Solution: Graded Quiz-9
(Bonus Quiz)

Q#1: Which of the following statements is/are correct?

- The columns of the transformation matrix A needs to be linearly independent.
- A over-determined system is a linear system where the number of conditions to be satisfied must not be less than the number of variable needed to express the system.
- The least-square method is applicable for over-determined system.
- ✓ All of the above. (Answer)

Q#2: Consider two vectors $u_1, u_2 \in \mathbb{R}^2$ with $u_1 = (1,0)$ and $u_2 = (1,1)$ and a_1 and a_2 are scalars or numbers. Which of the following statements about u_1, u_2 is/are correct?

- $|u_i| = 1$ for $i = 1, 2$ ✗
- $u_1^T u_2 = 0$ ✗
- ✓ $a_1 u_1 + a_2 u_2 = 0$ if and only if $a_1 = a_2 = 0$
- All of the above. ✗

$a_1 u_1 + a_2 u_2 = 0$
$\Rightarrow ((a+b), (b)) = 0 = (0, 0)$
$\therefore b = 0$ & $a = -b = 0$

Q#3: In a normal equation/system it is found that $\det(A^T A) = 1$, where A is the $m \times n$ transformation matrix that represent a linear system $Ax = b$ This implies that

- $A^T = A$ ✗
- $\det(A) = \pm 1$ ✗
- $\det(A) = \det(A^T)$ ✗
- ✓ None of the above. (Answer)

A is $m \times n$, not a square matrix if $m > n$. These are not correct in general.

Q#4: An overdetermined linear system three variables that need to satisfy four equations. The solution of the system by least-square method is a

- ◯ degree one polynomial. ✗
- ◯ degree four polynomial. ✗
- ✓ degree two polynomial. $n=2$ polynomial $\Rightarrow a_0, a_1, a_2 \Rightarrow$ Three unknown parameter.
- ◯ There is no solutions.

Q#5: Consider a set of three orthonormal vectors $S = \{\hat{i}, \hat{j}, \hat{k}\} \in \mathrm{R}^3$. Let us now define another set $S' = \{u_1, u_2, u_3\}$such that $u_1 = \hat{i} + \hat{j}$ $u_2 = \hat{j} + \hat{k}$and $u_3 = \hat{i} + \hat{k}$The set S' is now

- ✓ a linearly independent set of vectors.
- ◯ an orthogonal set of vectors. ✗
- ◯ a linearly dependent set of vectors.
- ◯ an orthonormal set of vectors. ✗

$c_1u_1 + c_2u_2 + c_3u_3 = 0$

$\Rightarrow c_1(1,1,0) + c_2(0,1,1) + c_3(1,0,1) = 0$

$\Rightarrow c_1 + c_2 = 0$; $c_2 + c_3 = 0$; $c_1 + c_3 = 0$

$\Rightarrow c_1 = -c_2$ $\quad$ $c_2 = -c_3$ $\quad$ $\Rightarrow c_1 + c_1 = 0$

$\therefore c_3 = c_1$ $\quad$ $\Rightarrow c_1 = 0$, $\therefore c_2 = 0$, $c_3 = 0$ $\Rightarrow$ linearly independent.

Q#6: Which of the following statement(s) about the QRdecomposition method is(are) true?

- ◯ The matrices $A$and Q must be of same orders. ✓
- ◯ The matrix Q must be a set of orthonormal vectors. ✓
- ◯ The matrix Rmust be an square matrix. ✓
- ✓ All of the above. (Answer)

Q#7: The matrix A in the QR decomposition method is of order $m \times n$ with $m \geq n$ and the hence the solution can of degree-$(n-1)$ polynomial. But if $m < n$ rather than $m > n$, then there will

- ◯ exist a solution of degree-$(m-1)$ polynomial. ✗
- ◯ still exist a solution of degree-$(n-1)$ polynomial. ✗
- ✓ be infinitely many solutions. {because there will be undetermined variable left over.
- ◯ It cannot be determined. ✗

Q#8: Which of the following set of vectors form(s) an orthonormal set?

- ◯ $\left\{\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\right\}$ ✓
- ◯ $\{(1,0),(0,1)\}$ ✓
- ◯ $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(1,-2,1)\right\}$ ✓
- ✓ All of the above. (Answer)

Check for each ~~At~~ option that. $u_1 \cdot u_2 = 0$ & $|u_1| = |u_2| = 1$

You have used 0 of 1 attempt

Consider the four data points: $f(-1) = 0$, $f(1) = 2$, $f(3) = 3$ and $f(5) = 4$. Now fit the least-squares straight line to these data by answering the following two questions:

Question-9: [1 Mark] Compute the 'normal matrix' of the system.

Question-10: [1 Mark] Find the equation of the straight line that fits the data using the normal matrix found in the previous question (that is find the coefficients a_0 and a_1, and then write the expression for $p_1(x)$).

Problem Solving

Here, we identify: $x_0 = -1 \Rightarrow f(x_0) = 0$

$x_1 = 1 \Rightarrow f(x_1) = 2$

$x_2 = 3 \Rightarrow f(x_2) = 3$

$x_4 = 5 \Rightarrow f(x_4) = 4$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix}$$

and $$b = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

#9 $$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 8 & 36 \end{pmatrix}$$

and $$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 31 \end{pmatrix}$$

Therefore the "normal" matrix (equation) of the system is

$$\boxed{\begin{pmatrix} 4 & 8 \\ 8 & 36 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 9 \\ 31 \end{pmatrix}}$$ Ans

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#10 The Augmented form of the normal equation is

$$\left(\begin{array}{cc|c} 4 & 8 & 9 \\ 8 & 36 & 31 \end{array}\right) \xrightarrow[\text{operation}]{\text{1st row}} \left(\begin{array}{cc|c} 4 & 8 & 9 \\ 0 & 20 & 13 \end{array}\right)$$

Therefore: $20a_1 = 13 \Rightarrow \boxed{a_1 = \frac{13}{20}}$

and $4a_0 + 8a_1 = 9 \Rightarrow a_0 = \frac{1}{4}(9 - 8a_1) = \frac{1}{4}\left(9 - 8 \times \frac{13}{20}\right)$

$\Rightarrow \boxed{a_0 = \frac{19}{20}}$

Therefore, $\boxed{P_1(x) = \frac{19}{20} + \frac{13}{20}x} \Rightarrow$ Solution of the system.
