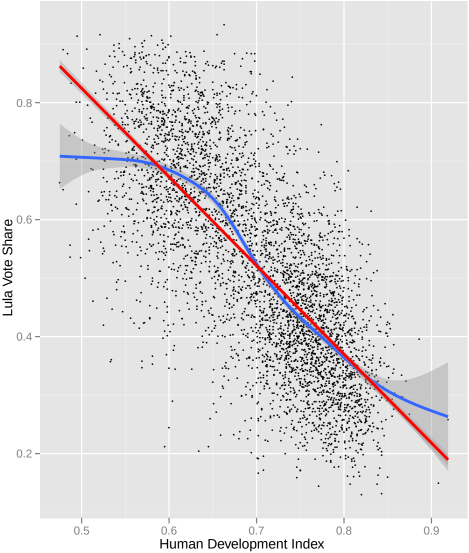
Maximum Likelihood

March 16, 2011

Two Models



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Human Development Index

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    - Estimate $\mathrm{P}(\theta|\{f(\cdot)y_i\})$ or, more simply, $\mathrm{P}(\theta|y_i)$
    - What is the probability of θ being a hypothesized value, given the assumed model and the observed data?

Can we estimate $\mathrm{P}(\theta|y_i)$?

Apply Bayes' theorem:

$$\mathrm{P}(\theta|y_i) = \frac{\mathrm{P}(\theta, y_i)}{P(y_i)} \tag{1}$$

$$= \frac{\mathrm{P}(\theta)\mathrm{P}(y|\theta)}{\mathrm{P}(y_i)} \tag{2}$$

We often focus on numerator, resulting in:

$$\mathrm{P}(\theta|y_i) \propto \mathrm{P}(\theta) \cdot \mathrm{P}(y_i|\theta)$$

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The posterior is proportional to the prior times the likelihood.

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- $L(\theta|y)$ is a function: for y fixed at the observed values, it gives the "likelihood" of any value of θ.
- Likelihood-model of uncertainty is a *relative* measure appropriate for making comparisons of parameter estimates with the the same dataset, not for making comparisons across datasets.

OLS Using MLE

- The *likelihood* of the probability of the data given the model and inputs:

$$p(y|\beta, \sigma, X) = \prod_{i=1}^{n} \mathrm{N}(y_i|X_i\beta, \sigma^2)$$

where $\mathrm{N}(\cdot|\cdot, \cdot)$ represents the normal probability density function $\mathrm{N}(y|m, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}}\exp(-\frac{1}{2}(\frac{Y-m}{\sigma})^2)$

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- The more general form of the above expression is:

$$p(y_i|\theta, X_i) = \prod_{i=1}^{n} p(y_i|\theta, X_i)$$

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  - Set the derivative equal to zero.
  - Solve for the parameters that make the derivative equal zero.
  - Check that the second derivative matrix is negative definite, proving that you've found a *maximum* rather than a *minimum*.

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- In the sample we have N observations, so sum over the N observations to get:

$$\log L(\beta|y_i) = -\frac{1}{2}\sum_{i=1}^{N}(y_i - \beta)^2 - n\log(\sqrt{2\pi})$$

Example, continued

- Next, take the derivative with respect to β.

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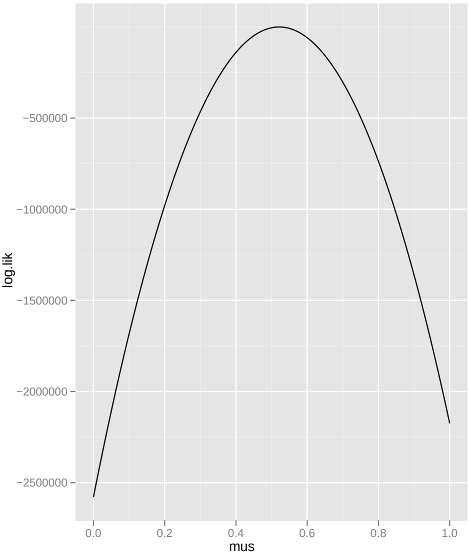
- Check if negative definite:

$$\frac{d^2 \log L}{d\beta^2} = -N$$

Likelihood Function for One Parameter



OLS in the MLE Framework, continued

- Likelihood for one unit:

$$L_i(\beta|y_i) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{(y_i - x_i\beta)^2}{\sigma^2}\right)$$

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- Log likelihood (in matrix form):

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2}\frac{(\mathbf{y} - \mathbf{x}\beta)'(\mathbf{y} - \mathbf{x}\beta)}{\sigma^2}$$

OLS in the MLE framework, continued

- Doing a bit of algebra, we get

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{y}'\mathbf{y} - \mathbf{2}\beta'\mathbf{x}'\mathbf{y} + \beta'\mathbf{x}'\mathbf{x}\beta)$$

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- Take the derivative with respect to β and we get:

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- Set derivative equal to zero and solve for the MLE:

$$0 = \frac{1}{\sigma^2}(\mathbf{x}'\mathbf{y} - \mathbf{x}'\mathbf{x}\beta) \quad (3)$$

$$\mathbf{x}'\mathbf{x}\beta = \mathbf{x}'\mathbf{y} \quad (4)$$

$$\hat{\beta} = \mathbf{x}'\mathbf{x}^{-1}\mathbf{x}'\mathbf{y} \quad (5)$$

OLS in the MLE framework, continued

- What about the variance, σ^2? Take the derivative of the log-likelihood with respect to σ^2

$$\frac{\partial \log L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}[(\mathbf{y} - \mathbf{x}\beta)'(\mathbf{y} - \mathbf{x}\beta))]$$

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- Since we've solved for $\hat{\beta}$, we can replace β with its estimate:

$$\begin{aligned}
\frac{1}{\sigma^2}[(\mathbf{y} - \mathbf{x}\hat{\beta})'(\mathbf{y} - \mathbf{x}\hat{\beta}))] &= n \\
\frac{1}{\sigma^2}[(\mathbf{y} - \hat{\mathbf{y}})'(\mathbf{y} - \mathbf{x}\hat{\mathbf{y}}))] &= n \\
\frac{1}{\sigma^2}[\mathbf{e}'\mathbf{e}] &= n \\
\hat{\sigma}^2 &= \frac{\mathbf{e}'\mathbf{e}}{n}
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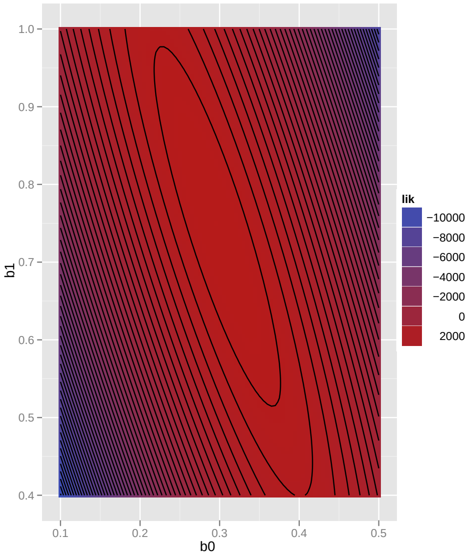

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- Let $S = X_1 + \cdots + X_n$. The likelihood function is as follows:

$$\begin{aligned} L_n(p) &= \sum_{i=1}^{n} [X_i \log p + (1 - X_i) \log(1-p)] && (6) \\ &= S \log p + (n - S) \log(1-p) && (7) \end{aligned}$$

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- Take the derivative $L_n'(p) = \frac{S}{p} - \frac{n-S}{1-p}$ and the second derivative $L_n''(p) = -\frac{S}{p^2} - \frac{n-S}{(1-p)^2}$. The MLE is $\hat{p} = \frac{S}{n}$.

Fisher Information

Theorem

Suppose $X_i, \cdots, X_n$ are IID with probability distribution governed by the parameter θ. Let θ_0 be the true value of θ. Under regularity conditions (which are omitted here), the MLE for θ is asymptotically normal. The asymptotic mean of the MLE is θ_0. The asymptotic variance can be computed as follows:

$$[-L_n''(\hat{\theta})]^{-1}$$

If $\hat{\theta}$ is the MLE and v_n is the asymptotic variance, the theorem says that $\frac{\hat{\theta}-\theta_0}{\sqrt{v_n}}$ is nearly $N(0,1)$ when the sample size n is large.

The Hessian

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- The *Hessian* is a matrix of second derivatives defined as

$$\mathbf{H}(\theta) : \frac{\partial^2 \log L(\theta)}{\partial\theta\partial\theta'}$$

which in our example is:

$$\mathbf{H}(\theta) = \begin{pmatrix} \frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\alpha} & \frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\beta} & \frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\sigma} \\ \frac{\partial^2 \log L(\theta)}{\partial\beta\partial\alpha} & \frac{\partial^2 \log L(\theta)}{\partial\beta\partial\beta} & \frac{\partial^2 \log L(\theta)}{\partial\beta\partial\sigma} \\ \frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\alpha} & \frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\alpha} & \frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\sigma} \end{pmatrix}$$

The Information Matrix

- The *information matrix* is defined as the negative of the expected value of the Hessian: $-E[\mathbf{H}(\theta)]$. Under very general conditions, the covariance matrix for the ML estimator is the inverse of the information matrix:

$$\mathrm{Var}(\hat{\theta}) = -E[\mathbf{H}(\theta)]^{-1}$$

. In our example,

$$\mathrm{Var}((\theta)) = \begin{pmatrix} -E\left(\frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\beta}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\alpha\partial\sigma}\right) \\ -E\left(\frac{\partial^2 \log L(\theta)}{\partial\beta\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\beta\partial\beta}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\beta\partial\sigma}\right) \\ -E\left(\frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\alpha}\right) & -E\left(\frac{\partial^2 \log L(\theta)}{\partial\sigma\partial\sigma}\right) \end{pmatrix}^{-1}$$

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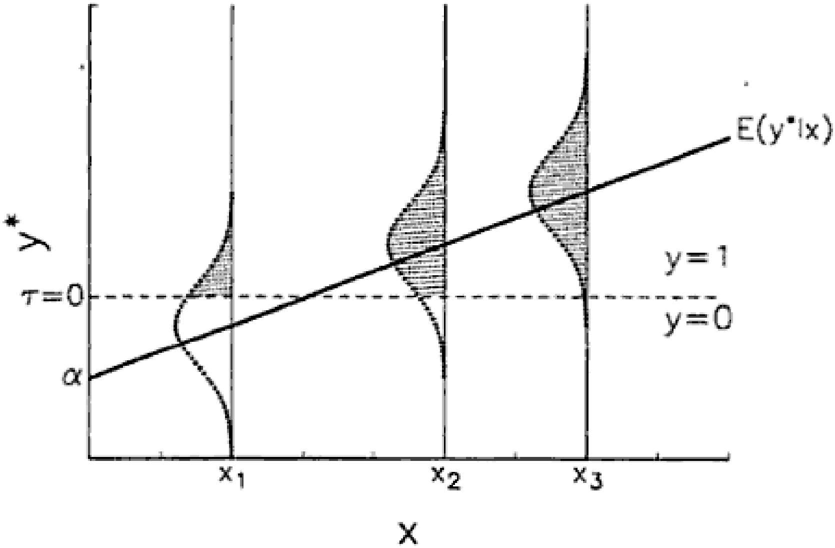
$$\prod_i^n Y_i \cdot \Phi(X_i\beta) \times (1 - Y_i) \cdot 1 - \Phi(X_i\beta)$$

- Log likelihood function:

$$L(\beta) = \sum_{i=1}^n (Y_i \log[\Phi(X_i\beta)] + (1 - Y_i) \log[1 - \Phi(X_i\beta)])$$

The Latent Variable Formulation



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- The latent variable y^* is linked to the observed binary variable y by the measurement equation:

$$y_i = \begin{cases} 1 & \text{if } Y_i^* > \tau \\ 0 & \text{if } Y_i^* \leq \tau \end{cases}$$

where τ is the *threshold* or *cutpoint*.

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- For probit, assume $E(U) = 0$, $\sigma = 1$ and is distributed normally.
- For logit, assume $E(U) = 0$, $\sigma = \pi^2/3$ which results in the following PDF:

$$\lambda(U) = \frac{\exp(U)}{(1 + \exp(U))^2}$$

and the following CDF:

$$\Lambda(U) = \frac{\exp(U)}{1 + \exp(U)}$$

Logistic vs Normal: PDF

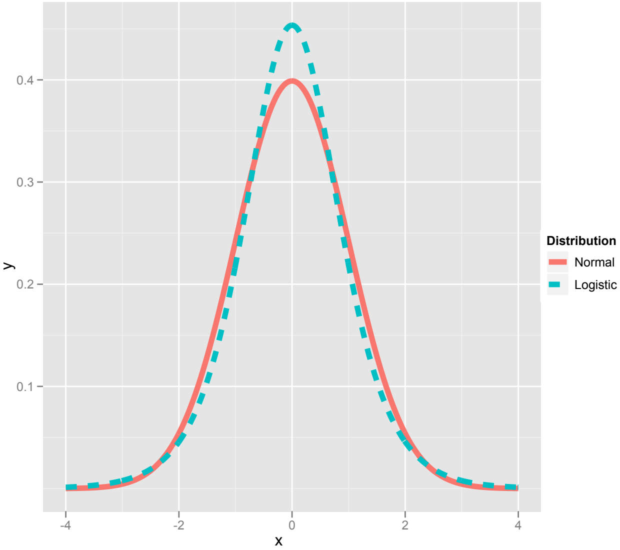
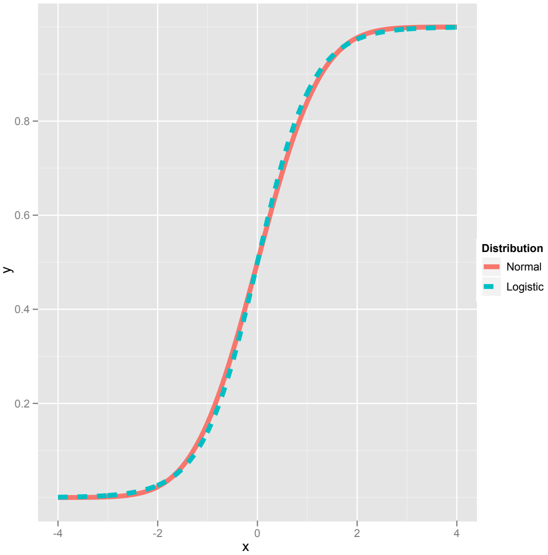
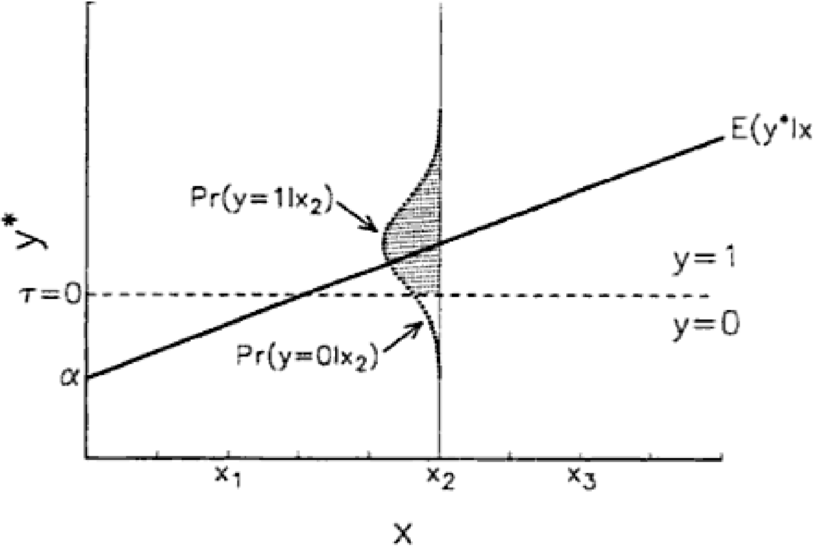


Figure: PDFs for Logistic and Normal Distribution

Logistic vs Normal: CDF



Probability of Observed Values



From Latent Variables to MLE

- Write $P(X_i\beta + U_i > 0) = P(U_i > -X_i\beta) = P(-U_i < X_i\beta)$ and because U_i is distributed symmetrically around 0, $P(-U_i < X_i\beta) = P(U_i < X_i\beta) = \Phi(X_i\beta)$, in the probit case.

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