标准模型及其费曼规则

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目录

本文各种约定主要遵从文献 [1].

度规张量

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1}$$

Pauli 矩阵

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{2}$$

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}). \tag{3}$$

手征表示下的 Dirac 矩阵

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}. \tag{4}$$

左右手投影算符

$$P_L \equiv \frac{1}{2}(1 - \gamma_5), \quad P_R \equiv \frac{1}{2}(1 + \gamma_5). \tag{5}$$

1 拉氏量

在粒子物理标准模型中, Higgs 场相关拉氏量

$$\mathcal{L}_{\mathrm{H}} = (D^\mu \Phi)^\dagger (D_\mu \Phi) - V_{\mathrm{H}}(\Phi), \quad V_{\mathrm{H}}(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2, \tag{6}$$

其中

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix} = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}[v + H(x) + i\chi(x)] \end{pmatrix}, \tag{7}$$

$$D_\mu \Phi = (\partial_\mu - ig' B_\mu Y_H - ig W_\mu^a T^a)\Phi, \quad Y_H = \frac{1}{2}, \quad T^a = \frac{\sigma^a}{2}. \tag{8}$$

由 Higgs 场势能 $V_{\mathrm{H}}(\Phi)$ 的极小值条件可知, 真空期待值 v 应满足

$$\mu^2 = \lambda v^2. \tag{9}$$

在幺正规范下,

$$\Phi(x) \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \quad \Phi^\dagger \Phi \to \frac{1}{2}(v + H)^2, \tag{10}$$

势能项

$$\begin{aligned} -V_{\mathrm{H}}(\Phi) &= \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \to \frac{1}{2}\mu^2 (v+H)^2 - \frac{1}{4}\lambda (v+H)^4 \\ &= \frac{1}{2}\mu^2 (v^2 + H^2 + 2vH) - \frac{1}{4}\lambda (v^4 + 4v^2H^2 + H^4 + 4v^3 H + 2v^2 H^2 + 4vH^3) \\ &= \frac{1}{4}\mu^2 v^2 + \frac{1}{4}(\mu^2 - \lambda v^2) v^2 + (\mu^2 - \lambda v^2) vH + \frac{1}{2}(\mu^2 - \lambda v^2) H^2 - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 \\ &= \frac{1}{4}\mu^2 v^2 - \frac{1}{2} m_H^2 H^2 - \frac{1}{2}\frac{m_H^2}{v} H^3 - \frac{1}{8}\frac{m_H^2}{v^2} H^4, \end{aligned} \tag{11}$$

其中 Higgs 粒子的质量为

$$m_H \equiv \sqrt{2}\mu, \quad m_H^2 = 2\mu^2 = 2\lambda v^2. \tag{12}$$

物理的光子场 $A_\mu(x)$, Z^0 玻色子矢量场 $Z_\mu(x)$ 和 $W^\pm$ 玻色子矢量场 $W_\mu^\pm(x)$ 与 $SU(2)_L \times U(1)_Y$ 规范场 $W_\mu^a(x)$ 和 $B_\mu(x)$ 之间的关系为

$$A_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}}(g' W_\mu^3 + g B_\mu) = s_W W_\mu^3 + c_W B_\mu, \tag{13}$$

$$Z_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}}(g W_\mu^3 - g' B_\mu) = c_W W_\mu^3 - s_W B_\mu, \tag{14}$$

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2), \tag{15}$$

或

$$B_\mu = c_W A_\mu - s_W Z_\mu, \quad W_\mu^3 = s_W A_\mu + c_W Z_\mu, \tag{16}$$

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-), \quad W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-). \tag{17}$$

参数间有如下关系,

$$s_W \equiv \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \quad c_W \equiv \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad e \equiv \frac{gg'}{\sqrt{g^2+g'^2}} = gs_W = g'c_W. \tag{18}$$

从而,

$$\begin{aligned} g'B_\mu + gW_\mu^3 &= g'(c_W A_\mu - s_W Z_\mu) + g(s_W A_\mu + c_W Z_\mu) = 2eA_\mu + \frac{g^2-g'^2}{\sqrt{g^2+g'^2}} Z_\mu \\ &= 2eA_\mu + \sqrt{g^2+g'^2}(c_W^2 - s_W^2) Z_\mu, \end{aligned} \tag{19}$$

则

$$\begin{aligned} g'B_\mu Y_H + gW_\mu^a T^a &= \frac{1}{2}\begin{pmatrix} g'B_\mu + gW_\mu^3 & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & g'B_\mu - gW_\mu^3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2eA_\mu + \sqrt{g^2+g'^2}(c_W^2 - s_W^2)Z_\mu & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & -\sqrt{g^2+g'^2}Z_\mu \end{pmatrix}. \end{aligned} \tag{20}$$

于是, 在幺正规范下,

$$\begin{aligned} & (D^\mu\Phi)^\dagger(D_\mu\Phi) \\ \to & \left| \left[\partial_\mu - \frac{i}{2}\begin{pmatrix} 2eA_\mu + \sqrt{g^2+g'^2}(c_W^2 - s_W^2)Z_\mu & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & -\sqrt{g^2+g'^2}Z_\mu \end{pmatrix} \right] \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix} \right|^2 \\ = & \frac{1}{2}\left(\frac{i}{\sqrt{2}}gW_\mu^-(v+H),\ \partial_\mu H - \frac{i}{2}\sqrt{g^2+g'^2}Z_\mu(v+H) \right) \begin{pmatrix} -\frac{i}{\sqrt{2}}gW_\mu^+(v+H) \\ \partial_\mu H + \frac{i}{2}\sqrt{g^2+g'^2}Z_\mu(v+H) \end{pmatrix} \\ = & \frac{1}{2}(\partial^\mu H)(\partial_\mu H) + (v+H)^2\left[\frac{1}{4}g^2 W_\mu^+ W^{-\mu} + \frac{1}{8}(g^2+g'^2)Z_\mu Z^\mu \right] \\ = & \frac{1}{2}(\partial^\mu H)(\partial_\mu H) + m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \\ & + 2\frac{m_W^2}{v}HW_\mu^+W^{-\mu} + \frac{m_Z^2}{v}HZ_\mu Z^\mu + \frac{m_W^2}{v^2}H^2 W_\mu^+ W^{-\mu} + \frac{m_Z^2}{2v^2}H^2 Z_\mu Z^\mu. \end{aligned} \tag{21}$$

故 $W^\pm$ 和 Z^0 玻色子通过 Brout–Englert–Higgs 机制获得质量, 分别为

$$m_W \equiv \frac{1}{2}gv, \quad m_Z \equiv \frac{1}{2}\sqrt{g^2+g'^2}v. \tag{22}$$

标准模型中的费米子, 包括 3 代中微子 $\nu_i = \nu_e, \nu_\mu, \nu_\tau$, 3 代带电轻子 $\ell_i = e, \mu, \tau$, 3 代上型夸克 $u_i = u, c, t$ 和 3 代下型夸克 $d_i = d, s, b$ $(i = 1, 2, 3)$. 左手的费米子构成 $SU(2)_L$ 二重态

$$L_{iL} = \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}_L, \quad Q_{iL} = \begin{pmatrix} u_i \\ d_i' \end{pmatrix}_L. \tag{23}$$

下型夸克的质量态 d_j 与规范态 d_j' 通过 CKM 矩阵 V_{ij} 联系起来,

$$d_i' = V_{ij}d_j. \tag{24}$$

标准模型费米子的量子数见表 1. 电荷数 Q, T^3 和弱超荷 Y 存在如下关系,

$$Q = T^3 + Y. \tag{25}$$

Table 1: 标准模型费米子的量子数.

			Q	T^3	Y	B	L
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	0 -1	$1/2$ $-1/2$	$-1/2$ $-1/2$	0 0	1 1
$\begin{pmatrix} u \\ d' \end{pmatrix}_L$	$\begin{pmatrix} c \\ s' \end{pmatrix}_L$	$\begin{pmatrix} t \\ b' \end{pmatrix}_L$	$2/3$ $-1/3$	$1/2$ $-1/2$	$1/6$ $1/6$	$1/3$ $1/3$	0 0
e_R	μ_R	τ_R	-1	0	-1	0	1
u_R	c_R	t_R	$2/3$	0	$2/3$	$1/3$	0
d'_R	s'_R	b'_R	$-1/3$	0	$-1/3$	$1/3$	0

$Y=-1/2$ 的 Higgs 场共轭态为

$$\tilde{\Phi}(x)=i\sigma^2\Phi^*(x)=\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix}\phi^-(x) \\ \phi^{0*}(x)\end{pmatrix}=\begin{pmatrix}\phi^{0*}(x) \\ -\phi^-(x)\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}[v+H(x)-i\chi(x)] \\ -\phi^-(x)\end{pmatrix}. \tag{26}$$

利用它可以写下 Yukawa 耦合项

$$\begin{aligned}\mathcal{L}_{\mathrm{Y}} &= -\lambda_{d_i}\bar{Q}_{iL}d'_{iR}\Phi-\lambda_{u_i}\bar{Q}_{iL}u_{iR}\tilde{\Phi}-\lambda_{\ell_i}\bar{L}_{iL}\ell_{iR}\Phi+\mathrm{h.c.}\\ &\to -\frac{\lambda_{d_i}}{\sqrt{2}}(v+H)\bar{d}'_{iL}d'_{iR}-\frac{\lambda_{u_i}}{\sqrt{2}}(v+H)\bar{u}_{iL}u_{iR}-\frac{\lambda_{\ell_i}}{\sqrt{2}}(v+H)\bar{\ell}_{iL}\ell_{iR}+\mathrm{h.c.}\\ &= -m_{d_i}\bar{d}_id_i-m_{u_i}\bar{u}_iu_i-m_{\ell_i}\bar{\ell}_i\ell_i-\frac{m_{d_i}}{v}H\bar{d}_id_i-\frac{m_{u_i}}{v}H\bar{u}_iu_i-\frac{m_{\ell_i}}{v}H\bar{\ell}_i\ell_i.\end{aligned} \tag{27}$$

通过这种耦合, 费米子获得了质量,

$$m_{d_i}\equiv\frac{1}{\sqrt{2}}\lambda_{d_i}v,\quad m_{u_i}\equiv\frac{1}{\sqrt{2}}\lambda_{u_i}v,\quad m_{\ell_i}\equiv\frac{1}{\sqrt{2}}\lambda_{\ell_i}v. \tag{28}$$

费米子与电弱规范玻色子的耦合源于拉氏量

$$\mathcal{L}_{\mathrm{EWF}}=\bar{Q}_{iL}i\not{D}Q_{iL}+\bar{u}_{iR}i\not{D}u_{iR}+\bar{d}'_{iR}i\not{D}d'_{iR}+\bar{L}_{iL}i\not{D}L_{iL}+\bar{\ell}_{iR}i\not{D}\ell_{iR}. \tag{29}$$

利用

$$\begin{aligned}g'YB_\mu+gT^3W^3_\mu &= g'Y(c_WA_\mu-s_WZ_\mu)+gT^3(s_WA_\mu+c_WZ_\mu)\\ &= e(Y+T^3)A_\mu+\left(gc_WT^3-\frac{gs_W}{c_W}s_WY\right)Z_\mu=QeA_\mu+\frac{g}{c_W}(T^3c_W^2-Ys_W^2)Z_\mu\\ &= QeA_\mu+\frac{g}{c_W}(T^3-Qs_W^2)Z_\mu,\end{aligned} \tag{30}$$

有

$$D_\mu Q_{iL} = (\partial_\mu-ig'B_\mu Y-igW^a_\mu T^a)Q_{iL}=\partial_\mu Q_{iL}-i\begin{pmatrix}g'YB_\mu+gT^3W^3_\mu & \frac{1}{2}g(W^1_\mu-iW^2_\mu)\\ \frac{1}{2}g(W^1_\mu+iW^2_\mu) & g'YB_\mu+gT^3W^3_\mu\end{pmatrix}Q_{iL}$$

$$
\begin{aligned}
&= \partial_\mu Q_{iL} - i \begin{pmatrix} QeA_\mu + \dfrac{g}{c_W}(T^3 - Qs_W^2)Z_\mu & \dfrac{1}{\sqrt{2}} g W_\mu^+ \\ \dfrac{1}{\sqrt{2}} g W_\mu^- & QeA_\mu + \dfrac{g}{c_W}(T^3 - Qs_W^2)Z_\mu \end{pmatrix} Q_{iL} \\
&= \partial_\mu Q_{iL} - i \begin{pmatrix} \left[QeA_\mu + \dfrac{g}{c_W}(T^3 - Qs_W^2)Z_\mu\right] u_{iL} + \dfrac{1}{\sqrt{2}} g W_\mu^+ d'_{iL} \\ \dfrac{1}{\sqrt{2}} g W_\mu^- u_{iL} + \left[QeA_\mu + \dfrac{g}{c_W}(T^3 - Qs_W^2)\right] d'_{iL} \end{pmatrix},
\end{aligned} \tag{31}
$$

故

$$
\begin{aligned}
\bar{Q}_{iL} i \not{D} Q_{iL} \supset{}& \left[QeA_\mu + \frac{g}{c_W}(T^3 - Qs_W^2)Z_\mu\right] \bar{u}_{iL}\gamma^\mu u_{iL} + \left[QeA_\mu + \frac{g}{c_W}(T^3 - Qs_W^2)\right] \bar{d}'_{iL}\gamma^\mu d'_{iL} \\
&+ \frac{1}{\sqrt{2}} g W_\mu^+ \bar{u}_{iL}\gamma^\mu d'_{iL} + \frac{1}{\sqrt{2}} g W_\mu^- \bar{d}'_{iL}\gamma^\mu u_{iL} \\
={}& \left(QeA_\mu + \frac{g}{c_W} g_L Z_\mu\right) \bar{u}_i \gamma^\mu \frac{1-\gamma_5}{2} u_i + \frac{1}{2}\left(QeA_\mu + \frac{g}{c_W} g_L\right) \bar{d}_i \gamma^\mu \frac{1-\gamma_5}{2} d_i \\
&+ \frac{1}{\sqrt{2}} g W_\mu^+ \bar{u}_i \gamma^\mu \frac{1-\gamma_5}{2} V_{ij} d_j + \frac{1}{\sqrt{2}} g W_\mu^- \bar{d}_j V_{ji}^\dagger \gamma^\mu \frac{1-\gamma_5}{2} u_i,
\end{aligned} \tag{32}
$$

其中

$$
g_L \equiv T^3 - Qs_W^2. \tag{33}
$$

另一方面,

$$
D_\mu d'_{iR} = (\partial_\mu - ig' B_\mu Y) d'_{iR} = \partial_\mu d'_{iR} - ig' Q(c_W A_\mu - s_W Z_\mu) d'_{iR} = \partial_\mu d'_{iR} - iQeA_\mu d'_{iR} + i\frac{g}{c_W} Q s_W^2 Z_\mu d'_{iR}, \tag{34}
$$

则

$$
\begin{aligned}
\bar{u}_{iR} i \not{D} u_{iR} + \bar{d}'_{iR} i \not{D} d'_{iR} \supset{}& \left(QeA_\mu - \frac{g}{c_W} Q s_W^2 Z_\mu\right) \bar{u}_{iR}\gamma^\mu u_{iR} + \left(QeA_\mu - \frac{g}{c_W} Q s_W^2 Z_\mu\right) \bar{d}'_{iR}\gamma^\mu d'_{iR} \\
={}& \left(QeA_\mu + \frac{g}{c_W} g_R Z_\mu\right) \bar{u}_i \gamma^\mu \frac{1+\gamma_5}{2} u_i + \left(QeA_\mu + \frac{g}{c_W} g_R Z_\mu\right) \bar{d}_i \gamma^\mu \frac{1+\gamma_5}{2} d_i,
\end{aligned} \tag{35}
$$

其中

$$
g_R \equiv -Qs_W^2. \tag{36}
$$

定义

$$
g_V \equiv g_L + g_R = T^3 - 2Qs_W^2, \quad g_A \equiv g_L - g_R = T^3, \tag{37}
$$

可得

$$
\begin{aligned}
&\bar{Q}_{iL} i \not{D} Q_{iL} + \bar{u}_{iR} i \not{D} u_{iR} + \bar{d}'_{iR} i \not{D} d'{}_{iR} \\
\supset{}& Qe\bar{u}_i\gamma^\mu u_i A_\mu + Qe\bar{d}\gamma^\mu d_i A_\mu + \frac{g}{2c_W}\bar{u}_i\gamma^\mu(g_V - g_A\gamma_5)u_i Z_\mu + \frac{g}{2c_W}\bar{d}_i\gamma^\mu(g_V - g_A\gamma_5)d_i Z_\mu \\
&+ \frac{1}{\sqrt{2}} g W_\mu^+ \bar{u}_i \gamma^\mu \frac{1-\gamma_5}{2} V_{ij} d_j + \frac{1}{\sqrt{2}} g W_\mu^- \bar{d}_j V_{ji}^\dagger \gamma^\mu \frac{1-\gamma_5}{2} u_i.
\end{aligned} \tag{38}
$$

同理, 有

$$
\begin{aligned}
\bar{L}_{iL} i \not{D} L_{iL} + \bar{\ell}_{iR} i \not{D} \ell_{iR} \supset{}& Qe\bar{\ell}_i\gamma^\mu \ell_i A_\mu + \frac{g}{2c_W}\bar{\ell}_i\gamma^\mu(g_V - g_A\gamma_5)\ell_i Z_\mu + \frac{g}{2c_W}\bar{\nu}_i\gamma^\mu(g_V - g_A\gamma_5)\nu_i Z_\mu \\
&+ \frac{1}{\sqrt{2}} g W_\mu^+ \bar{\nu}_i \gamma^\mu \frac{1-\gamma_5}{2} \ell_i + \frac{1}{\sqrt{2}} g W_\mu^- \bar{\ell}_i \gamma^\mu \frac{1-\gamma_5}{2} \nu_i.
\end{aligned} \tag{39}
$$

总结起来, 可以写成流耦合的形式,

$$\begin{aligned}\mathcal{L}_{\text{EWF}} &\supset \sum_f \left[Q_f e\bar{f}\gamma^\mu f A_\mu + \frac{g}{2c_W}\bar{f}\gamma^\mu(g_V^f - g_A^f\gamma_5)fZ_\mu\right] + g(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu}) \\ &= eA_\mu J_{\text{EM}}^\mu + g(Z_\mu J_Z^\mu + W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu}),\end{aligned} \tag{40}$$

其中, 流的定义为

$$\begin{aligned}J_{\text{EM}}^\mu &\equiv \sum_f Q_f\bar{f}\gamma^\mu f, \quad J_Z^\mu \equiv \frac{1}{2c_W}\sum_f \bar{f}\gamma^\mu(g_V^f - g_A^f\gamma_5)f = \frac{1}{c_W}\sum_f(g_L^f\bar{f}_L\gamma^\mu f_L + g_R^f\bar{f}_R\gamma^\mu f_R), \\ J_W^{+\mu} &\equiv \frac{1}{\sqrt{2}}(\bar{u}_{iL}\gamma^\mu V_{ij}d_{jL} + \bar{\nu}_{iL}\gamma^\mu\ell_{iL}), \quad J_W^{-\mu} \equiv \frac{1}{\sqrt{2}}(\bar{d}_{jL}V_{ji}^\dagger\gamma^\mu u_{iL} + \bar{\ell}_{iL}\gamma^\mu\nu_{iL}).\end{aligned} \tag{41}$$

对于各种费米子, 相关系数如下:

$$Q_{u_i} = \frac{2}{3}, \quad Q_{d_i} = -\frac{1}{3}, \quad Q_{\nu_i} = 0, \quad Q_{\ell_i} = -1; \tag{42}$$

$$g_V^{u_i} = \frac{1}{2} - \frac{4}{3}s_W^2, \quad g_A^{u_i} = \frac{1}{2}; \quad g_V^{d_i} = -\frac{1}{2} + \frac{2}{3}s_W^2, \quad g_A^{d_i} = -\frac{1}{2}; \tag{43}$$

$$g_V^{\nu_i} = \frac{1}{2}, \quad g_A^{\nu_i} = \frac{1}{2}; \quad g_V^{\ell_i} = -\frac{1}{2} + 2s_W^2, \quad g_A^{\ell_i} = -\frac{1}{2}; \tag{44}$$

$$g_L^{u_i} = \frac{1}{2} - \frac{2}{3}s_W^2, \quad g_R^{u_i} = -\frac{2}{3}s_W^2; \quad g_L^{d_i} = -\frac{1}{2} + \frac{1}{3}s_W^2, \quad g_R^{d_i} = \frac{1}{3}s_W^2; \tag{45}$$

$$g_L^{\nu_i} = \frac{1}{2}, \quad g_R^{\nu_i} = 0; \quad g_L^{\ell_i} = -\frac{1}{2} + s_W^2, \quad g_R^{\ell_i} = s_W^2. \tag{46}$$

纯电弱规范场相关的拉氏量

$$\mathcal{L}_{\text{EWG}} = -\frac{1}{4}W_{\mu\nu}^a W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{47}$$

其中

$$W^{a\mu\nu} \equiv \partial^\mu W^{a\nu} - \partial^\nu W^{a\mu} + g\varepsilon^{abc}W^{b\mu}W^{c\nu}, \quad B^{\mu\nu} \equiv \partial^\mu B^\nu - \partial^\nu B^\mu. \tag{48}$$

利用(16)式和(17)式, 可得

$$\begin{aligned}& W_\mu^2 W_\nu^3 - W_\mu^3 W_\nu^2 \\ =\ & \frac{i}{\sqrt{2}}[(W_\mu^+ - W_\mu^-)(s_W A_\nu + c_W Z_\nu) - (s_W A_\mu + c_W Z_\mu)(W_\nu^+ - W_\nu^-)] \\ =\ & \frac{i}{\sqrt{2}}[s_W(W_\mu^+ A_\nu - A_\mu W_\nu^+) + c_W(W_\mu^+ Z_\nu - Z_\mu W_\nu^+) - s_W(W_\mu^- A_\nu - A_\mu W_\nu^-) - c_W(W_\mu^- Z_\nu - Z_\mu W_\nu^-)],\end{aligned} \tag{49}$$

$$\begin{aligned}& W_\mu^3 W_\nu^1 - W_\mu^1 W_\nu^3 \\ =\ & \frac{1}{\sqrt{2}}[(s_W A_\mu + c_W Z_\mu)(W_\nu^+ + W_\nu^-) - (W_\mu^+ + W_\mu^-)(s_W A_\nu + c_W Z_\nu)] \\ =\ & -\frac{1}{\sqrt{2}}[s_W(W_\mu^+ A_\nu - A_\mu W_\nu^+) + c_W(W_\mu^+ Z_\nu - Z_\mu W_\nu^+) + s_W(W_\mu^- A_\nu - A_\mu W_\nu^-) + c_W(W_\mu^- Z_\nu - Z_\mu W_\nu^-)].\end{aligned} \tag{50}$$

从而,

$$\begin{aligned}W_{\mu\nu}^1 &= \partial_\mu W_\nu^1 - \partial_\nu W_\mu^1 + g\varepsilon^{1bc}W_\mu^b W_\nu^c = \partial_\mu W_\nu^1 - \partial_\nu W_\mu^1 + gW_\mu^2 W_\nu^3 - gW_\mu^3 W_\nu^2 \\ &= \frac{1}{\sqrt{2}}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+) + \frac{1}{\sqrt{2}}(\partial_\mu W_\nu^- - \partial_\nu W_\mu^-) + g(W_\mu^2 W_\nu^3 - gW_\mu^3 W_\nu^2) \\ &= \frac{1}{\sqrt{2}}\{\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + ig[s_W(W_\mu^+ A_\nu - A_\mu W_\nu^+) + c_W(W_\mu^+ Z_\nu - Z_\mu W_\nu^+)]\} \\ &\quad + \frac{1}{\sqrt{2}}\{\partial_\mu W_\nu^- - \partial_\nu W_\mu^- - ig[s_W(W_\mu^- A_\nu - A_\mu W_\nu^-) + c_W(W_\mu^- Z_\nu - Z_\mu W_\nu^-)]\}\end{aligned}$$

$$= \frac{1}{\sqrt{2}}(F^+_{\mu\nu} + F^-_{\mu\nu}), \tag{51}$$

其中,

$$F^+_{\mu\nu} \equiv \partial_\mu W^+_\nu - \partial_\nu W^+_\mu + ie(W^+_\mu A_\nu - A_\mu W^+_\nu) + igc_W(W^+_\mu Z_\nu - Z_\mu W^+_\nu), \tag{52}$$
$$F^-_{\mu\nu} \equiv \partial_\mu W^-_\nu - \partial_\nu W^-_\mu - ie(W^-_\mu A_\nu - A_\mu W^-_\nu) - igc_W(W^-_\mu Z_\nu - Z_\mu W^-_\nu). \tag{53}$$

另一方面,

$$\begin{aligned} W^2_{\mu\nu} &= \partial_\mu W^2_\nu - \partial_\nu W^2_\mu + g\varepsilon^{2bc}W^b_\mu W^c_\nu = \partial_\mu W^2_\nu - \partial_\nu W^2_\mu - gW^1_\mu W^3_\nu + gW^3_\mu W^1_\nu \\ &= \frac{i}{\sqrt{2}}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu) - \frac{i}{\sqrt{2}}(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) + g(W^3_\mu W^1_\nu - W^1_\mu W^3_\nu) \\ &= \frac{i}{\sqrt{2}}\{\partial_\mu W^+_\nu - \partial_\nu W^+_\mu + ig[s_W(W^+_\mu A_\nu - A_\mu W^+_\nu) + c_W(W^+_\mu Z_\nu - Z_\mu W^+_\nu)]\} \\ &\quad \times - \frac{i}{\sqrt{2}}\{\partial_\mu W^-_\nu - \partial_\nu W^-_\mu - ig[s_W(W^-_\mu A_\nu - A_\mu W^-_\nu) + c_W(W^-_\mu Z_\nu - Z_\mu W^-_\nu)]\} \\ &= \frac{i}{\sqrt{2}}(F^+_{\mu\nu} - F^-_{\mu\nu}). \end{aligned} \tag{54}$$

因此,

$$\begin{aligned} -\frac{1}{4}W^1_{\mu\nu}W^{1\mu\nu} - \frac{1}{4}W^2_{\mu\nu}W^{2\mu\nu} &= -\frac{1}{8}(F^+_{\mu\nu} + F^-_{\mu\nu})(F^{+\mu\nu} + F^{-\mu\nu}) + \frac{1}{8}(F^+_{\mu\nu} - F^-_{\mu\nu})(F^{+\mu\nu} - F^{-\mu\nu}) = -\frac{1}{2}F^+_{\mu\nu}F^{-\mu\nu} \\ &= -\frac{1}{2}[\partial_\mu W^+_\nu - \partial_\nu W^+_\mu + ie(W^+_\mu A_\nu - A_\mu W^+_\nu) + igc_W(W^+_\mu Z_\nu - Z_\mu W^+_\nu)] \\ &\quad \times[\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu} - ie(W^{-\mu}A^\nu - A^\mu W^{-\nu}) - igc_W(W^{-\mu}Z^\nu - Z^\mu W^{-\nu})] \\ &= -(\partial_\mu W^+_\nu)(\partial^\mu W^{-\nu}) + (\partial_\mu W^+_\nu)(\partial^\nu W^{-\mu}) \\ &\quad +ie[(\partial_\mu W^+_\nu)W^{-\mu}A^\nu - (\partial_\mu W^+_\nu)W^{-\nu}A^\mu - W^+_\mu(\partial^\mu W^{-\nu})A_\nu + W^+_\nu(\partial^\mu W^{-\nu})A_\mu] \\ &\quad +igc_W[(\partial_\mu W^+_\nu)W^{-\mu}Z^\nu - (\partial_\mu W^+_\nu)W^{-\nu}Z^\mu - W^+_\mu(\partial^\mu W^{-\nu})Z_\nu + W^+_\nu(\partial^\mu W^{-\nu})Z_\mu] \\ &\quad +e^2(W^+_\mu W^{-\nu}A_\nu A^\mu - W^+_\mu W^{-\mu}A_\nu A^\nu) + g^2c^2_W(W^+_\mu W^{-\nu}Z_\nu Z^\mu - W^+_\mu W^{-\mu}Z_\nu Z^\nu) \\ &\quad +egc_W(W^+_\mu W^{-\nu}A_\nu Z^\mu + W^+_\mu W^{-\nu}A^\mu Z_\nu - 2W^+_\mu W^{-\mu}A_\nu Z^\nu). \end{aligned} \tag{55}$$

由

$$W^1_\mu W^2_\nu - W^2_\mu W^1_\nu = \frac{i}{2}(W^+_\mu + W^-_\mu)(W^+_\nu - W^-_\nu) - \frac{i}{2}(W^+_\mu - W^-_\mu)(W^+_\nu + W^-_\nu) = -i(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu), \tag{56}$$

可得

$$\begin{aligned} W^3_{\mu\nu} &= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu + g\varepsilon^{3bc}W^b_\mu W^c_\nu = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu + gW^1_\mu W^2_\nu - gW^2_\mu W^1_\nu \\ &= s_W\partial_\mu A_v + c_W\partial_\mu Z_v - s_W\partial_\nu A_\mu + c_W\partial_\nu Z_\mu + g(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu) \\ &= s_W(\partial_\mu A_\nu - \partial_\nu A_\mu) + c_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu) - ig(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu), \end{aligned} \tag{57}$$
$$B_{\mu\nu} = \partial_\mu(c_W A_\nu - s_W Z_\nu) - \partial_\nu(c_W A_\mu - s_W Z_\mu) = c_W(\partial_\mu A_\nu - \partial_\nu A_\mu) - s_W(\partial_\mu Z_\nu - \partial_\nu Z_\mu). \tag{58}$$

于是,

$$\begin{aligned} &-\frac{1}{4}W^3_{\mu\nu}W^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ =& -\frac{1}{2}[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)] - \frac{1}{2}[(\partial_\mu Z_\nu)(\partial^\mu Z^\nu) - (\partial_\mu Z_\nu)(\partial^\nu Z^\mu)] \\ &+ie[W^{+\mu}W^{-\nu}(\partial_\mu A_\nu) - W^{+\nu}W^{-\mu}(\partial_\mu A_\nu)] + igc_W[W^{+\mu}W^{-\nu}(\partial_\mu Z_\nu) - W^{+\nu}W^{-\mu}(\partial_\mu Z_\nu)] \end{aligned}$$

$$+\frac{1}{2}g^2(W^+_\mu W^{+\mu}W^-_\nu W^{-\nu} - W^+_\mu W^{+\nu}W^-_\nu W^{-\mu}). \tag{59}$$

综合起来, 有

$$\begin{aligned}
\mathcal{L}_{\rm EWG} =\ & \frac{1}{2}[(\partial_\mu A_\nu)(\partial^\nu A^\mu) - (\partial_\mu A_\nu)(\partial^\mu A^\nu)] + \frac{1}{2}[(\partial_\mu Z_\nu)(\partial^\nu Z^\mu) - (\partial_\mu Z_\nu)(\partial^\mu Z^\nu)] \\
& +(\partial_\mu W^+_\nu)(\partial^\nu W^{-\mu}) - (\partial_\mu W^+_\nu)(\partial^\mu W^{-\nu}) + \frac{1}{2}g^2(W^+_\mu W^{+\mu}W^-_\nu W^{-\nu} - W^+_\mu W^{+\nu}W^-_\nu W^{-\mu}) \\
& +ie[(\partial_\mu W^+_\nu)W^{-\mu}A^\nu - (\partial_\mu W^+_\nu)W^{-\nu}A^\mu - W^{+\mu}(\partial_\mu W^-_\nu)A^\nu + W^{+\nu}(\partial_\mu W^-_\nu)A^\mu \\
& \quad +W^{+\mu}W^{-\nu}(\partial_\mu A_\nu) - W^{+\nu}W^{-\mu}(\partial_\mu A_\nu)] + e^2(W^+_\mu W^{-\nu}A_\nu A^\mu - W^+_\mu W^{-\mu}A_\nu A^\nu) \\
& +igc_W[(\partial_\mu W^+_\nu)W^{-\mu}Z^\nu - (\partial_\mu W^+_\nu)W^{-\nu}Z^\mu - W^{+\mu}(\partial_\mu W^-_\nu)Z^\nu + W^{+\nu}(\partial_\mu W^-_\nu)Z^\mu \\
& \quad\quad +W^{+\mu}W^{-\nu}(\partial_\mu Z_\nu) - W^{+\nu}W^{-\mu}(\partial_\mu Z_\nu)] + g^2c_W^2(W^+_\mu W^{-\nu}Z_\nu Z^\mu - W^+_\mu W^{-\mu}Z_\nu Z^\nu) \\
& +egc_W(W^+_\mu W^{-\nu}A_\nu Z^\mu + W^+_\mu W^{-\nu}A^\mu Z_\nu - 2W^+_\mu W^{-\mu}A_\nu Z^\nu).
\end{aligned} \tag{60}$$

QCD 的拉氏量可表达成

$$\mathcal{L}_{\rm QCD} = \sum_q \bar{q}(i\gamma^\mu D_\mu - m_q)q - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}, \quad q = u, d, s, c, b, t, \quad a = 1, \cdots, 8, \tag{61}$$

其中

$$D_\mu = \partial_\mu - ig_sG^a_\mu t^a, \quad G^{a\mu\nu} \equiv \partial^\mu G^{a\nu} - \partial^\nu G^{a\mu} + g_sf^{abc}G^{b\mu}G^{c\nu}. \tag{62}$$

$SU(3)_C$ 群基础表示生成元 $t^a = \lambda^a/2$, 其中 λ^a 为 Gell-Mann 矩阵. 生成元对易关系为 $[t^a, t^b] = if^{abc}t^c$. 结构常数 f^{abc} 是全反对称的, 其非零分量为

$$f_{123} = 1, \quad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \tag{63}$$

由

$$\begin{aligned}
-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} &= -\frac{1}{4}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_sf^{abc}G^b_\mu G^c_\nu)(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu} + g_sf^{ade}G^{d\mu}G^{e\nu}) \\
&= -\frac{1}{2}[(\partial_\mu G^a_\nu)(\partial^\mu G^{a\nu}) - (\partial_\mu G^a_\nu)(\partial^\nu G^{a\mu})] - g_sf^{abc}(\partial_\mu G^a_\nu)G^{b\mu}G^{c\nu} - \frac{1}{4}g_s^2f^{abc}f^{ade}G^b_\mu G^c_\nu G^{d\mu}G^{e\nu},
\end{aligned} \tag{64}$$

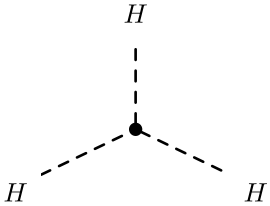
可得

$$\begin{aligned}
\mathcal{L}_{\rm QCD} =\ & \sum_q [\bar{q}(i\gamma^\mu\partial_\mu - m_q)q + g_sG^a_\mu\bar{q}\gamma^\mu t^a q] + \frac{1}{2}[(\partial_\mu G^a_\nu)(\partial^\nu G^{a\mu}) - (\partial_\mu G^a_\nu)(\partial^\mu G^{a\nu})] \\
& -g_sf^{abc}(\partial_\mu G^a_\nu)G^{b\mu}G^{c\nu} - \frac{1}{4}g_s^2f^{abc}f^{ade}G^b_\mu G^c_\nu G^{d\mu}G^{e\nu}.
\end{aligned} \tag{65}$$

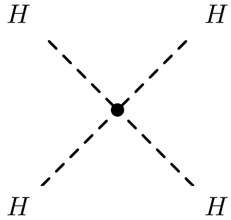
2 顶点费曼规则

下面列出幺正规范下的顶点费曼规则.

Higgs 粒子自耦合:

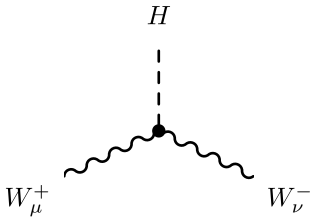


$$= -3i\frac{m_H^2}{v}$$

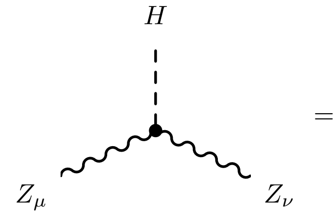


$$= -3i\frac{m_H^2}{v^2}$$

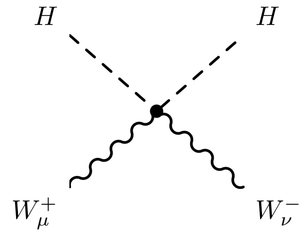
Higgs 粒子与电弱规范玻色子的耦合:



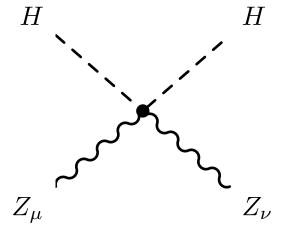
$= 2i\dfrac{m_W^2}{v}g^{\mu\nu}$



$= 2i\dfrac{m_Z^2}{v}g^{\mu\nu}$

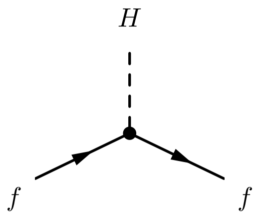


$= 2i\dfrac{m_W^2}{v^2}g^{\mu\nu}$



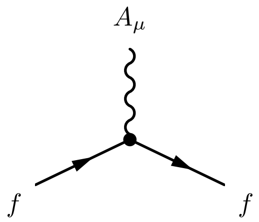
$= 2i\dfrac{m_Z^2}{v^2}g^{\mu\nu}$

Higgs 粒子与费米子的耦合:



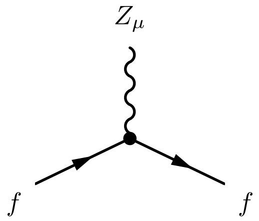
$= -i\dfrac{m_f}{v}$

QED 顶点:



$= iQ_f e\gamma^\mu$ (对于电子, $Q_f = -1$)

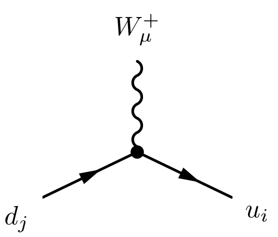
费米子与 Z 玻色子的耦合:



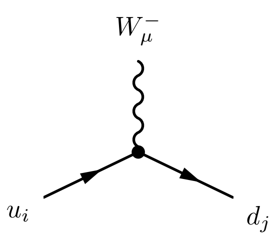
$= i\dfrac{g}{2c_W}\gamma^\mu(g_V^f - g_A^f\gamma_5)$

$$
\begin{aligned}
g_V^{u_i} &= \frac{1}{2} - \frac{4}{3}s_W^2, \quad g_A^{u_i} = \frac{1}{2}; \quad g_V^{d_i} = -\frac{1}{2} + \frac{2}{3}s_W^2, \quad g_A^{d_i} = -\frac{1}{2};\\
g_V^{\nu_i} &= \frac{1}{2}, \quad g_A^{\nu_i} = \frac{1}{2}; \quad g_V^{\ell_i} = -\frac{1}{2} + 2s_W^2, \quad g_A^{\ell_i} = -\frac{1}{2}.
\end{aligned}
$$

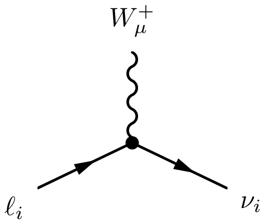
费米子与 W 玻色子的耦合:



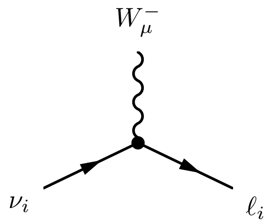
$= i\dfrac{g}{\sqrt{2}}V_{ij}\gamma^\mu\dfrac{1-\gamma_5}{2}$



$= i\dfrac{g}{\sqrt{2}}V_{ji}^\dagger\gamma^\mu\dfrac{1-\gamma_5}{2}$

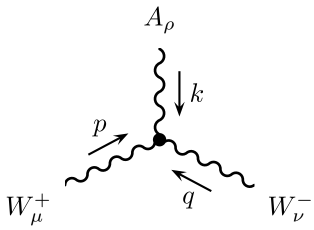


$$= i\frac{g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}$$



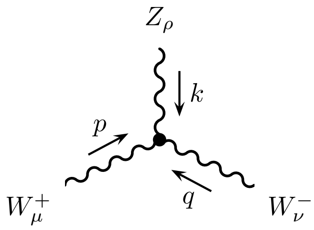
$$= i\frac{g}{\sqrt{2}}\gamma^\mu\frac{1-\gamma_5}{2}$$

电弱规范玻色子自耦合:

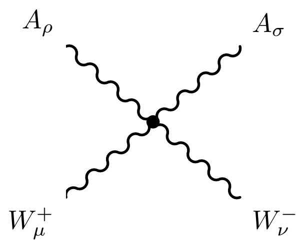


$$= -ie[(g^{\mu\nu}(p-q)^\rho + g^{\nu\rho}(q-k)^\mu + g^{\rho\mu}(k-p)^\nu]$$

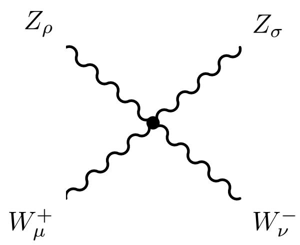
(对于指向顶点的动量 k, 时空导数 ∂_μ 在动量空间中贡献一个 $-ik_\mu$ 因子.)



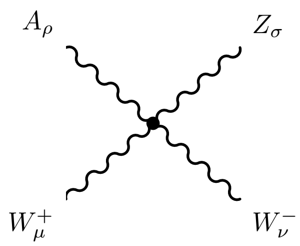
$$= -igc_W[(g^{\mu\nu}(p-q)^\rho + g^{\nu\rho}(q-k)^\mu + g^{\rho\mu}(k-p)^\nu]$$



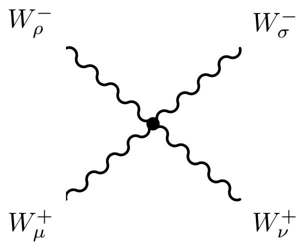
$$= ie^2(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2g^{\mu\nu}g^{\rho\sigma})$$



$$= ig^2c_W^2(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2g^{\mu\nu}g^{\rho\sigma})$$

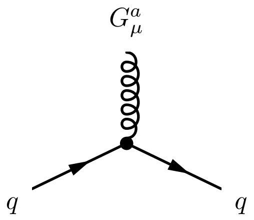


$$= iegc_W(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2g^{\mu\nu}g^{\rho\sigma})$$

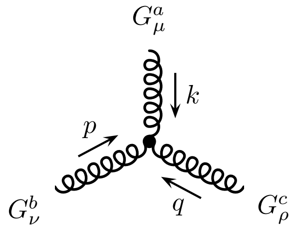


$$= -ig^2(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2g^{\mu\nu}g^{\rho\sigma})$$

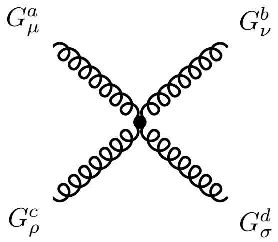
QCD 顶点:



$$= i g_s \gamma^\mu t^a$$



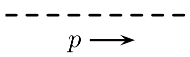
$$= g_s f^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$$



$$\begin{aligned} &= -i g_s^2 [f^{abe} f^{cde}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ &\qquad + f^{ade} f^{bce}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})] \end{aligned}$$

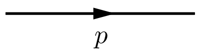
3 内外线费曼规则

标量玻色子传播子:



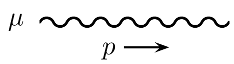
$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

Dirac 费米子传播子:

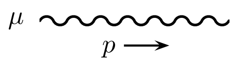


$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

无质量矢量玻色子 (光子) 传播子:

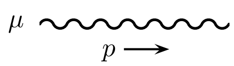


$$= \frac{-i g_{\mu\nu}}{p^2 + i\varepsilon} \quad \text{(费曼规范)}$$

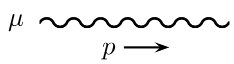


$$= \frac{-i(g_{\mu\nu} - p_\mu p_\nu / p^2)}{p^2 + i\varepsilon} \quad \text{(朗道规范)}$$

有质量矢量玻色子 ($W^\pm$, Z^0) 传播子:

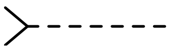


$$= \frac{-i(g_{\mu\nu} - p_\mu p_\nu / m^2)}{p^2 - m^2 + i\varepsilon} \quad \text{(幺正规范)}$$



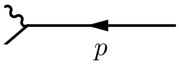
$$= \frac{-i g_{\mu\nu}}{p^2 - m^2 + i\varepsilon} \quad \text{(费曼规范)}$$

标量玻色子外线:

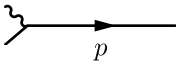


$$= 1 \quad \text{(初态或末态)}$$

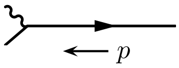
Dirac 费米子外线:



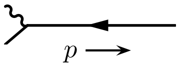
$= u^s(p)$ (正粒子初态)



$= \bar{u}^s(p)$ (正粒子末态)



$= \bar{v}^s(p)$ (反粒子初态)

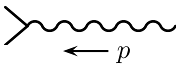


$= v^s(p)$ (反粒子末态)

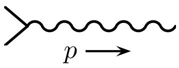
在计算非极化截面时, 可利用自旋求和关系

$$\sum_{s} u^s(p)\bar{u}^s(p) = \not{p} + m, \quad \sum_{s} v^s(p)\bar{v}^s(p) = \not{p} - m. \tag{66}$$

矢量玻色子外线:



$\mu \ = \varepsilon_\mu(p)$ (初态)



$\mu \ = \varepsilon^*_\mu(p)$ (末态)

在计算非极化截面时, 若包含无质量矢量玻色子外线, 可作替换

$$\sum_{\text{polarizations}} \varepsilon^*_\mu(p)\varepsilon_\nu(p) \to -g_{\mu\nu}; \tag{67}$$

若包含有质量矢量玻色子外线, 可作替换

$$\sum_{\text{polarizations}} \varepsilon^*_\mu(p)\varepsilon_\nu(p) \to -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{68}$$

参考文献

[1] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory,” Reading, USA: Addison-Wesley (1995) 842 p.