Bandwidth Selection of Kernel Estimator

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1 Kernel estimator

The bandwidth selection follows the lecture notes by Sun (2013).

If f is smooth in a small neighborhood $[x - h/2, x + h/2]$ of x, we can ustify the following approximation: $hf(x) \approx \int_{x-h/2}^{x+h/2} f(u)du$ by the mean theorem. The estimator of $f(x)$ is given by $\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} \{X_i \in (x - h/2, x + h/2)\}$

The kernel estimator is therefore

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K(\frac{x - X_i}{h}),$$

where $K(\cdot)$ is a kernel function with the assumptions. The example kernel functions can be found in Hansen's note on non-parametric estimation Hansen (2009).

Hansen, B. E. (2009). Lecture Notes on Nonparametrics. Retrieved from `https://www.ssc.wisc.edu/~bhansen/718/NonParametrics1.pdf`.

Higher-order kernels are obtained by multiplying a second-order kernel by an $(\nu/2 - 1)$-th order polynomial in u: Explicit formulae for the general polynomial family can be found in B. Hansen (Econometric Theory, 2005), and for the Gaussian family inWand and Schucany (Canadian Journal of Statistics, 1990).

Table 1: Kernel Functions

Kernel	$K(u)$	$R(k)$	$\kappa_2(k)$	$eff(k)$
Uniform	$\frac{1}{2}\mathbb{1}\{\lvert u\rvert \le 1\}$	1/2	1/3	1.0758
Triangle	$(1-\lvert u\rvert)\mathbb{1}\{\lvert u\rvert \le 1\}$	-	-	-
Epanechnikov	$\frac{3}{4}(1-u^2)\mathbb{1}\{\lvert u\rvert \le 1\}$	3/5	1/5	1.0
Quartic(Biweight)	$\frac{15}{16}(1-u^2)^2\mathbb{1}\{\lvert u\rvert \le 1\}$	5/7	1/7	1.0061
Triweight	$\frac{35}{32}(1-u^2)^3\mathbb{1}\{\lvert u\rvert \le 1\}$	350 / 429	1/9	1.0135
Gaussian	$\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$	$1/\sqrt{2\pi}$	1	1.0513
Cosinus	$\frac{\pi}{4}\cos(\frac{\pi}{2}u)\mathbb{1}\{\lvert u\rvert \le 1\}$	-	-	-

Table 2: Fourth Order Kernel Functions

Kernel	$K(u)$	$R(k)$	$\kappa_2(k)$	$eff(k)$
Epanechnikov	$\frac{15}{8}\frac{3}{4}(1-\frac{7}{3}u^2)(1-u^2)\mathbb{1}\{\lvert u\rvert \le 1\}$	5/4	-1/21	1.00
Biweight	$\frac{7}{4}\frac{15}{16}(1-3*u^2)(1-u^2)^2\mathbb{1}\{\lvert u\rvert \le 1\}$	805/572	-1/33	1.0056
Triweight	$\frac{27}{16}\frac{35}{32}(1-11/3u^2)(1-u^2)^3\mathbb{1}\{\lvert u\rvert \le 1\}$	3780 / 2431	-3/143	1.0134

2 Bandwidth Selection

2.1 Estimation Bias

The bias of a kernel density of order ν estimator is

$$Bias(\hat{f}(x)) = E\hat{f}(x) - f(x) = \frac{1}{\nu!}f^{(\nu)}(x)h^{\nu}\kappa_{\nu}(k) + o(h^{\nu}).$$

2.2 Estimation Variance

The kernel estimator is a linear estimator, and $\kappa(\frac{X_i - x}{u})$ is i.i.d, then

$$\begin{aligned} Var(\hat{f}(x)) &= \frac{1}{nh^2}Ek\left(\frac{X_i - x}{h}\right)^2 - \frac{1}{n}\left(\frac{1}{h}Ek\left(\frac{X_i - x}{h}\right)\right)^2 \\ &= \frac{f(x)R(k)}{nh} + O(1/n) \end{aligned}$$

2.3 Mean squared error

The measure of precision is the mean squared error

$$\begin{aligned} AMSE(\hat{f}(x)) &= E(\hat{f}(x) - f(x))^2 \\ &= \frac{\kappa_{\nu}^2(k)}{(\nu!)^2}f^{(\nu)}(x)^2h^{2\nu} + \frac{f(x)R(k)}{nh} \end{aligned}$$

Global measure of precision is the asymptotic mean integrated squared error(AMISE):

$$\begin{aligned} AMISE(\hat{f}(x)) &= \int_{-\inf}^{\inf} AMSE(\hat{f}(x))dx \\ &= \frac{\kappa_\nu^2(k)}{(\nu!)^2} R(f^{(\nu)}) h^{2\nu} + \frac{f(x)R(k)}{nh} \end{aligned}$$

2.4 Asymptotically optimal bandwidth

2.5 Asymptotically optimal kernel

2.6 Silverman Rule-of-thumb bandwidth selection

$$\begin{aligned} h &= \hat{\sigma} C_\nu(k) n^{-1/(2\nu+1)} \\ &= 2\hat{\sigma} n^{-1/(2\nu+1)} \left(\frac{\pi^{1/2}(\nu!)^3 R(k)}{2\nu(2\nu)!\kappa_\nu^2(k)} \right)^{1/(2\nu+1)} \end{aligned}$$

The rule of thumb constant constant can be looked up in the table.

Table 3: Rule of thumb constant $C_\nu(k)$ for single-variate

Kernel	$\nu = 2$	$\nu = 4$	$\nu = 6$
Epanechnikov	2.34	3.03	3.53
Biweight	2.78	3.39	3.84
Triweight	3.15	3.73	4.13
Gaussian	1.06	1.08	1.08

References

Hansen, B. E. (2009). Lecture Notes on Nonparametrics. Technical report.

Sun, Y. (2013). *Nonparametric and Semiparametric Econometrics*. Number August.