a marginal unit), how much does my utility go up? To find the marginal utility of x, MU_x, we find $\frac{dU}{dx}$:

$$\begin{aligned}\frac{d}{dx}\left(x^{\frac{3}{4}}y^{\frac{1}{4}}\right) &= \frac{3}{4}x^{\frac{-1}{4}}y^{\frac{1}{4}} \\ &= \frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}}\end{aligned}$$

Similarly, to find $MU_y = \frac{dU}{dy}$:

$$\begin{aligned}\frac{d}{dy}\left(x^{\frac{3}{4}}y^{\frac{1}{4}}\right) &= \frac{1}{4}x^{\frac{3}{4}}y^{\frac{-3}{4}} \\ &= \frac{1}{4}\left(\frac{x}{y}\right)^{\frac{3}{4}}\end{aligned}$$

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Next, we wish to calculate the *marginal rate of substitution*, $MRS_{x,y}$. The $MRS_{x,y}$ states how much y you would give up for another unit of x without reducing your utility. As shown in the book,

$$\Delta U = MU_x \Delta x + MU_y \Delta y = 0 \Longrightarrow \frac{\Delta y}{\Delta x} = -\frac{MU_x}{MU_y}$$

The amount of y you would give up for another unit of x is the ratio of the marginal utilities.[1] So if x is much more valuable to you than y (i.e., MU_x is much greater than MU_y), you would give up a lot of y to get more x. In our case,

$$\begin{aligned} MRS_{x,y} &\equiv \frac{MU_x}{MU_y} \\ &= \frac{\frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}}}{\frac{1}{4}\left(\frac{x}{y}\right)^{\frac{3}{4}}} \\ &= \frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}}\left(\frac{4}{1}\left(\frac{y}{x}\right)^{\frac{3}{4}}\right) \\ &= 3\frac{y}{x} \end{aligned}$$

2 Maximization

We wish to maximize utility *subject to* the budget constraint. To do this, we use the *Lagrangian equation.* The generic Lagrangian equation is

$$\mathcal{L} = \text{objective function } + \lambda(\text{constraint} = 0)$$

Think of this equation as taking the function that we wish to maximize and adding 0 to it—a perfectly acceptable mathematical "trick" that doesn't change the function at all, since the constraint must be satisfied and equal to 0. The symbol λ is called a *Lagrange multiplier*

[1] The "*give up*" is why there is a negative sign in there. The negative sign also reflects the fact that the slope of the indifference curve is the negative MRS.

and, of course, multiplying 0 by any number will return 0, so the Lagrangian is still equated to the objective function. For us, utility is our objective function and the budget constraint *equated to* 0 is our constraint. Now, maximizing the Lagrangian, we have

$$\max_{x,y} \mathcal{L} = \max_{x,y} x^{3/4}y^{1/4} + \lambda(M - p_x x - p_y y)$$

The x, y below max indicates that we are choosing both x and y to maximize the function. All other variables are beyond our control.[2] We take the following derivatives and set them equal to 0.[3] These are known as *first order conditions*.

$$\begin{aligned}
\frac{d\mathcal{L}}{dx} &= \frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}} - \lambda p_x = MU_x - \lambda p_x = 0 \\
\frac{d\mathcal{L}}{dy} &= \frac{1}{4}\left(\frac{x}{y}\right)^{\frac{3}{4}} - \lambda p_y = MU_y - \lambda p_y = 0 \\
\frac{d\mathcal{L}}{d\lambda} &= M - p_x x - p_y y = 0
\end{aligned}$$

Note that the derivative with respect to λ simply returns the budget constraint. One of the functions of the Lagrange multiplier is to ensure that we don't lose our constraint. We now have three equations defining our optimal bundle:

$$MU_x = \lambda p_x \quad (1)$$

$$MU_y = \lambda p_y \quad (2)$$

$$M = p_x x + p_y y \quad (3)$$

First see that

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\lambda p_x}{\lambda p_y} = \frac{p_x}{p_y}$$

[2] Since x and y are in our control, they are called *control variables*.

[3] This procedure only works if there is an *interior solution*—that is, the individual consumes some x and some y. If he does not consume one of these goods, there will be a *corner solution*. You know that there will be an interior solution if each marginal utility is a function of the quantity of the good and thus the first order conditions will be solvable. For example, $MU_x = 7$ is not a function of x and thus $MU_x = 7 \neq \lambda p_x$ in general. See Learning-by-Doing exercises 4.3 and 4.4 for examples.

Recall the interpretation of the marginal rate of substitution: it is how much y you are willing to give up to get one more unit of x, holding utility constant. The equation above states that, at the optimum, your willingness to exchange the goods must be equal to the ratio of the prices of the goods. So, if jeans cost three times more than shirts, when I am optimizing, my marginal utility of jeans must be three times that of shirts and my marginal rate of substitution of jeans for shirts must be three.

Here's another way to think of equations (1) and (2). Solve them this way:

$$\lambda = \frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

This tells us that the utility that we get from an additional unit (i.e., a *marginal* unit) of a good per dollar spent must be the same across every good that we consume. Let's say that this wasn't true; if I receive more utility per dollar that I spent on jeans than per dollar that I spend on shirts, then I should spend more on jeans to get higher utility. If marginal utility per dollar is equal across all goods, then I can't change my consumption in any way to get higher utility—that is, I am at a maximum of utility.

Now, let's solve the equations with our example utility function and prices. First solve equations (1) and (2) for λ:

$$\begin{aligned} MU_x &= \frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}} = \lambda p_x = 10\lambda \\ \lambda &= \frac{3}{40}\left(\frac{y}{x}\right)^{\frac{1}{4}} \\ MU_y &= \frac{1}{4}\left(\frac{x}{y}\right)^{\frac{3}{4}} = \lambda p_y = 30\lambda \\ \lambda &= \frac{1}{120}\left(\frac{x}{y}\right)^{\frac{3}{4}} \end{aligned}$$

Setting these two equations equal to one another and solving for y in terms of x:

$$
\begin{aligned}
\frac{3}{40}\left(\frac{y}{x}\right)^{\frac{1}{4}} &= \frac{1}{120}\left(\frac{x}{y}\right)^{\frac{3}{4}} \\
\left(\frac{y}{x}\right)^{\frac{1}{4}} &= \frac{1}{9}\left(\frac{x}{y}\right)^{\frac{3}{4}} \\
\frac{y}{x} &= \frac{1}{9^4}\left(\frac{x}{y}\right)^{3} \\
\left(\frac{y}{x}\right)^{4} &= \frac{1}{9^4} \\
\frac{y}{x} &= \frac{1}{9} \\
y &= \frac{x}{9}
\end{aligned}
$$

Taking this result and entering it into the budget constraint:

$$
\begin{aligned}
M &= p_x x + p_y y \\
360 &= 10x + 30\frac{x}{9} \\
x &= 27 \\
y &= \frac{x}{9} = 3
\end{aligned}
$$

At the optimum,

$$
\begin{aligned}
\frac{MU_x}{p_x} &= \frac{1}{p_x}\frac{3}{4}\left(\frac{y}{x}\right)^{\frac{1}{4}} = \frac{3}{4 \times 10 \times 9^4} \approx 0.0433 \\
\frac{MU_y}{p_y} &= \frac{1}{p_y}\frac{1}{4}\left(\frac{x}{y}\right)^{\frac{3}{4}} = \frac{9^{\frac{3}{4}}}{4 \times 30} \approx 0.0433
\end{aligned}
$$

As required, these values are the same, validating our results.

3 Tell me more about the Lagrange multiplier!

(Note: This section isn't required, but may help you better understand what λ is.)

As already mentioned, one of the important functions of the Lagrangian multiplier is to ensure that we don't lose our constraint during maximization (see page 3). Let's see what else we can learn from the Lagrangian equation. Take the derivative with respect to income, M:

$$\frac{d\mathcal{L}}{dM} = \frac{d}{dM}\left(U(x, y) + \lambda\left(M - p_x x - p_y y\right)\right) = \lambda$$

What does this mean? The Lagrangian multiplier tells us the increase in utility (that's what the Legrangian function is counting—utility) when we get an extra dollar of income. This has two interpretations. First, we could see it in the literal sense of the value of increasing our income. Alternatively, we could view it more generally as the value of relaxing our constraint. In other words, how much can we increase our objective function by making our constraint less restrictive? This interpretation can be used for any Lagrange function. It is the use in the context of utility maximization that gives the Lagrange multiplier its alternate name—a *shadow price*, the value (or price) of the budget constraint relaxed (i.e., increased) by a dollar.