Some Background On Computing Equilibria to Dynamic Games & Stochastic Algorithms.

by

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Basic Model (Ericson and Pakes, 1995, *Restud*).

Note. A more formal treatment of what follows is given in Doraszelski and Pakes (2007 *Handbook of Industrial Organation*). From an assumption point of view, the big difference between the material presented here and that in their article is that I will ignore random entry and exit fees. Proofs of existence require those sources of randomness. However most of the examples that I will go over do not have it, and still compute equilibria (or at least "ϵ-equilibria"). I ignore them here largely for pedagogical reasons; they make the notation much more complex.

States.

- $i \in \mathcal{Z}^+$.
- s_i will be the number of firms with efficiency level i,
- $s = [s_i;\ i \in \mathcal{Z}^+]$ is the "industry structure" (the number of firms at each different efficiency level).

Assumption. Given investment decisions, the distribution of future values of the state of the system (of (i, s)) are independent of the choice of prices or quantities.
$\Rightarrow$ changes in price (quantity) affect only current profits.
$\Rightarrow$ $\pi(i, s)$ can be calculated "off-line" (without needing to compute the value function).

Differentiated Product Example

Static Profit Function.

$$U_{i,j} = v_j - p_j^* + \epsilon(i,j)$$

where v_j is the "quality" of the good and p_j^* is the price (for $j = 1, \ldots, J$.

Consumer i chooses good j if and only if

$$\epsilon(i,j) - \epsilon(i,q) > [v_q - v_j] - [p_q^* - p_j^*]$$

$$\equiv [v_q - \zeta] - [p_q^* - p_0^*] - \{[v_j - \zeta] - [p_j^* - p_0^*]\}$$

where ζ is the quality of the outside alternative,

$$\equiv i_q - p_q - [i_j - p_j]$$

which implicitly defines "real prices" and "real quality" (i.e. both relative to the outside alternative).

Define $C(i_j; s, p) = \{\epsilon$: the consumer chooses good $j\}$ Assuming the distribution of ϵ is i.i.d. extreme value, then the market shares have the "logit form"

$$\sigma(i_j; s, p) = \frac{exp[i_j - p_j]}{[1 + \sum exp[i_q - p_q]}.$$

Profits, assuming a constant marginal cost of mc are then

$$[p_j - mc]M\sigma(i_j; s, p).$$

This implies that the Nash pricing equilibrium pricing vector satisfies

$$-[p_j - mc]\sigma_j[1 - \sigma_j] + \sigma_j = 0.$$

There is a unique solution to this system, say $p(i_j; s)$, (see Caplin and Nalebuff,1991) and profits become

$$\pi(i_j; s) = [p(i_j; s) - mc]M\sigma[i_j; s, p(s)].$$

Bellman Equation for Incumbent Behavior.

$$V(i,s) = \max\{\phi, \pi(i,s) +$$

$$\sup_{(x \geq 0)} [-cx + \beta \sum V(i',s')pr(i',s'|x,i,s)]\}.$$

- $pr(i',s'|x,i,s) = pr(i'|i,x)pr(s'|i,x,s)$. Games where my own investment only affects my own state variables are often called "capital accumulation games".

- $i_{t+1} - i_t \equiv \tau_{t+1}$

- $\tau_t \equiv \nu_t - \zeta_t$

- $\nu =$ firm's investment outcome.

  $$\mathcal{P} = \{p_\nu(\cdot|x), x \in \mathcal{R}^+\},$$

  stochastically increasing in x. Further we assume that $Pr\{\nu > 0|x = 0\} = 0$.

- $\zeta =$ common industry shock. Density $\mu(\zeta)$.

Let $\widehat{s}_i = s - e_i$ provide the states of the competitors of a firm at state i for a particular s, and $q[\widehat{s_i}'|i,s,\zeta]$ provide the firm's perceived probability of its competitors future states conditional on a particular value of ζ. Then given the above

$$pr(i' = i^*, s' = s^*|x,i,s) =$$

$$\Sigma_\zeta p(\nu = i^* - i - \zeta|x) q[\widehat{s_i}' = s^* - e(i^*)|i,s,\zeta]\mu(\zeta).$$

where $e(i)$ has a one in the i^{th} slot and zero elsewhere.

Note that $q[\cdot|i,s,\zeta]$ embodies the incumbent's beliefs about entry and exit.

Entry model.

- Must pay an amount x_e ($> \beta\phi$) to enter,

- Enters one period later at state $\imath^e \in \Omega^e \subset \mathcal{Z}^+$ with probability $p^e(\cdot)$.

- Only enters if the expected discounted value of future net cash flows from entering is greater than the cost of entry.

- The cost of entry can be specified as either x_e, or as a random variable which distributes uniformly on $[x_{e,l}, x_{e,u}]$. When random entry costs are used only the potential entrant knows the realization of the entry costs, the other incumbents know only that entry costs will be a random draw from this uniform distribution.

Dynamic Equilibrium:Characterization Results.

Explain "equilibrium".

- Every agent chooses optimal policies given its perceptions on likely future industry structures

- Those pereptions are consistent with the behavior of the agent's competitors.

This framework is due to E-P(1995). Doraszelski and Satterwaite (2003) prove that a Markov Perfect equilibrium exists for this model (at least if we have random entry fees and exit costs), and E-P show that any such equilibria has the following characteristics

1. It is “computable”, i.e..

   - Never more than $\overline{n}$ firms active.

   - Only observe “i” on $\Omega = \{1, ..., K\}$ (given the competitors states, there is a lower ω at which the given firm exits, and an upper ω at which the given firm stops investing – since there is a finite number of other firms possible you can then take the min of the lower state and the max of the upper state).

   $\Rightarrow$ need only compute equilibria for $(i, s) \in \Omega \times S$, where

   $$S \equiv \{s = [s_1, ..., s_k] : \sum s_j \leq \overline{n} < \infty\}$$

   so that the number of elements in S or $\#S \leq K^N$.

2. It is Markov. Indeed equilibrium policies generate a homogeneous Markov chain for

industry structures [for $\{s_t\}$], i.e.

$$Pr[s_{t+1} = s'|s^t] = Pr[s_{t+1} = s'|s_t] \equiv Q[s'|s_t].$$

with the Markov transition "kernel" $Q(\cdot,\cdot)$ on $S \times S$.

3. They provide conditions on the primitives such that insure that any equilibrium $Q[\cdot|\cdot]$ is *ergodic*. [Picture].

   - Note that the nature of states in R, how states cycle in R, and the transitions to R depend on primitives.

   - R is frequently much smaller than S (and the divergence is greatest for large markets with many state variables).

Brute Force Computation
(Pakes and McGuire,1994, *RAND*).

Assume temporarily that Ω and $\overline{n}$ are known. The first algorithm we consider is a "backward solution" algorithm that computes the value and policy functions pointwise. It is the multiple agent analogue of what we did to compute single agent dynamic problems.

- In memory. Estimates of the value function and policies associated with each $(i, s) \in \Omega \times S$.

- Updating. *Synchronous*; i.e. it circles through the points in S in some fixed order and updates all estimates associated with *every* $s \in S$ at each iteration (here updating estimates at s involves updating estimates at each (i, s) that has $s_i > 0$).

- Convergence. The values and policies from successive iterations are the same. Converged policies and values satisfy all the properties of equilibrium values and policies (see below).

Updating 1: Rewrite Bellman Equation.

$$V(i,s) = \max_{\chi \in \{0,1\}} \{[1-\chi]\phi + \qquad (1)$$

$$\chi\{\pi(i,s) - \sup_{x \geq 0}[-cx + \beta \sum_{\nu} w(\nu; i,s) p(\nu | x_1)]\},$$

where

$$w(\nu;\ i,s) \equiv$$

$$\sum_{(\widehat{s}_i^{'},\zeta)} V(i+\nu-\zeta, \widehat{s}_i^{'} + e(i+\nu-\zeta)|w) q[\widehat{s}_i^{'}|i,s,\zeta]\mu(\zeta), \quad (1a)$$

and

$$q[\widehat{s}_i^{'} = s_i^{*}|i,s,\zeta] \equiv$$

$$Pr\{\widehat{s}_i^{'} = \widehat{s}_i^{*}|i,s,\zeta, \text{equilibrium policies }\} \ (1b).$$

Here $w(\nu; i, s)$ is the expected discounted value of future net cash flow conditional on the current year's investment resulting in a particular value of ν, and the current state being (i, s) (it integrates out over the possible outcomes of both the investment strategies of competitors (the $\hat{s}_i'$), and over the outside alternative (the ζ)).

Note: Just as in the single agent problem $w(\nu; i, s)$ is all the firm needs to know in order to make decisions. It is thus a sufficient statistic for decision making purposes (note that it is sufficient for a very complicated object, the expected discounted value of future net cash flows given a realization for the investment process).

Updating Rules.

- calculates $w^{k-1}(\cdot|i,s)$ from the information in memory, i.e. from (x^{k-1}, V^{k-1}) (as in 1a),

- substitutes $w^{k-1}(\cdot)$ for $w(\cdot)$ in (1) and then solve the resultant *single agent* optimization problem for the j^{th} iteration's entry, exit and investment polices at (i,s). That is

  - Incumbents solve for (χ^k, x^k) that

$$\max_{\chi\in\{0,1\}}\{[1-\chi]\phi+$$

$$\chi \sup_{x\geq 0}[\pi(i,s)-cx+\beta\sum_{\nu} w^{k-1}(\nu;i,s)p(\nu|x)]$$

    I.e. we solve the Kuhn-Tucker problem for investment conditional on continuing which in the example works out to be

$$\frac{\partial p(x)}{\partial x}[w(1;i,s) - w(0;i,s)] - c \leq 0,$$

with stict inequality if and only if $x = 0$. We then substitute the solution for x in the continuation value above and determine whether it is greater than ϕ.

– Potential entrants compute

$$V_e^k(s) = \beta \sum_{\zeta} w^{k-1}(\zeta; i_e, s + e(i_e))\mu(\zeta).$$

and set $\chi_e^k = 1 \Leftrightarrow V_e^k(s) > x_e$,

- substitutes these policies and the w^{k-1} for the w, x and the max operator in (1), and labels the result $V^k(\cdot)$,

- calculates $V^k(\cdot) - V^{k-1}(\cdot)$ and then substitutes $V^k(\cdot)$ and the policies from iteration k, for the iteration $k-1$ values that were in memory.

Convergence.

At the end of the iteration calculate $||V^{k-1}(\cdot) - V^k(\cdot)||$ and $||x^{k-1}(\cdot) - x^k(\cdot)||$. If both are sufficiently small, stop. Else continue. The program prints out both the L^2 norm and the sup norm.

At fixed point each incumbent and potential entrant

- uses, as its perceived distribution of the future states of its competitors, the actual distribution of future states of those competitors, and

- chooses its policy to maximize its expected discounted value of future net cash flow given this distribution of the future of its competitors.

Star and Ho 1969, provide a proof that this is all that is needed for a MPE.

Setting K and $\overline{n}$

K.
Start with the monopoly problem ($\overline{n} = 1$) and an oversized K; $\rightarrow$ a lowest i at which the monopolist remains active and a highest i at which the monopolist invests. $\rightarrow$ 1 and K in Ω.

$\overline{n}$.
Set $\overline{n} = 2$ and do the iterative calculations again starting at $V^0(i_1, i_2) = V^*(i_1)$. Then set $\overline{n} = 3$ and set $V^0(i_1, i_2, i_3) = V^*(i_1, max(i_2, i_3))$. Continue until we reach an $\overline{n}$ so high that whenever there are $\overline{n} - 1$ firm's active there is no possible structure at which an entrant would want to enter. This is $\overline{n}$.

The computational burden is (essentially) the product of three factors,

- the number of points evaluated at each iteration;
- the time per point evaluated;
- the number of iterations.

Number of Points.

Since each of the $\overline{n}$ active firms can only be at K distinct states, the number of points we need to evaluate at each iteration, or

$$\#S \leq K^{\overline{n}}.$$

Exchangeability, of the value and the policy functions in the state variables of a firm's competitors implies that we do not need to differentiate between two vectors of competitors that are permutations of one another notation does not. Pakes (1993) shows that an upper bound for $\#S$ is given by the combinatoric

$$\binom{K+\overline{n}-1}{\overline{n}} << K^{\overline{n}}.$$

but for $\overline{n}$ large enough this bound is tight.

Burden per Point. Determined by

- the cost of calculating the expected value of future states conditional on outcomes (of obtaining the $w^j(\cdot; i, s)$ from the information in memory).

- The cost of obtaining the optimal polices and the new value function given $w^j(\cdot; i, s)$.

Take the simplest model and recall that

$$V^j(i,s) =$$

$$\max_{\chi\in\{0,1\}}\{[1-\chi]\phi+\chi\{\pi(i,s)-\sup_{x\geq 0}[-cx+\beta\sum_{\nu} w^{j-1}(\nu;i,s)p(\nu|x)]\},$$

where

$$w^j(\nu;\ i,s) \equiv \sum_{(\hat{s}_i^{'},\zeta)} V^{j-1}(i+\nu-\zeta,\hat{s}_i^{'}+e(i+\nu-\zeta)|w)q^{j-1}[\hat{s}_i^{'}|i,s,\zeta]\mu(\zeta),$$

and

$$q^{j-1}[\hat{s}_i^{'} = s_i^*|i,s,\zeta] \equiv$$

$$Pr\{\hat{s}_i^{'} = \hat{s}_i^*|i,s,\zeta,\text{policies at iteration "j-1" }\}.$$

Easiest to think of this as individual firms instead of a measure, i.e as a state being $(i_j, \underline{i})$. Assume that there is positive probability on each of κ points for each of the $m-1$ active competitors of a given firm. Then we need to sum over κ^m possible future states and there are $\kappa\times m$ values of $w^j(\cdot)$ needed at that s. Average m should increase in $\overline{n}$, and κ should be determined by the nature of the state space per firm (it typically goes up exponentially in the number of state variables per firm).

Conclusion

It is clear that the computational burden of the model grows quickly in both the number of firms ever simultaneously active (it grows geometrically in this dimension) and the number of state variables per firm (it grows exponentially in this dimension). This is the problem known as "The Curse of Dimensionality" in the computational literature.

Approximation Techniques.

- Pointwise algorithm has been used both as a tool for substantive problems and as a teaching device.

- Still many applied problems need more powerful computational tools.

- Three available. Each has their problems, but they compute equilibria with much less of a computational burden than the standard algorithm.
  - Stochastic algorithm (AI). Pakes and McGuire (2001) lead to EBE which I will turn to next
  - Continuous time algorithm. Doraszelski and Judd (2004).
  - Deterministic approximation techniques. Judd (book)

I will give you a paragraph on the continuous time and deterministic approximation methods, and then go into the detail of the stochastic algorithm. The latter is also one way to introduce learning models.

Continuous Time. Due to Doraszelski and Judd *Quantitative Economics, 2011*, is designed to ease the burden of computing the expectation over successor states, and hence decreases the computation time needed to update values and policies at a particular state (it makes no attempt to alleviate the burden imposed by the number of states). The set up is a continuous-time model in which at any particular instant only one firm experiences a change in its state. As a result if each firm's transition can go to one of K states, and there are n firms, we only need to sum over $(K-1)\times n$ states to compute continuation values. This is in contrast to the K^n possible future states that we need to sum over in the computational algorithm described earlier. This implies that the discrete- and continuous-time models have different implications. As a result it may (but need not) be the case that one of the models provides a better approximation to behavior in a particular setting than the other. For example in the continuous time model starting at t_1 which firm changes its state is a probabilistic function of all firm's investments. However if firm 1 changes its state at t_2 none of the investments of the other firms in the interim affect future sample paths. In the discrete time model the investments of other firms are totally capturred in the current state, so investment prior to t_1 do not matter conditional on the state, but investments in the interval do.

Deterministic Approximations ("Curve Fitting").

Judd's (2000) book provides a thorough introduction to such techniques. The techniques begin by specifying a set of functions considered rich enough to contain an element which provides a good approximation to the value function (e.g. polynomials of order d). Given some initial value functions for a small subset of the points in $\mathcal{S}$, they then find a member of the set of functions that approximates the value functions at the small set of points (in a polynomial they would find a number of points that is large and "varied" enough to enable them to set the polynomial coefficients). They then do the iteration on the small set of points using the approximating function to predict the value function at other points as needed. They then continue until convergence.

Experienced Based Equilibria and Stochastic Algorithms.

• The computational complexity of the standard model makes it difficult for research to use and requires a lot of agents in the market. We look for ways of easing these two burdens.

• The framework above assumes that the

1. state variables evolve as a Markov process

2. and the equililbrium is some form of Markov Perfection (no agent has an incentive to deviate at any value of the state variables).

On the Markov Assumption. Except in situations involving active experimentation to learn (where policies are transient), we are likely to stick with the assumption that states evolve as a time homogenous finite order Markov process. Reasons:

- It is a convenient and fits the data well.
- Realism suggests information access and retention conditions limit the memory used.
- We can bound unilateral deviations (Ifrach and Weintraub, 2014), and have conditions which insure those deviations can be made arbitrarily small by letting the length of the kept history grow (White and Scherer, 1994).

On 2: Perfection. The type of rationality built into Markov Perfection is more questionable; though it has been useful in the simple models used by our theory and computational colleagues to; i) explore possible outcomes in a structured way, and to ii) generate solutions to selection problems that appear in estimation problems.

I want to start from the premise that the complexity of Markov Perfection not only limits our ability to do dynamic analysis of market outcomes it also

- leads to a question of whether some other notion of equilibria will better approximate agents' behavior.

Question. If we abandon Markov Perfection can we both

- better approximate agents' behavior and,

- enlarge the set of dynamic questions we are able to analyze.

The complexity issue. When we try to incorporate "essential" institutional background we find that the agent is required to:

- Access a large amount of information (all state variables), and
- Either compute or learn an unrealistic number of strategies (one for each information set).

How demanding is this? Consider markets where consumer, as well as producer, choices are dynamic (e.g.'s; durable, experience, or network goods); need the distribution of; current stocks $\times$ household characteristics, production costs, ... In a symmetric information MPE an agent would have to access all state variables, and then either compute a doubly nested fixed point, or learn and retain, policies from each distinct information set.

Obvious Fix: Assume agents only have access to a subset of the state variables.

- Since agents presumably know their own characteristics and these tend to be persistent, we would need to allow for assymetric information: the "perfectness" notion would then lead us to a "Bayesian" Markov Perfect solution.

Is assuming "Bayesian MP" more realistic? It decreases the information access and retention conditions but increases the burden of computing the policies significantly over the burden of computing in symmetric information MPE models. The additional burden results from the need to compute posteriors, as well as optimal policies; and the requirement that they be consistent with one another.

Could agents learn these policies? I will come back to the issue of what the agents could and could not learn below.

Rest of Talk.

- I am going to introduce a notion of equilibrium that is less demanding than Markov Perfect for both the agents, and the analyst, to use and show how to
  (i) compute the equilibrium and
  (ii) estimate off of equilibrium conditions.

- Consider restrictions that mitigate multiplicity issues.

- Use the equilibrium notion to explore the impacts of information sharing in dynamic procurement auctions.

I start with strategies that are “rest points” to a dynamical system. Later I will consider institutional change, but only changes where it is reasonable to model responses to the change with a simple reinforcement learning process (I do not consider changes that lead to active experimentation). This makes my job much easier because:

- Strategies at the rest point likely satisfy a Nash condition of some sort; else someone has an incentive to deviate.

- However it still leaves opens the question: What is the form of the Nash Condition?

What Conditions Can We Assume for the Rest Point at States that are Visited Repeatedly?

We expect (and I believe should integrate into our modelling) that

1. Agents perceive that they are doing the best they can at each of these points, and

2. These perceptions are at least consistent with what they observe.

Note. It might be reasonable to assume more than this: that agents (i) know and/or (ii) explore, properties of outcomes of states not visited repeatedly. I come back to this below.

Formalization of Assumptions.

• Denote the information set of firm i in period t by $J_{i,t}$. $J_{i,t}$ will contain both public (ξ_t) and private ($\omega_{i,t}$) information, so $J_{i,t} = \{\xi_t, \omega_{i,t}\}$.

• Assume $(J_{1,t}, \dots J_{n_t,t})$ evolves as a finite state Markov process on $\mathcal{J}$ (or can be adequately approximated by one).

• Policies, say $m_{i,t} \in \mathcal{M}$, will be functions of $J_{i,t}$. For simplicity assume $\#\mathcal{M}$ is finite, and that it is a simple capital accumulation game, i.e. $\forall (m_i, m_{-i}) \in \mathcal{M}^n$, & $\forall\, \omega \in \Omega$

$$P_\omega(\cdot | m_i, m_{-i}, \omega) = P_\omega(\cdot | m_i, \omega),$$

(I will relax this when we consider auctions below).

• The public information, ξ, is used to predict competitor behavior and common demand and cost conditions (these evolve as an exogenous Markov process).

• A "state" of the system, is

$$s_t = \{J_{1,t}, \ldots, J_{n_t,t}\} \in \mathcal{S},$$

$\#\mathcal{S}$ is finite. $\Rightarrow$ any set of policies will insure that s_t will wander into a recurrent subset of $\mathcal{S}$, say $\mathcal{R} \subset \mathcal{S}$, in finite time, and after that $s_{t+\tau} \in \mathcal{R}$ w.p.1 forever.

• Note that the agents does not keep track of all of s_t, only $J_{i,t}$, and hopefully we can tell what is in $J_{i,t}$ by examining what determines dynamic policies.

• Let the agent's perception of the expected discounted value of current and future net cash flow were it to chose m at state J_i, be

$$W(m|J_i), \ \ \forall m \in \mathcal{M} \ \ \& \ \ \forall J_i \in \mathcal{J},$$

• and of expected profits be

$$\pi^E(m|J_i).$$

Our assumptions imply:

- Each agent choses an action which maximizes its perception of its expected discounted value, and

- For those states that are visited repeatedly (are in $\mathcal{R}$) these perceptions are consistent with observed outcomes.

Formally

A. $W(m^*|J_i) \geq W(m|J_i), \;\; \forall m \in \mathcal{M} \;\; \& \;\; \forall J_i \in \mathcal{J},$

B. &, $\forall J_i$ which is a component of an $s \in \mathcal{R}$

$$W(m(J_i)|J_i) = \pi^E(m|J_i) + \beta \sum_{J_i^{'}} W(m^*(J_i^{'})|J_i^{'}) p^e(J_i^{'}|J_i),$$

where, if $p^e(\cdot)$ provides the empirical probability (the fraction of periods the event occurs)

$$\pi^E(m|J_i) \equiv \sum_{J_{-i}} E[\pi(\cdot)|J_i, J_{-i}] p^e(J_{-i}|J_i),$$

and

$$\Big\{ p^e(J_{-i}|J_i) \equiv \frac{p^e(J_{-i}, J_i)}{p^e(J_i)} \Big\}_{J_{-i}, J_i},$$

while

$$\Big\{ p^e(J_i^{'}|J_i) \equiv \frac{p^e(J_i^{'}, J_i)}{p^e(J_i)} \Big\}_{J_i^{'}, J_i} . \;\; \spadesuit$$

"Experience Based Equilibrium"

These are the conditions of a (restricted) EBE (Fershtman and Pakes, 2012; for related earlier work see Fudenberg and Levine, 1993 on self confirming equilibria). Bayesian Perfect satisfy them, but so do weaker notions. We now turn to its :
(i) computational and estimation properties,
(ii) overcoming multiplicity issues, and then to
(iii) dynamic procurement auctions.

Computational Algorithm. Asynchronous "Reinforcement learning" algorithm (Pakes and McGuire, 2001). Can be viewed as a learning process. Makes it a candidate to:
(i) analyze (small) perturbations to the environment,
(ii) as well as to compute equilibrium.

• Formally it circumvents the two sources of the curse of dimensionality in computing equilibrium; but there is still lots of room for improvement.

Iterations defined by

• A location, say $L^k = (J_1^k, \ldots J_{n(k)}^k) \in \mathcal{S}$: is the information sets of active agents .

• Objects in memory (i.e. M^k):

(i) perceived evaluations, W^k,

(ii) No. of visits to each point, h^k.

Must update (L^k, W^k, h^k)**.** Computational burden determined by; memory constraint, and compute time. I use a simple (not neccesarily optimal) structure to memory.

Update Location.

• Calculate "greedy" policies for each agent

$$m_{i,k}^* = arg \max_{m \in \mathcal{M}} W^k(m|J_{i,k})$$

• Take random draws on outcomes conditional on $m_{i,k}^*$:

• E.g.; if we invest in "payoff relevant" $\omega_{i,k} \in J_{i,k}$, draw $\omega_{i,k+1}$ conditional on $(\omega_{i,k}, m^*_{i,k})$.

• Use outcomes to update $L^k \to L^{k+1}$.

Update W^k.

• "Learning" interpretation: Assume agent observes $b(m_{-i})$ and knows the primitives; $\pi_i(\cdot), p(\omega_{i,t+1}|\omega_{i,t}, m_{i,t})$.

• Its ex poste perception of what its value would have been had it chosen m is

$$V^{k+1}(J_{i,k}, m) =$$

$$\pi(\omega_{i,k}, m, b(m_{-i,k}), d_k) + \max_{\tilde{m} \in M} \beta W^k(\tilde{m}|J_{i,k+1}(m)),$$

where $J_i^{k+1}(m)$ is what the $k+1$ information would have been given m and *competitors actual play*.

Treat $V^{k+1}(J_{i,k})$ as a random draw from the possible realizations of $W(m|J_{i,k})$, and update W^k as in stochastic integration (Robbins and Monroe,1956)

$$W^{k+1}(m|J_{i,k}) =$$

$$\frac{1}{h^k(J_{i,k})}V^{k+1}(J_{i,k}, m) + \frac{(h^k(J_{i,k}) - 1)}{h^k(J_{i,k})}W^k(m|J_{i,k}),$$

or

$$W^{k+1}(m|J_{i,k}) - W^k(m|J_{i,k}) =$$

$$\frac{1}{h^k(J_{i,k})}[V^{k+1}(J_{i,k}, m) - W^k(m|J_{i,k})].$$

(other weights are more efficient, it would be good to know how to aggregate states ....)

Notes.

- If we have equilibrium valuations we tend to stay their, i.e. if * designates equilibrium

$$E[V^*(J_i, m^*)|W^*] = W^*(m^*|J_i).$$

• To learn equilibrium values we need to visit points repeatedly; only likely for states in $\mathcal{R}$.

• Agents (not only the analyst) could use the algorithm to find equilibrium policies or adjust to perturbations in the environment.

• Algorithm has no curse of dimensionality.
(i) Computing continuation values: integration is replaced by averaging two numbers.
(ii) States: algorithm eventually wanders into $\mathcal{R}$ and stays their, and $\#\mathcal{R} \leq \#\mathcal{J}$.

• $\#R$ need not be finite, but it typically grows linearly (not exponentially) in the number of states. Still the computational burden can be quite large.

• The stochastic approximation literature for single agent problems often augments this with functional form approximations ("TD learning"; Sutton and Barto,1998).

Convergence and Testing.

• Testing. The algorithm does not necessarily converge, but a test for convergence exists and does not involve a curse of dimensionality (Fershtman and Pakes, 2012).

• The test is based on simulation. It produces a consistent estimate of an $L^2(P(\mathcal{R}))$ norm of the percentage bias in the implied estimates of $V(m, J_i)$; where $P(\mathcal{R})$ is the invariant measure on the recurrent class.

• **Basis.** Simulate sample paths and $\forall(m, J_i)$ store mean $(\tilde{W}(m|J_i))$ and variance $(\tilde{V}(\tilde{W}(m|J_i))$ of EDV of playing m at J_i. $(\tilde{W}(m|J_i) - W(m|J_i))^2$ is the MSE of $\tilde{W}(m|J_i)$ as an estimate of $W(m|J_i)$.

$$\% Bias^2(m|J_i) = \frac{(\tilde{W}(m|J_i) - W(m|J_i))^2}{W(m|J_i)^2} - \frac{\tilde{V}(\tilde{W}(m|J_i)}{W(m|J_i)^2}.$$

Details. Any fixed W, say $\tilde{W}$, generates policies which define a finite state Markov process for $\{s_t\}$. Gather the transition probabilies into the Markov matrix, $Q(s', s|\tilde{W})$.

To test if the process satisfies our equilbirium conditions need:
(i) a candidate for $\mathcal{R}$, and checks for
(ii) optimality of policies and
(iii) consistency of W.

Candidate for $\mathcal{R}(\tilde{W})$. Start at any s^0 and use $Q(\cdot, \cdot|\tilde{W})$ to simulate a sample path $\{s^j\}_{j=1}^{J_1+J_2}$. Let $\mathcal{R}(J_1, J_2, \cdot)$ be the set of states visited at least once between $j = J_1$ and $j = J_2$.

$$(J_1, J_2) \to (\infty, \infty), \;\; \& \;\; J_2 - J_1 \;\; \to \infty$$

$$\Rightarrow \;\; \mathcal{R}(J_1, J_2, \cdot) \to \tilde{\mathcal{R}},$$

a recurrent class of $Q(\cdot, \cdot|\tilde{W})$ (C1 satisfied).

C2 (optimality of policies). Satisfied by construction, since we use the policies generated by $\tilde{W}$ to form $Q(\cdot, \cdot|\tilde{W})$.

C3 (consistency of $\tilde{W}$ with outcomes). Does

$$\tilde{W}(m^*|J_i) = \pi^E(J_i) + \beta \sum_{J_i'} \tilde{W}(m^*(J_i')|J_i')p^e(J_i'|J_i)$$

$(\forall J_i \in s \in \mathcal{R}.)$?

Direct summation. Computationally burdensome; indeed brings the curse of dimensionality back in.

Alternative. Check for consistency of simulated sample paths with evaluations.

• Start at $s_0 \in \mathcal{R}$ and forward simulate. At each J_i compute perceived values (our $V^{k+1}(\cdot)$), keep track of the average and the sample variance of those simulated perceived values, say

$$\Big(\widehat{\mu}(\tilde{W}(m^*(J_i)|J_i)), \widehat{\sigma}^2(\tilde{W}(m^*(J_i)|J_i))\Big).$$

• Let $E(\cdot)$ take expectations over the simulated random draws (where draws will be indexed by a tilde), let l index locations, and note that we can compute $\mathcal{T}_l$, where

$$\mathcal{T}_l \equiv E\Big(\frac{\widehat{\mu}(\tilde{W}_l) - \tilde{W}_l}{\tilde{W}_l}\Big)^2$$

$$= E\Big(\frac{\widehat{\mu}(\tilde{W}_l) - E[\widehat{\mu}(\tilde{W}_l)]}{\tilde{W}_l}\Big)^2 + \Big(\frac{E[\widehat{\mu}(\tilde{W}_l)] - \tilde{W}_l}{\tilde{W}_l}\Big)^2.$$

$$= \%Var(\widehat{\mu}(\tilde{W}_l)) \ + \ \%Bias^2(\widehat{\mu}(\tilde{W}_l)).$$

• $\mathcal{T}_l$ is observed, as is f_l, the fraction of visits to l. As the number of simulation draws grows

$$\sum_l f_l\Big(\frac{\widehat{\sigma}^2(\tilde{W}_l)}{\tilde{W}_l^2}\Big) - \sum_l f_l\Big(\frac{\widehat{\mu}(\tilde{W}_l) - E[\widehat{\mu}(\tilde{W}_l)]}{\tilde{W}_l}\Big)^2 \rightarrow_{a.s.} 0,$$

$\Rightarrow$

$$\sum_l f_l \mathcal{T}_l - \sum_l f_l \Big(\frac{\hat{\sigma}^2(\tilde{W}_l)}{\tilde{W}_l^2}\Big) \to_{a.s.} \sum f_l \Big(\frac{E[\hat{\mu}(\tilde{W}_l)] - \tilde{W}_l}{\tilde{W}_l}\Big)^2,$$

an $L^2(\mathcal{P}_{\mathcal{R}})$ norm in the percentage bias ($\mathcal{P}_{\mathcal{R}}$ is the invariant measure associated with $(\mathcal{R}, \tilde{W})$).

Estimation.

• Need a candidate for J_i. Either:
(i) empirically investigate determinants of controls (determinants of controls), and/or
(ii) ask actual participants.

• Does not require nested fixed point algorithm. Use estimation advances designed for MP equilibria (POB or BBL), or a perturbation (or "Euler" like) condition (below).

Euler-Like Condition.

• With assymetric information the equilibrium condition

$$W(m^*|J_i) \geq W(m|J_i)$$

is an inequality which can generate (set) estimators of parameters.

• J_i contains both public and private information. Let J^1 have the same public, but differnt private, information then J^2. If a firm is at J^1 it knows it could have played $m^*(J^2)$ and its competitors would respond by playing *on the equilbrium path* from J^2.

• If $m^*(J^2)$ results in outcomes in $\mathcal{R}$, we can simulate a sample path from J^2 using only observed equilibrium play. The Markov property insures it would intersect the sample path from the DGP at a random stopping time with probability one and from that time forward the two paths would generate the same profits.

- The conditional (on J_i) expectation of the difference in discounted profits between the simulated and actual path from the period of the deviation to the random stopping time, should, when evaluated at the true parameter vector, be positive. This yields moment inequalities for estimation as in Pakes, Porter, Ho and Ishii (2015), Pakes, (2010).

Multiplicity.

• $\mathcal{R}$ contains both "interior" and "boundary" points. Points at which there are feasible strategies which can lead outside of $\mathcal{R}$ are boundary points. Interior points are points that can only transit to other points in $\mathcal{R}$ no matter which (feasible) policy is chosen.

• Our conditions only insure that perceptions of outcomes are consistent with the results from actual play at interior points. Perceptions of outcomes for some feasible (but inoptimal) policy at boundary points are not tied down by actual outcomes.

• "MPBE" are a special case of (restricted) EBE and they have multiplicity. Here differing perceptions at boundary points can support a (possibly much) wider range of equilibria.

Narrowing the Set of Equilibria.

• In any empirical appliction the data will rule out equilibria. m^* is observable, at least for states in $\mathcal{R}$, and this implies inequalities on $W(m|\cdot)$. With enough data $W(m^*|\cdot)$ will also be observable up to a mean zero error.

• Use external information to constrain perceptions of the value of outcomes outside of $\mathcal{R}$. If available use it.

• Asker, Fershtman, Jihye, and Pakes, 2014 allow firms to experiment with $m_i \neq m_i^*$ at boundary points. Leads to a stronger notion of, and test for, equilibrium. We insure that perceptions are consistent with the results from **actual play** for each **feasible** action at boundary points (and hence on $\mathcal{R}$).

Boundary Consistency.

Let $B(J_i|\mathcal{W})$ be the set of actions at $J_i \in s \in \mathcal{R}$ which could generate outcomes which are not in the recurrent class (if it is not empty, J_i is a boundary point) and $B(\mathcal{W}) = \cup_{J_i \in \mathcal{R}} B(J_i|\mathcal{W})$. Then the extra condition needed to insure "Boundary Consistency" is:

Extra Condition. Let τ index future periods, then $\forall (m, J_i) \in B(\mathcal{W})$

$$W(m^*|J_i) \geq$$

$$E\Big[\sum_{\tau=0}^{\infty} \delta^{\tau} \pi(m(J_{i,\tau}), m(J_{-i,\tau}))|J_i = J_{i,0}, \mathcal{W}\Big],$$

where $E[\cdot|J_i, \mathcal{W}]$ takes expectations over future states starting at J_i using the policies generated by $\mathcal{W}$. ♠

Testing for Boundary Consistency.

Fix $(m, J_i) \in B(i)$. Simulate independent sample paths from it with initial J_{-i} drawn from the empirical distribution of $p^e(J_{-i}|J_i)$. Calculate mean, $\widehat{W}(m|J_i)$, and the variance, $\widehat{V}(\widehat{W}(m|J_i))$, of simulated sample path for each $(m, J_i) \in B(i)$.

Basis of Test. Average

$$\frac{\Big((W(m^*|J_i) - \widehat{W}(m|J_i))_-\Big)^2}{\widehat{V}(\widehat{W}(m|J_i))}$$

over $(m, J_i) \in B(i)$ and then a weighted average of these over boundary points. This is an Inequalities based test and one needs to simulate the test statistic's critical values.

Each path which we simulate either will or will not re-enter $\mathcal{R}$. Provided prior test is satisfied we have the correct expectation of the future value from any $(J_i, J_{-i}) = s \in \mathcal{R}$.

Let; r index simulation samples,
γ_r index the periods simulated for sample r,
γ_r^* be the first period when $s_{\gamma_r} \in \mathcal{R}$ (or some sufficiently large number if it does not enter),
$\{s_{\gamma_r}\}_{\gamma=1}^{\gamma_r^*}$ be the sequence of states simulated for sample path r.
Then an unbiased estimate of the actual value of the feasible play is

$$\widehat{W}_r(m|J_i) \equiv$$

$$\sum_{\gamma_r=1}^{\gamma_r^*-1} \delta^{\gamma_r} \pi(m(J_{i,\gamma_r}), m^*(J_{-i,\gamma_r})) + \delta^{\gamma_r^*} W(m^*|J_{i,\gamma_r^*}).$$

If there are R simulated paths, let $\overline{W}^R(m|J_i)$ be their average, and $Var[\overline{W}^R(m|J_i)]$ be the standard estimate of the variance of this average.

Let $B(J_i) = \{m : (m, J_i) \in B\}$ and $\#B(J_i)$ be the number of elements in $B(J_i)$. So

$$T(J_i) = \frac{1}{\#B(J_i)} \sum_{m \in B(J_i)} (\frac{[\overline{W}^R(m|J_i) - W(m^*|J_i)]_+}{W(m^*(J_i))}),$$

is a measure of the deviation of the boundary point from boundary consistency.

Let $\mathcal{J}_B = \{J_i : (b, J_i) \in B$ for at least one $b\}$, $h(J_i)$ be the number of times times the point J_i was visited in the test run, and

$$q(J_i) = \frac{\{J_i \in B\}h(J_i)}{\sum_{J_i \in B} h(J_i)}.$$

Then our test statistic is

$$T(B) = \sum_{J_i \in \mathcal{J}_B} q(J_i)T(J_i).$$

We have to simulate its distribution under the null that $W(m|J_i) = W(m^*|J_i)$ for each $(m, J_i) \in B$ (this insures the size of the test), and check whether the 95^th percentile of the simulated distribution is larger then $T(B)$. We accept

$$H_0 : \text{Boundary Consistency}$$

if and only if it is not.

The Impacts of Information Sharing in Dynamic Procurement Auctions. J. Asker, C. Fershtman, J. Jeon, & A. Pakes.

• Dynamic auctions are sequential auctions in which the state of the bidders, and therefore their evaluation of the good that is auctioned, changes endogenously depending on the history of auction.

• The value of winning an auction to produce aircraft or ships depends on whether the backlog or the order book of the firm, and the value of winning a highway repair project or a timber auction depends on the extent to which the outcomes of past auctions outcomes require the inputs currently under the control of the firm.

• We ask a traditional question, but in a dynamic environment: how does information sharing affect auction outcomes?

Institutional Treatment.

- Though explicit agreements to fix prices are *per se* violations of the U.S. antitrust laws, the legal treatment of information sharing among competitors is less clear.

- U.S. courts apply the rule of reason to decide whether the exchange of information constitutes a restraint of trade. The worry is information sharing that facilitates a cartel-like pricing. The courts recognize that efficiencies are possible & more likely from the sharing of cost info.

- The E.U., by contrast, has tended to take a harsher view of information sharing agreements. The exchange of information relating to future prices is considered a restriction of competition by object (equivalent to a *per se* offense in the U.S.). This may include non-price strategic information.

Structure of game.

- There is an auction for the right to harvest timber on a parcel of land in each period.

- Firms enter the period with a stock of lumber $\omega_{i,t}$. They harvest, process, and sell at a fixed price of one on the world market in each period. The harvest/processing outcome is stochastic.

- Firms decide need to pay a fee (F) to submit a bid. Those who do submit bids simultaneously, $b \in \{b_1, \ldots \bar{b}\} = \mathcal{B} \subset \mathcal{Z}_+$.

- If there is information exchange it occurs between the time the bids are submitted, and the outcome of the auction is announced. Shared information is truthful.

- The winner discovers the amount of timber on the plot $[(\theta + \eta);\ \eta \sim F_\eta(\cdot)]$, and each firm gets a random draw on harvest/processing $[(e + \epsilon);\ \epsilon \sim F_\epsilon(\cdot)]$.

- If $\{g_p(J_i)\} \in \{1, 0\}$ & $\{g_p(J_i)\} = 1$ if the firm participates, and $\{i_w(J_i, J_{-i})\} \in \{1, 0\}$ & $\{i_w(J_i, J_{-i})\} = 1$ when the firm wins, then

$$\pi(J_i, J_{-i}, \epsilon_i, \eta_i) =$$

$$min\{\omega_i + \{i_w(\cdot)\}(\theta + \eta), e + \epsilon_i\} - \{i_w(\cdot)\}b_i - \{g_p(\cdot)\}F.$$

Information Sets.

- Basic question: what are the implications of different information structures in dynamic auctions and do those implications depend on the extent to which we discount the future.

- Compare institutions which generate

  - revelation in each period,

  - revelation every $T > 1$ periods, and

  - every T periods firms chose whether to reveal in each of the next T periods. They both have to want to reveal before any of them reveals. The decision is made just after bidding.

• $J_{i,t} = (\xi_t, \omega_{i,t})$. Let τ_t be the time since last iteration, $i_w(t)$ provide the identity of the winning bidder $b_w(t)$ its bid, $g_p(t)$ be the participation decisions, and $\omega_t = (\omega_{i,t}, \omega_{-i,t})$. Each agent knows its $\omega_{i,t}$, but not $\omega_{-i,t}$.

• If there is no information revelation the public information evolves as

$$\xi_t = \{g_p(t), i_w(t), b_w(t), \tau_t\} \cup \xi_{t-1},$$

• if there is information revelation

$$\xi_t = \{\omega_{t-1}, i_w(t), b_w(t), \tau_t = 1\}.$$

• Note: this is not a capital accumulation game. I.e. one agent's choice of control will affect the evolution of the other firm's state. This complicates both the computation and the economics (e.g. an agent can refrain from bidding today in order to let its competitor build up its stock so that the competitor will be less aggressive in the future).

Value function

$$V(J_i) = max\{W(0|J_i), \max_{b \in \mathcal{B}} W(b|J_i)\}$$

Let

$$\pi^E(J_i) = \sum_{J_{-i},\eta,\epsilon_i} \pi(J_i, J_{-i}, \epsilon_i, \eta) p(J_{-i}|J_i) p(\epsilon_i) p(\eta).$$

Then if $b_i > 0$ the firms participate and

$$W(b \neq 0|J_i) = \pi^E(J_i) +$$

$$\beta p^w(b|J_i) \sum_{\epsilon_i,\eta,\xi'} V(\omega'(\omega_i, \eta, \epsilon_i), \xi') p(\xi'|J_i, b, i = i_w) p(\eta, \epsilon) +$$

$$\beta(1 - p^w(b|J_i)) \sum_{\epsilon_i,\xi'} V(\omega'(\omega_i, \epsilon_i), \xi') p(\xi'|J_i, b, i \neq i_w) p(\epsilon).$$

If $b_i = 0$ (the firm does not participate)

$$W(0|J_i) = \pi^E(J_i) + \beta \sum_{\epsilon_i,\xi'} V(\omega'(\omega_i, \epsilon_i), \xi') p(\xi'|J_i) p(\epsilon).$$

Parameter Values

		B	IE	VIE
Parameters:				
Periods between ω revelation	T	4	1	{1,4}
Common Parameters:				
Distribution of fixed cost of participation	F_i	U[0,1]		
Discount factor	β	0.9		
Mean timber in a lot	θ	3.5		
Disturbance around θ	η	{-0.5,0.5}		
Probability on η realizations		{0.5,0.5}		
Mean harvest capacity	e	2		
Disturbance around e	ϵ	{-1,0,1}		
Probability on ϵ realizations		{0.33,0.33,0.33}		
Bidding grid		{0.5,1,1.5,2}		
Number of firms/bidders		2		
Retail price of a unit of timber		1		

Computational Details.

Size of recurrent class:

B	IE	VIE
325,843	2,081	328,692

Number of all states visited during computation:

B	IE	VIE
7,495,307	2,724	7,908,122

Computation times per 5 million iterations (in hours):

B	IE	VIE
1:38	1:06	1:56

Computation times for testing for a REBE (5 million iterations, in hours):

B	IE	VIE
1:43	1:09	2:00

Computation times for testing for boundary consistency (100,000 iterations

B	IE	VIE
3:03	0:16	75:41

Notes: Computation was conducted in MATLAB version R2013a using (a Dell Precision T3610 desktop with) a 3.7 GHz Intel Xeon processor and 16GB RAM on Windows 7 Professional.

Six rounds of computation were required for B to pass the REBE test, eight for VIE and one for IE. We estimated models with several other parameter values. All that past the REBE test but one were boundary consistent, but we started with very high initial conditions.

Summary Statistiics

	B	IE	VIE	SP
Avg. bid	1.09	0.94	1.04	-
Avg. b_w (revenue for the auctioneer)	1.11	0.98	1.07	-
Avg. b_w when ≥ 1 firm	1.16	0.98	1.12	-
Avg. b_w with 1 firm	1.06	0.67	0.99	-
Avg. b_w with 2 firms	1.23	1.16	1.20	-
Avg. # of participants	1.52	1.63	1.52	1
Avg. # of participants, with ≥ 1 firm	1.59	1.63	1.59	1
Avg. participation rate	0.76	0.81	0.76	0.50
% of periods with no participation	4.39	0.15	3.85	0.004
Avg. total revenue	3.35	3.49	3.37	3.50
Avg. profit	0.81	0.87	0.84	-
% of periods; lowest omega wins	66.37	60.80	65.32	85.96
Average total social surplus	2.73	2.72	2.74	3.10

Procurement Revenue = winning bid.
$\pi_i(\cdot) = \min\{\omega_i + \{i_w\}(\theta + \eta), e + \epsilon_i\} - \{i_w\}b_i - \{g_p(\cdot)\}F$.
Revenue $\min\{\omega_i + \{i_w\}(\theta + \eta), e + \epsilon_i\}$.
Social surplus $\sum_i [\pi_i(\cdot) - \{g_p(\cdot)\}F]$.

B vs IE vs VIE.

- IE has lower (less aggressive) bids but more participation. In a static auction we expect less aggressive bidding to occur with less participation.
- The average profit and hence the average value is highest in IE.
- Despite this, VIE is very close to B, indicating that when there is a choice as to whether to exchange information most of the time they do not exchange.
- Social Surplus in $B \approx IE$.

States and Profits.

(ω_i, ω_{-i})	Prob. Dist. (%) B	IE	SP	Profit B	IE
$(\leq 4, \leq 4)$	65.51	32.59	90.12	0.68	0.52
$(\leq 4, 5-7)$	12.61	19.09	4.52	0.57	0.58
$(\leq 4, \geq 8)$	4.05	10.55	0.28	0.60	0.59
$(5-7, \leq 4)$	12.61	19.09	4.52	1.51	1.26
$(5-7, 5-7)$	0.88	5.72	0.22	1.49	1.46
$(5-7, \geq 8)$	0.14	1.12	0.02	1.49	1.13
$(\geq 8, \leq 4)$	4.05	10.55	0.28	1.62	1.58
$(\geq 8, 5-7)$	0.14	1.12	0.02	1.66	1.87
$(\geq 8, \geq 8)$	0.01	0.17	0.00	1.72	1.56

Notes: This table shows the probability of intervals of ω-tuples for B, IE and SP. Here the per-period profit is a probability weighted average, over the states underlying each ω-tuple.

- **B has higher profits** in just about every state, yet from prior table, **IE has higher value**.
- Reason: IE spends much less time in states where stocks are low. In those states bidding is more aggressive and profits lower in both the B and IE equilibria.
- The control here is bids & they effect; i) the evolution of stocks, and ii) current profits. In terms of value, the dominant effect of added information is not on the aggressiveness of bids, but on the evolution of the state.

Differences in Policies.

(ω_i, ω_{-i})	Bids							
	B				IE			
	0	0.5	1	1.5/2	0	0.5	1	1.5/ 2
$(\leq 4, \leq 4)$	0.22	0.13	0.27	0.38	0.07	0.13	**0.28**	**0.53**
$(\leq 4, 5-7)$	0.11	0.32	0.45	0.13	0.02	**0.53**	0.37	0.08
$(\leq 4, \geq 8)$	0.08	0.58	0.29	0.06	0.00	**0.88**	0.12	0.00
$(5-7, \leq 4)$	0.43	0.18	0.34	0.05	0.33	0.10	**0.52**	**0.05**
$(5-7, 5-7)$	0.37	0.50	0.09	0.03	**0.40**	**0.59**	0.01	0.00
$(5-7, \geq 8)$	0.39	0.53	0.06	0.02	0.11	**0.89**	0.00	0.00
$(\geq 8, \leq 4)$	0.51	0.25	0.22	0.02	**0.60**	0.14	**0.26**	0.00
$(\geq 8, 5-7)$	0.53	0.39	0.06	0.01	**0.84**	0.16	0.00	0.00
$(\geq 8, \geq 8)$	0.61	0.36	0.03	0.00	0.47	**0.53**	0.00	0.00

- Items in boldface: probabilities in IE greater than in B. On average "7" is two years supply.

- IE bids more intensely when both have low states, or one is in a middle states and its competitor is low. If, in IE, one is high and the other mid, the high cedes the market to the mid. In IE if both are high they toss a coin.

- The extra info in IE generates more intensive bidding at low, but less intensive bidding at high states.

- This generates dynamic incentives that lead IE to spend more time at higher states. At high states the bidder either $b \in \{0, .5\}$ which generates high profits.

Static Incentives, $\beta = 0$

	$\beta = 0.9$		$\beta = 0$	
	B	IE	B	IE
Avg. bid	1.09	0.94	0.61	0.59
Avg. b_w (revenue for the auctioneer)	1.11	0.98	0.54	0.53
Avg. b_w with $\geq$ 1 firm participating	1.16	0.98	0.62	0.60
Avg. b_w with 1 firm participating	1.06	0.67	0.55	0.53
Avg. b_w 2 firms participating	1.23	1.16	0.82	0.82
Avg. # of participants	1.52	1.63	1.10	1.10
Avg. # of participants with $\geq$ 1 firm	1.59	1.63	1.25	1.25
Avg. participation rate	0.76	0.81	0.55	0.55
% of periods with no participation	4.39	0.15	11.98	11.65
Avg. total revenue	3.35	3.49	3.08	3.09
Avg. profit	0.81	0.87	1.03	1.04
% of periods; lowest ω wins conditional on $\geq$ 1 firm participating	66.37	60.80	96.24	96.15
Average total social surplus	2.73	2.72	2.60	2.61

Notes

- History still matters here, as the information gathered from it is still a signal on competitor's ω.

- Very little difference between B and IE when firms do not care about the future.

- The advantage of extra information on a competitors' likely bid is that it incentivizes firms to spend more time at high states where bidding is less aggressive. However get to these high states there is a need to bid more aggressively at lower states, and without a future there is no incentive to do that.

- Also the revenue of the auctioneer goes down and profits from the auction increase.

Voluntary Information Exchange.

(ω_i, ω_{-i})	(%) VIE	$\Pr(\cup_i \chi_i \geq 1)$ VIE	$\Pr(\Pi_i \chi_i = 1)$	Profit B	IE
$(\leq 4, \leq 4)$	62.98	24.75	4.76	0.68	0.52
$(\leq 4, 5-7)$	13.17	24.57	4.47	0.57	0.58
$(\leq 4, \geq 8)$	4.58	28.06	6.09	0.60	0.59
$(5-7, \leq 4)$	13.17	21.38	4.47	1.51	1.26
$(5-7, 5-7)$	1.13	18.94	4.59	1.49	1.46
$(5-7, \geq 8)$	0.19	24.38	9.73	1.49	1.13
$(\geq 8, \leq 4)$	4.58	23.39	6.09	1.62	1.58
$(\geq 8, 5-7)$	0.19	24.60	9.73	1.66	1.87
$(\geq 8, \geq 8)$	0.02	38.14	20.34	1.72	1.56

$\chi_i \in \{0,1\}$, $\chi_i = 1$ indicates that firm i chose to reveal, so $\cup_i \chi_i \geq 1$ indicates that at least one firm chose to reveal and $\Pi_i \chi_i = 1$ indicates both firms chose to reveal. Only periods in which firms decide on information sharing (or periods with $\tau = 0$) are used in the calculation.

Notes

- Firms only chose to share info in 5% of the possible states (though one of the two shares 24% of time).
- Recall that value in IE is higher, so why only 5%?
- The propensity to share info. is only large at high ω states. If we are ever in B we will predominantly be in low ω states.
- In low states; $\pi_B > \pi_{IE} \Rightarrow$ to progress to IE we would have to give up current profits. If we could commit to IE in the future we might do this, but the lack of commitment rules that out.

Conclusions

- Dynamics are a part of many familiar auction markets. Then the distribution of states that determine the aggressiveness of bidding is endogenous. As a result dynamic auctions can generate very different incentives than static or repeated auctions.

- We illustrate this by comparing the role of added information in a model where there is no (explicit or implicit) collusion.

- Because sharing information has a different effect on different states, it leads to a different distribution of states, and must be analyzed with that effect in mind. It clearly is not necessarily harmful.