Solution: Assignment #9

Consider a set of four data points: $f(-4) = -3$, $f(-2) = -2$, $f(2) = 2$ and $f(4) = 3$. In the following, these data points are to be used to find the best fit polynomial of degree 2 by using Least-Squares method and also by QR-decomposition method.

Problem # 1: Find the best fit polynomial, $p_2(x)$ of the above data points by least=-squares method by answering the following:

1. [2 marks] Write down the matrices: A and b from the given data above.
2. [4 marks] Compute the normal matrix A^TA and A^Tb.
3. [4 marks] Use the results in the previous part to compute the column matrix $x = (a_0\ a_1\ a_2)^T$, where a_0, a_1 and a_2 are the coefficients of the polynomials p_2, and then write the expression of the polynomial p_2.

Problem # 2: We now find the solution by QR-decomposition method using the same four data points given at the top by answering the following:

1. I[1.5 marks] dentify the matrix A and b (Just copy from the previous problem). Now identify the linearly independent column vectors u_1, u_2 and u_3 from the matrix A.
2. [4.5 marks] Using Gram-Schmidt process construct the orthonormal column matrices (or vectors) q_1, q_2 and q_3 from the linearly independent column vectors obtained in the previous part, and then write down the Q matrix.
3. [2 marks] Now calculate the matrix elements of R, and write down the matrix R.
4. [1 mark] Compute Rx and Q^Tb, where $x = (a_0\ a_1\ a_2)$ which are the coefficients of the polynomial p_2.
5. [1 mark] Using the above result, find the values of (a_0, a_1 and a_2), and write the polynomial p_2.

#1

We identify: $x_0 = -4$; $x_1 = -2$; $x_2 = 2$ & $x_3 = 4$.

$\therefore f(x_0) = -3$; $f(x_1) = -2$; $f(x_2) = 2$ & $f(x_3) = 3$.

(1) Therefore:

$$A = \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 2 \\ 3 \end{pmatrix}$$

(2)

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -4 & -2 & 2 & 4 \\ 16 & 4 & 4 & 16 \end{pmatrix} \begin{pmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \Rightarrow A^TA = \begin{pmatrix} 4 & 0 & 40 \\ 0 & 40 & 0 \\ 40 & 0 & 544 \end{pmatrix}.$$

and

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -4 & -2 & 2 & 4 \\ 16 & 4 & 4 & 16 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 2 \\ 3 \end{pmatrix} \Rightarrow A^Tb = \begin{pmatrix} 0 \\ 32 \\ 0 \end{pmatrix}.$$

(3) Now the Augmented form is

$$\left(\begin{array}{ccc|c} 4 & 0 & 40 & 0 \\ 0 & 40 & 0 & 32 \\ 40 & 0 & 544 & 0 \end{array}\right) \xrightarrow[\text{operation}]{\text{2nd row}} \left(\begin{array}{ccc|c} 4 & 0 & 40 & 0 \\ 0 & 40 & 0 & 32 \\ 0 & 0 & 144 & 0 \end{array}\right)$$

[Note 1st row operation is not needed]

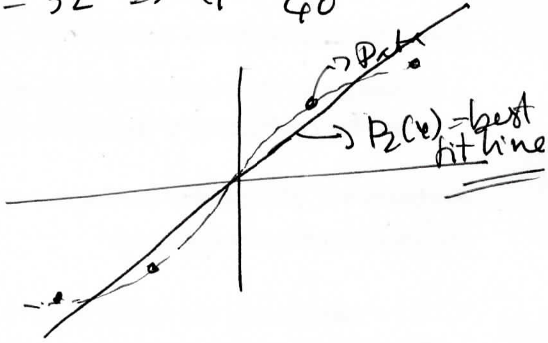
Therefore: we obtain, by Gaussian elimination method:

$144a_2 = 0 \Rightarrow a_2 = 0$; $40a_1 = 32 \Rightarrow a_1 = \frac{32}{40} = 0.80$

and $4a_0 + 40\cancel{a_2}^{\to 0} = 0 \Rightarrow a_0 = 0$

Therefore, the polynomial is:

$$\boxed{p_2(x) = 0.80x}$$



#2 (1) Here: $A = \begin{pmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ -2 \\ 2 \\ 3 \end{pmatrix}$

Therefore: $u_1 = (1\ 1\ 1\ 1)^T$; $u_2 = (-4\ -2\ 2\ 4)^T$
and $u_3 = (16\ 4\ 4\ 16)^T$

(2) $p_1 = u_1 = (1\ 1\ 1\ 1)^T$. Now, $|p_1| = (1+1+1+1)^{1/2} = 2$

$$\Rightarrow \boxed{q_1 = \frac{p_1}{|p_1|} = \frac{1}{2}(1\ 1\ 1\ 1)^T}$$

$$p_2 = u_2 - (u_2^T q_1) q_1 = \begin{bmatrix} -4 \\ -2 \\ 2 \\ 4 \end{bmatrix} - \underbrace{\left[(-4\ -2\ 2\ 4)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\frac{1}{2}\right]}_{=0} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ -2 \\ 2 \\ 4 \end{pmatrix} \Rightarrow |p_2| = (16+4+4+16)^{1/2} = \sqrt{40}$$

$$\therefore \boxed{q_2 = \frac{p_2}{|p_2|} = \frac{1}{\sqrt{40}}\begin{pmatrix} -4 \\ -2 \\ 2 \\ 4 \end{pmatrix}}$$

Now, $p_3 = u_3 - (u_3^T q_1)\, q_1 - (u_3^T q_2)\, q_2$

$$= \begin{pmatrix}16\\4\\4\\16\end{pmatrix} - \left(\frac{1}{2}\right)^2 \underbrace{\left[(16\ \ 4\ \ 4\ \ 16)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right]}_{=40} \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{1}{40}\underbrace{\left[(16\ \ 4\ \ 4\ \ 16)\begin{pmatrix}-4\\-2\\+2\\+4\end{pmatrix}\right]}_{=0}\begin{pmatrix}-4\\-2\\2\\4\end{pmatrix}$$

$$= \begin{pmatrix}16\\4\\4\\16\end{pmatrix} - \frac{1}{4}(40)\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \cancel{\frac{1}{40}(0)\begin{pmatrix}-4\\-2\\2\\4\end{pmatrix}} \to 0 = \begin{pmatrix}6\\-6\\-6\\6\end{pmatrix}$$

Now $|p_3| = (4\times 36)^{1/2} = 2\times 6 = 12$

Therefore: $q_3 = \dfrac{p_3}{|p_3|} = \dfrac{1}{12}\begin{pmatrix}6\\-6\\-6\\6\end{pmatrix} \Rightarrow \boxed{q_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}}$ 4

Hence: The matrix Q is.

$$\boxed{Q = \{q_1 \mid q_2 \mid q_3\} = \begin{pmatrix} \frac12 & -\frac{4}{\sqrt{40}} & \frac12 \\ \frac12 & -\frac{2}{\sqrt{40}} & -\frac12 \\ \frac12 & \frac{2}{\sqrt{40}} & -\frac12 \\ \frac12 & \frac{4}{\sqrt{40}} & \frac12 \end{pmatrix}}$$ 4

(3) Here:

$$\boxed{R = \begin{pmatrix} u_1^Tq_1 & u_1^Tq_2 & u_1^Tq_3 \\ 0 & u_2^Tq_2 & u_2^Tq_3 \\ 0 & 0 & u_3^Tq_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & 12 \end{pmatrix}}$$ 4

Elements

$u_1^Tq_1 = (1\ \ 1\ \ 1\ \ 1)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\frac12 = \frac12(4) = 2$; $u_1^Tq_2 = \frac{1}{\sqrt{40}}(1\ \ 1\ \ 1\ \ 1)\begin{pmatrix}-4\\-2\\2\\4\end{pmatrix} = 0$

$u_1^Tq_3 = \frac12(1\ \ 1\ \ 1\ \ 1)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = 0$; $u_2^Tq_2 = \frac{1}{\sqrt{40}}(-4\ \ -2\ \ 2\ \ 4)\begin{pmatrix}-4\\-2\\2\\4\end{pmatrix} = \sqrt{40}$

$u_2^Tq_3 = \frac12(-4\ \ -2\ \ 2\ \ 4)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = 0$; $u_3^Tq_3 = \frac12 \times (16\ \ 4\ \ 4\ \ 16)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = 12$

(4) Hence: $$Rx = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{40} & 0 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2a_0 \\ \sqrt{40}\, a_1 \\ 12 a_2 \end{pmatrix}$$

and $$Q^T b = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{4}{\sqrt{40}} & -\frac{2}{\sqrt{40}} & \frac{2}{\sqrt{40}} & \frac{4}{\sqrt{40}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{32}{\sqrt{40}} \\ 0 \end{pmatrix}$$

(5) Using the equations: $Rx = Q^T b$, we find

$$\begin{pmatrix} 2a_0 \\ \sqrt{40}\, a_1 \\ 12 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{32}{\sqrt{40}} \\ 0 \end{pmatrix}$$

Comparing: $2a_0 = 0 \Rightarrow a_0 = 0$

$\sqrt{40}\, a_1 = \frac{32}{\sqrt{40}} \Rightarrow a_1 = \frac{32}{40} = 0.80$

and $12 a_2 = 0 \Rightarrow a_2 = 0$.

Hence, the polynomial is:

$$\boxed{p_2(x) = 0.80\, x}$$

same as in Problem-9. (as it should be).