Algorithmic Analysis

Big O: $f(n) \in O(g(n)) \longleftrightarrow \exists c \in \mathbb{R}, n_0 \in \mathbb{Z}^{\geq 0}$ s.t. $\forall n \geq n_0, f(n) \leq c \cdot g(n)$

Big Ω: $f(n) \in \Omega(g(n)) \longleftrightarrow \exists c \in \mathbb{R}, n_0 \in \mathbb{Z}^{\geq 0}$ s.t. $\forall n \geq n_0, f(n) \geq c \cdot g(n)$

Big Θ: $f(n) \in \Theta(g(n)) \longleftrightarrow f(n) \in O(g(n)) \wedge f(n) \in \Omega(g(n))$

$$\longleftrightarrow \exists c_1, c_2 \in \mathbb{R}, n_0 \in \mathbb{Z}^{\geq 0} \text{ s.t. } \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

Growth Rates: $1 \to \log(n) \to n \to n\log(n) \to n^2 \to n^3 \to c^n \to n!$

Recursion Analysis

Case 1: work per call follows pattern - $T(n) = T(n-1) + O(1)$

Case 2: work per level is the same - $T(n) = 2T(\frac{n}{2}) + O(n)$

Case 3: work per call is the same - $T(n) = 2T(n-1) + O(1)$

Sorting Algorithms

Sort	Worst	Best	Expected
Bubble $A[i] \overset{swap}{\leftrightarrow} A[i+1]$	$O(n^2)$	$O(n)$ *pre-sorted*	$O(n^2)$
Insertion *by next unsorted*	$O(n^2)$	$O(n)$ *pre-sorted*	$O(n^2)$
Selection *by smallest*	$O(n^2)$	$O(n)$ *pre-sorted*	$O(n^2)$
Merge	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$
Quick *3-partition*	$O(n^2)$	$O(n)$ *uniform list*	$O(n \log n)$
Radix	$O(d(n+N))$	$O(d(n+N))$	$O(d(n+N))$

Basic Data Structures

Amortisation: $T(n) \div n$, where $n =$ no. of operations

	SPACE	*get*	*add*	*remove*
Dynamic Array	$O(n)$	$O(1)$	$O(n)$*	$O(n)$
Linked List	$O(n)$	$O(1)$	$O(n)$	$O(n)$
Stack - LIFO	*SPACE:* $O(n)$		$O(1)^1$	$O(1)$
Queue - FIFO	*SPACE:* $O(n)$		$O(1)$	$O(1)$

[1] *amortised if array-based implementation*

Binary Trees

Proper Binary Tree: internal nodes have exactly 2 children

Complete Binary Tree: levels $0 \to h-1$ full, level h left-most

Pre-Order	In-Order	Post-Order
self → left → right	left → self → right	left → right → self

Properties:

- Full level l has 2^l nodes (note $l \geq 0$)
- Max no. of nodes = $2^{l_{\text{MAX}}} - 1$, max internal nodes = $2^{l_{\text{MAX}}-1} - 1$
- h = no. of edges from lowest leaf = $\lfloor \log_2 n \rfloor$
  ⚠️ *care with null leaves in implementation vs. conceptual tree*
- l & d use 0-based index.

Heaps & Priority Queues

Binary Heap: complete binary tree with heap property order

$$key(node) \leq key(parent(node)) \text{ OR } key(node) \geq key(parent(node))$$

Insertion: insert at last node location then **upHeap** -

1. Check $k < parent \wedge k \neq root$
2. Swap $k \longleftrightarrow parent_{\text{greater than current}}$
3. Repeat until $k \geq parent \vee k = root \longrightarrow O(\log n)$

RemoveMin: swap root ⟷ last node then **downHeap** -

1. Check $k > child_{\text{left or right}}$
2. Swap $k \longleftrightarrow child_{\text{less than current}}$
3. Repeat until $k > child_{\text{both}} \vee k = leaf \longrightarrow O(\log n)$

Array-Based: left child @ $2i+1$ & right child @ $2i+2$

Heap Sort: for a in A → add to heap → removeMin back to A

Bottom Up Heap Construction: get $l_{\text{MAX}} = \lfloor \log_2 n \rfloor$ → get no. of leaves on bottom level (fill remaining with nulls) → add nodes to merge heaps

Maps - Hash Tables

General Summary: pre-hash → compress → handle collisions → rehash

Complexities: $O(1)$ expectation, $O(n)$ worst-case

Pre-hash to *Hash Code*:

- Component sum: e.g. sum of all char in string (collision risk)
- Polynomial accumulation: $p(z) = a_0 z^0 + a_1 z^1 + \ldots\ z \in \mathbb{Z}$
- Cyclic shift: replace z with bit-shifted version (e.g. $z \ll 5$)

Compress to *Hash Value*: fow below, $N =$ table size $\wedge N \in \mathbb{Z}^{\text{prime}}$

- Division: $h(k) = k \bmod N$
- MAD: $h(k) = ((ak+b) \bmod p) \bmod N$
  $p > N \wedge p \in \mathbb{Z}^{\text{prime}}, a \in [1, p-1], b \in [0, p-1]$

Collision Handling: separate chaining vs. open addressing

- Probe idx computed as $(h(k) + f(i)) \bmod N$ for $i = 0, 1, 2 \ldots$
- 1st probe is just $h(k)$ hence $i = 0$ for open addressing, below:

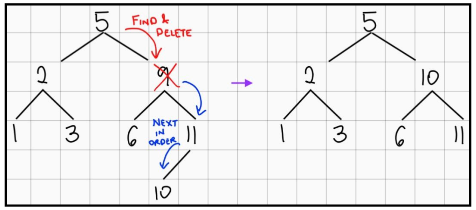
	Linear	Quadratic	Double Hash
$f(i)$	i	i^2	$i \times d(k)$

Load Factor $\alpha = \frac{n}{N} \to 0.8 \leq \alpha \leq 1$ chaining & $\alpha < \frac{2}{3}$ open addressing

- Expected no. of probes = $\frac{1}{1-\alpha}$

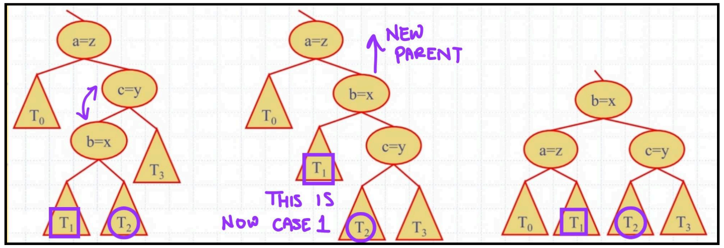
Search Trees

BST: $O(\log n)$ expected get, add, remove (in-order traversal), $O(n)$ worst



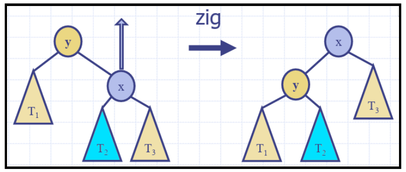
AVL Trees: BST that doesn't degrade to $O(n)$

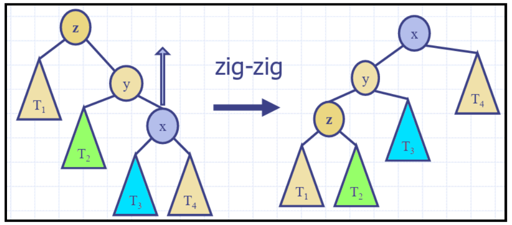
- Insertion: re-balance at **first** unbalanced node from bottom
- Deletion: after BST deletion, re-balance upwards from bottom
- Tri-node inputs: parent, child + grandchild of greater height

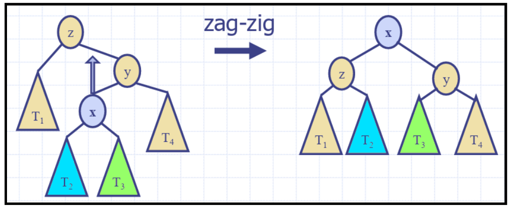


Splay Trees: $O(n)$ worst, $< O(\log n)$ best (popular nodes), $O(\log n)$*

- `get(K)` - splay found node, or last accessed node before null leaf
- `insert(K,V)` - splay inserted/updated node
- `remove(K)` - splay original parent of removal node







Graphs

- **Incident** edge = connected to vertex, **adjacent** vertices = connected by edge
- **Vertex Degree** = no. of incident edges (in vs out degree for directed)
- **Simple Graph** = no loops or multi-edges
- **Simple Path** = distinct edges AND vertices
- **Simple Path** = distinct edges AND vertices
- **Simple Cycle** = simple path with same start/end points
- **Subgraph** = subset of vertices/edges
- **Spanning Subgraph** of G = subgraph that contains *all* vertices of G
- **Connected Graph** = ∃ simple path between any 2 vertices
- **Unrooted Tree** = connected graph with no simple cycles (can be spanning)
- **Forest** = unconnected graph with no simple cycles (can be spanning)
- **Fully Dense Graph:** $\sum deg(v) = 2 \times |E|$ in undirected graph → $|E| \leq n(n-1) \div 2$
- **Graph Density:**
  - $\frac{2 \times |E|}{n(n-1)}$ undirected or $\frac{|E|}{n(n-1)}$ directed
  - $|E| \sim O(n)$ sparse or $|E| \sim O(n^2)$ dense
- **Graph Representations:**
  - Edge list: simple list of edge pointers only *[(e1), (e2), ...]* or *[(v1, v2), ...]*
  - Adjacency List (or *map*): for each vertex, store adjacent vertices OR edges - *[(v1: e1, e2), (v2: e1, e3) ...]* or *[(v1: v2, v3), (v2: v1, v4) ...]*
  - Adjacency Matrix: $V \times V$ grid where $(u, v) = 1 \leftrightarrow$ edge exists

Feature	Edge List	Adjacency List	Adjacency Matrix
Space	$n+m$	$n+m$	n^2
outgoingEdges(v)	m	deg(v)	n
incomingEdges(v)	m	deg(v)	n
getEdge(v, w)	m	min(deg(v),deg(w))	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	deg(v)	n^2
removeEdge(v)	1	1	1
Good for	small graphs	sparse graphs	dense graphs

- **DFS:** track unexplored + discovery + back edge, unexplored + explored vertex
  1. Initialize an empty stack, and push node 0 into it.
  2. Pop a vertex, mark it as visited only if not already visited.
  3. Add the visited vertex to the result collection.
  4. Push all unvisited adjacent vertices to the stack in reverse order.

Properties of DFS/BFS traversal:

- Back edge (DFS): edge to an ancestor (visited) vertex
- Cross edge (BFS): edge to a visited vertex, either same or earlier level
- Traversal visits all edges & vertices of connected component only
- Discovery edges from DFS/BFS form spanning tree of connected component
- DFS and BFS $O(n+m)$

Digraphs

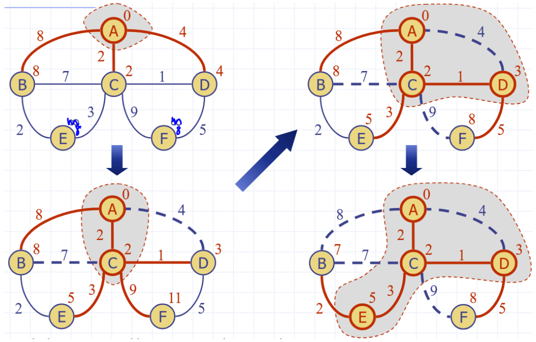
- If G is simple digraph $\rightarrow m \leq n(n-1)$
- Directed DFS algorithm tracks discovery, back, forward, cross edges
- **DAG** = Directed Acyclic Graph
- **Topological Sort** = linear ordering of vertices such that for every directed edge (u, v), $u < v$

Single Source Shortest Path (SSSP) for Weighted Graphs

- Subpath of a shortest path is itself a shortest path
- ∃ tree of shortest paths from a vertex to all others

Dijkstras Algorithm: assumes connected, undirected and non-negative edge weights

- Tracks distance from source $d(v)$ for each vertex
- Relaxation of edge e updates distance: $d(v) \leftarrow \min(d(v), d(u) + \text{weight}(e))$



Shortest Paths on DAGs

- Negative weights OK (no cycles to loop infinitely)
- Visit topological order, relax all outgoing edges for each vertex
- Does not require additional data structures

Minimum Spanning Trees: spanning tree w/ min edge weight sum

Prim-Jarnik's Algorithm: similar to Dikjstra's algorithm, except update step doesn't consider dist. from source - just edge weights. $d(v)$ = smallest edge weight connected v to the current cloud Add vertex u to the cloud which has smallest $d(u)$

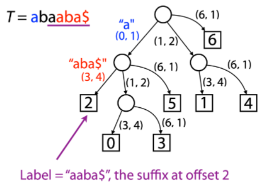
Kruskal's Algorithm: Start with single-vertex clusters. Store PQ of edge weights. Extract edges in increasing weight order, "accept" edge only if it connects distinct clusters.

Strings, Patterns, Tries

Tries: Edge = char, each node has a map/sorted list of edges for traversal. To check if string present, traverse until leaf hit (leaf not required for substring/prefix). $O(n)$ worst-case.

Suffix Tries: Append special char \$ to string end. Insert all suffixes of string into trie. $O(m)$ time to search for pattern of length m. $O(n^2)$ time to construct suffix trie for string of length n. Note: a substring is a prefix of a suffix. $O(n^2)$ worst space (no prefix sharing, all distinct chars), $O(n)$ best. Common uses:

- Check for substring P - traverse - true if no fall-off - $O(|P|)$
- Count substring occurrences - 0 if fall-off, else it is no. of leaf descendants of last node - $O(|P|)$
- Longest repeated substring - find **deepest** internal node with >1 children



Suffix Trees:

Compress paths

Replace edge labels by (offset, length) pairs

Keep T stored separately for ref

Store "final" offsets at leaves

$O(m^2)$ time to construct

$O(m)$ final space, $O(m^2)$ to build

ADT Methods

Stack	push(V)	pop()	top() or peek()
Queue	enqueue(V)	dequeue()	front() or peek()
Priority Q	insert(K, V)	removeMin()	min()
Entry	getKey()	getValue()	compareTo(Entry)
Map	get(K)	put(K, V)	remove(K)
	entrySet()	keySet()	values()
Graph	numVertices()	vertices()	
	numEdges()	edges()	
	outDegree(V)	inDegree(V)	
	outgoingEdges(V)	incomingEdges(V)	
	insertVertex(x)	insertEdge(V1, V2, x)	
	removeVertex(V)	removeEdge(E)	
	getEdge(V1, V2)	endVertices(E)	
	opposite(V, E)		

Pseudocode

Algorithm DFS-Recursive (G, v):
Input: Graph G **and** a vertex v of G
Output: Collection of vertices reachable from v + their discovery & back edges
Mark vertex v as visited
for all $e \in G.\text{outgoingEdges}(v)$ **do**
  if e is **not** explored **then**
    $w \leftarrow G.\text{opposite}(v, e)$
    if w has **not** been visited **then**
      Record edge e as discovery edge **for** vertex w
      DFS(G, w)
    else
      Mark e as a back edge **for** vertex w

Algorithm BFS (G, u):
Input: Graph G **and** a vertex u of G
Output: Collection of vertices reachable from u + their discovery & cross edges
$Q \leftarrow$ new empty queue
$Q.\text{enqueue}(u)$
Mark vertex u as visited
while $Q.\text{isEmpty}()$ **do**
  $v \leftarrow Q.\text{dequeue}()$
  for all e **in** $G.\text{incidentEdges}(v)$ **do**
    if e is **not** explored **then**
      $w \leftarrow G.\text{opposite}(v, e)$
      if w has **not** been visited **then**
        Record edge e as discovery edge **for** vertex w
        $Q.\text{enqueue}(w)$
        Mark vertex w as visited
      else
        Mark e as a cross edge

Algorithm TopologicalDFS (G, v):
Mark vertex v as visited
for all e **in** $G.\text{outgoingEdges}(v)$ **do**
  $w \leftarrow G.\text{opposite}(v, e)$
  if w has **not** been visited **then**
    { e is a discovery edge }
    $\text{topologicalDFS}(G, w)$
  else
    { e is a forward **or** cross edge }
    Label v with topological number n
    $n \leftarrow n - 1$

Algorithm DijkstraDistances (G, s):
$P \leftarrow$ new heap-based priority queue
for all v **in** $G.\text{vertices}()$ **do**
  if $v = s$ **then**
    $\text{setDistance}(v, 0)$
  else
    $\text{setDistance}(v, \infty)$
    $P.\text{insert}(\text{getDistance}(v), v)$
    while $P.\text{isEmpty}()$ **do**
      $u \leftarrow P.\text{removeMin}()$
      for all e **in** $G.\text{incidentEdges}(u)$ **do**
        { relax edge e }
        $z \leftarrow G.\text{opposite}(u, e)$
        $r \leftarrow \text{getDistance}(u) + \text{weight}(e)$
        if $r < \text{getDistance}(z)$ **then**
          $\text{setDistance}(z, r)$
          $P.\text{replaceKey}(\text{getLocator}(z), r)$