The Bootstrap

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The Sample Mean and the Sample Median

- Let X_i be IID for $i = 1, ..., n$, with mean μ and variance σ^2. We use the sample mean $\bar{X}$ to estimate μ.
- Is the estimator biased? What is it's standard error?
- Of course, we know it's unbiased and the SE is $\sigma/\sqrt{n}$, where $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$
- What about the median, particularly the difference in medians? Except for special circumstances, we don't have closed-form formulas for the uncertainty associated with these quantitites.

The Bootstrap Algorithm for estimating standard errors

1. Select B independent bootstrap samples $\mathbf{x}^{*\mathbf{1}}, \mathbf{x}^{*\mathbf{2}}, ..., \mathbf{x}^{*\mathbf{B}}$, each consisting of n data values draw **with replacement** from x.
2. Evaluate the bootstrap replication corresponding to each bootstrap sample,

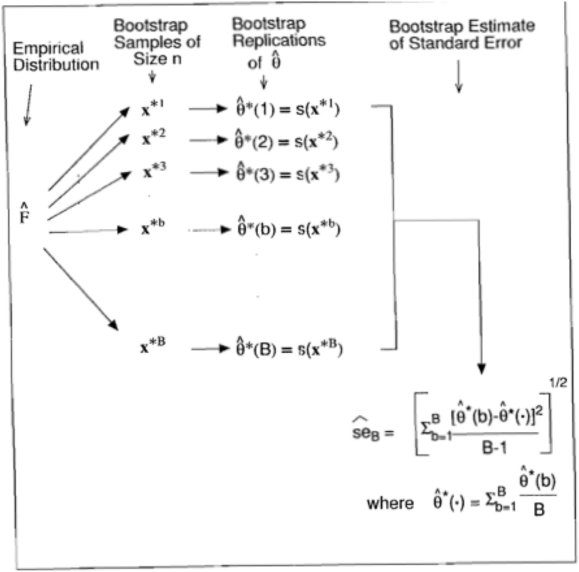
$$\hat{\theta}^*(b) = s(\mathbf{x}^{*\mathbf{b}}) \qquad b = 1, 2, \ldots, B.$$

3. Estimate the $\mathrm{se}_F(\hat{\theta})$ by the sample standard deviation of the B replications

$$\widehat{\mathrm{se}}_B = \left\{ \sum_{b=1}^{B} [\hat{\theta}^*(b) - \hat{\theta}^*(\cdot)]^2 / (B-1) \right\}^{1/2};$$

where $\theta^*(\cdot) = \sum_{b=1}^{B} \hat{\theta}^*(b)/B$

The Bootstrap Algorithm for SE



The Plug-in Principle

- We observe a random sample of size n from a probability distribution F,

$$F \to (x_1, X_2, \ldots, x_n)$$

the empirical distribution function $\hat{F}$ is defined to be the discrete distribution that puts probability $1/n$ on each value $x_1, i = 1, \ldots, n$.
- The plug-in estimate of a parameter $\theta = t(F)$ is defined to be $\hat{\theta} = t(\hat{F})$.

Confidence Intervals

- Standard confidence intervals depend on the large sample or asymptotic result:

$$\frac{\hat{\theta} - \theta}{\hat{\mathrm{se}}} \sim N(0, 1)$$

- In finite samples, we may not want to rely on that result and we instead can use bootstraped confidence intervals.

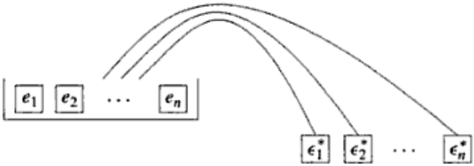
Percentile Method

- Many methods of bootstraped confidence intervals, but the **percentile** method is probably the easiest and most intuitive.
- To proceed we generate B independent bootstrap data sets $\mathbf{x}^{*\mathbf{1}}, \mathbf{x}^{*\mathbf{2}}, \ldots, \mathbf{x}^{*\mathbf{B}}$ and compute the bootstrap replications $\hat{\theta}^*(b) = s(\mathbf{x}^{*b})$ for $b = 1, 2, \ldots, B$.
- Let $\hat{\theta}_B^{*(a)}$ be the $100 \cdot \alpha$th empirical percentile of the $\hat{\theta}^*(b)$ values, that is the $B \cdot \alpha$th value in the ordered list of B replications of $\hat{\theta}^*$. Likewise let the $\hat{\theta}_B^{*(1-a)}$ be the $100 \cdot (1-\alpha)$th empirical percentile.
- The approximate $1 - 2\alpha$ percentile interval is

$$[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}] = [\hat{\theta}_B^{*(a)}, \hat{\theta}_B^{*(1-a)}]$$

Regression Models

- Suppose $Y = X\beta + \epsilon$, where the design matrix is $n \times p$, X is fixed and has full rank. The parameter vector β is $p \times 1$, unknown, to be estimated by OLS. The errors $\epsilon_1, \ldots, \epsilon_n$ are IID with mean 0 and variance σ^2.
- If we forgot the formulas and wanted to estimate the bias and variance of OLS, we could use the bootstrap. What's random here?
- In the model, the Y_i's are random, but not IID. The ϵ_i are random and IID but unobserved. What do we do?
- We can re-sample the residuals: $e = Y - X\hat{\beta}$.



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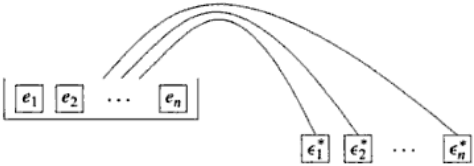

Regression Models

- We draw n times at random with replacement from this population to get bootstrap errors $\epsilon_1^*, \ldots \epsilon_n^*$. These are IID (because you sample them that way).
- Next we generate the Y_i^*:'

$$Y^* = X\hat{\beta} + \epsilon^*$$

- With the Y^* and X, we can directly examine the distribution of $\hat{\beta}^*$, where $\hat{\beta}^* = (X'X)^{-1}X'Y^*$.
- Note that this is known as the "parametric" bootstrap.
- The distribution of $\hat{\beta}^* - \hat{\beta}$ is a good approximation for the distribution of $\hat{\beta} - \beta$. In addition, the empirical covariance matrix of the $\hat{\beta}^*$ is a good approximation to the thoretical covariance matrix of $\hat{\beta}$.