Solution: Assignment #8

A linear system is described by the following equations

$$4x_1 - x_2 + x3 = 8$$
$$2x_1 + 5x_2 + 2x_3 = 3$$
$$x_1 + 2x_2 + 4x_3 = 11$$

Answer the following questions (1-5):

Questions-1: [2 Mark] Does this system has any unique solution? Explain or show calculation.

Question-2: [6 Marks] Solve the above linear system by Gaussian elimination method.

Now solve the same linear system above by the *LU*-decomposition method:

Question-3: [4 Marks] Construct the matrices $F^{(1)}$ and $F^{(2)}$.

Question-4: [4 Marks] Find the lower triangular matrix *L*.

Question-5: [4 Marks] Now find the solution of the linear system again using the matrix *L* found in the previous question. Is your solution agree with answer found in Question-2?

Let, rearrange for simplicity $\Rightarrow$
$$x_1 + 2x_2 + 4x_3 = 11 \rightarrow ①$$
$$2x_1 + 5x_2 + 2x_3 = 3 \rightarrow ②$$
$$4x_1 - x_2 + x_3 = 8 \rightarrow ③$$

Q#1 : Here: $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 2 \\ 4 & -1 & 1 \end{pmatrix} \Rightarrow \det A = 1(5+2) - 2(2-8) + 4(-2-20)$
$= 7 + 12 - \cancel{}88 \neq 0 \Rightarrow$ solution exists

Q#2: Here: $Aug(A) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 11 \\ 2 & 5 & 2 & 3 \\ 4 & -1 & 1 & 8 \end{array}\right)$; $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$

$m_{31} = \frac{a_{31}}{a_{11}} = \frac{4}{1} = 4$

$\therefore r_2 \to r_2' = r_2 - m_{21} r_1 = (2 \;\; 5 \;\; 2 \mid 3) - 2(1 \;\; 2 \;\; 4 \mid 11)$
$= (0 \;\; 1 \;\; -6 \mid -19)$

$r_3 \to r_3' = r_3 - m_{31} r_1 = (4 \;\; -1 \;\; 1 \;\; 8) - 4(1 \;\; 2 \;\; 4 \;\; 11)$
$= (0 \;\; -9 \;\; -15 \;\; -36)$

$\therefore Aug(A) \xrightarrow[\text{operation}]{\text{1st row}} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 11 \\ 0 & 1 & -6 & -19 \\ 0 & -9 & -15 & -36 \end{array}\right) \equiv A'$

For 2nd row operation. $m_{32} = \frac{a_{32}}{a_{22}} = \frac{-9}{1} = -9$

For 2nd row operation: $r_3 \to r_3' = r_3 - m_{32} r_2 = (0 \;\; -9 \;\; -15 \;\; -36) + 9(0 \;\; 1 \;\; -6 \;\; -19)$
$= (0 \;\; 0 \;\; -69 \;\; -207)$

So, $\cancel{Aug(A)}\, A' \xrightarrow[\text{operation}]{\text{2nd row}} \begin{pmatrix} 1 & 2 & 4 & 11 \\ 0 & 1 & -6 & -19 \\ 0 & 0 & -69 & -207 \end{pmatrix}$

In matrix form:

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -69 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -19 \\ -207 \end{pmatrix} \Rightarrow x_3 = \frac{-207}{-69} \Rightarrow \boxed{x_3 = 3}$$

Also, $x_2 - 6x_3 = -19 \Rightarrow x_2 = 6 \times 3 - 19 = 18 - 19 \Rightarrow \boxed{x_2 = -1}$

And $x_1 + 2x_2 + 4x_3 = 11 \Rightarrow x_1 = -2(-1) - 4(3) + 11 \Rightarrow \boxed{x_1 = 1}$

So, the solution is: $\boxed{x_1 = 1 \; ; \; x_2 = -1 \text{ and } x_3 = 3}$ LK

Q#3: We copy the row multipliers from the previous part.

$m_{21} = 2$; $m_{31} = 4$ and $m_{32} = -9$.

$$\therefore F^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} . \text{LK}$$

$$F^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 0 \end{pmatrix} . \text{LK}$$

Q#4: $L = (F^{(1)})^{-1}(F^{(2)})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{pmatrix} \Rightarrow \boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -9 & 1 \end{pmatrix}}$ LK

Q#5 From Q-1 $\Rightarrow U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -69 \end{pmatrix}$ & from Q-4 $\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -9 & 1 \end{pmatrix}$

By LU-decomposition we have: $LY = b \Rightarrow y$? & $Ux = y \Rightarrow \underline{x = ??}$.

Now, $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -9 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 8 \end{pmatrix} \Rightarrow \begin{cases} y_1 = 11. \\ 2y_1 + y_2 = 3 \Rightarrow y_2 = 3 - 22 = -19. \\ 4y_1 - 9y_2 + y_3 = 8 \Rightarrow y_3 = 8 - 44 - 171 = -207. \end{cases}$

Therefore:

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -69 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ -19 \\ -207 \end{pmatrix} \Rightarrow \begin{cases} -69x_3 = -207 \Rightarrow \underline{x_3 = 3}. \\ x_2 - 6x_3 = -19 \Rightarrow x_2 = -19 + 18 \Rightarrow \underline{x_2 = -1}. \\ x_1 + 2x_2 + 4x_1 = 11 \Rightarrow x_1 = 11 + 2 - 4(3) \Rightarrow \underline{x_1 = 1} \end{cases}$$

The solution is: $x_1 = 1$; $x_2 = -1$ and $x_3 = 3$ exactly same as Question-2. LK