Solutions: Assignment # 5
Course: CSE 330
Summer 2021 Semester

1. Question # 1 : A function is given by $f(x) = 6e^{-3x}$. Now Answer the following:
   (a) (1 mark) Mark] Calculate $f'(x)$ at $x = 0.5$ with $h = 0.32$ using the central difference formula.
   Solution: By using central difference formula, we get,
   $$f'(0.5) = \frac{6e^{-3(0.5+0.32)} - 6e^{-3(0.5-0.32)}}{2 \times 0.32} \approx -4.66231 .$$
   (b) (1 mark) Calculate $f'(x)$ at $x = 0.5$ with $h = 0.16$ using the central difference formula.
   Solution: By using central difference formula, we get,
   $$f'(0.5) = \frac{6e^{-3(0.5+0.16)} - 6e^{-3(0.5-0.16)}}{2 \times 0.32} \approx -4.17236 .$$
   (c) (3 marks) Now compute $D^{(1)}_{0.32}$ at $x = 0.5$ using Richardson extrapolation method.
   Solution: Richardson formula yields,
   $$D^{(1)}_{0.32} \equiv \frac{2^2 D_{0.16} - D_{0.32}}{3} = \frac{4 \times (-4.17236) - (-4.66231)}{3} \approx -4.00904 .$$
   (d) (2 marks) If the exact value of the derivative, $f'(0.5)$ is -4.01634, find the percentage error with extrapolated value found in the previous part.
   Solution: The percent error is, by definition,
   $$\%\text{Error} \equiv \left|\frac{\text{Exact Value - Approximate Value}}{\text{Exact Value}}\right| \times 100\% = \left|\frac{-4.01634 + 4.00904}{-4.01634}\right| \times 100\% \approx 0.18176\% .$$

2. Question # 2 : In the lecture note and also in the video lecture, we have shown the general expression for $D^{(1)}_h$, which is known as the Richardson Extrapolation method to find the numerical derivative of a function. Using the same method, answer the following:
   (a) (2 marks) Starting from the expression for$D^{(1)}_h$, write the expression for $D^{(1)}_{h/2}$ up to order of $\mathcal{O}(h^8)$.
   Solution: Using the formula/definition in the lecture note, we obtain (by keeping one more extra term than that given in the lecture notes),
   $$\begin{aligned} D^{(1)}_h &= f'(x) - \frac{h^4}{480} f^{(5)} - \frac{5h^6}{2^4 \times 7!} f^{(7)} + \mathcal{O}(h^8) . \\ \therefore\ D^{(1)}_{h/2} &= f'(x) - \frac{h^4}{2^4 \times 480} f^{(5)} - \frac{5h^6}{2^{10} \times 7!} f^{(7)} + \mathcal{O}(h^8) . \end{aligned}$$
   (b) (3 marks) Define the 6-th order approximation as the following
   $$D^{(2)}_h \equiv \frac{2^4 D^{(1)}_{h/2} - D^{(1)}_h}{2^4 - 1} .$$
   Now find an algebraic expression for $D^{(2)}_h$ up to terms of order $\mathcal{O}(h^8)$.
   Solution: From the given definition, we just plug in the expressions from the previous part, and obtain,
   $$\begin{aligned} D^{(2)}_h &= \frac{2^4\left(f'(x) - \frac{h^4}{2^4 \times 480} f^{(5)} - \frac{5h^6}{2^{10} \times 7!} f^{(7)}\right) - \left(f'(x) - \frac{h^4}{480} f^{(5)} - \frac{5h^6}{2^4 \times 7!} f^{(7)}\right) + \mathcal{O}(h^8)}{2^4 - 1} , \\ &= f'(x) + f^{(7)} \frac{h^6}{2^6 \times 7!} + \mathcal{O}(h^8) . \end{aligned}$$

3. Question # 3: A function $f(x) = 3x^3 + 12x - 20$ has a root in the interval $[0, 2]$. Now, answer the following:

   (a) (4 marks) Find the approximate root using Interval Bisection Method up to three iterations.
   Solution: Here we have to use the property that if a root of a function $f(x)$ is in the interval $[a, b]$, then we must have $f(a)f(b) < 0$. Using this and the idea of Bisection method, we obtain the following values:

k	$I_k = [a_k, b_k]$	a_k	$m_k = (a_k + b_k)/2$	b_k	$f(a_k)f(m_k)$	$f(m_k)f(b_k)$	$x_\star \in I_{k+1}$
0	$[0, 2]$	0	1	2	$100 > 0$	$-140 < 0$	$[1, 2]$
1	$[1, 2]$	1	1.5	2	$-40.6 < 0$	$227.6 > 0$	$[1, 1.5]$
2	$[1, 1, 5]$	1	1.25	1.5	$-4.29 < 0$	$6.98 > 0$	$[1, 1, 25]$
3	$[1, 1.25]$	1	1.125	1.25	$11.1 > 0$	$-1.92 < 0$	$[1.125, 1.25]$

   Therefore, after three iterations, the approximate root is: $x_\star \approx m_3 = 1.125$.

   (b) (2 marks) If the actual root is $x_\star = 1.2165$, calculate the percent error of the approximate result found in the previous part.
   Solution: The percent error is giben by,

$$\%\,\text{Error} \equiv \left|\frac{\text{Actual Root - Approximate Root}}{\text{Actual Root}}\right| \times 100\% = \left|\frac{1.2165 - 1.125}{1.2165}\right| \times 100\% \approx 7.52\% \;.$$

   (c) (2 marks) If the machine epsilon of the system is 1.6×10^{-8}, how many iterations are needed to find the root.
   Solution: The number of iteration required in Bisection method is given by,

$$n \geq \frac{\log|2-0| - \log\left(1.6 \times 10^{-8}\right)}{\log 2} - 1 \approx 26 \;.$$