Potential Outcomes and Causal Effects

September 13, 2012

Defining Causal Effects

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- How useful is ATE when effects are heterogeneous?

Identifying $\bar{\tau}$

- Define π to be the proportion of units in treatment

  Decompose $\bar{\tau}$:

$$
\begin{aligned}
\bar{\tau} = & \quad \pi\{E[Y_{i1}|T=1] - E[Y_{i0}|T=1]\} + \\
& (1-\pi)\{E[Y_{i1}|T=0] - E[Y_{i0}|T=0]\} \\
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- Expected bias of $\tilde{\tau}_{ATE} = E[Y_{i1}|T = 1] - E[Y_{i0}|T = 0]$:

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\tilde{\tau}_{ATE} &= \bar{\tau} + \{E[Y_{i0}|T = 1] - E[Y_{i0}|T = 0]\} \\
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- $\{E[Y_{i0}|T = 1] - E[Y_{i0}|T = 0]\}$ is *expected baseline bias*, difference in the average outcome in the absence of treatment.
- $(1 - \pi)\{E[\bar{\tau}|T = 1] - E[\bar{\tau}|T = 0]\}$ is *differential treatment effect* bias, i.e. the average difference in the treatment effect between those in the treatment group and those in the control group.

Identifying Assumptions

What assumptions do we need to identify $\bar{\tau}$?

- Assumption 1: $E[Y_{i1}|T=1] = E[Y_{i1}|T=0]$
- Assumption 2: $E[Y_{i0}|T=1] = E[Y_{i0}|T=0]$

A global assumption that gets us both assumption 1 and assumption 2 is independence between treatment assignment and potential outcomes:

$$\{Y_{i1}, Y_{i0}\} \perp\!\!\!\perp T$$

Beyond ATE

What about other estimands?

- Average effect of treatment on the treated (ATT): $\bar{\tau}|(T=1) = E[Y_{i1}|T=1] - E[Y_{i0}|T=1]$

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- Average effect of treatment on the treated (ATT):
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- Average effect of treatment on the controls (ATC):
  $\bar{\tau}|(T = 0) = E[Y_{i1}|T = 0] - E[Y_{i0}|T = 0]$

Selection on Observables

- Conditional independence:

$$\{Y_{i1}, Y_{i0}\} \perp\!\!\!\perp T | X$$

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- With this assumption, it follows that:
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E[ATE(x)] &= E\{E[Y|X=x, T=1] - E[Y|X=x, T=0]\} \\
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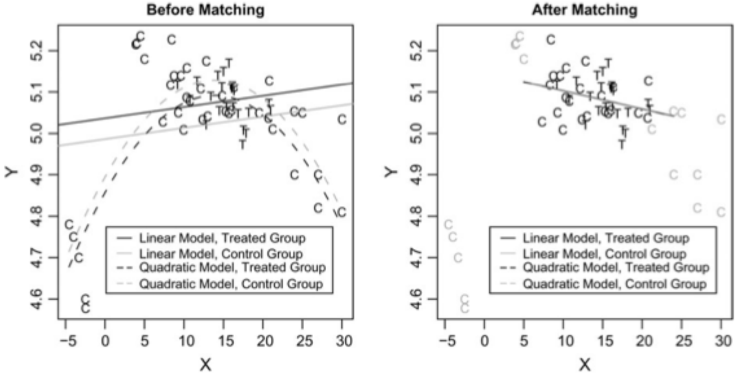
&= E[Y(1)] - E[Y(0)] \\
&= E[Y(1) - Y(0)] \\
&= ATE
\end{aligned}
$$

- $ATE(x)|(T=1)$

ATT


Before Matching
After Matching
Y
X
Linear Model, Treated Group
Linear Model, Control Group
Quadratic Model, Treated Group
Quadratic Model, Control Group
4.6
4.7
4.8
4.9
5.0
5.1
5.2
−5
0
5
10
15
20
25
30

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- "No interference between units" is "the observation on one unit should be unaffected the particular assignment of treatment to the other units"

No Interference

- Consider a uniform randomized experiment with two strata, four units in the first strata and two units in the second strata, for 6 units in total. Half the units in each stratum receive treatment.
- There are 12 possible treatment assignments contained in the set Ω.

$$
\Omega = \left\{
\begin{array}{cccccc}
\begin{bmatrix}1\\1\\0\\0\\1\\0\end{bmatrix} &
\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix} &
\begin{bmatrix}1\\0\\0\\1\\1\\0\end{bmatrix} &
\begin{bmatrix}0\\1\\1\\0\\1\\0\end{bmatrix} &
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\end{array}
\right\}.
$$

Causal Effects without SUTVA

- Without SUTVA, a causal effect is defined for every possible combination of the treatment assignment.
- The potential outcome for unit i might be $Y_{i100000000000}$ or $Y_{i010000000000}$, etc.
- How many potential outcomes will each unit have?
- Potential outcomes still well defined!

Sampling in Neyman Inference

The goal of a field experiment which consists of the comparison of ν varieties will be regarded as equivalent to the problem of comparing the numbers

$$a_1, a_2, \cdots, a_\nu$$

or their estimates by way of drawing several balls from an urn.

The simplest way of obtaining an estimate of the number a_i would be by drawing κ balls from the ith urn in such a way that after noting the expressions on the balls drawn, they would be returned to the urn. In this way we would obtain κ independent outcomes of an experiment, and their average X_i would, based on the law of large numbers, be an estimate of the mathematical expectation of the result of our trial. Let x

Average Treatment Effects

It should be emphasized that the problem of determining the difference between the yields of two varieties becomes more complicated in this case. Let us consider the scheme with ν urns. [From now on, x_i and x_j are the averages of κ trials corresponding to varieties i and j, sampled as in the scheme with ν urns.] It is easy to see that

$$\mathbb{E}(x_i - x_j) = a_i - a_j,$$

so that the expected value of the difference of the partial averages of yields from two different varieties is equal to the difference of their expectations. It can also be determined that this difference is an estimate of $a_i - a_j$, but

Where does Uncertainty Come From?

the expression for the standard deviation becomes more complicated:

$$
\begin{aligned}
\mu^2_{x_i - x_j} &= \mathbb{E}[x_i - x_j - (a_i - a_j)]^2 \\
&= \mathbb{E}(x_i - a_i)^2 + \mathbb{E}(x_j - a_j)^2 \\
&\quad - 2\mathbb{E}(x_i - a_i)(x_j - a_j) \\
&= \mu^2_{x_i} + \mu^2_{x_j} - 2[\mathbb{E}\, x_i x_j - a_i a_j].
\end{aligned}
$$

Fisher Permutation Inference

- What's uncertain here?
- Null hypotheses for Fisher and Neyman inference?
- Test distributions?
- Estimation v. Inference?
- Permutation Inference for a Population?