Randomization Inference

September 22, 2010

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- The Null hypothesis:

  > *It is evident that the null hypothesis must be exact, that is free from vagueness and ambiguity, because it must supply the basis of the "problem of distribution," of which the test of significance is the solution. A null hypothesis may, indeed contain arbitrary elements, and in more complicated cases often does so...*

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- Write $Z_{si} = 1$ if the ith unit in stratum s receives the treatment and write $Z_{si} = 0$ if this unit receives control.
- Write m_s for the number of treated units in stratum s, so $m_s = \sum_{i=0}^{n_s} Z_{si}$ and $0 \leq m_s \leq n_s$.

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- For example:

$$\Omega = \left\{ \begin{array}{cccccc} \begin{bmatrix}1\\1\\0\\0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\1\\0\end{bmatrix} & \begin{bmatrix}0\\1\\1\\0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\\1\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\\1\\1\\0\end{bmatrix} \\ \begin{bmatrix}1\\1\\0\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\\1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\1\\0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\\1\\0\\1\end{bmatrix} \end{array} \right\}.$$

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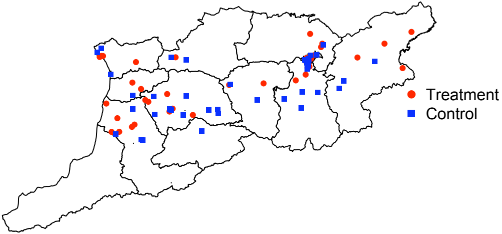
- We conducted a program evaluation of one such effort: a simple information campaign to give voters the information necessary for filing a formal complaint with civil society groups or election officials if they witnessed problems on election day.
- The intervention consisted of sending canvassers to knock on doors and hand out fliers in randomly selected precincts.

Example: Randomization Procedure



Ths structure of randomization was as follows.

- 36 rural precincts were in blocks of 2, one treatment and one control. So for these precincts, $m_s = 1$ and $n_s = 2$.
- 48 urban precincts were in blocks of 4, two in treatment and and two in control ($m_s = 2$ and $n_s = 4$).

Some R code

How big is Ω?

```
choose(2,1)^18 * choose(4,2)^12
[1] 5.706304e+14
```

Let's create a function that will assign treatment repeatedly.

```
treat.assign <- function(treat,blocks=NA){
  if(length(unique(blocks))==1){
    treat.vector <- sample(treat)
  }
  else{
  treat.vector <- tapply(treat,blocks,sample)
  treat.vector <- unlist(treat.vector)
  }
  return(treat.vector)
}
```

R Code

Let's create our distribution of treatment vectors. We could compute all 5.7×10^{14} treatment vectors, but to save on computing time, we can sample a large number of possible treatment vectors to get "close-to-exact" p-values. If our experiment were smaller, then exhaustive enumeration would be better.

Let's use the `replicate` function to assign treatment 5,000 times and generate our Ω:

```
omega <- replicate(5000,
                   treat.assign(treat,blocks))
omega <- unique(omega,MARGIN=2)
```

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- A unit labeled as "treated" will have the exact same outcome as a unit labeled as "control".
- Under the null, the units' responses are *fixed* and the only random element is the meaningless rotation of labels.
- When testing the null hypothesis of no effect, the response of the ith unit in stratum s can be written r_{si} and the vector of responses is $\mathbf{r}$.

The Test Statistic

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- The most commonly used test-statistic is the point estimate for the average treatment effect. In a block randomized experiment, the differences within blocks are summed, and each block difference is weighted by the proportion of units in the block:

$$\sum_{s=1}^{S} \frac{n_s}{N} \sum_{i=1}^{n_s} \left\{ \frac{Z_{si} r_{si}}{m_s} - \frac{(1 - Z_{si}) r_{si}}{n_s - m_s} \right\}$$

Significance Test

- To compute the p-value for any given test statistic, we simply calculate the proportion of treatment assignments $\mathbf{z}$ in Ω giving values of $t(\mathbf{z}, \mathbf{r})$ greater than or equal to the observed T, namely:

$$\mathrm{prob}\{t(\mathbf{Z}, \mathbf{r} \geq T\} = \frac{|\{\mathbf{z} \in \Omega : t(\mathbf{z}, \mathbf{r}) \geq T\}|}{K}$$

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- The above p-value is for a one-tailed test. What about a two-tailed test? There is some disagreement in the literature about this, but Rosenbaum recommends simply doubling the one-tailed p-value.

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- If all N responses were different numbers, the ranks woud be the numbers $1, 2, ..., N$. If some of the responses were equal, then the average of their ranks would be used.
- Write q_i for the rank of r_i, and write $\mathbf{q} = (q_i, ..., q_N)^T$. The rank sum statistic is simply the sum of the ranks of the treated observations, i.e. $t(\mathbf{z}, \mathbf{r}) = \mathbf{Z^T q}$.

Rank Tests

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- **Aligned rank test**: According to Hodges and Lehmann (1962), a more efficient rank test for block randomized experiment is the aligned rank statistic. For this statistic, subtract the mean of each stratum from the responses in that stratum, creating "aligned responses" . Rank the aligned responses without regard to block. The aligned rank statistic is the sum of the aligned ranks in the treated group.

Covariate Adjustment

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- Remember that under the null hypothesis, nothing is stochastic except for the shuffling of treatment assignment labels. As a result $\mathbf{e}$ is a fixed quantity, not a random variable or a by-product of estimation.
- $\mathbf{e}$, however, may be less dispersed than $\mathbf{r}$ because some of the variation in $\mathbf{r}$ will have been captured by $\mathbf{X}$.

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- So once can simply use the test statistic $t(\mathbf{z}, \mathbf{e})$ instead of $t(\mathbf{z}, \mathbf{r})$.
- With $\mathbf{e}$ in hand, just proceed as you would with $\mathbf{r}$.