Government 1000 Lecture Notes*

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Introduction

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The intellectual history of least squares is a glorious one. Many of the greatest minds of the 18th and 19th centuries contributed to its creation: De Moivre, several Bernoullis, Gauss, Laplace, Quetelet, Galton, Pearson, and Yule.

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The mystery is cleared up when one realizes that all of these scholars thought of themselves to be involved in the discovery of a method of statistical calculus that would do for social studies what Leibniz's and Newton's calculus did for physics.

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2. Rigorous mathematical theories such as Newtonian physics or the two theories of relativity.

It was hoped that statistical inference through the use of multiple regression, and other such methods, is able to provide to the social scientist what experiments and rigorous mathematical theories provide, respectively, to the micro-biologist and astronomer.

In essence, we are after the ability to control for multiple factors so that the scientist may observe the causal (or true) relationship serious problems between x and y.

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Least-squares also helped solve the problem of how to combine observations.

Mill's Methods of Inductive Inference

John Stuart Mill (in his *A System of Logic*) devised a set of five methods (or canons) by means of which to analyze and interpret our observations for the purpose of drawing conclusions about the causal relationships they exhibit. These methods have been used by generations of social science researchers.

Method of Agreement: "If two or more instances of the phenomenon under investigation have only one circumstance in common, the circumstance in which alone all the instances agree is the cause (or effect) of the given phenomenon."

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Method of Difference: "If an instance in which the phenomenon under investigation occurs, and an instance in which it does not occur, have every circumstance in common save one, that one occurring only in the former; the circumstance in which alone the two instances differ is the effect, or the cause, or an indispensable part of the cause, of the phenomenon."

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Mill himself realized many of these problems:

> “Nothing can be more ludicrous than the sort of parodies on experimental reasoning which one is accustomed to meet with, not in popular discussion only, but in grave treatises, when the affairs of nations are the theme. “How,” it is asked, “can an institution be bad, when the country has prospered under it?” “How can such or such causes have contributed to the prosperity of one country, when another has prospered without them?” Whoever makes use of an argument of this kind, not intending to deceive, should be sent back to learn the elements of some one of the more easy physical sciences.”

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A key and tricky concept in statistical inference is conditional probability.

Let's look at an example Mill himself brought up:

> "In England, westerly winds blow during about twice as great a portion of the year as easterly. If, therefore, it rains only twice as often with a westerly as with an easterly wind, we have no reason to infer that any law of nature is concerned in the coincidence. If it rains more than twice as often, we may be sure that some law is concerned; either there is some cause in nature which, in this climate, tends to produce both rain and a westerly wind, or a westerly wind has itself some tendency to produce rain."

$$\begin{aligned} \mathrm{H} :& \mathrm{P}(\text{rain}|\text{westerly wind},\ \Omega) > \\ & \mathrm{P}(\text{rain}|\textbf{not}\text{ westerly wind},\ \Omega), \end{aligned}$$

where Ω is a set of background conditions we consider necessary for a valid comparison.

But Conditional Probability is Tricky

This example was made famous by Monty Hall (Let's Make a Deal). Let us assume that Monty Hall presents to you three envelopes. One of the envelopes contains a $100 bill the other two are empty. Monty tells you that he put the money into an envelope by random (using a discrete uniform distribution $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$). You are asked to pick one envelope. You pick envelop A. Then, Monty tells you that he will open one of the other envelopes—one which does not contain any money. Monty opens envelope C. Monty then allows you the option of switching from the envelope you have chosen (A) to the remaining unopened envelope (B). Assume that Monty has been telling you the truth.

To be clear, let us assume the following:

1. Monty Hall would never open envelope you have chosen—i.e., A.
2. Monty would never open the envelope containing the money.
3. If the money is in envelope A, Monty will choose to open envelope B or C with equal probability.

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Probability also helps us make valid empirical inferences, both descriptive and causal.

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$$B = \{4, 5, 6\}. \tag{2}$$

It follows that $A \in \Omega$ and $B \in \Omega$, but $A \notin B$.

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   Proof:

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5. for any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   And $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Conditional Probability

Let A be an event such that $P(A) > 0$. The conditional probability of an event B given A occurs, denoted $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \tag{20}$$

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Let A be an event such that $\mathrm{P}(A) > 0$. The conditional probability of an event B given A occurs, denoted $\mathrm{P}(\mathrm{B}|A)$, is

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The probability measure $P_A(B) = P(B|A)$ is called the conditional distribution given A.

It is easily seen that our definition of the conditional distribution given A is sensible. Whatever prior information that led us to use P is not disturbed. We simply reassign the probabilities to exclude outcomes not in A since we know A occurs. Thus the probability of B occurring should now be proportional to $\mathrm{P}(A \cap \mathrm{B})$.

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$$\frac{N(A \cap B)/n}{N(A)/n} \approx \frac{P(A \cap B)}{P(A)} \tag{23}$$

Bayes' Rule

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Bayes Theorem is commonly ascribed to the Reverent Thomas Bayes (1701-1761) who left one hundred pounds in his will to Richard Price "now I suppose Preacher at Newington Green." Price discovered two unpublished essays among Bayes' papers which he forwarded to the Royal Society. This work made little impact, however, until it was independently discovered a few years later by the great French mathematician Laplace. English mathematicians then quickly rediscovered Bayes' work.

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The following is called Bayes Rule:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{24}$$

$$= \frac{P(B|A)P(A)}{P(B)} \tag{25}$$

Simple Card Example

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Monty Hall Problem: Solution

This example was made famous by Monty Hall (Let's Make a Deal). Let us assume that Monty Hall presents to you three envelopes. One of the envelopes contains a \$100 bill the other two are empty. Monty tells you that he put the money into an envelope by random (using a discrete uniform distribution $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$). You are asked to pick one envelope. You pick envelop A. Then, Monty tells you that he will open one of the other envelopes—one which does not contain any money. Monty opens envelope C. Monty then allows you the option of switching from the envelope you have chosen (A) to the remaining unopened envelope (B). Assume that Monty has been telling you the truth.

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Should you switch? Does it matter?

Let there be three envelopes: $\Omega \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$.

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Without loss of generality, let us assume that envelope C was revealed to be the one which is empty. Should you switch? *Let* $C = \$0$ *denote the event that "envelope C will be revealed to be empty."*

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The value of both of these probabilities can be obtaining using Bayes Rule. But let us first determine $P(C = \$0)$ because we will require this value to apply Bayes' Rule.

$$
\begin{aligned}
\mathrm{P}(\mathrm{C} = \$0) = {} & \mathrm{P}(\mathrm{C} = \$0 | A = \$100)\mathrm{P}(A = \$100) \\
& + \mathrm{P}(\mathrm{C} = \$0 | \mathrm{B} = \$100)\mathrm{P}(\mathrm{B} = \$100) \\
& + \mathrm{P}(\mathrm{C} = \$0 | \mathrm{C} = \$100)\mathrm{P}(\mathrm{C} = \$100)
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
P(C=\$0) &= P(C=\$0|A=\$100)P(A=\$100) && (31)\\
&+ P(C=\$0|B=\$100)P(B=\$100)\\
&+ P(C=\$0|C=\$100)P(C=\$100)\\
&= \frac{1}{2}\frac{1}{3} + 1\frac{1}{3} + 0\frac{1}{3} && (32)
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Let us now calculate $P(A = \$100|C = \$0)$:

$$P(A = \$100|C = \$0) = \frac{P(A = \$100 \cap C = \$0)}{P(C = \$0)} \tag{36}$$

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Therefore, the probability that envelope A contains the money remains unchanged.

Let us now calculate the probability that envelope B contains the money after envelope C has been revealed not to contain any money.

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Therefore, the probability that envelope A, our original choice, contains the money is, as before, $\frac{1}{3}$ while the probability that envelope B now contains the money is $\frac{2}{3}$. Therefore, we should switch!

Monty Hall, Version 2

What if we changed assumption 3 from:

> If the money is in envelope A, Monty will choose to open envelope B or C with equal probability.

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Let's work through the probabilities again. As before, *let* $\mathrm{C} = \$0$ *denote that event that "envelope C was revealed to be empty."* As before, let $A = \$100$ denote the event that envelope A contains \$100. And, as before, let $\mathrm{B} = \$100$ denote the event that envelope B contains \$100.

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We are interested in two conditional probabilities: $\mathrm{P}(A = \$100|\mathrm{C} = \$0)$ and $\mathrm{P}(\mathrm{B} = \$100|\mathrm{C} = \$0)$. We want to know the probability that A contains the money given that C was revealed to be empty and the probability that B contains the money given that C was revealed to be empty.

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&= \frac{1}{4}\frac{1}{3} + 1\frac{1}{3} + 0\frac{1}{3} && (46)\\
&= \frac{1}{12} + \frac{1}{3} + 0 && (47)\\
&= \frac{1}{12} + \frac{4}{12} && (48)
\end{aligned}
$$

As before, we know that we are going to need to use Bayes Rule. So let's calculate $P(C = \$0)$ first.

$$
\begin{aligned}
P(C = \$0) &= P(C = \$0|A = \$100)P(A = \$100) \\
&\quad + P(C = \$0|B = \$100)P(B = \$100) \\
&\quad + P(C = \$0|C = \$100)P(C = \$100) && (45) \\
&= \frac{1}{4}\frac{1}{3} + 1\frac{1}{3} + 0\frac{1}{3} && (46) \\
&= \frac{1}{12} + \frac{1}{3} + 0 && (47) \\
&= \frac{1}{12} + \frac{4}{12} && (48) \\
&= \frac{5}{12} && (49)
\end{aligned}
$$

$$P(A = \$100 | C = \$0) = \frac{P(A = \$100 \cap C = \$0)}{P(C = \$0)} \tag{50}$$

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You should clearly switch!

Before Monty Opens Anything

Say you choose envelope A, but you do not yet know which envelope Monty will open. Before you know this information, what is your probability of winning if you switch versus staying with A?

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> If the money is in envelope A, Monty will choose to open envelope B or C with equal probability.

To:

> If the money is in envelope A, Monty will choose to open envelope B with probability $\frac{3}{4}$ and envelope C with probability $\frac{1}{4}$.

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The two relevant probabilities are $\mathrm{P}(A = 1|\mathrm{B} = 0 \cup \mathrm{C} = 0)$ (i.e., the probability of A containing the money given that either B or C will be revealed to be empty and $\mathrm{P}(switch = 1)$ (i.e., the probability of the switched to envelope containing the money given that either B or C have been revealed to be empty).

It is straightforward to obtain both of these.

$$P(A = 1 | B = 0 \cup C = 0) = \frac{P\left(A = 1 \cap \left(B = 0 \cup C = 0\right)\right)}{P(B = 0 \cup C = 0)} \tag{59}$$

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&= \frac{P((B = 0 \cup C = 0)|A = 1)P(A = 1)}{P(B = 0 \cup C = 0)} && (60)\\
&= \frac{1\frac{1}{3}}{1} && (61)
\end{aligned}$$

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$$= \frac{1}{3} \tag{62}$$

$$P(switch = 1) = P(C = 1|B = 0)P(B = 0) + P(B = 1|C = 0)P(C = 0) \tag{63}$$

We have already calculated $P(B = 1|C = 0)$ (see Equations 54—58), but we do not yet know $P(C = 1|B = 0)$. In order to obtain this we must calculate $B = 0$.

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$$= \frac{7}{12} \tag{67}$$

We must still obtain $P(C = 1|B = 0)$.
pause

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We may now return to our story and obtain $\mathrm{P}(switch = 1)$.

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$$P(switch = 1) = P(C = 1|B = 0)P(B = 0) + P(B = 1|C = 0)P(C = 0) \tag{73}$$

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$$\begin{aligned} P(switch = 1) &= P(C = 1|B = 0)P(B = 0) + P(B = 1|C = 0)P(C = 0) && (73) \\ &= \frac{4}{7}\frac{7}{12} + \frac{4}{5}\frac{5}{12} && (74) \\ &= \frac{28}{84} + \frac{70}{60} && (75) \\ &= \frac{1}{3} + \frac{1}{3} && (76) \end{aligned}$$

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\mathrm{P}(switch = 1) &= \mathrm{P}(C = 1|B = 0)\mathrm{P}(B = 0) + \mathrm{P}(B = 1|C = 0)\mathrm{P}(C = 0) && (73)\\
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The Independence Of Monty Hall

What if Monty didn't always open another envelope?

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The answer is the importance of independence.

Exit Polls

In the California gubernatorial election in 1982, several TV stations predicted, on the basis of questioning people when they existed the polling place, that Tom Bradley, then mayor of Los Angeles, would win the election beating the only other candidate George Deukmejian.

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Since there were only two candidates, this makes the probability of voting for Deukmejian $\mathrm{P}(\mathrm{B}^c) = 0.55$.

Let $A =$ "The voter stops and answers a question about how she voted" and suppose that $\mathrm{P}(A|\mathrm{B}) = 0.4$, $\mathrm{P}(A|\mathrm{B}^c) = 0.3$. That is, 40% of Bradley voters will respond compared to 30% of the Deukmejian voters.

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Since $P(A) = P(B \cap A) + P(B^c \cap A)$,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.18}{0.18 + 0.165} = 0.5217 \tag{85}$$

Independence

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Independence

Intuitively, two events A and B are independent if the occurrence of A has no influence on the probability of occurrence of B.

The formal definition is: A and B are independent if $P(A \cap B) = P(A)P(B)$.

A and B are considered independent if the occurrence of one in no way depends on the occurrence of the other. In other words:

$$P(A|B) = P(A) \tag{86}$$

$$P(B|A) = P(B). \tag{87}$$

Bayes Rule is always true. That is, regardless of independence

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

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Example 2 Roll two dice. $A =$ "The first die shows 4," $\mathrm{B} =$ "The second die shows 2." $\mathrm{P}(A) = \frac{1}{6}$, $\mathrm{P}(\mathrm{B}) = \frac{1}{6}$, $\mathrm{P}(A \cap \mathrm{B}) = \frac{1}{36}$.

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In general, if A and B are disjoint events that have positive probability, they are not independent since $P(A)P(B) > 0$ (by definition), but $P(A \cap B) = 0$ (because they are disjoint).

A finite sequence $A_1, A_2, \cdots, A_n$ or an infinite sequence $A_1, A_2, \cdots$ of events is said to be disjoint if $A_i \cap A_j = \phi$, for all $i \neq j$.

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Independence is about the probability of the occurrence of events and disjoint is a description of the sample space. It may be useful to think of "independence" as a property of random variables and "disjoint" as a property of events.

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The three events A, B, and C are not independent, however, since $A \cap B = A \cap B \cap C$ and hence

$$P(A \cap B \cap C) = \frac{1}{365^2} \neq \left(\frac{1}{365}\right)^3 = P(A)P(B)P(C) \tag{91}$$

The moral of this example is that in order to determine if several events are independent, you pair-wise comparisons are not enough.

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Answer: $P(spending\ money|Z) \neq P(spending\ money)$, where Z is a long list of baseline variables such as House voting record, general quality of the candidate, the constituency service the candidate performs. The upshot is that there is very little agreement on the effect of money.

How to obtain conditional independence is a central concern of only this class but of all empirical research.

The most used “solution” in the social sciences is (ordinary least squares) regression. This is a particular way of modeling the conditional mean: $P(Y|X)$, where X are some variables we wish to condition on.

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Foreshadowing Note: When a regression model is not consistent with the Neyman-Rubin-Holland Causal Model, it cannot be interpreted as offering direct causal estimates. But it may still offer useful information as in the "Peasants or Bankers? The American Electorate and the U.S. Economy" article we will be discussing later in the course.

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For example: in the "Quality Meets Quantity" article we are interesting in comparing the following two conditional probabilities:

$$\mathrm{P}(\text{revolution}|\text{foreign threat},\ Z), \tag{92}$$

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where Z is the set of background conditions we consider necessary for valid comparisons (such as village autonomy and dominant classes who are economically independent). In the article Z is denoted by Ω but that may cause confusion given our definition of Ω.

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Recall that Equation 92 $= \dfrac{1}{8}$ and Equation 93 $= \dfrac{2}{69}$. Note We have rounded Equation 93 from the number reported in the article.

We wish to know if Equation 92 is SIGNIFICANTLY larger than 93. In other words, we need to rule out that Equation 92 is larger than Equation 93 just by chance.

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The use of many purely algebraic concepts such as the mean crucially depend on the distribution which is assumed. For example, the use of the mean did not become widespread in society until the normal distribution was discovered and until it became generally believed. This did not occur until the late 19th century. Without an implied distribution, the mean may be a completely uninformative concept. This will become clear in a lecture or two.

Random Variables I

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There are more fine grained distinctions.

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4. Ratio If there exists a meaningful zero value and the ratio between two measures is meaningful, a ratio scale can be used. A task that takes four days to complete is twice as long as a task that takes two days to complete.

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9. and to use an illustration form classical probability, the number of heads obtained in tossing a coin 100 times (discrete, count).

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The normal distribution has two parameters: the mean (μ) and variance (σ^2). The standard normal distribution is assumed to have a mean of zero and a variance of one. The concepts of mean and variance obviously are not limited to just this distribution.

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Random variables actually follow certain (mathematical) rules.

The most important distribution in statistics is the normal distribution. It is continuous, unimodal, symmetric and unbounded. Another important distribution is the binomial, which we have already seen—e.g., "the number of heads in several tosses of a coin."

The normal distribution has two parameters: the mean (μ) and variance (σ^2). The standard normal distribution is assumed to have a mean of zero and a variance of one. The concepts of mean and variance obviously are not limited to just this distribution.

We shall discuss probability densities and then examine the concepts of mean, variance, covariance and correlation without explicit reference to any distribution.

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Let's look at some examples.

Simple Discrete Example

Let $Y = 1$ denote a vote for the Republican Party and $Y = 0$ denote a vote for the Democratic Party. $\Omega = \{0, 1\}$. A valid probability distribution for Y is:

- P(Y=1) = 0.6
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Another valid distribution for Y could be:

- P(Y=1) = 0.9
- P(Y=0) = 0.1

A generalized form for such distributions is called the Bernoulli Distribution.

Bernoulli Distribution

This distribution corresponds to our previous voting example or to the flip of one possibly unfair coin.

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- The parameter π can be interpreted as a probability:
  - ⋆ $P(Y = 1) = \pi$
  - ⋆ $P(Y = 0) = 1 - \pi$

- The common summary of the Bernoulli distribution is:

$$P(Y = y|\pi) = \pi^y (1 - \pi)^{1-y} \tag{94}$$

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For example, if $\pi = .4$ our probability of obtaining $y = 1$ is:

$$P(Y = 1|\pi = .4) \quad = .4^1(1-.4)^{1-1} \quad = 0.4 \tag{95}$$

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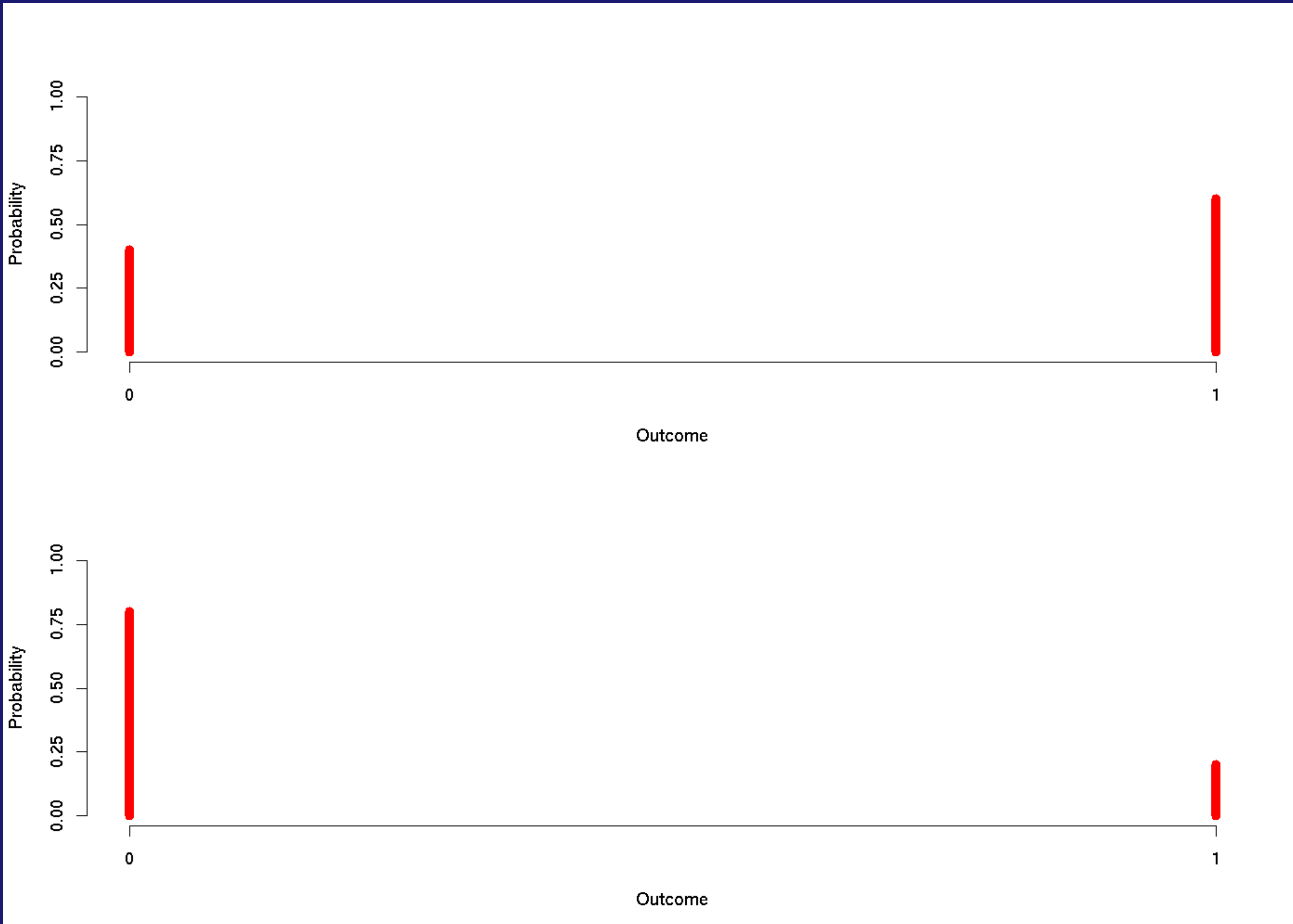
And our probability of obtaining $y = 0$ is:

$$P(Y = 0|\pi = .4) \quad = .4^0(1-.4)^{1-0} \quad = 0.6 \tag{96}$$

Graphical Summary of Two Bernoulli Distributions



Binomial Distribution

This is the most common discrete distribution. It results when there are many independent Bernoulli trials with the same π. The number of trails is denoted by n and the number of successes (i.e., $y = 1$) by s. n could be the number of voters and s the number of votes for the Republican candidate.

To get from the Bernoulli to the Binomial we need to assume:

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The binomial distribution is covered in Wannacott and Wonnacott chapter 4 and Freedman et al. chapter 15. The Freedman discussion of distributions is generally better.

If the previous three conditions hold, then S is called a binomial variable. The binomial PDF which gives the probability of exactly s successes in n trails when each trail has probability π of a success is:

$$P(s) = \binom{n}{s}\pi^{s}(1-\pi)^{n-s} \tag{97}$$

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where, in turn, the factorial $n!$ is defined by:

$$n! = n(n-1)(n-2)\cdots 1 \tag{99}$$

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$$P(2) \quad = \quad \binom{69}{2}\left[\frac{2}{69}\right]^2(1-\frac{2}{69})^{69-2} = 0.2746708 \tag{102}$$

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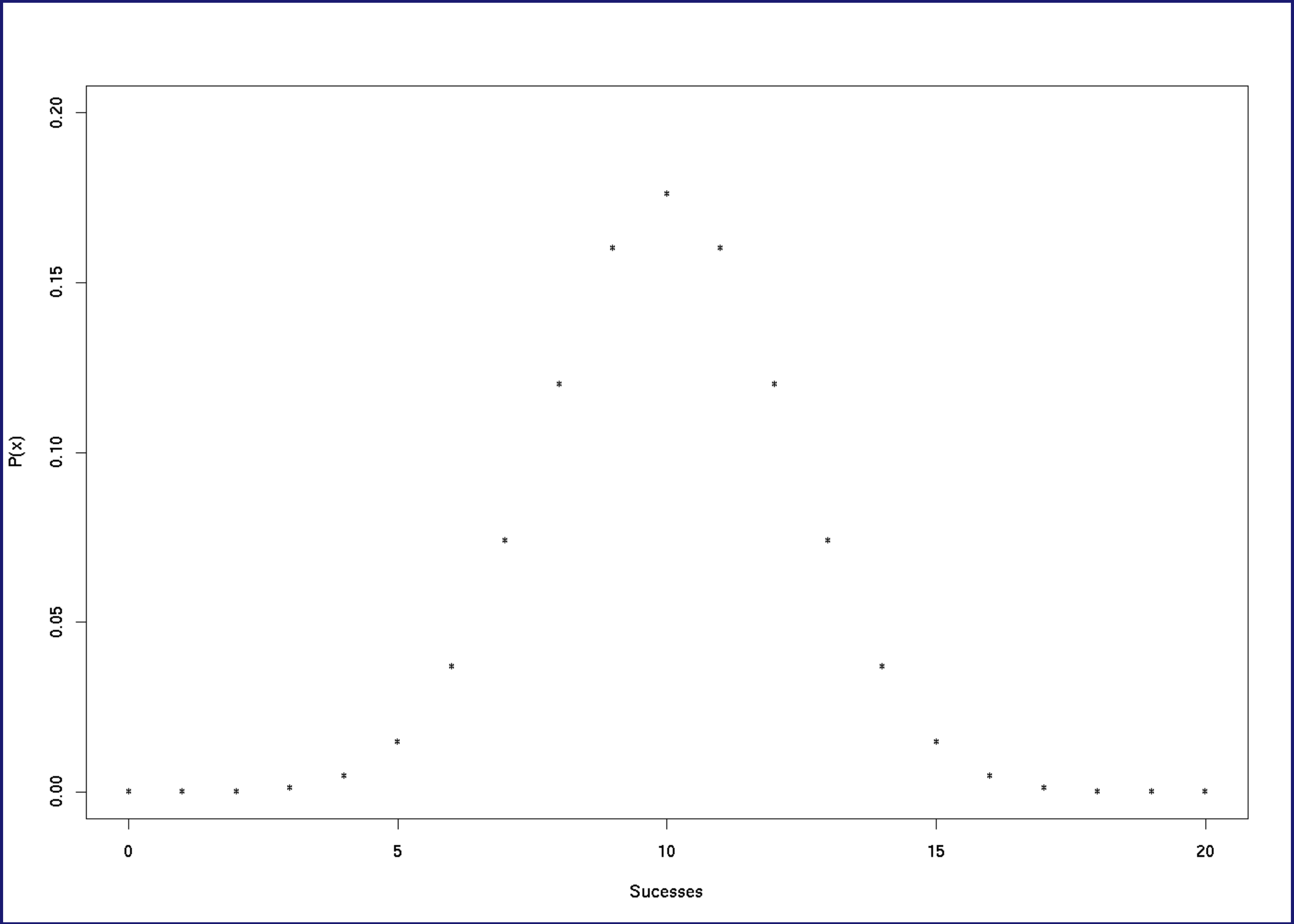
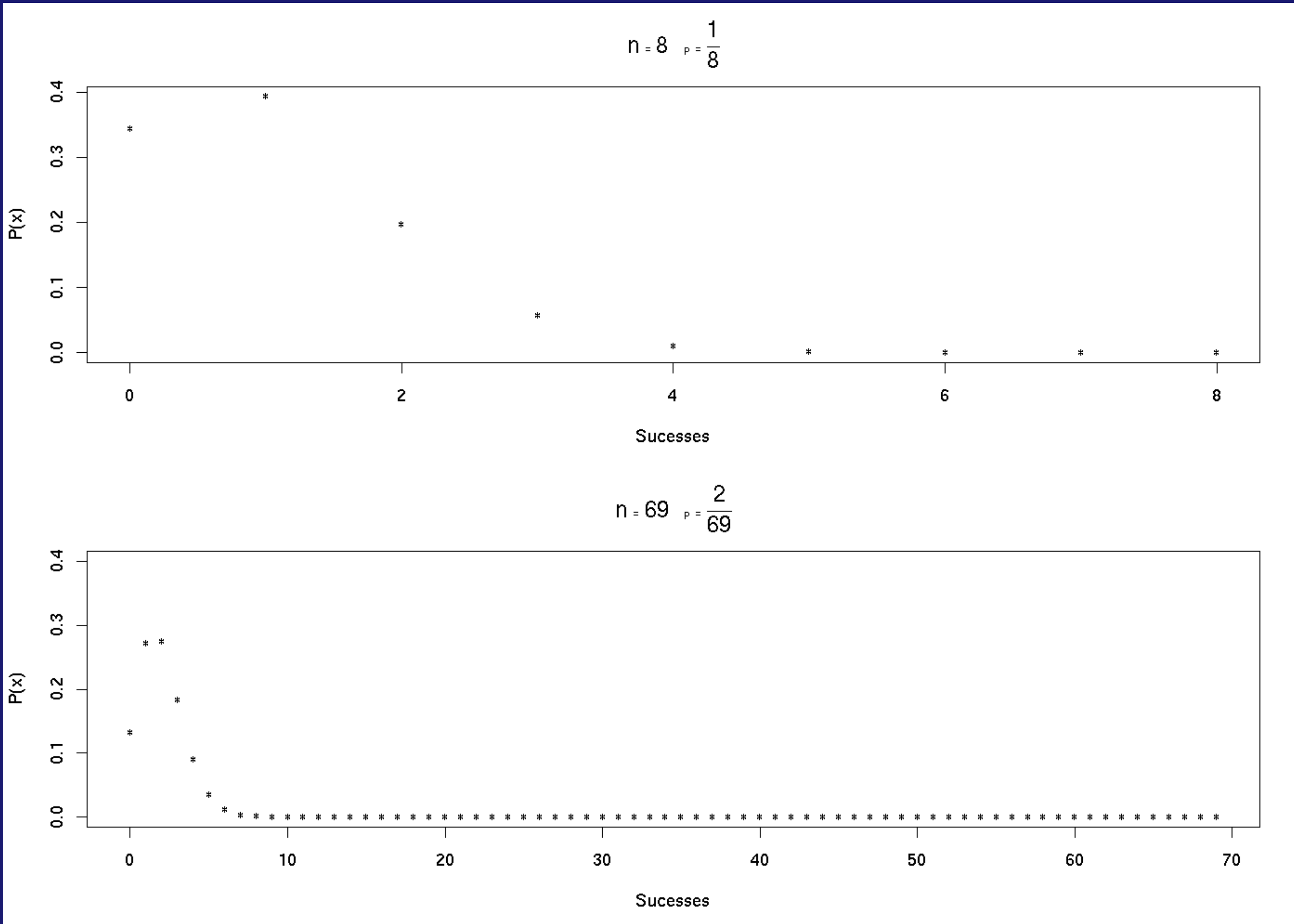
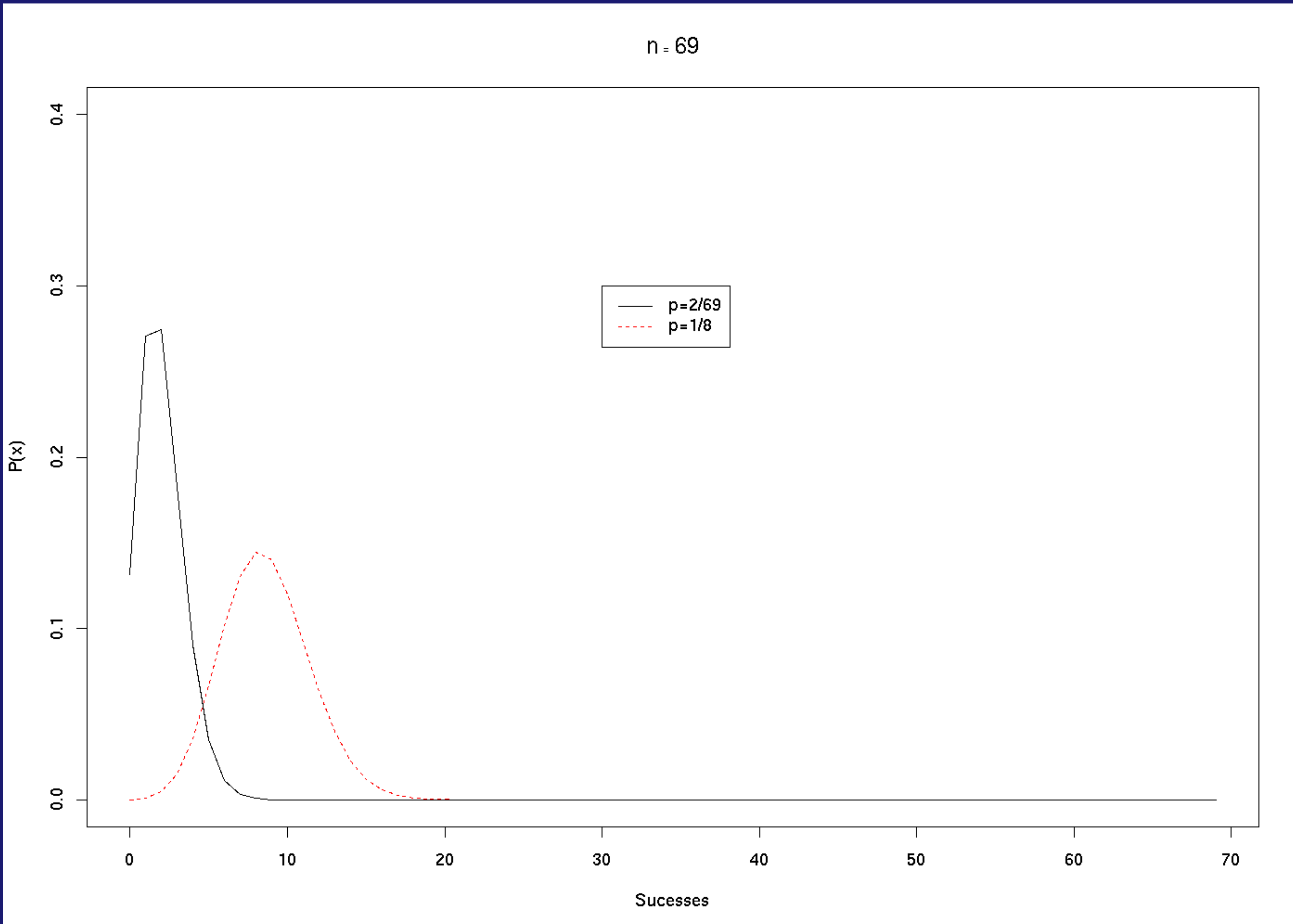
Binomial PDF with $pi = 0.5$ and $n = 20$



Binomial PDFs of Data From “Quality Meets Quantity”)



Binomial PDFs of Data From “Quality Meets Quantity” RESCALED



Binomial: More Details

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A lecture on how to use these **R** functions will be presented in section.

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- With the work of Quetelet and others (such as Francis Galton) the normal distribution became an ideal defended by data and data defended by the ideal.
- "Everybody believes in the [normal approximation], the experimenters because they think it is a mathematical theorem, the mathematicians because they think it is an experimental fact" G. Lippmann (French Physicist, 1845-1921).

- The univariate normal density:

$$N(y_i|\mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left(\frac{-(y_i-\mu)^2}{2\sigma^2}\right) \tag{105}$$

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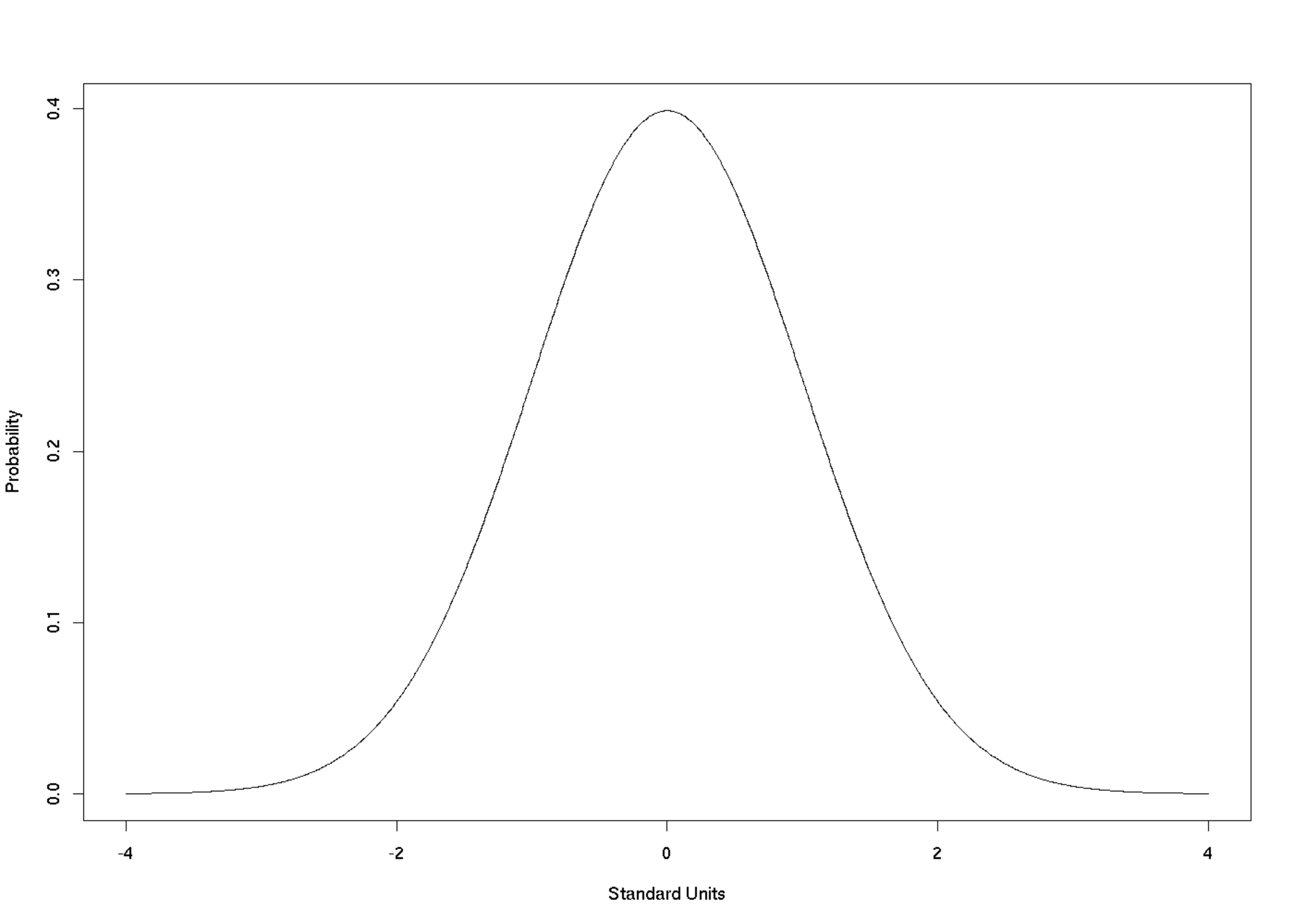
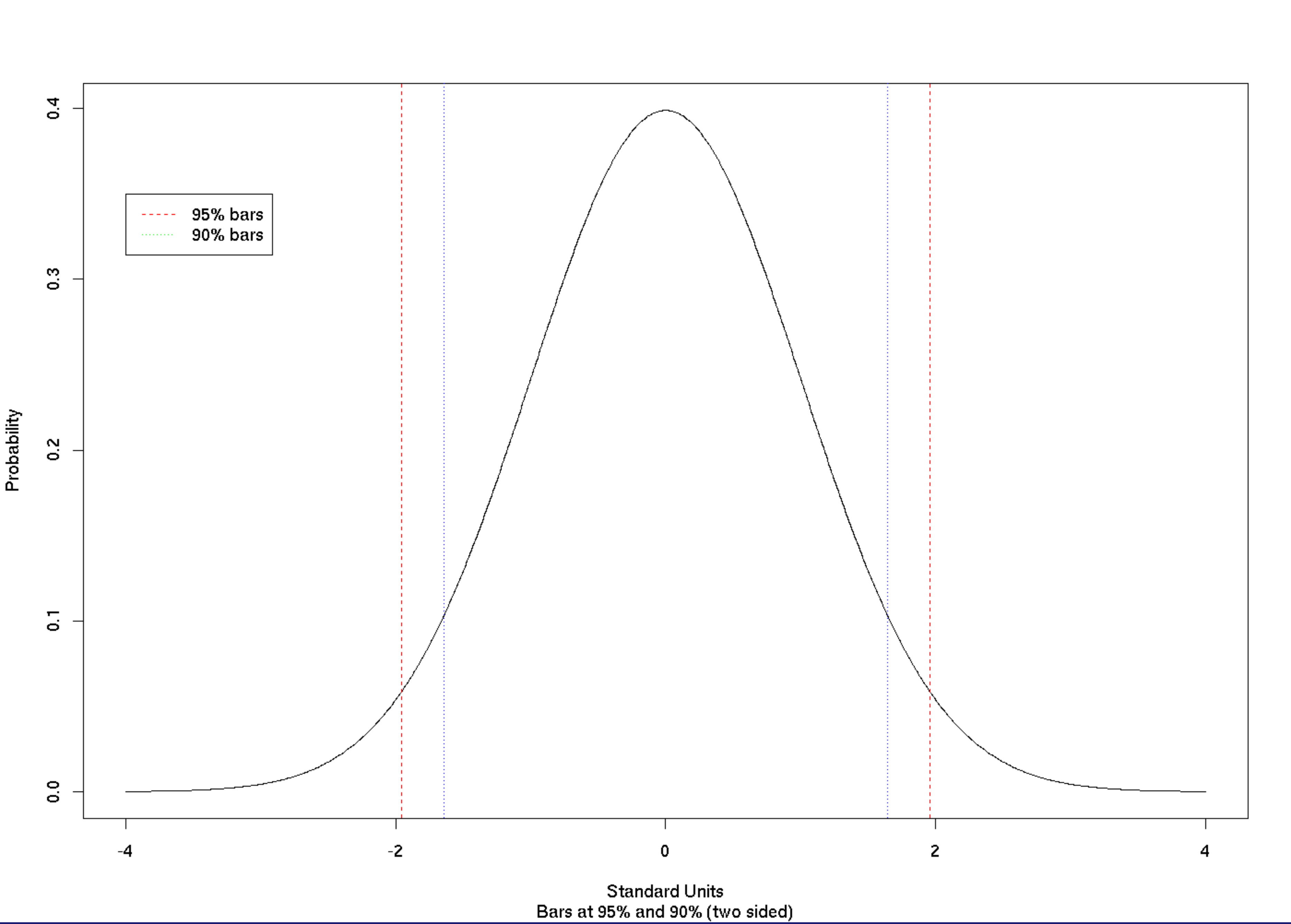
- The standardized univariate normal density:

$$N(y_i|0, 1) = (2\pi)^{-1/2}\exp\left(\frac{-y_i^2}{2}\right) \tag{106}$$

Standard Normal PDF



Standard Normal PDF, two-sided CI



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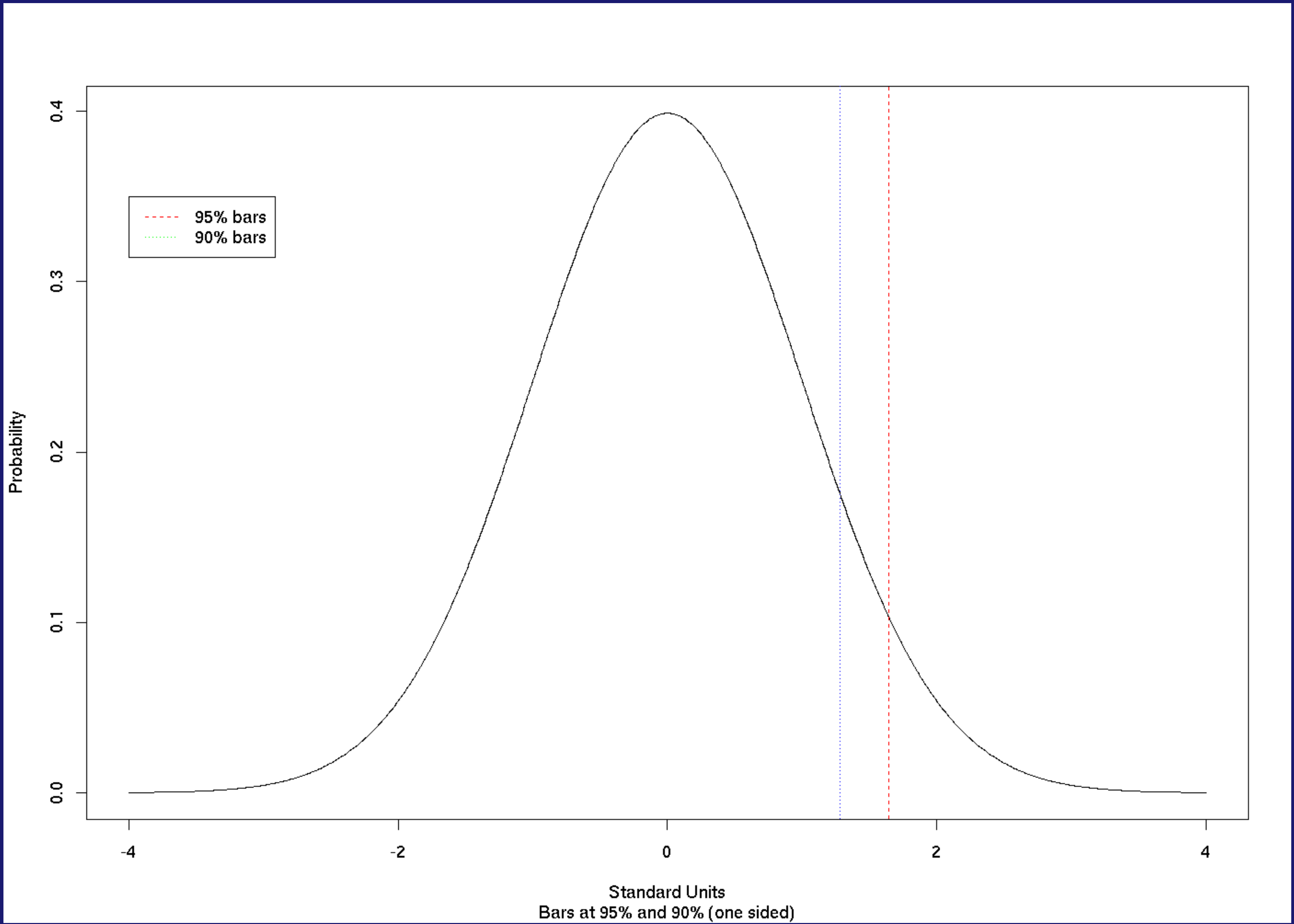

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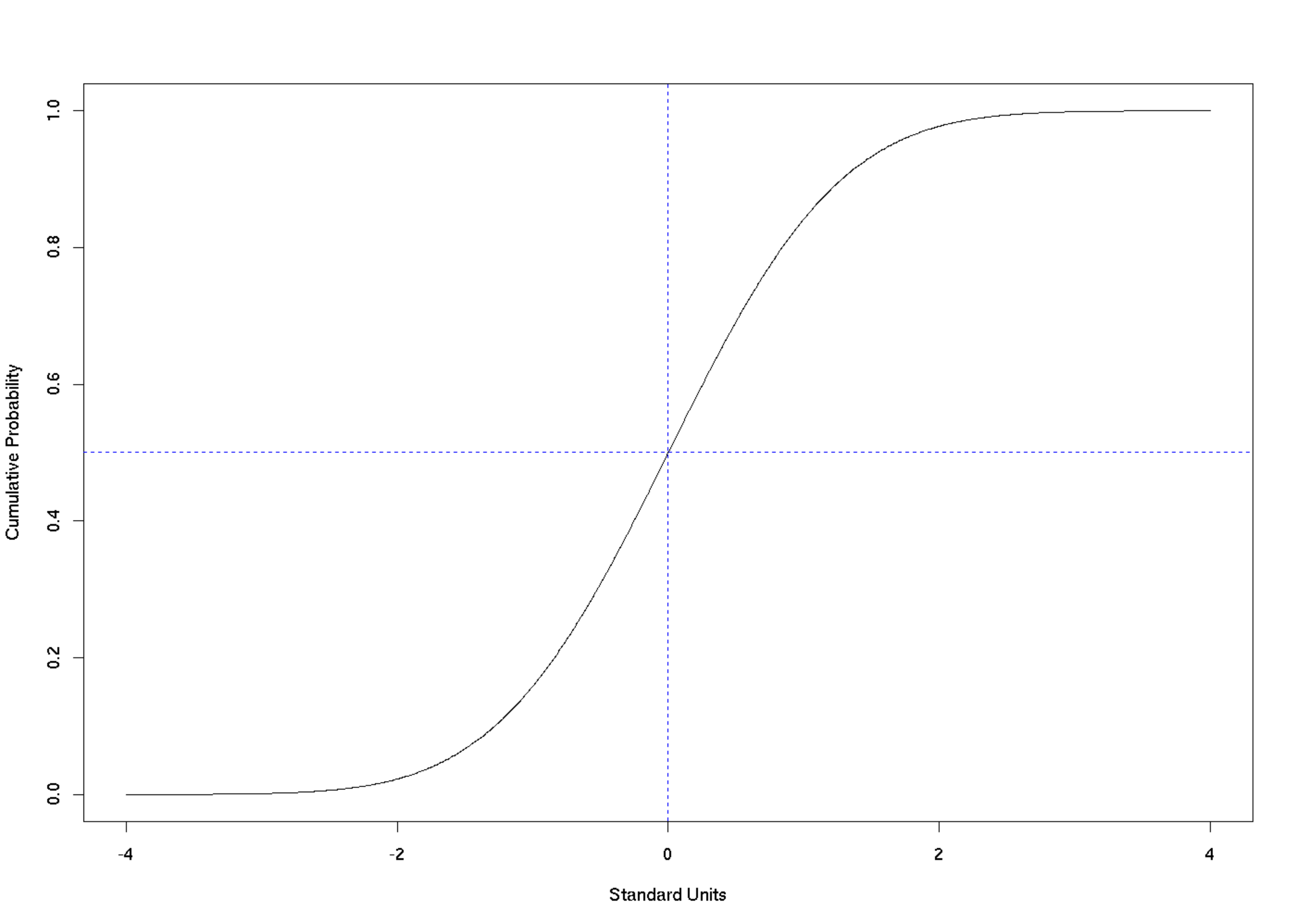
Hence, there is a mapping from the probability distribution function (PDF) and the cumulative distribution function (CDF):

$$
\begin{aligned}
P(a \leq Y \leq b) &= \int_{a}^{b} f(y)dy && (107) \\
&= \int_{-\infty}^{b} f(y)dy - \int_{-\infty}^{a} f(y)dy && (108) \\
&= F(b) - F(a) && (109)
\end{aligned}
$$

Cumulative Standard Normal PDF (from the bottom)



Normal Distribution R

R has various functions associated with the normal, they will be discussed in section:

- `rnorm()`: generates pseudo-random draws from a normal distribution
- `dnorm()`: the probability distribution function for a normal distribution
- `pnorm()`: the cumulative distribution function
- `qnorm()`: the quantile distribution function. You tell this function the probability you want and it returns the quantile. This is the reverse of what `pnorm` does.

For examples see http://jsekhon.fas.harvard.edu/gov1000/normal1.R

Mean and Median

Estimating the mean ought to be familiar to everyone. Let μ denote the mean of n realizations of the random variable X: $x_1, x_2, \cdots, x_n$

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{110}$$

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Whether the mean or median is a better measure depends on the underlying distribution of the variable of interest. Social scientists, and lay people, usually (often implicitly) assume the normal distribution. Therefore, they generally use the mean.

Rules of Summation; Variance and Covariance

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These three rules can be used to to derive some other rules.

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$$\mathrm{cov}(X, Y) = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right) \tag{122}$$

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Please see page 167 in Wonnacott and Wonnacott and chapters 8 and 9 in Freedman et al. for more details.

Rule 5

The covariance between X and Y is equal to the mean of the products of observations on X and Y minus the product of their means:

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \bar{X}\bar{Y} \tag{124}$$

Proof:

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})=\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\frac{1}{n}\sum_{i=1}^{n}\bar{X}Y_i-\frac{1}{n}\sum_{i=1}^{n}X_i\bar{Y}+\frac{1}{n}\sum_{i=1}^{n}\bar{X}\bar{Y} \tag{125}$$

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$$=\frac{1}{n}\sum_{i=1}^{n}X_iY_i-\bar{X}\bar{Y}-\bar{Y}\bar{X}+\frac{1}{n}\sum_{i=1}^{n}\bar{X}\bar{Y} \tag{127}$$

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Proof:

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Rule 6

The variance of X is equal to the mean of the squares of observations on X minus its mean squared. Rule 6 follows from Rule 5 since it applies to the case in which X and X are the two variables (instead of X and Y).

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - \bar{X}^2 \tag{130}$$

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It is interesting to note that when X and Y have a mean of zero, the definitions of covariance and variance become:

$$cov(x, y) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i \tag{131}$$

$$var(x) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{132}$$

In certain situations it will be necessary to use summations which apply to two random variables, called double summations. Specifically, let X_{ij} be a random variable which takes on N values for each outcome of i and j. There will, of course, be N^2 total outcomes. New we define the double summation of these N^2 outcomes as:

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We here list two rules of double-summation.

Rule 7

$$\sum_{i=1}^{n} \sum_{j=1}^{n} X_i Y_j = \left(\sum_{i=1}^{n} X_i \right) \left(\sum_{j=1}^{n} Y_j \right) \tag{135}$$

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Rule 8

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (X_{ij} + Y_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} Y_{ij} \tag{136}$$

Mathematical Expectation

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Consider the experiment of rolling a single die. Let X be the value that shows on the die. The probability distribution of X is

x	1	2	3	4	5	6
f(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

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What would the average value of X be if the experiment were repeated an infinite number of times? Intuitively, you would expect $X = 1$ on $\frac{1}{6}$ of the throws, $X = 2$ on $\frac{1}{6}$ of the throws, and so on.

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What would the average value of X be if the experiment were repeated an infinite number of times? Intuitively, you would expect $X = 1$ on $\frac{1}{6}$ of the throws, $X = 2$ on $\frac{1}{6}$ of the throws, and so on. So, on average, the value of X is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5 \tag{137}$$

That is, 3.5 is the average value of X that occurs in infinitely many trials of the experiment. This average is the *expected value* of the random variable X, despite the fact that X cannot actually take the value 3.5.

Variance

The variance of a random variable provides a measure of the spread, or dispersion, around the mean. It is denoted σ^2, and (in the discrete case) it is defined as

$$var(X) = \sigma^2 = \sum_{i=1}^{n} p_i \left[X_i - E(X)\right]^2 \tag{138}$$

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The positive square root of the variance is called the standard deviation and is denoted by σ.

Properties of the Expectations Operator

Result 1

$$E(aX + b) = aE(X) + b, \tag{140}$$

where X is a random variable and a and b are constants.

Result 2

$$E[(aX)^2] = a^2E(X^2) \tag{141}$$

Note that $E(X^2) \neq [E(X)]^2$.

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For example, let $X = 1$ where a coin appears heads and $X = 0$ when it appears tails. Then for a fair coin, $p_1 = \dfrac{1}{2}$ and $p_0 = \dfrac{1}{2}$, so that

$$E(X^2) = \frac{1}{2}(1^2) + \frac{1}{2}(0^2)$$

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PROOF

By definition

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By Result 1 we have that $E(aX+b) = aE(X)+b$, therefore

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PROOF

By definition

$$var(aX + b) = E\left[(aX + b) - E(aX + b)\right]^2 \tag{146}$$

By Result 1 we have that $E(aX + b) = aE(X) + b$, therefore

$$var(aX + b) = E\left[aX - E(aX)\right]^2 \tag{147}$$

$$= E\left[aX - aE(X)\right]^2 \tag{148}$$

$$= E\left[a(X - E(X))\right]^2 \tag{149}$$

Result 3

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Now, we can use the expectations operator to prove some results concerning the covariance between two random variables.

Result 4

If X and Y are random variables, then

$$E(X+Y) = E(X) + E(Y) \tag{152}$$

Result 5

$$var(X+Y) = var(X) + var(Y) + 2cov(X,Y) \tag{153}$$

PROOF

$$var(X+Y) = \mathrm{E}[(X+Y) - \mathrm{E}(X+Y)]^2 \tag{154}$$

Result 5

$$var(X+Y) = var(X) + var(Y) + 2cov(X,Y) \tag{153}$$

PROOF

$$var(X+Y) = E[(X+Y) - E(X+Y)]^2 \tag{154}$$

$$\text{by Result 4}$$

$$= E[(X+Y) - E(X) - E(Y)]^2 \tag{155}$$

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Result 6

If X and Y are independent, then $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.

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PROOF

$$cov(X, Y) = E[X - E(X)][Y - E(Y)] \tag{159}$$

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If X and Y are independent, then $E(XY) = E(X)E(Y)$.

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If X and Y are independent, then $cov(X, Y) = 0$.

PROOF

$$\begin{aligned} cov(X, Y) &= E[X - E(X)][Y - E(Y)] && (159) \\ &= E[XY - E(X)Y - XE(Y) + E(X)E(Y)] && (160) \end{aligned}$$

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by Result 6

$$= 0. \tag{162}$$

Result 8

$$var(\bar{X}) = \frac{\sigma^2}{n}, \tag{163}$$

where $\bar{X}$ is the sample mean of a random variable with mean μ and variance σ^2.

PROOF:

$$var(\bar{X}) = var(\frac{1}{n}\sum_{i=1}^{n} X_i) \tag{164}$$

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$$var(\bar{X}) = var(\frac{1}{n}\sum_{i=1}^{n} X_i) \tag{164}$$

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by Results 5 and 7

and the assumption that all X_i are independent

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$$= \frac{\sigma^2}{n}. \tag{167}$$

Result 8 shows that the variance of the estimator of the mean $\bar{X}$ falls as the sample size increases. Thus, with more and more information, we get more and more accuracy in our estimates of the mean μ. What happens to this variance as we get infinite data?

Result 9

$$\sigma^2 = \mathrm{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \tag{168}$$

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$$= Var(X) + [E(X)]^2 \tag{170}$$

Sampling

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Designs other than simple random sampling may give each unit equal probability of being included in the sample, but only with simple random sampling does each possible sample of n units have the same probability.

Estimating The Population Mean

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In fact, $\bar{y}$ is not only biased, but also *design-unbiased*.

It is called *design-unbiased* because the unbiasedness of the sample mean for the population mean with simple random sampling does not depend on any assumptions about the population itself. This is true because the probability with respect to which the expectation is evaluated arises from the probabilities, due to the design, of selecting different samples.

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Note that as sample size n approaches the population size N in simple random sampling, the finite population correction factor approaches zero, so that the variance of the estimator $\bar{y}$ approaches zero.

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$$\hat{\tau} = N\bar{y} \tag{181}$$

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However, several special features are worth noting:

1. The formulae simplify considerably with attribute (i.e., zero, one) data.;
2. exact confidence intervals are possible;
3. a sample size sufficient for a desired absolute precision may be chosen without any information about population parameters. This is possible because the population parameters are bounded.

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$$p = \frac{1}{N} \sum_{i=1}^{N} y_i = \mu \tag{185}$$

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$$= \frac{Np - Np^2}{N-1} \tag{188}$$

$$= \frac{N}{N-1}p(1-p) \tag{189}$$

Letting $\hat{p}$ denote the proportion in the sample with the attribute,

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Thus, the relevant statistics can be computed from the sample proportion alone.

Since the sample proportion is the sample mean of a simple random sample, it is unbiased for the population proportion, and has variance

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Confidence Intervals For A Proportion

An approximate confidence interval for p based on a normal distribution is given by

$$\hat{p} \pm t\sqrt{\widehat{var}(\hat{p})}, \tag{196}$$

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The normal approximation on which this interval is based improves the larger the sample size and the closer p is to 0.5.

Confidence limits may also be obtained based on the exact hypergeometric distribution of the number of units in the sample with the attribute. I will not discuss the exact method.

Sample Size for Estimating a Proportion

To obtain an estimator $\hat{p}$ having probability at least $1-\alpha$ of being no farther than d from the population proportion, the sample size formula based on the normal approximation gives

$$n = \frac{Np(1-p)}{(N-1)\frac{d^2}{z^2} + p(1-p)}, \tag{197}$$

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The quantity $p(1-p)$, and hence the value of n required by the formula, assumes its maximum value when p is one-half.

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The textbooks do a good job of discussing the details of the methods involved. But they lose focus on the big picture.

Presidential Approval

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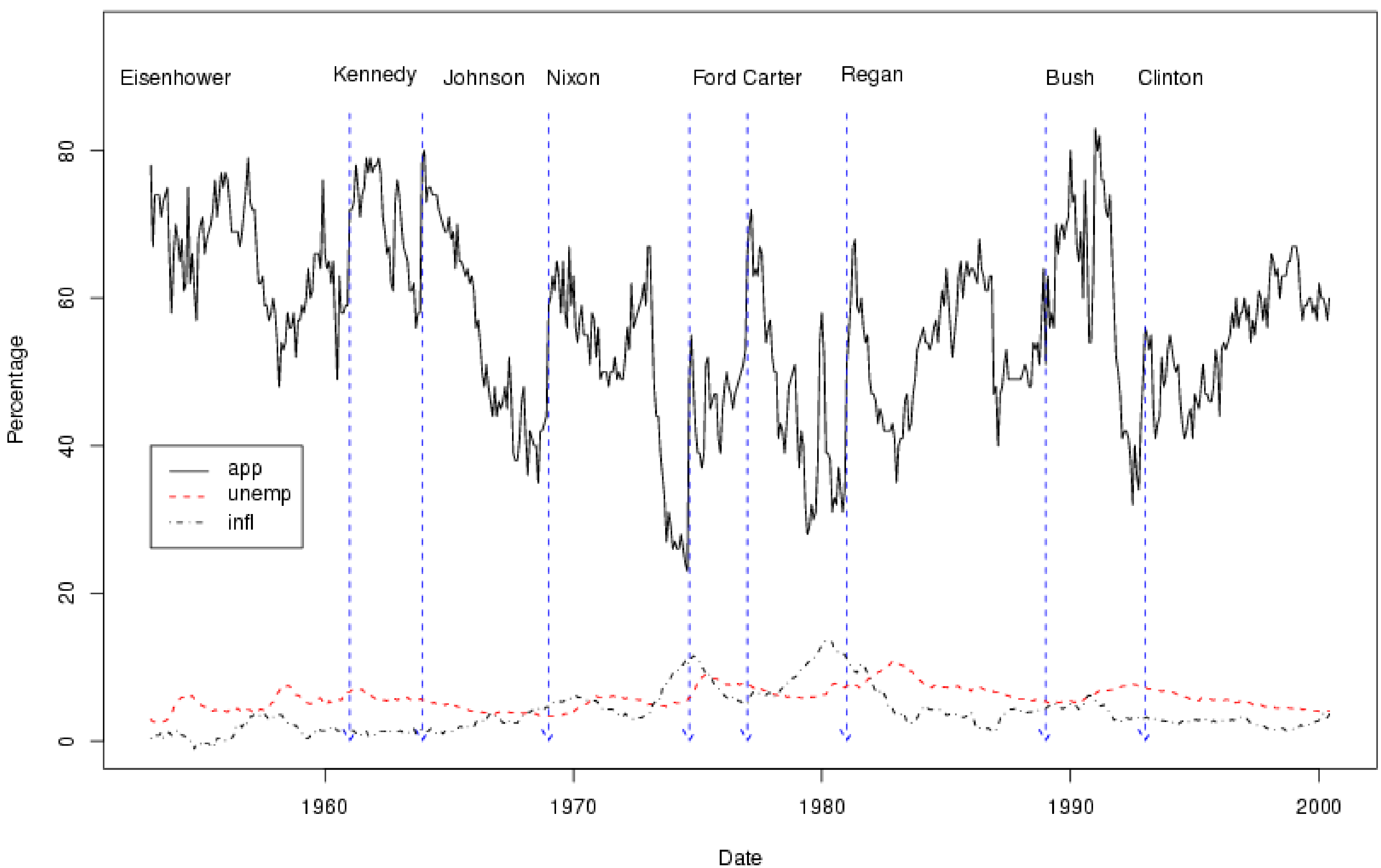
Figure 1 plots all three series.

Figure 1: U.S. Presidential Approval, Unemployment and Inflation



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Are voters forward looking or retrospective?

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P is determined by the nature of the world, it is not known. The problem of estimation and inference arises precisely because it is unknown.

If we observe a realization of the sequence Z, then we can infer some knowledge of $\mathbb{P}$ from this realization. In practice, observation of the entire sequence is impossible. Instead, we have a realization $z^n = (z_1, z_2, \cdots, z_n)$ of a finite history. We call z^n a sample of size n. We usually hope that this sample is $random$.

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We can, however, learn P arbitrarily well as the sample size n goes to ∞.

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In practice, a linear function is most often used:

$$f(X) = \alpha + \beta X, \tag{224}$$

where α is the intercept and β is the slope.

Models

We are interested in the relationship between Y and X—i.e., in explaining the behavior of Y using X. A function of X, $f(X)$, is used to approximate Y. This function is called a model or a predictor for Y.

In practice, a linear function is most often used:

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This is where the concept of a loss function comes in.

Loss Function

How well the model $f(X)$ will explain Y is described by a what is called a "loss function." In general, there exists a discrepancy between $f(X)$ and Y. When $f(X) \neq Y$, a "loss" will occur. A function which tells us how big this "loss" will be, is called a loss function.

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A loss function $l(Y, f(X))$ is a real-valued function that describes how well the model $f(X)$ can explain Y.

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These least square predictor is the loss function where $p = 2$. This is an arbitrary choice. But it is a choice with some nice properties.

A perfectly good loss function which has some nice properties (some of them better than least-squares) is:

$$l(Y, f(X)) = \mathrm{median}\, (Y_i - f(X_i))^2. \tag{226}$$

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We wish to minimized the loss function. The estimator which minimizes the MSE is called the least squares estimator.

Regression Models

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A regression model consisting of Y, X and ϵ satisfies the following property:

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where the disturbance has the property $E(\epsilon|X) = 0$.

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$$\begin{aligned} Y &= E(Y|X) + \epsilon && (229) \\ \text{Then} \\ \epsilon &= Y - E(Y|X). && (230) \\ \text{and} \\ E(\epsilon|X) &= E\left[(Y - E(Y|X)|X\right] && (231) \\ &= E(Y|X) - E\left[E(Y|X)|X\right] && (232) \\ &= E(Y|X) - E(Y|X) = 0 && (233) \end{aligned}$$

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2. The term ϵ is called the "regression disturbance." The fact $E(\epsilon|X) = 0$ implies that ϵ contains no systematic information of X in predicted Y. In other words, all information of X that is useful to predict Y has been summarized by $E(Y|X)$.

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Under these assumptions, OLS is BLUE: the Best Linear Unbiased Estimator.

Assumptions A1—A4 taken together are sufficient for unbiasedness, and assumptions A1—A5 taken together are sufficient to prove efficiency.

Classical Assumptions

A1. $Y_t = \sum_k X_{kt}\beta_k + \epsilon_t,$ $t = 1, 2, 3, \cdots, n \quad k = 1, \cdots, K,$ where t indexes the observations and k the variables. This is a very strong assumption.

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A5. The variance of the random error, ϵ is equal to a constant, σ^2, for all values of every X (i.e., $var[\epsilon_t] = \sigma^2$), and ϵ is normally distributed. This assumption implies that The errors associated with any two observations are independent and identically distributed. This assumption can be significantly weakened, but the assumption of normality plays a key role.

Simple Regression, Approval Example

Simple regression is a way to obtain the total effect of one variable on another. For example, if we estimate the following simple regression model:

$$\text{Approval}_t = \beta_0 + \beta_1 \text{Unemployment}_t + \epsilon_t, \qquad (234)$$

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we observe that $\beta_0 = 69.303$ and $\beta_1 = -2.280$. We may then interpret β_1 to be the total effect of one percent of unemployment on approval—it is the slope of unemployment's effect on approval.

What is the relationship between our estimate of $\widehat{\beta}_1$ and cov(approval,unemployment) and cor(approval,unemployment)?

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Recall that The covariance between two random variables (X, Y) is defined to be:

$$cov(X, Y) = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}\right) \left(Y_i - \bar{Y}\right) \tag{235}$$

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$$cor(X, Y) = \frac{cov(X, Y)}{\sqrt{(var(X)}\sqrt{(var(Y)}} \tag{236}$$

Note that:

- mean(approval) $=$ 56.0579
- mean(unemployment rate) $=$ 5.809825

- var(approval) $=$ 143.6153
- var(unemployment rate) $=$ 2.332944

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And this is the same as our slope estimate: $\beta_1 = -2.280$!

Derivation of Simple Least-Squares Parameter Estimates

In this section we explore how we obtain our estimates of α and β. This section requires some knowledge of calculus. It is **not** essential to understand this section to understand subsequent sections **nor** will you be tested on this material.

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Our goal is to minimize $\sum_i^n (Y_i - \hat{Y}_i)^2$, where $\hat{Y}_i = \alpha + \beta X_i$ is the fitted value of Y_i corresponding to a particular observation X_i.

We minimize the expression by taking the partial derivatives with respect to α and β, setting each equal to 0, and solving the resulting pair of simultaneous equations:

$$\nabla_\alpha \sum_i^n (Y_i - \alpha - \beta X_i)^2 = -2 \sum_i^n (Y_i - \alpha - \beta X_i) \tag{237}$$

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Equating these two derivatives to zero and dividing by -2, we obtain:

$$\sum_i^n (Y_i - \hat{\alpha} - \hat{\beta} X_i) = 0 \tag{239}$$

$$\sum_i^n X_i(Y_i - \hat{\alpha} - \hat{\beta} X_i) = 0 \tag{240}$$

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We may now rewrite these two equations to obtain what are called the **normal equations**:

$$\sum_{i}^{n} Y_i = \hat{\alpha} n + \hat{\beta} \sum_{i}^{n} X_i \tag{242}$$

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We can solve for $\hat{\alpha}$ and $\hat{\beta}$ simultaneously by multiplying Equation 242 by $\sum_i^n X_i$ and multiplying Equation 243 by n:

$$\sum_i^n X_i \sum_i^n Y_i = \hat{\alpha} n \sum_i^n X_i + \hat{\beta} (\sum_i^n X_i)^2 \tag{244}$$

$$n \sum_i^n X_i Y_i = \hat{\alpha} n \sum_i^n X_i + \hat{\beta} n \sum_i^n X_i^2 \tag{245}$$

Subtracting Equation 244 from Equation 245, we obtain

$$n \sum X_i Y_i - \sum_i^n X_i Y_i = \hat{\beta} \left[n \sum_i^n X_i^2 - (\sum_i^n X_i)^2 \right] \tag{246}$$

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It follows that:

$$\hat{\beta} = \frac{n \sum_i^n X_i Y_i - \sum_i^n X_i \sum_i^n Y_i}{n \sum_i^n X_i^2 - (\sum_i^n X_i)^2} \tag{247}$$

You are responsible for knowing that $\hat{\beta}$ is. You should also know that it results from minimizing the least squares loss function. You don't need to know the exact derivation.

$$\widehat{\beta} = \frac{n \sum_i^n X_i Y_i - \sum_i^n X_i \sum_i^n Y_i}{n \sum_i^n X_i^2 - (\sum_i^n X_i)^2} \tag{248}$$

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Note that $n\sum_i^n X_i^2 - (\sum_i^n X_i)^2 = n\sum_i^n (X_i - \bar{X})^2$. One may use this note which comes from the rules of summation (remember those?) to greatly simplify the equation which defines β if the mean of X and the mean of Y is zero:

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Note the differences between the simple measure of covariance and the regression coefficient. There is a lot of intuition to be found examining the expression about, particularly Equation 250.

Given our solution of $\hat{\beta}$, we may obtain our solution for $\hat{\alpha}$ from Equation 242

$$\hat{\alpha} = \frac{\sum_i^n Y_i}{n} - \hat{\beta}\frac{\sum_i^n X_i}{n} \tag{251}$$

The Differences Between Simple and Multiple Regression

As you read this discussion you may want to jump ahead and take a look at Figure 2 in these notes, which shows the direct and indirect relationships between inflation, unemployment and presidential approval.

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The direct effect of, say, unemployment on presidential approval is the effect that unemployment has on approval if everything else is held constant.

The indirect effect of unemployment on approval is the direct effect that unemployment has on other variables times the direct effect these other variables have on approval.

Therefore, the indirect effect of unemployment on approval is the effect that unemployment has on approval when unemployment moves variables which themselves have a direct effect on approval.

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It is often assumed that multiple regression is a way to obtain the direct effect of one variable on another.

But the use of multiple regression is much more complicated. The research issues we spoke about at the beginning of the term are very important to consider and are often overlooked.

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Our goal is to minimize $\sum_i^n (Y_i - \hat{Y}_i)^2$, where $\hat{Y}_i = \hat{\alpha} + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i}$.

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To further simplify the algebra we deviate the observed variables by their means. These mean deviated variables are denoted, as before, by y_i, x_{1i} and x_{2i}.

Therefore,

$$\mathrm{ESS} = \sum_i^n (y_i - \alpha + \beta_1 X_{1i} + \beta_2 X_{2i}) \tag{252}$$

Then,

$$\frac{\partial ESS}{\partial \beta_1} = \hat{\beta}_1 \sum_{i}^{n} x_{1i}^2 + \hat{\beta}_2 \sum_{i}^{n} x_{1i}x_{2i} - \sum_{i}^{n} x_{1i}y_i \tag{253}$$

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These can be rewritten as:

$$\sum_i^n x_{1i}y_i = \hat{\beta}_1 \sum_i^n x_{1i}^2 + \hat{\beta}_2 \sum_i^n x_{1i}x_{2i} \tag{255}$$

$$\sum_i^n x_{2i}y_i = \hat{\beta}_1 \sum_i^n x_{1i}x_{2i} + \hat{\beta}_2 \sum_i^n x_{2i}^2 \tag{256}$$

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To solve, we multiply Equation 255 by $\sum_i^n x_{2i}^2$ and Equation 256 by $\sum_i^n x_{1i}x_{2i}$ and substract the latter from the former.

Then,

$$\sum_{i}^{n} x_{1i} y_i \sum_{i}^{n} x_{x2i}^2 - \sum_{i}^{n} x_{2i} y_i \sum_{i}^{n} x_{1i} x_{2i} \quad = \quad \hat{\beta}_1 \left[\sum_{i}^{n} x_{1i}^2 x_{2i}^2 - (\sum_{i}^{n} x_{1i} x_{2i})^2 \right]$$

Then,

$$\sum_i^n x_{1i}y_i \sum_i^n x_{x2i}^2 - \sum_i^n x_{2i}y_i \sum_i^n x_{1i}x_{2i} \quad = \quad \hat{\beta}_1 \left[\sum_i^n x_{1i}^2 x_{2i}^2 - \left(\sum_i^n x_{1i}x_{2i}\right)^2 \right]$$

Thus,

$$\hat{\beta}_1 \quad = \quad \frac{(\sum_i^n x_{1i}y_i)(\sum_i^n x_{x2i}^2) - (\sum_i^n x_{2i}y_i)(\sum_i^n x_{1i}x_{2i})}{(\sum_i^n x_{1i}^2)(\sum_i^n x_{2i}^2) - (\sum_i^n x_{1i}x_{2i})^2} \tag{257}$$

Then,

$$\sum_i^n x_{1i}y_i \sum_i^n x_{x2i}^2 - \sum_i^n x_{2i}y_i \sum_i^n x_{1i}x_{2i} \quad = \quad \hat{\beta}_1 \left[\sum_i^n x_{1i}^2 x_{2i}^2 - (\sum_i^n x_{1i}x_{2i})^2 \right]$$

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And

$$\hat{\beta}_2 = \frac{(\sum_i^n x_{2i}y_i)(\sum_i^n x_{x1i}^2) - (\sum_i^n x_{1i}y_i)(\sum_i^n x_{1i}x_{2i})}{(\sum_i^n x_{1i}^2)(\sum_i^n x_{2i}^2) - (\sum_i^n x_{1i}x_{2i})^2} \tag{258}$$

Then,

$$\sum_i^n x_{1i} y_i \sum_i^n x_{x2i}^2 - \sum_i^n x_{2i} y_i \sum_i^n x_{1i} x_{2i} \quad = \quad \hat{\beta}_1 \left[\sum_i^n x_{1i}^2 x_{2i}^2 - (\sum_i^n x_{1i} x_{2i})^2 \right]$$

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If we do the same for α we find that:

$$\hat{\alpha} = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 X_2 \tag{259}$$

The equations for the estimates of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ can be rewritten as:

$$\widehat{\beta}_1 = \frac{cov(X_{1i}, Y_i)\ var(X_{2i}) - cov(X_{2i}, Y_i)\ cov(X_{1i}, X_{2i})}{var(X_{1i})\ var(X_{2i}) - [cov(X_{1i}, X_{2i})]^2} \tag{260}$$

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And,

$$\hat{\beta}_2 = \frac{cov(X_{2i}, Y_i)\, var(X_{1i}) \; - \; cov(X_{1i}, Y_i)\, cov(X_{1i}, X_{2i})}{var(X_{1i})\, var(X_{2i}) \; - \; [cov(X_{1i}, X_{2i})]^2} \tag{261}$$

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For an example using **R** code see `http://jsekhon.fas.harvard.edu/gov1000/mr1.R` and its output file `http://jsekhon.fas.harvard.edu/gov1000/mr1.Rout`.

Multiple Regression, Approval Example

$$\text{Approval}_t = \alpha_0 + \alpha_1 \text{inflation}_t + \alpha_2 \text{Unemployment}_t + \epsilon_t, \tag{262}$$

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we observe that $\alpha_0 = 69.7785$, $\alpha_1 = -2.1394$ and $\alpha_2 = -0.9258$.

Multiple Regression, Approval Example

$$\text{Approval}_t = \alpha_0 + \alpha_1 \text{inflation}_t + \alpha_2 \text{Unemployment}_t + \epsilon_t, \qquad (262)$$

we observe that $\alpha_0 = 69.7785$, $\alpha_1 = -2.1394$ and $\alpha_2 = -0.9258$. It is clear that the slope associated with unemployment has *greatly* changed from -2.280 in the simple regression to -0.9258. In other words, in the simple regression model a 1 unit increase in the unemployment level decreases presidential approval by 2.28 units, but in the multiple regression model a 1 unit increase in the unemployment level decreases approval by only .9258 units. What's going on?

The multiple regression is giving us the effect of unemployment on approval *holding inflation constant*. In other words, the indirect effect of unemployment on approval which works through inflation is not taken into consideration. However, in the simple regression model the direct and indirect effect of unemployment—i.e., the total effect—is estimated.

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Recall that the multiple regression model gives the direct effect, and the simple regression model the total effect which is equal to the direct effect plus the indirect effect.

We know the direct effect of unemployment on approval is the α_2 coefficient in equation 262, $\alpha_2 = -0.9258$. The indirect effect is equal to the direct effect of inflation on approval (which is $\alpha_1 = -2.1394$) **times** the effect of unemployment on inflation, which we have not calculated.

Given that, aside from the intercept α_0, we are only considering two independent variables in our multiple regression model (equation 262), the direct effect of unemployment on inflation can be found by estimating the following simple regression:

$$\text{Inflation}_t = \gamma_0 + \gamma_1 \text{Unemployment}_t + \epsilon_t. \tag{263}$$

Given that, aside from the intercept α_0, we are only considering two independent variables in our multiple regression model (equation 262), the direct effect of unemployment on inflation can be found by estimating the following simple regression:

$$\text{Inflation}_t \quad = \gamma_0 + \gamma_1 \text{Unemployment}_t + \epsilon_t. \tag{263}$$

If we estimate this model we find that the intercept, γ_0, equals 0.2224 and the slope of the effect of unemployment on inflation, γ_1, equals 0.6329.

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If we estimate this model we find that the intercept, γ_0, equals 0.2224 and the slope of the effect of unemployment on inflation, γ_1, equals 0.6329.

Therefore, the total effect of unemployment on approval must equal $-0.9258 + (0.6329 * -2.1394) = -2.280$. And this is exactly what we found when we estimated Equation 234.

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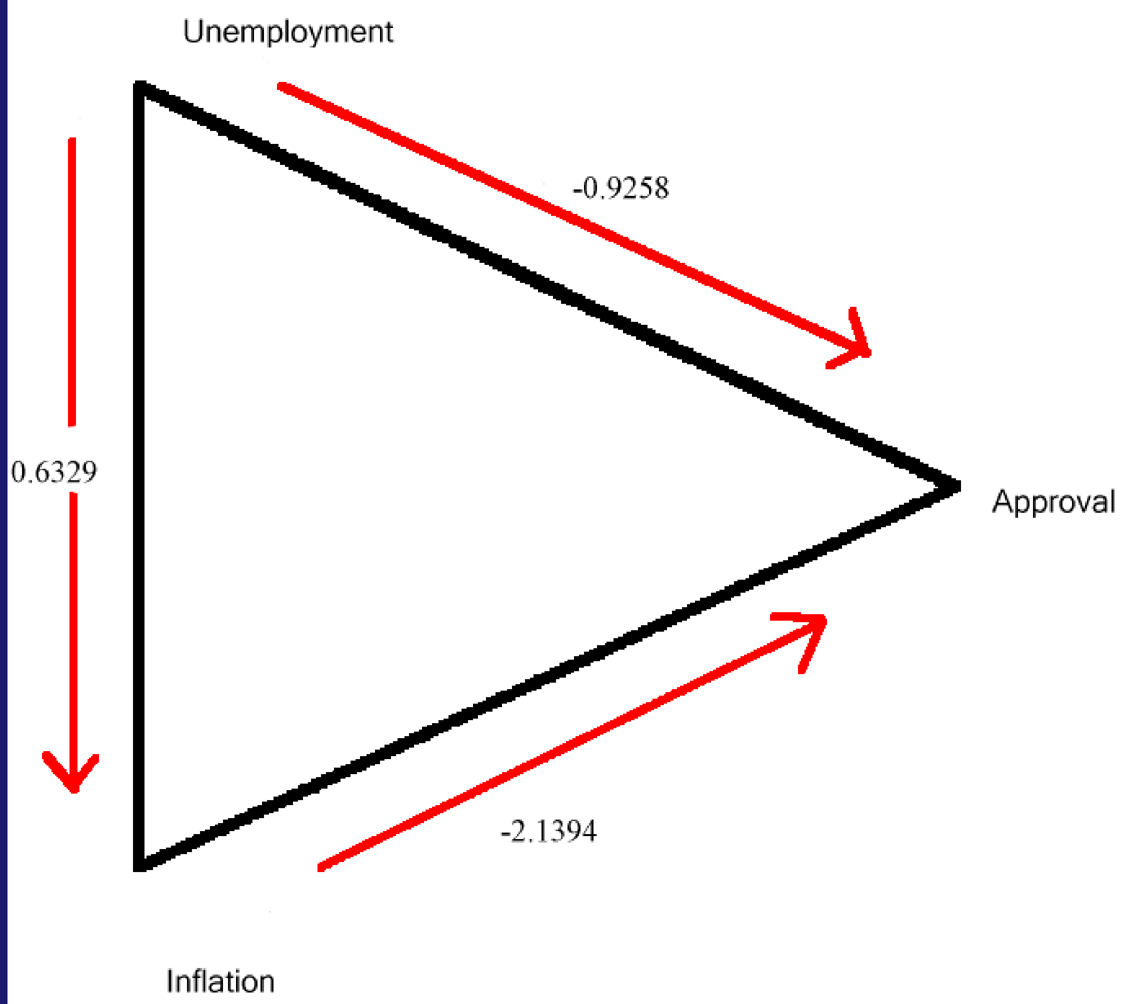
Figure 2, on the next page, shows the direct and indirect relationships between inflation, unemployment and presidential approval.

Figure 2: The Relationship Between Unemployment and Presidential Approval Given Inflation



Direct and Indirect Effects of Unemployment on Approval

Caution about the Direct/Indirect Distinction

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The central statistical assumption is the correct specification assumption previously mentioned: $E(\epsilon|X) = 0$.

What does the correct specification assumption imply?

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It implies that ϵ, the residual, contains no systematic information of X in predicted Y. In other words, all information of X that is useful to predict Y has been summarized by $E(Y|X)$.

Hypothesis Testing

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The hypothesis that the restriction or set of restriction to be tested does in fact hold is called the null hypothesis and is usually denoted H_0.

The model in which the restrictions do not hold is usually called the alternative hypothesis, or sometimes the maintained hypothesis, and is usually denoted H_1.

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Hypothesis tests usually involve the use of a test statistic.

A test statistic, such as T, is a random variable of which the probability distribution is known under the null hypothesis.

We then see how likely the observed value of T is to have occurred, according to that probability distribution.

If T is a number that could easily have occurred by chance, then we have no evidence against the null hypothesis H_0.

However, if it is a number that would occur by chance only rarely, we do have evidence against the null.

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2. Type II accepting the null when it is false—$P(H_0^A|H_0^F)$

How To Conduct These Tests

Please see section 12-2 in Wonnacott and Wonnacott.

Key Assumptions of the Linear Model

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  ⋆ b. The random variables ϵ_i are statistically independent. Thus, $E(\epsilon_i \epsilon_j) = 0$, for all $i \neq j$.

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- ⋆ a. The error term has zero expected value and constant variance for all observations; that is, $E(\epsilon_i) = 0$ and $E(\epsilon_i^2) = \sigma^2$.
  - ⋆ b. The random variables ϵ_i are statistically independent. Thus, $E(\epsilon_i \epsilon_j) = 0$, for all $i \neq j$.
  - ⋆ c. The error term is normally distributed. Thus $\epsilon \sim N(0, \sigma^2)$.

Key Assumptions of the Linear Model

There are three key assumptions that we need. All of them can be weakened, but that involves advanced material.

- 1. The relationship between Y and X is linear:

$$Y_i = \alpha + \beta X_i + \epsilon_i. \tag{264}$$

- 2. The X's are nonstochastic variables whose values are fixed.

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Not all of these assumptions are required for all actions involved with the linear model.

Properties of Beta

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Note that our estimate for β is defined as follows:

$$\hat{\beta} = \frac{n \sum_i^n X_i Y_i - \sum_i^n X_i \sum_i^n Y_i}{n \sum_i^n X_i^2 - (\sum_i^n X_i)^2} \tag{266}$$

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The first result is that:

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The second result is that

$$Var(\hat{\beta}) = \frac{\sigma^2}{\sum x_i^2}, \tag{269}$$

so that the variance of $\hat{\beta}$ depends solely on the error variance (σ^2), the variance of the X's, and the number of observations.

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Finally, the covariance between $\hat{\alpha}$ and $\hat{\beta}$ is given by:

$$Cov(\hat{\alpha}, \hat{\beta}) = \frac{-\bar{X}\sigma^2}{\sum x_i^2} \tag{272}$$

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Note that another name for $\hat{\sigma}^2$ is simply s^2.

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Roughly speaking the central-limit theorem states that the distribution of the sample mean of an independently distributed variable will tend toward normality as the sample size gets infinitely large.

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Thus, the larger the variance of X_i, the better you are likely to do in estimating β.

Back to the Approval Example

Recall the Presidential Approval Figure. Let us consider five Presidential Approval Models.

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$$\begin{aligned}
\text{Approval}_t = {} & \alpha + \beta_1 \text{Unemp}_t + \beta_2 \text{Infl}_t + \beta_3 \text{Approval}_{t-1} + \beta_4 \text{Eisenhower} \\
& + \beta_5 \text{Kennedy} + \beta_6 \text{Johnson} + \beta_7 \text{Nixon} \\
& + \beta_8 \text{Ford} + \beta_9 \text{Carter} + \beta_{10} \text{Regan} + \beta_{11} \text{Bush} + \epsilon_t
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Dummy Variables

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- Recall that the nominal scale is the least powerful. It only maps the attributes of the object into a name. This mapping is simply a classification of entities. The only relationship is whether the measure of two attributes are the same or different.

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- We can't, because we would have the same variable in our model twice.

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- Leaving one of the indicator variables out, allows us to interpret the remaining coefficients relative to the indicator left out.

Results of the 5 Models (coefficients only)

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Compare with the findings previously presented based on summary data.

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- It is called the $t-$statistic because for small sample sizes we use the $t-$distribution (see Section 8-2 in Wonnacott and Wonnacott). But in this class we use the normal approximation.

- To test the null hypothesis that $\beta = \beta_0$ we calculate the $t-$statistic:

$$t = \frac{\beta - \beta_0}{s_{\hat{\beta}}},$$

where $s^2_{\hat{\beta}} = \frac{s^2}{\sum_{i=1}^{n} x_i^2}$, and $s^2 = \frac{\sum \hat{e}_i^2}{n-k}$.

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- When $n-k$ is larger than 30 and we assume that the null hypothesis is correct, the test statistic t follows a normal distribution with mean zero and variance 1.

- If the test statistic t is significantly larger than we should expect under the null hypothesis, we reject the null hypothesis.

- Since we know that 95% of the normal distribution falls within 1.96 standard units of its mean, we obtain a 95% confidence interval for β by:

$$\hat{\beta} \quad \pm \quad 1.96 * s_{\hat{\beta}} \tag{277}$$

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- There is nothing magical about a 95% confidence interval, or a hypothesis test of power 0.05. We could easily be interested in a test of size .1. In which case, our critical value is no longer 1.96, but 1.645.

Full Results of the 5 Models A

Variables	Model 1	Model 2	Model 3	Model 4	Model 5
Intercept	69.30***	65.03***	69.78***	9.86***	13.76***
	(1.89)	(0.70)	(1.63)	(1.66)	(2.06)

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Significance Codes: if p-value ≈ 0, ***; if p-value < 0.001, ***; if p-value < 0.001, **; if p-value < 0.01, *; if p-value < 0.05, $^{\$}$.

Standard errors in parenthesizes.

Full Results of the 5 Models B

Variables	Model 1	Model 2	Model 3	Model 4	Model 5
Eisenhower					$1.30^{\$}$
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Kennedy					2.50^{*}
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$$\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2. \tag{278}$$

- Another way to consider the sum of squared errors is the root mean squared error:

$$\text{RMSE} = \sqrt{\frac{1}{n-k}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2} \tag{279}$$

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R^2 and $\bar{R}^2$

For each observation, we can break down the difference between Y_i and its mean $\bar{Y}$ as follows:

$$(Y_i - \bar{Y}) = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \bar{Y}) \tag{280}$$

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- R^2 does not account for the number of degrees of freedom.

- One idea is to use variances, not variations, thus (in part) accounting for the number of independent variables in the model. The correction is based on the fact that variance equals variation divided by degrees of freedom.

$\bar{R}^2$ or corrected R^2 is defined as

$$\bar{R}^2 = 1 - \frac{\widehat{var}(\epsilon)}{\widehat{var}(Y)} \tag{285}$$

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It is very important NOT to use R^2 to compare the validity of alternative regression models when the dependent variable varies from regression to regression.

R^2 can be misleading in part because it chooses models which are too large. This is also true of $\bar{R}^2$ even though $\bar{R}^2$ is obviously likely to choose smaller models than R^2.

Fit Summaries of the 5 Model

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If the null hypothesis is true, then we would expect RSS, R^2, and therefore F to be close to 0. Thus, a large value of the F statistic is a rationale for rejecting the null hypothesis.

F Statistic

The F statistic calculated by most regression programs can be used in the multiple regression model to test the significance of the R^2 statistic.

The F statistic with $k-1$ and $n-k$ degrees of freedom allows us to test the hypothesis that none of the explanatory variables helps explain the variation of Y about its mean.

The F statistic tests the joint hypothesis that $\beta_2 = \beta_3 = \cdots = \beta_k = 0$.

It can be shown that:

$$F_{k-1,n-k} = \frac{RSS}{ESS} \frac{n-k}{k-1} \tag{291}$$

$$= \frac{R^2}{1-R^2} \frac{n-k}{k-1} \tag{292}$$

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An F statistic not significantly different from 0 lets us conclude that the explanatory variables do little to explain the variation of Y about its mean.

The F test of the significance of a regression equation may allow for rejection of the null hypothesis even though none of the regression coefficients are found to be significant according to individual t tests.

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This situation may arise if the independent variables are highly correlated with each other.

The $\mathbb{F}$ Distribution

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Note that unlike the normal distribution, the shape of the F-distribution radically changes depending on its parameters.

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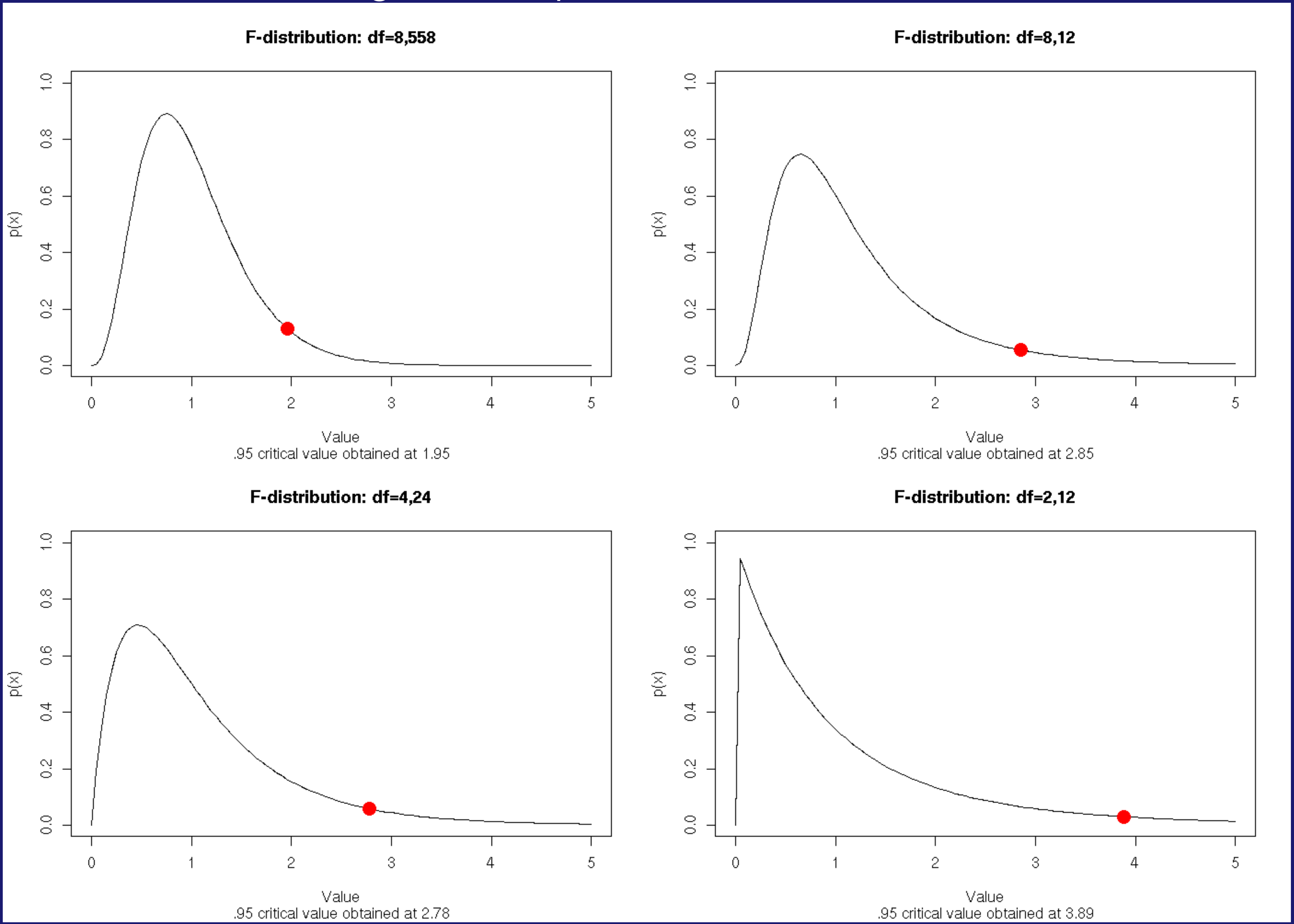
See page 328 of W&W for more information.

Figure 3: Graphs of the F−distribution



The General $\mathbb{F}$ Statistic

Consider the following unrestricted multiple regression model:

$$Y = \beta_1 + \beta_2 X_2 + \cdots + \beta_k X_k + \epsilon. \tag{293}$$

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The restricted model, which would be correct if all of the last q variables really where equal to 0, is:

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The test is based on the error (or residual) sum of squares associated with the unrestricted and restricted models.

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$$= \frac{\left[\sum_{i=1}^{n}(Y_{R,i} - \widehat{Y}_R)^2 - \sum_{i=1}^{n}(Y_{UR,i} - \widehat{Y}_{UR})^2\right]/q}{\left[\sum_{i=1}^{n}(Y_{UR,i} - \widehat{Y}_{UR})^2\right]/(n-k)} \tag{297}$$

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The F statistic can be written in a simpler form:

$$F_{q,n-k} = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k)} \tag{298}$$

If we seek to determine if the extra variables in Model 5 (the model with the lags and indicator variables) significantly improve the fit when compared to the simpler Model 4 (the model with only the lag), we do the following:

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Which translates into a p-value of 0.0317. What at conventional test levels, is significant.