Solutions: Grade Quiz # 5
Course: CSE 330
Summer 2021 Semester

---

1. Q#1: How many roots of $\cos(x)$ are there in the interval $[0, 42]$? **Answer: 13**
Solution: Roots of $\cos(x)$ within the interval $[0, 42]$ implies to find and integer n such that

$$\cos(2n-1)\pi/2 \leq 42 \Longrightarrow n \leq \frac{42}{\pi} + \frac{1}{2} = 13.8 \ .$$

Since n must be an integer and less than 143.8, we must have: $n = 13$.

2. Q#2: A function $f(x)$ has only one root in the interval from $a = 0.3$ to $b = 0.4$. And it is also given that $f(a) = 1.4$ and $f(b) = -1.1$. If $f(c) > 0$, where $c = (a+b)/2$, in which of the following sub-interval does the root exist? **Answer:** $[0.35, 0.4]$**.**
Solution: If a root exists in an interval $[p, q]$, then we must have $f(p)f(q) < 0$. Here, we have: $f(a) > 0$, $f(b) < 0$ and $f(c) < 0$ with $c \in [a, b]$. Since $f(c)f(b) = f(0.35)f(0.4) < 0$, the root must be in the sub-interval $[0.35, 0.4]$.

3. Q#3: In the interval $[-5, -4.5]$ there lies only one root of a given polynomial. What is the number of iterations needed to successfully find the root using the interval bisection method if given that machine epsilon $= 5.96 \times 10^{-8}$? **Answer: 22**.
Solution: No. of Iteration $= \dfrac{\log 0.5 - \log(5.96 \times 10^{-8})}{\log 2} - 1 = 22$.

4. Q#4: In an interval $[5, 6.5]$, it takes 63 iterations to find the root using the bisection method. What is a possible value of the machine epsilon? **Answer:** 8.1×10^{-20}.
Solution: For the bisection method, the machine epsilon is given by

$$\epsilon_M \equiv \delta \geq \frac{|b-a|}{2^{n+1}} = \frac{1.5}{2^{64}} = 8.1 \times 10^{-20} \ .$$

5. Q#5: Which of the following is/are not a non-linear function(s)? **Answer:** $f(x) = 3x + 7$**.**
Solution: Not a non-linear function means a linear function or straight line equation which is $f(x) = 3x + 7$.

6. Q#6: Let $f(x) = e^x \cos(0.5x)$. Find the value of $f'(1)$ with $h = 0.1$ using the central difference method. **Answer: 1.7319**.
Solution: the answer must be computed in 'radian' modes because of the cosine function. Therefore, we get,

$$f'(1) = \frac{e^{1.1}\cos(0.5 \times 1.1) - e^{0.9}\cos(0.5 \times 0.9)}{2 \times 0.1} = 1.7319 \ .$$

7. Q#7: Let $f(x) = e^x \cos(0.5x)$. Find the value of $f'(1)$ with $h = 0.05$ using the central difference method.**Answer: 1.7334**.
Solution: the answer must be computed in 'radian' modes because of the cosine function. Therefore, we get,

$$f'(1) = \frac{e^{1.05}\cos(0.5 \times 1.05) - e^{0.9}\cos(0.5 \times 0.95)}{2 \times 0.95} = 1.7334 \ .$$

8. Q#8: Let $f(x) = e^x \cos(0.5x)$. Now, compute $D_h^{(1)}$ using the Richardson Extrapolation formula with $h = 0.1$ and $h/2 = 0.05$? (If you are confident, you may also use the results found in the previous two questions instead of recalculating). **Answer: 1.7339**.
Solution: By the Richardson extrapolation formula, we find,

$$D_{0.1}^{(1)} = \frac{2^2 D_{0.05} - D_{0.1}}{3} = \frac{4 \times 1.7334 - 1.7319}{3} = 1.7339 \ .$$

---

9. Question #9 :Read the following question carefully. The question is long, but the answer is very short and can be completed in three lines only.

In Richardson extrapolation method, starting from the expressions

$$D_h \equiv \frac{f(x+h) - f(x-h)}{2h}$$

and using Taylor expansion and replacing h by $h/2$ in D_h, we derived the following formula

$$D_h^{(1)} \equiv \frac{2^2 D_{h/2} - D_h}{3} = f'(x) - \frac{h^4}{480} f^{(5)}(x) + \mathcal{O}(h^6) \ ,$$

which is 4th-order accurate. This is done is lecture note and also in the video lecture. Here in this question we will repeat the above procedure by replacing the parameter h by $h/3$ (instead of $h/2$), and find the formula for $D_h^{(1)}$. To do so follow the following steps:

(a) (1 mark) Write down the Taylor expanded forms of D_h and $D_{h/3}$ upto order of $\mathcal{O}(h^8)$.
Solution: Using the Central-difference formula and the Taylor expansion for $h << x$, we find:

$$\begin{aligned} D_h &= f'(x) + \frac{h^2}{6} f'''(x) + \frac{h^4}{120} f^{(5)}(x) + \frac{h^6}{7!} f^{(7)}(x) + \mathcal{O}(h^8) \ , \\ \text{and } D_{h/3} &= f'(x) + \frac{h^2}{3^2 \times 6} f'''(x) + \frac{h^4}{3^4 \times 120} f^{(5)}(x) + \frac{h^6}{3^6 \times 7!} f^{(7)}(x) + \mathcal{O}(h^8) \ . \end{aligned}$$

(b) (1 mark) Now define $D_h^{(1)}$ in terms of D_h and $D_{h/3}$ and use the expressions in the previous part to simplify. Is your result 4th-order accurate? Your answer should be in the following form

$$D_h^{(1)} = f'(x) + \text{New term} + \mathcal{O}(\text{Powers of } h) \ ,$$

and you should find an algebraic expression for the 'New term' and indicate the power of h in the last term, like the expression as in the right-hand side of the second equation above.
Solution: Since we are using $h/3$ instead of $h/2$, the numerical first derivative needs to be defined by,

$$D_h^{(1)} \equiv \frac{3^2 D_{h/3} - D_h}{3^2 - 1} = \frac{9 D_{h/3} - D_h}{8} = f'(x) - \frac{h^4}{1080} f^{(5)}(x) - \frac{10h^6}{81 \times 7!} f^{(7)}(x) + \mathcal{O}(h^8) \ .$$

Yes. It is 4-th order accurate.