

Bandwidth Selection of Kernel Estimator

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1 Kernel estimator

The bandwidth selection follows the lecture notes by Sun (2013).

If f is smooth in a small neighborhood $[x - h/2, x + h/2]$ of x , we can justify the following approximation: $hf(x) \approx \int_{x-h/2}^{x+h/2} f(u)du$ by the mean theorem. The estimator of $f(x)$ is given by $\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n \{X_i \in (x - h/2, x + h/2)\}$

The kernel estimator is therefore

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right),$$

where $K(\cdot)$ is a kernel function with the assumptions. The example kernel functions can be found in Hansen's note on non-parametric estimation Hansen (2009).

Hansen, B. E. (2009). Lecture Notes on Nonparametrics. Retrieved from <https://www.ssc.wisc.edu/~bhansen/718/NonParametrics1.pdf>.

Higher-order kernels are obtained by multiplying a second-order kernel by an $(\nu/2 - 1)$ -th order polynomial in u : Explicit formulae for the general polynomial family can be found in B. Hansen (Econometric Theory, 2005), and for the Gaussian family in Wand and Schucany (Canadian Journal of Statistics, 1990).

Table 1: Kernel Functions

Kernel	$K(u)$	$R(k)$	$\kappa_2(k)$	$eff(k)$
Uniform	$\frac{1}{2}\mathbb{1}\{ u \leq 1\}$	1/2	1/3	1.0758
Triangle	$(1 - u)\mathbb{1}\{ u \leq 1\}$	-	-	-
Epanechnikov	$\frac{3}{4}(1 - u^2)\mathbb{1}\{ u \leq 1\}$	3/5	1/5	1.0
Quartic(Biweight)	$\frac{15}{16}(1 - u^2)^2\mathbb{1}\{ u \leq 1\}$	5/7	1/7	1.0061
Triweight	$\frac{35}{32}(1 - u^2)^3\mathbb{1}\{ u \leq 1\}$	350 / 429	1/9	1.0135
Gaussian	$\frac{1}{\sqrt{2\pi}}\exp(-\frac{u^2}{2})$	$1/\sqrt{2\pi}$	1	1.0513
Cosinus	$\frac{\pi}{4}\cos(\frac{\pi}{2}u)\mathbb{1}\{ u \leq 1\}$	-	-	-

Table 2: Fourth Order Kernel Functions

Kernel	$K(u)$	$R(k)$	$\kappa_2(k)$	$eff(k)$
Epanechnikov	$\frac{15}{8}\frac{3}{4}(1 - \frac{7}{3}u^2)(1 - u^2)\mathbb{1}\{ u \leq 1\}$	5/4	-1/21	1.00
Biweight	$\frac{7}{4}\frac{15}{16}(1 - 3 * u^2)(1 - u^2)^2\mathbb{1}\{ u \leq 1\}$	805/572	-1/33	1.0056
Triweight	$\frac{27}{16}\frac{35}{32}(1 - 11/3u^2)(1 - u^2)^3\mathbb{1}\{ u \leq 1\}$	3780 / 2431	-3/143	1.0134

2 Bandwidth Selection

2.1 Estimation Bias

The bias of a kernel density of order ν estimator is

$$Bias(\hat{f}(x)) = E\hat{f}(x) - f(x) = \frac{1}{\nu!}f^{(\nu)}(x)h^\nu\kappa_\nu(k) + o(h^\nu).$$

2.2 Estimation Variance

The kernel estimator is a linear estimator, and $\kappa(\frac{X_i - x}{u})$ is i.i.d, then

$$\begin{aligned} Var(\hat{f}(x)) &= \frac{1}{nh^2}Ek\left(\frac{X_i - x}{h}\right)^2 - \frac{1}{n}\left(\frac{1}{h}Ek\left(\frac{X_i - x}{h}\right)\right)^2 \\ &= \frac{f(x)R(k)}{nh} + O(1/n) \end{aligned}$$

2.3 Mean squared error

The measure of precision is the mean squared error

$$\begin{aligned} AMSE(\hat{f}(x)) &= E(\hat{f}(x) - f(x))^2 \\ &= \frac{\kappa_\nu^2(k)}{(\nu!)^2}f^{(\nu)}(x)^2h^{2\nu} + \frac{f(x)R(k)}{nh} \end{aligned}$$

Global measure of precision is the asymptotic mean integrated squared error(AMISE):

$$\begin{aligned} AMISE(\hat{f}(x)) &= \int_{-\inf}^{\inf} AMSE(\hat{f}(x))dx \\ &= \frac{\kappa_{\nu}^2(k)}{(\nu!)^2} R(f^{(\nu)})h^{2\nu} + \frac{f(x)R(k)}{nh} \end{aligned}$$

2.4 Asymptotically optimal bandwidth

2.5 Asymptotically optimal kernel

2.6 Silverman Rule-of-thumb bandwidth selection

$$\begin{aligned} h &= \hat{\sigma}C_{\nu}(k)n^{-1/(2\nu+1)} \\ &= 2\hat{\sigma}n^{-1/(2\nu+1)} \left(\frac{\pi^{1/2}(\nu!)^3 R(k)}{2\nu(2\nu)! \kappa_{\nu}^2(k)} \right)^{1/(2\nu+1)} \end{aligned}$$

The rule of thumb constant constant can be looked up in the table.

Table 3: Rule of thumb constant $C_{\nu}(k)$ for single-variate

Kernel	$\nu = 2$	$\nu = 4$	$\nu = 6$
Epanechnikov	2.34	3.03	3.53
Biweight	2.78	3.39	3.84
Triweight	3.15	3.73	4.13
Gaussian	1.06	1.08	1.08

References

- Hansen, B. E. (2009). Lecture Notes on Nonparametrics. Technical report.
- Sun, Y. (2013). *Nonparametric and Semiparametric Econometrics*. Number August.