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## Parameter Estimation

Q.1. Let $(x_1, x_2, \ldots, x_n)$ be a random sample of size n taken from a Normal population with parameters mean $= \theta_1$ and variance $= \theta_2$. Find the Maximum Likelihood Estimates of these two parameters.

⇒ PDF of Normal distribution :-

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{\theta_2}} e^{-1/2\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

where $\theta_2 = \sigma^2$
$\theta_1 = \mu$

According to question, $x_1, \ldots x_n$ are random values from the distribution, which makes likelihood function as follows

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-1/2 \frac{(x_i-\theta_1)^2}{\theta_2}}$$

Taking log on both sides

$$\log(L) = \log\left(\sqrt{2\pi\theta_2}^{\,-n} \prod_{i=1}^{n} e^{-1/2 \frac{(x_i-\theta_1)^2}{\theta_2}}\right)$$

$$\log(L) = -\frac{n}{2}\log(2\pi\theta_2) + \left(\frac{-1}{2\theta_2}\right)\sum_{i=1}^{n}(x_i-\theta_1)^2$$

Differentiate wrt $\theta_1$

$$\frac{1}{L}\frac{\partial L}{\partial \theta_1} = \frac{-1}{2\theta_2}\sum_{i=1}^{n} 2(x_i-\theta_1)(-1)$$

Date :

$$\frac{1}{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{1}{2\theta_2} \sum_{i=1}^{n} 2(x_i - \theta_1)$$

Equating $\frac{\partial L}{\partial \theta_1} = 0$

$$\mathcal{L} \cdot \frac{1}{2\theta_2} \sum_{i=1}^{n} 2(x_i - \theta_1) = 0$$

Either $\mathcal{L} = 0$

or $\frac{1}{2\theta_2} \sum_{i=1}^{n} 2(x_i - \theta_1) = 0$

$\mathcal{L} = 0$ can't be possible

$$\therefore \frac{1}{2\theta_2} \sum_{i=1}^{n} 2(x_i - \theta_1) = 0$$

$$\sum_{i=1}^{n} \cancel{2} x_i = \cancel{2} \sum_{i=1}^{n} \theta_1$$

$$n\theta_1 = \sum_{i=1}^{n} x_i$$

$$\theta_1 = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{\theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i}$$ $\theta_1 =$ sample mean

Now differentiate wrt $\theta_2$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} \left(\frac{1}{\mathcal{L}}\right) = \frac{-n}{2} \frac{2\pi}{2\pi \theta_2} + \sum_{i=1}^{n} (x_i - \theta_1)^2 \frac{1}{(2\theta_2^2)}$$

Putting $\frac{\partial \mathcal{L}}{\partial \theta_2} = 0$

$$\frac{-n}{2\theta_2} + \sum_{i=1}^{n} (x_i - \theta_1)^2 \cdot \frac{1}{2\theta_2^2} = 0$$

Date :

$$\sum_{i=1}^{n} (x_i - \theta_1)^2 = n\theta_2$$

$$\theta_2 = \frac{\sum_{i=1}^{n} (x_i - \theta_1)^2}{n}$$

$$\boxed{\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2}$$

$\theta_2$ = Sample variance

Q2. Let $X_1, X_2 \ldots X_n$ be a random sample from $B(m, \theta)$ distribution where $\theta \in (0, 1)$ is unknown and $m$ is a known positive integer. Compute value of $\theta$ using MLE.

$\Rightarrow$ PMF of Binomial distribution :-

$$P(X = k) = {}^{m}C_{k}\, \theta^{k} (1-\theta)^{m-k}$$

Let $X_1, X_2, \ldots X_n$ be random sample from $B(m, \theta)$ distribution where for a $X_i$, it represents number of successes in ith trial.

$$L(\theta) = \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

Taking log on both sides

$$\log L = \log \left( \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i} \right)$$

$$\log L = \sum_{i=1}^{n} \left[ \log {}^{m}C_{x_i} + x_i \log \theta + (m - x_i) \log(1-\theta) \right]$$

Differentiate w.r.t $\theta$ and equate to 0.

$$\frac{1}{L} \frac{dL}{d\theta} = \frac{1}{\theta} \sum_{i=1}^{n} x_i - \frac{1}{1-\theta} \sum_{i=1}^{n} (m - x_i)$$

Date :

$$\frac{\partial L}{\partial \theta} = 0$$

$$L\left(\frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\sum_{i=1}^{n} m - x_i\right) = 0$$

$L$ can't be zero

$$\frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\sum_{i=1}^{n} m - x_i$$

$$(1-\theta)\sum_{i=1}^{n} x_i = \theta n m - \theta\sum_{i=1}^{n} x_i$$

$\theta = \frac{\Sigma x_i}{nm}$ where $i$ goes from 1 to $n$

$$\boxed{\theta = \frac{1}{m}\left(\frac{1}{n}\Sigma x_i\right)}$$

$$\theta = \frac{\text{Sample mean}}{m}$$