Bisection Method Additional Notes

Complete Algorithm for the Bisection Method

- Step #1: Choose x_l and x_u as two guesses for the root such that $f(x_\ell)f(x_u) < 0$, in other words, *f(x)* changes sign between x_l and x_u.
- Step #2: Estimate the root, x_m of the equation as the mid-point between x_l and x_u as, $x_m = \dfrac{x_\ell + x_u}{2}$
- Step #3: Now check the following
  - If $f(x_\ell)f(x_m) < 0$ then the root lies between x_l and x_m then $x_l = x_l$ and $x_u = x_m$.
  - If $f(x_\ell)f(x_m) > 0$ then the root lies between x_m and x_u then $x_l = x_m$ and $x_u = x_u$.
  - If $f(x_\ell)f(x_m) = 0$ then the root is x_m and stop the iteration.

- Step #4: Find the new estimate of the root $x_m = \dfrac{x_\ell + x_u}{2}$
- Step #5: Find the absolute relative approximate error as

$$\left|\in_a\right| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$

where,

= x_m^{new} estimated root from present iteration

= x_m^{old} estimated root from previous iteration

- Step #6: Compare the absolute relative approximate error $|\in_a|$ with the pre-specified relative error tolerance $\in_s$
- Step #7: If $|\in_a| > \in_s$ then go to Step 3, else stop the algorithm.
- Note: one should also check whether the number of iterations is more than the maximum number of iterations allowed.
- If so, one needs to terminate the algorithm and notify the user about it.

An Exercise

- A ceramic company that makes floats for commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth x to which the ball is submerged when floating in water.

- The equation that gives the depth to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

- Use the bisection method of finding roots of equations to find the depth to which the ball is submerged under water.
- Conduct three iterations to estimate the root of the above equation.
- Find the absolute relative approximate error at the end of each iteration.

Boundary of the Solution

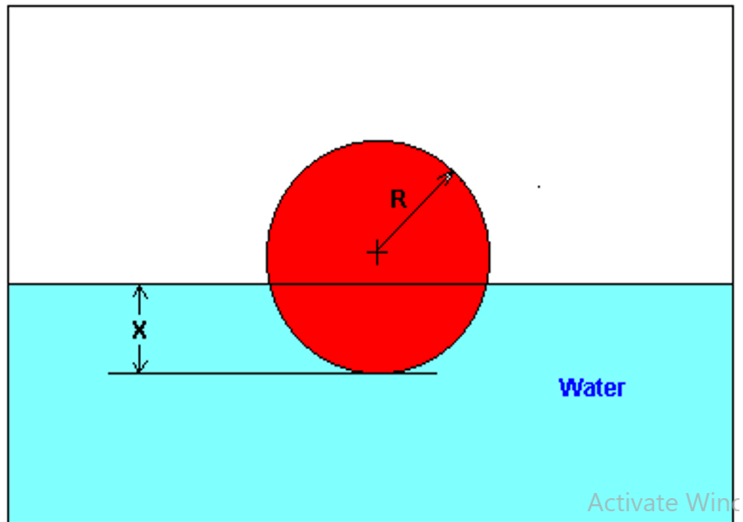
- From the physics of the problem, the ball would be submerged between x= 0 and x=2R

where, R = radius of the ball

that is,

$0 \leq x \leq 2R$ or

$0 \leq x \leq 0.11$



Test for the boundaries of the root

- Lets us assume, $x_\ell = 0, x_u = 0.11$
- Check if the function changes sign between x_l and x_u.

$$f(x_\ell) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$
$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

- Hence,

$$f(x_\ell)f(x_u) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

- So there is at least one root between x_l and x_u that is between 0 and 0.11.

Iteration 1

- The estimate of the root is x_m=(0+0.11)/2=0.055

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$
$$f(x_\ell)f(x_m) = f(0)f(0.055) = (3.993\times10^{-4})(6.655\times10^{-4}) > 0$$

- Hence the root is bracketed between x_m and x_u that is between 0.055 and 0.11. So, the lower and upper limit of the new bracket is x_l = 0.055 and x_u = 0.11
- At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation

Iteration 2

- Next estimate of the root is, $x_m = \dfrac{x_\ell + x_u}{2}$ =0.082

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times10^{-4} = -1.622\times10^{-4}$$

$$f(x_\ell)f(x_m) = f(0.055)f(0.0825) = \left(6.655\times10^{-5}\right)\times\left(-1.622\times10^{-4}\right) < 0$$

- Hence, the root is bracketed between x_l and x_m that is, between 0.055 and 0.0825. So, the lower and upper limit of the new bracket is x_l=0.055 and x_u=0.0825

Iteration 2 (continued)

- The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$

- $|\in_a|$ =33.33%
- Let us assume that acceptable error is less than 5%. But because the absolute relative approximate error after 2nd iteration is greater than 5%, so the error is not acceptable.

Iteration 3

- x_m=0.06875

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993\times10^{-4} = -5.563\times10^{-5}$$

$$f(x_\ell)f(x_m) = f(0.055)f(0.06875) = (6.655\times10^{5})\times(-5.563\times10^{-5}) < 0$$

- Hence, the root is bracketed between and , that is, between 0.055 and 0.06875. So the lower and upper limit of the new bracket is x_l = 0.055 and x_u = 0.06875
- The absolute relative approximate error $|\in_a|$ at the ends of Iteration 3 is 20%
- Still the absolute relative approximate error is greater than 5%

Convergence after ten iterations

Table 1 Root of as function of number of iterations for bisection method.

Iterations	x_l	x_u	x_m	% error	$f(x_m)$
1	0.00000	0.11	0.055	----------	$6.655X10^{-5}$
2	0.055	0.11	0.0825	33.33	$-1.622X10^{-4}$
3	0.055	0.0825	0.06875	20.00	$-5.5.63X10^{-5}$
4	0.055	0.06875	0.06188	11.11	$4.484X10^{-6}$
5	0.06188	0.06875	0.06531	5.263	$-2.593X10^{-5}$
6	0.06188	0.06531	0.06359	2.702	$-1.080X10^{-5}$
7	0.06188	0.06359	0.06273	1.370	$-3.176X10^{-6}$
8	0.06188	0.06273	0.0623	0.6897	$6.497X10^{-7}$
9	0.0623	0.06273	0.06252	0.3436	$-1.265X10^{-6}$
10	0.0623	0.06252	0.06241	0.1721	$-3.077X10^{-7}$

Advantages of bisection method

- Since the method brackets the root, the method is guaranteed to converge.
- As iterations are conducted, the interval gets halved. So one can guarantee the error in the solution of the equation.

Drawbacks of bisection method

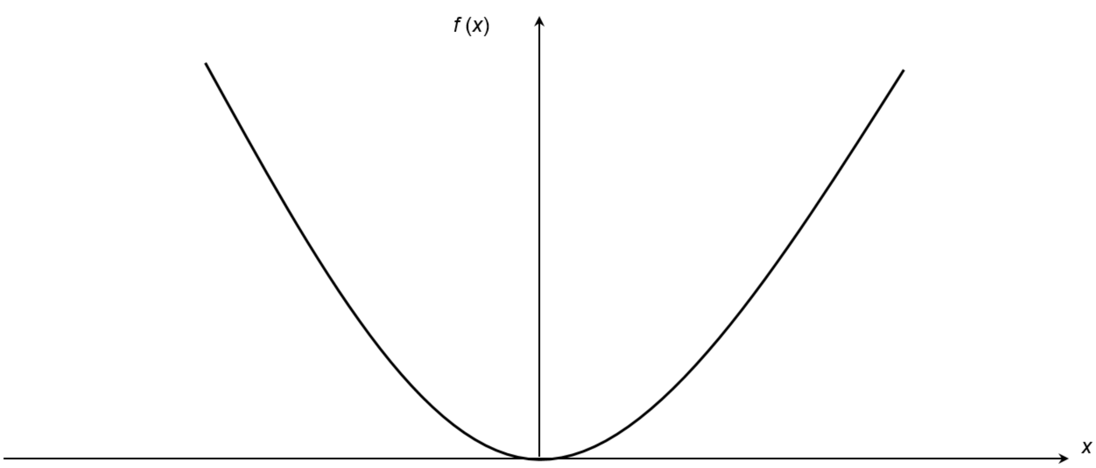
- The convergence of the bisection method is slow as it is simply based on halving the interval.
- If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.
- If a function is such that it just touches the x-axis (Figure 6) such as

$$f(x) = x^2 = 0$$

it will be unable to find the lower guess, x_l, and upper guess, x_u, such that

$$f(x_\ell)f(x_u) < 0$$

Figure 6 : The equation $f(x) = x^2 = 0$ has a single root and that cannot be bracketed



Drawbacks of bisection method

- A singularity in a function is defined as a point where the function becomes infinite.

- For functions where there is a singularity and it reverses sign at the singularity, the bisection method may not converge on the singularity (Figure 7). An example includes

$$f(x) = \frac{1}{x}$$

where x_l=-2 and x_u=3 are valid initial guesses which satisfy

$$f(x_\ell)f(x_u) < 0$$

- However, the function is not continuous and the theorem that a root exists is also not applicable.