Using Euler Equation to Estimate Non-Finite-Dependent Dynamic Discrete Choice Model with Unobserved Heterogeneity

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Abstract

In the dynamic discrete choice analysis, controlling for unobserved heterogeneity is an important issue, and finite mixture models provide flexible ways to account for it. The previous discussion of incorporating finite mixture model in the dynamic discrete choice model focuses on a class of models where the difference in future value terms depends on a few conditional choice probabilities(finite dependence property). In models that do not exhibit finite dependence property, it is computationally costly to estimate finite mixture models with the expectation-maximization(EM) algorithm. Following the previous discussion of the finite mixture in dynamic discrete choice with finite dependence property, this paper adopts the EM algorithm to incorporate unobserved heterogeneity for a broader range of dynamic discrete choice model that does not require the finite dependence property. Following the Euler Equation(EE) representation of dynamic discrete decision problems, we provide an alternative conditional choice probability (CCP) value function representation that relies only on the CCP of one action. Contrasting to the Hotz-Miller CCP representation that relies on all the conditional choice probabilities, this characterization avoids the matrix inversion in each EM iteration. The matrix inversion can be computed outside the EM iterations and therefore is computationally attractive. The characterization provides unbiased estimator for models with and without finite dependence property. We illustrate the computational gains with Monte Carlo simulations.

Keywords: dynamic discrete choice; finite mixture; finite dependence; unobserved heterogeneity

1 Introduction

Many discrete decisions are made with concern about the future impacts. One example is that in labor economics, choices over levels of education are in part driven by how these decisions affect future earnings. In industrial organization, firms are willing to pay the entry costs in anticipation of future profit flows. In labor economics, agents choose their career paths taking future payoffs into consideration. To analyze the effects of theses dynamic decisions on particular outcomes can be done by using descriptive empirical methods that rely on randomization and quasi-randomization. In this case, understanding how the decision was made is not relevant except in how it forms the researcher's identification strategy. The randomization, regression discontinuity, and natural experiments provide exogenous source of variation in the data such that the predicted effect of a college degree on earnings is not being driven by problems of selection. To analyze the structural models, the decision behind these descriptive results need to be formally modelled. While the structural methods are often compared against their descriptive counterparts, the two methods can serve as complements. Ideally, structural models can replicate results obtained from randomized experiments or attempts to exploit quasi-randomization and tell us ow individuals will respond to counter-factual policies. One strategy of estimation is to use the first order condition. Another strategy is to use the Euler equation. The development of Euler equation-GMM approach by Hansen and Singleton (1982) establish the framework of estimation of dynamic structural models. One of the main advantages of this method is that it avoids the curse of dimensionality associated with the computation of present values. The computational cost of the estimating structural parameters from Euler equations increases with the sample size but not with the dimension of the state space. However, the Euler equation based approach has some limitations. First, the conventional wisdom in the literature is that this method cannot be applied to models of discrete choice because optimal decisions cannot be characterized in terms of marginal conditions in these models. Second, while the Euler equation-GMM significantly reduces the computational burden associated with estimating structural model by avoiding full solution, the counter-factual requires full solution. Among different setups in the dynamic decision process, the discrete decisions cannot be modelled by taking the first order derivatives due to the fact that the utility function is not differentiable with respect to the decision variables. The paper by Rust (1987),Rust (1994),Pakes and McGuire (1994) showed that under certain restrictions, estimating these dynamic discrete choice models are both feasible and important for answering key economic questions. The paper exploit Bellman's representation of the dynamic discrete choice discrete choice by decomposing the payoff from a particular choice into the component of flow payoff received today and a future utility term that is constructed by assuming the agent will make optimal decision in all future time. The dynamic discrete choices have been used in many economic research areas. In industrial organizations, this has been used to model the forward looking nature of firms' and consumers' behaviors,(CollardWexler (2013),Aguirregabiria and Ho (2012),Berry and Tamer (2006),Ellickson and Misra (2015),Yakovlev (2016),Sweeting (2013),Bronnenberg et al. (2012),Bronnenberg et al. (2012)).There are also various applications in health economics to model the competition between medical institutes and insurance institutes Beauchamp (2015),Gaynor and Town (2012), Gowrisankaran and Town (1997),Gowrisankaran et al. (2011); in the marketing, the researchers use this model to describe the advertisement campaign Dubé et al. (2005),Doraszelski and Pakes (2007),Doganoglu and Klapper (2006); and in Labor economics, researchers use dynamic discrete choice to model the forward looking nature of employment choice(Keane et al. (2011)), school choice(Todd and Wolpin (2006)) and mammography choices(Fang and Wang (2009)).

However, modeling these dynamic decision processes is complicated, requiring calculations of the present discounted value of lifetime utility or profits across all possible choices. Formally modeling this decision process requires identifying the optimal decision rule for each period and explicitly modelling expectations regarding future events. Recent surveys by Aguirregabiria and Mira (2007) and Keane et al. (2011) show the complications that arise in the formulation and estimation of dynamic discrete choice problems, and provide overviews of the methods that exist to handle them.(Review these paper). Rust (1987) develop the full solution method by iterating the Bellman operator until convergence for each candidate structural parameter. The structural parameter is updated after solving the fixed point. Hotz and Miller (1993) exploit the inverse mapping between the choice-dependent payoff and the conditional choice probability. Aguirregabiria and Mira (2007) develop the nested pseudo likelihood estimator. Arcidiacono and Ellickson (2011) provide a convenient estimator for models with the finite dependence property. They also show that the model can include unobserved state variables. Aguirregabiria and Magesan (2016) redefine the dynamic discrete choice problem into the decision problem in probability space. The paper uses two-period deviation rule of the Euler Equation and establish the Euler Equation mapping for model that exhibit finite 2 period single action dependence.

In this paper, I follow two streams of literature. The first one is the dynamic discrete choice problems with finite-dependence properties by Arcidiacono and Ellickson (2011). The paper develop the estimation strategy that exploits the finite-dependence property, which constructs the contraction mapping on the conditional choice probabilities without solving explicitly the value function. Following the method in Aguirregabiria and Magesan (2016), I redefine the decision problem for non-finite-dependent models in the probability space and derived the optimal decision rule condition. I extend their results by developing the Euler Equation contraction mapping in a non-finite-dependent model. I show that with no additional restrictions, a contraction mapping can be constructed with knowing the conditional choice probability of one of the many choices. In addition, I show that with the Euler Equation contraction mapping, I can incorporate unobserved state variables in the estimation.

2 Dynamic Discrete Choice Model

Assume the agent makes a decision choice after observing a realization of state and choice-dependent idiosyncratic shock $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \ldots, \epsilon_D)^\top$. For each time, the agent observe a state $z \in \mathcal{Z} = \{1, ..., |\mathcal{Z}|\}$. Assume that the support for state variable $\mathcal{Z}$ is finite. The agent then observe a choice-dependent idiosyncratic shock ϵ_d for each $d \in \mathcal{D} = \{0, 1, \ldots, D\}$ and make a discrete choice.

Assumption 1 (Private Information)**.** *[The private shocks are independent across d. (This can be relaxed if we consider the Generalized extreme value distribution)] The private shocks have a known distribution function $\boldsymbol{\epsilon} \sim G(\cdot)$. Furthermore, the distribution has a finite first moment and support almost everywhere on $\mathbb{R}^{D+1}$.*

Assumption 2 (Additive Separability)**.** *The payoff of choosing action d at state z at period t is given by $U_t(d, z, \boldsymbol{\epsilon}) = u_t(d, z) + \epsilon_d$.*

Assumption 3 (State Transitions)**.** *Assume the transition of state variable at period t is defined by $f_t(z'|z, d)$, where z is the current the state, d is the current choice and z' is the future state. Furthermore, $\sum_{z' \in \mathcal{Z}} f_t(z'|z, d) = 1$ for any $z \in \mathcal{Z}$ and $d \in \mathcal{D}$.*

For each time period t, the agent observe a state z_t and make a decision d_t to maximize the total discounted payoff:

$$E\Big[\sum_{\tau=t}^{T} \beta^{\tau-t} U_t(d_\tau, z_\tau, \boldsymbol{\epsilon}_\tau)|z_t\Big]. \tag{1}$$

The Bellman equation is given by

$$V_t(z_t) = \int \max_{d \in \mathcal{D}} \{v_t(d, z_t) + \epsilon_d\}\, dG(\boldsymbol{\epsilon}) \quad \text{for } t = 0, 1, ..., T, \tag{2}$$

where

$$v_t(d, z_t) := u_t(d, z_t) + \beta \sum_{z_{t+1} \in \mathcal{Z}} f_t(z_{t+1}|z_t, d) V_{t+1}(z_{t+1}) \tag{3}$$

is the choice-dependent value function. Define the the value differences as $\tilde{v}_t(d, z_t) := v_t(d, z_t) - v_t(0, z_t) = \tilde{u}_t(d, z_t) + \beta \sum_{z_{t+1} \in \mathcal{Z}} \tilde{f}_t(z_{t+1}|z_t, d) V_{t+1}(z_{t+1})$ for $d = \mathcal{D} \setminus \{0\}$, where $\tilde{u}_t(d, z) = u_t(d, z) - u_t(0, z)$ and $\tilde{f}_t(z'|z, d) = f_t(z'|z, d) - f_t(z'|z, 0)$. Let $\tilde{\boldsymbol{v}}_t(z) = (\tilde{v}_t(1, z), \ldots, \tilde{v}_t(D, z))^\top$. Given $\tilde{\boldsymbol{v}}_t(z)$, the conditional choice probabilities of the action d, given the state z can be computed as

$$\Lambda(d, \tilde{\boldsymbol{v}}_t(z)) = \int \mathbf{1}\{\tilde{v}_t(d, z) + \epsilon_d \geq \tilde{v}_t(d', z) + \epsilon_{d'}\ \forall d' \in \mathcal{D}\} dG(\epsilon) \quad \text{for } d = 1, ..., D$$

and $\Lambda(0, \tilde{\boldsymbol{v}}_t(z)) := 1 - \sum_{d'=1}^{D} \Lambda(d', \tilde{\boldsymbol{v}}_t(z))$. We call $\{\Lambda(d, \tilde{\boldsymbol{v}}_t(z)) : d \in \mathcal{D}\}$ as the optimal conditional choice probability (OCP) mapping.

Let $\boldsymbol{\Lambda}(\tilde{\boldsymbol{v}}_t(z)) = (\Lambda(1, \tilde{\boldsymbol{v}}_t(z)), \ldots, \Lambda(D, \tilde{\boldsymbol{v}}_t(z)))^\top$ and $\boldsymbol{p}_t(z) = (p_t(1, z), p_t(2, z), ..., p_t(D, z))^\top$. Given the solution to the Bellman equation (2) and the corresponding value differences $\{\tilde{\boldsymbol{v}}_t(z) : z \in \mathcal{Z}\}$, the vector of choice probabilities is said to be optimal if and only if $\boldsymbol{p}_t(z) = \boldsymbol{\Lambda}(\tilde{\boldsymbol{v}}_t(z))$ for all $z \in \mathcal{Z}$. The following proposition is a well known result from Hotz and Miller (1993).

Proposition 1 (Hotz-Miller Inversion). *Under Assumption 1-3, for any vector of differences in choice-dependent value functions $\tilde{\boldsymbol{v}}_t(z) \in \mathbb{R}^D$, the OCP mapping is invertible such that there is a one-to-one relationship between the vector of value differences and the vector of optimal choice probabilities, i.e., $\tilde{\boldsymbol{v}}_t(z) = \boldsymbol{\Lambda}^{-1}(\boldsymbol{p}_t(z))$.*

In many empirical applications, $\boldsymbol{\epsilon}$ is assumed to follow a Type I extreme value distribution.

Assumption 4 (Type I Extreme Value). *$\boldsymbol{\epsilon}_t = (\epsilon_{0t}, ..., \epsilon_{Dt})^\top$ is independently and identically distributed under a Type-I extreme value distribution with $g(\boldsymbol{\epsilon}_t) = \prod_{d \in \mathcal{D}} \exp(\epsilon_{dt} - \exp(\epsilon_{dt}))$.*

While we don't necessarily impose Assumption 4 to derive our main theoretical results, Assumption 4 simplifies our analysis and it plays an important role to implementing our proposed estimation procedure in practice.

Example 1 (Hotz-Miller Inversion under Type I Extreme Value assumption). *When $\boldsymbol{\epsilon}$ is drawn from a Type I extreme value distribution under Assumption 4, the OCP mapping and its inverse mapping has a closed form expression as follows:*

$$\Lambda(d, \tilde{\boldsymbol{v}}_t(z)) = \frac{\exp(\tilde{v}_t(d, z))}{1 + \sum_{d'=1}^{D} \exp(\tilde{v}_t(d', z))} \quad \text{for } d \in \mathcal{D} \quad \text{and} \quad \boldsymbol{\Lambda}^{-1}(\boldsymbol{p}_t(z)) = \log\left((\boldsymbol{B}(\boldsymbol{p}_t(z)))^{-1} \boldsymbol{p}_t(z)\right)$$

where

$$\boldsymbol{B}(\boldsymbol{p}_t(z)) = \boldsymbol{I}_D - \boldsymbol{p}_t(z)\iota_D^\top = \begin{bmatrix} 1 - p_{1t}(z) & -p_{2t}(z) & \cdots & -p_{Dt}(z) \\ -p_{1t}(z) & 1 - p_{2t}(z) & \cdots & -p_{Dt}(z) \\ \vdots & \vdots & \ddots & \vdots \\ -p_{1t}(z) & -p_{2t}(z) & \cdots & 1 - p_{Dt}(z) \end{bmatrix},$$

where $\boldsymbol{I}_D = diag(\iota_D)$ and $\iota_D = (1, ..., 1)^\top$ are D-dimensional identity matrix and unit vector, respectively. Also,

$$\int \epsilon_d \prod_{d' \neq d} \mathbb{1}\{\Lambda^{-1}(d, \boldsymbol{p}_t(z)) + \epsilon_d \geq \Lambda^{-1}(d', \boldsymbol{p}_t(z)) + \epsilon_{d'}\} d\boldsymbol{\epsilon} = \gamma - \ln(p_t(d, z)).$$

Following Aguirregabiria and Magesan (2016), we the reformulate dynamic decision problem

(2) in probability space. Define

$$\bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} := \begin{bmatrix} \bar{U}_t^{\boldsymbol{p}_t(1)} \\ \vdots \\ \bar{U}_t^{\boldsymbol{p}_t(|\mathcal{Z}|)} \end{bmatrix} := \begin{bmatrix} \sum_{d=0}^{D} p_{d,t}(1)\big(u_t(d,1) + e^{\boldsymbol{p}_t(1)}(d,1)\big) \\ \vdots \\ \sum_{d=0}^{D} p_{d,t}(|\mathcal{Z}|)\big(u_t(d,|\mathcal{Z}|) + e^{\boldsymbol{p}_t(|\mathcal{Z}|)}(d,|\mathcal{Z}|)\big) \end{bmatrix},$$

where

$$e^{\boldsymbol{p}_t(z)}(d,z) = \int \epsilon_d \prod_{d' \neq d} \mathbb{1}\{\Lambda^{-1}(d,\boldsymbol{p}_t(z)) + \epsilon_d \geq \Lambda^{-1}(d',\boldsymbol{p}_t(z)) + \epsilon_{d'}\} d\boldsymbol{\epsilon}.$$

Under Assumption 4, $e^{\boldsymbol{p}_t(z)}(d,z) = \gamma - \ln(p_t(d,z))$ for any $d \in \mathcal{D}$ as stated in Example 1.

Define a $|\mathcal{Z}| \times |\mathcal{Z}|$ matrix

$$\bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} := \begin{bmatrix} (\bar{\boldsymbol{f}}_t^{\boldsymbol{p}_t(1)})^\top \\ \vdots \\ (\bar{\boldsymbol{f}}_t^{\boldsymbol{p}_t(|\mathcal{Z}|)})^\top \end{bmatrix} = \begin{bmatrix} \sum_{d=0}^{D} p_{d,t}(1) f_t(1|1,d) & \cdots & \sum_{d=0}^{D} p_{d,t}(1) f_t(|\mathcal{Z}||1,d) \\ \vdots & \ddots & \vdots \\ \sum_{d=0}^{D} p_{d,t}(|\mathcal{Z}|) f_t(1||\mathcal{Z}|,d) & \cdots & \sum_{d=0}^{D} p_{d,t}(|\mathcal{Z}|) f_t(|\mathcal{Z}|||\mathcal{Z}|,d) \end{bmatrix}.$$

Then, we can write the integrated Bellman equation as follows:

$$V_t(j) = \max_{\boldsymbol{p}_t(j) \in \mathcal{P}_j} \bar{U}_t^{\boldsymbol{p}_t(j)} + \beta(\bar{\boldsymbol{f}}_t^{\boldsymbol{p}_t(j)})^\top \boldsymbol{V}_{t+1} \text{ for } j = 1, \ldots, |\mathcal{Z}|$$

or, in matrix form,

$$\boldsymbol{V}_t = \max_{\boldsymbol{P}_t \in \mathcal{P}} \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} \boldsymbol{V}_{t+1}, \tag{4}$$

where $\boldsymbol{V}_t = [V_t(1), \ldots, V_t(|\mathcal{Z}|)]^\top$.

The solution to (4) is characterized by a sequence of value function $\{\boldsymbol{V}_t^*\}_{t=1}^T$ and optimal decision rules $\{\boldsymbol{P}_t^*\}_{t=1}^T$, where $\boldsymbol{P}_t^* = (\boldsymbol{p}_t^*(1), \ldots, \boldsymbol{p}_t^*(|\mathcal{Z}|))$. We also denote the choice-dependent value function and the value differences associated with $\{\boldsymbol{V}_t^*\}_{t=1}^T$ by $\{v_t(d,z) : (d,z) \in \mathcal{D} \times \mathcal{Z}\}$ and $\{\tilde{\boldsymbol{v}}_t^*(z) : z \in \mathcal{Z}\}$, respectively.

Define

$$\nabla \bar{\boldsymbol{e}}^{\boldsymbol{P}_t} = \begin{bmatrix} \boldsymbol{p}_t(1)^\top \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(1)}(1) \\ \vdots \\ \boldsymbol{p}_t(|\mathcal{Z}|)^\top \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(|\mathcal{Z}|)}(|\mathcal{Z}|) \end{bmatrix}, \quad \boldsymbol{u}_{d,t} := \begin{bmatrix} u_t(d,1) \\ \vdots \\ u_t(d,|\mathcal{Z}|) \end{bmatrix}, \quad \boldsymbol{e}_d^{\boldsymbol{P}_t} := \begin{bmatrix} e^{\boldsymbol{p}_t(1)}(d,1) \\ \vdots \\ e^{\boldsymbol{p}_t(|\mathcal{Z}|)}(d,|\mathcal{Z}|) \end{bmatrix},$$

$$\text{and} \quad \boldsymbol{F}_{d,t} := \begin{bmatrix} (\boldsymbol{f}_{d,t}(1))^\top \\ \vdots \\ (\boldsymbol{f}_{d,t}(|\mathcal{Z}|))^\top \end{bmatrix},$$

with $\nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(j)}(j) := \sum_{d=0} p_{d,t}(j) \frac{\partial e^{\boldsymbol{p}_t(j)}(d,j)}{\partial \boldsymbol{p}_t(j)}$ and $\boldsymbol{f}_{d,t}(i) := (f_t(1|i,d), ..., f_t(|\mathcal{Z}||i,d))^\top$.

Proposition 2. *Suppose that Assumptions 1-3 hold. Then, (a) for any $d \in \mathcal{D}$,*

$$\boldsymbol{V}_t^* = \nabla \bar{\boldsymbol{e}}^{\boldsymbol{P}_t^*} + \boldsymbol{u}_{d,t} + \boldsymbol{e}_d^{\boldsymbol{P}_t^*} + \beta \boldsymbol{F}_{d,t} \boldsymbol{V}_{t+1}^*.$$

(b) If, in addition, Assumption 4 holds, then $\nabla \bar{\boldsymbol{e}}^{\boldsymbol{P}_t^*} = \mathbf{0}$ so that

$$\boldsymbol{V}_t^* = \boldsymbol{u}_{d,t} + \boldsymbol{e}_d^{\boldsymbol{P}_t^*} + \beta \boldsymbol{F}_{d,t} \boldsymbol{V}_{t+1}^*.$$

[For the T1EV case, we have $\nabla \bar{e}^{\boldsymbol{p}_t(j)}(j) = 0$. However, if we change the Is this generally the case? Need to discuss about generalized extreme value case.]

Proof. We prove the stated result when $d = 0$. For $d \neq 0$, the proof is similar. For any distribution of Z_t, $\boldsymbol{\kappa} = (\kappa(1), ..., \kappa(|\mathcal{Z}|))^\top$ with $\kappa(j) = \Pr(Z_t = j)$, define

$$W_t(\boldsymbol{\kappa}) := \boldsymbol{\kappa}^\top \boldsymbol{V}_t.$$

Denote the derivative of $W_t(\boldsymbol{\kappa})$ with respect to $\boldsymbol{\kappa}$ by

$$W_t'(\boldsymbol{\kappa}) := \frac{dW_t(\boldsymbol{\kappa})}{d\boldsymbol{\kappa}} = \boldsymbol{V}_t. \tag{5}$$

Then we may rewrite the Bellman equation (4) in terms of $W(\boldsymbol{\kappa}_t)$ as

$$\begin{aligned} W_t(\boldsymbol{\kappa}_t) &= \boldsymbol{\kappa}_t^\top \left(\max_{\boldsymbol{P}_t \in \mathcal{P}} \{ \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} \boldsymbol{V}_{t+1} \} \right) \\ &= \max_{\boldsymbol{P}_t \in \mathcal{P}} \{ \boldsymbol{\kappa}_t^\top \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \boldsymbol{\kappa}_t^\top \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} \boldsymbol{V}_{t+1} \} \\ &= \max_{\boldsymbol{P}_t \in \mathcal{P}} \{ \boldsymbol{\kappa}_t^\top \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta W_{t+1}(\boldsymbol{\kappa}_t^\top \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t}) \}, \end{aligned} \tag{6}$$

where the second equality holds because, given $\boldsymbol{P}_t = (\boldsymbol{p}_t(1), ..., \boldsymbol{p}_t(|\mathcal{Z}|)) \in \mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2, \ldots \oplus \mathcal{P}_{|\mathcal{Z}|}$, we have

$$\max_{\boldsymbol{P}_t \in \mathcal{P}} \boldsymbol{\kappa}_t^\top (\bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} \boldsymbol{V}_{t+1}) = \sum_{j=1}^{|\mathcal{Z}|} \kappa_t(j) \left\{ \max_{\boldsymbol{p}_t(j) \in \mathcal{P}_j} \bar{U}_t^{\boldsymbol{p}_t(j)}(j) + \beta (\bar{\boldsymbol{f}}_t^{\boldsymbol{p}_t(j)})^\top \boldsymbol{V}_{t+1} \right\}.$$

Note that

$$\frac{\partial \bar{U}_t^{\boldsymbol{p}_t(j)}(j)}{\partial \boldsymbol{p}_t(j)} = \tilde{\boldsymbol{u}}_t(j) + \tilde{\boldsymbol{e}}^{\boldsymbol{p}_t(j)}(j) + \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(j)}(j), \tag{7}$$

where

$$\tilde{\boldsymbol{u}}_t(j) := (u_t(1,j) - u_t(0,j), ..., u_t(D,j) - u_t(0,j))^\top,$$
$$\tilde{\boldsymbol{e}}^{\boldsymbol{p}_t(j)}(j) := (e^{\boldsymbol{p}_t(j)}(1,j) - e^{\boldsymbol{p}_t(j)}(0,j), ..., e^{\boldsymbol{p}_t(j)}(D,j) - e^{\boldsymbol{p}_t(j)}(0,j))^\top,$$
$$\nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(j)}(j) := \sum_{d=0} p_{d,t}(j) \frac{\partial e^{\boldsymbol{p}_t(j)}(d,j)}{\partial \boldsymbol{p}_t(j)}.$$

Then, the first order condition with respect to the decision $\boldsymbol{p}_t(i)$ in (6) is given by

$$\kappa_t(i) \left(\tilde{\boldsymbol{u}}_t(i) + \tilde{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + \beta \boldsymbol{H}_t(i) W'_{t+1}(\boldsymbol{\kappa}_t^\top \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t}) \right) = \boldsymbol{0},$$

where $\boldsymbol{H}_t(i)$ is a $D \times |\mathcal{Z}|$ matrix of which (d,j) element is given by $f_t(j|i,d) - f_t(j|i,0)$. On the other hand, the envelop condition in (6) is given by

$$W'_t(\boldsymbol{\kappa}_t) = \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} W'_{t+1}(\boldsymbol{\kappa}_t^\top \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t}),$$

Therefore, for $\kappa_t(j) > 0$, it follows from (5), the first order condition, and the envelop condition that

$$\boldsymbol{0} = \tilde{\boldsymbol{u}}_t(i) + \tilde{\boldsymbol{e}}^{\boldsymbol{p}_t}(i) + \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + \beta \boldsymbol{H}_t(i) \boldsymbol{V}_{t+1}, \tag{8}$$
$$\boldsymbol{V}_t = \bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t} + \beta \bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t} \boldsymbol{V}_{t+1}. \tag{9}$$

Multiplying (8) by $\boldsymbol{p}_t(i)^\top$ and substituting (9) give

$$\begin{aligned}
0 &= \boldsymbol{p}_t(i)^\top \left(\tilde{\boldsymbol{u}}_t(i) + \tilde{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + \beta \boldsymbol{H}_t(i) \boldsymbol{V}_{t+1} \right) \\
&= \bar{U}_t^{\boldsymbol{p}_t(i)}(i) + \beta (\bar{\boldsymbol{f}}_t^{\boldsymbol{p}_t(i)})^\top \boldsymbol{V}_{t+1} - \left(\boldsymbol{p}_t(i)^\top \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + u_t(0,i) + e^{\boldsymbol{p}_t(i)}(0,i) + \beta (\boldsymbol{f}_{0,t}(i))^\top \boldsymbol{V}_{t+1} \right) \\
&= V_t(i) - \left(\boldsymbol{p}_t(i)^\top \nabla \bar{\boldsymbol{e}}^{\boldsymbol{p}_t(i)}(i) + u_t(0,i) + e^{\boldsymbol{p}_t(i)}(0,i) + \beta (\boldsymbol{f}_{0,t}(i))^\top \boldsymbol{V}_{t+1} \right).
\end{aligned}$$

Therefore,

$$\boldsymbol{V}_t = \nabla \bar{\boldsymbol{e}}^{\boldsymbol{P}_t} + \boldsymbol{u}_{0,t} + \boldsymbol{e}_0^{\boldsymbol{P}_t} + \beta \boldsymbol{F}_{0,t} \boldsymbol{V}_{t+1},$$

and the stated result for part (a) follows.

Under Assumption 4, we have $e^{\boldsymbol{p}_t(j)}(d,j) = \gamma - \ln p_{d,t}(j)$ for $d = 1, ..., D$ and $e^{\boldsymbol{p}_t(j)}(0,j) = \gamma - \ln(1 - \sum_{d'=1}^D p_{d',t}(j))$. Therefore, for any $d = 1, ..., D$,

$$\sum_{d'=0} p_{d',t}(j) \frac{\partial e^{\boldsymbol{p}_t(j)}(d',j)}{\partial p_{d,t}(j)} = p_{0,t}(j) \frac{1}{1 - \sum_{d'=1}^D p_{d',t}(j)} - p_{d,t}(j) \frac{1}{p_{d,t}(j)} = 0,$$

and $\nabla \bar{e}^{\boldsymbol{P}_t} = \mathbf{0}$ follows.

□

3 Finite Dependence

This section follows the discussion of Arcidiacono and Ellickson (2011). For brevity, we assume Assumption 4 in this section. Consider the vector fo value differences relative to the choice $d_t^\dagger$ by

$$\tilde{\boldsymbol{v}}_{d_t^\dagger,t}(d) := \begin{bmatrix} \tilde{v}_{d_t^\dagger,t}(d,1) \\ \vdots \\ \tilde{v}_{d_t^\dagger,t}(d,|\mathcal{Z}|) \end{bmatrix} := \tilde{\boldsymbol{u}}_{d_t^\dagger,t}(d) + \beta \tilde{\boldsymbol{F}}_{d_t^\dagger,t}(d) \boldsymbol{V}_{t+1}, \tag{10}$$

where

$$\tilde{\boldsymbol{u}}_{d^\dagger,t}(d) := \begin{bmatrix} \tilde{u}_{d^\dagger,t}(d,1) \\ \vdots \\ \tilde{u}_{d^\dagger,t}(d,|\mathcal{Z}|) \end{bmatrix} := \begin{bmatrix} u_t(d,1) - u_t(d^\dagger,1) \\ \vdots \\ u_t(d,|\mathcal{Z}|) - u_t(d^\dagger,|\mathcal{Z}|) \end{bmatrix}.$$

and

$$\tilde{\boldsymbol{F}}_{d^\dagger,t}(d) := \begin{bmatrix} (\tilde{\boldsymbol{f}}_{d^\dagger,t}(d,1))^\top \\ \vdots \\ (\tilde{\boldsymbol{f}}_{d^\dagger,t}(d,|\mathcal{Z}|))^\top \end{bmatrix} := \begin{bmatrix} f_t(1|1,d) - f_t(1|1,d^\dagger) & \dots & f_t(|\mathcal{Z}||1,d) - f_t(|\mathcal{Z}||1,d^\dagger) \\ \vdots & \ddots & \vdots \\ f_t(1||\mathcal{Z}|,d) - f_t(1||\mathcal{Z}|,d^\dagger) & \dots & f_t(|\mathcal{Z}|||\mathcal{Z}|,d) - f_t(|\mathcal{Z}|||\mathcal{Z}|,d^\dagger) \end{bmatrix}.$$

As in the previous section, we define $\tilde{\boldsymbol{v}}_t(d)$ and $\tilde{\boldsymbol{F}}_t(d)$ relative to the choice of $d^\dagger = 0$ as $\tilde{\boldsymbol{v}}_t(d) := \tilde{\boldsymbol{v}}_{0,t}(d)$ and $\tilde{\boldsymbol{F}}_t(d) := \tilde{\boldsymbol{F}}_{0,t}(d)$.

From Proposition 2 under Assumption 4, we have $\boldsymbol{V}_{t+1} = \boldsymbol{u}_{d_{t+1}^\dagger,t+1} + \boldsymbol{e}_{d_{t+1}^\dagger}^{\boldsymbol{P}_{t+1}} + \beta \boldsymbol{F}_{d_{t+1}^\dagger,t+1} \boldsymbol{V}_{t+2}$ for any $d_{t+1}^\dagger \in \mathcal{D}$, and substituting this to (10) evaluated at $d_t^\dagger = 0$ gives

$$\tilde{\boldsymbol{v}}_t(d) = \tilde{\boldsymbol{u}}_t(d) + \beta \tilde{\boldsymbol{F}}_t(d) \left(\boldsymbol{u}_{d_{t+1}^\dagger,t+1} + \boldsymbol{e}_{d_{t+1}^\dagger}^{\boldsymbol{P}_{t+1}} \right) + \beta^2 \tilde{\boldsymbol{F}}_t(d) \boldsymbol{F}_{d_{t+1}^\dagger,t+1} \boldsymbol{V}_{t+2}. \tag{11}$$

If $\tilde{\boldsymbol{F}}_t(d) \boldsymbol{F}_{d_{t+1}^\dagger,t+1} = \mathbf{0}$ for some $d_{t+1}^\dagger \in \mathcal{D}$, then $\tilde{\boldsymbol{v}}_t(d) = \tilde{\boldsymbol{u}}_t(d) + \beta \tilde{\boldsymbol{F}}_t(d) \left(\boldsymbol{u}_{d_{t+1}^\dagger,t+1} + \boldsymbol{e}_{d_{t+1}^\dagger}^{\boldsymbol{P}_{t+1}} \right)$ so that the evaluation of $\tilde{\boldsymbol{v}}_t(d)$ can be done with $\boldsymbol{P}_t$ and $\boldsymbol{P}_{t+1}$ but requires neither $\{\boldsymbol{P}_{t+\tau} : \tau \geq 2\}$ nor $\{\boldsymbol{V}_{t+\tau} : \tau \geq 2\}$ as $\tilde{\boldsymbol{v}}_t(d) = \tilde{\boldsymbol{u}}_t(d) + \beta \tilde{\boldsymbol{F}}_t(d) \left(\boldsymbol{u}_{d_{t+1}^\dagger,t+1} + \boldsymbol{e}_{d_{t+1}^\dagger}^{\boldsymbol{P}_{t+1}} \right)$. In this case, the model is said to exhibit the *2-period finite dependence*.

Definition 1 (Finite Dependence). *When the value differences $\{\tilde{\boldsymbol{v}}_t(d) : d \in \mathcal{D}\}$ can be evaluated with $\{\boldsymbol{P}_{t+\tau}\}_{\tau=0}^{\rho-1}$ but without using neither $\{\boldsymbol{P}_{t+\tau} : \tau \geq \rho\}$ nor $\{\boldsymbol{V}_{t+\tau} : \tau \geq \rho\}$, then the model is said to* *exhibit the ρ-period finite dependence.*

[To Jasmine: this definition has a room for improvement.]

Proposition 3 (Characterization of Finite Dependence)**.** *Suppose that there exists a sequence $\{d^{\dagger}_{t+\tau}(z)\}_{\tau=1}^{\rho}$ such that $(\tilde{\boldsymbol{f}}_t(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau},t+\tau}=\boldsymbol{0}$ for all $(d,z)\in\mathcal{D}\times\mathcal{Z}$, then the model exhibits the $(\rho+1)$-period single action finite dependence with*

$$\tilde{v}_t(d,z)=\tilde{u}_t(d,z)+(\tilde{\boldsymbol{f}}_t(d,z))^{\top}\sum_{\tau=1}^{\rho}\beta^{\tau}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}\left(\nabla\boldsymbol{u}_{d^{\dagger}_{t+\tau}(z)}+\boldsymbol{e}^{\boldsymbol{P}_{t+\tau}}_{d^{\dagger}_{t+\tau}(z)}\right).$$

Proof. For any sequence $\{d^{\dagger}_{t+\tau}(z)\}_{\tau=0}^{\rho}$, repeatedly substituting $\boldsymbol{V}_{t+\tau}=\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}_{t+\tau}}+\boldsymbol{u}_{d^{\dagger}_{t+\tau}(z)}+\boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{d^{\dagger}_{t+\tau}(z)}+\beta\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}\boldsymbol{V}_{t+\tau+1}$ to (10) for $\tau=1,...,\rho$ gives

$$\begin{aligned}\tilde{v}_t(d,z)=\tilde{u}_t(d,z)&+(\tilde{\boldsymbol{f}}_t(d,z))^{\top}\sum_{\tau=1}^{\rho}\beta^{\tau}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}\left(\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}_{t+1}}+\boldsymbol{u}_{d^{\dagger}_{t+\tau}(z)}+\boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{d^{\dagger}_{t+\tau}(z)}\right)\\&+(\tilde{\boldsymbol{f}}_t(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}V_{t+\rho+1}\quad\text{for } z=1,...,|\mathcal{Z}|.\end{aligned}$$

Therefore, the stated result follows with $(\tilde{\boldsymbol{f}}_t(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}=\boldsymbol{0}$. □

Remark 1. *We set $d^{\dagger}_t=0$ as the default choice at period t. This is without loss of generality because, given z, $(\tilde{\boldsymbol{f}}_{0,t}(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}$ holds for all $d\in\mathcal{D}$ if and only if $(\tilde{\boldsymbol{f}}_{d^{\dagger}_t,t}(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}=0$ holds for all $(d,d^{\dagger}_t)\in\mathcal{D}^2$. To see this, fix z and suppose that $(\tilde{\boldsymbol{f}}_{0,t}(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}$ holds for all $d\in\mathcal{D}$. Then, for any $(d,d^{\dagger}_t)\in\mathcal{D}^2$, we have $0=(\tilde{\boldsymbol{f}}_{0,t}(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}-(\tilde{\boldsymbol{f}}_{0,t}(d^{\dagger}_t,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}=((\tilde{\boldsymbol{f}}_{0,t}(d,z)-\tilde{\boldsymbol{f}}_{0,t}(d^{\dagger}_t,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}=(\tilde{\boldsymbol{f}}_{d^{\dagger}_t,t}(d,z))^{\top}\prod_{\tau=1}^{\rho}\boldsymbol{F}_{d^{\dagger}_{t+\tau}(z),t+\tau}=0$.*

When the model exhibits finite dependence, given the initial estimate for $\{\boldsymbol{p}_{t+\tau}:\tau=1,...,\rho\}$, the conditional choice probabilities can be evaluated without solving the Bellman equation.

Example 2. *Suppose that Assumption 4 holds. If $\tilde{\boldsymbol{F}}_{0,t}(d)\boldsymbol{F}_{0,t+1}=\boldsymbol{0}$, then*

$$\Lambda(d,\tilde{\boldsymbol{v}}_{0,t}(z))=\frac{\exp(\tilde{v}_{0,t}(d,z))}{\sum_{d'=0}^{D}\exp(\tilde{v}_{0,t}(d',z))}=\frac{\exp\left(\tilde{u}_{0,t}(d,z)+\beta(\tilde{\boldsymbol{f}}_{0,t}(d,z))^{\top}\left(\boldsymbol{u}_{0,t+1}+\gamma-\ln\boldsymbol{p}_{0,t+1}\right)\right)}{\sum_{d'=0}^{D}\exp\left(\tilde{u}_{0,t}(d',z)+\beta(\tilde{\boldsymbol{f}}_{0,t}(d',z))^{\top}\left(\boldsymbol{u}_{0,t+1}+\gamma-\ln\boldsymbol{p}_{0,t+1}\right)\right)},$$

where $\boldsymbol{p}_{0,t+1}=(p_{0,t+1}(1),...,p_{0,t+1}(|\mathcal{Z}|))^{\top}$. In general, if we can find a sequence $\{d^{\dagger}_{t+\tau}(z)\}_{\tau=0}^{\rho}$ such that

$(\tilde{\boldsymbol{f}}_{d_t^\dagger(z),t}(d,z))^\top \prod_{\tau=1}^{\rho} \boldsymbol{F}_{d_{t+\tau}^\dagger(z),t+\tau} = \boldsymbol{0}$*, then we can evaluate* $\Lambda(d, \tilde{\boldsymbol{v}}_{d_t^\dagger(z),t}(z))$ *as*

$$\Lambda(d, \tilde{\boldsymbol{v}}_{d_t^\dagger(z),t}(z)) = \frac{\exp(\tilde{v}_{d_t^\dagger(z),t}(d,z))}{\sum_{d'=0}^{D} \exp(\tilde{v}_{d_t^\dagger(z),t}(d',z))},$$

where

$$\tilde{v}_{d_t^\dagger(z),t}(d,z)) = \tilde{u}_{d_t^\dagger(z),t}(d,z) + (\tilde{\boldsymbol{f}}_{d_t^\dagger(z),t}(d,z))^\top \sum_{\tau=1}^{\rho} \beta^\tau \boldsymbol{F}_{d_{t+\tau}^\dagger(z),t+\tau} \left(\boldsymbol{u}_{d_{t+\tau}^\dagger(z),t+\tau} + \gamma - \ln \boldsymbol{p}_{d_{t+\tau}^\dagger(z),t+\tau} \right)$$

with $\boldsymbol{p}_{d_t^*,t} = (p_{d_t^*,t}(0), ..., p_{d_t^*,t}(d_t^* - 1), p_{d_t^*,t}(d_t^* + 1), ..., p_{d_t^*,t}(|\mathcal{Z}|))^\top$.

We can extend the above analysis by considering the value relative to the weighted average of values across choices. Let $\boldsymbol{\omega} := (\omega(0), \omega(1), ..., \omega(D))^\top \in \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} := \{\boldsymbol{\omega} \in \mathbb{R}^{D+1} : \sum_{d^\dagger \in \mathcal{D}} \omega(d^\dagger) = 1\}$.

Then, from Proposition 2, we also have

$$\boldsymbol{V}_{t+1} = \boldsymbol{u}_{\boldsymbol{\omega}_{t+1},t+1} + \boldsymbol{e}_{\boldsymbol{\omega}_{t+1}}^{\boldsymbol{P}_{t+1}} + \beta \boldsymbol{F}_{\boldsymbol{\omega}_{t+1},t+1} \boldsymbol{V}_{t+2},$$

where

$$\boldsymbol{u}_{\boldsymbol{\omega},t+1} := \sum_{d^\dagger \in \mathcal{D}} \omega(d^\dagger) \boldsymbol{u}_{d^\dagger,t+1}, \quad \boldsymbol{e}_{\boldsymbol{\omega}}^{\boldsymbol{P}_{t+1}} := \sum_{d^\dagger \in \mathcal{D}} \omega(d^\dagger) \boldsymbol{e}_{d^\dagger}^{\boldsymbol{P}_{t+1}}, \quad \text{and} \quad \boldsymbol{F}_{\boldsymbol{\omega},t+1} := \sum_{d^\dagger \in \mathcal{D}} \omega(d^\dagger) \boldsymbol{F}_{d^\dagger,t+1}.$$

The following proposition characterizes the condition under which the model exhibits finite dependence.

Proposition 4 (Characterization of Finite Dependence)**.** *Suppose that, for each* $z \in \mathcal{Z}$*, there exists a sequence* $\{\boldsymbol{\omega}_\tau(z)\}_{\tau=1}^{\rho}$ *such that* $(\tilde{\boldsymbol{f}}_t(d,z))^\top \prod_{\tau=1}^{\rho} \boldsymbol{F}_{\boldsymbol{\omega}_{t+\tau}(z),t+\tau} = \boldsymbol{0}$ *for all* $d \in \mathcal{D}$*, then the model exhibits the* $(\rho+1)$*-period finite dependence with*

$$\tilde{v}_t(d,z) = \tilde{u}_t(d,z) + (\tilde{\boldsymbol{f}}_t(d,z))^\top \sum_{\tau=1}^{\rho} \beta^\tau \boldsymbol{F}_{\boldsymbol{\omega}_{t+\tau}(z),t+\tau} \left(\boldsymbol{u}_{\boldsymbol{\omega}_{t+\tau}(z)} + \boldsymbol{e}_{\boldsymbol{\omega}_{t+\tau}(z)}^{\boldsymbol{P}_{t+\tau}} \right).$$

Example 3 (2-period finite dependence)**.** *If the model exhibits the 2-period finite dependence, then, for each* z*,*

$$(\tilde{\boldsymbol{f}}_t(d,z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1} = \boldsymbol{0} \quad \text{for } d \in \mathcal{D},$$

which provides $|\mathcal{Z}| \times (D+1)$ *restrictions for* D *unknowns,* $\boldsymbol{\omega}_{t+1}(z)$*. If we are able find such an* $\boldsymbol{\omega}_{t+1}(z)$*,* *then we may evaluate the conditional choice probability as*

$$\Lambda(d, \tilde{\boldsymbol{v}}_t(d, z)) = \frac{\exp\left(\tilde{u}_t(d, z) + \beta(\tilde{\boldsymbol{f}}_t(d, z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1}\left(\boldsymbol{u}_{\boldsymbol{\omega}_{t+1}(z)} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{\boldsymbol{\omega}_{t+1}(z)}\right)\right)}{\sum_{d'=0}^{D} \exp\left(\tilde{u}_t(d', z) + \beta(\tilde{\boldsymbol{f}}_t(d', z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1}\left(\boldsymbol{u}_{\boldsymbol{\omega}_{t+1}(z)} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{\boldsymbol{\omega}_{t+1}(z)}\right)\right)}. \quad (12)$$

Even if the model does not exhibits the 2-period finite dependence, we may possibly develop an estimation procedure by choosing $\boldsymbol{\omega}_{t+1}(z)$ for each $z \in \mathcal{Z}$ so that $(\tilde{\boldsymbol{f}}_t(d, z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1}$ is close to zero for all $d \in \mathcal{D}$.

4 An estimator based on finite dependence property

The above characterization of finite dependence can be numerically exploited to develop a computationally attractive estimator. The idea is to choose $\boldsymbol{\omega}_{t+1}(z)$ for each z such that $(\tilde{\boldsymbol{f}}_t(d, z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1}$ is close to zero for all $d \in \mathcal{D}$; then the conditional choice probabilities are approximated by (12) which is easy to compute.

For each z, define

$$r(z; \boldsymbol{\omega}_{t+1}(z)) = \sum_{d \in \mathcal{D}} ||(\tilde{\boldsymbol{f}}_t(d, z))^\top \boldsymbol{F}_{\boldsymbol{\omega}_{t+1}(z),t+1}||$$

and let

$$\hat{\boldsymbol{\omega}}_{t+1}(z) = \underset{\boldsymbol{\omega}_{t+1}(z) \in \Omega}{\arg\min}\; r(z; \boldsymbol{\omega}_{t+1}(z)). \quad (13)$$

Then,

$$\Lambda(d, \tilde{\boldsymbol{v}}_t(d, z)) \approx \frac{\exp\left(\tilde{u}_{\hat{\boldsymbol{\omega}}_t(z),t}(d, z) + \beta(\tilde{\boldsymbol{f}}_{\hat{\boldsymbol{\omega}}_t(z),t}(d, z))^\top \boldsymbol{F}_{\hat{\boldsymbol{\omega}}_{t+1}(z),t+1}\left(\boldsymbol{u}_{\hat{\boldsymbol{\omega}}_{t+1}(z)} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{\hat{\boldsymbol{\omega}}_{t+1}(z)}\right)\right)}{\sum_{d'=0}^{D} \exp\left(\tilde{u}_{\hat{\boldsymbol{\omega}}_t(z),t}(d', z) + \beta(\tilde{\boldsymbol{f}}_{\hat{\boldsymbol{\omega}}_t(z),t}(d', z))^\top \boldsymbol{F}_{\hat{\boldsymbol{\omega}}_{t+1}(z),t+1}\left(\boldsymbol{u}_{\hat{\boldsymbol{\omega}}_{t+1}(z)} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{\hat{\boldsymbol{\omega}}_{t+1}(z)}\right)\right)}.$$

In practice, it may be computationally costly to solve the minimization problem (13) across different values of z. Then, we may choose the same value of $\boldsymbol{\omega}_{t+1}(z)$ across z by solving the following minimization problem:

$$\hat{\boldsymbol{\omega}}_{t+1} = \underset{\boldsymbol{\omega}_{t+1} \in \Omega}{\arg\min} \sum_{z \in \mathcal{Z}} \lambda(z) r(z; \boldsymbol{\omega}_{t+1}),$$

where we may use the empirical frequency of $Z = z$ for the weight $\lambda(z)$.

5 An infinite horizon model under stationarity

We now consider an infinite horizon model under stationarity.

Assumption 5 (Infinite Horizon under Stationarity)**.** *The agent maximizes the total sum of discounted payoffs,* $E\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(u(d_\tau,z_\tau)+\boldsymbol{\epsilon}_\tau\right)|z_t\right]$*, where* $\beta\in(0,1)$*, the payoff function and the transition function are given by* $u(d,z)$ *and* $f(z'|z,d)$*, respectively, for all* $(d,z,z')\in\mathcal{D}\times\mathcal{Z}^2$ *and do not change over time.*

We define $\boldsymbol{V}$, $\tilde{\boldsymbol{v}}$, $\boldsymbol{u}_d$, $\boldsymbol{e}_d^{\boldsymbol{P}}$, $\boldsymbol{F}_d$, $\bar{\boldsymbol{U}}^{\boldsymbol{P}}$, $\bar{\boldsymbol{F}}^{\boldsymbol{P}}$ etc. similarly to $\boldsymbol{V}_t$, $\tilde{\boldsymbol{v}}_t$, $\boldsymbol{u}_{d,t}$, $\boldsymbol{e}_{d,t}^{\boldsymbol{P}}$, $\boldsymbol{F}_{d,t}$, $\bar{\boldsymbol{U}}_t^{\boldsymbol{P}_t}$, $\bar{\boldsymbol{F}}_t^{\boldsymbol{P}_t}$ etc. but using $u(d,z)$ and $f(z'|z,d)$ in place of $u_t(d,z)$ and $f_t(z'|z,d)$ under Assumption 5. Then, the integrated Bellman equation in probability space is given by

$$\boldsymbol{V}=\max_{\boldsymbol{P}\in\mathcal{P}}\bar{\boldsymbol{U}}^{\boldsymbol{P}}+\beta\bar{\boldsymbol{F}}^{\boldsymbol{P}}\boldsymbol{V}. \tag{14}$$

Define

$$\boldsymbol{u}(\{d_i\}_{i=1}^{|\mathcal{Z}|}):=\begin{bmatrix}u_t(d_1,1)\\ \vdots\\ u_t(d_{|\mathcal{Z}|},|\mathcal{Z}|)\end{bmatrix},\quad \boldsymbol{e}^{\boldsymbol{P}}(\{d_i\}_{i=1}^{|\mathcal{Z}|}):=\begin{bmatrix}e^{\boldsymbol{p}(1)}(d_1,1)\\ \vdots\\ e^{\boldsymbol{p}(|\mathcal{Z}|)}(d_{|\mathcal{Z}|},|\mathcal{Z}|)\end{bmatrix},\quad\text{and}\quad \boldsymbol{F}(\{d_i\}_{i=1}^{|\mathcal{Z}|}):=\begin{bmatrix}(\boldsymbol{f}_{d_1}(1))^\top\\ \vdots\\ (\boldsymbol{f}_{d_{|\mathcal{Z}|}}(|\mathcal{Z}|))^\top\end{bmatrix}.$$

With this notation, we have $\boldsymbol{u}_d=\boldsymbol{u}(\{d_i\}_{i=1}^{|\mathcal{Z}|})$, $\boldsymbol{e}_d^{\boldsymbol{P}}=\boldsymbol{e}^{\boldsymbol{P}}(\{d_i\}_{i=1}^{|\mathcal{Z}|})$, and $\boldsymbol{F}_d=\boldsymbol{F}(\{d_i\}_{i=1}^{|\mathcal{Z}|})$ when $d_i=d$ for all $i=1,...,|\mathcal{Z}|$.

The next proposition follows from Proposition 2.

Proposition 5. *Suppose that Assumptions 1-3 and 5 hold. Then,*

$$\begin{aligned}\boldsymbol{V}^*&=(\boldsymbol{I}-\beta\bar{\boldsymbol{F}}^{\boldsymbol{P}^*})^{-1}\bar{\boldsymbol{U}}^{\boldsymbol{P}^*}\\ &=(\boldsymbol{I}-\beta\boldsymbol{F}_d)^{-1}\left(\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}^*}+\boldsymbol{u}_d+\boldsymbol{e}_d^{\boldsymbol{P}^*}\right)\quad \textit{for any } d\in\mathcal{D}\\ &=(\boldsymbol{I}-\beta\boldsymbol{F}(\{d_i\}_{i=1}^{|\mathcal{Z}|}))^{-1}\left(\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}^*}+\boldsymbol{u}(\{d_i\}_{i=1}^{|\mathcal{Z}|})+\boldsymbol{e}^{\boldsymbol{P}^*}(\{d_i\}_{i=1}^{|\mathcal{Z}|})\right)\quad \textit{for any } \{d_i\}_{i=1}^{|\mathcal{Z}|}\in\mathcal{D}^{|\mathcal{Z}|}.\end{aligned}$$

Proof. The first and the second equalities follow from $\boldsymbol{V}^*=\bar{\boldsymbol{U}}^{\boldsymbol{P}^*}+\beta\bar{\boldsymbol{F}}^{\boldsymbol{P}^*}\boldsymbol{V}^*$ and $\boldsymbol{V}^*=\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}^*}+\boldsymbol{u}_d+\boldsymbol{e}_d^{\boldsymbol{P}^*}+\beta\boldsymbol{F}_d\boldsymbol{V}^*$, respectively, where the latter follows from Proposition 2, given that both $(\boldsymbol{I}-\beta\bar{\boldsymbol{F}}^{\boldsymbol{P}^*})$ and $(\boldsymbol{I}-\beta\boldsymbol{F}_d)$ are non-singular. Evaluating the i-th row of $\boldsymbol{V}^*=\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}^*}+\boldsymbol{u}_d+\boldsymbol{e}_d^{\boldsymbol{P}^*}+\beta\boldsymbol{F}_d\boldsymbol{V}^*$ at $d=d_i$ for $i=1,...,|\mathcal{Z}|$ and collecting them into a vector gives $\boldsymbol{V}^*=\nabla\bar{\boldsymbol{e}}^{\boldsymbol{P}^*}+\boldsymbol{u}(\{d_i\}_{i=1}^{|\mathcal{Z}|})+\boldsymbol{e}^{\boldsymbol{P}^*}(\{d_i\}_{i=1}^{|\mathcal{Z}|})+\beta\boldsymbol{F}(\{d_i\}_{i=1}^{|\mathcal{Z}|})\boldsymbol{V}^*$, and the third equality follows. □

Example 4. *Suppose that Assumptions 1-5 holds and that* $\boldsymbol{u}_0$ *is known. Then,* $\boldsymbol{V}=(\boldsymbol{I}-\beta\boldsymbol{F}_0)^{-1}\left(\boldsymbol{u}_0+\gamma-\log\boldsymbol{p}_0\right)$*, where* $\boldsymbol{p}_0=(p(0,1),...,p(0,|\mathcal{Z}|))^\top$*. In this case, given* $\boldsymbol{F}_0$ *and* $\boldsymbol{p}_0$*, we may* *evaluate $\Lambda(d, \tilde{\boldsymbol{v}}_0(z))$ as*

$$\Lambda(d, \tilde{\boldsymbol{v}}_0(z)) = \frac{\exp\Big(\tilde{u}_0(d,z) + \beta(\tilde{\boldsymbol{f}}_0(d,z))^\top(\boldsymbol{I} - \beta\boldsymbol{F}_0)^{-1}\left(u(0,z) + \gamma - \ln \boldsymbol{p}_0\right)\Big)}{\sum_{d'=0}^{D}\exp\Big(\tilde{u}_0(d',z) + \beta(\tilde{\boldsymbol{f}}_0(d',z))^\top(\boldsymbol{I} - \beta\boldsymbol{F}_0)^{-1}\left(u(0,z) + \gamma - \ln \boldsymbol{p}_0\right)\Big)}.$$

When both $u(0,z)$ and the transition function are known or can be estimated without solving the Bellman equation, then we may compute $(\tilde{\boldsymbol{f}}_0(d,z))^\top(\boldsymbol{I} - \beta\boldsymbol{F}_0)^{-1}\left(u(0,z) + \gamma - \ln \boldsymbol{p}_0\right)$ only once outside of the optimization routine, and $\Lambda(d, \tilde{\boldsymbol{v}}_0(z))$ can be evaluated across different parameter values without repeatedly evaluate $(\tilde{\boldsymbol{f}}_0(d,z))^\top(\boldsymbol{I} - \beta\boldsymbol{F}_0)^{-1}\left(u(0,z) + \gamma - \ln \boldsymbol{p}_0\right)$.

6 Monte Carlo Simulation

In this session, we show the consistency using a Monte Carlo simulation under homogeneous and heterogeneous agent assumption. The baseline firm entry-exit model follows that of Aguirregabiria and Magesan (2016) with flexible assumption that firms may have a higher profitability given past entry. Assume the firm's structural vector is represented by $\theta = (\theta_0^{VP}, \theta_1^{VP}, \theta_2^{VP}, \theta_0^{FC}, \theta_1^{FC}, \theta_0^{EC}, \theta_1^{EC})$. The state variables that are observed by the econometrician is $z = (z_1, z_2, z_3, z_4, \omega, y)$, where y is the current market entry status of the firm. In addition, the state variable that is observed by the firm but unobserved by the econometrician is ν. This state variable introduces the heterogeneity among the firms. The firm observe the state variables (z, ν) and make a decision $a \in \{0, 1\}$ at each state. $a = 1$ indicates that the firm operates in the market and $a = 0$ indicates that the firm does not operate in the market.

At each time t, the firm's state variable is $(z_{1,t}, \ldots, z_{4,t}, \omega_t, y_t, \nu_t)$, where $y_t = a_{t-1}$. The firm's flow payoff is defined as equation (46). The flow payoff comprises three component, the variable profit(VP), the fixed cost for operating in the market(FC) and the entry cost when entering the market (EC). If $y_{t-1} = 1$ and $a_t = 1$, the firm only pays the fixed cost to operate in this market. If $y_{t-1} = 0$ and $a_t = 1$, the firm pays the fixed cost as well as the entry cost to operate in this market.

$$\begin{aligned} u(d_t, z_t, \nu_t; \theta) &= a_t(VP_t - EC_t - FC_t) \\ \text{where } VP_t &= \exp(\omega + \nu)[\theta_0^{VP} + \theta_1^{VP} z_{1t} + \theta_2^{VP} z_{2t}] \\ FC_t &= [\theta_0^{FC} + \theta_1^{FC} z_{3t}] \\ EC_t &= (1 - y_t)[\theta_0^{EC} + \theta_1^{EC} z_{4t}]. \end{aligned} \tag{15}$$

The exogenous shocks $(z_1, z_2, z_3, z_4, \omega)$ follow independent $AR(1)$ process. To discretize the state space, I use Tauchen's method to construct the transition probabilities of theses discrete state variables(Tauchen, 1986). Suppose K represents the number of grid of the exogenous state variables. Each of the exogeneous state variables takes K values. The dimension of the state space

$|\mathcal{X}| = |\mathcal{Z}| * |\mathcal{Y}| = 2 * K^5$. Let $\{z_j^{(k)} : k = 1, \dots, K\}$ be the support of the state variable z_j, and define the width values $w_j^{(k)} = z_j^{(k+1)} - z_j^k$. Let $\tilde{z}_{jt}$ be a continuous latent variable that follows the $AR(1)$ process $\tilde{z}_{jt} = \gamma_0^j + \gamma_1^j \tilde{z}_{j,t-1} + e_{jt}$, with $e_{jt} \overset{i.i.d}{\sim} N(0, \sigma_j^2)$.

Consider the case where the transition density does not display the finite-dependence property. The transition density is defined as

$$f(z_j'|z_j) = \begin{cases} \Phi([z_j^{(1)} + (\omega_j^{(1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(1)}; \\ \Phi([z_j^{(k)} + (\omega_j^{(k)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) - & \\ \qquad \Phi([z_j^{(k-1)} + (\omega_j^{(k-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(k)}, k = 2, \dots, K-1; \\ 1 - \Phi([z_j^{(K-1)} + (\omega_j^{(K-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(K)}. \end{cases}$$

$$f(\omega'|\omega, a) = \begin{cases} \Phi([\omega^{(1)} + (\omega^{(1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(1)}; \\ \Phi([\omega^{(k)} + (\omega^{(k)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) - & \\ \qquad \Phi([\omega^{(k-1)} + (\omega^{(k-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(k)}, k = 2, \dots, K-1; \\ 1 - \Phi([\omega^{(K-1)} + (\omega^{(K-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(K)}. \end{cases}$$

6.1 Model without unobserved states

The estimation of model with homogeneous agents have been discussed in the past literature. Rust (1987) proposed a nested fixed point estimator to estimate the bus engine replacement cost. This method solve the value function for every candidate parameters. The method uses iteration to solve for the value function. However, this method suffers from computational cost when the state space is large because iteration takes a long time. Since the inversion of transition probability density matrix is involved. It is in-feasible when the state space is too large. Hotz and Miller (1993) proposed a two-step estimator also known as the Hotz-Miller inversion. This method utilize the property of the assumption that the unobserved idiosyncratic shock is Type I extreme value distributed. It shows that we can back out the continuation value associated with each choice. Aguirregabiria (2012) proposed the sequential version of the Hotz-Miller inversion. The idea is to update the value function only for each candidate of parameters. The method avoids solving for the full solution of the value function and there for is proven faster compared to the full information method, which is the nested fixed point(NFXP) as proposed by Rust (1987).

6.1.1 Nested fixed point estimator

Define the inverse mapping from the conditional choice probability to the value function as

$$\begin{aligned}&\Phi_{NFXP}(P;\theta) = (I - \beta \boldsymbol{F}_{\boldsymbol{\omega}_{t+1},t+1})^{-1}\left(\boldsymbol{u}_{\boldsymbol{\omega}_{t+1},t+1} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{\boldsymbol{\omega}_{t+1}}\right)\\ &\text{where } F^e(P) = \left[\sum_{d\in\mathcal{D}} p(d,z) f(z'|z,d)\right]_{z,z'\in\mathcal{Z}} \text{ and } u^e(P;\theta) = \left[\sum_{d\in\mathcal{D}} p(d,z) u(d,z;\theta)\right]_{z\in\mathcal{Z}},\end{aligned} \tag{16}$$

where $F^e(P)$ is the collection of mixture of transition densities with the conditional choice probabilities as the mixture probabilities and $u^e(P;\theta)$ is the vector of expected flow payoffs over the conditional choice probabilities. Therefore in equilibrium, the conditional choice probability P satisfies the fixed point constraint $P = \Psi_{NFXP}(P;\theta) = \Lambda_{NFXP}(\Phi(P;\theta);\theta)$. Suppose we have an initial consistent estimator of P denoted by $\hat{P}_0$. The nested pseudo likelihood (NPL) estimator start from the initial estimation $\tilde{P}_0$. At each iteration k, update the estimator of θ and the estimator of conditional choice probability:

Step 1: Given $\hat{P}_{k-1}$, update the estimator of θ by $\hat{\theta}_k = \arg\max_\theta \sum_i \sum_t \log \Psi(\hat{P}_{k-1};\theta)(d_{it}, z_{it})$.

Step 2: Update the conditional choice probability by estimated $\hat{\theta}_k$. $\hat{P}_k = \Psi(\hat{\theta}_k, \hat{P}_{d-1})(a_{it}, z_{it})$.

Alternatively, with the constraint given by proposition **??**, I construct another fixed point mapping on the conditional choice probability as

$$\begin{aligned}&\Phi_{EE}(P;\theta) = (I - \beta F_0)^{-1}(u_0(\theta) + e_0(P)),\\ &\text{where } F_0 = [f(z'|z,0)]_{z,z'\in\mathcal{Z}}, u_0(\theta) = [u(0,z;\theta)]_{z\in\mathcal{Z}} \text{ and } e_0(P) = [\gamma - \log(p(0,z))]_{z\in\mathcal{Z}}.\end{aligned} \tag{17}$$

6.1.2 Sequential estimator

Now consider a sequential likelihood estimator as discussed in Aguirregabiria and Mira (2002) and Aguirregabiria and Mira (2007). In addition to the fixed point estimation method, we can also define a sequential version of estimator with the alternative fixed point mapping is that $P = \Psi_{EE}(P;\theta) = \Gamma(\Phi_{EE}(P;\theta);\theta)$. By replacing the contraction mapping using the Euler equation contraction mapping, we can define an alternative estimator similar to the sequential estimator of one sequential estimator defined as in Aguirregabiria and Ho (2012). Similar to the NPL estimator, I start from an initial estimator of the conditional choice probability $\hat{P}_0$. At each iteration of k, update the estimator of θ and the estimator of conditional choice probability sequentially:

Step 1: Given $\hat{P}_{k-1}$, update the estimator of θ by $\hat{\theta}_k = \arg\max_\theta \sum_i \sum_t \log \Psi_A(\hat{P}_{k-1};\theta)(d_{it}, z_{it})$.

Step 2: Update the conditional choice probability by estimated $\hat{\theta}_k$. $\hat{P}_k = \Psi_{EE}(\hat{\theta}_k, \hat{P}_{d-1})(a_{it}, z_{it})$.

There are several candidates for updating the value function. The Bellman operator to update

V is

$$\Gamma_{VF}(z,V;\theta) = \gamma + \log\left(\sum_{d\in\mathcal{D}} \exp\left(u(d,z;\theta) + \beta f(z'|z,d)V(z')\right)\right), \tag{18}$$

with $\Gamma_{VF}(V;\theta) = \{\Gamma_{VF}(z,V;\theta)\}_{z\in\mathcal{Z}}$. As shown by proposition **??**, a similar operator to the Bellman operator can be defined as

$$\Gamma_{VF}(z,V;\theta) = \gamma + \log\left(\sum_{d\in\mathcal{D}} \exp\left(u(d,z;\theta) + \beta f(z'|z,d)V(z')\right)\right), \tag{19}$$

The MPEC estimator as discussed in Su and Judd (2012) has the insight that the estimation of θ can be viewed as a constrained optimization problem:

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{N}\sum_{t=1}^{T} \log\Lambda(d_{it}, z_{it}, V;\theta) \text{ subject to } V = \Gamma_{VF}(V;\theta). \tag{20}$$

Define a similar value function update equation using alternative fixed point mapping

$$\Gamma_{EE}(z,V,P;\theta) = u(0,z) + \gamma - log(p(0,z)) + \beta\sum_{z'} f(z'|z,0)V(z'). \tag{21}$$

6.1.3 MPEC estimator

A similar estimation to the MPEC estimator as Su and Judd (2012) can be defined as

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{N}\sum_{t=1}^{T} \log\Lambda(d_{it}, z_{it}, V;\theta) \text{ subject to } V = \Gamma_{EE}(V,P;\theta), P = \Lambda(V;\theta). \tag{22}$$

6.2 Model with unobserved state variables

Now consider the case where the agents are subject to unobserved heterogeneous states. In this section, I show that the mixture of unobserved states can be identified as well. Previous work of Arcidiacono and Ellickson (2011) has show that the unobserved states can be identified when the model exhibit finite-dependence. This paper show that even if the model does not have finite-dependence property, the unobserved states can be identified.

The idea is to use EM algorithm to identify the mixture probability. The general relationship between mixtures and the EM algorithm has been covered in a number of literature. The following section provide a brief introduction to apply the EM algorithm in this application. Consider a panel data set of I individuals, where for each individual i we observe T realizations from the data $\{d_{it}, z_{it}\}_{t=1}^{T}$.

If there exist an unobserved state variable $s_{it} \in \mathcal{S}$, where $\mathcal{S}$ is finite. To estimate the model, I follow the estimation strategy of Arcidiacono and Ellickson (2011) that combines the expectation-maximization (EM) algorithm and the conditional choice probability estimation when there is no unobserved heterogeneity. With unobserved heterogeneity, the estimation procedure include the additional step of updating the unobserved state variable. Let π_s denote the probability of state. With

$$\{\hat{\theta}, \hat{\pi}\} = \arg\max_{\theta,\pi} = \sum_{n=1}^{N} \log\left\{\sum_{s=1}^{S} \pi_s \Pi_{t=1}^{T} l(d_{it}, z_{it}, s; \theta)\right\}, \tag{23}$$

where $\hat{P}$ is a consistent estimator of the conditional choice probability. Given the ML estimates $(\hat{\theta}, \hat{\pi})$ and using Baye's rule, we can calculate $\hat{q}_{is}$, the probability n is in unobserved state s, as

$$\hat{q}_{is} = Pr(s_i = s | d_i, z_i; \hat{\theta}, \hat{\pi}, \hat{P}) = \frac{\hat{\pi}_s \Pi_{t=1}^{T} l(d_{it}, z_{it}, s, \hat{P}, \hat{\theta})}{\sum_{s'=1}^{S} \hat{\pi}_{s'} \Pi_{t=1}^{T} l(d_{it}, z_{it}, s', \hat{P}, \hat{\theta})}. \tag{24}$$

Then use $\hat{q}_i s$ to update π:

$$\hat{\pi}_s = \frac{1}{N} \sum_{i=1}^{N} \hat{q}_{is}. \tag{25}$$

The EM algorithm is a computationally attractive alternative to directly maximizing equation (23) To obtain this estimation, first fix some mixing probabilities $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_M)$ for the distribution of unobserved heterogeneity. Then estimate the structural parameter from the likelihood function with $\hat{pi}$ fixed. For each k-th round of iteration, obtain the estimation of θ given the unconditional probability. Next, update π_k by replacing $\hat{q}_{is}$ with $\hat{q}_{is,k}$. The algorithm begin with setting the initial values for $\hat{\theta}_0$, $\hat{\pi}_0$ and $\hat{P}_0$.

Step 1: Compute $\hat{q}_{is,k}$ as

$$\hat{q}_{is,k} = \frac{\hat{\pi}_{s,k-1} \Pi_{t=1}^{T} l(d_{it}, z_{it}, s, \hat{P}_{k-1}, \hat{\theta}_{k-1})}{\sum_{s' \in \mathcal{S}} \hat{\pi}_{s',k-1} \Pi_{t=1}^{T} l(d_{it}, z_{it}, s', \hat{P}_{k-1}, \hat{\theta}_{k-1})}. \tag{26}$$

Step 2: Using $\hat{q}_{is,k}$ to compute $\hat{\pi}_{s,k}$ using the equation:

$$\hat{\pi}_{s,k} = \frac{1}{N} \sum_{i=1}^{N} \hat{q}_{is,k}. \tag{27}$$

Step 3: Using $\hat{q}_{is,k}$ to update $\hat{P}_k$ with the equation:

$$\hat{p}_k(d, z, s) = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} q_{is,k} \mathbb{1}\{d_{it} = d\} \mathbb{1}\{z_{it} = z\}}{\sum_{i=1}^{N} \sum_{t=1}^{T} q_{is,k} \mathbb{1}\{z_{it} = z\}}. \tag{28}$$

Step 4: Update estimator of θ with the equation

$$\hat{\theta}_k = \arg\max_{\theta} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s \in \mathcal{S}} \hat{\pi}_{s,k-1} \log l(d_{it}, x_{it}, s, \hat{P}_{k-1}, \hat{\theta}_{k-1}). \tag{29}$$

6.3 The dynamic discrete choice model with unobserved states

The section lays out a general class of dynamic discrete choice models with unobserved state and derive a representation of the conditional valuation function upon which the identification and estimation based on. This class of models extend the result of Arcidiacono and Ellickson (2011).

In each period, an individual observes a state from finite support and chooses among $|D|$ mutually exclusive actions similar to section 2. In addition to the state $z \in \mathcal{Z}$ observed by the researcher, the agent also make the choice conditional the state $s \in \mathcal{S}$ unobserved by the researcher.

The flow utility is conditional on d, z, s. The individual agent chooses the discrete choice in each time period to maximize the discounted payoff

$$E\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U(d_\tau, z_\tau, s_\tau, \epsilon_\tau)\right]. \tag{30}$$

The transition of z follows the transition density $f(z'|z,d)$. Define the transition of the unobserved state as $\pi(s'|s)$ and the initial distribution of the unobserved state $\pi(s_1|z_1)$.

In the estimation, we need to specify the likelihood function for a given θ. First define the likelihood of agent i choose action d given state z as

$$l(d, z, s; \theta, P) = Pr\left\{d = \arg\max_{d' \in \mathcal{D}} \{v(d', z) + \epsilon'_d\} \,|z, s; \theta, \pi, P\right\}. \tag{31}$$

The corresponding likelihood of observing $\{d_{it}, z_{t+1}\}_{t=1}^{T}$ is then defined as

$$\begin{aligned} L(\{d_{it}, z_{t+1}\}_{t=1}^{T}; \theta, \pi, P) = & \sum_{s_{i1} \in \mathcal{S}} \dots \sum_{s_{iT} \in \mathcal{S}} \pi(s_{i1}|z_{i1}) l(d_{i1}, z_{i1}, s_{i1}; \theta, P) \\ & \prod_{t=2}^{T} \big\{\pi(s_{it}|s_{i,t-1}) f(z_{it}|z_{i,t-1}, d_{i,t-1}) l(d_{it}, z_{it}, s_{it}; \theta, P)\big\}. \end{aligned} \tag{32}$$

The log-likelihood of the sample is then given by

$$\sum_{i=1}^{N} L(\{d_{it}, z_{t+1}\}_{t=1}^{T}; \theta, \pi, P). \tag{33}$$

Therefore, the estimator following the example as shown in section 4, can be defined by the following equation. Given the estimation $\hat{P}$, define the estimators of (θ, π) as

$$(\hat{\theta}, \hat{\pi}) = \arg\max_{\theta,\pi} \sum_{i=1}^{N} \log L(\{d_{it}, z_{t+1}\}_{t=1}^{T}; \theta, \pi, \hat{P}). \tag{34}$$

With the result, we can construct two-step estimator as in Hotz and Miller (1993), the sequential estimator as in Aguirregabiria (2012) or the MPEC estimator as in Su and Judd (2012).

The flow utility function is indexed by parameter θ. Therefore, $u(d, z; \theta)$ is the payoff of action d at state z given parameter value θ. Let the $\Lambda(V; \theta)$ be the optimal probability mapping (OCP) given parameter value θ.

$$\Lambda(d, z, V; \theta) = \frac{\exp(u(d, z; \theta) + \sum_{z'} f(z'|z, d)V(z'))}{\sum_{d' \in \mathcal{D}} \exp(u(d', z; \theta) + \sum_{z'} f(z'|z, d')V(z'))}, \tag{35}$$

and $\Lambda(V; \theta) = \{\Lambda(d, z, V; \theta)\}_{d \in \mathcal{D}, z \in \mathcal{Z}}$.

To estimate unobserved state variable, the EM algorithm is time costly. Arcidiacono and Ellickson (2011) exploit the finite-dependence property to estimate the unobserved state variable. The model that exhibit finite dependence does not require solving the value function, therefore the EM algorithm is much faster. The algorithm follows that modified EM algorithm from Arcidiacono and Ellickson (2011), except that in the updating procedure, the algorithm does not depend on the finite-dependence property.

Step 1: The first step of the k-th iteration is to calculate the conditional probability of being in each unobserved state in each time period given the values of the structural parameters and conditional choice probabilities from the $(k-1)$-th iteration $\{\hat{\theta}_{k-1}, \hat{\pi}_{k-1}, \hat{P}_{k-1}\}$. Define the sample analogue estimator similar to that as equation (32):

$$L_{i,k-1} = L(\{d_{it}, z_{t+1}\}_{t=1}^{T}; \hat{\theta}_{k-1}, \hat{\pi}_{k-1}, \hat{P}_{k-1}) = \sum_{s_{i1} \in \mathcal{S}} \cdots \sum_{s_{iT} \in \mathcal{S}} \hat{\pi}_{k-1}(s_{i1}|z_{i1}) l(d_{i1}, z_{i1}, s_{i1}; \hat{\theta}_{k-1}, \hat{P}_{k-1})$$
$$\prod_{t=2}^{T} \left\{ \hat{\pi}_{k-1}(s_{it}|s_{i,t-1}) f(z_{it}|z_{i,t-1}, d_{i,t-1}) l(d_{it}, z_{it}, s_{it}; \hat{\theta}_{k-1}, \hat{P}_{k-1}) \right\}. \tag{36}$$

Define the joint likelihood function of the data and unobserved state s occurring at time t as

$$L_{i,k-1}(s_{it} = s) \tag{37}$$

With the likelihood function, update probability of i being in unobserved state s at time t $\hat{q}_{its,k}$ with the equation

$$\hat{q}_{its,k} = \frac{L_{i,k-1}(s_{it} = s)}{L_{i,k-1}}. \tag{38}$$

Step 2: The next step is to update the initial distribution of $\pi(s_{i1}|z_{i1})$. Setting $t = 1$ in equation (38) yields the conditional probability of the i-th individual being in unobserved state s in the first time period. When the state variables are exogeneous at $t = 1$, we can update the probabilities for the initial states by averaging the conditional probabilities obtained from the previous iteration over the sample population: To allow for situations where the distribution of the unobserved states in the first period depends on the values of the observed state variables, I form the averages over $q_{its,k}$ for each value of x.

$$\hat{\pi}_k(s|z) = \frac{\sum_{i=1}^N \hat{q}_{i1s,k} \mathbb{1}\{z_{i1} = z\}}{\sum_{i=1}^N \mathbb{1}\{z_{i1} = z\}}. \tag{39}$$

Step 3: The next step is to update the transition probabilities of the unobserved states. The joint probability of i being in state s at time $t-1$ and state s' at time t can be expressed as the product of $q_{i,t-1,s,k}$ and $q_{i,t,s',k}$. Update the transition probability using the formula

$$\hat{\pi}_k(s'|s) = \frac{\sum_{i=1}^N \sum_{t=2}^T q_{i,t-1,s,k} q_{i,t-1,s',k}}{\sum_{i=1}^N \sum_{t=2}^T q_{i,t-1,s,k}}. \tag{40}$$

Step 4: The next step is to update the conditional choice probabilities.

$$p_k(d, x, s) = \frac{\sum_{i=1}^N \sum_{t=1}^T q_{its,k} \mathbb{1}\{d_{it} = d\} \mathbb{1}\{z_{it} = z\}}{\sum_{i=1}^N \sum_{t=1}^T q_{its,k} \mathbb{1}\{z_{it} = z\}}. \tag{41}$$

Step 5: The maximization step of the EM algorithm update the estimation of θ. Obtain $\hat{\theta}_k$ by maximizing the log likelihood function:

$$\hat{\theta}_k = \arg\max_\theta \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^S \sum_{d=0}^T q_{its,k} \log l(d_{it}, z_{it}, s_{it}; \theta, \hat{P}_k). \tag{42}$$

There are several alternative candidate for likelihood function. This paper provide an alternative method that can estimate unobserved state of models that does not exhibit finite dependence. As shown as in (55), we can estimate the value function from the conditional choice probability using $\Phi_{EE}(P;\theta) = (I - \beta F_0)^{-1}(u_0(\theta) + e_0(P))$ as shown in equation (17). First obtain the non-parametric estimation of $\hat{F}_0$, with

$$\hat{f}(z'|z, 0) = \frac{\sum_{i=1}^N \sum_{t=1}^{T-1} \mathbb{1}\{z_{i,t+1} = z'\} \mathbb{1}\{z_{it} = z\} \mathbb{1}\{d_{it} = 0\}}{\sum_{i=1}^N \sum_{t=1}^{T-1} \mathbb{1}\{z_{it} = z\} \mathbb{1}\{d_{it} = d\}}.$$

Note that the likelihood function $l(d_{i1}, z_{i1}, s_{i1}; \hat{\theta}_{k-1}, \hat{P}_{k-1})$ can take variety of forms. Arcidiacono and Ellickson (2011) defined the likelihood function using finite-dependence property.

Where the mapping defined on probability space is

$$\Psi_{FD}(P;\theta)(d,z,s) = \frac{\tilde{u}(d,z,s) + \beta \sum_{z'} \sum_{s'} \pi(s'|s)\tilde{f}(z'|z,d)\{u(0,z',s') + \gamma - \log(p(0,z'))\}}{1 + \sum_{d'} \tilde{u}(d',z,s) + \beta \sum_{z'} \sum_{s'} \pi(s'|s)\tilde{f}(z'|z,d')\{u(0,z',s') + \gamma - \log(p(0,z'))\}}. \tag{43}$$

Likewise, we can define the mapping in probability space in a similar pattern to that in the previous section. Define $\Pi = [\pi(s'|s)]_{s',s}$ as the transition matrix of the unobserved state variable.

$$\begin{aligned} \Psi_{SEQ}(P;\theta) &= \Gamma(\Phi(P;\theta);\theta), \\ \Phi(P;\theta) &= (I - \beta F^e(P) \otimes \Pi)^{-1}(u^e(P;\theta) + \gamma - log(P)), \\ \text{where } F^e(P) &= \left[\sum_{d\in\mathcal{D}} p(d,z)f(z'|z,d)\right]_{z,z'\in\mathcal{Z}} \text{ and } u^e(P;\theta) = \left[\sum_{d\in\mathcal{D}} p(d,z)u(d,z;\theta)\right]_{z\in\mathcal{Z}}. \end{aligned} \tag{44}$$

With the Euler Equation, we can define the contraction mapping as

$$\begin{aligned} \Psi_{SEQ-EE}(P;\theta) &= \Gamma(\Phi_{EE}(P;\theta);\theta), \\ \Phi_{EE}(P;\theta) &= (I - \beta F_0 \otimes \Pi)^{-1}(u_0(\theta) + \gamma - log(P)). \end{aligned} \tag{45}$$

7 Application in stationary game with unobserved heterogeneity

The baseline model follows that of Aguirregabiria and Magesan (2016). Assume the firm's structural vector is represented by $\theta = (\theta_0^{VP}, \theta_1^{VP}, \theta_2^{VP}, \theta_0^{FC}, \theta_1^{FC}, \theta_0^{EC}, \theta_1^{EC})$. The state variables that are observed by the econometrician is $z = (z_1, z_2, z_3, z_4, \omega, y)$, where y is the current market entry status of the firm. In addition, the state variable that is observed by the firm but unobserved by the econometrician is ν. This state variable introduces the heterogeneity among the firms. The firm observe the state variables (z, ν) and make a decision $a \in \{0, 1\}$ at each state. $a = 1$ indicates that the firm operates in the market and $a = 0$ indicates that the firm does not operate in the market.

At each time t, the firm's state variable is $(z_{1,t}, \ldots, z_{4,t}, \omega_t, y_t, \nu_t)$, where $y_t = a_{t-1}$. The firm's flow payoff is defined as equation (46). The flow payoff comprises three component, the variable profit(VP), the fixed cost for operating in the market(FC) and the entry cost when entering the market (EC). If $y_{t-1} = 1$ and $a_t = 1$, the firm only pays the fixed cost to operate in this market. If $y_{t-1} = 0$ and $a_t = 1$, the firm pays the fixed cost as well as the entry cost to operate in this market.

$$\begin{aligned} u(d_t, z_t, \nu_t; \theta) &= a_t(VP_t - EC_t - FC_t) \\ \text{where } VP_t &= \exp(\omega + \nu)[\theta_0^{VP} + \theta_1^{VP} z_{1t} + \theta_2^{VP} z_{2t}] \\ FC_t &= [\theta_0^{FC} + \theta_1^{FC} z_{3t}] \\ EC_t &= (1 - y_t)[\theta_0^{EC} + \theta_1^{EC} z_{4t}]. \end{aligned} \tag{46}$$

The exogenous shocks $(z_1, z_2, z_3, z_4, \omega)$ follow independent $AR(1)$ process. To discretize the state space, I use Tauchen's method to construct the transition probabilities of theses discrete state variables(Tauchen, 1986). Suppose K represents the number of grid of the exogenous state variables. Each of the exogeneous state variables takes K values. The dimension of the state space $|\mathcal{X}| = |\mathcal{Z}| * |\mathcal{Y}| = 2 * K^5$. Let $\{z_j^{(k)} : k = 1, \ldots, K\}$ be the support of the state variable z_j, and define the width values $w_j^{(k)} = z_j^{(k+1)} - z_j^k$. Let $\tilde{z}_{jt}$ be a continuous latent variable that follows the $AR(1)$ process $\tilde{z}_{jt} = \gamma_0^j + \gamma_1^j \tilde{z}_{j,t-1} + e_{jt}$, with $e_{jt} \overset{i.i.d}{\sim} N(0, \sigma_j^2)$. With proposition (**??**), the transition density function determines whether the model shows finite dependence property. I now consider the two models where one has the finite dependence property and the other does not. I show that when the model does not show the finite-dependence property, the estimator proposed by Aguirregabiria and Magesan (2016) and Arcidiacono and Ellickson (2011) is biased. The nexted fixed point estimator (NFXP) is unbiased, and my proposed Euler equation (EE) method is superior to the NFXP method.

7.1 Model with finite dependence

For the first experiment, I consider a model with the finite-dependence property. In this experiment, I choose the transition density such that the model satisfy the condition specified in proposition **??**. The transition probability for the discrete state variable z_{jt} is not dependent on the action chosen:

$$
f(z_j'|z_j) = \begin{cases}
\Phi([z_j^{(1)} + (\omega_j^{(1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(1)}; \\
\Phi([z_j^{(k)} + (\omega_j^{(k)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) - & \\
\quad \Phi([z_j^{(k-1)} + (\omega_j^{(k-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(k)}, k = 2, \ldots, K-1; \\
1 - \Phi([z_j^{(K-1)} + (\omega_j^{(K-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(K)}.
\end{cases}
$$

$$
f(\omega'|\omega, a) = \begin{cases}
\Phi([\omega^{(1)} + (\omega^{(1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(1)}; \\
\Phi([\omega^{(k)} + (\omega^{(k)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega]/\sigma) - & \\
\quad \Phi([\omega^{(k-1)} + (\omega^{(k-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega]/\sigma) & \omega' = \omega^{(k)}, k = 2, \ldots, K-1; \\
1 - \Phi([\omega^{(K-1)} + (\omega^{(K-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega]/\sigma) & \omega' = \omega^{(K)}.
\end{cases}
$$

As show in examples, this transitioned density is finite 2 period single action dependent. The data generating process parameters used in the numerical and Monte Carlo experiments is summarize in the following table 1. In this model, firms have heterogeneous unobserved profitability shocks ν. The unobserved profitability shock enters the pay-off function as additive term to the observed profitability shock as described in equation (46). To simplify the experiment, assume that the unobserved shock does not vary through time. If one firm has a lower unobserved profitability shock at the $t = 0$, the firm will always have a lower profitability.

Table 1: Parameters in DGP

Payoff Parameters:	$\theta_0^{VP} = 0.5$	$\theta_1^{VP} = 1.0$	$\theta_2^{VP} = -1.0$
	$\theta_0^{FC} = 0.5$	$\theta_1^{FC} = 1.0$	
	$\theta_0^{EC} = 1.0$	$\theta_1^{EC} = 1.0$	
Each z_k state variable	z_{kt} is AR(1),	$\gamma_0^k = 0,$	$\gamma_1^k = 0.6$
Productivity	ω_t is AR(1),	$\gamma_0^\omega = 0,$	$\gamma_1^\omega = 0.9$
Discount Factor	$\beta = 0.95.$		

In the simulation, I compare four estimation method to estimate the model: nested fixed point estimator(NFXP)(Rust (1987)), the finite-dependence(FD)(Aguirregabiria and Magesan (2016),Arcidiacono and Ellickson (2011)), the sequential estimator(SEQ)(Aguirregabiria and Mira (2002)) and my proposed Euler equation estimator. With the property of finite-dependence, the FD estimator performs as well as the NFXP, which is also known as the full estimation method. With the Euler equation estimator(EE), we also solve for the full solution. The EE estimator can solve for the solution faster than the NFXP estimator. In this experiment, the EE estimator does not show much advantages, because FD estimator can consistently estimate the parameters yet is much faster than other methods. This is because that when the model shows finite-dependence property, the FD estimator can estimate the parameters without solving for the value function as discussed in section 3.

Table 2: Simulation results for the market Entry Exit Problem(Two type mixture)

	DGP	NFXP	EE	FD	SEQ
θ_0^{VP}	0.5	0.50209	0.5021	0.5021	0.5021
		(0.03493)	(0.03486)	(0.03486)	(0.03486)
θ_1^{VP}	1	0.99894	0.99894	0.99894	0.99894
		(0.04759)	(0.04757)	(0.04757)	(0.04757)
θ_2^{VP}	-1	-1.00098	-1.00097	-1.00097	-1.00097
		(0.04672)	(0.04668)	(0.04668)	(0.04668)
θ_0^{FC}	0.5	0.50067	0.50084	0.50084	0.50084
		(0.05186)	(0.05187)	(0.05187)	(0.05187)
θ_1^{FC}	1	1.00257	1.00257	1.00257	1.00257
		(0.06503)	(0.06503)	(0.06503)	(0.06503)
θ_0^{EC}	1	0.99706	0.99698	0.99698	0.99698
		(0.07129)	(0.0715)	(0.0715)	(0.0715)
θ_1^{EC}	1	1.00517	1.00551	1.00551	1.00551
		(0.10378)	(0.10396)	(0.10396)	(0.10396)

Table 3: Average time used in two type mixture

	NFXP	EE	FD	SEQ
Time	9.60614	1.00513	0.83298	0.88540
Iteration	5.97000	5.98200	5.98200	5.98200

7.2 The model without finite Dependence property

Now consider the case where the transition density does not display the finite-dependence property. The transition density is defined as

$$f(z_j'|z_j) = \begin{cases} \Phi([z_j^{(1)} + (\omega_j^{(1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(1)}; \\ \Phi([z_j^{(k)} + (\omega_j^{(k)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) - & \\ \quad \Phi([z_j^{(k-1)} + (\omega_j^{(k-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(k)}, k = 2, \ldots, K-1; \\ 1 - \Phi([z_j^{(K-1)} + (\omega_j^{(K-1)}/2) - \gamma_0^j - \gamma_1^j z]/\sigma_j) & z' = z_j^{(K)}. \end{cases}$$

$$f(\omega'|\omega, a) = \begin{cases} \Phi([\omega^{(1)} + (\omega^{(1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(1)}; \\ \Phi([\omega^{(k)} + (\omega^{(k)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) - & \\ \quad \Phi([\omega^{(k-1)} + (\omega^{(k-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(k)}, k = 2, \ldots, K-1; \\ 1 - \Phi([\omega^{(K-1)} + (\omega^{(K-1)}/2) - \gamma_0^\omega - \gamma_1^\omega \omega - \gamma_a a]/\sigma) & \omega' = \omega^{(K)}. \end{cases}$$

The data generating process used in the numerical and Monte Carlo experiments is summarize in the following table 4. With all other parameters remain the same as the first experiment, the action taken at each time permanently impact the transition of profitability. This can be viewed as the firm's learning effect of operating in the market. When a firm operate in the market for one time period, the firm will permanently increase the future profitability shocks.

Table 4: Parameters in DGP

Payoff Parameters:	$\theta_0^{VP} = 0.5$	$\theta_1^{VP} = 1.0$	$\theta_2^{VP} = -1.0$
	$\theta_0^{FC} = 0.5$	$\theta_1^{FC} = 1.0$	
	$\theta_0^{EC} = 1.0$	$\theta_1^{EC} = 1.0$	$\gamma_a = 0.5$
Each z_k state variable	z_{kt} is AR(1),	$\gamma_0^k = 0,$	$\gamma_1^k = 0.6$
Productivity	ω_t is AR(1),	$\gamma_0^\omega = 0,$	$\gamma_1^\omega = 0.9$
Past action on productivity	$\gamma_a = 0.5$		
Discount Factor	$\beta = 0.95.$		

Without the finite dependence property, the FD estimator as discussed in Aguirregabiria and Magesan (2016) and Arcidiacono and Ellickson (2011) cannot consistently estimate the parameters because the update mapping will be incorrectly formed as discussed in section 4. In this experiment, I include the FD estimation to show the estimators are biased. The simulation results in table **??**. From the table, we can see that the FD estimator is not consistent with the data generating parameters. The EE estimator has less biasness compare to the FD estimator.

7.3 Non-finite-dependent model with larger state space

Now consider a dynamic discrete choice model with larger state space. In this experiment, I increase the number of grids in this model. Estimating using NFXP is not feasible, because as the state space increases, the computational cost grows exponentially. In this experiment, I use the finite dependence(FD) estimator, the sequential estimator using value function mapping to update(SEQ) and the sequential estimator using Euler equation mapping to update(SEQ-EE). The results in table **??** show that both SEQ and SEQ-EE has smaller bias than FD estimator. As table **??** shows, the time using SEQ-EE estimator is less than the time using SEQ. With large state space where NFXP is not feasible, SEQ-EE serves as a substitute for SEQ.

7.4 Simulation results

Table 5: The norm of the differences in transition densities when $nGrid = 2$

γ_a	0	1	2	3	4	5
$\lvert\tilde{\boldsymbol{F}}\boldsymbol{F_0}\rvert$ norm before modified	0.0	0.5253	0.8997	1.0127	1.0261	1.0267
$\lvert\tilde{\boldsymbol{F}}\boldsymbol{F^{\omega}}\rvert$ norm after modified	0.0	0.2521	0.1164	0.0143	0.0007	0.0

† This experiment uses 2 step estimator with different values ofγ_a.

Table 6: Dynamic discrete choice two-step estimators

	FD	$FD2$	AFD	$AFD2$	HM	EE	$HM(true)$	$EE(true)$	$FD2(BEL)$	$AFD2(BEL)$
					$Market = 200, Time = 20, \gamma_a = 0$					
θ_0^{VP}	0.4845	0.4845	0.4845	0.4845	0.5016	0.4845	0.5040	0.5041	0.4845	0.4845
	(0.0706)	(0.0706)	(0.0706)	(0.0706)	(0.0350)	(0.0706)	(0.0350)	(0.0400)	(0.0706)	(0.0706)
θ_0^{FC}	0.5447	0.5447	0.5447	0.5447	0.5098	0.5447	0.5171	0.5244	0.5447	0.5447
	(0.0904)	(0.0904)	(0.0904)	(0.0904)	(0.0627)	(0.0904)	(0.0628)	(0.0851)	(0.0904)	(0.0904)
					$Market = 200, Time = 120, \gamma_a = 0$					
θ_0^{VP}	0.4963	0.4963	0.4963	0.4963	0.4983	0.4963	0.4987	0.4986	0.4963	0.4963
	(0.0189)	(0.0189)	(0.0189)	(0.0189)	(0.0140)	(0.0189)	(0.0140)	(0.0162)	(0.0189)	(0.0189)
θ_0^{FC}	0.4990	0.4990	0.4990	0.4990	0.4954	0.4990	0.4966	0.4955	0.4990	0.4990
	(0.0301)	(0.0301)	(0.0301)	(0.0301)	(0.0279)	(0.0301)	(0.0280)	(0.0384)	(0.0301)	(0.0301)
					$Market = 200, Time = 20, \gamma_a = 1$					
θ_0^{VP}	0.4439	0.5631	0.4975	0.5524	0.5002	0.5638	0.5034	0.5023	0.5437	0.5546
	(0.0769)	(0.1006)	(0.0769)	(0.0774)	(0.0557)	(0.1008)	(0.0561)	(0.0502)	(0.0909)	(0.0861)
θ_0^{FC}	0.2538	0.7058	0.3829	0.6421	0.5095	0.7082	0.5136	0.5112	0.6319	0.6487
	(0.1963)	(0.2178)	(0.2046)	(0.1962)	(0.1301)	(0.2181)	(0.1317)	(0.1284)	(0.2072)	(0.2360)
					$Market = 200, Time = 120, \gamma_a = 1$					
θ_0^{VP}	0.4000	0.5047	0.4486	0.5083	0.5004	0.5053	0.5011	0.5013	0.4942	0.4999
	(0.0232)	(0.0273)	(0.0216)	(0.0242)	(0.0233)	(0.0273)	(0.0234)	(0.0208)	(0.0266)	(0.0243)
θ_0^{FC}	0.1254	0.5221	0.2530	0.5244	0.4998	0.5242	0.5013	0.4985	0.4822	0.4963
	(0.0554)	(0.0631)	(0.0546)	(0.0609)	(0.0602)	(0.0632)	(0.0605)	(0.0595)	(0.0624)	(0.0635)
					$Market = 200, Time = 20, \gamma_a = 5$					
θ_0^{VP}	0.3434	0.5679	0.4925	0.5067	0.5307	0.5691	0.5213	0.5105	0.5004	0.5071
	(0.0790)	(0.1457)	(0.0860)	(0.0908)	(0.0800)	(0.1460)	(0.0796)	(0.0496)	(0.1027)	(0.0882)
θ_0^{FC}	-0.0155	0.7095	0.4432	0.4751	0.5833	0.7134	0.5512	0.5229	0.4912	0.4763
	(0.2228)	(0.3321)	(0.2402)	(0.2518)	(0.2209)	(0.3330)	(0.2199)	(0.1491)	(0.2482)	(0.2455)
					$Market = 200, Time = 120, \gamma_a = 5$					
θ_0^{VP}	0.3058	0.4965	0.4829	0.4954	0.4982	0.4975	0.4964	0.4970	0.4536	0.4955
	(0.0333)	(0.0484)	(0.0436)	(0.0453)	(0.0395)	(0.0485)	(0.0396)	(0.0250)	(0.0418)	(0.0445)
θ_0^{FC}	-0.1239	0.4920	0.4583	0.4860	0.4977	0.4953	0.4915	0.4957	0.3533	0.4862
	(0.0845)	(0.1237)	(0.1096)	(0.1140)	(0.1036)	(0.1239)	(0.1036)	(0.0706)	(0.1069)	(0.1119)
					$Market = 100, Time = 20, \gamma_a = 2$					
θ_0^{VP}	0.4558	0.6841	0.6506	0.5315	0.5135	0.6853	0.5105	0.5070	0.6216	0.6204
	(0.1125)	(0.1821)	(0.1628)	(0.1670)	(0.1180)	(0.1825)	(0.1193)	(0.0786)	(0.1411)	(0.1720)
θ_0^{FC}	0.2639	1.0214	0.7998	0.4830	0.5415	1.0253	0.5208	0.5142	0.8138	0.7792
	(0.3007)	(0.4546)	(0.4448)	(0.4667)	(0.3146)	(0.4557)	(0.3181)	(0.2219)	(0.3535)	(0.4790)
					$Market = 1000, Time = 20, \gamma_a = 2$					
θ_0^{VP}	0.3324	0.5056	0.5459	0.5064	0.4972	0.5065	0.4968	0.4979	0.4720	0.5120
	(0.0295)	(0.0395)	(0.0355)	(0.0350)	(0.0330)	(0.0395)	(0.0331)	(0.0226)	(0.0360)	(0.0374)
θ_0^{FC}	-0.0535	0.5214	0.5729	0.5174	0.4947	0.5244	0.4926	0.4961	0.4095	0.5361
	(0.0926)	(0.1122)	(0.0994)	(0.0955)	(0.0929)	(0.1124)	(0.0930)	(0.0662)	(0.1066)	(0.1064)

† The data generating process is $\theta_0^{VP} = 0.5, \theta_0^{FC} = 0.5$. The sample size is $Market \cdot Time$.

† The table shows a subset of the parameters. The misspecified estimator(FD,AFD) generate larger biasness on these two estimators.

† The results are based on 100 Monte Carlo simulations.

† $HM(true)$ and $EE(true)$ use the data generating conditional choice probabilities rather than the estimated conditional choice probabilities.

† The complete estimation results are in the appendix.

Table 7: The mean and standard deviation of sequential estimators

	FD	$FD2$	AFD	$AFD2$	$FD2(BEL)$	$AFD2(BEL)$	HM	EE	$HM(true)$	$EE(true)$	$SEQ(1)$	$SEQ(2)$	$SEQ(5)$
$Market = 200, Time = 20, \gamma_a = 0$													
θ_0^{VP}	0.5163	0.5079	0.5163	0.4799	0.5079	0.5079	0.5080	0.5079	0.5080	0.5079	0.5079	0.5079	0.5079
	(0.0376)	(0.0369)	(0.0376)	(0.0672)	(0.0369)	(0.0369)	(0.0368)	(0.0369)	(0.0368)	(0.0369)	(0.0369)	(0.0369)	(0.0369)
θ_0^{FC}	0.4203	0.5146	0.4203	0.5516	0.5146	0.5146	0.5148	0.5146	0.5148	0.5146	0.5146	0.5146	0.5146
	(0.0635)	(0.0591)	(0.0635)	(0.0804)	(0.0591)	(0.0591)	(0.0593)	(0.0591)	(0.0593)	(0.0591)	(0.0591)	(0.0591)	(0.0591)
$Market = 200, Time = 20, \gamma_a = 1$													
θ_0^{VP}	0.3721	0.5010	0.3722	0.5525	0.5010	0.5009	0.5010	0.5010	0.5010	0.5010	0.4987	0.5010	0.5010
	(0.0482)	(0.0551)	(0.0482)	(0.0665)	(0.0551)	(0.0552)	(0.0552)	(0.0551)	(0.0552)	(0.0551)	(0.0549)	(0.0551)	(0.0551)
θ_0^{FC}	-0.1279	0.4971	-0.1277	0.6387	0.4971	0.4969	0.4972	0.4971	0.4972	0.4971	0.4883	0.4970	0.4971
	(0.1194)	(0.1381)	(0.1195)	(0.1722)	(0.1381)	(0.1382)	(0.1383)	(0.1381)	(0.1383)	(0.1381)	(0.1370)	(0.1381)	(0.1381)
$Market = 200, Time = 20, \gamma_a = 5$													
θ_0^{VP}	0.3128	0.5084	-0.1775	0.4940	0.5085	0.5083	0.5096	0.5084	0.5087	0.5084	0.5043	0.5084	0.5084
	(0.0661)	(0.0925)	(0.1838)	(0.1151)	(0.0925)	(0.0929)	(0.0938)	(0.0925)	(0.0926)	(0.0925)	(0.0921)	(0.0925)	(0.0925)
θ_0^{FC}	-0.2597	0.5167	-1.9434	0.4391	0.5168	0.5166	0.5207	0.5167	0.5175	0.5167	0.5034	0.5167	0.5167
	(0.1703)	(0.2506)	(0.6454)	(0.3056)	(0.2506)	(0.2513)	(0.2567)	(0.2506)	(0.2510)	(0.2506)	(0.2493)	(0.2506)	(0.2506)
$Market = 100, Time = 20, \gamma_a = 2$													
θ_0^{VP}	0.3252	0.5132	-0.2742	0.5232	0.5132	0.5129	0.5130	0.5132	0.5130	0.5132	0.5092	0.5132	0.5132
	(0.0925)	(0.1213)	(0.1705)	(0.1599)	(0.1213)	(0.1212)	(0.1213)	(0.1213)	(0.1213)	(0.1213)	(0.1206)	(0.1213)	(0.1213)
θ_0^{FC}	-0.2318	0.5275	-2.1828	0.4410	0.5275	0.5273	0.5275	0.5275	0.5275	0.5275	0.5140	0.5275	0.5275
	(0.2302)	(0.3184)	(0.5916)	(0.4069)	(0.3184)	(0.3181)	(0.3183)	(0.3184)	(0.3183)	(0.3184)	(0.3158)	(0.3184)	(0.3184)
$Market = 1000, Time = 20, \gamma_a = 2$													
θ_0^{VP}	0.3165	0.5037	-0.3303	0.5129	0.5037	0.5036	0.5036	0.5037	0.5036	0.5037	0.5011	0.5037	0.5037
	(0.0261)	(0.0347)	(0.0412)	(0.0385)	(0.0347)	(0.0347)	(0.0346)	(0.0347)	(0.0346)	(0.0347)	(0.0347)	(0.0348)	(0.0347)
θ_0^{FC}	-0.2583	0.5050	-2.3650	0.5274	0.5050	0.5049	0.5048	0.5050	0.5048	0.5050	0.4965	0.5050	0.5050
	(0.0698)	(0.0947)	(0.1425)	(0.0993)	(0.0947)	(0.0946)	(0.0944)	(0.0947)	(0.0944)	(0.0947)	(0.0946)	(0.0947)	(0.0947)

† The AFD does not converge when $\gamma_a = 5$.

† The data generating process is $\theta_0^{VP} = 0.5, \theta_0^{FC} = 0.5$.

† The table shows a subset of the parameters. The misspecified estimator(FD,AFD) generate larger biasness on these two estimators.

† The results are based on 100 Monte Carlo simulations.

† $HM(true)$ and $EE(true)$ use the data generating conditional choice probabilities rather than the estimated conditional choice probabilities.

† The complete estimation results are in the appendix.

Table 8: Median Time and Iteration in homogeneous dynamic discrete choice model

Algorithm	$Market$	200									100	1000
	$Time$	10			20			120			20	
	γ_a	0	1	5	0	1	5	0	1	5	2	
FD	*Time*	1.5129	1.6507	1.6903	2.1023	2.3227	2.3186	8.4469	8.6624	10.4187	1.5574	7.763
	Iteration	10	11	11	10	11	11	10	11	12	11	12
$FD2$	*Time*	1.4178	2.7577	2.4841	1.9872	3.9388	3.1754	7.1096	13.0962	12.8322	2.548	10.4367
	Iteration	8	17	15	8	17	14	8	15	14	16	15
AFD	*Time*	1.4182	2.4181	168.7573	1.9945	3.3076	230.3824	7.4735	13.3311	-	165.9659	733.3145
	Iteration	10	16	1000	10	16	1000	10	17	-	1000	1000
$AFD2$	*Time*	1.446	2.5157	17.3078	1.9922	3.4684	23.578	7.3938	11.5065	95.8954	13.4769	54.8919
	Iteration	8	15.5	89.5	8	15	93	8	14	94	76	71
$FD2(BEL)$	*Time*	1.3881	2.7503	2.4339	1.9033	3.8328	3.3144	7.1037	14.68	14.1806	2.4519	11.8584
	Iteration	8	17	14	8	17	14	8	18	15	16	17
$AFD2(BEL)$	*Time*	1.3612	2.4898	18.1304	1.903	3.6167	24.0672	7.1416	12.9835	101.6634	14.8182	57.7984
	Iteration	8	15	96	8	15	93	8	16	103	84	73
HM	*Time*	0.6303	0.9931	1.0745	0.8046	1.3868	1.4521	2.9969	5.006	5.6845	1.0147	4.4518
	Iteration	4	6	6	4	6	6	4	6	6	6	6
EE	*Time*	1.3052	1.3444	1.4505	1.8161	1.9366	2.0039	6.5651	7.0136	7.828	1.339	5.9889
	Iteration	8	9	9	8	9	9	8	9	9	9	9
$HM(true)$	*Time*	0.5519	0.9905	1.0647	0.7534	1.3766	1.4432	2.97	4.932	5.6584	1.0086	4.4237
	Iteration	4	6	6	4	6	6	4	6	6	6	6
$EE(true)$	*Time*	1.1287	1.2471	1.2919	1.5569	1.7061	1.7815	5.8386	6.2601	7.1882	1.205	5.3692
	Iteration	8	8	8	7	8	8	7	8	8	8	8
$SEQ(1)$	*Time*	1.2576	1.4264	1.4666	1.7802	2.051	2.0541	7.1809	7.6537	8.6539	1.3853	6.5836
	Iteration	9	9	9	9	9.5	10	9	10	10	9	10
$SEQ(2)$	*Time*	1.642	1.9249	2.4262	2.2997	2.7204	3.4804	9.5244	10.7994	14.5555	2.227	10.6575
	Iteration	11	13	16	11	13	16	12	14	17	15	16
$SEQ(5)$	*Time*	1.9422	1.8247	2.2418	2.6517	2.6229	3.0686	10.947	10.0698	13.2766	1.8728	9.1645
	Iteration	13	12		13	12	14.5	14	13	15	13	14

† The Iteration shows the number of Iterations in the sequentiao estimation algorithm.

† The results shows the time and Iteration used in the estimation based on 100 Monte Carlo simulations of different market size and time length when the number of grid(nGrid) is 2.

† The stopping crIterationion is 10^{-6} with log-likelihood function.

A Appendix

Proof of Proposition 2. **[To Jasmine: let's use bold capital letter for matrix while bold lower-case letter for vector. Need to have an assumption that $\kappa_t(z_{t+1}|z_0) > 0$ for all $z_{t+1} \in \mathcal{Z}$ (high level assumption) which holds if $f(z_{t+1}|z_t, d) > 0$ and $p_t(d_t, z_t) > 0$. We also need to add more discussion on the "transversality condition" $\lim_{t\to\infty} \beta^t \boldsymbol{F}_0^t \boldsymbol{\lambda}_t = 0$ as well as $\lim_{t\to\infty} \beta^t (\boldsymbol{F}^P)^t \boldsymbol{\lambda}_t = 0$, which should hold if $\boldsymbol{\lambda}_t$ does not go to ∞, i.e., the integrated value function is bounded. Is there any direct way to show that $\boldsymbol{\lambda} = \boldsymbol{V}$? Not sure if the proof is complete for the statement in Proposition 2 given that the relationship between $\boldsymbol{\lambda}_t$, $\boldsymbol{\lambda}_{t+1}$ and $\Lambda^{-1}(P)$ and V is not clear. I guess you have implicitly used some results from Aguirregabiria and Magesan (2016)?]** The first order conditions for the Lagrangian (**??**) with respect to $\boldsymbol{p}_t(z_t)$ and $\boldsymbol{\kappa}_t(z_0) := (\kappa_t(z^{(1)}|z_0), ..., \kappa_t(z^{(|\mathcal{Z}|)}|z_0))^\top$ are given by

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{p}_t(z_t)} = \beta^t \kappa_t(z_t|z_0) \left(\tilde{\boldsymbol{u}}(z_t) + \tilde{\boldsymbol{e}}(z_t) + \beta \tilde{\boldsymbol{F}}(z_t) \boldsymbol{\lambda}_{t+1} \right) = \boldsymbol{0} \text{ and} \tag{47}$$

$$\frac{\partial \mathcal{L}}{\partial \kappa_t(z_t|z_0)} = \beta^t \left(U^P(z_t) - \lambda_t(z_t) + \beta \boldsymbol{f}^P(z_t)^\top \boldsymbol{\lambda}_{t+1} \right) = 0, \tag{48}$$

where

$$\tilde{\boldsymbol{F}}(z_t) := \begin{bmatrix} \tilde{\boldsymbol{f}}(z^{(1)}|z_t) & \dots & \tilde{\boldsymbol{f}}(z^{(|\mathcal{Z}|)}|z_t) \end{bmatrix}, \ \boldsymbol{\lambda}_{t+1} := \begin{bmatrix} \lambda(z_{t+1}^{(1)}) \\ \vdots \\ \lambda(z^{(|\mathcal{Z}|)}{}_{t+1}) \end{bmatrix},$$

$$\boldsymbol{f}_t^P(z_t) := \begin{bmatrix} \sum_{j=0}^D p_t(j, z_t) f(z_{t+1}^{(1)}|z_t, j) \\ \vdots \\ \sum_{j=0}^D p_t(j, z_t) f(z_{t+1}^{(|\mathcal{Z}|)}|z_t, j) \end{bmatrix} = \tilde{\boldsymbol{F}}(z_t)^\top \boldsymbol{p}_t(z_t) + \boldsymbol{f}_0(z_t)$$

with $\tilde{\boldsymbol{f}}(z|z_t) := (f(z|z_t, 1) - f(z|z_t, 0), ..., f(z|z_t, D) - f(z|z_t, 0))^\top$ and $\boldsymbol{f}_0(z_t) := (f(z^{(1)}|z_t, 0), ..., f(z^{(|\mathcal{Z}|)}|z_t, 0))^\top$.

From equation (48) and $\kappa_t(z_t|z_0) > 0$, we can write the relationship between $\boldsymbol{\lambda}_t$ and $\boldsymbol{\lambda}_{t+1}$ as

$$\boldsymbol{\lambda}_t = \boldsymbol{U}^P + \beta \boldsymbol{F}^P \boldsymbol{\lambda}_{t+1}, \tag{49}$$

where $\boldsymbol{U}^P = [(\boldsymbol{U}^P(z^{(1)}))' \dots, (\boldsymbol{U}^P(z^{(|\mathcal{Z}|)}))']$ and $\boldsymbol{F}^P = [(\boldsymbol{f}^P(z^{(1)})', \dots, (\boldsymbol{f}^P(z^{(|\mathcal{Z}|)})']$. When $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_{t+1} = \boldsymbol{\lambda}$ for all t, we have $\boldsymbol{\lambda} = (\boldsymbol{I} - \beta \boldsymbol{F}^P)^{-1} \boldsymbol{U}^P$.

From equation (47) and $\kappa_t(z_t|z_0) > 0$,

$$\begin{aligned} 0 &= \boldsymbol{p}_t(z_t)^\top (\tilde{\boldsymbol{u}}(z_t) + \tilde{\mathbf{e}}(z_t)) + \beta \boldsymbol{p}_t(z_t)^\top \tilde{\boldsymbol{F}}(z_t) \boldsymbol{\lambda}_{t+1} \\ &= U^P(z_t) + \beta \boldsymbol{f}_t^P(z_t)^\top \boldsymbol{\lambda}_{t+1} - \left\{ u_0(z_t) + e_0(z_t) + \beta \boldsymbol{f}_0(z_t)^\top \boldsymbol{\lambda}_{t+1} \right\}. \end{aligned} \tag{50}$$

Then, it follows from (48) and (50), we have $\lambda_t(z_t) = u_0(z_t)+e_0(z_t)+\beta \boldsymbol{f}_0(z_t)^\top \boldsymbol{\lambda}_{t+1}$, or equivalently,

$$\boldsymbol{\lambda}_t = \boldsymbol{u}_0 + \boldsymbol{e}_0 + \beta \boldsymbol{F}_0 \boldsymbol{\lambda}_{t+1} \tag{51}$$

where $\boldsymbol{u}_0$, $\boldsymbol{e}_0$, and $\boldsymbol{F}_0$ are defined by stacking $u_0(z)$, $e_0(z)$, and $\boldsymbol{f}_0(z)^\top$ vertically across different values of zs. By recursive substitutions,

$$\boldsymbol{\lambda}_t = \sum_{\tau=t}^{T} \beta^{\tau-t} \boldsymbol{F}_0^{\tau-t}(\boldsymbol{u}_0 + \boldsymbol{e}_0) + \beta^{T-t+1} \boldsymbol{F}_0^{T-t+1} \boldsymbol{\lambda}_{T+1}. \tag{52}$$

To solve for λ_t, we need to make assumption on λ_{T+1}. If the time horizon is finite, assume $\lambda_{T+1} = 0$. If the time horizon is infinite, $\lim_{t\to\infty} \beta^t F_0^t \lambda_t = 0$. The solution is that $\lambda_t = \sum_{\tau=t}^{\infty} \beta^{\tau-t} F_0^{\tau-t}(u_0 + e_0)$ for $t = 0, \dots, T$.

The interpretation of $\boldsymbol{\lambda}$: Consider the stationary problem. Then, the Bellman equation that corresponds to the optimization problem (**??**) is given by

$$V(z) = \max_{\boldsymbol{P}, \kappa(z'|z_t)} U^{\boldsymbol{P}}(z) + \beta \sum_{z'} \kappa(z'|z) V^{\boldsymbol{P}}(z')$$

subject to $\kappa(z'|z) = \sum_d p_d(z) f(z'|z,d)$, where $V^{\boldsymbol{P}}(z) = U^{\boldsymbol{P}}(z) + \beta \sum_{z'} \kappa(z'|z) V^{\boldsymbol{P}}(z')$. The corresponding Lagrangian is

$$\mathcal{L} = U^{\boldsymbol{P}}(z) + \beta \sum_{z'} \kappa(z'|z) V^{\boldsymbol{P}}(z') - \beta \sum_{z'} \lambda(z') \left(\kappa(z'|z) - \sum_d p_d(z) f(z'|z,d) \right).$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{p}(z)} = \tilde{\boldsymbol{u}}(z) + \tilde{\boldsymbol{e}}(z) + \beta \sum_{z'} \lambda(z') \tilde{\boldsymbol{f}}(z'|z) + \beta \sum_{z'} \kappa(z'|z) \frac{\partial V^{\boldsymbol{P}}(z')}{\partial \boldsymbol{p}(z')} = \boldsymbol{0}, \tag{53}$$

$$\frac{\partial \mathcal{L}}{\partial \kappa(z'|z)} = \beta(-\lambda(z') + V^{\boldsymbol{P}}(z')) = 0. \tag{54}$$

Equation (54) gives $\lambda(z) = V^{\boldsymbol{P}}(z)$. Therefore, $\lambda(z)$ represents the value function. Evaluated at $\boldsymbol{P} = \boldsymbol{P}^*$, we have $\frac{\partial V^{\boldsymbol{P}}(z')}{\partial \boldsymbol{p}(z')}|_{\boldsymbol{P}=\boldsymbol{P}^*} = \boldsymbol{0}$. Therefore, (53)-(54) gives

$$\tilde{\boldsymbol{u}}(z) + \tilde{\boldsymbol{e}}(z) + \beta \sum_{z'} \tilde{\boldsymbol{f}}(z'|z) V^{\boldsymbol{P}^*}(z') = \boldsymbol{0}.$$

□

Proof of Proposition 3. The integrated value function for state z is that

$$\begin{aligned} V(|\mathcal{Z}|) &= \int \sum_{d\in\mathcal{D}} (v(d,z)+\epsilon_d)\mathbb{1}\{v(d,z)+\epsilon_d \geq v(d',z)+\epsilon'_d\} dG(\epsilon_1)\dots dG(\epsilon_{|D|}) \\ &= \sum_{d\in\mathcal{D}} \int (v(d,z)+\epsilon_d)\mathbb{1}\{v(d,z)+\epsilon_d \geq v(d',z)+\epsilon'_d\} dG(\epsilon_d) \\ &= \sum_{d\in\mathcal{D}} p(d,z)\left(v(d,z)+e(d,z)\right) = \sum_{d\in\mathcal{D}} p(d,z)\left(v(0,z)+e(0,z)\right) = v(0,z)+e(0,z). \end{aligned} \tag{55}$$

[To Jasmine: add a proof for $v(d,z)+e(d,z)=v(0,z)+e(0,z)$. I know that this holds for extreme value case but do you know the general proof?] □

Proof of proposition 4. If a model has a terminal or renewal choice, then the model exhibit 2-period single action finite dependence. Therefore, by the definition we have that $\kappa^*_{\rho-1}(x_{t+\rho+1}|x_t,j) = \kappa^*_{\rho-1}(x_{t+\rho+1}|x_t,0)$ with $\delta(x,\tau)=0$. By the definition of $\kappa^*_{\rho-1}(x_{t+\rho+1}|x_t,d)$, define $\mathcal{K}^*_{\rho-1}(d)$ as the collection of the probabilities of reaching $x_{t+\rho}$ starting from x_t. Define the ρ-period forward transition probability matrix conditional on choosing action d as
$\mathcal{K}^*_{\rho-1}(d) = \begin{bmatrix} \kappa^*_{\rho-1}(z^{(1)}_{t+\rho+1}|z^{(1)}_t,d),\dots,\kappa^*_{\rho-1}(z^{(|Z|)}_{t+\rho+1}|z^{(1)}_t,d) \\ \vdots,\ddots,\vdots \\ \kappa^*_{\rho-1}(z^{(1)}_{t+\rho+1}|z^{(|Z|)}_t,d),\dots,\kappa^*_{\rho-1}(z^{(|Z|)}_{t+\rho+1}|z^{(|Z|)}_t,d) \end{bmatrix}$. Then $\mathcal{K}^*_{\rho-1}(d) = F_d F_0^{\rho-1}$. If the model exhibit single action ρ-dependence, then $\mathcal{K}^*_{\rho-1}(d) = \mathcal{K}^*_{\rho-1}(0)$, for all $d=1,\dots,D$.

Therefore, if the model exhibit finite 2-dependence $F_dF_0 = F_0^2$ for all $d=1,\dots,D$ implies that $\tilde{F}_dF_0=0$. Recall that $\tilde{F} = \begin{bmatrix} \tilde{F}_1 \\ \vdots \\ \tilde{F}_D \end{bmatrix}$. Then $\tilde{F}F_0 = 0$. □

Remark 2 (The time complexity with respect to $|\mathcal{X}|$). *In the estimation methods, there are 4 steps that can be time-costly.*

1. *The initial estimation of conditional choice probability and value function, and update of A and B. This step varies across different algorithms.*
2. *Estimation of $\hat{\theta}$. The complexity of the estimation does not depend on the dimension of the state space.*
3. *Update conditional choice probability given V and P. This step involves matrix multiplication, therefore the complexity is $O(d)-O(d^2)$.*
4. *Compute V. For HM the complexity is $O(d^3)$.*

Among the 4 steps, 1 is computed one time for each monte carlo simulation, 2-4 is computed for each EM iteration.

B Computational Details

B.1 Optimal weight for almost finite dependent estimator

Proposition 6 (Characterization of Finite Dependence)**.** *Suppose that there exists a sequence $\{d^*_{t+\tau}(z)\}_{\tau=0}^{\rho}$ such that $(\tilde{\boldsymbol{f}}_{d^*_t(z),t}(d,z))^\top \prod_{\tau=1}^{\rho} \boldsymbol{F}_{d^*_{t+\tau},t+\tau} = \boldsymbol{0}$ for all $(d,z) \in \mathcal{D} \times \mathcal{Z}$, then the model exhibits the $(\rho+1)$-period finite dependence with*

$$\tilde{v}_{d^*_t(z),t}(d,z) = \tilde{u}_{d^*_t(z),t}(d,z) + (\tilde{\boldsymbol{f}}_{d^*_t(z),t}(d,z))^\top \sum_{\tau=1}^{\rho} \beta^\tau \boldsymbol{F}_{d^*_{t+\tau}(z),t+\tau} \left(\nabla \bar{\boldsymbol{e}}^{\boldsymbol{P}_{t+1}} + \boldsymbol{u}_{d^*_{t+\tau}(z)} + \boldsymbol{e}^{\boldsymbol{P}_{t+1}}_{d^*_{t+\tau}(z)}\right).$$

When the model exhibits finite dependence, given the initial estimate for $\{\boldsymbol{p}_{t+\tau} : \tau = 1, ..., \rho\}$, the conditional choice probabilities can be evaluated without solving the Bellman equation.

[Add examples for finite dependence here]

Lemma 1. *The model displays finite dependence if and only if $\left(\tilde{\boldsymbol{F}}^T \otimes \tilde{\boldsymbol{F}}\right) vec(\tilde{\boldsymbol{P}}) = vec(-\tilde{\boldsymbol{F}}\boldsymbol{F_0})$ has a solution $\tilde{\boldsymbol{P}}$.*

Proof. Suppose the model displays finite dependence property, for each $z^{(i)}$, we have

$$\left(\boldsymbol{P_t}(\boldsymbol{z^{(i)}}) - \boldsymbol{P'_t}(\boldsymbol{z^{(i)}})\right) \boldsymbol{F}(\boldsymbol{z^{(i)}}) \left(\boldsymbol{P_{t+1}F}\right) = \boldsymbol{0}.$$

Define $\mathcal{K} = \{\boldsymbol{D} \in \mathbb{R}^{D+1}, \sum_i \boldsymbol{D}_i = 0\}$ as the linear space of vectors that sum up to 0. As $\boldsymbol{P_t}$ and $\boldsymbol{P'_t}$ are arbitrary, we have that

$$\boldsymbol{DF}(\boldsymbol{z^{(i)}}) \left(\boldsymbol{P_{t+1}F}\right) = \boldsymbol{0}. \; \forall \; \boldsymbol{P} \in \mathcal{S}$$

With $\boldsymbol{PF} \in \mathcal{K}$ as well, we have that $rank(\boldsymbol{P}_{t+1}\boldsymbol{F}) = 1$, and $dim(null(\boldsymbol{P}_{t+1}\boldsymbol{F})) = |\mathcal{Z}| - 1$. From the above results, we have that

$$\tilde{\boldsymbol{F}} \left(\boldsymbol{P_{t+1}F}\right) = \boldsymbol{0}$$
$$\tilde{\boldsymbol{F}} \left(\tilde{\boldsymbol{P}}_{\boldsymbol{t+1}}\tilde{\boldsymbol{F}} + \boldsymbol{F_0}\right) = \boldsymbol{0}.$$

Let $A \otimes B$ be the Kronecker product of two matrix A and B and $\mathrm{vec}(A)$ be the vectorization of matrix A. Then by the property of vectorization

$$\left(\tilde{\boldsymbol{F}}^T \otimes \tilde{\boldsymbol{F}}\right) \mathrm{vec}(\tilde{\boldsymbol{P}}_{\boldsymbol{t+1}}) = \mathrm{vec}(-\tilde{\boldsymbol{F}}\boldsymbol{F_0}).$$

Thus the model displays finite dependence property if and only if above equation has a suitable solution for $\boldsymbol{P_{t+1}}$. ☐

To find the optimal weight that minimizes $\left(\tilde{\boldsymbol{F}}^T \otimes \tilde{\boldsymbol{F}}\right) (\tilde{\boldsymbol{P}}_{\boldsymbol{t+1}}) + \mathrm{vec}(\tilde{\boldsymbol{F}}\boldsymbol{F_0})$, we solve the linear equation of $\left(\tilde{\boldsymbol{F}}^T \otimes \tilde{\boldsymbol{F}}\right) \text{vec}(\tilde{\boldsymbol{P}}_{\boldsymbol{t+1}}) = \text{vec}(-\tilde{\boldsymbol{F}}\boldsymbol{F_0})$.

Note that $vec(\tilde{\boldsymbol{P}})$ has many zeros, then we need to select the non-zero rows that corresponds to the conditional choice probabilities.

A conditional choice probabilities $\tilde{\boldsymbol{P}} \in \mathcal{P}$ is

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} p(1, z^{(1)}) & \dots & p(D, z^{(1)}) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & p(1, z^{(|\mathcal{Z}|)}) & \dots & p(D, z^{(|\mathcal{Z}|)}) \end{bmatrix}_{|\mathcal{Z}| \times |\mathcal{Z}|D}$$

(According to wikipedia), if $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, then $vec(A) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. The rows that are non-zero in $\tilde{\boldsymbol{P}}$ are $\{(k-1) * |D\mathcal{Z}| + (m-1)|\mathcal{Z}| + k\}$ where $k = 1, \dots, |\mathcal{Z}|$,$m = 1, \dots, D$.

In computation, we can find the corresponding columns in the matrix $\tilde{\boldsymbol{F}}^\top \otimes \tilde{\boldsymbol{F}}$ such that the rows in $vec(\tilde{\boldsymbol{P}})$ are non-zeros. In order to compute the optimal weight, we write the linear constraint as $\mathcal{A}\, vec(\boldsymbol{P}) = \mathcal{B}$, where $\mathcal{A}$ is the desginated columns of matrix $\tilde{\mathcal{F}}\mathcal{F}_0$ and $\mathcal{B}$ is the designated rows of $vec(\tilde{\boldsymbol{F}}\boldsymbol{F}_0)$.

- Then we can compute the conditional choice probability with OLS using the selected rows. However, this may suffer from the fact that, the parameters are not bounded between 0 and 1.
- I tested the method of direct OLS in matlab when the number of state is 64. The optimized value is the same that solved using matlab(I test different values of γ_as).
- Since only very small amount of columns of the kronecker product will be used, I can write my own version of computational method to compute the selected columns of $\tilde{\boldsymbol{F}}' \otimes \tilde{\boldsymbol{F}}$.
- Note that $\tilde{\boldsymbol{F}}$ is a $|\mathcal{Z}D| \times |\mathcal{Z}|$ matrix. $\tilde{\boldsymbol{F}}' \otimes \tilde{\boldsymbol{F}}$ is a $|\mathcal{Z}|^2 D \times |\mathcal{Z}|^2 D$ matrix. When number of grid is 2, the state space size $|\mathcal{Z}| = 2 * 2^5 = 64$, the kronecker product is still feasible. When the number of grid is 3, $|\mathcal{Z}| = 2 * 3^5 = 486$. In that case, the kronecker product is not feasible.

In the simulation, when the number of grids in each exogenous variable is greater than 4, which make a state space of $4^5 * 2 = 2048$, the gradient descent method requires more than 15.4Gb of memory, and therefore is not feasible. When the state number is too large, we cannot rely on this gradient descent method to solve for the optimal weight for the almost finite dependence estimator.

B.2 Sequential Estimation

This section explains the computational details of the algorithms. In the sequential estimation, we obtain the estimator of θ in each iteration given the value function and conditional choice probability of each iteration. In order to optimize the estimation in computer, we write in code the objective function as a logit form

$$Q(\theta, A, B) = \sum_{i=1}^{N} \sum_{t=1}^{T} \log \left(\frac{\exp(A_{i,t}\theta + B_{i,t})}{1 + \exp(A_{i,t}\theta + B_{i,t})} \right),$$

where $A_{i,t}$ and $B_{i,t}$ are computed based on the conditional choice probability and value functions prior to the optimization. The algorithms differs in the calculation of the value functions. The benefit of such form is that optimization depend only on the sample size but not the size of the state space.

In each of the estimators, we first obtain the sample analogue estimated conditional choice probabilities

$$\hat{p}^{(0)}(d, z) = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{d_{it} = d, z_{it} = z\}}{\sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{z_{it} = z\}}.$$

B.2.1 Sequential Estimators (SEQ)

Step 1: Update the value function given the conditional choice probabilities. Define the Bellman operator to update the value function as

$$\Gamma(V, \theta)(z) = \gamma + \log \left(\sum_{d \in \mathcal{D}} \exp \left(u(d, z; \theta) + \beta \sum_{z' \in \mathcal{Z}} f(z'|z, d) V^{(k)}(z') \right) \right).$$

Define the q-fold Bellman operator as

$$\Gamma^q(V, \theta) = \overbrace{\Gamma(\Gamma(\dots, \theta), \theta)}^{q \text{ times}}.$$

For q-fold sequential estimator, update the value function with

$$V^{(k)} = \Gamma^q(V^{(k-1)}, \theta^{(k-1)}).$$

Step 2: Update the optimization functions A and B. Let $\tilde{u}_z$ be the first order derivative of $\tilde{u}$

with respect to z. For $SEQ(q)$ estimator, update A and B with

$$A_{i,t}^{SEQ,(k)} = (2 \cdot \mathbb{1}\{a_{i,t} = 1\} - 1) \{\tilde{u}_z(d_{it}, z_{it})\},$$

$$B_{i,t}^{SEQ,(k)} = (2 \cdot \mathbb{1}\{a_{i,t} = 1\} - 1)\beta \left\{ \sum_{z' \in \mathcal{Z}} \tilde{f}(z'|z_{it}, d_{it}) \left(V^{(k)}\right)(z') \right\}.$$

Step 4: Estimate θ with updated A and B.

$$\theta^{(k)} = \arg\max_{\theta} Q(\theta, A^{SEQ,(k)}, B^{SEQ,(k)}).$$

B.2.2 Hotz-Miller (HM) estimators

Step 1: Update the conditional choice probabilities. Given the estimator $\theta^{(k-1)}$, the value function from last iteration $V^{(k-1)}$, update the conditional choice probabilities.

$$p^{(k)}(d, z) = \Lambda(V^{(k-1)}, \theta^{(k-1)}) = \frac{\exp\left(u(d, z; \theta^{(k-1)}) + \beta \sum_{z' \in \mathcal{Z}} f(z'|d, z) V^{(k-1)}(z')\right)}{\sum_{d' \in \mathcal{D}} \exp\left(u(d', z; \theta^{(k-1)}) + \beta \sum_{z' \in \mathcal{Z}} f(z'|d', z) V^{(k-1)}(z')\right)}.$$

Step 2: Update the value function given the conditional choice probabilities. For HM algorithm, V is computed

$$\boldsymbol{V}^{(k)} = (I - \beta \boldsymbol{F}^{P^{(k)}})^{-1}(\boldsymbol{u}^{P^{(k)}}).$$

Step 3: Update the optimization functions A and B. Let $\tilde{u}_z$ be the first order derivative of $\tilde{u}$ with respect to z. For HM estimator, let $\boldsymbol{G}^{(k)} = \left(I - \beta \boldsymbol{F}^{\boldsymbol{P}^{(k)}}\right)^{-1}$. Then A and B are

$$A_{i,t}^{HM,(k)} = (2 \cdot \mathbb{1}\{a_{i,t} = 1\} - 1) \{\tilde{u}_z(d_{it}, z_{it})\},$$

$$B_{i,t}^{HM,(k)} = (2 \cdot \mathbb{1}\{a_{i,t} = 1\} - 1)\beta \left\{ \sum_{z' \in \mathcal{Z}} \tilde{f}(z'|z_{it}) \left(\boldsymbol{G}^{(k)} V^{(k-1)}\right)(z') \right\}.$$

Step 4: Estimate θ with updated A and B.

$$\theta^{(k)} = \arg\max_{\theta} Q(\theta, A^{HM,(k)}, B^{HM,(k)}).$$

B.2.3 Finite dependence ($FD2$,$AFD2$) estimators

For the $AFD2$ and $FD2$ estimators, we do the following steps:

Step 1: Update the conditional choice probabilities.

Given the estimator $\theta^{(k-1)}$, the value function from last iteration $V^{(k-1)}$, update the conditional choice probabilities.

$$p^{(k)}(d,z) = \Lambda(V^{(k-1)}, \theta^{(k-1)}) = \frac{\exp\left(u(d,z;\theta^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d,z)V^{(k-1)}(z')\right)}{\sum_{d'\in\mathcal{D}}\exp\left(u(d',z;\theta^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d',z)V^{(k-1)}(z')\right)}.$$

Step 2: Update the value function given the conditional choice probabilities.

For the updating the value function, the $AFD2$ algorithm uses the following the contraction mapping defined as follow

$$\begin{aligned}\boldsymbol{V}^{(k)} = \Gamma^{AFD2}(V^{(k-1)}, p^{(k)}, \theta^{(k-1)}) =& \gamma + \boldsymbol{\omega}(-\log(\boldsymbol{p}^{(k)}) + u_1(\theta^{(k-1)}) + \beta\boldsymbol{F}_1\boldsymbol{V}^{(k-1)}) \\ &+ (1-\boldsymbol{\omega})(-\log(1-\boldsymbol{p}^{(k)}) + u_0(\theta^{(k-1)}) + \beta\boldsymbol{F}_0\boldsymbol{V}^{(k-1)}).\end{aligned}$$

where ω is the weight. For $FD2$ algorithm, the value function is updated using

$$\boldsymbol{V}^{(k)} = \Gamma^{FD2}(V^{(k-1)}, p^{(k)}, \theta^{(k-1)}) = \gamma - \log(1-p^{(k)}) + u_0(\theta^{(k-1)}) + \beta\boldsymbol{F}_0\boldsymbol{V}^{(k-1)}).$$

Step 3: Update the optimization functions A and B. Let $\tilde{u}_z$ be the first order derivative of $\tilde{u}$ with respect to z. For the $FD2$ estimator,

$$\begin{aligned}A_{i,t}^{FD2,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\left\{\tilde{u}_z(d_{it},z_{it}) + \beta\sum_{z'\in\mathcal{Z}}\tilde{f}(z'|z_{it})u_{0,z}(z')\right\}, \\ B_{i,t}^{FD2,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\beta\left\{\sum_{z'\in\mathcal{Z}}\tilde{f}(z'|z_{it})(u(z') + \gamma - \log(p_0(z')) + \beta f(z''|z',0)V^{(k)}(z''))\right\}.\end{aligned}$$

For $AFD2$ estimator

$$\begin{aligned}A_{i,t}^{AFD2,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\left\{\tilde{u}_z(d_{it},z_{it}) + \sum_{z'\in\mathcal{Z}}\tilde{f}(z'|z_{it})(\omega(z')u_{1,z}(z') + (1-\omega(z'))u_{0,z}(z'))\right\}, \\ B_{i,t}^{AFD2,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\,\beta\Big\{\sum_{z'\in\mathcal{Z}}\tilde{f}(z'|z_{it})\big\{(1-\omega(z'))(u_0(z') + \gamma - \log(p_0(z'))+ \\ &\quad \beta f(z''|z',0)V^{(k)}(z'')) + \omega(z')(u_1(z') + \gamma - \log(p_1(z')) + \beta f(z''|z',1)V^{(k-1)}(z''))\big\}\Big\}.\end{aligned}$$

Step 4: Estimate θ with updated A and B.

$$\theta^{(k)} = \arg\max_{\theta} Q(\theta, A^{FD2,(k)}, B^{FD2,(k)}) \text{ or } \theta^{(k)} = \arg\max_{\theta} Q(\theta, A^{AFD2,(k)}, B^{AFD2,(k)}).$$

B.2.4 Euler equation (EE) estimators

Step 1: Update the conditional choice probabilities.

Given the estimator $\theta^{(k-1)}$, the value function from last iteration $V^{(k-1)}$, update the conditional choice probabilities.

$$p^{(k)}(d,z) = \Lambda(V^{(k-1)},\theta^{(k-1)}) = \frac{\exp\left(u(d,z;\theta^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d,z)V^{(k-1)}(z')\right)}{\sum_{d'\in\mathcal{D}}\exp\left(u(d',z;\theta^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d',z)V^{(k-1)}(z')\right)}.$$

Step 2: Update the value function given the conditional choice probabilities. For EE algorithm, V is computed using

$$V^{(k)} = (I-\beta\boldsymbol{F}_0)^{-1}\left(\gamma - \log(\boldsymbol{p}_0^{(k)})\right).$$

Step 3: Update the optimization functions A and B. Let $\tilde{u}_z$ be the first order derivative of $\tilde{u}$ with respect to z. For EE estimator, let $\boldsymbol{G}$ be defined as $(I-\beta\boldsymbol{F}_0)^{-1}$. Then A and B are

$$\begin{aligned}
A_{i,t}^{EE,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\left\{\tilde{u}_z(d_{it},z_{it})\right\},\\
B_{i,t}^{EE,(k)} &= (2\cdot\mathbb{1}\{a_{i,t}=1\}-1)\beta\left\{\sum_{z'\in\mathcal{Z}}\tilde{f}(z'|z_{it})\left(\boldsymbol{G}V^{(k-1)}\right)(z')\right\},
\end{aligned}$$

where $\left(\boldsymbol{G}V^{(k-1)}\right)(z')$ is the component in the vector $\left(\boldsymbol{G}V^{(k-1)}\right)$ that corresponds to z'.

Step 4: Estimate θ with updated A and B.

$$\theta^{(k)} = \arg\max_{\theta} Q(\theta, A^{EE,(k)}, B^{EE,(k)}).$$

B.3 EM algorithm

For the EM algorithm, assume there are M types. We observe the state for agent $i = 1,\ldots,N$ for time $t = 1,\ldots,T$. For the EM algorithm, we start with the weight $\{w_{i,m}^{(0)}\}_{i=1,\ldots,N}$, where $\sum_{m=1,\ldots,M} w_{i,m}^{(0)} = 1$. We define the prior probability of each types as $q_m^{(0)} = \frac{1}{N}\sum_{i=1}^N w_{i,m}^{(0)}$, for $m = 1,\ldots,M$. Similar to the homogeneous agent model, we estimate the conditional choice probabilities

$$\hat{p}_m^{(0)}(d,z) = \frac{\sum_{i=1}^N\sum_{t=1}^T\mathbb{1}\{d_{it}=d, z_{it}=z\}}{\sum_{i=1}^N\sum_{t=1}^T\mathbb{1}\{z_{it}=z\}}, m = 1,\ldots,M.$$

M-step:

For each iteration k, given the estimated parameters for each type $\theta_m^{(k-1)}$, the updated value function for each type $V_m^{(k-1)}$, we compute the conditional choice probabilities

$$p_m^{(k)} = \Lambda(V_m^{(k-1)},\theta_m^{(k-1)}) = \frac{\exp\left(u(d,z;\theta_m^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d,z)V_m^{(k-1)}(z')\right)}{\sum_{d'\in\mathcal{D}}\exp\left(u(d',z;\theta_m^{(k-1)}) + \beta\sum_{z'\in\mathcal{Z}} f(z'|d',z)V_m^{(k-1)}(z')\right)}.$$

Then we update the type-dependent value function with the mapping

$$V_m^{(k)} = \Gamma^h(V_m^{(k-1)}, p_m^{(k)}, \theta_m^{(k-1)}), \quad \text{where } h \in \{AFD2, FD2, EE, HM, SEQ\}.$$

Then we compute the $A_{it,m}^{(k)}$'s and $B_{it,m}^{(k)}$'s using the same method mentioned in section B.2.

Estimate the parameter with

$$\theta_m^{(k)} = \arg\max_\theta Q(\theta, A_{it,m}^{(k)}, B_{it,m}^{(k)}), m = 1, \ldots, M$$

$$Q(\theta, A_{it,m}^{(k)}, B_{it,m}^{(k)}) = \sum_{i=1}^N w_{i,m}^{(k-1)} \sum_{t=1}^T \log\left(\frac{\exp(A_{it,m}^{(k)}\theta + B_{it,m}^{(k)})}{1 + \exp(A_{it,m}^{(k)}\theta + B_{it,m}^{(k)})}\right).$$

E-step Given the estimation, we update the weight for each agent using the following formula

$$w_{i,m}^{(k)} = \frac{q_m^{(k-1)} \sum_{t=1}^T \log\left(\frac{\exp(A_{it,m}^{(k)}\theta + B_{it,m}^{(k)})}{1+\exp(A_{it,m}^{(k)}\theta + B_{it,m}^{(k)})}\right)}{\sum_{m'=1}^M q_{m'}^{(k-1)} \sum_{t=1}^T \log\left(\frac{\exp(A_{it,m'}^{(k)}\theta + B_{it,m'}^{(k)})}{1+\exp(A_{it,m'}^{(k)}\theta + B_{it,m'}^{(k)})}\right)},$$

$$q_m^{(k)} = \frac{1}{N}\sum_{i=1}^N w_{i,m}^{(k)}.$$

C Simulation parameters

The data generating process used in the numerical and Monte Carlo experiments is summarize in the following table 9. With all other parameters remain the same as the first experiment, the action taken at each time permanently impact the transition of profitability. This can be viewed as the firm's learning effect of operating in the market. When a firm operate in the market for one time period, the firm will permanently increase the future profitability shocks.

Table 9: Parameters in DGP

Flow-Payoff Parameters	$\theta_0^{VP} = 0.5$	$\theta_1^{VP} = 1.0$	$\theta_2^{VP} = -1.0$
	$\theta_0^{FC} = 0.5$	$\theta_1^{FC} = 1.0$	
	$\theta_0^{EC} = 1.0$	$\theta_1^{EC} = 1.0$	
State Variable Transition	z_{kt} is AR(1), $\gamma_0^k = 0$, $\gamma_1^k = 0.6$		
Productivity Transition	ω_t is AR(1), $\gamma_0^\omega = 0$, $\gamma_1^\omega = 0.9$		
Past action on productivity	$\gamma_a \in [0, 5]$		
Discount Factor	$\beta = 0.95$		

Table 10: Parameters in DGP

Flow-Payoff Parameters θ^1	$\theta_0^{VP} = 0$ $\theta_1^{VP} = 1.0$ $\theta_2^{VP} = -1.0$ $\theta_0^{FC} = 0.5$ $\theta_1^{FC} = 1.0$ $\theta_0^{EC} = 1.0$ $\theta_1^{EC} = 1.0$
Flow-Payoff Parameters θ^2	$\theta_0^{VP} = 1$ $\theta_1^{VP} = 1.0$ $\theta_2^{VP} = -1.0$ $\theta_0^{FC} = 0.5$ $\theta_1^{FC} = 1.0$ $\theta_0^{EC} = 1.0$ $\theta_1^{EC} = 1.0$
Mixing Probability	$(0.5, 0.5)$
State Variable Transition	z_{kt} is AR(1), $\gamma_0^k = 0$, $\gamma_1^k = 0.6$
Productivity Transition	ω_t is AR(1), $\gamma_0^\omega = 0$, $\gamma_1^\omega = 0.9$
Past action on productivity	$\gamma_a = 2$
Discount Factor	$\beta = 0.95$

D Additional Simulation Results

Table 11: $nGrid = 3$: The mean and standard deviation of the estimators for N=100, T=20 with 100 Monte Carlo Simulations.

$Parameter$	DGP	$FD2$	$AFD2$	$FD2(FV)$	$AFD2(FV)$	EE	HM	$SEQ(1)$	$FD2(BEL)$	$AFD2(BEL)$
$\theta_0^{VP,1}$	1.0000	1.0116	1.0106	0.8960	0.6547	1.0116	1.0108	1.0116	1.0116	1.0106
	-	(0.3438)	(0.3438)	(0.2955)	(0.2734)	(0.3438)	(0.3442)	(0.3438)	(0.3438)	(0.3438)
$\theta_1^{VP,1}$	1.0000	1.0118	1.0067	1.0074	0.9973	1.0118	1.0089	1.0118	1.0118	1.0067
	-	(0.1383)	(0.1354)	(0.1359)	(0.1363)	(0.1383)	(0.1335)	(0.1383)	(0.1383)	(0.1354)
$\theta_2^{VP,1}$	-1.0000	-1.0135	-1.0093	-1.0134	-1.0008	-1.0135	-1.0128	-1.0135	-1.0135	-1.0093
	-	(0.1223)	(0.1230)	(0.1218)	(0.1222)	(0.1223)	(0.1210)	(0.1223)	(0.1223)	(0.1230)
$\theta_0^{FC,1}$	0.5000	0.5204	0.5255	0.1809	-0.8791	0.5204	0.5515	0.5204	0.5204	0.5255
	-	(0.8442)	(0.8277)	(0.6653)	(0.6613)	(0.8442)	(0.7909)	(0.8442)	(0.8442)	(0.8277)
$\theta_1^{FC,1}$	1.0000	0.9859	0.9791	0.9806	0.9609	0.9859	0.9820	0.9859	0.9859	0.9791
	-	(0.2555)	(0.2480)	(0.2513)	(0.2479)	(0.2555)	(0.2515)	(0.2555)	(0.2555)	(0.2480)
$\theta_0^{EC,1}$	1.0000	0.8906	0.8893	0.8871	0.9076	0.8906	0.8870	0.8906	0.8905	0.8893
	-	(0.4752)	(0.4600)	(0.4742)	(0.4616)	(0.4752)	(0.4539)	(0.4752)	(0.4752)	(0.4600)
$\theta_1^{EC,1}$	1.0000	1.1568	1.1609	1.1717	1.1308	1.1568	1.1600	1.1568	1.1568	1.1609
	-	(0.5932)	(0.5939)	(0.5920)	(0.5847)	(0.5932)	(0.5787)	(0.5932)	(0.5932)	(0.5939)
$\theta_0^{VP,2}$	0.0000	0.0029	0.0028	0.0312	-0.1811	0.0029	0.0034	0.0029	0.0029	0.0028
	-	(0.1102)	(0.1126)	(0.1297)	(0.1312)	(0.1102)	(0.1108)	(0.1102)	(0.1102)	(0.1126)
$\theta_1^{VP,2}$	1.0000	1.0202	1.0256	1.0094	1.0164	1.0202	1.0230	1.0202	1.0202	1.0256
	-	(0.0988)	(0.0980)	(0.1010)	(0.0960)	(0.0988)	(0.1006)	(0.0988)	(0.0988)	(0.0980)
$\theta_2^{VP,2}$	-1.0000	-1.0133	-1.0176	-0.9977	-1.0121	-1.0133	-1.0138	-1.0133	-1.0133	-1.0176
	-	(0.1075)	(0.1052)	(0.1057)	(0.1055)	(0.1075)	(0.1069)	(0.1075)	(0.1075)	(0.1052)
$\theta_0^{FC,2}$	0.5000	0.4618	0.4561	0.7508	-0.1444	0.4618	0.4317	0.4618	0.4618	0.4561
	-	(0.4671)	(0.4984)	(0.6164)	(0.4707)	(0.4671)	(0.5440)	(0.4671)	(0.4671)	(0.4984)
$\theta_1^{FC,2}$	1.0000	1.0327	1.0401	1.0367	1.0266	1.0327	1.0369	1.0327	1.0327	1.0401
	-	(0.1811)	(0.1884)	(0.1834)	(0.1833)	(0.1811)	(0.1847)	(0.1811)	(0.1811)	(0.1884)
$\theta_0^{EC,2}$	1.0000	1.0036	0.9977	0.9892	0.9924	1.0036	1.0071	1.0036	1.0036	0.9977
	-	(0.2536)	(0.2305)	(0.2553)	(0.2303)	(0.2536)	(0.2450)	(0.2536)	(0.2536)	(0.2305)
$\theta_1^{EC,2}$	1.0000	1.0390	1.0474	1.0777	1.0549	1.0390	1.0379	1.0390	1.0390	1.0474
	-	(0.3254)	(0.3190)	(0.3363)	(0.3221)	(0.3254)	(0.3134)	(0.3254)	(0.3254)	(0.3190)

- One issue with HM estimator is that the inversion of $(I - \beta \boldsymbol{F}^P)^{-1}$ is sometimes singular.(Observed in simulations).
- I was not able to find the alternative stopping criterion. The current stopping criterion is 10^{-6} on log likelihood.

Table 12: $nGrid = 4$: The mean and standard deviation of the estimators for N=100, T=20 with 100 Monte Carlo Simulations.

$Parameter$	DGP	$FD2$	$FD2(FV)$	EE	HM	$SEQ(1)$	$FD2(BEL)$
$\theta_0^{VP,1}$	1.0000	1.0205	0.9945	1.0205	1.0201	1.0205	1.0205
	-	(0.3381)	(0.2939)	(0.3381)	(0.3346)	(0.3381)	(0.3381)
$\theta_1^{VP,1}$	1.0000	1.0320	1.0320	1.0320	1.0316	1.0320	1.0320
	-	(0.1655)	(0.1651)	(0.1655)	(0.1656)	(0.1655)	(0.1655)
$\theta_2^{VP,1}$	-1.0000	-1.0059	-1.0090	-1.0059	-1.0063	-1.0059	-1.0059
	-	(0.1275)	(0.1273)	(0.1275)	(0.1273)	(0.1275)	(0.1275)
$\theta_0^{FC,1}$	0.5000	0.5203	0.4844	0.5203	0.5197	0.5203	0.5203
	-	(0.8725)	(0.7091)	(0.8725)	(0.8623)	(0.8725)	(0.8725)
$\theta_1^{FC,1}$	1.0000	1.0154	1.0178	1.0154	1.0153	1.0154	1.0154
	-	(0.2832)	(0.2822)	(0.2832)	(0.2841)	(0.2832)	(0.2832)
$\theta_0^{EC,1}$	1.0000	0.9627	0.9606	0.9627	0.9608	0.9627	0.9627
	-	(0.4467)	(0.4470)	(0.4467)	(0.4312)	(0.4467)	(0.4467)
$\theta_1^{EC,1}$	1.0000	1.0262	1.0357	1.0262	1.0346	1.0262	1.0262
	-	(0.6441)	(0.6438)	(0.6441)	(0.5892)	(0.6441)	(0.6441)
$\theta_0^{VP,2}$	0.0000	0.0162	0.1145	0.0162	0.0163	0.0162	0.0162
	-	(0.1232)	(0.1378)	(0.1232)	(0.1232)	(0.1232)	(0.1232)
$\theta_1^{VP,2}$	1.0000	1.0143	0.9969	1.0143	1.0135	1.0143	1.0143
	-	(0.0972)	(0.0962)	(0.0972)	(0.0970)	(0.0972)	(0.0972)
$\theta_2^{VP,2}$	-1.0000	-1.0084	-0.9884	-1.0084	-1.0081	-1.0084	-1.0084
	-	(0.1140)	(0.1116)	(0.1140)	(0.1139)	(0.1140)	(0.1140)
$\theta_0^{FC,2}$	0.5000	0.5675	1.2019	0.5675	0.5674	0.5675	0.5675
	-	(0.3185)	(0.3669)	(0.3185)	(0.3185)	(0.3185)	(0.3185)
$\theta_1^{FC,2}$	1.0000	1.0145	1.0135	1.0145	1.0143	1.0145	1.0145
	-	(0.2238)	(0.2234)	(0.2238)	(0.2238)	(0.2238)	(0.2238)
$\theta_0^{EC,2}$	1.0000	0.9872	0.9575	0.9872	0.9870	0.9872	0.9872
	-	(0.2373)	(0.2405)	(0.2373)	(0.2385)	(0.2373)	(0.2373)
$\theta_1^{EC,2}$	1.0000	0.9975	1.0567	0.9975	0.9962	0.9975	0.9975
	-	(0.2971)	(0.3120)	(0.2971)	(0.2968)	(0.2971)	(0.2971)

References

Aguirregabiria, V. (2012). A method for implementing counterfactual experiments in models with multiple equilibria. *Economics Letters*, 114(2):190–194.

Aguirregabiria, V. and Ho, C. Y. (2012). A dynamic oligopoly game of the US airline industry: Estimation and policy experiments. *Journal of Econometrics*, 168(1):156–173.

Aguirregabiria, V. and Magesan, A. (2016). Solution and Estimation of Dynamic Discrete Choice Structural Models Using Euler Equations. *SSRN Electronic Journal*.

Aguirregabiria, V. and Mira, P. (2002). Swapping the Nested Fixed Point Algorithm: A Class of Estimators for Discrete Markov Decision Models. *Econometrica*, 70(4):1519–1543.

Aguirregabiria, V. and Mira, P. (2007). Sequential estimation of dynamic discrete games. *Econometrica*, 75(1):1–53.

Arcidiacono, P. and Ellickson, P. (2011). Practical Methods for Estimation of Dynamic Discrete Choice Models. *Annual Review of Economics*, 3(1).

Beauchamp, A. (2015). Regulation, Imperfect competition, and the U.S. abortion market. *International Economic Review*, 56(3).

Berry, S. and Tamer, E. (2006). Identification in models of oligopoly entry. *Econometric Society Monographs*, Berry, S.(August 2005):1–46.

Bronnenberg, B. J., Dubé, J.-P. H. P. H., and Gentzkow, M. (2012). The Evolution of Brand Preferences: Evidence from Consumer Migration. *American Economic Review*, 102(6):2472–2508.

CollardWexler, A. (2013). Demand Fluctuations in the Ready-Mix Concrete Industry. *Econometrica*, 81(3):1003–1037.

Doganoglu, T. and Klapper, D. (2006). Goodwill and dynamic advertising strategies. *Quantitative Marketing and Economics*, 4(1):5–29.

Doraszelski, U. and Pakes, A. (2007). A Framework for Applied Dynamic Analysis in IO. *Handbook of Industrial Organization*, 3(December 2007):1887–1966.

Dubé, J. P., Hitsch, G. J., and Manchanda, P. (2005). An empirical model of advertising dynamics. *Quantitative Marketing and Economics*, 3(2):107–144.

Ellickson, P. and Misra, S. (2015). Supermarket Pricing Strategies. *Food Manufacture*, 27(5):811–828.

Fang, H. and Wang, Y. (2009). Estimating Dynamic Discrete Choice Models with Hyperbolic Discounting, with an Application to Mammography Decisions.

Gaynor, M. and Town, R. (2012). Competition in Health Care Markets. In *Handbook of Health Economics*, number 2, chapter 2, pages 499–637. Elsevier.

Gowrisankaran, G., Lucarelli, C., Schmidt-Dengler, P., and Town, R. (2011). Government policy and the dynamics of market structure : Evidence from Critical Access Hospitals.

Gowrisankaran, G. and Town, R. J. (1997). Dynamic equilibrium in the hospital industry. *Journal of Economics and Management Strategy*, 6(1):45–74.

Hansen, L. P. and Singleton, K. J. (1982). Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models. *Econometrica*, 50(5):1269.

Hotz, J. and Miller, R. (1993). Conditional Choice Probabilities and the Estimation of Dynamic Models. *The Review of Economic Studies*, 60(3):497–529.

Keane, M., Todd, P., and Wolpin, K. (2011). The Structural Estimation of Behavioral Models: Discrete Choice Dynamic Programming Methods and Applications. In *Handbook of Labor Economics*, volume 4, pages 331–461.

Pakes, A. and McGuire, P. (1994). Computing Markov perfect Nash equilibria: Numerical implications of a dynamic differentiated product model. *The RAND Journal of Economics*, 25(4):555–589.

Rust, J. (1987). Optimal Replacement of GMC Bus Engines : An Empirical Model of Harold Zurcher. *Econometrica*, 55(5):999–1033.

Rust, J. (1994). Structural estimation of Markov Decision Processes. *Handbook of econometrics*, 4:3081–3143.

Su, C.-L. and Judd, K. L. (2012). Constrained optimization approaches to estimation of structural models. *Econometrica*, 80(5):2213–2230.

Sweeting, A. (2013). Dynamic product positioning in differentiated product markets: The effect of fees for musical performance rights on the commercial radio industry. *Econometrica*, 81(5):1763–1803.

Todd, P. E. and Wolpin, K. I. (2006). Assessing the Impact of a School Subsidy Program in Mexico : Using a Soc Experiment to Validate a Dynam Fertility Behavioral Model of Child Schooling. *The American Economic Review*, 96(5):1384–1417.

Yakovlev, E. (2016). Demand for Alcohol Consumption and Implication for Mortality: Evidence from Russia.