Section 2 : Introduction to potential outcome and causal relationships

(and Monte – Carlo simulations)

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Definitions

- Let T_i be an indicator variable whether individual i received treatment ($T_i = 1$) or control ($T_i = 0$)
- Let Y_{i1} be the potential outcome of individual i with treatment and Y_{i0} the potential outcome without treatment
- The observed outcomes are,

$$Y_i = T_i Y_i 1 + (1 - T_i) Y_{i0}$$

Group	Y_{i1}	Y_{i0}
$T = 1$	Observable: $Y_{i1} \mid T = 1$	Counterfactual: $Y_{i0} \mid T = 1$
$T = 0$	Counterfactual: $Y_{i1} \mid T = 0$	Observable: $Y_{i0} \mid T = 0$

- The treatment effect on individual i is,

$$\tau_i = Y_{i1} - Y_{i0}$$

- There can be many parameters of interest. A few common parameters are,

$$ATE = \mathbb{E}(Y_{i1} - Y_{i0})$$

$$ATT = \mathbb{E}(Y_{i1} - Y_{i0} | T_i = 1)$$

$$ATC = \mathbb{E}(Y_{i1} - Y_{i0} | T_i = 0)$$

- We can also be interested in the treatment effect conditional on a certain value of Y_{i0}, for example:

$$ATT' = \mathbb{E}(Y_{i1} - Y_{i0} | Y_{i0} \leq K)$$

Definition

Parameter: A number or vector that indexes a family of distributions

Example: the rate parameter in a Poisson distribution, or the potential outcomes in our causal model.

Definition

Identifiability: Let P_θ be a family of distributions indexed by θ. A function of θ is identifiable if $f(\theta_1) \neq f(\theta_2)$ implies $P_{\theta_1} \neq P_{\theta_2}$ for all θ_1, θ_2.

Definition

Estimability: A function $f(\theta)$ is estimable if there exist an estimator of $f(\theta)$ that is unbiased.

Theorem

If $f(\theta)$ is estimable then $f(\theta)$ is identifiable

The other direction does not hold. Estimability implies Identifiability, but Identifiability does imply estimability.

Example: Let $0 < p < 1$ and x be binomial with $P_p(x = 1) = p$. The function $f(\theta) = \sqrt{p}$ is identifiable, however $\sqrt{p}$ is not estimable.

Let $g(x)$ be some estimator. Then,

$$\mathbb{E}_p[g(x)] = (1-p)g(0) + pg(1)$$

This is a linear function in p, however $\sqrt{p}$ is not a linear function of p. So, $\mathbb{E}_p[g(x)] \neq \sqrt{p}$.

Median treatment effect

- Is the median treatment effect, $median(Y_{i1} - Y_{i0})$ identifiable? No
- Consider the following two populations of units:
  Population 1:

$$Pr(Y_{i1} = 6, Y_{i0} = 4) = 1/3, Pr(Y_{i1} = 8, Y_{i0} = 6) = 1/3,$$

$$Pr(Y_{i1} = 10, Y_{i0} = 8) = 1/3$$

  Population 2:

$$Pr(Y_{i1} = 10, Y_{i0} = 4) = 1/3, Pr(Y_{i1} = 8, Y_{i0} = 8) = 1/3,$$

$$Pr(Y_{i1} = 6, Y_{i0} = 6) = 1/3$$

Median treatment effect

- The distribution of treatment effects is:
  Population 1: $(2, 2, 2)$ with probability (1/3,1/3,1/3), hence the effect of the treatment is always 2!
  Population 2: $(6, 0, 0)$ with probability (1/3,1/3,1/3), hence the median treatment effect is 0
- The marginal distributions of Y_{i1} and Y_{i0} are the same in both populations
- **However** the treatment effect is determined by the joint distribution of (Y_{i1}, Y_{i0}) and the joint is different between the two populations
- Imagine the ideal experiment, can we ever observe the joint distribution of potential outcome? *No*

Median treatment effect: Another example

- Consider the following two populations:
  Population 1:

$$Pr(Y_{i1} = 1, Y_{i0} = 0) = 1/3, Pr(Y_{i1} = 3, Y_{i0} = 1) = 1/3,$$

$$Pr(Y_{i1} = 4, Y_{i0} = 3) = 1/3$$

  Population 2:

$$Pr(Y_{i1} = 4, Y_{i0} = 0) = 1/3, Pr(Y_{i1} = 3, Y_{i0} = 1) = 1/3,$$

$$Pr(Y_{i1} = 1, Y_{i0} = 3) = 1/3$$

- In population 1 the treatment effect is, $(1, 2, 1)$ and in population 2 the treatment effect is, $(4, 2, -2)$

Median treatment effect: Continuous variable example

- Let the joint distribution of the potential outcome be,

$$(Y_1, Y_0) \sim N((1, 0),\ \Sigma),$$

$$\Sigma = \begin{pmatrix} \mathbb{V}(Y_1) & Cov(Y_1, Y_0) \\ Cov(Y_1, Y_0) & \mathbb{V}(Y_0) \end{pmatrix}$$

- A binary treatment T is assigned at random.
- Can we identify the ATE? Can we identify the median treatment effect? can we identify percentiles of the treatment effect?

Median treatment effect: Continuous variable example

- Can we distinguish between this two distributions of the potential outcomes?
- Distribution 1,

$$\Sigma_1 = \begin{pmatrix} \mathbb{V}(Y_1) & Cov(Y_1, Y_0) \\ Cov(Y_1, Y_0) & \mathbb{V}(Y_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Distribution 2,

$$\Sigma_1 = \begin{pmatrix} \mathbb{V}(Y_1) & Cov(Y_1, Y_0) \\ Cov(Y_1, Y_0) & \mathbb{V}(Y_0) \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

Median treatment effect: Continuous variable example

- Distribution 1,

$$\tau_1 = Y_1 - Y_0 \sim N(1, \mathbb{V}(Y_0) + \mathbb{V}(Y_1)) = N(1, 2)$$

- Distribution 2,

$$\tau_2 = Y_1 - Y_0 \sim N(1, \mathbb{V}(Y_0) + \mathbb{V}(Y_1) - 2Cov(Y_1, Y_0) = N(1, 1)$$

- The ATE is identified, and also the median treatment effect, as both τ_1 and τ_2 are symmetric distributions centred at 1 (the ATE and the median are equal).
  However all the other moments are not identified

The difference in means is an unbiased estimator of the ATE, when $(Y_{i1}, Y_{i0} \perp T_i)$

$$\mathbb{E}\left(\frac{1}{m}\sum_{i=1}^{N} T_i Y_i - \frac{1}{N-m}\sum_{i=1}^{N}(1-T_i)Y_i\right) =$$

$$\frac{1}{m}\sum_{i=1}^{N}\mathbb{E}\left(Y_i T_i\right) - \sum_{i=1}^{N}\frac{1}{N-m}\mathbb{E}\left((1-T_i)Y_i\right) =$$

$$\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left(Y_{i1}|\, T_i = 1\right) - \sum_{i=1}^{N-m}\frac{1}{N-m}\mathbb{E}\left(Y_{i0}|\, T_i = 0\right) = ATE$$

$$\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left(Y_{i1}\right) - \sum_{i=1}^{N-m}\frac{1}{N-m}\mathbb{E}\left(Y_{i0}\right) = \mathbb{E}\left(Y_{i1}\right) - \mathbb{E}\left(Y_{i0}\right)$$

$$\mathbb{E}\left(Y_{i1} - Y_{i0}\right) = ATE$$

SUTVA

Definition

No interference between units: the observation on one unit should be unaffected by the particular assignment of treatment to the other units.

- *No-interference* is the assumption that the allocation of treatment to unit i has no effect on the outcome of unit j for all i, j
- SUTVA is a slightly stronger assumption than *no-interference*, hence SUTVA implies *no-interference*, and the opposite does not hold
- In this course we refer to SUTVA and *no-interference* as equivalent terms

SUTVA

- Consider a uniform randomized experiment with two strata, four units in the first strata and two units in the second strata, for 6 units in total. Half the units in each stratum receive treatment.
- There are 12 possible treatment assignments contained in the set Ω.

$$\Omega = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right.$$
$$\left. \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Causal Effects without assuming SUTVA

- Without SUTVA, a causal effect is defined for every possible combination of the treatment assignment.
- The potential outcome for unit i might be $Y_{i100000000000}$ or $Y_{i010000000000}$, etc.
- How many potential outcomes will each unit have in a sample with N observation? 2^N
- Potential outcomes are still well defined when SUTVA is not satisfied!

SUTVA: Rubin (1986)

Statistics and Causal Inference: Comment: Which If's Have Causal Answers

- In a comment to Holland (1986) Rubin provides a formal definition of SUTVA.
- There are N units indexed by $u = 1, \ldots, N$, T treatments indexed by $t = 1, \ldots, T$, and an outcome variable Y_{tu}
- Rubin's definition: *"SUTVA is simply the a priori assumption that the value of Y for unit u when exposed to treatment t will be the same no matter what mechanism is used to assign treatment t to unit u and no matter what treatments the other units receive"* $\forall_t, \forall_u$
- Examples when SUTVA is violated:
  1. There exist unrepresented versions of treatments: *Y depends on which version of treatment t was received*
  2. interference between units: *the outcome, Y, of unit u depends on whether unit u' received treatment t or t'*

SUTVA: Rubin (1986)

Statistics and Causal Inference: Comment: Which If's Have Causal Answers

- Does the following statement has a causal meaning?
  If the females at firm f had been male, their starting salaries would have averaged 20% *higher*
  No, the statement is causal meaningless
- Rubin's answer:
  "the statement, by itself, is too vague to have a clear formulation satisfying SUTVA and thus is too vague to admit a clear causal answer. What are the units, treatments, and outcomes such that SUTVA is satisfied? I am not at all sure how to define anything except Y, which clearly involves starting salary"
- See Rubin (1986) for a variety of ways to make the statement have a causal meaning

SUTVA: Example

- Assume the following DGP (data generating process):

$$Y_i = \alpha + \tau T_i + X_i\beta + \epsilon_i$$

- Is SUTVA satisfied in this model? Yes
- If $Cov(X_i, \epsilon_i) \neq 0$, X_i is endogenous. Is SUTVA satisfied? Yes

- Consider the following model of the treatment effect (multiplicative treatment effect)

$$Y_{i1} = \tau Y_{i0}$$

- What is the *ATE* effect?
  Answer: $\mathbb{E}(Y_{i1} - Y_{i0}) = \mathbb{E}(\tau Y_{i0} - Y_{i0}) = \mathbb{E}(Y_{i0})(\tau - 1)$
- How can we estimate τ?
- One solution is to employ the following transformation on the data, *log*:

$$log(Y_{i1}) = \tau + log(Y_{i0})$$

- Now τ is the *ATE* of the treatment after the transformation, and can be estimated by the difference in means

- Prior to the *log* transformation, what is the variance of the potential outcomes with the treatment? Is it equal to the variance under control?

$$\mathbb{V}(Y_{i1}) = \tau^2 \mathbb{V}(Y_{i0})$$

- After the *log* transformation, the variance in both groups is the same,

$$\mathbb{V}(Y_{i1}) = \mathbb{V}(Y_{i0} + \tau) = \mathbb{V}(Y_{i0})$$

Conditional independence assumption (CIA)

- The CIA implies that:

$$\mathbb{E}\left(Y_{i1}|X_i, T_i = 1\right) = \mathbb{E}\left(Y_{i1}|X_i, T_i = 0\right) = \mathbb{E}\left(Y_{i1}|X_i\right)$$

and

$$\mathbb{E}\left(Y_{i0}|X_i, T_i = 1\right) = \mathbb{E}\left(Y_{i0}|X_i, T_i = 0\right) = \mathbb{E}\left(Y_{i0}|X_i\right)$$

- Assuming CIA holds,

$$ATE = \mathbb{E}_{X_i}\left(\mathbb{E}_{Y_{i1}|X_i}\left(Y_{i1}|X_i, T_i = 1\right)\right) - \mathbb{E}_{X_i}\left(\mathbb{E}_{Y_{i0}|X_i}\left(Y_{i0}|X_i, T_i = 0\right)\right)$$

Conditional assumption (CIA)

- Assuming the following model (linear regression),

$$y_i = \alpha + \tau_1 T_i + X_i\beta + \epsilon$$

- Then,

$$\mathbb{E}(Y_i|T_i = 1, X_i) = \alpha + \tau_1 + X_i\beta, \quad \mathbb{E}(Y_i|T_i = 0, X_i) = \alpha + X_i\beta$$

- In a regression model the standard assumption is that X_i is fixed (not a random variable), and therefore,

$$\mathbb{E}_{X_i}\left(\mathbb{E}_{Y_{i1}|X_i}\left(Y_{i1}|X_i, T_i = 1\right)\right) = \mathbb{E}_{Y_{i1}|X_i}\left(Y_{i1}|X_i, T_i = 1\right)$$

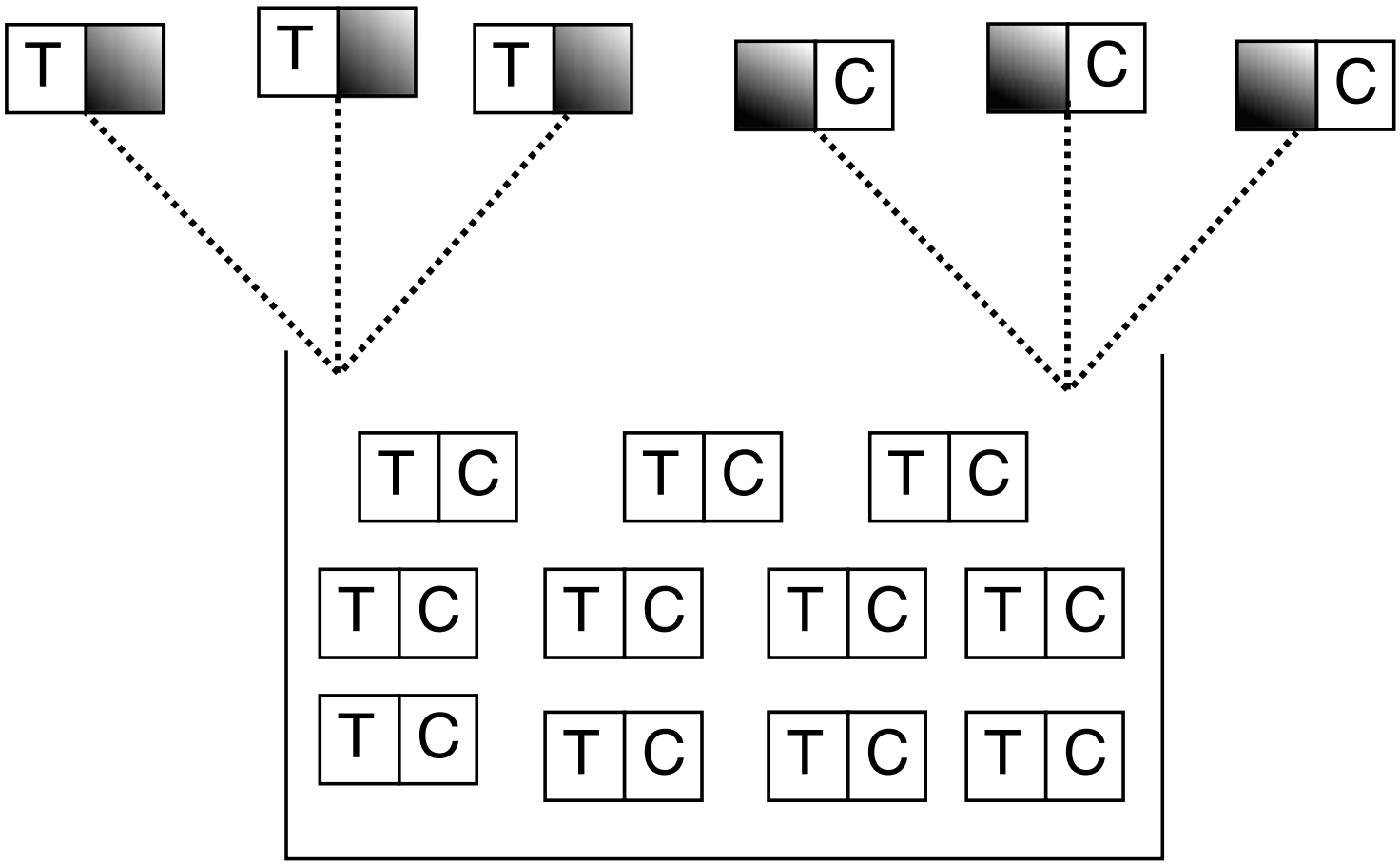
- Therefore the parameter τ_1 can be estimated by a regression adjustment, $\hat{\beta}^T_{OLS}$
- There are also many other ways of estimating τ_1, such as matching

Treatment assignment mechanisms

- There are many possible random treatment assignment mechanisms. The must common is selecting m observations to be assigned treatment out of N possible units

- In this approach, m, is fixed, it is not a random variable. The source of randomization is the random assignment of treatment

Treatment assignment mechanisms



Treatment assignment mechanisms

- There are N units, and m units are assigned a binary treatment at random
- Let Z_i be an indicator variable whether unit i was assigned treatment or control
- Is Z_i and Z_j independent? *No*
- What is $Cov(Z_i, Z_j) =$? Is it positive or negative?
  $Cov(Z_i, Z_j) < 0$, If unit i is assigned treatment the probability of unit j to receive treatment decreases. There is a negative relationship

Treatment assignment mechanisms

- What is, $Pr(Z_i = 1|m)$? $Pr(Z_i = 1|m) = \frac{m}{N}$
- Is Z_i and Z_j independent? What is $cov(Z_i, Z_j)$?
- When there are m units to be assigned treatment among N remaining units, the probability of $Z_i = 1$ conditional on Z_j is? $Pr(Z_i = 1|z_j = 0) = \frac{m}{N-1}$, $Pr(Z_i = 1|z_j = 1) = \frac{m-1}{N-1}$
- When $N \to \infty$:
  $Pr(Z_i = 1|z_j = 1) = Pr(Z_i = 1|z_j = 0) = Pr(Z_i = 1)$
- When $N \to \infty$, Z_i and Z_j are independent and $cov(Z_i, Z_j) = 0$

Calculating $Cov(Z_i, Z_j)$ Analytically

As Z_i is an indicator variable it follows that,

$$\mathbb{E}(Z_i) = Pr(Z_i = 1) = \frac{m}{N},\ \forall_i, j$$

$$\mathbb{E}(Z_i \cdot Z_j) = 0 \times 0 \times Pr(Z_i = 0, Z_j = 0) + 1 \times 0 \times Pr(Z_i = 1, Z_j = 0) +$$
$$0 \times 1 \times Pr(Z_i = 0, Z_j = 1) + 1 \times 1 \times Pr(Z_i = 1, Z_j = 1)$$

$$= Pr(Z_i = 1, Z_j = 1) = \frac{m}{N} \cdot \frac{m-1}{N-1}$$

Hence,

$$Cov(Z_i, Z_j) = \mathbb{E}(Z_i \cdot Z_j) - \mathbb{E}(Z_i) \cdot \mathbb{E}(Z_j)$$
$$= \frac{m}{N}\left(\frac{m-1}{N-1} - \frac{m}{N}\right) < 0$$

Monte Carlo simulations

- An alternative approach for estimating $Cov(Z_i, Z_j)$ is by a Monte-Carlo approximation

- The data generating process is known, a treatment was assigned at random, m units where chosen out of N. We can construct a simulation which performs exactly this process a multiple number of time and using the repetitions approximate the random component of the assignment mechanism.

Monte Carlo simulations: code

```
m=4
R=10000 #or 500000
n.vec = c(c(5:20),seq(21,100,by=5)) # sample sizes, N
cov.real1 <- cov.approx1 <- rep(999,length(n.vec))
for (i in c(1:length(n.vec))){
  N = n.vec[i]
 ## analytical:
 cov.real1[i] <- (m/N)*((m-1)/(N-1)-(m/N))
 ### Simulation:
  z1<-z2<-rep(999,R)
  for (j in c(1:R)){
    id.treat = sample(c(1:N),m,replace=FALSE)
    treat0 = rep(0,N)
    treat0[id.treat]=1
    z1[j] = treat0[1]
    z2[j] = treat0[2]
  }
  cov.approx1[i] <- cov(z1,z2)
}
```

Monte Carlo simulations: Results