

Electroweak oblique parameters

Ref: Peskin & Takeuchi, PRD **46**, 381 (1992)

Vacuum polarization amplitude of gauge bosons I and J :

$$i\Pi_{IJ}^{\mu\nu}(p) = ig^{\mu\nu}\Pi_{IJ}(p^2) + (p^\mu p^\nu \text{ terms})$$

$$\Pi_{IJ}(p^2) \equiv \Pi_{IJ}(0) + p^2\Pi'_{IJ}(p^2) = \Pi_{IJ}(0) + p^2\Pi'_{IJ}(0) + \mathcal{O}(p^4), \quad \Pi'_{IJ}(0) = \left. \frac{\partial \Pi_{IJ}(p^2)}{\partial p^2} \right|_{p^2=0}$$

$$\Pi_{AA}(p^2) = e^2\Pi_{QQ}(p^2), \quad \Pi_{ZA}(p^2) = \frac{e^2}{s_W c_W} [\Pi_{3Q}(p^2) - s_W^2 \Pi_{QQ}(p^2)]$$

$$\Pi_{ZZ}(p^2) = \frac{e^2}{s_W^2 c_W^2} [\Pi_{33}(p^2) - 2s_W^2 \Pi_{3Q}(p^2) + s_W^4 \Pi_{QQ}(p^2)], \quad \Pi_{WW}(p^2) = \frac{e^2}{s_W^2} \Pi_{11}(p^2)$$

$$\text{QED Ward identity} \Rightarrow \Pi_{3Q}(0) = \Pi_{QQ}(0) = 0, \quad \Pi_{AA}(0) = \Pi_{ZA}(0) = 0$$

$$S \equiv \frac{4e^2}{\alpha} [\Pi'_{33}(0) - \Pi'_{3Q}(0)] = \frac{4s_W^2 c_W^2}{\alpha} \left[\Pi'_{ZZ}(0) - \frac{c_W^2 - s_W^2}{s_W c_W} \Pi'_{ZA}(0) - \Pi'_{AA}(0) \right]$$

$$T \equiv \frac{e^2}{\alpha s_W^2 c_W^2 m_Z^2} [\Pi_{11}(0) - \Pi_{33}(0)] = \frac{1}{\alpha} \left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} \right]$$

$$U \equiv \frac{4e^2}{\alpha} [\Pi'_{11}(0) - \Pi'_{33}(0)] = \frac{4s_W^2}{\alpha} [\Pi'_{WW}(0) - c_W^2 \Pi'_{ZZ}(0) - 2s_W c_W \Pi'_{ZA}(0) - s_W^2 \Pi'_{AA}(0)]$$

$$c_W \equiv \cos \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad s_W \equiv \sin \theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g s_W = g' c_W, \quad m_W = m_Z c_W$$

Fan, Reece & Wang, 1411.1054:

S is contributed by $SU(2)_L$ multiplets that are split by the electroweak symmetry breaking

T and U are contributed by $SU(2)_L$ multiplets that are split by the electroweak symmetry

breaking with custodial symmetry [firstly called in Sikivie et al., NPB **173**, 189 (1980)] violating effects

EFT viewpoint [Zhenyu Han, 0807.0490]

$$\frac{1}{\Lambda^2} H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu} \rightarrow S, \quad \frac{1}{\Lambda^2} H^\dagger D_\mu H (D^\mu H)^\dagger H \rightarrow T, \quad \frac{1}{\Lambda^4} H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu} \sigma^b H \rightarrow U$$

U corresponds to higher dimensional operators and is typically much smaller than S and T

$H^\dagger W_{\mu\nu}^a \sigma^a HB^{\mu\nu}$ operator

$$H \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad W_{\mu\nu}^3 = s_W A_{\mu\nu} + c_W Z_{\mu\nu} - g W_\mu^1 W_\nu^2 + g W_\mu^2 W_\nu^1, \quad B_{\mu\nu} = c_W A_{\mu\nu} - s_W Z_{\mu\nu}$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$H^\dagger W_{\mu\nu}^a \sigma^a H \supset H^\dagger W_{\mu\nu}^3 \sigma^3 H \rightarrow \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} W_{\mu\nu}^3 = -\frac{v^2}{2} W_{\mu\nu}^3$$

$$\begin{aligned} \mathcal{O}_{WB} &= a_{WB} H^\dagger W_{\mu\nu}^a \sigma^a HB^{\mu\nu} \rightarrow -\frac{a_{WB} v^2}{2} W_{\mu\nu}^3 B^{\mu\nu} \supset -\frac{a_{WB} v^2}{2} (s_W A_{\mu\nu} + c_W Z_{\mu\nu}) (c_W A^{\mu\nu} - s_W Z^{\mu\nu}) \\ &= -\frac{a_{WB} v^2}{2} [s_W c_W (A_{\mu\nu} A^{\mu\nu} - Z_{\mu\nu} Z^{\mu\nu}) + (c_W^2 - s_W^2) Z_{\mu\nu} A^{\mu\nu}] \end{aligned}$$

[Note: Feynman rule for momenta pointing into the vertex : $\partial_\mu \rightarrow -ip_\mu$]

Feynman rule for $Z_\mu(p) - \times - Z_\nu(p)$ from $Z_{\mu\nu} Z^{\mu\nu}$

$$\begin{aligned} Z_{\mu\nu} Z^{\mu\nu} &= 2(\partial_\mu Z_\nu \partial^\mu Z^\nu - \partial_\mu Z_\nu \partial^\nu Z^\mu) \\ &\rightarrow 2[g^{\rho\sigma} g^{\mu\nu} \partial_\rho Z_\mu(p) \partial_\sigma Z_\nu(p) + g^{\rho\sigma} g^{\nu\mu} \partial_\rho Z_\nu(p) \partial_\sigma Z_\mu(p) \\ &\quad - g^{\rho\nu} g^{\mu\sigma} \partial_\rho Z_\mu(p) \partial_\sigma Z_\nu(p) - g^{\rho\mu} g^{\nu\sigma} \partial_\rho Z_\nu(p) \partial_\sigma Z_\mu(p)] \\ &\rightarrow 2i[(g^{\rho\sigma} g^{\mu\nu} - g^{\rho\nu} g^{\mu\sigma})(-ip_\rho)(ip_\sigma) + (g^{\rho\sigma} g^{\nu\mu} - g^{\rho\mu} g^{\nu\sigma})(ip_\rho)(-ip_\sigma)] = 4i(g^{\mu\nu} p^2 - p^\mu p^\nu) \end{aligned}$$

Contribution to $\Pi_{ZZ}^{\mu\nu}$ from \mathcal{O}_{WB} :

$$i\Pi_{ZZ}^{\mu\nu}(p^2) = \frac{s_W c_W a_{WB} v^2}{2} 4i(g^{\mu\nu} p^2 - p^\mu p^\nu) = 2is_W c_W a_{WB} v^2 (g^{\mu\nu} p^2 - p^\mu p^\nu)$$

$$\Pi_{ZZ}(p^2) = 2s_W c_W a_{WB} v^2 p^2 \Rightarrow \Pi'_{ZZ}(0) = 2s_W c_W a_{WB} v^2$$

Feynman rule for $A_\mu(p) - \times - A_\nu(p)$ from $A_{\mu\nu} A^{\mu\nu}$: $A_{\mu\nu} A^{\mu\nu} \rightarrow 4i(g^{\mu\nu} p^2 - p^\mu p^\nu)$

Contribution to $\Pi_{AA}^{\mu\nu}$ from \mathcal{O}_{WB} :

$$i\Pi_{AA}^{\mu\nu}(p^2) = -\frac{s_W c_W a_{WB} v^2}{2} 4i(g^{\mu\nu} p^2 - p^\mu p^\nu) = -2is_W c_W a_{WB} v^2 (g^{\mu\nu} p^2 - p^\mu p^\nu)$$

$$\Pi_{AA}(p^2) = -2s_W c_W a_{WB} v^2 p^2 \Rightarrow \Pi'_{AA}(0) = -2s_W c_W a_{WB} v^2$$

Feynman rule for $Z_\mu(p) - \times - A_\nu(p)$ from $Z_{\mu\nu} A^{\mu\nu}$:

$$\begin{aligned} Z_{\mu\nu} A^{\mu\nu} &= 2(\partial_\mu Z_\nu \partial^\mu A^\nu - \partial_\mu Z_\nu \partial^\nu A^\mu) \rightarrow 2[g^{\rho\sigma} g^{\mu\nu} \partial_\rho Z_\mu(p) \partial_\sigma A_\nu(p) - g^{\rho\nu} g^{\mu\sigma} \partial_\rho Z_\mu(p) \partial_\sigma A_\nu(p)] \\ &\rightarrow 2i(g^{\rho\sigma} g^{\mu\nu} - g^{\rho\nu} g^{\mu\sigma})(-ip_\rho)(ip_\sigma) = 2i(g^{\mu\nu} p^2 - p^\mu p^\nu) \end{aligned}$$

Contribution to $\Pi_{ZA}^{\mu\nu}$ from \mathcal{O}_{WB} :

$$i\Pi_{ZA}^{\mu\nu}(p^2) = -\frac{a_{WB} v^2}{2} (c_W^2 - s_W^2) 2i(g^{\mu\nu} p^2 - p^\mu p^\nu) = -i(c_W^2 - s_W^2) a_{WB} v^2 (g^{\mu\nu} p^2 - p^\mu p^\nu)$$

$$\Pi_{ZA}(p^2) = -(c_W^2 - s_W^2) a_{WB} v^2 p^2 \Rightarrow \Pi'_{ZA}(0) = -(c_W^2 - s_W^2) a_{WB} v^2$$

$$S = \frac{4s_W^2 c_W^2}{\alpha} \left[\Pi'_{ZZ}(0) - \frac{c_W^2 - s_W^2}{s_W c_W} \Pi'_{ZA}(0) - \Pi'_{AA}(0) \right] = \frac{4s_W^2 c_W^2}{\alpha} a_{WB} v^2 \left[2s_W c_W + \frac{(c_W^2 - s_W^2)^2}{s_W c_W} + 2s_W c_W \right] = \frac{4s_W c_W}{\alpha} a_{WB} v^2$$

$$T = \frac{1}{\alpha} \left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} \right] = -\frac{\Pi_{ZZ}(0)}{\alpha m_Z^2} = 0$$

$$U = \frac{4s_W^2}{\alpha} \left[\Pi'_{WW}(0) - c_W^2 \Pi'_{ZZ}(0) - 2s_W c_W \Pi'_{ZA}(0) - s_W^2 \Pi'_{AA}(0) \right] = \frac{4s_W^2}{\alpha} a_{WB} v^2 \left[-2s_W c_W^3 + 2s_W c_W (c_W^2 - s_W^2) + 2s_W^3 c_W \right] = 0$$

$$\underline{H^\dagger D_\mu H (D^\mu H)^\dagger H \text{ operator}}$$

$$D_\mu H = (\partial_\mu - ig' B_\mu Y_H - ig W_\mu^a t_D^a) H, \quad Y_H = \frac{1}{2}, \quad t_D^a = \frac{\sigma^a}{2}$$

$$D_\mu H \supset -\frac{i}{2} \begin{pmatrix} g' B_\mu + g W_\mu^3 & \sqrt{2} g W_\mu^+ \\ \sqrt{2} g W_\mu^- & g' B_\mu - g W_\mu^3 \end{pmatrix} H \rightarrow -\frac{iv}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} g W_\mu^+ \\ g' B_\mu - g W_\mu^3 \end{pmatrix}, \quad H^\dagger D_\mu H \rightarrow -\frac{iv}{4} (g' B_\mu - g W_\mu^3)$$

$$g' B_\mu - g W_\mu^3 = -\sqrt{g^2 + g'^2} Z_\mu, \quad m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}$$

$$\mathcal{O}_h = a_h H^\dagger D_\mu H (D^\mu H)^\dagger H \rightarrow \frac{a_h v^2}{16} (g' B_\mu - g W_\mu^3)^2 = \frac{a_h v^2}{16} (g^2 + g'^2) Z_\mu Z^\mu = \frac{a_h v^2}{4} m_Z^2 Z_\mu Z^\mu$$

$$\text{Contribution to } \Pi_{ZZ}^{\mu\nu} \text{ from } \mathcal{O}_{WB}: \quad i\Pi_{ZZ}^{\mu\nu}(p^2) = i \frac{a_h v^2}{2} m_Z^2 g^{\mu\nu}$$

$$\Pi_{ZZ}(p^2) = \frac{a_h v^2}{2} m_Z^2$$

$$S = \frac{4s_W^2 c_W^2}{\alpha} \Pi'_{ZZ}(0) = 0, \quad T = -\frac{1}{\alpha} \frac{1}{m_Z^2} \frac{a_h v^2}{2} m_Z^2 = -\frac{1}{2\alpha} a_h v^2, \quad U = -\frac{4s_W^2}{\alpha} c_W^2 \Pi'_{ZZ}(0) = 0$$

$$\underline{H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu} \sigma^b H \text{ operator}}$$

$$H^\dagger W_{\mu\nu}^a \sigma^a H \rightarrow -\frac{v^2}{2} W_{\mu\nu}^3$$

$$\mathcal{O}_{WW} = a_{WW} H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu} \sigma^b H \rightarrow \frac{a_{WW} v^4}{4} W_{\mu\nu}^3 W^{3,\mu\nu}$$

$$= \frac{a_{WW} v^4}{4} (s_W A_{\mu\nu} + c_W Z_{\mu\nu}) (s_W A^{\mu\nu} + c_W Z^{\mu\nu}) = \frac{a_{WW} v^4}{4} (s_W^2 A_{\mu\nu} A^{\mu\nu} + 2s_W c_W Z_{\mu\nu} A^{\mu\nu} + c_W^2 Z_{\mu\nu} Z^{\mu\nu})$$

$$\Pi_{ZZ}(p^2) = c_W^2 a_{WW} v^4 p^2, \quad \Pi_{ZA}(p^2) = s_W c_W a_{WW} v^4 p^2, \quad \Pi_{AA}(p^2) = s_W^2 a_{WW} v^4 p^2$$

$$\Pi'_{ZZ}(0) = c_W^2 a_{WW} v^4, \quad \Pi'_{ZA}(0) = s_W c_W a_{WW} v^4, \quad \Pi'_{AA}(0) = s_W^2 a_{WW} v^4$$

$$S = \frac{4s_W^2 c_W^2}{\alpha} a_{WW} v^4 \left[c_W^2 - \frac{c_W^2 - s_W^2}{s_W c_W} s_W c_W - s_W^2 \right] = 0, \quad T = -\frac{\Pi_{ZZ}(0)}{\alpha m_Z^2} = 0$$

$$U = \frac{4s_W^2}{\alpha} a_{WW} v^4 (0 - c_W^4 - 2s_W^2 c_W^2 - s_W^4) = -\frac{4s_W^2}{\alpha} a_{WW} v^4$$

Normal and χ^2 distributions

$$n\text{-dim normal distribution } f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\mathbf{V}|^{1/2}} e^{-Q/2}$$

$$\text{Quadratic form } Q = (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{x} - \boldsymbol{\mu}) = (x_i - \mu_i) V_{ij}^{-1} (x_j - \mu_j)$$

\mathbf{V} is the positive definite symmetric covariance matrix

$$V_{ij} = \text{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \rho_{1n}\sigma_1\sigma_n & \cdots & & \sigma_n^2 \end{pmatrix}$$

$$\text{Correlation coefficients } \rho_{ij} = \frac{\text{cov}(X_i, X_j)}{\sqrt{V(X_i)V(X_j)}}$$

$$\text{Probability inside the } n\text{-dim ellipsoid } Q = Q_0: P(Q < Q_0) = F_{\chi^2(n)}(Q_0)$$

$F_{\chi^2(n)}(x)$ is the cumulative χ^2 distribution function with n d.o.f.

$$F_{\chi^2(1)}(1) = 68.3\%, \quad F_{\chi^2(1)}(4) = 95.4\%, \quad F_{\chi^2(1)}(9) = 99.7\%$$

$$F_{\chi^2(2)}(2.295749) = 68.3\%, \quad F_{\chi^2(2)}(6.180074) = 95.4\%, \quad F_{\chi^2(2)}(11.82916) = 99.7\%$$

$$F_{\chi^2(3)}(3.526741) = 68.3\%, \quad F_{\chi^2(3)}(8.024882) = 95.4\%, \quad F_{\chi^2(3)}(14.15641) = 99.7\%$$

2-dim normal distribution

$$V_{ij} = \text{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad V_{ij}^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-Q/2}$$

$$Q = \frac{1}{1-\rho^2} \left[\left(\frac{x_1 - \mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2} \right)^2 - 2\rho \frac{x_1 - \mu_1}{\sigma_1} \frac{x_2 - \mu_2}{\sigma_2} \right]$$

Ellipse parametric equation with parameter t :

$$\begin{cases} x_1 = \mu_1 + a \cos \phi \cos t - b \sin \phi \sin t \\ x_2 = \mu_2 + a \sin \phi \cos t + b \cos \phi \sin t \end{cases}$$

$$\phi = \frac{1}{2} \tan^{-1} \left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} \right), \quad a = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2 \cos^2 \phi - 2\rho\sigma_1\sigma_2 \sin \phi \cos \phi + \sigma_1^2 \sin^2 \phi}}$$

$$b = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2 \sin^2 \phi + 2\rho\sigma_1\sigma_2 \sin \phi \cos \phi + \sigma_1^2 \cos^2 \phi}}$$

Precision directly extracted from previous works

Gfitter, 1407.3792

Current constraints on S and T with $U = 0$ fixed [below Eq. (4)]:

$$S = 0.06 \pm 0.09, \quad T = 0.10 \pm 0.07, \quad \rho_{ST} = +0.91$$

Current precision on S and T [Fig. 7]:

$$\sigma_S = 0.093, \quad \sigma_T = 0.082, \quad \rho_{ST} = +0.93$$

Fan, Reece & Wang, 1411.1054

Current precision [Fig. 1]:

$$\sigma_S = 0.086, \quad \sigma_T = 0.074, \quad \rho_{ST} = +0.91$$

CEPC optimistic baseline precision [Fig. 3]:

$$\sigma_S = 0.024, \quad \sigma_T = 0.019, \quad \rho_{ST} = +0.84$$

CEPC precision with improvements of m_t , Γ_Z and $\sin^2 \theta_{\text{eff}}^\ell$ [Fig. 3]:

$$\sigma_S = 0.011, \quad \sigma_T = 0.0072, \quad \rho_{ST} = +0.78$$

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CEPC baseline precision [Fig. 4.2]:

$$\sigma_S = 0.016, \quad \sigma_T = 0.015, \quad \rho_{ST} = +0.82$$

CEPC precision with improvements of m_t , m_Z and Γ_Z [Fig. 4.2]:

$$\sigma_S = 0.011, \quad \sigma_T = 0.0073, \quad \rho_{ST} = +0.78$$

Tree level:

$$m_W = 80.385 \text{ GeV}, \quad m_Z = 91.1876 \text{ GeV}, \quad c_W^2 = \frac{m_W^2}{m_Z^2} = 0.77737, \quad s_W^2 = 1 - c_W^2 = 0.22263$$

Loop level:

$$m_W = m_W^{\text{SM}} + \Delta m_W, \quad \Gamma_W = \Gamma_W^{\text{SM}} + \Delta \Gamma_W, \quad \sin^2 \theta_{\text{eff}}^\ell = (\sin^2 \theta_{\text{eff}}^\ell)^{\text{SM}} + \Delta \sin^2 \theta_{\text{eff}}^\ell, \quad \Gamma_Z = \Gamma_Z^{\text{SM}} + \Delta \Gamma_Z$$

$$F_1(S, T, U) \equiv S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U = S - 1.55T - 1.24U$$

$$F_2(S, T) \equiv S - 4s_W^2 c_W^2 T = S - 0.693T$$

$$F_3(S, T) \equiv -10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T = -12.2S + 32.9T = -12.2(S - 2.71T)$$

$$\Delta m_W = -\frac{\alpha m_W^{\text{SM}}}{4(c_W^2 - s_W^2)} \left(S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U \right) = -\frac{\alpha m_W^{\text{SM}}}{4(c_W^2 - s_W^2)} F_1(S, T, U)$$

$$\Delta \Gamma_W = -\frac{3\alpha \Gamma_W^{\text{SM}}}{4(c_W^2 - s_W^2)} \left(S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U \right) = -\frac{3\alpha \Gamma_W^{\text{SM}}}{4(c_W^2 - s_W^2)} F_1(S, T, U)$$

$$\mathcal{L} \supset \sum_f \frac{g}{2c_W} \bar{f} \gamma^\mu (g_V^f - g_A^f \gamma_5) f Z_\mu, \quad g_V^f \equiv T_f^3 - 2Q_f s_W^2, \quad g_A^f \equiv T_f^3$$

$$\left[\begin{array}{l} Q_{\ell_i} = -1, \quad Q_{\nu_i} = 0, \quad Q_{d_i} = -\frac{1}{3}, \quad Q_{u_i} = \frac{2}{3} \\ g_V^{\ell_i} = -\frac{1}{2} + 2s_W^2, \quad g_A^{\ell_i} = -\frac{1}{2}; \quad g_V^{\nu_i} = \frac{1}{2}, \quad g_A^{\nu_i} = \frac{1}{2} \\ g_V^{d_i} = -\frac{1}{2} + \frac{2}{3}s_W^2, \quad g_A^{d_i} = -\frac{1}{2}; \quad g_V^{u_i} = \frac{1}{2} - \frac{4}{3}s_W^2, \quad g_A^{u_i} = \frac{1}{2} \end{array} \right]$$

$$\delta g_V^f = \frac{g_V^f \alpha}{2} T + \frac{(g_V^f - g_A^f) \alpha}{4s_W^2 (c_W^2 - s_W^2)} (S - 4s_W^2 c_W^2 T), \quad \delta g_A^f = \frac{g_A^f \alpha}{2} T$$

$$\begin{aligned} \Delta \sin^2 \theta_{\text{eff}}^\ell &= -\frac{g_A^e \delta g_V^e - g_V^e \delta g_A^e}{4(g_A^e)^2} = -\frac{1}{4(g_A^e)^2} \left[g_A^e \left(\frac{g_V^e \alpha}{2} T + \frac{(g_V^e - g_A^e) \alpha}{4s_W^2 (c_W^2 - s_W^2)} (S - 4s_W^2 c_W^2 T) \right) - g_V^e \frac{g_A^e \alpha}{2} T \right] \\ &= -\frac{(g_V^e - g_A^e) \alpha}{16g_A^e s_W^2 (c_W^2 - s_W^2)} (S - 4s_W^2 c_W^2 T) = \frac{\alpha}{4(c_W^2 - s_W^2)} F_2(S, T) \end{aligned}$$

$$G_f \equiv (g_V^f)^2 + (g_A^f)^2$$

$$\delta G_f \equiv 2(g_V^f \delta g_V^f + g_A^f \delta g_A^f) = \frac{g_V^f (g_V^f - g_A^f) \alpha}{2s_W^2 (c_W^2 - s_W^2)} (S - 4s_W^2 c_W^2 T) + [(g_V^f)^2 + (g_A^f)^2] \alpha T$$

$$\left[\begin{array}{l} g_V^{\ell_i} (g_V^{\ell_i} - g_A^{\ell_i}) = -s_W^2 + 4s_W^4, \quad (g_V^{\ell_i})^2 + (g_A^{\ell_i})^2 = \frac{1}{2} - 2s_W^2 + 4s_W^4 \\ g_V^{\nu_i} (g_V^{\nu_i} - g_A^{\nu_i}) = 0, \quad (g_V^{\nu_i})^2 + (g_A^{\nu_i})^2 = \frac{1}{2} \\ g_V^{d_i} (g_V^{d_i} - g_A^{d_i}) = -\frac{1}{3}s_W^2 + \frac{4}{9}s_W^4, \quad (g_V^{d_i})^2 + (g_A^{d_i})^2 = \frac{1}{2} - \frac{2}{3}s_W^2 + \frac{4}{9}s_W^4 \\ g_V^{u_i} (g_V^{u_i} - g_A^{u_i}) = -\frac{2}{3}s_W^2 + \frac{16}{9}s_W^4, \quad (g_V^{u_i})^2 + (g_A^{u_i})^2 = \frac{1}{2} - \frac{4}{3}s_W^2 + \frac{16}{9}s_W^4 \end{array} \right]$$

$$f = e, \nu_e, \mu, \nu_\mu, \tau, \mu_\tau, d, u, s, c, b$$

$$\sum_f N_c^f g_V^f (g_V^f - g_A^f) = 3(-s_W^2 + 4s_W^4) + 3 \cdot 3 \left(-\frac{1}{3}s_W^2 + \frac{4}{9}s_W^4 \right) + 2 \cdot 3 \left(-\frac{2}{3}s_W^2 + \frac{16}{9}s_W^4 \right) = -\frac{10}{3}s_W^2 (3 - 8s_W^2)$$

$$\sum_f N_c^f [(g_V^f)^2 + (g_A^f)^2] = \left[3 \left(\frac{1}{2} + \frac{1}{2} - 2s_W^2 + 4s_W^4 \right) + 3 \cdot 3 \left(\frac{1}{2} - \frac{2}{3}s_W^2 + \frac{4}{9}s_W^4 \right) + 2 \cdot 3 \left(\frac{1}{2} - \frac{4}{3}s_W^2 + \frac{16}{9}s_W^4 \right) \right] = \frac{21}{2} - 20s_W^2 + \frac{80}{3}s_W^4$$

$$\begin{aligned} \sum_f N_c^f \delta G_f &= \frac{\alpha}{2s_W^2 (c_W^2 - s_W^2)} (S - 4c_W^2 s_W^2 T) \sum_f N_c^f g_V^f (g_V^f - g_A^f) + \alpha T \sum_f N_c^f [(g_V^f)^2 + (g_A^f)^2] \\ &= -\frac{5\alpha}{3(c_W^2 - s_W^2)} (S - 4c_W^2 s_W^2 T) (3 - 8s_W^2) + \alpha T \left(\frac{21}{2} - 20s_W^2 + \frac{80}{3}s_W^4 \right) \\ &= -\frac{5(3 - 8s_W^2)}{3(c_W^2 - s_W^2)} \alpha S + \frac{20\alpha T}{3(c_W^2 - s_W^2)} \left[c_W^2 s_W^2 (3 - 8s_W^2) + \frac{3(c_W^2 - s_W^2)}{20} \left(\frac{21}{2} - 20s_W^2 + \frac{80}{3}s_W^4 \right) \right] \\ &= \frac{\alpha}{6(c_W^2 - s_W^2)} [-10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T] \end{aligned}$$

$$\Delta \Gamma_Z = \frac{\alpha m_Z}{12s_W^2 c_W^2} \sum_f N_c^f \delta G_f = \frac{\alpha^2 m_Z}{72s_W^2 c_W^2 (c_W^2 - s_W^2)} [-10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T] = \frac{\alpha^2 m_Z}{72s_W^2 c_W^2 (c_W^2 - s_W^2)} F_3(S, T)$$

Parametrization of the SM correction to m_W

Ref: Awramik et al., hep-ph/0311148

$$\text{Loop level: } m_W^2 \left(1 - \frac{m_W^2}{m_Z^2} \right) = \frac{\pi\alpha}{\sqrt{2}G_F} (1 + \Delta r) \Rightarrow m_W^2 = m_Z^2 \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2}G_F m_Z^2} (1 + \Delta r)} \right]$$

Parametrization formula:

$$m_W^{\text{SM}} = m_W^0 - c_1 dH - c_2 dH^2 + c_3 dH^4 + c_4 (dh - 1) - c_5 d\alpha + c_6 dt - c_7 dt^2 - c_8 dHdt + c_9 dhdt - c_{10} d\alpha_s + c_{11} dZ$$

$$dH = \ln \frac{m_h}{100 \text{ GeV}}, \quad dh = \left(\frac{m_h}{100 \text{ GeV}} \right)^2, \quad dt = \left(\frac{m_t}{174.3 \text{ GeV}} \right)^2 - 1$$

$$dZ = \frac{m_Z}{91.1875 \text{ GeV}} - 1, \quad d\alpha = \frac{\Delta\alpha}{0.05907} - 1, \quad d\alpha_s = \frac{\alpha_s(m_Z)}{0.119} - 1$$

$$\Delta\alpha \equiv \Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{had}}^{(5)}, \quad \Delta\alpha_{\text{lep}} = 0.0314977$$

Coefficient values can be found in Eq.(9) for $m_h > 100 \text{ GeV}$

Parametrization of the SM correction to $\sin^2\theta_{\text{eff}}^\ell$

Ref: Awramik et al., hep-ph/0608099

$$\text{Loop level: } \sin^2\theta_{\text{eff}}^\ell = \left(1 - \frac{m_W^2}{m_Z^2} \right) (1 + \Delta\kappa)$$

Parametrization formula:

$$(\sin^2\theta_{\text{eff}}^\ell)^{\text{SM}} = s_0 + d_1 L_H + d_2 L_H^2 + d_3 L_H^4 + d_4 (\Delta_H^2 - 1) + d_5 \Delta_\alpha + d_6 \Delta_t + d_7 \Delta_t^2 + d_8 \Delta_t (\Delta_H - 1) + d_9 \Delta_{\alpha_s} + d_{10} \Delta_Z$$

$$L_H = \ln \frac{m_h}{100 \text{ GeV}}, \quad \Delta_H = \frac{m_h}{100 \text{ GeV}}, \quad \Delta_\alpha = \frac{\Delta\alpha}{0.05907} - 1$$

$$\Delta_t = \left(\frac{m_t}{178 \text{ GeV}} \right)^2 - 1, \quad \Delta_{\alpha_s} = \frac{\alpha_s(m_Z)}{0.117} - 1, \quad \Delta_Z = \frac{m_Z}{91.1876 \text{ GeV}} - 1$$

Coefficient values can be found in the 2nd column of Table 5

Parametrization of the SM correction to Γ_Z

Ref: Freitas, 1401.2447

Parametrization formula:

$$\Gamma_Z^{\text{SM}} = \Gamma_Z^0 + c_1 L_H + c_2 \Delta_t + c_3 \Delta_{\alpha_s} + c_4 \Delta_{\alpha_s}^2 + c_5 \Delta_{\alpha_s} \Delta_t + c_6 \Delta_\alpha + c_7 \Delta_Z$$

$$L_H = \ln \frac{m_h}{125.7 \text{ GeV}}, \quad \Delta_t = \left(\frac{m_t}{173.2 \text{ GeV}} \right)^2 - 1$$

$$\Delta_{\alpha_s} = \frac{\alpha_s(m_Z)}{0.1184} - 1, \quad \Delta_\alpha = \frac{\Delta\alpha}{0.059} - 1, \quad \Delta_Z = \frac{m_Z}{91.1876 \text{ GeV}} - 1$$

$$\Delta\alpha \equiv \Delta\alpha_{\text{lep}} + \Delta\alpha_{\text{had}}^{(5)}, \quad \Delta\alpha_{\text{lep}} = 0.0314976$$

Coefficient values can be found in the 9th row of Table 5

Likelihood for n normal variables $x_i \sim N(\mu_i, \sigma_i)$

$$L = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2\right]$$

$$\ln L = \sum_{i=1}^n \left[-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 - \ln(\sqrt{2\pi}\sigma_i) \right]$$

$$\chi^2 \equiv \sum_{i=1}^n \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 \Rightarrow \chi^2 = -2\ln L + \text{const.}$$

Assume a flat theoretical uncertainty δ_j for a observable O_j with theoretical parameters $\{\alpha_k\}$:

$$P(O_j | \{\alpha_k\}) = \begin{cases} \frac{1}{2\delta_j}, & |O_j - O_j^{\text{pred}}(\{\alpha_k\})| \leq \delta_j \\ 0, & |O_j - O_j^{\text{pred}}(\{\alpha_k\})| > \delta_j \end{cases}$$

O_j = true value, O_j^{pred} = theoretically predicted value

Assume a normal experimental uncertainty σ_j for the measured value O_j^{meas} :

$$P(O_j^{\text{meas}} | O_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}} - O_j}{\sigma_j}\right)^2\right]$$

Extract the probability for O_j^{meas} in terms of $\{\alpha_k\}$ as a convolution, integrating out the unknown true value O_j :

$$P(O_j^{\text{meas}} | \{\alpha_k\}) = \int dO_j P(O_j^{\text{meas}} | O_j) P(O_j | \{\alpha_k\}) = q(O_j^{\text{meas}}, O_j^{\text{pred}}(\{\alpha_k\}), \sigma_j, \delta_j) \Rightarrow \chi_{\text{mod}}^2 \supset -2\ln q(O_j^{\text{meas}}, O_j^{\text{pred}}(\{\alpha_k\}), \sigma_j, \delta_j)$$

$$q(x; \mu, \sigma, \delta) \equiv \frac{1}{4\delta} \left[\text{erf}\left(\frac{x - \mu + \delta}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{x - \mu - \delta}{\sqrt{2}\sigma}\right) \right]$$

$$\left[\begin{aligned} &\text{Cumulative distribution function for } x \sim N(\mu, \sigma): \quad \text{CDF}(x; \mu, \sigma) = \Phi\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{x - \mu}{\sqrt{2}\sigma}\right), \quad \text{erf}(-x) = -\text{erf}(x) \\ &\int dO_j P(O_j^{\text{meas}} | O_j) P(O_j | \{\alpha_k\}) = \int dO_j \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}} - O_j}{\sigma_j}\right)^2\right] \frac{\theta(\delta_j - |O_j - O_j^{\text{pred}}|)}{2\delta_j} \\ &= \frac{1}{2\delta_j} \int_{O_j^{\text{pred}} - \delta_j}^{O_j^{\text{pred}} + \delta_j} dO_j \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j - O_j^{\text{meas}}}{\sigma_j}\right)^2\right] = \frac{1}{2\delta_j} \left[\Phi\left(\frac{O_j^{\text{pred}} + \delta_j - O_j^{\text{meas}}}{\sigma_j}\right) - \Phi\left(\frac{O_j^{\text{pred}} - \delta_j - O_j^{\text{meas}}}{\sigma_j}\right) \right] \\ &= \frac{1}{4\delta_j} \left[\text{erf}\left(\frac{O_j^{\text{pred}} + \delta_j - O_j^{\text{meas}}}{\sqrt{2}\sigma_j}\right) - \text{erf}\left(\frac{O_j^{\text{pred}} - \delta_j - O_j^{\text{meas}}}{\sqrt{2}\sigma_j}\right) \right] = \frac{1}{4\delta_j} \left[\text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} + \delta_j}{\sqrt{2}\sigma_j}\right) - \text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} - \delta_j}{\sqrt{2}\sigma_j}\right) \right] \end{aligned} \right]$$

Define modified χ^2 for n observables O_i without theoretical uncertainties

and m observables O_j with flat theoretical uncertainties:

$$\chi_{\text{mod}}^2 = \sum_{i=1}^n \left(\frac{O_i^{\text{meas}} - O_i^{\text{pred}}}{\sigma_i}\right)^2 + \sum_{j=1}^m \left\{ -2\ln \left[\text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} + \delta_j}{\sqrt{2}\sigma_j}\right) - \text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} - \delta_j}{\sqrt{2}\sigma_j}\right) \right] \right\}$$

Current data

$$\alpha_s(m_Z^2) = 0.1185 \pm 0.0006 \text{ [PDG 2014]}$$

$$\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = 0.02765 \pm 0.00008 \text{ [1209.4802]}$$

$$m_Z = 91.1875 \pm 0.0021 \text{ GeV [hep-ex/0509008]}$$

$$m_t = 173.34 \pm 0.76_{\text{exp}} \text{ [1403.4427]} \pm 0.5_{\text{the}} \text{ [1412.4435]} \text{ GeV}$$

$$m_h = 125.09 \pm 0.24 \text{ GeV [1503.07589]}$$

$$m_W = 80.385 \pm 0.015_{\text{exp}} \text{ [PDG 2014]} \pm 0.004_{\text{the}} \text{ GeV [hep-ph/0311148]}$$

$$\sin^2 \theta_{\text{eff}}^\ell = 0.23153 \pm 0.00016 \text{ [hep-ex/0509008]}$$

$$\Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV [hep-ex/0509008]}$$

$$\sigma_S = 0.10, \quad \sigma_T = 0.12, \quad \sigma_U = 0.094, \quad \rho_{ST} = +0.89, \quad \rho_{SU} = -0.55, \quad \rho_{TU} = -0.80$$

$$U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.085, \quad \sigma_T = 0.072, \quad \rho_{ST} = +0.90}$$

$$S=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.054, \quad \sigma_U = 0.078, \quad \rho_{TU} = -0.81}$$

$$T=U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.037}$$

$$S=U=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.032}$$

CEPC baseline precision

$$\alpha_s(m_Z^2): \pm 1.0 \times 10^{-4} \text{ [1404.0319]}$$

$$\Delta\alpha_{\text{had}}^{(5)}(m_Z^2): \pm 4.7 \times 10^{-5} \text{ [1407.3792]}$$

$$m_Z: \pm 5 \times 10^{-4} \text{ GeV [CEPC-SPPC pre-CDR]}$$

$$m_t: \pm 0.2_{\text{exp}} \text{ [CMS-PAS-FTR-13-017]} \pm 0.5_{\text{the}} \text{ [1412.4435]} \text{ GeV}$$

$$m_h: \pm 5.9 \times 10^{-3} \text{ GeV [CEPC-SPPC pre-CDR]}$$

$$m_W: (\pm 3_{\text{exp}} \text{ [CEPC-SPPC pre-CDR]} \pm 1_{\text{the}} \text{ [1307.3962]}) \times 10^{-3} \text{ GeV}$$

$$\sin^2 \theta_{\text{eff}}^\ell: (\pm 2.3_{\text{exp}} \text{ [CEPC-SPPC pre-CDR]} \pm 1.5_{\text{the}} \text{ [1307.3962]}) \times 10^{-5}$$

$$\Gamma_Z: (\pm 5_{\text{exp}} \text{ [CEPC-SPPC pre-CDR]} \pm 0.8_{\text{the}} \text{ [1411.1054, Mishima's talk]}) \times 10^{-4} \text{ GeV}$$

$$\sigma_S = 0.021, \quad \sigma_T = 0.026, \quad \sigma_U = 0.020, \quad \rho_{ST} = +0.90, \quad \rho_{SU} = -0.68, \quad \rho_{TU} = -0.84$$

$$U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.015, \quad \sigma_T = 0.014, \quad \rho_{ST} = +0.83}$$

$$S=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.011, \quad \sigma_U = 0.015, \quad \rho_{TU} = -0.72}$$

$$T=U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.0085}$$

$$S=U=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.0079}$$

Potential improvements for CEPC

Reduced systematic uncertainty in the CEPC measurement

$$\rightarrow m_Z: \pm 1 \times 10^{-4} \text{ GeV [CEPC-SPPC pre-CDR]}$$

$$\Gamma_Z: (\pm 1_{\text{exp}} \text{ [CEPC-SPPC pre-CDR]} \pm 0.8_{\text{the}} \text{ [1411.1054, Mishima's talk]}) \times 10^{-4} \text{ GeV}$$

$$\text{ILC top threshold scan} \rightarrow m_t: \pm 0.03_{\text{exp}} \pm 0.1_{\text{the}} \text{ GeV [1306.6352]}$$

m_Z, Γ_Z & m_t improvements:

$$\sigma_S = 0.011, \quad \sigma_T = 0.0071, \quad \sigma_U = 0.010, \quad \rho_{ST} = +0.74, \quad \rho_{SU} = +0.15, \quad \rho_{TU} = -0.21$$

$$U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.011, \quad \sigma_T = 0.0069, \quad \rho_{ST} = +0.80}$$

$$S=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.0048, \quad \sigma_U = 0.010, \quad \rho_{TU} = -0.48}$$

$$T=U=0 \text{ fixed} \Rightarrow \underline{\sigma_S = 0.0068}$$

$$S=U=0 \text{ fixed} \Rightarrow \underline{\sigma_T = 0.0042}$$

DM models

Convention:

$$v^i \in \mathbf{2}, \quad v_i \in \bar{\mathbf{2}}, \quad \varepsilon^{12} = +1, \quad \varepsilon_{12} = -1, \quad \varepsilon^{ij} = -\varepsilon^{ji}, \quad \varepsilon_{ij} = -\varepsilon_{ji}$$

$$\text{SM Higgs field} \quad H = H^i = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} G^+(x) \\ \frac{1}{\sqrt{2}}[v + h(x) + iG^0(x)] \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2} \right) \text{ under } (\text{SU}(2)_L, \text{U}(1)_Y)$$

$$H_i = \varepsilon_{ij} H^j = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} -H^0 \\ H^+ \end{pmatrix} \in \left(\bar{\mathbf{2}}, \frac{1}{2} \right)$$

$$H_i^\dagger = (H^i)^* = \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} \in \left(\bar{\mathbf{2}}, -\frac{1}{2} \right)$$

$$\tilde{H} = H^{\dagger i} = \varepsilon^{ij} H_j^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2} \right)$$

$$\left[\text{Note: } H^\dagger H = H_i^\dagger H^i = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = H^+ H^- + H^0 H^{0*} = -H^{\dagger i} H_i \right]$$

Singlet-Doublet Fermionic Dark Matter (SDFDM)

Ref: D'Eramo, 0705.4493; Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604

Left-handed Weyl fermions:

$$S \in (\mathbf{1}, 0), \quad D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2} \right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2} \right)$$

$$\mathcal{L}_S = iS^\dagger \bar{\sigma}^\mu \partial_\mu S - \frac{1}{2}(m_S SS + \text{h.c.}), \quad \mathcal{L}_D = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.})$$

$$\mathcal{L}_{\text{HSD}} = y_1 H_i SD_1^i - y_2 H_i^\dagger SD_2^i + \text{h.c.}$$

$$-m_D \varepsilon_{ij} D_1^i D_2^j = m_D D_1^0 D_2^0 - m_D D_1^- D_2^+$$

$$y_1 H_i SD_1^i = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} S \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = -y_1 H^0 SD_1^0 + y_1 H^+ SD_1^- \rightarrow -\frac{1}{\sqrt{2}} y_1 (v + h) SD_1^0$$

$$-y_2 H_i^\dagger SD_2^i = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} S \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 H^- SD_2^+ - y_2 H^{0*} SD_2^0 \rightarrow -\frac{1}{\sqrt{2}} y_2 (v + h) SD_2^0$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_N \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} - m_D D_1^- D_2^+ + \text{h.c.} = -\frac{1}{2} \sum_{i=1}^3 m_{\chi_i^0} \chi_i^0 \chi_i^0 - m_{\chi^\pm} \chi^- \chi^+ + \text{h.c.}$$

$$\mathcal{M}_N = \begin{pmatrix} m_S & \frac{1}{\sqrt{2}} y_1 v & \frac{1}{\sqrt{2}} y_2 v \\ \frac{1}{\sqrt{2}} y_1 v & 0 & -m_D \\ \frac{1}{\sqrt{2}} y_2 v & -m_D & 0 \end{pmatrix}, \quad m_{\chi^\pm} = m_D, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$$

$$\mathcal{N}^\top \mathcal{M}_N \mathcal{N} = \text{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}$$

$$y = y_1 = y_2 \Rightarrow \underline{\text{Custodial SU}(2)_R \text{ global symmetry}}$$

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an SU}(2)_R \text{ indice}$$

$$H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = -\frac{1}{2}[\varepsilon_{12} \varepsilon^{ij} (\mathcal{H}^1)_i (\mathcal{H}^2)_j + \varepsilon_{21} \varepsilon^{ij} (\mathcal{H}^2)_i (\mathcal{H}^1)_j] = -\frac{1}{2} \varepsilon_{AB} \varepsilon^{ij} (\mathcal{H}^A)_i (\mathcal{H}^B)_j$$

$$\mathcal{L}_{\text{HSD}} = y(H_i S D_1^i - H_i^\dagger S D_2^i) + \text{h.c.} = y \varepsilon_{AB} (\mathcal{H}^A)_i S (\mathcal{D}^B)^i + \text{h.c.}$$

$$\mathcal{L}_{\text{D}} = i D_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + i D_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = i D_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A - \frac{1}{2} [m_D \varepsilon_{AB} \varepsilon_{ij} (\mathcal{D}^A)^i (\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_S \Rightarrow \chi_1^0 = \frac{1}{\sqrt{2}}(-D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_{\chi^\pm} = m_D \\ m_{\chi_2^0} = \frac{1}{2} \left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} + m_D - m_S \right] \\ m_{\chi_3^0} = \frac{1}{2} \left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} - m_D + m_S \right] \end{cases}$$

$$m_D > m_S \Rightarrow \begin{cases} m_{\chi_1^0} = \frac{1}{2} \left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} - m_D + m_S \right] \\ m_{\chi_2^0} = m_{\chi^\pm} = m_D \\ m_{\chi_3^0} = \frac{1}{2} \left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} + m_D - m_S \right] \end{cases} \quad \text{for } |y v| < \sqrt{2m_D(m_D - m_S)}$$

$$\left[\begin{aligned} y = y_1 = -y_2 &\Rightarrow \text{Another custodial symmetry limit} \\ (\mathcal{D}^A)^i &= \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} -H_i^\dagger \\ H_i \end{pmatrix} \\ H^\dagger H &= H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = \frac{1}{2}[\varepsilon_{12} \varepsilon^{ij} (\mathcal{H}^1)_i (\mathcal{H}^2)_j + \varepsilon_{21} \varepsilon^{ij} (\mathcal{H}^2)_i (\mathcal{H}^1)_j] = \frac{1}{2} \varepsilon_{AB} \varepsilon^{ij} (\mathcal{H}^A)_i (\mathcal{H}^B)_j \\ \mathcal{L}_{\text{HSD}} &= y(H_i S D_1^i + H_i^\dagger S D_2^i) + \text{h.c.} = y \varepsilon_{AB} (\mathcal{H}^A)_i S (\mathcal{D}^B)^i + \text{h.c.} \end{aligned} \right]$$

Gauge interactions

$$D_\mu D_i = (\partial_\mu - i g' B_\mu Y_{D_i} - i g W_\mu^a t_D^a) D_i$$

$$Y_{D_1} = -\frac{1}{2}, \quad Y_{D_2} = \frac{1}{2}, \quad t_D^1 = \frac{\sigma^1}{2} = \frac{1}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad t_D^2 = \frac{\sigma^2}{2} = \frac{1}{2} \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad t_D^3 = \frac{\sigma^3}{2} = \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$g' B_\mu Y_{D_1} + g W_\mu^a t_D^a = \frac{1}{2} \begin{pmatrix} -g' B_\mu + g W_\mu^a & g(W_\mu^1 - i W_\mu^2) \\ g(W_\mu^1 + i W_\mu^2) & -g' B_\mu - g W_\mu^a \end{pmatrix} = \begin{pmatrix} \frac{g}{2c_W} Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -e A_\mu + \frac{g}{2c_W} (s_W^2 - c_W^2) Z_\mu \end{pmatrix}$$

$$g' B_\mu Y_{D_2} + g W_\mu^a t_D^a = \frac{1}{2} \begin{pmatrix} g' B_\mu + g W_\mu^a & g(W_\mu^1 - i W_\mu^2) \\ g(W_\mu^1 + i W_\mu^2) & g' B_\mu - g W_\mu^a \end{pmatrix} = \begin{pmatrix} e A_\mu - \frac{g}{2c_W} (s_W^2 - c_W^2) Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{g}{2c_W} Z_\mu \end{pmatrix}$$

$$\mathcal{L}_{\text{D}} \supset i D_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + i D_2^\dagger \bar{\sigma}^\mu D_\mu D_2$$

$$\begin{aligned} &= \frac{g}{2c_W} Z_\mu (D_1^0)^\dagger \bar{\sigma}^\mu D_1^0 + \left[-e A_\mu + \frac{g}{2c_W} (s_W^2 - c_W^2) Z_\mu \right] (D_1^-)^\dagger \bar{\sigma}^\mu D_1^- + \frac{g}{\sqrt{2}} [W_\mu^+ (D_1^0)^\dagger \bar{\sigma}^\mu D_1^- + W_\mu^- (D_1^-)^\dagger \bar{\sigma}^\mu D_1^0] \\ &\quad + \left[e A_\mu - \frac{g}{2c_W} (s_W^2 - c_W^2) Z_\mu \right] (D_2^+)^\dagger \bar{\sigma}^\mu D_2^+ - \frac{g}{2c_W} Z_\mu (D_2^0)^\dagger \bar{\sigma}^\mu D_2^0 + \frac{g}{\sqrt{2}} [W_\mu^+ (D_2^+)^\dagger \bar{\sigma}^\mu D_2^0 + W_\mu^- (D_2^0)^\dagger \bar{\sigma}^\mu D_2^+] \end{aligned}$$

$$X_i^0 = \begin{pmatrix} (\chi_{iL}^0)_\alpha \\ (\chi_{iR}^0)^{\dagger\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \chi_{iL}^0 \\ (\chi_{iR}^0)^\dagger \end{pmatrix}, \quad \chi_L^0 = \chi_R^0 = \mathcal{N}^\dagger \psi_L^0 = \mathcal{N}^\dagger \psi_R^0 = \begin{pmatrix} \chi_1^0 & \chi_2^0 & \chi_3^0 \end{pmatrix}^T, \quad \bar{X}_i^0 = \begin{pmatrix} \chi_{iR}^0 & (\chi_{iL}^0)^\dagger \end{pmatrix}$$

$$\Psi_i^0 = \begin{pmatrix} \psi_{iL}^0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix}, \quad \psi_L^0 = \psi_R^0 = \mathcal{N} \chi_L^0 = \mathcal{N} \chi_R^0 = \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix}^T, \quad \bar{\Psi}_i^0 = \begin{pmatrix} \psi_{iR}^0 & (\psi_{iL}^0)^\dagger \end{pmatrix}$$

$$X^+ = \begin{pmatrix} \chi^+ \\ (\chi^-)^\dagger \end{pmatrix}, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$$

$$\Psi_{iL}^0 = \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{N} \chi_L^0)_i \\ 0 \end{pmatrix} = \mathcal{N}_{ij} X_{jL}^0, \quad \Psi_{iR}^0 = \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{N} \chi_R^0)_i^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* X_{jR}^0$$

$$\bar{\Psi}_{iL}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* \bar{X}_{jL}^0, \quad \bar{\Psi}_{iR}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} = \mathcal{N}_{ij} \bar{X}_{jR}^0$$

$$\bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = (\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \psi_{iR}^0 \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^0 = -(\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\begin{aligned} \mathcal{L}_{Z\Psi_i^0\Psi_i^0} &= \frac{1}{2} a_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 + \frac{1}{2} b_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 \\ &= \frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + \frac{1}{2} b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0 = \frac{1}{2} (a_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0) \end{aligned}$$

$$a_{Z\Psi_1^0\Psi_1^0} = b_{Z\Psi_1^0\Psi_1^0} = 0, \quad a_{Z\Psi_2^0\Psi_2^0} = -b_{Z\Psi_2^0\Psi_2^0} = \frac{g}{2c_W}, \quad a_{Z\Psi_3^0\Psi_3^0} = -b_{Z\Psi_3^0\Psi_3^0} = -\frac{g}{2c_W}$$

$$a_{ZX_i^0 X_j^0} = a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}, \quad b_{ZX_i^0 X_j^0} = b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*$$

$$\left(\begin{array}{l} \text{Note: } a_{ZX_j^0 X_i^0} = a_{ZX_i^0 X_j^0}^*, \quad b_{ZX_j^0 X_i^0} = b_{ZX_i^0 X_j^0}^*, \quad a_{ZX_i^0 X_j^0} = -b_{ZX_j^0 X_i^0} \\ C^{-1} = -C, \quad X_i^0 = C(\bar{X}_i^0)^T, \quad \bar{X}_i^0 = (X_i^0)^T C \\ C^{-1}(\gamma^\mu P_L)^T C = \frac{1}{2} C^{-1}(\gamma^\mu)^T C - \frac{1}{2} C^{-1}(\gamma^\mu \gamma_5)^T C = -\frac{1}{2} \gamma^\mu - \frac{1}{2} \gamma^\mu \gamma_5 = -\gamma^\mu P_R \\ a_{ZX_1^0 X_2^0} \bar{X}_{1L}^0 \gamma^\mu X_{2L}^0 = a_{ZX_1^0 X_2^0} (\bar{X}_1^0 \gamma^\mu P_L X_2^0)^T = -a_{ZX_1^0 X_2^0} (X_2^0)^T C C^{-1}(\gamma^\mu P_L)^T C^{-1} C(\bar{X}_1^0)^T = -a_{ZX_1^0 X_2^0} \bar{X}_2^0 C^{-1}(\gamma^\mu P_L)^T C^{-1} X_1^0 \\ = a_{ZX_1^0 X_2^0} \bar{X}_2^0 C^{-1}(\gamma^\mu P_L)^T C X_1^0 = -a_{ZX_1^0 X_2^0} \bar{X}_2^0 \gamma^\mu P_R X_1^0 = b_{ZX_2^0 X_1^0} \bar{X}_{2R}^0 \gamma^\mu X_{1R}^0 \\ \frac{1}{2} a_{ZX_1^0 X_2^0} Z_\mu \bar{X}_{1L}^0 \gamma^\mu X_{2L}^0 \text{ and } \frac{1}{2} b_{ZX_2^0 X_1^0} Z_\mu \bar{X}_{2R}^0 \gamma^\mu X_{1R}^0 \text{ give an identical vertex!} \end{array} \right)$$

$$\bar{X}_L^+ \gamma^\mu X_L^+ = \begin{pmatrix} 0 & (\chi^+)^{\dagger} \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} \chi^+ \\ 0 \end{pmatrix} = (D_2^+)^{\dagger} \bar{\sigma}^\mu D_2^+$$

$$\bar{X}_R^+ \gamma^\mu X_R^+ = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} 0 \\ (\chi^-)^{\dagger} \end{pmatrix} = \chi^- \sigma^\mu (\chi^-)^{\dagger} = -(\chi^-)^{\dagger} \bar{\sigma}^\mu \chi^- = -(D_1^-)^{\dagger} \bar{\sigma}^\mu D_1^-$$

$$\mathcal{L}_{AX^+X^+} = a_{AX^+X^+} A_\mu \bar{X}_L^+ \gamma^\mu X_L^+ + b_{AX^+X^+} A_\mu \bar{X}_R^+ \gamma^\mu X_R^+$$

$$a_{AX^+X^+} = b_{AX^+X^+} = e$$

$$\mathcal{L}_{ZX^+X^+} = a_{ZX^+X^+} Z_\mu \bar{X}_L^+ \gamma^\mu X_L^+ + b_{ZX^+X^+} Z_\mu \bar{X}_R^+ \gamma^\mu X_R^+$$

$$a_{ZX^+X^+} = b_{ZX^+X^+} = -\frac{g}{2c_W} (s_W^2 - c_W^2)$$

$$\bar{X}_L^+ \gamma^\mu \Psi_{iL}^0 = \begin{pmatrix} 0 & (\chi^+)^{\dagger} \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = (\chi^+)^{\dagger} \bar{\sigma}^\mu \psi_{iL}^0$$

$$\bar{X}_R^+ \gamma^\mu \Psi_{iR}^0 = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} \sigma^\mu \\ \bar{\sigma}^\mu \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{iR}^0)^{\dagger} \end{pmatrix} = \chi^- \sigma^\mu (\psi_{iR}^0)^{\dagger} = -(\psi_{iR}^0)^{\dagger} \bar{\sigma}^\mu \chi^-$$

$$\mathcal{L}_{WX^+\Psi_i^0} = a_{WX^+\Psi_i^0} (W_\mu^+ \bar{X}_L^+ \gamma^\mu \Psi_{iL}^0 + \text{h.c.}) + b_{WX^+\Psi_i^0} (W_\mu^+ \bar{X}_R^+ \gamma^\mu \Psi_{iR}^0 + \text{h.c.})$$

$$= a_{WX^+\Psi_i^0} [\mathcal{N}_{ij} W_\mu^+ \bar{X}_L^+ \gamma^\mu X_{jL}^0 + \text{h.c.}] + b_{WX^+\Psi_i^0} [\mathcal{N}_{ij}^* W_\mu^+ \bar{X}_R^+ \gamma^\mu X_{jR}^0 + \text{h.c.}]$$

$$= a_{WX^+X_i^0} W_\mu^+ \bar{X}_L^+ \gamma^\mu X_{iL}^0 + a_{WX^+X_i^0}^* W_\mu^- \bar{X}_{iL}^0 \gamma^\mu X_L^+ + b_{WX^+X_i^0} W_\mu^+ \bar{X}_R^+ \gamma^\mu X_{iR}^0 + b_{WX^+X_i^0}^* W_\mu^- \bar{X}_{iR}^0 \gamma^\mu X_R^+$$

$$b_{WX^+\Psi_2^0} = -\frac{g}{\sqrt{2}}, \quad a_{WX^+\Psi_3^0} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0$$

$$a_{WX^+X_i^0} = a_{WX^+\Psi_i^0} \mathcal{N}_{ij}, \quad b_{WX^+X_i^0} = b_{WX^+\Psi_i^0} \mathcal{N}_{ij}^*$$

$$\bar{\Psi}_{iR}^0 \Psi_{jL}^0 = \begin{pmatrix} \psi_{jL}^0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = \psi_{iR}^0 \psi_{jL}^0, \quad \bar{\Psi}_{iL}^0 \Psi_{jR}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^{\dagger} \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{jR}^0)^{\dagger} \end{pmatrix} = (\psi_{iL}^0)^{\dagger} (\psi_{jR}^0)^{\dagger}$$

$$\mathcal{L}_{h\Psi_i^0\Psi_j^0} = \frac{1}{2} a_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iR}^0 \Psi_{jL}^0 + \frac{1}{2} b_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iL}^0 \Psi_{jR}^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj} h \bar{X}_{iR}^0 X_{jL}^0 + \frac{1}{2} b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^* h \bar{X}_{iL}^0 X_{jR}^0 = \frac{1}{2} (a_{hX_i^0X_j^0} h \bar{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0X_j^0} h \bar{X}_{iL}^0 X_{jR}^0)$$

$$a_{h\Psi_1^0\Psi_2^0} = b_{h\Psi_1^0\Psi_2^0} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi_2^0\Psi_1^0} = b_{h\Psi_2^0\Psi_1^0}, \quad a_{h\Psi_1^0\Psi_3^0} = b_{h\Psi_1^0\Psi_3^0} = -\frac{y_2}{\sqrt{2}} = a_{h\Psi_3^0\Psi_1^0} = b_{h\Psi_3^0\Psi_1^0}, \quad \text{others} = 0$$

$$a_{hX_i^0X_j^0} = a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj}, \quad b_{hX_i^0X_j^0} = b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^*$$

$$\left(\begin{array}{l} \text{Note: } a_{hX_j^0X_i^0} = a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{kj} \mathcal{N}_{li} = a_{h\Psi_l^0\Psi_k^0} \mathcal{N}_{kj} \mathcal{N}_{li} = a_{hX_i^0X_j^0}, \quad b_{hX_j^0X_i^0} = b_{hX_i^0X_j^0}, \quad a_{hX_i^0X_j^0} = b_{hX_i^0X_j^0}^* \\ C^{-1} (P_L)^T C = \frac{1}{2} C^{-1} (1 - \gamma_5)^T C = \frac{1}{2} (1 - \gamma_5) = P_L \\ a_{hX_1^0X_2^0} \bar{X}_{1R}^0 X_{2L}^0 = a_{hX_1^0X_2^0} (\bar{X}_1^0 P_L X_2^0)^T = -a_{hX_1^0X_2^0} (X_2^0)^T (P_L)^T (\bar{X}_1^0)^T = -a_{hX_1^0X_2^0} (X_2^0)^T C C^{-1} (P_L)^T C^{-1} C (\bar{X}_1^0)^T = -a_{hX_1^0X_2^0} \bar{X}_2^0 C^{-1} (P_L)^T C^{-1} X_1^0 \\ = a_{hX_1^0X_2^0} \bar{X}_2^0 C^{-1} (P_L)^T C X_1^0 = a_{hX_1^0X_2^0} \bar{X}_2^0 P_L X_1^0 = a_{hX_2^0X_1^0} \bar{X}_{2R}^0 X_{1L}^0 \\ \frac{1}{2} a_{hX_1^0X_2^0} h \bar{X}_{1R}^0 X_{2L}^0 \left(\frac{1}{2} b_{hX_1^0X_2^0} h \bar{X}_{1L}^0 X_{2R}^0 \right) \text{ and } \frac{1}{2} a_{hX_2^0X_1^0} h \bar{X}_{2R}^0 X_{1L}^0 \left(\frac{1}{2} b_{hX_2^0X_1^0} h \bar{X}_{2L}^0 X_{1R}^0 \right) \text{ give an identical vertex!} \end{array} \right)$$

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$a_{h\Psi_l^0\Psi_j^0} = b_{h\Psi_l^0\Psi_j^0}$$

$$\mathcal{L}_{hX_i^0 X_j^0} = \frac{1}{2}(a_{hX_i^0 X_j^0} h\bar{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0 X_j^0} h\bar{X}_{iL}^0 X_{jR}^0) = \frac{1}{2} h\bar{X}_i^0 (a_{hX_i^0 X_j^0} P_L + b_{hX_i^0 X_j^0} P_R) X_j^0$$

$$= \frac{1}{4}(a_{hX_i^0 X_j^0} + b_{hX_i^0 X_j^0}) h\bar{X}_i^0 X_j^0 + \frac{1}{4}(b_{hX_i^0 X_j^0} - a_{hX_i^0 X_j^0}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki} \mathcal{N}_{lj} + \mathcal{N}_{ki}^* \mathcal{N}_{lj}^*) h\bar{X}_i^0 X_j^0 + \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}^* \mathcal{N}_{lj}^* - \mathcal{N}_{ki} \mathcal{N}_{lj}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \text{Re}(\mathcal{N}_{ki} \mathcal{N}_{lj}) h\bar{X}_i^0 X_j^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \text{Im}(\mathcal{N}_{ki} \mathcal{N}_{lj}) h\bar{X}_i^0 i\gamma_5 X_j^0$$

$$\text{Im}(\mathcal{N}_{ki} \mathcal{N}_{li}) = 0$$

$$\mathcal{L}_{hX_1^0 X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \text{Re}(\mathcal{N}_{k1} \mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \text{Im}(\mathcal{N}_{k1} \mathcal{N}_{l1}) h\bar{X}_1^0 i\gamma_5 X_1^0$$

$$= [a_{h\Psi_1^0\Psi_2^0} \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= -\frac{1}{\sqrt{2}} [y_1 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + y_2 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0 X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0 X_1^0} = \frac{1}{2} (a_{hX_1^0 X_1^0} + b_{hX_1^0 X_1^0}) = -\sqrt{2} [y_1 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + y_2 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})]$$

$$\text{Effective operators: } \mathcal{L}_{S,q} = \sum_q G_{S,q} \bar{X}_1^0 X_1^0 \bar{q} q, \quad \mathcal{L}_{S,N} = \sum_{N=p,n} G_{S,N} \bar{X}_1^0 X_1^0 \bar{N} N$$

$$G_{S,N} = m_N \left(\sum_{q=u,d,s} \frac{G_{S,q}}{m_q} f_q^N + \sum_{q=c,b,t} \frac{G_{S,q}}{m_q} f_q^N \right), \quad f_Q^N = \frac{2}{27} \left(1 - \sum_{q=u,d,s} f_q^N \right)$$

hep-ph/0001005 :

$$f_u^p = 0.020 \pm 0.004, \quad f_d^p = 0.026 \pm 0.005, \quad f_u^n = 0.014 \pm 0.003, \quad f_d^n = 0.036 \pm 0.008, \quad f_s^p = f_s^n = 0.118 \pm 0.062$$

$$\Rightarrow f_Q^p = 0.0619, \quad f_Q^n = 0.0616$$

$$G_{S,q} = -\frac{g_{hX_1^0 X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0 X_1^0} m_N}{2vm_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \Rightarrow G_{S,n} \simeq G_{S,p}$$

$$\sigma_{\chi N}^{\text{SI}} = \frac{4}{\pi} \mu_{\chi N}^2 G_{S,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$a_{Z\Psi_k^0\Psi_k^0} = -b_{Z\Psi_k^0\Psi_k^0}$$

$$\mathcal{L}_{Z\chi_1^0 X_j^0} = \frac{1}{2}(a_{Z\chi_1^0 X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{Z\chi_1^0 X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0) = \frac{1}{2}(a_{Z\chi_1^0 X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_L X_j^0 + b_{Z\chi_1^0 X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_R X_j^0)$$

$$= \frac{1}{4}(a_{Z\chi_1^0 X_j^0} + b_{Z\chi_1^0 X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{Z\chi_1^0 X_j^0} - a_{Z\chi_1^0 X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4}(a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} + b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* - a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki}^* \mathcal{N}_{kj} - \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 - \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki} \mathcal{N}_{kj}^* + \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$\mathcal{L}_{Z\chi_1^0 X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{Z\chi_1^0 X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{Z\chi_1^0 X_1^0} = \frac{1}{2} (b_{Z\chi_1^0 X_1^0} - a_{Z\chi_1^0 X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W} (|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

$$\text{Effective operators: } \mathcal{L}_{A,q} = \sum_q G_{A,q} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{q} \gamma^\mu \gamma_5 q, \quad \mathcal{L}_{A,N} = \sum_{N=p,n} G_{A,N} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{N} \gamma^\mu \gamma_5 N$$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q} \Delta_q^N$$

hep-ex/0609039:

$$\Delta_u^p = \Delta_d^n = 0.842 \pm 0.012, \quad \Delta_d^p = \Delta_u^n = -0.427 \pm 0.013, \quad \Delta_s^p = \Delta_s^n = -0.085 \pm 0.018$$

$$G_{A,q} = \frac{gg_A^q g_{Z\chi_1^0 X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$\sigma_{\chi N}^{\text{SD}} = \frac{12}{\pi} \mu_{\chi N}^2 G_{A,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Left-handed Weyl fermions:

$$D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2} \right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2} \right), \quad T = \begin{pmatrix} T^+ \\ T^0 \\ T^- \end{pmatrix} \in (\mathbf{3}, 0)$$

$$\mathcal{L}_D = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}), \quad \mathcal{L}_T = iT^\dagger \bar{\sigma}^\mu D_\mu T - \frac{1}{2}(m_T T^a T^a + \text{h.c.})$$

$$\mathcal{L}_{\text{HDT}} = y_1 H_i T^a (\sigma^a)^i_j D_1^j - y_2 H_i^\dagger T^a (\sigma^a)^i_j D_2^j + \text{h.c.}, \quad i, j = 1, 2, \quad a = 1, 2, 3$$

$$m_D \varepsilon_{ij} D_1^i D_2^j = -m_D D_1^0 D_2^0 + m_D D_1^- D_2^+$$

$$T^\pm = -\frac{1}{\sqrt{2}}(T^1 \mp iT^2), \quad T^0 = T^3, \quad T^1 = -\frac{1}{\sqrt{2}}(T^+ + T^-), \quad T^2 = -\frac{i}{\sqrt{2}}(T^+ - T^-)$$

$$-\frac{1}{2}m_T T^a T^a = -\frac{1}{2}m_T \left[\frac{1}{2}(T^+ + T^-)^2 - \frac{1}{2}(T^+ - T^-)^2 + T^0 T^0 \right] = -m_T T^- T^+ - \frac{1}{2}m_T T^0 T^0$$

$$T^a \sigma^a = \begin{pmatrix} T^3 & T^1 - iT^2 \\ T^1 + iT^2 & -T^3 \end{pmatrix} = \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix}$$

$$\begin{aligned} y_1 H_i T^a (\sigma^a)^i_j D_1^j &= y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 D_1^0 - \sqrt{2}T^+ D_1^- \\ -\sqrt{2}T^- D_1^0 - T^0 D_1^- \end{pmatrix} \\ &= y_1 (-H^0 T^0 D_1^0 + \sqrt{2}H^0 T^+ D_1^- - \sqrt{2}H^+ T^- D_1^0 - H^+ T^0 D_1^-) \rightarrow -\frac{1}{\sqrt{2}}y_1 (v+h)T^0 D_1^0 + y_1 (v+h)T^+ D_1^- \\ -y_2 H_i^\dagger T^a (\sigma^a)^i_j D_2^j &= -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 D_2^+ - \sqrt{2}T^+ D_2^0 \\ -\sqrt{2}T^- D_2^+ - T^0 D_2^0 \end{pmatrix} \\ &= -y_2 (H^- T^0 D_2^+ - \sqrt{2}H^- T^+ D_2^0 - \sqrt{2}H^{0*} T^- D_2^+ - H^{0*} T^0 D_2^0) \rightarrow y_2 (v+h)T^- D_2^+ + \frac{1}{\sqrt{2}}y_2 (v+h)T^0 D_2^0 \end{aligned}$$

$$\left[\begin{aligned} \text{Note: } T_j^i &= u^i v_j - \frac{1}{2} \delta_j^i u^k v_k = \frac{1}{\sqrt{2}} T^a (\sigma^a)^i_j \\ T^+ &= -T_2^1, \quad T^- = -T_1^2, \quad T^0 = \sqrt{2}T_1^1 = -\sqrt{2}T_2^2 \Rightarrow -\frac{1}{2}m_T T_i^j T_j^i = -m_T T^- T^+ - \frac{1}{2}m_T T^0 T^0 \\ \sqrt{2}y_1 H_i T_j^i D_1^j &= \sqrt{2}y_1 (H_1 T_1^1 D_1^1 + H_1 T_2^1 D_1^2 + H_2 T_1^2 D_1^1 + H_2 T_2^2 D_1^2) \\ &= \sqrt{2}y_1 \left(-\frac{1}{\sqrt{2}}H^0 T^0 D_1^0 + H^0 T^+ D_1^- - H^+ T^- D_1^0 - \frac{1}{\sqrt{2}}H^+ T^0 D_1^- \right) \\ &= y_1 (-H^0 T^0 D_1^0 + \sqrt{2}H^0 T^+ D_1^- - \sqrt{2}H^+ T^- D_1^0 - H^+ T^0 D_1^-) \\ -\sqrt{2}y_2 H_i^\dagger T_j^i D_2^j &= -\sqrt{2}y_2 (H_1^\dagger T_1^1 D_2^1 + H_1^\dagger T_2^1 D_2^2 + H_2^\dagger T_1^2 D_2^1 + H_2^\dagger T_2^2 D_2^2) \\ &= -\sqrt{2}y_2 \left(\frac{1}{\sqrt{2}}H^- T^0 D_2^+ - H^- T^+ D_2^0 - H^{0*} T^- D_2^+ - \frac{1}{\sqrt{2}}H^{0*} T^0 D_2^0 \right) \\ &= -y_2 (H^- T^0 D_2^+ - \sqrt{2}H^- T^+ D_2^0 - \sqrt{2}H^{0*} T^- D_2^+ - H^{0*} T^0 D_2^0) \end{aligned} \right]$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} T^0 & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_N \begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} - \begin{pmatrix} T^- & D_1^- \end{pmatrix} \mathcal{M}_C \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} + \text{h.c.} = -\frac{1}{2} \sum_{i=1}^3 m_{\chi_i^0} \chi_i^0 \chi_i^0 - \sum_{i=1}^2 m_{\chi_i^\pm} \chi_i^- \chi_i^+ + \text{h.c.}$$

$$\mathcal{M}_N = \begin{pmatrix} m_T & \frac{1}{\sqrt{2}}y_1 v & -\frac{1}{\sqrt{2}}y_2 v \\ \frac{1}{\sqrt{2}}y_1 v & 0 & m_D \\ -\frac{1}{\sqrt{2}}y_2 v & m_D & 0 \end{pmatrix}, \quad \mathcal{M}_C = \begin{pmatrix} m_T & -y_2 v \\ -y_1 v & -m_D \end{pmatrix}$$

$$\mathcal{N}^T \mathcal{M}_N \mathcal{N} = \text{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L = \text{diag}(m_{\chi_1^\pm}, m_{\chi_2^\pm}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \mathcal{C}_L^{-1} = \mathcal{C}_L^\dagger, \quad \mathcal{C}_R^{-1} = \mathcal{C}_R^\dagger$$

$$\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{M}_C \mathcal{C}_L = (\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{C}_R^*) (\mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L) = \text{diag}(m_{\chi_1^\pm}^2, m_{\chi_2^\pm}^2), \quad \mathcal{C}_R^T \mathcal{M}_C \mathcal{M}_C^\dagger \mathcal{C}_R^* = (\mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L) (\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{C}_R^*) = \text{diag}(m_{\chi_1^\pm}^2, m_{\chi_2^\pm}^2)$$

$$\begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}, \quad \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} = \mathcal{C}_L \begin{pmatrix} \chi_1^+ \\ \chi_2^+ \end{pmatrix}, \quad \begin{pmatrix} T^- \\ D_1^- \end{pmatrix} = \mathcal{C}_R \begin{pmatrix} \chi_1^- \\ \chi_2^- \end{pmatrix}$$

$$y = y_1 = y_2 \Rightarrow \underline{\text{Custodial SU(2)}_{\text{R}} \text{ global symmetry}}$$

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an SU(2)}_{\text{R}} \text{ indice}$$

$$\mathcal{L}_{\text{HDT}} = y[H_i T^a (\sigma^a)_j^i D_1^j - H_i^\dagger T^a (\sigma^a)_j^i D_2^j] + \text{h.c.} = y \varepsilon_{AB} (\mathcal{H}^A)_i T^a (\sigma^a)_j^i (\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_{\text{D}} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A + \frac{1}{2} [m_D \varepsilon_{AB} \varepsilon_{ij} (\mathcal{D}^A)^i (\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_T \Rightarrow \chi_1^0 = \frac{1}{\sqrt{2}}(D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_D \\ m_{\chi_2^0} = m_{\chi_1^\pm} = \frac{1}{2} \left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T \right] \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2} \left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T \right] \end{cases}$$

$$m_D > m_T \Rightarrow \begin{cases} m_{\chi_1^0} = m_{\chi_1^\pm} = \frac{1}{2} \left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T \right] \\ m_{\chi_2^0} = m_D \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2} \left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T \right] \end{cases} \quad \text{for } |yv| < \sqrt{2m_D(m_D - m_T)}$$

Gauge interactions

$$D_\mu T = (\partial_\mu - ig W_\mu^a t_{\text{T}}^a) T$$

$$t_{\text{T}}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix}, \quad t_{\text{T}}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} & -i & \\ i & & i \\ & -i & \end{pmatrix}, \quad t_{\text{T}}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$g W_\mu^a t_{\text{T}}^a = \begin{pmatrix} g W_\mu^3 & g(W_\mu^1 - i W_\mu^2)/\sqrt{2} & 0 \\ g(W_\mu^1 + i W_\mu^2)/\sqrt{2} & 0 & -g(W_\mu^1 - i W_\mu^2)/\sqrt{2} \\ 0 & -g(W_\mu^1 + i W_\mu^2)/\sqrt{2} & -g W_\mu^3 \end{pmatrix}$$

$$= \begin{pmatrix} e A_\mu + g c_{\text{W}} Z_\mu & g W_\mu^+ & 0 \\ g W_\mu^- & 0 & -g W_\mu^+ \\ 0 & -g W_\mu^- & -e A_\mu - g c_{\text{W}} Z_\mu \end{pmatrix}$$

$$\mathcal{L}_{\text{T}} \supset T^\dagger \bar{\sigma}^\mu g W_\mu^a t_{\text{T}}^a T$$

$$\begin{aligned} &= g[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\ &\quad + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\ &\quad - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-] \\ &= (e A_\mu + g c_{\text{W}} Z_\mu) (T^+)^\dagger \bar{\sigma}^\mu T^+ + g W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\ &\quad + g W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - g W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\ &\quad - g W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - (e A_\mu + g c_{\text{W}} Z_\mu) (T^-)^\dagger \bar{\sigma}^\mu T^- \end{aligned}$$

$$\begin{aligned}
X_i^0 &= \begin{pmatrix} \chi_{iL}^0 \\ (\chi_{iR}^0)^\dagger \end{pmatrix}, \quad \chi_L^0 = \chi_R^0 = \mathcal{N}^\dagger \psi_L^0 = \mathcal{N}^\dagger \psi_R^0 = (\chi_1^0 \quad \chi_2^0 \quad \chi_3^0)^\top, \quad \bar{X}_i^0 = (\chi_{iR}^0 \quad (\chi_{iL}^0)^\dagger) \\
X_i^+ &= \begin{pmatrix} \chi_{iL}^+ \\ (\chi_{iR}^-)^\dagger \end{pmatrix}, \quad \chi_L^+ = \mathcal{C}_L^\dagger \psi_L^+ = (\chi_1^+ \quad \chi_2^+)^\top, \quad \chi_R^- = \mathcal{C}_R^\dagger \psi_R^- = (\chi_1^- \quad \chi_2^-)^\top, \quad \bar{X}_i^+ = (\chi_{iR}^- \quad (\chi_{iL}^+)^\dagger) \\
\Psi_i^0 &= \begin{pmatrix} \psi_{iL}^0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix}, \quad \psi_L^0 = \psi_R^0 = \mathcal{N} \chi_L^0 = \mathcal{N} \chi_R^0 = (T^0 \quad D_1^0 \quad D_2^0)^\top, \quad \bar{\Psi}_i^0 = (\psi_{iR}^0 \quad (\psi_{iL}^0)^\dagger) \\
\Psi_i^+ &= \begin{pmatrix} \psi_{iL}^+ \\ (\psi_{iR}^-)^\dagger \end{pmatrix}, \quad \psi_L^+ = \mathcal{C}_L \chi_L^+ = (T^+ \quad D_2^+)^\top, \quad \psi_R^- = \mathcal{C}_R \chi_R^- = (T^- \quad D_1^-)^\top, \quad \bar{\Psi}_i^+ = (\psi_{iR}^- \quad (\psi_{iL}^+)^\dagger)
\end{aligned}$$

$$\begin{aligned}
\Psi_{iL}^0 &= \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{N} \chi_L^0)_i \\ 0 \end{pmatrix} = \mathcal{N}_{ij} X_{jL}^0, \quad \Psi_{iR}^0 = \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{N} \chi_R^0)_i^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* X_{jR}^0 \\
\bar{\Psi}_{iL}^0 &= (0 \quad (\psi_{iL}^0)^\dagger) = \mathcal{N}_{ij}^* \bar{X}_{jL}^0, \quad \bar{\Psi}_{iR}^0 = (\psi_{iR}^0 \quad 0) = \mathcal{N}_{ij} \bar{X}_{jR}^0 \\
\Psi_{iL}^+ &= \begin{pmatrix} \psi_{iL}^+ \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{C}_L \chi_L^+)_i \\ 0 \end{pmatrix} = (\mathcal{C}_L)_{ij} X_{jL}^+, \quad \Psi_{iR}^+ = \begin{pmatrix} 0 \\ (\psi_{iR}^-)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{C}_R \chi_R^-)_i^\dagger \end{pmatrix} = (\mathcal{C}_R)_{ij}^* X_{jR}^+ \\
\bar{\Psi}_{iL}^+ &= (0 \quad (\psi_{iL}^+)^\dagger) = (\mathcal{C}_L)_{ij}^* \bar{X}_{jL}^+, \quad \bar{\Psi}_{iR}^+ = (\psi_{iR}^- \quad 0) = (\mathcal{C}_R)_{ij} \bar{X}_{jR}^+
\end{aligned}$$

$$\begin{aligned}
\bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 &= (\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0, \quad \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \psi_{iR}^0 \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^0 = -(\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0 \\
\mathcal{L}_{Z\Psi_i^0\Psi_i^0} &= \frac{1}{2} a_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 + \frac{1}{2} b_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \frac{1}{2} (a_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0) \\
a_{Z\Psi_1^0\Psi_1^0} &= b_{Z\Psi_1^0\Psi_1^0} = 0, \quad a_{Z\Psi_2^0\Psi_2^0} = -b_{Z\Psi_2^0\Psi_2^0} = \frac{g}{2c_W}, \quad a_{Z\Psi_3^0\Psi_3^0} = -b_{Z\Psi_3^0\Psi_3^0} = -\frac{g}{2c_W} \\
a_{ZX_i^0 X_j^0} &= a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}, \quad b_{ZX_i^0 X_j^0} = b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*
\end{aligned}$$

$$\begin{aligned}
\bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{iL}^0 &= (\psi_{iL}^+)^\dagger \bar{\sigma}^\mu \psi_{iL}^0, \quad \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{iR}^0 = \psi_{iR}^- \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^- \\
\mathcal{L}_{W\Psi_i^+\Psi_j^0} &= a_{W\Psi_i^+\Psi_j^0} (W_\mu^+ \bar{\Psi}_i^+ \gamma^\mu \Psi_{jL}^0 + \text{h.c.}) + b_{W\Psi_i^+\Psi_j^0} (W_\mu^+ \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{jR}^0 + \text{h.c.}) \\
&= a_{W\Psi_k^+\Psi_l^0} [(\mathcal{C}_L)_{ki}^* \mathcal{N}_{lj} W_\mu^+ \bar{X}_{iL}^+ \gamma^\mu X_{jL}^0 + \text{h.c.}] + b_{W\Psi_k^+\Psi_l^0} [(\mathcal{C}_R)_{ki} \mathcal{N}_{lj}^* W_\mu^+ \bar{X}_{iR}^+ \gamma^\mu X_{jR}^0 + \text{h.c.}] \\
&= a_{WX_i^+ X_j^0} W_\mu^+ \bar{X}_{iL}^+ \gamma^\mu X_{jL}^0 + a_{WX_i^+ X_j^0}^* W_\mu^- \bar{X}_{jL}^0 \gamma^\mu X_{iL}^+ + b_{WX_i^+ X_j^0} W_\mu^+ \bar{X}_{iR}^+ \gamma^\mu X_{jR}^0 + b_{WX_i^+ X_j^0}^* W_\mu^- \bar{X}_{jR}^0 \gamma^\mu X_{iR}^+ \\
a_{W\Psi_1^+\Psi_1^0} &= b_{W\Psi_1^+\Psi_1^0} = g, \quad b_{W\Psi_2^+\Psi_2^0} = -\frac{g}{\sqrt{2}}, \quad a_{W\Psi_2^+\Psi_3^0} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0 \\
a_{WX_i^+ X_j^0} &= a_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_L)_{ki}^* \mathcal{N}_{lj}, \quad b_{WX_i^+ X_j^0} = b_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_R)_{ki} \mathcal{N}_{lj}^*
\end{aligned}$$

$$\bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{iL}^+ = (\Psi_{iL}^+)^{\dagger} \bar{\sigma}^\mu \Psi_{iL}^+, \quad \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{iR}^+ = \Psi_{iR}^- \sigma^\mu (\Psi_{iR}^-)^{\dagger} = -(\Psi_{iR}^-)^{\dagger} \bar{\sigma}^\mu \Psi_{iR}^-$$

$$\begin{aligned} \mathcal{L}_{A\Psi_i^+\Psi_i^+} &= a_{A\Psi_i^+\Psi_i^+} A_\mu \bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{iL}^+ + b_{A\Psi_i^+\Psi_i^+} A_\mu \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{iR}^+ \\ &= a_{A\Psi_k^+\Psi_k^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_L)_{kj} A_\mu \bar{X}_{iL}^+ \gamma^\mu X_{jL}^+ + b_{A\Psi_k^+\Psi_k^+} (\mathcal{C}_R)_{ki}^* (\mathcal{C}_R)_{kj} A_\mu \bar{X}_{iR}^+ \gamma^\mu X_{jR}^+ \\ &= a_{AX_i^+X_j^+} A_\mu \bar{X}_{iL}^+ \gamma^\mu X_{jL}^+ + b_{AX_i^+X_j^+} A_\mu \bar{X}_{iR}^+ \gamma^\mu X_{jR}^+ \\ a_{A\Psi_1^+\Psi_1^+} &= b_{A\Psi_1^+\Psi_1^+} = a_{A\Psi_2^+\Psi_2^+} = b_{A\Psi_2^+\Psi_2^+} = e \\ a_{AX_i^+X_j^+} &= a_{A\Psi_k^+\Psi_k^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_L)_{kj} = e\delta_{ij}, \quad b_{AX_i^+X_j^+} = b_{A\Psi_k^+\Psi_k^+} (\mathcal{C}_R)_{ki}^* (\mathcal{C}_R)_{kj} = e\delta_{ij} \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{Z\Psi_i^+\Psi_i^+} &= a_{Z\Psi_i^+\Psi_i^+} Z_\mu \bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{iL}^+ + b_{Z\Psi_i^+\Psi_i^+} Z_\mu \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{iR}^+ \\ &= a_{Z\Psi_k^+\Psi_k^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_L)_{kj} Z_\mu \bar{X}_{iL}^+ \gamma^\mu X_{jL}^+ + b_{Z\Psi_k^+\Psi_k^+} (\mathcal{C}_R)_{ki}^* (\mathcal{C}_R)_{kj} Z_\mu \bar{X}_{iR}^+ \gamma^\mu X_{jR}^+ \\ &= a_{ZX_i^+X_j^+} Z_\mu \bar{X}_{iL}^+ \gamma^\mu X_{jL}^+ + b_{ZX_i^+X_j^+} Z_\mu \bar{X}_{iR}^+ \gamma^\mu X_{jR}^+ \end{aligned}$$

$$a_{Z\Psi_1^+\Psi_1^+} = b_{Z\Psi_1^+\Psi_1^+} = g c_W, \quad a_{Z\Psi_2^+\Psi_2^+} = b_{Z\Psi_2^+\Psi_2^+} = -\frac{g}{2c_W} (s_W^2 - c_W^2)$$

$$a_{ZX_i^+X_j^+} = a_{Z\Psi_k^+\Psi_k^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_L)_{kj}, \quad b_{ZX_i^+X_j^+} = b_{Z\Psi_k^+\Psi_k^+} (\mathcal{C}_R)_{ki}^* (\mathcal{C}_R)_{kj}$$

$$\bar{\Psi}_{iR}^0 \Psi_{jL}^0 = \psi_{iR}^0 \psi_{jL}^0, \quad \bar{\Psi}_{iL}^0 \Psi_{jR}^0 = (\psi_{iL}^0)^{\dagger} (\psi_{jR}^0)^{\dagger}$$

$$\begin{aligned} \mathcal{L}_{h\Psi_i^0\Psi_j^0} &= \frac{1}{2} a_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iR}^0 \Psi_{jL}^0 + \frac{1}{2} b_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iL}^0 \Psi_{jR}^0 \\ &= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj} h \bar{X}_{iR}^0 X_{jL}^0 + \frac{1}{2} b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^* h \bar{X}_{iL}^0 X_{jR}^0 = \frac{1}{2} (a_{hX_i^0X_j^0} h \bar{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0X_j^0} h \bar{X}_{iL}^0 X_{jR}^0) \\ a_{h\Psi_1^0\Psi_2^0} &= b_{h\Psi_1^0\Psi_2^0} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi_2^0\Psi_1^0} = b_{h\Psi_2^0\Psi_1^0}, \quad a_{h\Psi_1^0\Psi_3^0} = b_{h\Psi_1^0\Psi_3^0} = \frac{y_2}{\sqrt{2}} = a_{h\Psi_3^0\Psi_1^0} = b_{h\Psi_3^0\Psi_1^0}, \quad \text{others} = 0 \\ a_{hX_i^0X_j^0} &= a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj}, \quad b_{hX_i^0X_j^0} = b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^* \end{aligned}$$

$$\bar{\Psi}_{iR}^+ \Psi_{jL}^+ = (\psi_{iR}^- \quad 0) \begin{pmatrix} \psi_{jL}^+ \\ 0 \end{pmatrix} = \psi_{iR}^- \psi_{jL}^+, \quad \bar{\Psi}_{iL}^+ \Psi_{jR}^+ = (0 \quad (\psi_{iL}^+)^{\dagger}) \begin{pmatrix} 0 \\ (\psi_{jR}^-)^{\dagger} \end{pmatrix} = (\psi_{iL}^+)^{\dagger} (\psi_{jR}^-)^{\dagger}$$

$$\begin{aligned} \mathcal{L}_{h\Psi_i^+\Psi_j^+} &= a_{h\Psi_i^+\Psi_j^+} h \bar{\Psi}_{iR}^+ \Psi_{jL}^+ + b_{h\Psi_i^+\Psi_j^+} h \bar{\Psi}_{iL}^+ \Psi_{jR}^+ \\ &= a_{h\Psi_k^+\Psi_l^+} (\mathcal{C}_R)_{ki} (\mathcal{C}_L)_{lj} h \bar{X}_{iR}^+ X_{jL}^+ + b_{h\Psi_k^+\Psi_l^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_R)_{lj}^* h \bar{X}_{iL}^+ X_{jR}^+ = a_{hX_i^+X_j^+} h \bar{X}_{iR}^+ X_{jL}^+ + b_{hX_i^+X_j^+} h \bar{X}_{iL}^+ X_{jR}^+ \\ a_{h\Psi_1^+\Psi_2^+} &= b_{h\Psi_2^+\Psi_1^+} = y_2, \quad a_{h\Psi_2^+\Psi_1^+} = b_{h\Psi_1^+\Psi_2^+} = y_1, \quad \text{others} = 0 \\ a_{hX_i^+X_j^+} &= a_{h\Psi_k^+\Psi_l^+} (\mathcal{C}_R)_{ki} (\mathcal{C}_L)_{lj}, \quad b_{hX_i^+X_j^+} = b_{h\Psi_k^+\Psi_l^+} (\mathcal{C}_L)_{ki}^* (\mathcal{C}_R)_{lj}^* \end{aligned}$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$\begin{aligned}
\mathcal{L}_{hX_1^0 X_1^0} &= \frac{1}{2} a_{h\Psi_k^0 \Psi_l^0} \text{Re}(\mathcal{N}_{k1} \mathcal{N}_{l1}) h \bar{X}_1^0 X_1^0 = [a_{h\Psi_1^0 \Psi_2^0} \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + a_{h\Psi_1^0 \Psi_3^0} \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})] h \bar{X}_1^0 X_1^0 \\
&= \frac{1}{\sqrt{2}} [-y_1 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + y_2 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})] h \bar{X}_1^0 X_1^0 \\
&\equiv \frac{1}{2} g_{hX_1^0 X_1^0} h \bar{X}_1^0 X_1^0 \\
g_{hX_1^0 X_1^0} &= \frac{1}{2} (a_{hX_1^0 X_1^0} + b_{hX_1^0 X_1^0}) = \sqrt{2} [-y_1 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{21}) + y_2 \text{Re}(\mathcal{N}_{11} \mathcal{N}_{31})] \\
G_{S,q} &= -\frac{g_{hX_1^0 X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0 X_1^0} m_N}{2vm_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right), \quad \sigma_{\chi N}^{\text{SI}} = \frac{4}{\pi} \mu_{\chi N}^2 G_{S,N}^2
\end{aligned}$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$\begin{aligned}
\mathcal{L}_{ZX_1^0 X_1^0} &= -\frac{1}{2} a_{Z\Psi_k^0 \Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0 X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \\
g_{ZX_1^0 X_1^0} &= \frac{1}{2} (b_{ZX_1^0 X_1^0} - a_{ZX_1^0 X_1^0}) = -a_{Z\Psi_k^0 \Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W} (|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2) \\
G_{A,q} &= \frac{gg_A^q g_{ZX_1^0 X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2} \\
G_{A,N} &= \sum_{q=u,d,s} G_{A,q} \Delta_q^N, \quad \sigma_{\chi N}^{\text{SD}} = \frac{12}{\pi} \mu_{\chi N}^2 G_{A,N}^2
\end{aligned}$$

CP-even real scalar singlet S and complex scalar doublet Φ :

$$S \in (\mathbf{1}, 0), \quad \Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2} \right)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)\partial^\mu S + (D_\mu \Phi)^\dagger D^\mu \Phi - V(S, \Phi)$$

$$V(S, \Phi) = \frac{1}{2}m_S^2 S^2 + m_D^2 |\Phi|^2 + (\kappa S \Phi^\dagger H + h.c.) + \frac{1}{2}\lambda_{Sh} S^2 |H|^2 + \lambda_1 |H|^2 |\Phi|^2 \\ + (\lambda_2 \Phi^\dagger H \Phi^\dagger H + h.c.) + \lambda_3 |\Phi^\dagger H|^2 + (\text{irrelevant terms})$$

$$|H|^2 \rightarrow \frac{1}{2}(v+h)^2, \quad |\Phi|^2 = \phi^+ \phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2, \quad \kappa S \Phi^\dagger H + h.c. \rightarrow \kappa(v+h)S\phi^0$$

$$\frac{1}{2}\lambda_{Sh} S^2 |H|^2 \rightarrow \frac{\lambda_{Sh}}{4} S^2 (v+h)^2, \quad \lambda_1 |H|^2 |\Phi|^2 = \frac{\lambda_1}{2}(v+h)^2 \left[\phi^+ \phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 \right]$$

$$\lambda_2 \Phi^\dagger H \Phi^\dagger H + h.c. \rightarrow \frac{\lambda_2}{2}(v+h)^2 [(\phi^0)^2 - (a^0)^2], \quad \lambda_3 |\Phi^\dagger H|^2 = \frac{\lambda_3}{4}(v+h)^2 [(\phi^0)^2 + (a^0)^2]$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \begin{pmatrix} S & \phi^0 \end{pmatrix} M_0^2 \begin{pmatrix} S \\ \phi^0 \end{pmatrix} - \frac{1}{2} m_a^2 (a^0)^2 - m_c^2 |\phi^+|^2$$

$$M_0^2 = \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{Sh}v^2 & \kappa v \\ \kappa v & m_D^2 + \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)v^2 \end{pmatrix}, \quad m_a^2 = m_D^2 + \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)v^2, \quad m_c^2 = m_D^2 + \frac{1}{2}\lambda_1 v^2$$

$$U^T M_0^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix} = U \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} c_\theta X_1 - s_\theta X_2 \\ s_\theta X_1 + c_\theta X_2 \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2} \left\{ (M_0^2)_{11} + (M_0^2)_{22} \mp \sqrt{[(M_0^2)_{11} - (M_0^2)_{22}]^2 + 4(M_0^2)_{12}^2} \right\}$$

$$\text{DM candidate} = \begin{cases} X_1, & m_1^2 < m_a^2, m_c^2 \\ a^0, & m_a^2 < m_1^2, m_c^2 \end{cases}$$

$$\mathcal{L}_{hXX, haa} = -\kappa h S \phi^0 - \frac{\lambda_{Sh} v}{2} h S^2 - \frac{\lambda_1}{2} v h [(\phi^0)^2 + (a^0)^2] - \lambda_2 v h [(\phi^0)^2 - (a^0)^2] - \frac{\lambda_3}{2} v h [(\phi^0)^2 + (a^0)^2]$$

$$= -\kappa h (c_\theta X_1 - s_\theta X_2)(s_\theta X_1 + c_\theta X_2) - \frac{\lambda_{Sh} v}{2} h (c_\theta X_1 - s_\theta X_2)^2$$

$$- \frac{1}{2} v h (\lambda_1 + 2\lambda_2 + \lambda_3)(s_\theta X_1 + c_\theta X_2)^2 - \frac{1}{2} (\lambda_1 - 2\lambda_2 + \lambda_3) v h (a^0)^2$$

$$= \frac{1}{2} \lambda_{hX_1 X_1} v h X_1^2 + \frac{1}{2} \lambda_{hX_2 X_2} v h X_2^2 + \lambda_{hX_1 X_2} v h X_1 X_2 + \frac{1}{2} \lambda_{haa} v h (a^0)^2$$

$$\lambda_{hX_1 X_1} = -2 \frac{\kappa}{v} s_\theta c_\theta - \lambda_{Sh} c_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3) s_\theta^2, \quad \lambda_{hX_2 X_2} = 2 \frac{\kappa}{v} s_\theta c_\theta - \lambda_{Sh} s_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3) c_\theta^2$$

$$\lambda_{hX_1 X_2} = -\frac{\kappa}{v} (c_\theta^2 - s_\theta^2) - (-\lambda_{Sh} + \lambda_1 + 2\lambda_2 + \lambda_3) s_\theta c_\theta, \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

Direct detection

Effective operators: $\mathcal{L}_{S,q} = \frac{1}{2} \sum_q F_{S,q} \chi^2 \bar{q} q$, $\mathcal{L}_{S,N} = \frac{1}{2} \sum_{N=p,n} F_{S,N} \chi^2 \bar{N} N$, $\chi = X_1, a^0$

$$F_{S,N} = m_N \left(\sum_{q=u,d,s} \frac{F_{S,q}}{m_q} f_q^N + \sum_{q=c,b,t} \frac{F_{S,q}}{m_q} f_Q^N \right), \quad f_Q^N = \frac{2}{27} \left(1 - \sum_{q=u,d,s} f_q^N \right)$$

$$\chi = X_1 \Rightarrow F_{S,q} = \frac{1}{i} \lambda_{hX_1 X_1} v \frac{i}{-m_h^2} \left(-i \frac{m_q}{v} \right) = -\frac{\lambda_{hX_1 X_1} m_q}{m_h^2}, \quad F_{S,N} = -\frac{\lambda_{hX_1 X_1} m_N}{m_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right)$$

$$\chi = a^0 \Rightarrow F_{S,q} = -\frac{\lambda_{haa} m_q}{m_h^2}, \quad F_{S,N} = -\frac{\lambda_{haa} m_N}{m_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right)$$

$$\sigma_{\chi N}^{\text{SI}} = \frac{m_N^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}$$

Gauge interactions

$$D^\mu \Phi = \left(\partial_\mu - \frac{1}{2} i g' B_\mu - \frac{1}{2} i g W_\mu^a \sigma^a \right) \Phi$$

$$\frac{1}{2} (g' B_\mu + g W_\mu^a \sigma^a) \Phi = \begin{pmatrix} eA_\mu - \frac{g}{2c_W} (s_W^2 - c_W^2) Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{g}{2c_W} Z_\mu \end{pmatrix} \Phi = \begin{pmatrix} \left[eA_\mu + \frac{g}{2c_W} (c_W^2 - s_W^2) Z_\mu \right] \phi^+ + \frac{g}{2} W_\mu^+ (\phi^0 + ia^0) \\ \frac{g}{\sqrt{2}} W_\mu^- \phi^+ - \frac{g}{2\sqrt{2}c_W} Z_\mu (\phi^0 + ia^0) \end{pmatrix}$$

$$\varphi_1 \overleftrightarrow{\partial}^\mu \varphi_2 \equiv \varphi_1 \partial^\mu \varphi_2 - (\partial^\mu \varphi_1) \varphi_2 = -\varphi_2 \overleftrightarrow{\partial}^\mu \varphi_1, \quad (\varphi_1 i \overleftrightarrow{\partial}^\mu \varphi_2)^\dagger = \varphi_1 i \overleftrightarrow{\partial}^\mu \varphi_2$$

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\frac{i}{2} (\partial^\mu \Phi^\dagger) (g' B_\mu + g W_\mu^a \sigma^a) \Phi + \frac{i}{2} [(g' B_\mu + g W_\mu^a \sigma^a) \Phi]^\dagger \partial^\mu \Phi + \left| \frac{1}{2} (g' B_\mu + g W_\mu^a \sigma^a) \Phi \right|^2 \\ &= -i \partial^\mu \phi^- \left\{ \left[eA_\mu + \frac{g}{2c_W} (c_W^2 - s_W^2) Z_\mu \right] \phi^+ + \frac{g}{2} W_\mu^+ (\phi^0 + ia^0) \right\} - \frac{i}{\sqrt{2}} (\partial^\mu \phi^0 - i \partial^\mu a^0) \left[\frac{g}{\sqrt{2}} W_\mu^- \phi^+ - \frac{g}{2\sqrt{2}c_W} Z_\mu (\phi^0 + ia^0) \right] \\ &\quad + i \partial^\mu \phi^+ \left\{ \left[eA_\mu + \frac{g}{2c_W} (c_W^2 - s_W^2) Z_\mu \right] \phi^- + \frac{g}{2} W_\mu^- (\phi^0 - ia^0) \right\} + \frac{i}{\sqrt{2}} (\partial^\mu \phi^0 + i \partial^\mu a^0) \left[\frac{g}{\sqrt{2}} W_\mu^+ \phi^- - \frac{g}{2\sqrt{2}c_W} Z_\mu (\phi^0 - ia^0) \right] \\ &\quad + \left| \left[eA_\mu + \frac{g}{2c_W} (c_W^2 - s_W^2) Z_\mu \right] \phi^+ + \frac{g}{2} W_\mu^+ (\phi^0 + ia^0) \right|^2 + \left| \frac{g}{\sqrt{2}} W_\mu^- \phi^+ - \frac{g}{2\sqrt{2}c_W} Z_\mu (\phi^0 + ia^0) \right|^2 \\ &= \frac{g}{2} [W_\mu^+ \phi^- i \overleftrightarrow{\partial}^\mu (\phi^0 + ia^0) + h.c.] + eA_\mu \phi^- i \overleftrightarrow{\partial}^\mu \phi^+ + \frac{g}{2c_W} Z_\mu [ia^0 i \overleftrightarrow{\partial}^\mu \phi^0 + (c_W^2 - s_W^2) \phi^- i \overleftrightarrow{\partial}^\mu \phi^+] \\ &\quad + \frac{g^2}{4} W_\mu^+ W^{-\mu} [2\phi^+ \phi^- + (\phi^0)^2 + (a^0)^2] + e^2 A_\mu A^\mu \phi^+ \phi^- + \frac{g^2}{4c_W^2} Z_\mu Z^\mu \left[(c_W^2 - s_W^2)^2 \phi^+ \phi^- + \frac{1}{2} (\phi^0)^2 + \frac{1}{2} (a^0)^2 \right] \\ &\quad + \left[\frac{eg}{2} W_\mu^+ A^\mu \phi^- (\phi^0 + ia^0) - \frac{g^2 s_W^2}{2c_W} W_\mu^+ Z^\mu \phi^- (\phi^0 + ia^0) + h.c. \right] + \frac{eg}{c_W} (c_W^2 - s_W^2) A_\mu Z^\mu \phi^+ \phi^- \end{aligned}$$

Custodial symmetry: $T = U = 0$

1) $\lambda_3 = 2\lambda_2$

Bidoublet formalism: $\mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix}$

$$\tilde{H}^\dagger \tilde{H} = \begin{pmatrix} H^0 & -H^+ \end{pmatrix} \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = H^0 H^{0*} + H^+ H^- = H^\dagger H$$

$$\text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = \text{tr} \left(\begin{pmatrix} \tilde{H}^\dagger \\ H^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2|H|^2, \quad \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2|\Phi|^2$$

$$\tilde{\Phi}^\dagger \tilde{H} = \left(\frac{1}{\sqrt{2}}(\phi^0 + ia^0) \quad -\phi^+ \right) \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)H^{0*} + \phi^+ H^- = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} = H^\dagger \Phi$$

$$\text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \text{tr} \left(\begin{pmatrix} \tilde{\Phi}^\dagger \\ \Phi^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{\Phi}^\dagger \tilde{H} + \Phi^\dagger H = \Phi^\dagger H + H^\dagger \Phi$$

$$[\text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2 = (\Phi^\dagger H + H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi + 2|\Phi^\dagger H|^2$$

$$V(S, \Phi) \supset \frac{1}{2} m_D^2 \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \kappa S \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) + \frac{1}{4} \lambda_{sh} S^2 \text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4} \lambda_1 \text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 [\text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2$$

$$\left[\begin{array}{l} \text{Tensor formalism: } \mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \mathcal{F}_i^A = \begin{pmatrix} \Phi_i^\dagger \\ \Phi_i \end{pmatrix}, \quad A \text{ is an } \text{SU}(2)_R \text{ indice} \\ \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B = \varepsilon^{ij} (\varepsilon_{12} \mathcal{H}_i^1 \mathcal{H}_j^2 + \varepsilon_{21} \mathcal{H}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij} (-H_i^\dagger H_j + H_i H_j^\dagger) \\ = -H_i^\dagger H^i - H^j H_j^\dagger = -2|H|^2 \\ \text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = -\varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B, \quad \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = -\varepsilon_{AB} \varepsilon^{ij} \mathcal{F}_i^A \mathcal{F}_j^B \\ \varepsilon_{AB} \varepsilon^{ij} \mathcal{F}_i^A \mathcal{H}_j^B = \varepsilon^{ij} (\varepsilon_{12} \mathcal{F}_i^1 \mathcal{H}_j^2 + \varepsilon_{21} \mathcal{F}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij} (-\Phi_i^\dagger H_j + \Phi_i H_j^\dagger) \\ = -\Phi_i^\dagger H^i - \Phi^j H_j^\dagger = -\Phi^\dagger H - H^\dagger \Phi = -\text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) \end{array} \right]$$

2) $\lambda_3 = -2\lambda_2$ and $\kappa = 0$

$$\mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} -\tilde{\Phi} & \Phi \end{pmatrix}$$

$$\text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = 2|H|^2, \quad \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2|\Phi|^2, \quad \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \Phi^\dagger H - H^\dagger \Phi$$

$$\text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2 = (\Phi^\dagger H - H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi - 2|\Phi^\dagger H|^2$$

$$V(S, \Phi) \supset \frac{1}{2} m_D^2 \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \frac{1}{4} \lambda_{sh} S^2 \text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4} \lambda_1 \text{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 \text{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2$$

$\lambda_2 = \lambda_3 = 0$ and $\kappa = 0$: $S = T = U = 0$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \left(m_S^2 + \frac{1}{2} \lambda_{sh} v^2 \right) S^2 - \left(m_D^2 + \frac{1}{2} \lambda_1 v^2 \right) \left[\frac{1}{2} (\phi^0)^2 + \frac{1}{2} (a^0)^2 + |\phi^+|^2 \right]$$

\Rightarrow No mixing, components of Φ have exactly degenerate masses

$\Rightarrow S = T = U = 0$

$\kappa = 0$ and $m_S \rightarrow \infty$: inert Higgs doublet model [Deshpande & Ma, PRD **18**, 2574 (1978)]

$$s_\theta = 0, \quad c_\theta = 1$$

$$m_{\phi^0}^2 = m_D^2 + \frac{1}{2} (\lambda_1 + 2\lambda_2 + \lambda_3) v^2, \quad m_a^2 = m_D^2 + \frac{1}{2} (\lambda_1 - 2\lambda_2 + \lambda_3) v^2, \quad m_C^2 = m_D^2 + \frac{1}{2} \lambda_1 v^2$$

$$\lambda_3 < 2|\lambda_2|, \quad \text{DM candidate} = \begin{cases} \phi^0, & \lambda_2 < 0 \\ a^0, & \lambda_2 > 0 \end{cases}$$

$$\lambda_{h\phi^0\phi^0} = -(\lambda_1 + 2\lambda_2 + \lambda_3), \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

CP-even real scalar singlet S and complex scalar doublet Φ :

$$S \in (\mathbf{1}, 0), \quad \Delta = \begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} \in (\mathbf{3}, 0), \quad \Delta^0 = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)$$

$$\Delta^+ = \Delta_2^1, \quad \Delta^- = \Delta_1^2, \quad \Delta^0 = \sqrt{2}\Delta_2^2 = -\sqrt{2}\Delta_1^1$$

$$\sum_i \Delta_i^i = 0, \quad \Delta_j^i = \varepsilon^{ik} \varepsilon_{jl} \Delta_k^l, \quad (\Delta^\dagger)_i^j = (\Delta_j^i)^\dagger$$

$$(\Delta^+)^* = (\Delta_1^\dagger)^2, \quad (\Delta^-)^* = (\Delta_2^\dagger)^1, \quad (\Delta^0)^* = \sqrt{2}(\Delta^\dagger)_2^2 = -\sqrt{2}(\Delta^\dagger)_1^1$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)\partial^\mu S + (D_\mu \Delta)^\dagger D^\mu \Delta - V(S, \Delta)$$

$$V(S, \Delta) = \frac{1}{2}m_S^2 S^2 + m_\Delta^2 |\Delta|^2 + \frac{1}{2}\lambda_{sh} S^2 |H|^2 + \lambda_0 |H|^2 |\Delta|^2 + \lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k + \lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k \\ + (-\lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k - \lambda_3' |H|^2 \Delta_j^i \Delta_i^j - \lambda_4 S H_i^\dagger \Delta_j^i H^j + h.c.) + (\text{irrelevant terms})$$

$$|\Delta|^2 = (\Delta^\dagger)_i^j \Delta_j^i = (\Delta^+)^* \Delta^+ + \Delta^- (\Delta^-)^* + (\Delta^0)^* \Delta^0, \quad (\Delta^0)^* \Delta^0 = \frac{1}{2}[(\phi^0)^2 + (a^0)^2], \quad (\Delta^0)^2 + h.c. = (\phi^0)^2 - (a^0)^2$$

$$H^1 \rightarrow 0, \quad H^2 \rightarrow \frac{v+h}{\sqrt{2}}, \quad H_1^\dagger \rightarrow 0, \quad H_2^\dagger \rightarrow \frac{v+h}{\sqrt{2}}$$

$$\lambda_0 |H|^2 |\Delta|^2 \rightarrow \frac{\lambda_0}{2}(v+h)^2 [(\Delta^+)^* \Delta^+ + \Delta^- (\Delta^-)^* + (\Delta^0)^* \Delta^0]$$

$$\lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k \rightarrow \lambda_1 H_2^\dagger \Delta_j^2 (\Delta^\dagger)_2^j H^2 = \frac{\lambda_1}{2}(v+h)^2 \left[\Delta^- (\Delta^-)^* + \frac{1}{2}(\Delta^0)^* \Delta^0 \right]$$

$$\lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k \rightarrow \lambda_2 H_2^\dagger (\Delta^\dagger)_2^2 \Delta_2^2 H^2 = \frac{\lambda_2}{2}(v+h)^2 \left[(\Delta^+)^* \Delta^+ + \frac{1}{2}(\Delta^0)^* \Delta^0 \right]$$

$$-\lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k \rightarrow -\lambda_3 H_2^\dagger \Delta_j^2 \Delta_2^j H^2 = -\frac{\lambda_3}{2}(v+h)^2 \left[\Delta^- \Delta^+ + \frac{1}{2}(\Delta^0)^2 \right]$$

$$-\lambda_3' |H|^2 \Delta_j^i \Delta_i^j \rightarrow -\frac{\lambda_3'}{2}(v+h)^2 [2\Delta^- \Delta^+ + (\Delta^0)^2] \propto \lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k \quad [\lambda_3' \text{ can be absorbed into } \lambda_3 \text{ in the unitary gauge}]$$

$$-\lambda_4 S H_i^\dagger \Delta_j^i H^j \rightarrow -\lambda_4 S H_2^\dagger \Delta_2^2 H^2 = -\frac{\lambda_4}{2\sqrt{2}}(v+h)^2 S \Delta^0 = -\frac{\lambda_4}{4}(v+h)^2 S(\phi^0 + ia^0)$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}m_a^2 (a^0)^2 - \frac{1}{2}(S \quad \phi^0) M_0^2 \begin{pmatrix} S \\ \phi^0 \end{pmatrix} - ((\Delta^+)^* \quad \Delta^-) M_C^2 \begin{pmatrix} \Delta^+ \\ (\Delta^-)^* \end{pmatrix}$$

$$m_a^2 = m_\Delta^2 + \frac{1}{4}(2\lambda_0 + \lambda_1 + \lambda_2 + 2\lambda_3 + 4\lambda_3')v^2$$

$$M_0^2 = \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{sh}v^2 & -\frac{1}{2}\lambda_4 v^2 \\ -\frac{1}{2}\lambda_4 v^2 & m_\Delta^2 + \frac{1}{4}(2\lambda_0 + \lambda_1 + \lambda_2 - 2\lambda_3 - 4\lambda_3')v^2 \end{pmatrix}, \quad M_C^2 = \begin{pmatrix} m_\Delta^2 + \frac{1}{2}(\lambda_0 + \lambda_2)v^2 & -\frac{1}{2}(\lambda_3 + 2\lambda_3')v^2 \\ -\frac{1}{2}(\lambda_3 + 2\lambda_3')v^2 & m_\Delta^2 + \frac{1}{2}(\lambda_0 + \lambda_1)v^2 \end{pmatrix}$$

$$\lambda_\pm \equiv \lambda_1 \pm \lambda_2$$

$$\lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k + \lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k \rightarrow \frac{\lambda_+}{4}(v+h)^2 [(\Delta^+)^* \Delta^+ + \Delta^- (\Delta^-)^* + (\Delta^0)^* \Delta^0] + \frac{\lambda_-}{4}(v+h)^2 [\Delta^- (\Delta^-)^* - (\Delta^+)^* \Delta^+]$$

$$\Rightarrow \lambda_0 \text{ can be absorbed into } \lambda_+ \text{ in the unitary gauge}$$

$$m_a^2 \rightarrow m_\Delta^2 + \frac{1}{4}(\lambda_+ + 2\lambda_3)v^2, \quad M_0^2 \rightarrow \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{sh}v^2 & -\frac{1}{2}\lambda_4 v^2 \\ -\frac{1}{2}\lambda_4 v^2 & m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\lambda_3)v^2 \end{pmatrix}, \quad M_C^2 \rightarrow \begin{pmatrix} m_\Delta^2 + \frac{1}{4}(\lambda_+ - \lambda_-)v^2 & -\frac{1}{2}\lambda_3 v^2 \\ -\frac{1}{2}\lambda_3 v^2 & m_\Delta^2 + \frac{1}{4}(\lambda_+ + \lambda_-)v^2 \end{pmatrix}$$

$$U^{\text{T}}M_{\text{C}}^2U=\begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U=\begin{pmatrix} c_{\theta} & -s_{\theta} \\ s_{\theta} & c_{\theta} \end{pmatrix}, \quad \begin{pmatrix} \Delta^+ \\ (\Delta^-)^* \end{pmatrix}=U\begin{pmatrix} \Delta_1^+ \\ \Delta_2^+ \end{pmatrix}$$

$$m_{1,2}^2=\frac{1}{2}\Big\{(M_{\text{C}}^2)_{11}+(M_{\text{C}}^2)_{22}\mp\sqrt{[(M_{\text{C}}^2)_{11}-(M_{\text{C}}^2)_{22}]^2+4(M_{\text{C}}^2)_{12}^2}\Big\}=m_{\Delta}^2+\frac{v^2}{4}\Big(\lambda_+\mp\sqrt{\lambda_-^2+4\lambda_3^2}\Big)$$

$$V^{\text{T}}M_0^2V=\begin{pmatrix} \mu_1^2 & \\ & \mu_2^2 \end{pmatrix}, \quad U=\begin{pmatrix} c_{\alpha} & -s_{\alpha} \\ s_{\alpha} & c_{\alpha} \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix}=V\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}=\begin{pmatrix} c_{\alpha}X_1-s_{\alpha}X_2 \\ s_{\alpha}X_1+c_{\alpha}X_2 \end{pmatrix}$$

$$\mu_{1,2}^2=\frac{1}{2}\Big\{(M_0^2)_{11}+(M_0^2)_{22}\mp\sqrt{[(M_0^2)_{11}-(M_0^2)_{22}]^2+4(M_0^2)_{12}^2}\Big\}$$

$$\text{DM candidate}=\begin{cases} X_1, & \mu_1^2<m_a^2,m_1^2 \\ a^0, & m_a^2<\mu_1^2,m_C^2 \end{cases}$$

$$\begin{aligned}\mathcal{L}_{hXX,haa} &= -\frac{1}{2}\lambda_{Sh}vhS^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(\phi^0)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4vhS\phi^0 \\ &= -\frac{1}{2}\lambda_{Sh}vh(c_{\alpha}X_1 - s_{\alpha}X_2)^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(s_{\alpha}X_1 + c_{\alpha}X_2)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4vh(c_{\alpha}X_1 - s_{\alpha}X_2)(s_{\alpha}X_1 + c_{\alpha}X_2) \\ &= \frac{1}{2}\lambda_{hX_1X_1}vhX_1^2 + \frac{1}{2}\lambda_{hX_2X_2}vhX_2^2 + \lambda_{hX_1X_2}vhX_1X_2 + \frac{1}{2}\lambda_{haa}vh(a^0)^2\end{aligned}$$

$$\lambda_{hX_1X_1}=-\left[\lambda_{Sh}c_{\alpha}^2+\frac{1}{2}(\lambda_+-2\lambda_3)s_{\alpha}^2-2\lambda_4s_{\alpha}c_{\alpha}\right], \quad \lambda_{hX_2X_2}=-\left[\lambda_{Sh}s_{\alpha}^2+\frac{1}{2}(\lambda_+-2\lambda_3)c_{\alpha}^2+2\lambda_4s_{\alpha}c_{\alpha}\right]$$

$$\lambda_{hX_1X_2}=\left[\lambda_{Sh}-\frac{1}{2}(\lambda_+-2\lambda_3)\right]s_{\alpha}c_{\alpha}+\lambda_4(c_{\alpha}^2-s_{\alpha}^2), \quad \lambda_{haa}=-\frac{1}{2}(\lambda_++2\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{\text{SI}}=\frac{m_N^2F_{S,N}^2}{4\pi(m_{\chi}+m_N)^2},\quad\left\{\begin{array}{l} \text{DM candidate}\;\;\chi=X_1\quad\Rightarrow\quad F_{S,N}=-\frac{\lambda_{hX_1X_1}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N+3f_Q^N\right) \\ \text{DM candidate}\;\;\chi=a^0\quad\Rightarrow\quad F_{S,N}=-\frac{\lambda_{haa}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N+3f_Q^N\right) \end{array}\right.$$

Gauge interactions

$$D^{\mu}\Delta=(\partial_{\mu}-igW_{\mu}^at_{\text{T}}^a)\Delta$$

$$t_{\text{T}}^1=\frac{1}{\sqrt{2}}\begin{pmatrix} & -1 \\ -1 & & 1 \\ & & 1 \end{pmatrix}, \quad t_{\text{T}}^2=\frac{1}{\sqrt{2}}\begin{pmatrix} & i \\ -i & & -i \\ & & i \end{pmatrix}, \quad t_{\text{T}}^3=\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$gW_{\mu}^at_{\text{T}}^a\Delta=\begin{pmatrix} gW_{\mu}^3 & -g(W_{\mu}^1-iW_{\mu}^2)/\sqrt{2} & 0 \\ -g(W_{\mu}^1+iW_{\mu}^2)/\sqrt{2} & 0 & g(W_{\mu}^1-iW_{\mu}^2)/\sqrt{2} \\ 0 & g(W_{\mu}^1+iW_{\mu}^2)/\sqrt{2} & -gW_{\mu}^3 \end{pmatrix}\begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix}=\begin{pmatrix} (eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^+-gW_{\mu}^+\Delta^0 \\ -gW_{\mu}^-\Delta^++gW_{\mu}^+\Delta^- \\ gW_{\mu}^-\Delta^0-(eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^- \end{pmatrix}$$

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -i(\partial^{\mu}\Delta^{\dagger})gW_{\mu}^at_{\text{T}}^a\Delta+i(gW_{\mu}^at_{\text{T}}^a\Phi)^{\dagger}\partial^{\mu}\Delta+|gW_{\mu}^at_{\text{T}}^a\Delta|^2 \\ &= -i\partial^{\mu}(\Delta^+)^*[(eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^+-gW_{\mu}^+\Delta^0]-i\partial^{\mu}(\Delta^0)^*(-gW_{\mu}^-\Delta^++gW_{\mu}^+\Delta^-)-i\partial^{\mu}(\Delta^-)^*[gW_{\mu}^-\Delta^0-(eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^-] \\ &\quad +i\partial^{\mu}\Delta^+[(eA_{\mu}+gc_{\text{W}}Z_{\mu})(\Delta^+)^*-gW_{\mu}^-(\Delta^0)^*]+i\partial^{\mu}\Delta^0[-gW_{\mu}^+(\Delta^+)^*+gW_{\mu}^-(\Delta^-)^*]+i\partial^{\mu}\Delta^-[gW_{\mu}^+(\Delta^0)^*-(eA_{\mu}+gc_{\text{W}}Z_{\mu})(\Delta^-)^*] \\ &\quad +|(eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^+-gW_{\mu}^+\Delta^0|^2+|-gW_{\mu}^-\Delta^++gW_{\mu}^+\Delta^-|^2+|gW_{\mu}^-\Delta^0-(eA_{\mu}+gc_{\text{W}}Z_{\mu})\Delta^-|^2 \\ &= \frac{g}{\sqrt{2}}[W_{\mu}^+(\phi^0+ia^0)i\overline{\partial^{\mu}}(\Delta^+)^*+W_{\mu}^+(\phi^0-ia^0)i\overline{\partial^{\mu}}\Delta^-+h.c.]+(eA_{\mu}+gc_{\text{W}}Z_{\mu})[(\Delta^+)^*i\overline{\partial^{\mu}}\Delta^++\Delta^-i\overline{\partial^{\mu}}(\Delta^-)^*] \\ &\quad +g^2W_{\mu}^+W^{-\mu}[|\Delta^+|^2+|\Delta^-|^2+(\phi^0)^2+(a^0)^2]-g^2[W_{\mu}^+W^{+\mu}(\Delta^+)^*\Delta^-+h.c.] \\ &\quad +(e^2A_{\mu}A^{\mu}+g^2c_{\text{W}}^2Z_{\mu}Z^{\mu}+2egc_{\text{W}}A_{\mu}Z^{\mu})(|\Delta^+|^2+|\Delta^-|^2) \\ &\quad -\frac{g}{\sqrt{2}}[W_{\mu}^+(\Delta^+)^*(\phi^0+ia^0)+W_{\mu}^+\Delta^-(\phi^0-ia^0)+h.c.](eA^{\mu}+gc_{\text{W}}Z^{\mu})\end{aligned}$$

Custodial symmetry limit $\lambda_- = \lambda_4 = 0$: $T = U = 0$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}$$

$$\varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B = \varepsilon_{12} \mathcal{H}_i^1 \mathcal{H}_j^2 + \varepsilon_{21} \mathcal{H}_i^2 \mathcal{H}_j^1 = -H_i^\dagger H_j + H_i H_j^\dagger$$

$$\varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B = \varepsilon^{ij} (-H_i^\dagger H_j + H_i H_j^\dagger) = -H_i^\dagger H^i - H^j H_j^\dagger = -2 |H|^2$$

Custodial symmetric potential:

$$V_{\text{cust}} \supset \varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B \left[-\frac{1}{4} \lambda_{Sh} S^2 \varepsilon^{ij} - \frac{1}{2} \lambda_0 |\Delta|^2 \varepsilon^{ij} + \frac{1}{2} \lambda'_3 \Delta_n^m \Delta_m^n \varepsilon^{ij} + (\lambda_a S \Delta_k^i \varepsilon^{kj} + h.c.) + \lambda_b \Delta_k^i (\Delta^\dagger)_l^j \varepsilon^{kl} + (\lambda_c \Delta_k^i \Delta_l^j \varepsilon^{kl} + h.c.) \right]$$

$$\left[\begin{array}{l} \text{For scalars } A^i \text{ and } B^i, \quad A^i B_i = A^i \varepsilon_{ij} B^j = -A^i \varepsilon_{ji} B^j = -B^j \varepsilon_{ji} A^i = -B^i A_i = -A_i B^i \\ \Rightarrow \quad \text{Raising an index and lowering the index contracted to it give a minus sign} \end{array} \right]$$

$$V_a = \varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B [\lambda_a S \Delta_k^i \varepsilon^{kj}] = \lambda_a S (-H_i^\dagger H_j + H_i H_j^\dagger) \Delta_k^i \varepsilon^{kj} = \lambda_a S (-H_i^\dagger \Delta_j^i H^j + H^{\dagger i} \Delta_i^j H_j) = \lambda_a S (-H_i^\dagger \Delta_j^i H^j + H_i^\dagger \Delta_j^i H^j) = 0$$

$$\begin{aligned} V_b &= \varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B [\lambda_b \Delta_k^i (\Delta^\dagger)_l^j \varepsilon^{kl}] = \lambda_b (-H_i^\dagger H_j + H_i H_j^\dagger) \Delta_k^i (\Delta^\dagger)_l^j \varepsilon^{kl} = \lambda_b [-H_i^\dagger \Delta_k^i \varepsilon^{kl} (\Delta^\dagger)_l^j H_j - H_j^\dagger (\Delta^\dagger)_l^j \varepsilon^{lk} \Delta_k^i H_i] \\ &= \lambda_b [H_i^\dagger \Delta_k^i (\Delta^\dagger)_j^k H^j + H_j^\dagger (\Delta^\dagger)_l^j \Delta_l^i H^i] = \lambda_b [H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k + H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k] \end{aligned}$$

$$\begin{aligned} V_c &= \varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B [\lambda_c \Delta_k^i \Delta_l^j \varepsilon^{kl}] = \lambda_c (-H_i^\dagger H_j + H_i H_j^\dagger) \Delta_k^i \Delta_l^j \varepsilon^{kl} = \lambda_c (-H_i^\dagger \Delta_k^i \varepsilon^{kl} \Delta_l^j H_j - H_j^\dagger \Delta_l^j \varepsilon^{lk} \Delta_k^i H_i) \\ &= \lambda_c (H_i^\dagger \Delta_k^i \Delta_j^k H^j + H_j^\dagger \Delta_l^j \Delta_i^l H^i) = 2 \lambda_c H_i^\dagger \Delta_i^j \Delta_j^k H^k \end{aligned}$$

$$\lambda_4 = 0, \quad \lambda_1 = \lambda_2 = \lambda_b (\Rightarrow \lambda_- = 0), \quad -\lambda_3 = 2\lambda_c \quad \Rightarrow \quad \text{Custodial symmetry}$$

$\lambda_- = \lambda_4 = \lambda_3 = 0$: components of Δ have exactly degenerate masses

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \left(m_S^2 + \frac{1}{2} \lambda_{Sh} v^2 \right) S^2 - \left(m_\Delta^2 + \frac{1}{4} \lambda_+ v^2 \right) \left[\frac{1}{2} (\phi^0)^2 + \frac{1}{2} (a^0)^2 + |\Delta^+|^2 + |\Delta^-|^2 \right]$$

$\lambda_4 = 0$ and $m_S \rightarrow \infty$: complex triplet model

$$s_\alpha = 0, \quad c_\alpha = 1$$

$$m_{\phi^0}^2 = m_\Delta^2 + \frac{1}{4} (\lambda_+ - 2\lambda_3) v^2, \quad m_a^2 = m_\Delta^2 + \frac{1}{4} (\lambda_+ + 2\lambda_3) v^2$$

$$m_1^2 = m_\Delta^2 + \frac{1}{4} \lambda_+ v^2 - \frac{1}{4} \sqrt{\lambda_-^2 + 4\lambda_3^2} v^2 \leq m_\Delta^2 + \frac{1}{4} (\lambda_+ - 2|\lambda_3|) v^2 \leq \min(m_{\phi^0}^2, m_a^2)$$

$$\lambda_- = 0 \quad \Rightarrow \quad m_1^2 = m_\Delta^2 + \frac{1}{4} (\lambda_+ - 2|\lambda_3|) v^2 = \min(m_{\phi^0}^2, m_a^2)$$

$$\lambda_- \neq 0 \quad \Rightarrow \quad \text{unstable DM candidate}$$

Complex scalar quadruplet X :

$$X = \begin{pmatrix} X^{++} \\ X^+ \\ X^0 \\ X^- \end{pmatrix} \in \left(\mathbf{4}, \frac{1}{2} \right), \quad X^0 = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)$$

$$X^{++} = X_2^{11}, \quad X^+ = \sqrt{3}X_1^{11} = -\sqrt{3}X_2^{12} = -\sqrt{3}X_2^{21}, \quad X^0 = \sqrt{3}X_2^{22} = -\sqrt{3}X_1^{12} = -\sqrt{3}X_1^{21}, \quad X^- = X_1^{22}$$

$$X_k^{ij} = X_k^{ji}, \quad \sum_k X_k^{ik} = \sum_k X_k^{kj} = 0, \quad (X^\dagger)_{ij}^k = (X_k^{ij})^\dagger$$

$$(X^{++})^* = (X^\dagger)_{11}^2, \quad (X^+)^* = \sqrt{3}(X^\dagger)_{11}^1 = -\sqrt{3}(X^\dagger)_{12}^2 = -\sqrt{3}(X^\dagger)_{21}^2, \quad (X^0)^* = \sqrt{3}(X^\dagger)_{22}^2 = -\sqrt{3}(X^\dagger)_{12}^1 = -\sqrt{3}(X^\dagger)_{21}^1, \quad (X^-)^* = (X^\dagger)_{22}^1$$

$$\mathcal{L} = (D_\mu X)^\dagger D^\mu X - V(X)$$

$$V(X) = m_X^2 |X|^2 + \lambda_0 |H|^2 |X|^2 + \lambda_1 H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l - (\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) + (\text{irrelevant terms})$$

$$|X|^2 = (X^\dagger)_{ij}^k X_k^{ij} = (X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-, \quad (X^0)^* X^0 = \frac{1}{2}[(\phi^0)^2 + (a^0)^2], \quad (X^0)^2 + h.c. = (\phi^0)^2 - (a^0)^2$$

$$H^1 \rightarrow 0, \quad H^2 \rightarrow \frac{v+h}{\sqrt{2}}, \quad H_1^\dagger \rightarrow 0, \quad H_2^\dagger \rightarrow \frac{v+h}{\sqrt{2}}$$

$$\lambda_0 |H|^2 |X|^2 \rightarrow \frac{\lambda_0}{2} (v+h)^2 [(X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-]$$

$$\lambda_1 H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l \rightarrow \lambda_1 H_2^\dagger X_k^{2j} (X^\dagger)_{j2}^k H^2 = \frac{\lambda_1}{2} (v+h)^2 [X_1^{21} (X^\dagger)_{12}^1 + X_1^{22} (X^\dagger)_{22}^1 + X_2^{21} (X^\dagger)_{12}^2 + X_2^{22} (X^\dagger)_{22}^2]$$

$$= \frac{\lambda_1}{2} (v+h)^2 \left[\frac{1}{3} (X^+)^* X^+ + \frac{2}{3} (X^0)^* X^0 + (X^-)^* X^- \right]$$

$$\lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l \rightarrow \lambda_2 H_2^\dagger (X^\dagger)_{jk}^2 X_2^{jk} H^2 = \frac{\lambda_2}{2} (v+h)^2 [(X^\dagger)_{11}^2 X_2^{11} + (X^\dagger)_{12}^2 X_2^{12} + (X^\dagger)_{21}^2 X_2^{21} + (X^\dagger)_{22}^2 X_2^{22}]$$

$$= \frac{\lambda_2}{2} (v+h)^2 [(X^{++})^* X^{++} + \frac{2}{3} (X^+)^* X^+ + \frac{1}{3} (X^0)^* X^0]$$

$$-\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} \rightarrow -\lambda_3 H_2^\dagger H_2^\dagger X_l^{2k} X_k^{2l} = -\frac{\lambda_3}{2} (v+h)^2 (X_1^{21} X_1^{21} + X_2^{21} X_1^{22} + X_1^{22} X_2^{21} + X_2^{22} X_2^{22}) = -\frac{\lambda_3}{2} (v+h)^2 \left[\frac{2}{3} (X^0)^2 - \frac{2}{\sqrt{3}} X^+ X^- \right]$$

$$\mathcal{L}_{\text{mass}} = -m_{++}^2 |X^{++}|^2 - ((X^+)^* \quad X^-) M_C^2 \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} - \frac{1}{2} m_\phi^2 (\phi^0)^2 - \frac{1}{2} m_a^2 (a^0)^2$$

$$m_{++}^2 = m_X^2 + \frac{1}{2} (\lambda_0 + \lambda_2) v^2, \quad m_\phi^2 = m_X^2 + \left(\frac{1}{2} \lambda_0 + \frac{1}{3} \lambda_1 + \frac{1}{6} \lambda_2 - \frac{2}{3} \lambda_3 \right) v^2, \quad m_a^2 = m_X^2 + \left(\frac{1}{2} \lambda_0 + \frac{1}{3} \lambda_1 + \frac{1}{6} \lambda_2 + \frac{2}{3} \lambda_3 \right) v^2$$

$$M_C^2 = \begin{pmatrix} m_X^2 + \left(\frac{\lambda_0}{2} + \frac{\lambda_1}{6} + \frac{\lambda_2}{3} \right) v^2 & \frac{\lambda_3}{\sqrt{3}} v^2 \\ \frac{\lambda_3}{\sqrt{3}} v^2 & m_X^2 + \frac{1}{2} (\lambda_0 + \lambda_1) v^2 \end{pmatrix}$$

$$\lambda_{\pm} \equiv \lambda_1 \pm \lambda_2$$

$$\begin{aligned} \lambda_1 H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l &\rightarrow \frac{\lambda_+}{4} (v+h)^2 \left[(X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^- + (X^{++})^* X^{++} \right] \\ &\quad + \frac{\lambda_-}{4} (v+h)^2 \left[-(X^{++})^* X^{++} - \frac{1}{3} (X^+)^* X^+ + \frac{1}{3} (X^0)^* X^0 + (X^-)^* X^- \right] \end{aligned}$$

$\Rightarrow \lambda_0$ can be absorbed into λ_+ in the unitary gauge

$$m_{++}^2 \rightarrow m_X^2 + \frac{1}{4}(\lambda_+ - \lambda_-)v^2, \quad m_\phi^2 \rightarrow m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- - 8\lambda_3)v^2, \quad m_a^2 \rightarrow m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- + 8\lambda_3)v^2$$

$$M_C^2 \rightarrow \begin{pmatrix} m_X^2 + \frac{1}{12}(3\lambda_+ - \lambda_-)v^2 & \frac{\lambda_3}{\sqrt{3}}v^2 \\ \frac{\lambda_3}{\sqrt{3}}v^2 & m_X^2 + \frac{1}{4}(\lambda_+ + \lambda_-)v^2 \end{pmatrix}$$

$$U^T M_C^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} = U \begin{pmatrix} X_1^+ \\ X_2^+ \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2} \left\{ (M_C^2)_{11} + (M_C^2)_{22} \mp \sqrt{[(M_C^2)_{11} - (M_C^2)_{22}]^2 + 4(M_C^2)_{12}^2} \right\} = m_X^2 + \frac{v^2}{12} \left(3\lambda_+ + \lambda_- \mp 2\sqrt{\lambda_-^2 + 12\lambda_3^2} \right)$$

$$\text{DM candidate} = \begin{cases} \phi^0, & \lambda_3 > 0 \text{ and } |\lambda_-| < 2\lambda_3 \\ a^0, & \lambda_3 < 0 \text{ and } |\lambda_-| < -2\lambda_3 \end{cases}$$

$$\mathcal{L}_{h\phi\phi, haa} = \frac{1}{2} \lambda_{h\phi\phi} v h (\phi^0)^2 + \frac{1}{2} \lambda_{haa} v h (a^0)^2, \quad \lambda_{h\phi\phi} = -\frac{1}{6}(3\lambda_+ + \lambda_- - 8\lambda_3), \quad \lambda_{haa} = -\frac{1}{6}(3\lambda_+ + \lambda_- + 8\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{\text{SI}} = \frac{m_N^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}, \quad \begin{cases} \text{DM candidate } \chi = \phi^0 & \Rightarrow F_{S,N} = -\frac{\lambda_{h\phi\phi} m_N}{m_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \\ \text{DM candidate } \chi = a^0 & \Rightarrow F_{S,N} = -\frac{\lambda_{haa} m_N}{m_h^2} \left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \end{cases}$$

$$\underline{\lambda_- = \lambda_3 = 0: S = T = U = 0}$$

$$\mathcal{L}_{\text{mass}} = - \left(m_X^2 + \frac{1}{4} \lambda_+ v^2 \right) \left[|X^{++}|^2 + |X^+|^2 + |X^-|^2 + \frac{1}{2} (\phi^0)^2 + \frac{1}{2} (a^0)^2 \right]$$

\Rightarrow No mixing between X^+ and $(X^-)^*$, components of X have exactly degenerate masses

$\Rightarrow S = T = U = 0$

$$D^\mu X = (\partial_\mu - \frac{1}{2}ig'B_\mu - igW_\mu^a t_Q^a)X$$

$$t_Q^1 = \begin{pmatrix} & -\sqrt{3}/2 & & \\ -\sqrt{3}/2 & & -1 & \\ & -1 & & -\sqrt{3}/2 \\ & & -\sqrt{3}/2 & \end{pmatrix}, \quad t_Q^2 = \begin{pmatrix} & \sqrt{3}i/2 & & \\ -\sqrt{3}i/2 & & i & \\ & -i & & \sqrt{3}i/2 \\ & & -\sqrt{3}i/2 & \end{pmatrix}, \quad t_Q^3 = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix}$$

$$\left(\frac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X = \begin{pmatrix} 2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu & -\frac{\sqrt{6}}{2}gW_\mu^+ & 0 & 0 \\ -\frac{\sqrt{6}}{2}gW_\mu^- & eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu & -\sqrt{2}gW_\mu^+ & 0 \\ 0 & -\sqrt{2}gW_\mu^- & -\frac{g}{2c_W}Z_\mu & -\frac{\sqrt{6}}{2}gW_\mu^+ \\ 0 & 0 & -\frac{\sqrt{6}}{2}gW_\mu^- & -eA_\mu - \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu \end{pmatrix} \begin{pmatrix} X^{++} \\ X^+ \\ X^0 \\ X^- \end{pmatrix}$$

$$= \begin{pmatrix} \left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+X^+ \\ -\frac{\sqrt{6}}{2}gW_\mu^-X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+X^0 \\ -\sqrt{2}gW_\mu^-X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+X^- \\ -\frac{\sqrt{6}}{2}gW_\mu^-X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^- \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= \left[-i(\partial^\mu \Delta^\dagger) \left(\frac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X + h.c.\right] + \left|\left(\frac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X\right|^2 \\ &= \left\{-i\partial^\mu (X^{++})^* \left[\left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+X^+\right] - i\partial^\mu (X^+)^* \left[-\frac{\sqrt{6}}{2}gW_\mu^-X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+X^0\right] \right. \\ &\quad \left.- i\partial^\mu (X^0)^* \left[-\sqrt{2}gW_\mu^-X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+X^-\right] - i\partial^\mu (X^-)^* \left[-\frac{\sqrt{6}}{2}gW_\mu^-X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^-\right] + h.c.\right\} \\ &\quad + \left|\left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+X^+\right|^2 + \left|-\frac{\sqrt{6}}{2}gW_\mu^-X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+X^0\right|^2 \\ &\quad + \left|-\sqrt{2}gW_\mu^-X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+X^-\right|^2 + \left|-\frac{\sqrt{6}}{2}gW_\mu^-X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^-\right|^2 \\ &= g \left[\frac{\sqrt{6}}{2}W_\mu^+X^+i\overleftrightarrow{\partial}^\mu (X^{++})^* + W_\mu^+(\phi^0 + ia^0)i\overleftrightarrow{\partial}^\mu (X^+)^* + \frac{\sqrt{3}}{2}W_\mu^+X^-i\overleftrightarrow{\partial}^\mu (\phi^0 - ia^0) + h.c.\right] \\ &\quad + eA_\mu[2(X^{++})^*i\overleftrightarrow{\partial}^\mu X^{++} + (X^+)^*i\overleftrightarrow{\partial}^\mu X^+ - (X^-)^*i\overleftrightarrow{\partial}^\mu X^-] \\ &\quad + \frac{g}{2c_W}Z_\mu[(3c_W^2 - s_W^2)(X^{++})^*i\overleftrightarrow{\partial}^\mu X^{++} + (c_W^2 - s_W^2)(X^+)^*i\overleftrightarrow{\partial}^\mu X^+ + ia^0i\overleftrightarrow{\partial}^\mu \phi^0 - (3c_W^2 + s_W^2)(X^-)^*i\overleftrightarrow{\partial}^\mu X^-] \\ &\quad + g^2W_\mu^+W^{-\mu} \left[\frac{3}{2}|X^{++}|^2 + \frac{7}{2}|X^+|^2 + \frac{7}{4}(\phi^0)^2 + \frac{7}{4}(a^0)^2 + \frac{3}{2}|X^-|^2\right] + g^2 \left[\sqrt{3}W_\mu^+W^{+\mu}(X^{++})^*X^0 + \sqrt{3}W_\mu^+W^{+\mu}(X^+)^*X^- + h.c.\right] \\ &\quad + e^2A_\mu A^\mu (4|X^{++}|^2 + |X^+|^2 + |X^-|^2) + \frac{eg}{c_W}A_\mu Z^\mu [2(3c_W^2 - s_W^2)|X^{++}|^2 + (c_W^2 - s_W^2)|X^+|^2 + (3c_W^2 + s_W^2)|X^-|^2] \\ &\quad + \frac{g^2}{4c_W^2}Z_\mu Z^\mu \left[(3c_W^2 - s_W^2)^2|X^{++}|^2 + (c_W^2 - s_W^2)^2|X^+|^2 + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + (3c_W^2 + s_W^2)^2|X^-|^2\right] \\ &\quad + \left\{-\frac{\sqrt{6}}{2} \left[2egA^\mu + \frac{g^2(3c_W^2 - s_W^2)}{2c_W}Z^\mu\right]W_\mu^+X^+(X^{++})^* - \left[egA^\mu + \frac{g^2(c_W^2 - s_W^2)}{2c_W}Z^\mu\right]W_\mu^+ \left(\frac{\sqrt{6}}{2}X^+(X^{++})^* + (X^+)^*(\phi^0 + ia^0)\right) \right. \\ &\quad \left. + \frac{\sqrt{3}g^2}{4c_W}Z^\mu W_\mu^+X^-(\phi^0 - ia^0) + \frac{g^2}{2c_W}Z^\mu W_\mu^+(X^+)^*(\phi^0 + ia^0) + \frac{\sqrt{3}}{2} \left[egA^\mu + \frac{g^2(3c_W^2 + s_W^2)}{2c_W}Z^\mu\right]W_\mu^+X^-(\phi^0 - ia^0) + h.c.\right\} \end{aligned}$$

Custodial symmetry: $T = U = 0$

$$1) \lambda_- = -2\lambda_3$$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \varepsilon_{AB} \mathcal{H}_i^A \mathcal{H}_j^B = -H_i^\dagger H_j + H_i H_j^\dagger, \quad \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B = -2 |H|^2$$

$$(\mathcal{X}^A)_k^{ij} = \begin{pmatrix} (X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}, \quad (X^\dagger)_k^{ij} = \varepsilon^{il} (X^\dagger)_{lk}^j$$

$$\begin{aligned} \varepsilon_{AB} (\mathcal{X}^A)_l^{ik} (\mathcal{X}^B)_k^{jl} &= \varepsilon_{12} (\mathcal{X}^1)_l^{ik} (\mathcal{X}^2)_k^{jl} + \varepsilon_{21} (\mathcal{X}^2)_l^{ik} (\mathcal{X}^1)_k^{jl} = - (X^\dagger)_l^{ik} X_k^{jl} + X_l^{ik} (X^\dagger)_k^{jl} \\ \varepsilon_{AB} \varepsilon_{ij} (\mathcal{X}^A)_l^{ik} (\mathcal{X}^B)_k^{jl} &= \varepsilon_{ij} [- (X^\dagger)_l^{ik} X_k^{jl} + X_l^{ik} (X^\dagger)_k^{jl}] = (X^\dagger)_{jl}^k X_k^{jl} + X_l^{ik} (X^\dagger)_{ik}^l = 2 |X|^2 \end{aligned}$$

Custodial symmetric potential:

$$\begin{aligned} V_{\text{cust}} &\supset \lambda_{0a} \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B \varepsilon_{CD} \varepsilon_{mn} (\mathcal{X}^C)_l^{mk} (\mathcal{X}^D)_k^{nl} + \lambda_{0b} \varepsilon_{AB} \varepsilon_{CD} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^C \varepsilon_{mn} (\mathcal{X}^B)_l^{mk} (\mathcal{X}^D)_k^{nl} \\ &\quad + \lambda_a \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^B (\mathcal{X}^C)_l^{ik} (\mathcal{X}^D)_k^{jl} + \lambda_b \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^C (\mathcal{X}^B)_l^{ik} (\mathcal{X}^D)_k^{jl} \end{aligned}$$

[Note: raising an index and lowering the index contracted to it give a minus sign]

$$V_{0a} = \lambda_{0a} \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B \varepsilon_{CD} \varepsilon_{mn} (\mathcal{X}^C)_l^{mk} (\mathcal{X}^D)_k^{nl} = -4\lambda_{0a} |H|^2 |X|^2$$

$$V_{0b} = \lambda_{0b} \varepsilon_{AB} \varepsilon_{CD} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^C \varepsilon_{mn} (\mathcal{X}^B)_l^{mk} (\mathcal{X}^D)_k^{nl}$$

$$= \lambda_{0b} [\varepsilon_{12} \varepsilon_{12} \varepsilon^{ij} \mathcal{H}_i^1 \mathcal{H}_j^1 \varepsilon_{mn} (\mathcal{X}^2)_l^{mk} (\mathcal{X}^2)_k^{nl} + \varepsilon_{12} \varepsilon_{21} \varepsilon^{ij} \mathcal{H}_i^1 \mathcal{H}_j^2 \varepsilon_{mn} (\mathcal{X}^2)_l^{mk} (\mathcal{X}^1)_k^{nl}]$$

$$+ \varepsilon_{21} \varepsilon_{12} \varepsilon^{ij} \mathcal{H}_i^2 \mathcal{H}_j^1 \varepsilon_{mn} (\mathcal{X}^1)_l^{mk} (\mathcal{X}^2)_k^{nl} + \varepsilon_{21} \varepsilon_{21} \varepsilon^{ij} \mathcal{H}_i^2 \mathcal{H}_j^2 \varepsilon_{mn} (\mathcal{X}^1)_l^{mk} (\mathcal{X}^1)_k^{nl}]$$

$$= \lambda_{0b} [\varepsilon^{ij} H_i^\dagger H_j^\dagger \varepsilon_{mn} X_l^{mk} X_k^{nl} - \varepsilon^{ij} H_i^\dagger H_j \varepsilon_{mn} X_l^{mk} (X^\dagger)_k^{nl} - \varepsilon^{ij} H_i H_j^\dagger \varepsilon_{mn} (X^\dagger)_l^{mk} X_k^{nl} + \varepsilon^{ij} H_i H_j \varepsilon_{mn} (X^\dagger)_l^{mk} (X^\dagger)_k^{nl}]$$

$$= \lambda_{0b} [-H_i^\dagger H^i (X^\dagger)_{mk}^l X_l^{mk} - H_j^\dagger H^j \varepsilon_{nm} (X^\dagger)_{nl}^k X_k^{nl}] = -2\lambda_{0b} |H|^2 |X|^2$$

$$V_a = \lambda_a \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^B (\mathcal{X}^C)_l^{ik} (\mathcal{X}^D)_k^{jl} = \lambda_a (-H_i^\dagger H_j + H_i H_j^\dagger) [- (X^\dagger)_l^{ik} X_k^{jl} + X_l^{ik} (X^\dagger)_k^{jl}]$$

$$= \lambda_a [H_i^\dagger (X^\dagger)_l^{ik} X_k^{jl} H_j - H_i^\dagger X_l^{ik} (X^\dagger)_k^{jl} H_j - H_j^\dagger X_k^{jl} (X^\dagger)_l^{ik} H_i + H_j^\dagger (X^\dagger)_k^{jl} X_l^{ik} H_i]$$

$$= 2\lambda_a [H_i^\dagger (X^\dagger)_l^{ik} X_k^{jl} H_j - H_i^\dagger X_l^{ik} (X^\dagger)_k^{jl} H_j] = 2\lambda_a [H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l + H_i^\dagger X_k^{ij} (X^\dagger)_k^j H^l]$$

$$V_b = \lambda_b \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^C (\mathcal{X}^B)_l^{ik} (\mathcal{X}^D)_k^{jl}$$

$$= \lambda_b [\varepsilon_{12} \varepsilon_{12} \mathcal{H}_i^1 \mathcal{H}_j^1 (\mathcal{X}^2)_l^{ik} (\mathcal{X}^2)_k^{jl} + \varepsilon_{12} \varepsilon_{21} \mathcal{H}_i^1 \mathcal{H}_j^2 (\mathcal{X}^2)_l^{ik} (\mathcal{X}^1)_k^{jl} + \varepsilon_{21} \varepsilon_{12} \mathcal{H}_i^2 \mathcal{H}_j^1 (\mathcal{X}^1)_l^{ik} (\mathcal{X}^2)_k^{jl} + \varepsilon_{21} \varepsilon_{21} \mathcal{H}_i^2 \mathcal{H}_j^2 (\mathcal{X}^1)_l^{ik} (\mathcal{X}^1)_k^{jl}]$$

$$= \lambda_b [H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} - H_i^\dagger H_j X_l^{ik} (X^\dagger)_k^{jl} - H_i H_j^\dagger (X^\dagger)_l^{ik} X_k^{jl} + H_i H_j (X^\dagger)_l^{ik} (X^\dagger)_k^{jl}]$$

$$= \lambda_b [(H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) + 2H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset -(4\lambda_{0a} + 2\lambda_{0b}) |H|^2 |X|^2 + 2(\lambda_a + \lambda_b) H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l + 2\lambda_a H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l + \lambda_b (H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.)$$

$$\text{Custodial symmetry} \Rightarrow \begin{cases} \lambda_0 = -(4\lambda_{0a} + 2\lambda_{0b}) \\ \lambda_1 = 2(\lambda_a + \lambda_b) \\ \lambda_2 = 2\lambda_a \\ -\lambda_3 = \lambda_b \end{cases} \Rightarrow \lambda_- = \lambda_1 - \lambda_2 = 2\lambda_b = -2\lambda_3$$

$$2) \lambda_- = 2\lambda_3$$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad (\mathcal{X}^A)_k^{ij} = \begin{pmatrix} -(X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}$$

$$\varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B = -2 |H|^2, \quad \varepsilon_{AB} \varepsilon_{ij} (\mathcal{X}^A)_l^{ik} (\mathcal{X}^B)_k^{jl} = -2 |X|^2$$

$$V_{\text{cust}} \supset \lambda_{0a} \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B \varepsilon_{CD} \varepsilon_{mn} (\mathcal{X}^C)_l^{mk} (\mathcal{X}^D)_k^{nl} + \lambda_{0b} \varepsilon_{AB} \varepsilon_{CD} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^C \varepsilon_{mn} (\mathcal{X}^B)_l^{mk} (\mathcal{X}^D)_k^{nl} \\ + \lambda_a \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^B (\mathcal{X}^C)_l^{ik} (\mathcal{X}^D)_k^{jl} + \lambda_b \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^C (\mathcal{X}^B)_l^{ik} (\mathcal{X}^D)_k^{jl}$$

$$V_{0a} = \lambda_{0a} \varepsilon_{AB} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^B \varepsilon_{CD} \varepsilon_{mn} (\mathcal{X}^C)_l^{mk} (\mathcal{X}^D)_k^{nl} = 4\lambda_{0a} |H|^2 |X|^2$$

$$V_{0b} = \lambda_{0b} \varepsilon_{AB} \varepsilon_{CD} \varepsilon^{ij} \mathcal{H}_i^A \mathcal{H}_j^C \varepsilon_{mn} (\mathcal{X}^B)_l^{mk} (\mathcal{X}^D)_k^{nl} = 2\lambda_{0b} |H|^2 |X|^2$$

$$V_a = \lambda_a \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^B (\mathcal{X}^C)_l^{ik} (\mathcal{X}^D)_k^{jl} = -2\lambda_a [H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l + H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l]$$

$$V_b = \lambda_b \varepsilon_{AB} \varepsilon_{CD} \mathcal{H}_i^A \mathcal{H}_j^C (\mathcal{X}^B)_l^{ik} (\mathcal{X}^D)_k^{jl} = \lambda_b [(H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) - 2H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset (4\lambda_{0a} + 2\lambda_{0b}) |H|^2 |X|^2 - 2(\lambda_a + \lambda_b) H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l - 2\lambda_a H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l + \lambda_b (H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.)$$

$$\text{Custodial symmetry} \Rightarrow \begin{cases} \lambda_0 = 4\lambda_{0a} + 2\lambda_{0b} \\ \lambda_1 = -2(\lambda_a + \lambda_b) \\ \lambda_2 = -2\lambda_a \\ -\lambda_3 = \lambda_b \end{cases} \Rightarrow \lambda_- = \lambda_1 - \lambda_2 = -2\lambda_b = 2\lambda_3$$

Passarino-Veltman scalar functions

D -dim one-loop integrals defined by Denner, 0709.1075:

$$T_{\mu_1 \cdots \mu_P}^N(p_1, \cdots, p_{N-1}, m_0, \cdots, m_{N-1}) = F \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \cdots D_{N-1}}, \quad F \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2}$$

N = number of propagator factors in the denominator, P = number of integration momenta in the numerator

$$D_0 = q^2 - m_0^2 + i\varepsilon, \quad D_i = (q + p_i)^2 - m_i^2 + i\varepsilon, \quad i = 1, \cdots, N-1, \quad \varepsilon = \frac{4-D}{2}, \quad D = 4 - 2\varepsilon$$

These integrals give rise to a UV-divergent term $\Delta = \frac{1}{\varepsilon} - \gamma_E + \log 4\pi$

Subtracting Δ corresponds to the $\overline{\text{MS}}$ scheme

$$(2\pi)^{4-D} = (2\pi)^{2\varepsilon} = [1 + 2\varepsilon \log 2\pi + \mathcal{O}(\varepsilon^2)], \quad F = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} = \frac{\mu^{2\varepsilon}}{i\pi^2} [1 + 2\varepsilon \log 2\pi + \mathcal{O}(\varepsilon^2)]$$

Conventionally, $T^1 \equiv A$, $T^2 \equiv B$, $T^3 \equiv C$, \cdots

$$A(m_0^2) = A_0(m_0^2), \quad B(p_1^2, m_0^2, m_1^2) = B_0(p_1^2, m_0^2, m_1^2), \quad B_\mu(p_1^2, m_0^2, m_1^2) = p_{1\mu} B_1(p_1^2, m_0^2, m_1^2)$$

$$B_{\mu\nu}(p_1^2, m_0^2, m_1^2) = g_{\mu\nu} B_{00}(p_1^2, m_0^2, m_1^2) + p_{1\mu} p_{1\nu} B_{11}(p_1^2, m_0^2, m_1^2)$$

$$A_0(m^2) \sim m^2 \Delta, \quad B_0(p^2, m_1^2, m_2^2) \sim \Delta, \quad B_1(p^2, m_1^2, m_2^2) \sim -\frac{1}{2} \Delta, \quad B_{00}(p^2, m_1^2, m_2^2) \sim -\frac{1}{12} (p^2 - 3m_1^2 - 3m_2^2) \Delta$$

D -dim one-loop integrals defined by LoopTools User's Guide:

$$T_{\mu_1 \cdots \mu_P}^N(p_1, \cdots, p_{N-1}, m_0, \cdots, m_{N-1}) = F' \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \cdots D_{N-1}}, \quad F' \equiv \frac{\mu^{4-D}}{i\pi^{D/2} r_\Gamma}, \quad r_\Gamma = \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{1}{r_\Gamma} = 1 + \gamma_E \varepsilon + \mathcal{O}(\varepsilon^2), \quad \frac{1}{\pi^{D/2}} = \frac{1}{\pi^{2-\varepsilon}} = \frac{1}{\pi^2} [1 + \varepsilon \log \pi + \mathcal{O}(\varepsilon^2)]$$

$$F' = \frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon} r_\Gamma} = \frac{\mu^{2\varepsilon}}{i\pi^2} [1 + \varepsilon \log \pi + \mathcal{O}(\varepsilon^2)] [1 + \gamma_E \varepsilon + \mathcal{O}(\varepsilon^2)] = \frac{\mu^{2\varepsilon}}{i\pi^2} [1 + \gamma_E \varepsilon + \varepsilon \log \pi + \mathcal{O}(\varepsilon^2)]$$

$$\frac{T_{\mu_1 \cdots \mu_P}^N}{T_{\mu_1 \cdots \mu_P}^N} = \frac{F'}{F} = \frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon} r_\Gamma} \frac{i\pi^2}{(2\pi\mu)^{2\varepsilon}} = \frac{1}{(2\pi)^{2\varepsilon} \pi^{-\varepsilon} r_\Gamma} = 1 + (\gamma_E - \log 4\pi) \varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\Delta' = \frac{F'}{F} \Delta = \Delta + \Delta(\gamma_E - \log 4\pi) \varepsilon + \mathcal{O}(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \frac{1}{\varepsilon} (\gamma_E - \log 4\pi) \varepsilon + \mathcal{O}(\varepsilon) = \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)$$

The UV-divergent term from $T_{\mu_1 \cdots \mu_P}^N$ is Δ' , and subtracting Δ' corresponds to the $\overline{\text{MS}}$ scheme

$$\frac{1}{K^{2-D/2}} = 1 - \frac{4-D}{2} \ln K + \mathcal{O}((4-D)^2), \quad \Gamma(2-D/2) = \frac{2}{4-D} - \gamma_E + \mathcal{O}(4-D)$$

$$\frac{1}{(4\pi)^{D/2}} = \frac{1}{(4\pi)^2 (4\pi)^{(D-4)/2}} = \frac{1}{16\pi^2} (4\pi)^{2-D/2} = \frac{1}{16\pi^2} \left[1 + \frac{4-D}{2} \ln 4\pi + \mathcal{O}((4-D)^2) \right]$$

$$\frac{\Gamma(2-D/2)}{(4\pi)^{D/2} K^{2-D/2}} = \frac{1}{16\pi^2} \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D) \right]$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - K)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}} \frac{\Gamma(n-D/2)}{\Gamma(n)} \frac{1}{K^{n-D/2}}$$

$$\Gamma(2-D/2) = (1-D/2)\Gamma(1-D/2), \quad \Gamma(1) = 1, \quad \frac{1}{1-(4-D)/2} = 1 + \frac{4-D}{2} + \mathcal{O}((4-D)^2)$$

$$\begin{aligned} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - K} &= \frac{-i}{(4\pi)^{D/2}} \frac{\Gamma(1-D/2)}{\Gamma(1)} \frac{1}{K^{1-D/2}} = \frac{-iK}{1-D/2} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2} K^{2-D/2}} \\ &= \frac{iK}{1-(4-D)/2} \frac{1}{16\pi^2} \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D) \right] \\ &= \frac{iK}{16\pi^2} \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + 1 + \mathcal{O}(4-D) \right] \end{aligned}$$

$$\mu^{4-D} = 1 + \frac{4-D}{2} \ln \mu^2 + \mathcal{O}((4-D)^2)$$

$$\begin{aligned} A_0(m^2) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{q^2 - m^2 + i\varepsilon} = -16i\pi^2 \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 - (m^2 - i\varepsilon)} \\ &= (m^2 - i\varepsilon) \mu^{4-D} \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln(m^2 - i\varepsilon) + 1 + \mathcal{O}(4-D) \right] = m^2 \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi + \ln \mu^2 - \ln m^2 + 1 + \mathcal{O}(4-D) \right] \\ &= m^2 \left(\Delta + 1 - \ln \frac{m^2}{\mu^2} \right) + \mathcal{O}(4-D) \end{aligned}$$

$$\Delta \equiv \frac{2}{4-D} - \gamma_E + \ln 4\pi$$

$$\begin{aligned} x(q^2 - m_1^2 + i\varepsilon) + (1-x)[(q+p)^2 - m_2^2 + i\varepsilon] &= q^2 + 2(1-x)q \cdot p + (1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon \\ &= [q + (1-x)p]^2 + x(1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon \\ &= \ell^2 - K \end{aligned}$$

$$\ell \equiv q + (1-x)p, \quad K = -x(1-x)p^2 + xm_1^2 + (1-x)m_2^2 - i\varepsilon = x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon$$

$$\frac{1}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]} = \int_0^1 dx \frac{1}{\{x[q^2 - m_1^2 + i\varepsilon] + (1-x)[(q+p)^2 - m_2^2 + i\varepsilon]\}^2} = \int_0^1 dx \frac{1}{(\ell^2 - K)^2}$$

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - K)^2} = i \frac{\Gamma(2-D/2)}{(4\pi)^{D/2} K^{2-D/2}} = \frac{i}{16\pi^2} \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D) \right]$$

$$\begin{aligned} B_0(p^2, m_1^2, m_2^2) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]} = \frac{16\pi^2 \mu^{4-D}}{i} \int_0^1 dx \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - K)^2} \\ &= \mu^{4-D} \int_0^1 dx \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D) \right] = \int_0^1 dx \left[\frac{2}{4-D} - \gamma_E + \ln 4\pi + \ln \mu^2 - \ln K + \mathcal{O}(4-D) \right] \\ &= \Delta - \int_0^1 dx \ln \frac{x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon}{\mu^2} + \mathcal{O}(4-D) \end{aligned}$$

$$x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon = p^2 (x - x_+)(x - x_-)$$

$$x_{\pm} \equiv \frac{1}{2p^2} \left[p^2 - m_1^2 + m_2^2 \pm \sqrt{(p^2 - m_1^2 + m_2^2)^2 - 4p^2(m_2^2 - i\varepsilon)} \right]$$

$$\int_0^1 dx \ln \frac{x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon}{\mu^2} = \int_0^1 dx \ln \frac{p^2 (x - x_+)(x - x_-)}{\mu^2} = \ln \frac{p^2}{\mu^2} + \int_0^1 dx [\ln(x - x_+) + \ln(x - x_-)]$$

$$= \ln \frac{p^2}{\mu^2} + [(x - x_+) \ln(x - x_+) - x + (x - x_-) \ln(x - x_-) - x] \Big|_0^1 = \ln \frac{p^2}{\mu^2} + f(x_+) + f(x_-)$$

$$f(z) \equiv (1-z) \ln(1-z) + z \ln(-z) - 1 = \ln(1-z) - z \ln(1-z^{-1}) - 1$$

$$B_0(p^2, m_1^2, m_2^2) = \Delta - \ln \frac{p^2}{\mu^2} - f(x_+) - f(x_-) + \mathcal{O}(4-D)$$

$$p_\mu B_1(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_\mu}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$p^2 B_1(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q \cdot p}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$2q \cdot p = [(q+p)^2 - m_2^2 + i\varepsilon] - [q^2 - m_1^2 + i\varepsilon] - p^2 - m_1^2 + m_2^2$$

$$\frac{2q \cdot p}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]} = \frac{1}{q^2 - m_1^2 + i\varepsilon} - \frac{1}{(q+p)^2 - m_2^2 + i\varepsilon} - \frac{p^2 + m_1^2 - m_2^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$2p^2 B_1(p^2, m_1^2, m_2^2) = A_0(m_1^2) - A_0(m_2^2) - (p^2 + m_1^2 - m_2^2) B_0(p^2, m_1^2, m_2^2)$$

$$B_1(p^2, m_1^2, m_2^2) = \frac{1}{2p^2} [A_0(m_1^2) - A_0(m_2^2) - (p^2 + m_1^2 - m_2^2) B_0(p^2, m_1^2, m_2^2)]$$

$$g_{\mu\nu} B_{00}(p^2, m_1^2, m_2^2) + p_\mu p_\nu B_{11}(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_\mu q_\nu}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$DB_{00}(p^2, m_1^2, m_2^2) + p^2 B_{11}(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$q^2 = (q^2 - m_1^2 + i\varepsilon) + m_1^2 - i\varepsilon$$

$$\frac{q^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]} = \frac{1}{(q+p)^2 - m_2^2 + i\varepsilon} + \frac{m_1^2 - i\varepsilon}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$DB_{00}(p^2, m_1^2, m_2^2) + p^2 B_{11}(p^2, m_1^2, m_2^2) = A_0(m_2^2) + m_1^2 B_0(p^2, m_1^2, m_2^2)$$

$$4Dp^2 B_{00}(p^2, m_1^2, m_2^2) + 4p^4 B_{11}(p^2, m_1^2, m_2^2) = 4p^2 A_0(m_2^2) + 4p^2 m_1^2 B_0(p^2, m_1^2, m_2^2)$$

$$p^2 B_{00}(p^2, m_1^2, m_2^2) + p^4 B_{11}(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{(q \cdot p)^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$4(q \cdot p)^2 = \{[(q+p)^2 - m_2^2 + i\varepsilon] - [q^2 - m_1^2 + i\varepsilon] - [p^2 + m_1^2 - m_2^2]\}^2$$

$$= [(q+p)^2 - m_2^2 + i\varepsilon] \{[(q+p)^2 - m_2^2 + i\varepsilon] - 2[q^2 - m_1^2 + i\varepsilon] - 2[p^2 + m_1^2 - m_2^2]\}$$

$$+ [q^2 - m_1^2 + i\varepsilon] \{[q^2 - m_1^2 + i\varepsilon] + 2[p^2 + m_1^2 - m_2^2]\} + [p^2 + m_1^2 - m_2^2]^2$$

$$= [(q+p)^2 - m_2^2 + i\varepsilon](-q^2 + 2q \cdot p - p^2 + m_2^2 - i\varepsilon) + (q^2 - m_1^2 + i\varepsilon)(q^2 + 2p^2 + m_1^2 - 2m_2^2 + i\varepsilon) + (p^2 + m_1^2 - m_2^2)^2$$

$$\frac{4(q \cdot p)^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]} = \frac{-q^2 + 2q \cdot p - p^2 + m_2^2 - i\varepsilon}{q^2 - m_1^2 + i\varepsilon} + \frac{q^2 + 2p^2 + m_1^2 - 2m_2^2 + i\varepsilon}{(q+p)^2 - m_2^2 + i\varepsilon} + \frac{(p^2 + m_1^2 - m_2^2)^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$= -1 + \frac{2q \cdot p - p^2 - m_1^2 + m_2^2}{q^2 - m_1^2 + i\varepsilon} + 1 + \frac{-2q \cdot p + p^2 + m_1^2 - m_2^2}{(q+p)^2 - m_2^2 + i\varepsilon} + \frac{(p^2 + m_1^2 - m_2^2)^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$= \frac{2q \cdot p - (p^2 + m_1^2 - m_2^2)}{q^2 - m_1^2 + i\varepsilon} + \frac{-2q' \cdot p + 3p^2 + m_1^2 - m_2^2}{q'^2 - m_2^2 + i\varepsilon} + \frac{(p^2 + m_1^2 - m_2^2)^2}{[q^2 - m_1^2 + i\varepsilon][(q+p)^2 - m_2^2 + i\varepsilon]}$$

$$q' = q + p$$

$$4p^2 B_{00}(p^2, m_1^2, m_2^2) + 4p^4 B_{11}(p^2, m_1^2, m_2^2) = -(p^2 + m_1^2 - m_2^2) A_0(m_1^2) + (3p^2 + m_1^2 - m_2^2) A_0(m_2^2) + (p^2 + m_1^2 - m_2^2)^2 B_0(p^2, m_1^2, m_2^2)$$

$$\begin{aligned}
4(D-1)p^2B_{00}(p^2, m_1^2, m_2^2) &= 4p^2A_0(m_2^2) + 4p^2m_1^2B_0(p^2, m_1^2, m_2^2) \\
&\quad + (p^2 + m_1^2 - m_2^2)A_0(m_1^2) - (3p^2 + m_1^2 - m_2^2)A_0(m_2^2) - (p^2 + m_1^2 - m_2^2)^2B_0(p^2, m_1^2, m_2^2) \\
&= p^2A_0(m_1^2) + p^2A_0(m_2^2) + 2p^2(m_1^2 + m_2^2)B_0(p^2, m_1^2, m_2^2) - p^4B_0(p^2, m_1^2, m_2^2) \\
&\quad + (m_1^2 - m_2^2)A_0(m_1^2) - (m_1^2 - m_2^2)A_0(m_2^2) - (m_1^2 - m_2^2)^2B_0(p^2, m_1^2, m_2^2)
\end{aligned}$$

$$\begin{aligned}
[1 - (4-D)/3]6B_{00}(p^2, m_1^2, m_2^2) &= \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2, m_1^2, m_2^2) \\
&\quad + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)]
\end{aligned}$$

$$\frac{1}{1 - (4-D)/3} = 1 + \frac{1}{3}(4-D) + \mathcal{O}((4-D)^2)$$

$$\begin{aligned}
6B_{00}(p^2, m_1^2, m_2^2) &= \frac{1}{1 - (4-D)/3} \left\{ \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2, m_1^2, m_2^2) \right. \\
&\quad \left. + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)] \right\} \\
&= \frac{1}{3}(m_1^2 + m_2^2) + \frac{2}{3} \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right) + \frac{m_1^2 - m_2^2}{3p^2}[m_1^2 - m_2^2 - (m_1^2 - m_2^2)] \\
&\quad + \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2, m_1^2, m_2^2) \\
&\quad + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)] + \mathcal{O}(4-D)
\end{aligned}$$

$$\begin{aligned}
B_{00}(p^2, m_1^2, m_2^2) &= \frac{1}{6} \left\{ m_1^2 + m_2^2 - \frac{1}{3}p^2 + \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2, m_1^2, m_2^2) \right. \\
&\quad \left. + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)] \right\}
\end{aligned}$$

$$J_1(p^2, m_1^2, m_2^2) \equiv A_0(m_1^2) + A_0(m_2^2) - (p^2 - m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2) - 4B_{00}(p^2, m_1^2, m_2^2)$$

$$J_1'(p^2, m_1^2, m_2^2) = -B_0(p^2, m_1^2, m_2^2) - (p^2 - m_1^2 - m_2^2)B_0'(p^2, m_1^2, m_2^2) - 4B_{00}'(p^2, m_1^2, m_2^2)$$

$$B_0'(p^2, m_1^2, m_2^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2}$$

$$B_{00}'(p^2, m_1^2, m_2^2) = \frac{1}{6} \left\{ -\frac{1}{3} - \frac{1}{2} B_0(p^2, m_1^2, m_2^2) + \left(m_1^2 + m_2^2 - \frac{1}{2} p^2 \right) B_0'(p^2, m_1^2, m_2^2) \right. \\ \left. - \frac{m_1^2 - m_2^2}{2p^4} [A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)] - \frac{(m_1^2 - m_2^2)^2}{2p^2} B_0'(p^2, m_1^2, m_2^2) \right\}$$

$$\underline{m = m_1 = m_2}$$

$$-\ln \frac{x^2 p^2 - x p^2 + m^2}{\mu^2} = -\ln \frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$$B_0(p^2, m^2, m^2) = \Delta - \int_0^1 dx \ln \frac{x^2 p^2 - x p^2 + m^2 - i\varepsilon}{\mu^2} = \Delta - \ln \frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$$-\frac{x^2 - x}{x^2 p^2 - x p^2 + m^2} = -\frac{x^2 - x}{m^2} + \mathcal{O}(p^2)$$

$$B_0'(p^2, m^2, m^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - x p^2 + m^2} = \frac{1}{6m^2} + \mathcal{O}(p^2)$$

$$B_{00}(p^2, m^2, m^2) = \frac{1}{3} m^2 - \frac{1}{18} p^2 + \frac{1}{6} A_0(m^2) + \left(\frac{1}{3} m^2 - \frac{1}{12} p^2 \right) B_0(p^2, m^2, m^2)$$

$$B_{00}'(p^2, m^2, m^2) = -\frac{1}{18} - \frac{1}{12} B_0(p^2, m^2, m^2) + \left(\frac{1}{3} m^2 - \frac{1}{12} p^2 \right) B_0'(p^2, m^2, m^2)$$

$$J_1(p^2, m^2, m^2) = 2A_0(m^2) - (p^2 - 2m^2)B_0(p^2, m^2, m^2) - 4B_{00}(p^2, m^2, m^2)$$

$$J_1'(p^2, m^2, m^2) = -B_0(p^2, m^2, m^2) - (p^2 - 2m^2)B_0'(p^2, m^2, m^2) + \frac{2}{9} + \frac{1}{3} B_0(p^2, m^2, m^2) - \left(\frac{4}{3} m^2 - \frac{1}{3} p^2 \right) B_0'(p^2, m^2, m^2)$$

$$= \frac{2}{9} - \frac{2}{3} B_0(p^2, m^2, m^2) + \frac{2}{3} (m^2 - p^2) B_0'(p^2, m^2, m^2)$$

$$= \frac{2}{9} - \frac{2}{3} \left(\Delta - \ln \frac{m^2}{\mu^2} \right) + \frac{2}{3} m^2 \frac{1}{6m^2} + \mathcal{O}(p^2) = -\frac{2}{3} \Delta + \frac{1}{3} + \frac{2}{3} \ln \frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$$\underline{m_1 = 0}$$

$$-\ln \frac{x^2 p^2 - x(p^2 + m_2^2) + m_2^2}{\mu^2} = -\ln \frac{(1-x)m_2^2}{\mu^2} + \frac{x p^2}{m_2^2} + \frac{x^2 p^4}{2m_2^4} + \mathcal{O}(p^6)$$

$$B_0(p^2, 0, m_2^2) = \Delta - \int_0^1 dx \ln \frac{x^2 p^2 - x(p^2 + m_2^2) + m_2^2 - i\varepsilon}{\mu^2} = \Delta + 1 - \ln \frac{m_2^2}{\mu^2} + \frac{p^2}{2m_2^2} + \frac{p^4}{6m_2^4} + \mathcal{O}(p^6)$$

$$-\frac{x^2 - x}{x^2 p^2 - x(p^2 + m_2^2) + m_2^2} = \frac{x}{m_2^2} + \frac{x^2 p^2}{m_2^4} + \frac{x^3 p^4}{m_2^6} + \mathcal{O}(p^6)$$

$$B_0'(p^2, 0, m_2^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - x(p^2 + m_2^2) + m_2^2 - i\varepsilon} = \frac{1}{2m_2^2} + \frac{p^2}{3m_2^4} + \frac{p^4}{4m_2^6} + \mathcal{O}(p^6)$$

$$-A_0(m_2^2) + m_2^2 B_0(p^2, 0, m_2^2) = \frac{p^2}{2} + \frac{p^4}{6m_2^2} + \mathcal{O}(p^6)$$

$$B_{00}'(p^2, 0, m_2^2) = \frac{1}{6} \left[-\frac{1}{3} - \frac{1}{2} B_0(p^2, 0, m_2^2) + \left(m_2^2 - \frac{1}{2} p^2 \right) B_0'(p^2, 0, m_2^2) + \frac{m_2^2}{2p^4} [-A_0(m_2^2) + m_2^2 B_0(p^2, 0, m_2^2)] - \frac{m_2^4}{2p^2} B_0'(p^2, 0, m_2^2) \right]$$

$$= \frac{1}{6} \left[-\frac{1}{3} - \frac{1}{2} \left(\Delta + 1 - \ln \frac{m_2^2}{\mu^2} \right) + m_2^2 \frac{1}{2m_2^2} + \frac{m_2^2}{2p^4} \left(\frac{p^2}{2} + \frac{p^4}{6m_2^2} \right) - \frac{m_2^4}{2p^2} \left(\frac{1}{2m_2^2} + \frac{p^2}{3m_2^4} \right) \right] + \mathcal{O}(p^2)$$

$$= -\frac{1}{12} \Delta - \frac{5}{72} + \frac{1}{12} \ln \frac{m_2^2}{\mu^2} + \mathcal{O}(p^2)$$

$$J_1'(p^2, 0, m_2^2) = -B_0(p^2, 0, m_2^2) - (p^2 - m_2^2)B_0'(p^2, 0, m_2^2) - 4B_{00}'(p^2, 0, m_2^2)$$

$$= -\left(\Delta + 1 - \ln \frac{m_2^2}{\mu^2} \right) + m_2^2 \frac{1}{2m_2^2} - 4 \left(-\frac{1}{12} \Delta - \frac{5}{72} + \frac{1}{12} \ln \frac{m_2^2}{\mu^2} \right) + \mathcal{O}(p^2)$$

$$= -\frac{2}{3} \Delta - \frac{2}{9} + \frac{2}{3} \ln \frac{m_2^2}{\mu^2} + \mathcal{O}(p^2)$$

Custodial symmetry

Ref: Jose Santiago's lecture note "the Physics of Electroweak Symmetry Breaking", 2009;

Montero & Pleitez, hep-ph/0607144; Branco, Ferreira, Lavoura, Rebelo, Sher & Silva, 1106.0034

$$\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}} \text{ bidoublet } \mathbf{H} = \begin{pmatrix} \tilde{H} & H \\ -H^- & H^0 \end{pmatrix}$$

$$\tilde{H}^\dagger \tilde{H} = H^0 H^{0*} + H^+ H^- = H^\dagger H, \quad \text{tr}(\mathbf{H}^\dagger \mathbf{H}) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2 |H|^2$$

$$U_{\text{L}} \in \text{global SU}(2)_{\text{L}}, \quad U_{\text{R}} \in \text{global SU}(2)_{\text{R}}, \quad \mathbf{H} \rightarrow U_{\text{L}} \mathbf{H} U_{\text{R}}^\dagger \Rightarrow \text{tr}(\mathbf{H}^\dagger \mathbf{H}) \rightarrow \text{tr}(U_{\text{R}} \mathbf{H}^\dagger U_{\text{L}}^\dagger U_{\text{L}} \mathbf{H} U_{\text{R}}^\dagger) = \text{tr}(\mathbf{H}^\dagger \mathbf{H})$$

$$\text{SM scalar potential } V = -\mu^2 |H|^2 + \lambda |H|^4 = -\frac{\mu^2}{2} \text{tr}(\mathbf{H}^\dagger \mathbf{H}) + \frac{\lambda}{4} [\text{tr}(\mathbf{H}^\dagger \mathbf{H})]^2$$

$$W_\mu \equiv W_\mu^a \frac{\sigma^a}{2}, \quad (\sigma^2)^2 = 1, \quad \sigma^a \sigma^2 = -\sigma^2 (\sigma^a)^*$$

$$\begin{aligned} |(\partial_\mu - igW_\mu) \tilde{H}|^2 &= [(\partial^\mu - igW^\mu) i\sigma^2 H^*]^\dagger (\partial_\mu - igW_\mu) i\sigma^2 H^* = H^\top \sigma^2 (\overline{\partial^\mu} + igW^\mu) (\partial_\mu - igW_\mu) \sigma^2 H^* \\ &= H^\top (\overline{\partial^\mu} - igW^{\mu*}) (\partial_\mu + igW_\mu^*) H^* = H^\dagger (\overline{\partial^\mu} + igW^\mu) (\partial_\mu - igW_\mu) H = |(\partial_\mu - igW_\mu) H|^2 \end{aligned}$$

$$\mathbf{D}_\mu \mathbf{H} \equiv (\partial_\mu - igW_\mu) \mathbf{H} = \begin{pmatrix} (\partial_\mu - igW_\mu) \tilde{H} & (\partial_\mu - igW_\mu) H \end{pmatrix}$$

$$\text{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}] = |(\partial_\mu - igW_\mu) \tilde{H}|^2 + |(\partial_\mu - igW_\mu) H|^2 = 2 |(\partial_\mu - igW_\mu) H|^2$$

$$W_\mu \rightarrow U_{\text{L}} W_\mu U_{\text{L}}^\dagger$$

$$\mathbf{D}_\mu \mathbf{H} \rightarrow (\partial_\mu - igU_{\text{L}} W_\mu U_{\text{L}}^\dagger) U_{\text{L}} \mathbf{H} U_{\text{R}}^\dagger = U_{\text{L}} \mathbf{D}_\mu \mathbf{H} U_{\text{R}}^\dagger$$

$$\text{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}] \rightarrow \text{tr}(U_{\text{R}} \mathbf{D}^\mu \mathbf{H}^\dagger U_{\text{L}}^\dagger U_{\text{L}} \mathbf{D}_\mu \mathbf{H} U_{\text{R}}^\dagger) = \text{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}]$$

An $\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$ global symmetry exists in the potential and the $\text{SU}(2)_{\text{L}}$ gauge interaction of the Higgs field

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \mathbf{H} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \rightarrow U_{\text{L}} \langle \mathbf{H} \rangle U_{\text{R}}^\dagger = \frac{1}{\sqrt{2}} U_{\text{L}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} U_{\text{R}}^\dagger = \frac{v}{\sqrt{2}} U_{\text{L}} U_{\text{R}}^\dagger$$

$$U_{\text{L}} = U_{\text{R}} \Rightarrow U_{\text{L}} \langle \mathbf{H} \rangle U_{\text{R}}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \langle \mathbf{H} \rangle$$

\Rightarrow the vacuum is invariant under the $\text{SU}(2)_{\text{L+R}}$ global symmetry (equal left and right rotations)

$\Rightarrow \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}} \rightarrow \underline{\text{SU}(2)_{\text{L+R}}}$ custodial symmetry

$\Rightarrow W_\mu^a$ transform as a triplet under $\text{SU}(2)_{\text{L+R}}$

\Rightarrow identical contributions to the masses of W_μ^a from electroweak symmetry breaking

\Rightarrow the custodial symmetry protects the parameter $\rho = \frac{m_W^2}{m_Z^2 c_W^2}$, and leads to $T = U = 0$

Custodial symmetry violation in the SM

1) $\text{U}(1)_{\text{Y}}$ gauge interaction

$$\text{U}(1)_{\text{Y}} \text{ gauge transformation } H \rightarrow e^{iY_H \theta(x)} H, \quad \tilde{H} \rightarrow e^{-iY_H \theta(x)} \tilde{H}, \quad Y_H = \frac{1}{2}$$

$$\exp[-iY_H \theta(x) \sigma^3] = \sum_{k=0}^{\infty} \frac{1}{k!} [-iY_H \theta(x) \sigma^3]^k = \sum_{k=0}^{\infty} \frac{1}{(2k)!} [-iY_H \theta(x)]^{2k} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} [-iY_H \theta(x)]^{2k+1} \sigma^3$$

$$= \cos[Y_H \theta(x)] - i[Y_H \theta(x)] \sigma^3 = \begin{pmatrix} \cos[Y_H \theta(x)] - i \sin[Y_H \theta(x)] & \\ & \cos[Y_H \theta(x)] + i \sin[Y_H \theta(x)] \end{pmatrix} = \begin{pmatrix} e^{-iY_H \theta(x)} & \\ & e^{iY_H \theta(x)} \end{pmatrix}$$

$$\mathbf{H} \rightarrow \begin{pmatrix} e^{-iY_H \theta(x)} \tilde{H} & e^{iY_H \theta(x)} H \end{pmatrix} = \mathbf{H} \exp[-iY_H \theta(x) \sigma^3], \quad \frac{\sigma^3}{2} \text{ is just one } \text{SU}(2)_{\text{R}} \text{ generator}$$

$Y_H \neq 0$ violates custodial symmetry

2) Difference between Yukawa couplings of up-type and down-type fermions

$$\mathcal{Q}_{\text{L}} \equiv \begin{pmatrix} t_{\text{L}} \\ b'_{\text{L}} \end{pmatrix} \rightarrow U_{\text{L}} \mathcal{Q}_{\text{L}}, \quad \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix} \rightarrow U_{\text{R}} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix}, \quad \bar{\mathcal{Q}}_{\text{L}} \mathbf{H} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix} \rightarrow \bar{\mathcal{Q}}_{\text{L}} U_{\text{L}}^\dagger U_{\text{L}} \mathbf{H} U_{\text{R}}^\dagger U_{\text{R}} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix} = \bar{\mathcal{Q}}_{\text{L}} \mathbf{H} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix}$$

$$-y \bar{\mathcal{Q}}_{\text{L}} \mathbf{H} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix} = -y \bar{\mathcal{Q}}_{\text{L}} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \begin{pmatrix} t_{\text{R}} \\ b'_{\text{R}} \end{pmatrix} = -y (\bar{\mathcal{Q}}_{\text{L}} \tilde{H} t_{\text{R}} + \bar{\mathcal{Q}}_{\text{L}} H b'_{\text{R}})$$

$$\text{Yukawa couplings in the SM: } -y_t \bar{\mathcal{Q}}_{\text{L}} \tilde{H} t_{\text{R}} - y_b \bar{\mathcal{Q}}_{\text{L}} H b'_{\text{R}} = -\bar{\mathcal{Q}}_{\text{L}} \mathbf{H} \begin{pmatrix} y_t t_{\text{R}} \\ y_b b'_{\text{R}} \end{pmatrix}$$

$y_t \neq y_b$ violates custodial symmetry

U(1)_Y gauge symmetry

$$\mathbf{H} \rightarrow \mathbf{H} \exp\left(-i\theta(x)\frac{\sigma^3}{2}\right), \quad B_\mu \rightarrow B_\mu + \frac{1}{g'}\partial_\mu\theta$$

$$D_\mu H = \left(\partial_\mu - igW_\mu - \frac{i}{2}g'B_\mu\right)H, \quad D_\mu \tilde{H} = \left(\partial_\mu - igW_\mu + \frac{i}{2}g'B_\mu\right)\tilde{H}$$

$$D_\mu \mathbf{H} = \partial_\mu \mathbf{H} - igW_\mu \mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu$$

$$\begin{aligned} D_\mu \mathbf{H} &\rightarrow \left[\partial_\mu \mathbf{H} + \mathbf{H}\left(-i\partial_\mu\theta\frac{\sigma^3}{2}\right) - igW_\mu \mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu + ig'\mathbf{H}\frac{1}{g'}\partial_\mu\theta\frac{\sigma^3}{2}\right]\exp\left(-i\theta(x)\frac{\sigma^3}{2}\right) \\ &= D_\mu \mathbf{H} \exp\left(-i\theta(x)\frac{\sigma^3}{2}\right) \end{aligned}$$

$H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu}$ operator

$$W_{\mu\nu} = W_{\mu\nu}^a \frac{\sigma^a}{2} = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc}W_\mu^b W_\nu^c) \frac{\sigma^a}{2}$$

SU(2)_L × SU(2)_R global transformation:

$$\mathbf{H} \rightarrow U_L \mathbf{H} U_R^\dagger, \quad W_\mu \rightarrow U_L W_\mu U_L^\dagger, \quad W_{\mu\nu} \rightarrow U_L W_{\mu\nu} U_L^\dagger$$

$$\text{tr}(\mathbf{H}^\dagger W_{\mu\nu} \mathbf{H}) B^{\mu\nu} \rightarrow \text{tr}(U_R \mathbf{H}^\dagger U_L^\dagger U_L W_{\mu\nu} U_L^\dagger U_L \mathbf{H} U_R^\dagger) B^{\mu\nu} = \text{tr}(\mathbf{H}^\dagger W_{\mu\nu} \mathbf{H}) B^{\mu\nu}$$

$$\tilde{H}^\dagger W_{\mu\nu} \tilde{H} = H^T \sigma^2 W_{\mu\nu} \sigma^2 H^* = -H^T W_{\mu\nu}^* H^* = -H^\dagger W_{\mu\nu} H$$

$$\begin{aligned} \text{tr}(\mathbf{H}^\dagger W_{\mu\nu} \mathbf{H}) B^{\mu\nu} &= \text{tr}\left[\begin{pmatrix} \tilde{H}^\dagger \\ H^\dagger \end{pmatrix} W_{\mu\nu} \begin{pmatrix} \tilde{H} & H \end{pmatrix}\right] B^{\mu\nu} = \text{tr}\begin{pmatrix} \tilde{H}^\dagger W_{\mu\nu} \tilde{H} & \tilde{H}^\dagger W_{\mu\nu} H \\ H^\dagger W_{\mu\nu} \tilde{H} & H^\dagger W_{\mu\nu} H \end{pmatrix} B^{\mu\nu} \\ &= (\tilde{H}^\dagger W_{\mu\nu} \tilde{H} + H^\dagger W_{\mu\nu} H) B^{\mu\nu} = 0 \end{aligned}$$

\Rightarrow The operator $H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu}$ does not respect the custodial symmetry

$\left\{ \begin{array}{l} \text{However, scalar potential terms or Yukawa terms for EW multiplets that respect} \\ \text{the custodial symmetry, along with the U(1)_Y gauge interaction that violates the} \\ \text{custodial symmetry, can contribute to this operator and hence to } S \end{array} \right\}$

Triplet in SU(2) as an example

Matrix notation

$$U(x) = \exp[i\theta^a(x)t_{\text{T}}^a], \quad U^\dagger = \exp(-i\theta^a t_{\text{T}}^a)$$

$$\underline{D_\mu T \equiv (\partial_\mu - igW_\mu^a t_{\text{T}}^a)T}$$

$$T \rightarrow UT, \quad W_\mu^a t_{\text{T}}^a \rightarrow UW_\mu^a t_{\text{T}}^a U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

$$\begin{aligned} D_\mu T &\rightarrow \left[\partial_\mu - ig \left(UW_\mu^a t_{\text{T}}^a U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \right) \right] UT \\ &= UU^\dagger (\partial_\mu U) T + U \partial_\mu T - ig UW_\mu^a t_{\text{T}}^a T + U (\partial_\mu U^\dagger) UT \\ &= U (\partial_\mu - igW_\mu^a t_{\text{T}}^a) T + U \partial_\mu (U^\dagger U) UT = UD_\mu T \end{aligned}$$

$$\underline{D_\mu T \text{ transforms as } T}$$

$$\text{Infinitesimal transformation } U \simeq 1 + i\theta^a t_{\text{T}}^a$$

$$\delta T = i\theta^a t_{\text{T}}^a T, \quad \delta(W_\mu^a t_{\text{T}}^a) = i\theta^b [t_{\text{T}}^b, W_\mu^a t_{\text{T}}^a] + \frac{1}{g} (\partial_\mu \theta^a) t_{\text{T}}^a$$

$$\begin{aligned} \delta(D_\mu T) &= (\partial_\mu - igW_\mu^a t_{\text{T}}^a) \delta T - ig \delta(W_\mu^a t_{\text{T}}^a) T \\ &= (\partial_\mu - igW_\mu^a t_{\text{T}}^a) i\theta^b t_{\text{T}}^b T - ig \left\{ i\theta^b [t_{\text{T}}^b, W_\mu^a t_{\text{T}}^a] + \frac{1}{g} (\partial_\mu \theta^a) t_{\text{T}}^a \right\} T \\ &= i(\partial_\mu \theta^a) t_{\text{T}}^a T + i\theta^a t_{\text{T}}^a \partial_\mu T + g\theta^b W_\mu^a t_{\text{T}}^a t_{\text{T}}^b T + g\theta^b [t_{\text{T}}^b, W_\mu^a t_{\text{T}}^a] T - i(\partial_\mu \theta^a) t_{\text{T}}^a T \\ &= i\theta^a t_{\text{T}}^a \partial_\mu T + g\theta^b t_{\text{T}}^b W_\mu^a t_{\text{T}}^a T = i\theta^a t_{\text{T}}^a (\partial_\mu - igW_\mu^b t_{\text{T}}^b) T = i\theta^a t_{\text{T}}^a D_\mu T \end{aligned}$$

$$D_\mu D_\nu T = \partial_\mu \partial_\nu T - ig(\partial_\mu W_\nu^a) t_{\text{T}}^a T - igW_\nu^a t_{\text{T}}^a \partial_\mu T - igW_\mu^a t_{\text{T}}^a (\partial_\nu T - igW_\nu^b t_{\text{T}}^b T)$$

$$[D_\mu, D_\nu] T = -ig(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) t_{\text{T}}^a T - g^2 [W_\mu^a t_{\text{T}}^a, W_\nu^b t_{\text{T}}^b] T$$

$$W_{\mu\nu}^a t_{\text{T}}^a T \equiv \frac{i}{g} [D_\mu, D_\nu] T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) t_{\text{T}}^a T - ig[W_\mu^a t_{\text{T}}^a, W_\nu^b t_{\text{T}}^b] T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc} W_\mu^b W_\nu^c) t_{\text{T}}^a T$$

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc} W_\mu^b W_\nu^c$$

$$[D_\mu, D_\nu] T \rightarrow U[D_\mu, D_\nu] T = U[D_\mu, D_\nu] U^\dagger U T \Rightarrow W_{\mu\nu}^a t_{\text{T}}^a \rightarrow UW_{\mu\nu}^a t_{\text{T}}^a U^\dagger$$

$$\text{tr}(W_{\mu\nu}^a t_{\text{T}}^a W^{b\mu\nu} t_{\text{T}}^b) \text{ is a gauge invariant}$$

Tensor notation

$$V_j^i(x) = \exp[i\theta^a(x)\tau^a]_j^i, \quad (V^\dagger)_j^i = \exp(-i\theta^a\tau^a)_j^i, \quad \tau^a \equiv \frac{\sigma^a}{2}$$

$$(D_\mu T)_j^i \equiv \partial_\mu T_j^i - ig(W_\mu)_k^i T_j^k + igT_k^i(W_\mu)_j^k$$

[Ref: Ta-Pei Cheng & Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, Eq.(4.140)]

$$T_j^i \rightarrow V_k^i T_l^k (V^\dagger)_j^l, \quad (W_\mu)_j^i \rightarrow V_k^i (W_\mu)_l^k (V^\dagger)_j^l + \frac{i}{g} V_k^i \partial_\mu (V^\dagger)_j^k$$

$$\text{Infinitesimal transformation } V_j^i \simeq \delta_j^i + i\theta^a (\tau^a)_j^i, \quad (V^\dagger)_j^i \simeq \delta_j^i - i\theta^a (\tau^a)_j^i$$

$$\delta T_j^i = i\theta^a (\tau^a)_k^i T_j^k - i\theta^a T_l^i (\tau^a)_j^l = i\theta^a [(\tau^a)_k^i T_j^k - T_k^i (\tau^a)_j^k]$$

$$\delta (W_\mu)_j^i = i\theta^a [(\tau^a)_k^i (W_\mu)_j^k - (W_\mu)_k^i (\tau^a)_j^k] + \frac{1}{g} (\partial_\mu \theta^a) (\tau^a)_j^i$$

$$\begin{aligned} \delta (D_\mu T)_j^i &= \delta (\partial_\mu T_j^i) - ig \delta (W_\mu)_l^i T_j^l + ig \delta (W_\mu)_j^l T_l^i - ig (W_\mu)_l^i \delta T_j^l + ig \delta T_l^i (W_\mu)_j^l \\ &= i(\partial_\mu \theta^a) [(\tau^a)_k^i T_j^k - T_k^i (\tau^a)_j^k] + i\theta^a [(\tau^a)_k^i \partial_\mu T_j^k - (\partial_\mu T_k^i) (\tau^a)_j^k] \\ &\quad + g\theta^a [(\tau^a)_k^i (W_\mu)_l^k - (W_\mu)_k^i (\tau^a)_l^k] T_j^l - i(\partial_\mu \theta^a) (\tau^a)_l^i T_j^l \\ &\quad - g\theta^a T_l^i [(\tau^a)_k^l (W_\mu)_j^k - (W_\mu)_k^l (\tau^a)_j^k] + i(\partial_\mu \theta^a) T_l^i (\tau^a)_j^l \\ &\quad + g(W_\mu)_l^i \theta^a [(\tau^a)_k^l T_j^k - T_k^l (\tau^a)_j^k] - g\theta^a [(\tau^a)_k^i T_l^k - T_k^i (\tau^a)_l^k] (W_\mu)_j^l \\ &= i\theta^a (\tau^a)_k^i \partial_\mu T_j^k + g\theta^a (\tau^a)_k^i (W_\mu)_l^k T_j^l - g\theta^a (\tau^a)_k^i T_l^k (W_\mu)_j^l \\ &\quad - i\theta^a (\partial_\mu T_k^i) (\tau^a)_j^k - g\theta^a (W_\mu)_l^i T_k^l (\tau^a)_j^k + g\theta^a T_l^i (W_\mu)_k^l (\tau^a)_j^k \\ &= i\theta^a (\tau^a)_k^i [\partial_\mu T_j^k - ig(W_\mu)_l^k T_j^l + igT_l^k (W_\mu)_j^l] - i\theta^a [\partial_\mu T_k^i - ig(W_\mu)_l^i T_k^l + igT_l^i (W_\mu)_k^l] (\tau^a)_j^k \\ &= i\theta^a [(\tau^a)_k^i (D_\mu T)_j^k - (D_\mu T)_k^i (\tau^a)_j^k] \end{aligned}$$

$$(D_\mu T)_j^i \text{ transforms as } T_j^i$$

$$\begin{aligned} (D_\mu D_\nu T)_j^i &= \partial_\mu \partial_\nu T_j^i - ig \partial_\mu (W_\nu)_k^i T_j^k - ig (W_\nu)_k^i \partial_\mu T_j^k + ig (\partial_\mu T_k^i) (W_\nu)_j^k + ig T_k^i \partial_\mu (W_\nu)_j^k \\ &\quad - ig (W_\mu)_l^i [\partial_\nu T_j^l - ig (W_\nu)_k^l T_j^k + ig T_k^l (W_\nu)_j^k] + ig [\partial_\nu T_l^i - ig (W_\nu)_k^l T_l^k + ig T_k^i (W_\nu)_l^k] (W_\mu)_j^l \\ ([D_\mu, D_\nu] T)_j^i &= -ig [\partial_\mu (W_\nu)_k^i - \partial_\nu (W_\mu)_k^i] T_j^k + ig T_k^i [\partial_\mu (W_\nu)_j^k - \partial_\nu (W_\mu)_j^k] \\ &\quad - g^2 [(W_\mu)_l^i (W_\nu)_k^l - (W_\nu)_l^i (W_\mu)_k^l] T_j^k + g^2 T_k^i [(W_\mu)_l^k (W_\nu)_j^l - (W_\nu)_l^k (W_\mu)_j^l] \\ (W_{\mu\nu})_k^i T_j^k - T_k^i (W_{\mu\nu})_j^k &\equiv \frac{i}{g} ([D_\mu, D_\nu] T)_j^i = \{\partial_\mu (W_\nu)_k^i - \partial_\nu (W_\mu)_k^i - ig [(W_\mu)_l^i (W_\nu)_k^l - (W_\nu)_l^i (W_\mu)_k^l]\} T_j^k \\ &\quad - T_k^i \{\partial_\mu (W_\nu)_j^k - \partial_\nu (W_\mu)_j^k - ig [(W_\mu)_l^k (W_\nu)_j^l - (W_\nu)_l^k (W_\mu)_j^l]\} \\ (W_{\mu\nu})_k^i &= \partial_\mu (W_\nu)_k^i - \partial_\nu (W_\mu)_k^i - ig [(W_\mu)_l^i (W_\nu)_k^l - (W_\nu)_l^i (W_\mu)_k^l] \end{aligned}$$

$$\begin{aligned} (W_{\mu\nu})_k^i T_j^k - T_k^i (W_{\mu\nu})_j^k &\rightarrow V_m^i [(W_{\mu\nu})_k^m T_n^k - T_k^m (W_{\mu\nu})_n^k] (V^\dagger)_j^n \\ &= V_m^i (W_{\mu\nu})_k^m (V^\dagger)_l^k V_p^l T_n^p (V^\dagger)_j^n - V_m^i T_k^m (V^\dagger)_l^k V_p^l (W_{\mu\nu})_n^p (V^\dagger)_j^n \end{aligned}$$

$$\Rightarrow (W_{\mu\nu})_j^i \text{ transforms as } T_j^i$$

$$(W_{\mu\nu})_j^i (W^{\mu\nu})_i^j \text{ is a gauge invariant}$$

Gauge interactions

For an SU(2) doublet D , $D_\mu D = (\partial_\mu - igW_\mu^a \tau^a)D$, or equivalently, $\underline{(D_\mu D)^i = \partial_\mu D^i - ig(W_\mu)_j^i D^j}$

$$\Rightarrow (W_\mu)_j^i = W_\mu^a (\tau^a)_j^i = \frac{1}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}$$

$$W_\mu^+ = \sqrt{2}(W_\mu)_2^1, \quad W_\mu^- = \sqrt{2}(W_\mu)_1^2, \quad W_\mu^3 = 2(W_\mu)_1^1 = -2(W_\mu)_2^2$$

For an SU(2) triplet T , $\underline{(D_\mu T)_j^i = \partial_\mu T_j^i - ig(W_\mu)_k^i T_j^k + igT_k^i (W_\mu)_j^k}$

$$T^+ = -T_2^1, \quad T^- = -T_1^2, \quad T^0 = \sqrt{2}T_1^1 = -\sqrt{2}T_2^2$$

$$(T^+)^\dagger = -(T^\dagger)_1^2, \quad (T^-)^\dagger = -(T^\dagger)_2^1, \quad (T^0)^\dagger = \sqrt{2}(T^\dagger)_1^1 = -\sqrt{2}(T^\dagger)_2^2$$

$$\begin{aligned} (T^\dagger)_j^i \bar{\sigma}^\mu (W_\mu)_k^j T_i^k &= (T^\dagger)_1^1 \bar{\sigma}^\mu (W_\mu)_1^1 T_1^1 + (T^\dagger)_1^1 \bar{\sigma}^\mu (W_\mu)_2^1 T_1^2 + (T^\dagger)_2^1 \bar{\sigma}^\mu (W_\mu)_1^2 T_1^1 + (T^\dagger)_2^1 \bar{\sigma}^\mu (W_\mu)_2^2 T_1^2 \\ &\quad + (T^\dagger)_1^2 \bar{\sigma}^\mu (W_\mu)_1^1 T_2^1 + (T^\dagger)_1^2 \bar{\sigma}^\mu (W_\mu)_2^1 T_2^2 + (T^\dagger)_2^2 \bar{\sigma}^\mu (W_\mu)_1^2 T_2^1 + (T^\dagger)_2^2 \bar{\sigma}^\mu (W_\mu)_2^2 T_2^2 \end{aligned}$$

$$= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^+ T^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^- T^0 - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^3 T^-$$

$$+ \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^3 T^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^+ T^0 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^- T^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0$$

$$= \frac{1}{2}[W_\mu^3(T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+(T^+)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^-(T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+(T^0)^\dagger \bar{\sigma}^\mu T^- - W_\mu^-(T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3(T^-)^\dagger \bar{\sigma}^\mu T^-]$$

$$\begin{aligned} (T^\dagger)_j^i \bar{\sigma}^\mu T_k^j (W_\mu)_i^k &= (T^\dagger)_1^1 \bar{\sigma}^\mu T_1^1 (W_\mu)_1^1 + (T^\dagger)_1^1 \bar{\sigma}^\mu T_2^1 (W_\mu)_1^2 + (T^\dagger)_2^1 \bar{\sigma}^\mu T_1^2 (W_\mu)_1^1 + (T^\dagger)_2^1 \bar{\sigma}^\mu T_2^2 (W_\mu)_1^2 \\ &\quad + (T^\dagger)_1^2 \bar{\sigma}^\mu T_1^1 (W_\mu)_2^1 + (T^\dagger)_1^2 \bar{\sigma}^\mu T_2^1 (W_\mu)_2^2 + (T^\dagger)_2^2 \bar{\sigma}^\mu T_1^2 (W_\mu)_2^1 + (T^\dagger)_2^2 \bar{\sigma}^\mu T_2^2 (W_\mu)_2^2 \end{aligned}$$

$$= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^+ W_\mu^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^- W_\mu^3 + \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^0 W_\mu^-$$

$$- \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^0 W_\mu^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^+ W_\mu^3 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^- W_\mu^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3$$

$$= \frac{1}{2}[-W_\mu^3(T^+)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+(T^+)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^-(T^0)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+(T^0)^\dagger \bar{\sigma}^\mu T^- + W_\mu^-(T^-)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^3(T^-)^\dagger \bar{\sigma}^\mu T^-]$$

$$\mathcal{L} \supset g[(T^\dagger)_j^i \bar{\sigma}^\mu (W_\mu)_k^j T_i^k - (T^\dagger)_j^i \bar{\sigma}^\mu T_k^j (W_\mu)_i^k]$$

$$\begin{aligned} &= g[W_\mu^3(T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+(T^+)^\dagger \bar{\sigma}^\mu T^0 \\ &\quad + W_\mu^-(T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+(T^0)^\dagger \bar{\sigma}^\mu T^- \\ &\quad - W_\mu^-(T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3(T^-)^\dagger \bar{\sigma}^\mu T^-] \end{aligned}$$

For a general SU(2) multiplet $\psi_{j_1 \dots j_q}^{i_1 \dots i_p}$,

$$(D_\mu \psi)_{j_1 \dots j_q}^{i_1 \dots i_p} = \partial_\mu \psi_{j_1 \dots j_q}^{i_1 \dots i_p} - ig \left[\sum_{m=1}^p (W_\mu)_{k_m}^{i_m} \psi_{j_1 \dots j_q}^{i_1 \dots k_m \dots i_p} - \sum_{n=1}^q \psi_{j_1 \dots i_n \dots j_q}^{i_1 \dots i_p} (W_\mu)_{j_n}^{i_n} \right]$$

[Note: in differential geometry, the gauge field $(W_\mu)_j^i$ is a connection form. The gauge connection defines a principal bundle whose base space is the spacetime and structure group is the gauge group.]