Lab 2: Mine Crafting

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I. Introduction

When tackling any project, it is important to determine whether the plan in place is feasible. The goal of this project is to determine the true depth of our four kilometer mine shaft by dropping a one kilogram test mass in it and measuring how long it takes for it to reach the bottom. To understand how plausible this goal is, we theoretically synthesized different conditions for this test mass to undergo, slowly building up to the most realistic situation. We considered drag, gravitational, and coriolis forces, as well as varying densities and uniformity of the Earth. In this report, we will discuss the fall times calculated in different situations, the factors that impact this time, and whether or not this can actually be completed.

II. Calculation of Fall Time

A projectile facing a constant gravitational force and a drag force can be described with Equation 1.

$$\frac{d^2y}{dt^2} = -g + \alpha\left|\frac{dy}{dt}\right|^{\gamma} \quad [1]$$

Where g is the acceleration due to gravity, α is the drag coefficient, and γ is the speed dependence of drag.

For the first fall time calculation, we considered the ideal case: no drag and constant gravity. In this situation, kinematics can be used to determine a theoretical fall time using Equation 2 below.

$$t = \sqrt{\frac{2\Delta x}{a}} \quad [2]$$

Where Δx and a are displacement and acceleration, respectively. Theoretical fall time was found to be 28.6 seconds. When computing this number experimentally by solving Equation 1, assuming constant g (Equation 6), the fall time was found to agree with the theoretical time.

The second case considered was linear gravity and no drag. Assuming the Earth's mass is homogeneous throughout, the gravitational constant will depend on the distance from the center of the Earth as described in Equation 3.

$$g(r) = g_0\left(\frac{r}{R_{Earth}}\right) \quad [3]$$

Where g_0 is the gravity at the surface and r is the distance from Earth's center.

Plugging this into Equation 1 and still ignoring drag, we calculated that the mass reaches the bottom at the same time the theoretical and ideal case did. However, if we look to the later decimal places past what is necessary, a linear gravity does slightly increase the fall time, however, it is not significant.

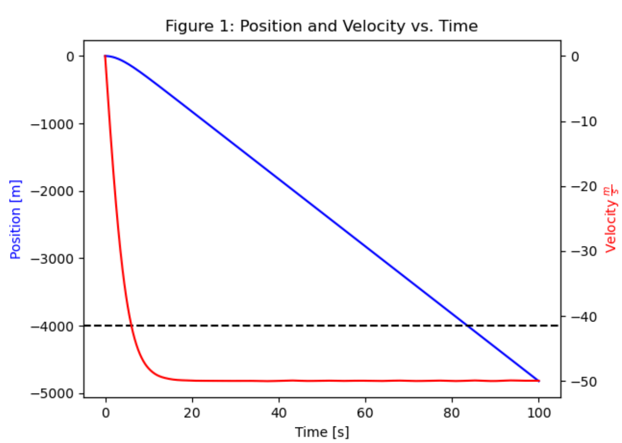

Figure 1: Plot of the position and velocity as a function of time assuming linear gravity and drag. The dashed line is the bottom of the mine shaft.

In the final situation the second case was reconsidered with drag now included, as shown in Figure 1. It can be seen that velocity decreases with depth and eventually levels off at terminal velocity (fifty meters per second). To do involve drag, the drag coefficient must be calculated using Equation 1 and making a few assumptions, which will be specified in later sections. This new free fall time was found to be 83.5 seconds. This calculation is different from the previous because the second term of the differential is now included. Adding drag increases the fall time by about 2.9 times due to the fact that there is a force pushing upward on the object as it falls.

III. Feasibility of the Depth Measurement Approach

Due to the Earth's rotation, the test mass will face a Coriolis force as it falls causing the object to hit the walls of the shaft, assuming it is not very wide. It is important, then, to understand if the test mass is more likely to hit the side of the mine or the bottom first. If it hits the side first, the value found for the total fall time will not be as accurate and the test mass will be fragmented. Including drag force adds an additional term to Equation 1, resulting in the new equation below.

$$\frac{d^2y}{dt^2} = -g(r) - 2\Omega v_x \qquad [4]$$

Where Ω is the Earth's rotation rate at the Equation and V_x is the x-component of velocity. By solving this differential equation and imposing the initial conditions, it was found that the test mass hits the bottom at 28.6 seconds. By completing that same calculation with different initial conditions we can determine when the test mass will hit the side of the wall, which occurs at 21.9 seconds and at a depth of 2352.9 meters, meaning the test mass will bump the wall before it reaches the bottom.

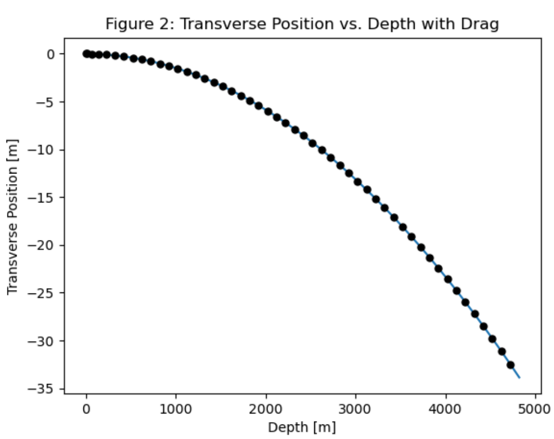

Figure 2: Plot of transverse position as a function of depth with drag included. When there is more mass interior to the object, it moves a smaller distance in three seconds than when there is less mass interior, as shown when observing the distance between the dots at depth 1000 meters and depth 4000 meters.

Adding drag to this situation keeps the same definitive result but changes the times at which these events occur. With drag included, the object hits the side of the shaft at 29.6 seconds and the bottom at 83.5 seconds. It will also hit the side of the wall at an earlier depth, about 1302.9 seconds. The effects of drag are shown in Figure 2, the object travels less distance every two seconds. With these results, it is not recommended to proceed with this method of depth measurement as the coriolis force prevents the mass from hitting the bottom in one piece.

IV. Calculation of Crossing Times for Homogenous and Non-Homogenous Earth

Homogeneity affects the crossing time of the test mass, specifically due to how uniform/non-uniform the Earth is. Density relies on Equation 5 where ρ_n is a normalization constant.

$$\rho(r) = \rho_n\left(1 - \frac{r^2}{R_{Earth}}\right)^n \qquad [5]$$

This equation shows that when n is anything but zero the density is no longer uniform. Mass then depends on density with the equation $M = \int \rho(r)r^2 dr$. Using this relationship and Equation 5 and Equation 6 as shown below, where G is the gravitational constant, M is the mass of the object and R is the radius of the object,

$$a = \frac{GM}{R^2} \qquad [6]$$

The relationship between density and acceleration can be found. The acceleration an object undergoes decides the crossing time. In the case of uniform density, $n = 0$, the crossing time to the center of the Earth was found to be 1267.3 seconds. In the most extreme case with most of the mass at the center, $n = 9$, the crossing time was calculated as 943.8 seconds. The implications of these various n are shown in Figure 3.

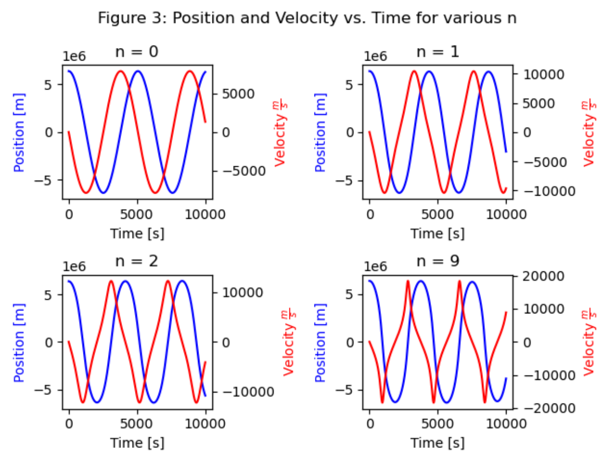

Figure 3: Plot of position and velocity as a function of time for varying n (changing density). Shows that density impacts the speed at which the object oscillates.

This is because when most of the mass is at the center, the object is accelerating for longer due to more mass being interior to the object for a longer time. Similar crossing times can be found on different bodies, such as the Moon. Using equations 5 and 6, we can determine the trans-Moon crossing time to be 3250.2 seconds, which is longer than that of the Earth due to the fact that it is less dense.

V. Discussion and Future Work

As demonstrated through this report, it is understood that this type of project is implausible. The test mass would hit the side of the wall before it reaches the bottom and we cannot accurately calculate the vertical depth of the mine shaft. Throughout this project, many assumptions were made. In the ideal case, acceleration was assumed to be constant despite the changing distance from the center of the Earth. We also assumed acceleration could be found with Equation 6 rather than using the known value of 9.81 on Earth. There were several times in each section where we assumed that drag could be ignored to understand the underlying situation before making it more realistic. When calculating the fall time with linear gravity and no drag/drag, terminal velocity was also assumed to be the terminal velocity a skydiver reaches ($50 \frac{m}{s}$). Lastly, when drag was included we assumed gamma to be two. It is also important to understand that as an object moves through the Earth, the density changes as much as 40% moving from one layer to the next, showing how inaccurate assuming uniform density is. While this project was not possible with the assumptions we initially made, in the future, we can make our calculations more accurate by assuming a non-spherical Earth. The Earth is not a perfect sphere due to the bulging it experiences because of the tidal forces from the Moon. This affects the acceleration the object faces because the equations used throughout this lab assume a spherical mass.