Regression and Causal Inference

September 8, 2010

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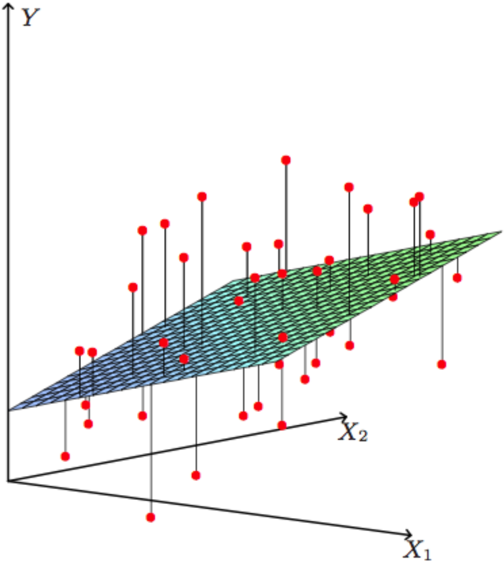
- We can pick the coefficients $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^T$ in a variety of ways but OLS is by far the most common, which minimizes the **residual sum of squares** (RSS):

$$\begin{aligned} RSS(\beta) &= \sum_{i=1}^{N} (y_i - f(x_i))^2 \\ &= \sum_{i=1}^{N} (y_i - \beta_0 - \sum_{j=1}^{P} x_{ij} \beta_j)^2 \end{aligned}$$

OLS in a Picture



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- Solve for β:

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Making a Prediction

- The *hat matrix*, or *projection matrix*

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- If $\mathbf{HY}$ yields part of $\mathbf{Y}$ that projects into $\mathbf{X}$, this means that $\tilde{\mathbf{H}}\mathbf{Y}$ is the part of $\mathbf{Y}$ that does not project into $\mathbf{X}$, which is the *residual* part of $\mathbf{Y}$. Therefore, $\tilde{\mathbf{H}}\mathbf{Y}$ makes the residuals.

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7. *Normal Errors* (optional): $Y \sim \mathbb{N}(X\beta, \sigma^2)$

Deriving σ^2

- Recall:

$$
\begin{aligned}
\hat{\beta} &= (X^T X)^{-1} X^T Y \\
&= (X^T X)^{-1} X^T (X\beta + \epsilon) \\
\Rightarrow \quad \hat{\beta} - \beta &= (X^T X)^{-1} X^T \epsilon
\end{aligned}
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Deriving σ^2

- Recall:

$$\begin{aligned} \hat{\beta} &= (X^TX)^{-1}X^TY \\ &= (X^TX)^{-1}X^T(X\beta + \epsilon) \\ \Rightarrow \quad \hat{\beta} - \beta &= (X^TX)^{-1}X^T\epsilon \end{aligned}$$

- Plugging this into the covariance equation:

$$\begin{aligned} cov(\hat{\beta}|X) &= E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X] \\ &= E\left[\left((X^TX)^{-1}X^T\epsilon\right)\left((X^TX)^{-1}X^T\epsilon\right)'|X\right] \\ &= E[(X^TX)^{-1}X^T\epsilon\epsilon^TX(X^TX)^{-1}|X] \\ &= (X^TX)^{-1}X^TE(\epsilon\epsilon^T|X)X(X^TX)^{-1} \\ &\qquad \text{where } E(\epsilon\epsilon^T|X) = \sigma^2 I_{p\times p} \\ &= (X^TX)^{-1}X^T\sigma^2 I_{p\times p}X(X^TX)^{-1} \\ &= \sigma^2(X^TX)^{-1}X^TX(X^TX)^{-1} \\ &= \sigma^2(X^TX)^{-1} \end{aligned}$$

Deriving σ^2

We estimate σ^2 dividing the residuals squared by the degrees of freedom because the e_i are generally smaller than the ϵ_i due to the fact that $\hat{\beta}$ was chosen to make the sum of square residuals as small as possible.

$$\hat{\sigma}^2 = \frac{1}{n-p} \sum_{i=1}^{n} e_i^2$$

Unbiasedness

Recall:

$$\begin{aligned}
\hat{\beta} &= (X^T X)^{-1} X^T Y \\
&= (X^T X)^{-1} X^T (X\beta + \epsilon) \\
&= (X^T X)^{-1} X^T X\beta + (X^T X)^{-1} X^T \epsilon \\
&= \beta + (X^T X)^{-1} X^T \epsilon
\end{aligned}$$

We know that $\hat{\beta}$ is unbiased if $E(\hat{\beta}) = \beta$

$$\begin{aligned}
E(\hat{\beta}) \quad &= E(\beta + (X^T X)^{-1} X^T \epsilon | X) \\
&= E(\beta | X) + E((X^T X)^{-1} X^T \epsilon | X) \\
&= \beta + (X^T X)^{-1} E(\epsilon | X) \\
&\qquad \text{where } E(\epsilon | X) = E(\epsilon) = 0 \\
E(\hat{\beta}) \quad &= \beta
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- In the simple bivariate case:

$$\beta_1 = \frac{\mathrm{Cov}(Y_i, X_i)}{\mathrm{Var}(X_i)}$$

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- The multiple regression coefficient $\hat{\beta}_j$ represents the additional contribution of x_j on y, after x_j has been adjusted for $x_o, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$

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- What happens when x_j is highly correlated with some of the other x_k's?

Regression in Causal Analysis

- Imagine we are analyzing a *randomized* experiment with a regression using the following model:

$$Y_i = \alpha + \beta_1 \cdot T_i + \mathbf{X}_i^T \cdot \beta_2 + \epsilon_i$$

  where T_i is an indicator variable for treatment status and $\mathbf{X}_i$ is a vector of *pre-treatment characteristics*

- Under this model, what is random?

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- How do we interpret the coefficient β_1?

Regression in an Observational Study