

离散对称性 P, T, C

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度规及 Dirac 矩阵约定与文献 [1] 相同. 标量场与旋量场部分主要参考文献 [2].
度规张量

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(+1, -1, -1, -1). \quad (1)$$

Pauli 矩阵

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad (2)$$

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}). \quad (3)$$

手征表示下的 Dirac 矩阵

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}. \quad (4)$$

固有 Lorentz 群变换对时空坐标的作用为

$$x^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (5)$$

$\Lambda^\mu{}_\nu$ 满足

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}, \quad \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}, \quad (\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho. \quad (6)$$

1 标量场

固有 Lorentz 群变换 $\Lambda^\mu{}_\nu$ 相应么正算符 $U(\Lambda)$ 对标量场 $\phi(x)$ 的作用为

$$U(\Lambda)^{-1} \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x). \quad (7)$$

然而, 宇称变换

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \mathcal{P}_\nu{}^\mu = \text{diag}(+1, -1, -1, -1) \quad (8)$$

和时间反演变换

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \mathcal{T}_\nu{}^\mu = \text{diag}(-1, +1, +1, +1) \quad (9)$$

不属于固有 Lorentz 群. 由于宇称变换和时间反演变换均为自身的逆变换, 作用两次不会改变任意可观测量. 标量场 $\phi(x)$ 在原理上是一个可观测量. 记 $P \equiv U(\mathcal{P})$, $T \equiv U(\mathcal{T})$, 则对标量场有

$$P^{-2}\phi(x)P^2 = \phi(x), \quad T^{-2}\phi(x)T^2 = \phi(x). \quad (10)$$

从而, 标量场的 P 变换可以有两种形式:

$$\text{对于标量场, } P^{-1}\phi(x)P = +\phi(\mathcal{P}x); \quad (11)$$

$$\text{对于赝标量场, } P^{-1}\phi(x)P = -\phi(\mathcal{P}x). \quad (12)$$

标量场的 T 变换也有两种形式:

$$T^{-1}\phi(x)T = +\phi(\mathcal{T}x); \quad (13)$$

$$T^{-1}\phi(x)T = -\phi(\mathcal{T}x). \quad (14)$$

若拉氏量满足 $P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x)$ 和 $T^{-1}\mathcal{L}(x)T = +\mathcal{L}(\mathcal{T}x)$, 则对 d^4x 积分而得的作用量 S 对 P 变换和 T 变换也是不变的, 此时宇称和时间反演守恒.

在固有 Lorentz 变换下, 能量动量矢量 $P^\mu = (H, P^i)$ 的变换为

$$U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu. \quad (15)$$

如果哈密顿量 H 在 P 变换和 T 变换下不变, 即 $P^{-1}HP = +H$ 和 $T^{-1}HT = +H$, 则应要求

$$P^{-1}P^\mu P = \mathcal{P}^\mu{}_\nu P^\nu, \quad T^{-1}P^\mu T = -\mathcal{T}^\mu{}_\nu P^\nu. \quad (16)$$

值得注意的是, 时间反演算符 T 是反幺正的 (注 1), 即

$$T^{-1}iT = -i. \quad (17)$$

注 1: 时空平移算符 $\mathbb{T}(a) = \exp(-iP \cdot a)$ 对标量场的作用为

$$\mathbb{T}(a)^{-1}\phi(x)\mathbb{T}(a) = \phi(x - a), \quad (18)$$

由其构造方式易知它是 Lorentz 标量:

$$U(\Lambda)^{-1}\mathbb{T}(a)U(\Lambda) = \mathbb{T}(\Lambda^{-1}a). \quad (19)$$

将上式展开至第 1 阶得 $U(\Lambda)^{-1}(1 - ia_\mu P^\mu)U(\Lambda) = 1 - i(\Lambda^{-1})^\mu{}_\nu a_\mu P^\nu = 1 - i\Lambda^\mu{}_\nu a_\mu P^\nu$, 对于 T 变换, 此式化为

$$T^{-1}(1 - ia_\mu P^\mu)T = 1 - i\mathcal{T}^\mu{}_\nu a_\mu P^\nu. \quad (20)$$

可见, 若要求 $T^{-1}P^\mu T = -\mathcal{T}^\mu{}_\nu P^\nu$, 则 T 变换必须满足 $T^{-1}iT = -i$.

考虑复标量场 $\phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$ 和拉氏量

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{4} \lambda (\phi^\dagger \phi)^2$$

$$= \frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 + \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2. \quad (21)$$

则 \mathcal{L} 除了具有 $U(1)$ 对称性 (对两个实标量场而言是 $SO(2)$ 对称性)

$$\phi(x) \rightarrow e^{-i\alpha} \phi(x), \quad \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \quad (22)$$

之外, 还具有离散对称性

$$\phi(x) \rightarrow \phi^\dagger(x), \quad \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} +1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}. \quad (23)$$

此离散对称性被称为 电荷共轭 对称性, 它总是伴随 $U(1)$ 连续对称性而来.

可以将标量场的电荷共轭算符 C 定义为

$$C^{-1} \phi(x) C = \phi^\dagger(x), \quad (24)$$

亦即

$$C^{-1} \varphi_1(x) C = +\varphi_1(x), \quad C^{-1} \varphi_2(x) C = -\varphi_2(x), \quad (25)$$

从而 $C^{-1} \mathcal{L}(x) C = \mathcal{L}(x)$ 对应着电荷共轭对称性.

算符 $\phi^\dagger \phi$ 的变换性质为

$$P^{-1} \phi^\dagger(x) \phi(x) P = +\phi^\dagger(\mathcal{P}x) \phi(\mathcal{P}x), \quad (26)$$

$$T^{-1} \phi^\dagger(x) \phi(x) T = +\phi^\dagger(\mathcal{T}x) \phi(\mathcal{T}x), \quad (27)$$

$$C^{-1} \phi^\dagger(x) \phi(x) C = +\phi^\dagger(x) \phi(x). \quad (28)$$

由于

$$P^{-1} \partial^\mu P = \mathcal{P}^\mu{}_\nu \partial^\nu, \quad T^{-1} \partial^\mu T = \mathcal{T}^\mu{}_\nu \partial^\nu, \quad C^{-1} \partial^\mu C = \partial^\mu, \quad (29)$$

算符 $\phi^\dagger i \overleftrightarrow{\partial}^\mu \phi$ 的变换性质为

$$P^{-1} \phi^\dagger(x) i \overleftrightarrow{\partial}^\mu \phi(x) P = +\mathcal{P}^\mu{}_\nu \phi^\dagger(\mathcal{P}x) i \overleftrightarrow{\partial}^\nu \phi(\mathcal{P}x), \quad (30)$$

$$T^{-1} \phi^\dagger(x) i \overleftrightarrow{\partial}^\mu \phi(x) T = -\mathcal{T}^\mu{}_\nu \phi^\dagger(\mathcal{T}x) i \overleftrightarrow{\partial}^\nu \phi(\mathcal{T}x), \quad (31)$$

$$C^{-1} \phi^\dagger(x) i \overleftrightarrow{\partial}^\mu \phi(x) C = -\phi^\dagger(x) i \overleftrightarrow{\partial}^\mu \phi(x). \quad (32)$$

动能项 $\partial^\mu \phi^\dagger \partial_\mu \phi$ 的变换性质为

$$P^{-1} \partial^\mu \phi^\dagger(x) \partial_\mu \phi(x) P = \mathcal{P}^\mu{}_\sigma \mathcal{P}_\mu{}^\rho \partial^\sigma \phi^\dagger(\mathcal{P}x) \partial_\rho \phi(\mathcal{P}x) = +\partial^\mu \phi^\dagger(\mathcal{P}x) \partial_\mu \phi(\mathcal{P}x), \quad (33)$$

$$T^{-1} \partial^\mu \phi^\dagger(x) \partial_\mu \phi(x) T = \mathcal{T}^\mu{}_\sigma \mathcal{T}_\mu{}^\rho \partial^\sigma \phi^\dagger(\mathcal{T}x) \partial_\rho \phi(\mathcal{T}x) = +\partial^\mu \phi^\dagger(\mathcal{T}x) \partial_\mu \phi(\mathcal{T}x), \quad (34)$$

$$C^{-1} \partial^\mu \phi^\dagger(x) \partial_\mu \phi(x) C = \partial^\mu \phi(x) \partial_\mu \phi^\dagger(x) = +\partial^\mu \phi^\dagger(x) \partial_\mu \phi(x). \quad (35)$$

2 旋量场

固有 Lorentz 群变换 $\Lambda^\mu{}_\nu$ 相应么正算符 $U(\Lambda)$ 对 (Dirac 或 Majorana) 旋量场 $\psi(x)$ 的作用为

$$U(\Lambda)^{-1} \psi(x) U(\Lambda) = D(\Lambda) \psi(\Lambda^{-1}x). \quad (36)$$

无穷小变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ 对应于 $D(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$, 其中 $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

现在, 对于宇称变换, 有

$$P^{-1}\psi(x)P = D(\mathcal{P})\psi(\mathcal{P}x), \quad (37)$$

作用两次得 $P^{-2}\psi(x)P^2 = D(\mathcal{P})^2\psi(x)$. 由于需要偶数个旋量场才能构造可观测量, 要求 $D(\mathcal{P})^2 = \pm 1$. 由于

$$P^{-1}\mathbf{x}P = -\mathbf{x}, \quad P^{-1}\mathbf{P}P = -\mathbf{P}, \quad P^{-1}\mathbf{J}P = +\mathbf{J}, \quad (38)$$

宇称变换反转 3 动量的方向, 但会保持自旋方向不变, 故对产生算符的变换形式为

$$P^{-1}b_s^\dagger(\mathbf{p})P = \eta b_s^\dagger(-\mathbf{p}), \quad P^{-1}d_s^\dagger(\mathbf{p})P = \eta d_s^\dagger(-\mathbf{p}). \quad (39)$$

依照上述讨论, $\eta^2 = \pm 1$. 这里对算符 b 和 d 用了同一个 η , 这样宇称变换就与 Majorana 条件 $b_s(\mathbf{p}) = d_s(\mathbf{p})$ 一致了.

将旋量场 $\psi(x)$ 展开为

$$\psi(x) = \sum_s \int \widetilde{dp} [b_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}], \quad s = \pm, \quad \widetilde{dp} \equiv \frac{d^3\mathbf{p}}{(2\pi^3)\sqrt{2E_{\mathbf{p}}}}, \quad (40)$$

则

$$\begin{aligned} P^{-1}\psi(x)P &= \sum_s \int \widetilde{dp} [\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + \eta d_s^\dagger(-\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}] \\ &= \sum_s \int \widetilde{dp} [\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})e^{-ip\cdot \mathcal{P}x} + \eta d_s^\dagger(\mathbf{p})v_s(-\mathbf{p})e^{ip\cdot \mathcal{P}x}]. \end{aligned} \quad (41)$$

由平面波解

$$u_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}, \quad v_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_s \\ -\sqrt{p\cdot\bar{\sigma}}\eta_s \end{pmatrix}, \quad (42)$$

有

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p}), \quad v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p}), \quad \beta \equiv \gamma^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (43)$$

如果我们取 $\eta = -i$, 就可以得到与 $P^{-1}\psi(x)P = D(\mathcal{P})\psi(\mathcal{P}x)$ 对应的表达式

$$P^{-1}\psi(x)P = \sum_s \int \widetilde{dp} [ib_s(\mathbf{p})\beta u_s(\mathbf{p})e^{-ip\cdot \mathcal{P}x} - id_s^\dagger(\mathbf{p})(-\beta)v_s(\mathbf{p})e^{ip\cdot \mathcal{P}x}] = i\beta\psi(\mathcal{P}x). \quad (44)$$

此时

$$D(\mathcal{P}) = i\beta, \quad (45)$$

即

$$P^{-1}\psi(x)P = i\beta\psi(\mathcal{P}x). \quad (46)$$

η 为纯虚数具有物理意义. 在质心系中考虑一对正反费米子的态 $|\Phi\rangle = \int \widetilde{dp} \Phi(\mathbf{p})b_s^\dagger(\mathbf{p})d_{s'}^\dagger(-\mathbf{p})|0\rangle$. 假设动量空间波函数 $\Phi(\mathbf{p})$ 具有确定的轨道宇称 $\Phi(-\mathbf{p}) = (-)^l\Phi(\mathbf{p})$, 而真空是宇称变换不变的, 则

$$\begin{aligned} P^{-1}|\Phi\rangle &= \int \widetilde{dp} \Phi(\mathbf{p})P^{-1}b_s^\dagger(\mathbf{p})PP^{-1}d_{s'}^\dagger(-\mathbf{p})PP^{-1}|0\rangle = \eta^2 \int \widetilde{dp} \Phi(\mathbf{p})b_s^\dagger(-\mathbf{p})d_{s'}^\dagger(\mathbf{p})|0\rangle \\ &= (-i)^2 \int \widetilde{dp} \Phi(-\mathbf{p})b_s^\dagger(\mathbf{p})d_{s'}^\dagger(-\mathbf{p})|0\rangle = -(-)^l|\Phi\rangle. \end{aligned} \quad (47)$$

可见, 一对正反费米子的 内禀宇称 为 -1 , 这一结论对 Majorana 费米子也成立.

用 Weyl 场表示 Dirac 场, 有

$$\psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \xi^a & \chi_a^\dagger \end{pmatrix}. \quad (48)$$

对 (37) 式应用 $D(\mathcal{P}) = i\beta$, 得

$$P^{-1}\chi_a(x)P = i\xi^{\dagger\dot{a}}(\mathcal{P}x), \quad P^{-1}\xi^{\dagger\dot{a}}(x)P = i\chi_a(\mathcal{P}x). \quad (49)$$

可见, 宇称变换使左手场与右手场互换. 对上式取厄米共轭, 并利用

$$\varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab}, \quad \varepsilon^{\dot{a}\dot{b}}(\chi_b)^\dagger = \varepsilon^{\dot{a}\dot{b}}\chi_b^\dagger = \chi^{\dagger\dot{a}}, \quad \varepsilon^{\dot{a}\dot{b}}(\xi^{\dagger\dot{b}})^\dagger = -\varepsilon_{ab}\xi^b = -\xi_a, \quad (50)$$

有

$$P^{-1}\chi^{\dagger\dot{a}}(x)P = i\xi_a(\mathcal{P}x), \quad P^{-1}\xi_a(x)P = i\chi^{\dagger\dot{a}}(\mathcal{P}x), \quad (51)$$

这与 Majorana 条件 $\chi_a(x) = \xi_a(x)$ 是一致的.

下面讨论旋量场双线性型 $\bar{\psi}A\psi$ 的变换性质, 其中 A 是 Dirac 矩阵的组合. 记 $\bar{A} \equiv \beta A^\dagger \beta$, 若 $\bar{A} = A$, 则

$$(\bar{\psi}A\psi)^\dagger = \psi^\dagger A^\dagger \beta \psi = \psi^\dagger \beta \beta A^\dagger \beta \psi = \bar{\psi} \bar{A} \psi = \bar{\psi} A \psi, \quad (52)$$

此时 $\bar{\psi}A\psi$ 是厄米的. 由

$$P^{-1}\bar{\psi}(x)P = P^{-1}\psi^\dagger(x)\beta P = [P^{-1}\beta\psi(x)P]^\dagger = [i\beta\beta\psi(\mathcal{P}x)]^\dagger = -i\bar{\psi}(\mathcal{P}x)\beta \quad (53)$$

可知

$$P^{-1}\bar{\psi}(x)A\psi(x)P = \bar{\psi}(\mathcal{P}x)\beta A\beta\psi(\mathcal{P}x). \quad (54)$$

因此, $\bar{\psi}A\psi$ 的 P 变换性质由 $\beta A\beta$ 的形式所决定.

利用

$$\begin{aligned} \beta 1 \beta &= +1, \quad \beta i\gamma_5 \beta = -i\gamma_5, \quad \beta \gamma^0 \beta = +\gamma^0, \quad \beta \gamma^i \beta = -\gamma^i, \\ \beta \gamma^0 \gamma_5 \beta &= -\gamma^0 \gamma_5, \quad \beta \gamma^i \gamma_5 \beta = +\gamma^i \gamma_5, \quad \beta \gamma^0 \gamma^0 \beta = \gamma^0 \gamma^0, \quad \beta \gamma^0 \gamma^i \beta = -\gamma^0 \gamma^i, \quad \beta \gamma^i \gamma^j \beta = \gamma^i \gamma^j, \end{aligned} \quad (55)$$

有

$$\beta \gamma^\mu \beta = \mathcal{P}^\mu{}_\nu \gamma^\nu, \quad \beta \gamma^\mu \gamma_5 \beta = -\mathcal{P}^\mu{}_\nu \gamma^\nu \gamma_5, \quad \beta \gamma^\mu \gamma^\nu \beta = +\mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma \gamma^\rho \gamma^\sigma, \quad (56)$$

而对于 $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$,

$$\beta \sigma^{\mu\nu} \beta = +\frac{i}{2}(\mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma \gamma^\rho \gamma^\sigma - \mathcal{P}^\nu{}_\rho \mathcal{P}^\mu{}_\sigma \gamma^\rho \gamma^\sigma) = +\frac{i}{2}(\mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma \gamma^\rho \gamma^\sigma - \mathcal{P}^\nu{}_\sigma \mathcal{P}^\mu{}_\rho \gamma^\sigma \gamma^\rho) = +\mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma \sigma^{\rho\sigma}, \quad (57)$$

于是,

$$P^{-1}\bar{\psi}(x)\psi(x)P = +\bar{\psi}(\mathcal{P}x)\psi(\mathcal{P}x), \quad (58)$$

$$P^{-1}\bar{\psi}(x)i\gamma_5\psi(x)P = -\bar{\psi}(\mathcal{P}x)i\gamma_5\psi(\mathcal{P}x), \quad (59)$$

$$P^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)P = +\mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\psi(\mathcal{P}x), \quad (60)$$

$$P^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)P = -\mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\gamma_5\psi(\mathcal{P}x), \quad (61)$$

$$P^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)P = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\bar{\psi}(\mathcal{P}x)\sigma^{\rho\sigma}\psi(\mathcal{P}x). \quad (62)$$

由此可知, $\bar{\psi}\psi$ 是标量, $\bar{\psi}i\gamma_5\psi$ 是赝标量, $\bar{\psi}\gamma^\mu\psi$ 是(极) 矢量, $\bar{\psi}\gamma^\mu\gamma_5\psi$ 是轴矢量.

左手流 $\bar{\psi}_L\gamma^\mu\psi_L \equiv \frac{1}{2}\bar{\psi}\gamma^\mu(1 - \gamma_5)\psi$ 和右手流 $\bar{\psi}_R\gamma^\mu\psi_R \equiv \frac{1}{2}\bar{\psi}\gamma^\mu(1 + \gamma_5)\psi$ 的变换关系为

$$P^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)P = \mathcal{P}^\mu{}_\nu\bar{\psi}_R(\mathcal{P}x)\gamma^\nu\psi_R(\mathcal{P}x), \quad (63)$$

$$P^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)P = \mathcal{P}^\mu{}_\nu\bar{\psi}_L(\mathcal{P}x)\gamma^\nu\psi_L(\mathcal{P}x). \quad (64)$$

带有电荷 Q 的 Dirac 场与电磁场相互作用时, 运动方程为 (参考文献 [3])

$$[\gamma^\mu(i\partial_\mu + QeA_\mu) - m]\psi = 0, \quad (65)$$

取厄米共轭并右乘 γ^0 , 得

$$0 = \psi^\dagger[\gamma^{\mu\dagger}(-i\partial_\mu + QeA_\mu) - m]\gamma^0 = \bar{\psi}[-\gamma^\mu(i\partial_\mu - QeA_\mu) - m], \quad (66)$$

转置, 得

$$[-(\gamma^\mu)^T(i\partial_\mu - QeA_\mu) - m]\bar{\psi}^T = 0. \quad (67)$$

设 $\mathcal{C} \equiv i\gamma^0\gamma^2$, 它有如下性质:

$$\mathcal{C}^T = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C}, \quad \mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^T, \quad \mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5. \quad (68)$$

由此, 可得

$$0 = [\mathcal{C}^{-1}\gamma^\mu\mathcal{C}(i\partial_\mu - QeA_\mu) - m]\bar{\psi}^T = \mathcal{C}^{-1}[\gamma^\mu(i\partial_\mu - QeA_\mu) - m]\mathcal{C}\bar{\psi}^T. \quad (69)$$

令

$$\psi_c \equiv \mathcal{C}\bar{\psi}^T = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix}, \quad (70)$$

则

$$[\gamma^\mu(i\partial_\mu - QeA_\mu) - m]\psi_c = 0. \quad (71)$$

与 (65) 式比较, 可知 ψ_c 是与 ψ 带相反电荷的场. 因此, Dirac 场的电荷共轭变换为

$$C^{-1}\psi(x)C = \mathcal{C}\bar{\psi}^T(x), \quad (72)$$

而 \mathcal{C} 是旋量空间的电荷共轭变换矩阵. 从而,

$$\begin{aligned} C^{-1}\bar{\psi}C &= C^{-1}\psi^\dagger(x)\beta C = [C^{-1}\beta\psi C]^\dagger = [\beta\mathcal{C}\bar{\psi}^T]^\dagger = \{[\bar{\psi}\mathcal{C}^T\beta^T]^\dagger\}^T = \{[\psi^\dagger\beta\mathcal{C}^T\beta^T]^\dagger\}^T \\ &= [\beta^*\mathcal{C}^*\beta\psi]^\dagger = \psi^T\beta^T\mathcal{C}^\dagger\beta = \psi^T\beta^T\mathcal{C}^{-1}\beta = -\psi^T\mathcal{C}^{-1}\beta\beta = \psi^T\mathcal{C}. \end{aligned} \quad (73)$$

于是, 注意到转置时交换两个费米子场应多出一个额外的负号, 有

$$C^{-1}\bar{\psi}A\psi C = \psi^T\mathcal{C}A\mathcal{C}\bar{\psi}^T = -\bar{\psi}\mathcal{C}^TA^T\mathcal{C}^T\psi, \quad (74)$$

亦即

$$C^{-1}\bar{\psi}(x)A\psi(x)C = \bar{\psi}(x)\mathcal{C}^{-1}A^T\mathcal{C}\psi(x). \quad (75)$$

利用

$$\begin{aligned} C^{-1}1^TC &= +1, \quad C^{-1}(i\gamma_5)^TC = +i\gamma_5, \quad C^{-1}(\gamma^\mu)^TC = -\gamma^\mu, \\ C^{-1}(\gamma^\mu\gamma_5)^TC &= +\gamma^\mu\gamma_5, \quad C^{-1}(\gamma^\mu\gamma^\nu)^TC = \gamma^\nu\gamma^\mu, \quad C^{-1}(\sigma^{\mu\nu})^TC = -\sigma^{\mu\nu}, \end{aligned} \quad (76)$$

有

$$C^{-1}\bar{\psi}(x)\psi(x)C = +\bar{\psi}(x)\psi(x), \quad (77)$$

$$C^{-1}\bar{\psi}(x)i\gamma_5\psi(x)C = +\bar{\psi}(x)i\gamma_5\psi(x), \quad (78)$$

$$C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C = -\bar{\psi}(x)\gamma^\mu\psi(x), \quad (79)$$

$$C^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)C = +\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x), \quad (80)$$

$$C^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)C = -\bar{\psi}(x)\sigma^{\mu\nu}\psi(x), \quad (81)$$

$$C^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)C = -\bar{\psi}_R(x)\gamma^\mu\psi_R(x), \quad (82)$$

$$C^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)C = -\bar{\psi}_L(x)\gamma^\mu\psi_L(x). \quad (83)$$

对于 Majorana 场, $\psi = \mathcal{C}\bar{\psi}^T$, 则

$$\bar{\psi} = (\mathcal{C}\bar{\psi}^T)^\dagger\beta = [(\bar{\psi}\mathcal{C}^T)^\dagger]^T\beta = [(\psi^\dagger\beta\mathcal{C}^T)^\dagger]^T\beta = (\mathcal{C}\beta\psi)^T\beta = \psi^T\beta^T\mathcal{C}^T\beta = \psi^T\mathcal{C}, \quad (84)$$

于是,

$$C^{-1}\psi(x)C = \psi(x), \quad C^{-1}\bar{\psi}(x)C = \bar{\psi}(x), \quad (85)$$

从而,

$$C^{-1}\bar{\psi}(x)A\psi(x)C = \bar{\psi}(x)A\psi(x). \quad (86)$$

由此可见, 对于 Majorana 场, 必有

$$\bar{\psi}(x)\gamma^\mu\psi(x) = 0, \quad \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) = 0. \quad (87)$$

对于时间反演变换, 有

$$T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x), \quad (88)$$

同理 $D(\mathcal{T})^2 = \pm 1$. 注意到 (16) 式, 有

$$T^{-1}\mathbf{x}T = +\mathbf{x}, \quad T^{-1}\mathbf{P}T = -\mathbf{P}, \quad T^{-1}\mathbf{J}T = -\mathbf{J}, \quad (89)$$

因而时间反演变换同时反转 3 动量方向和自旋方向, 故对产生算符的变换形式为

$$T^{-1}b_s^\dagger(\mathbf{p})T = \zeta_sb_{-s}^\dagger(-\mathbf{p}), \quad T^{-1}d_s^\dagger(\mathbf{p})T = \zeta_sd_{-s}^\dagger(-\mathbf{p}). \quad (90)$$

注意到 T 变换的反么正性 $T^{-1}iT = -i$, 可得

$$\begin{aligned} T^{-1}\psi(x)T &= \sum_s \int \widetilde{dp} [\zeta_s^*b_{-s}(-\mathbf{p})u_s^*(\mathbf{p})e^{ip\cdot x} + \zeta_sd_{-s}^\dagger(-\mathbf{p})v_s^*(\mathbf{p})e^{-ip\cdot x}] \\ &= \sum_s \int \widetilde{dp} [\zeta_{-s}^*b_s(\mathbf{p})u_{-s}^*(-\mathbf{p})e^{-ip\cdot \mathcal{T}x} + \zeta_{-s}d_s^\dagger(\mathbf{p})v_{-s}^*(-\mathbf{p})e^{ip\cdot \mathcal{T}x}]. \end{aligned} \quad (91)$$

由于 (注 2)

$$u_{-s}^*(-\mathbf{p}) = -s\mathcal{C}\gamma_5u_s(\mathbf{p}), \quad v_{-s}^*(-\mathbf{p}) = -s\mathcal{C}\gamma_5v_s(\mathbf{p}), \quad (92)$$

如果我们取 $\zeta_s = s$, 就可以得到与 $T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$ 对应的表达式

$$T^{-1}\psi(x)T = \sum_s \int \widetilde{dp} [(-s)^2b_s(\mathbf{p})\mathcal{C}\gamma_5u_s(\mathbf{p})e^{-ip\cdot \mathcal{T}x} + (-s)^2d_s^\dagger(\mathbf{p})\mathcal{C}\gamma_5v_s(\mathbf{p})e^{ip\cdot \mathcal{T}x}] = \mathcal{C}\gamma_5\psi(\mathcal{T}x). \quad (93)$$

此时

$$D(\mathcal{T}) = \mathcal{C}\gamma_5, \quad (94)$$

即

$$T^{-1}\psi(x)T = \mathcal{C}\gamma_5\psi(\mathcal{T}x). \quad (95)$$

注 2: 对于动量方向矢量 $\hat{\mathbf{p}} = (s_\theta c_\phi, s_\theta s_\phi, c_\theta)$, 其中 $s_\theta \equiv \sin \theta$, $c_\theta \equiv \cos \theta$, 螺旋度算符可用矩阵表示为

$$\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} = \begin{pmatrix} c_\theta & e^{-i\phi} s_\theta \\ e^{i\phi} s_\theta & -c_\theta \end{pmatrix}. \quad (96)$$

螺旋态基底取为

$$\xi_+ = \begin{pmatrix} c_{\theta/2} \\ e^{i\phi} s_{\theta/2} \end{pmatrix}, \quad \xi_- = \begin{pmatrix} -e^{-i\phi} s_{\theta/2} \\ c_{\theta/2} \end{pmatrix}, \quad (97)$$

就可以满足

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_s = s\xi_s, \quad s = \pm. \quad (98)$$

定义 $\tilde{\xi}_s \equiv -is\sigma^2\xi_s^*$, 则有 $\tilde{\xi}_+ = \xi_-$, $\tilde{\xi}_- = \xi_+$, 于是

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\tilde{\xi}_s = -s\tilde{\xi}_s. \quad (99)$$

因此, 基底 $\tilde{\xi}_s$ 是对应于螺旋度 $-s$.

利用 $\sigma^i\sigma^2 = -\sigma^2(\sigma^i)^*$ 和

$$\mathcal{C} = i\gamma^0\gamma^2 = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}, \quad \mathcal{C}\gamma_5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}, \quad (100)$$

便可以得到

$$\begin{aligned} u_{-s}^*(-\mathbf{p}) &= \left(\frac{\sqrt{p \cdot \bar{\sigma}} \tilde{\xi}_s}{\sqrt{p \cdot \bar{\sigma}} \tilde{\xi}_s} \right)^* = \left(\frac{-is\sigma^2 \sqrt{p \cdot \sigma^*} \xi_s^*}{-is\sigma^2 \sqrt{p \cdot \sigma^*} \xi_s^*} \right)^* = \begin{pmatrix} -is\sigma^2 \sqrt{p \cdot \sigma} \xi_s \\ -is\sigma^2 \sqrt{p \cdot \sigma} \xi_s \end{pmatrix} \\ &= -s \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \sigma} \xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ \sqrt{p \cdot \sigma} \xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 u_s(\mathbf{p}), \end{aligned} \quad (101)$$

$$\begin{aligned} v_{-s}^*(-\mathbf{p}) &= \left(\frac{\sqrt{p \cdot \bar{\sigma}} \tilde{\xi}_s}{-\sqrt{p \cdot \bar{\sigma}} \tilde{\xi}_s} \right)^* = \left(\frac{-is\sigma^2 \sqrt{p \cdot \sigma^*} \xi_s^*}{+is\sigma^2 \sqrt{p \cdot \sigma^*} \xi_s^*} \right)^* = \begin{pmatrix} -is\sigma^2 \sqrt{p \cdot \sigma} \xi_s \\ +is\sigma^2 \sqrt{p \cdot \sigma} \xi_s \end{pmatrix} \\ &= -s \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ -\sqrt{p \cdot \sigma} \xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 \begin{pmatrix} \sqrt{p \cdot \sigma} \xi_s \\ -\sqrt{p \cdot \sigma} \xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 v_s(\mathbf{p}). \end{aligned} \quad (102)$$

用 Weyl 场表示 Dirac 场, 留意到 $\mathcal{C}\gamma_5 = \text{diag}(i\sigma^2, i\sigma^2) = \text{diag}(\varepsilon^{ab}, -\varepsilon_{\dot{a}\dot{b}})$, 有

$$T^{-1}\chi_a(x)T = +\chi^a(\mathcal{T}x), \quad T^{-1}\xi^{\dagger\dot{a}}(x)T = -\xi_a^\dagger(\mathcal{T}x). \quad (103)$$

可见, T 变换不改变 Weyl 场的手征. 对上式取厄米共轭, 利用 $\varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}}$, 和 $\varepsilon^{ab} = -\varepsilon_{ab}$, 可得

$$T^{-1}\chi^{\dagger\dot{a}}(x)T = -\chi_a^\dagger(\mathcal{T}x), \quad T^{-1}\xi_a(x)T = +\xi^a(\mathcal{T}x), \quad (104)$$

这与 Majorana 条件 $\chi_a(x) = \xi_a(x)$ 一致.

由 $T^{-1}\psi(x)T = \mathcal{C}\gamma_5\psi(\mathcal{T}x)$ 可得

$$T^{-1}\bar{\psi}(x)T = T^{-1}\psi^\dagger(x)\beta T = [\mathcal{C}\gamma_5\psi(\mathcal{T}x)]^\dagger\beta = \psi^\dagger(\mathcal{T}x)\gamma_5\mathcal{C}^\dagger\beta = \psi^\dagger(\mathcal{T}x)\beta\gamma_5\mathcal{C}^\dagger = \bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}, \quad (105)$$

因而, 利用 $T^{-1}AT = A^*$, 有

$$T^{-1}\bar{\psi}(x)A\psi(x)T = \bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\psi(\mathcal{T}x). \quad (106)$$

注意到

$$\begin{aligned} \gamma_5\mathcal{C}^{-1}1^*\mathcal{C}\gamma_5 &= +1, & \gamma_5\mathcal{C}^{-1}(i\gamma_5)^*\mathcal{C}\gamma_5 &= -i\gamma_5, \\ \gamma_5\mathcal{C}^{-1}(\gamma^0)^*\mathcal{C}\gamma_5 &= +\gamma^0, & \gamma_5\mathcal{C}^{-1}(\gamma^i)^*\mathcal{C}\gamma_5 &= -\gamma^i, \\ \gamma_5\mathcal{C}^{-1}(\gamma^0\gamma_5)^*\mathcal{C}\gamma_5 &= +\gamma^0\gamma_5, & \gamma_5\mathcal{C}^{-1}(\gamma^i\gamma_5)^*\mathcal{C}\gamma_5 &= -\gamma^i\gamma_5, \\ \gamma_5\mathcal{C}^{-1}(\gamma^0\gamma^0)^*\mathcal{C}\gamma_5 &= +\gamma^0\gamma^0, & \gamma_5\mathcal{C}^{-1}(\gamma^0\gamma^i)^*\mathcal{C}\gamma_5 &= -\gamma^0\gamma^i, & \gamma_5\mathcal{C}^{-1}(\gamma^i\gamma^j)^*\mathcal{C}\gamma_5 &= +\gamma^i\gamma^j, \end{aligned} \quad (107)$$

可得

$$\begin{aligned} \gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*\mathcal{C}\gamma_5 &= -\mathcal{T}^\mu{}_\nu\gamma^\nu, & \gamma_5\mathcal{C}^{-1}(\gamma^\mu\gamma_5)^*\mathcal{C}\gamma_5 &= -\mathcal{T}^\mu{}_\nu\gamma^\nu\gamma_5, \\ \gamma_5\mathcal{C}^{-1}(\sigma^{\mu\nu})^*\mathcal{C}\gamma_5 &= -\frac{i}{2}\gamma_5\mathcal{C}^{-1}[\gamma^\mu, \gamma^\nu]^*\mathcal{C}\gamma_5 = -\frac{i}{2}\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[\gamma^\rho, \gamma^\sigma] = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\sigma^{\rho\sigma}. \end{aligned} \quad (108)$$

于是

$$T^{-1}\bar{\psi}(x)\psi(x)T = +\bar{\psi}(\mathcal{T}x)\psi(\mathcal{T}x), \quad (109)$$

$$T^{-1}\bar{\psi}(x)i\gamma_5\psi(x)T = -\bar{\psi}(\mathcal{T}x)i\gamma_5\psi(\mathcal{T}x), \quad (110)$$

$$T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\psi(\mathcal{T}x), \quad (111)$$

$$T^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\gamma_5\psi(\mathcal{T}x), \quad (112)$$

$$T^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\bar{\psi}(\mathcal{T}x)\sigma^{\rho\sigma}\psi(\mathcal{T}x), \quad (113)$$

$$T^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}_L(\mathcal{T}x)\gamma^\nu\psi_L(\mathcal{T}x), \quad (114)$$

$$T^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}_R(\mathcal{T}x)\gamma^\nu\psi_R(\mathcal{T}x). \quad (115)$$

动能项 $\bar{\psi}i\gamma^\mu\partial_\mu\psi$ 的 P, T 变换性质为

$$\begin{aligned} P^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)P &= \mathcal{P}_\mu{}^\rho\bar{\psi}(\mathcal{P}x)i\beta\gamma^\mu\beta\partial_\rho\psi(\mathcal{P}x) = \mathcal{P}_\mu{}^\rho\mathcal{P}^\mu{}_\sigma\bar{\psi}(\mathcal{P}x)i\gamma^\sigma\partial_\rho\psi(\mathcal{P}x) \\ &= +\bar{\psi}(\mathcal{P}x)i\gamma^\mu\partial_\mu\psi(\mathcal{P}x), \end{aligned} \quad (116)$$

$$\begin{aligned} T^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)T &= -i\mathcal{T}_\mu{}^\rho\bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*\mathcal{C}\gamma_5\partial_\rho\psi(\mathcal{T}x) = i\mathcal{T}_\mu{}^\rho\mathcal{T}^\mu{}_\sigma\bar{\psi}(\mathcal{T}x)\gamma^\sigma\partial_\rho\psi(\mathcal{T}x) \\ &= +\bar{\psi}(\mathcal{T}x)i\gamma^\mu\partial_\mu\psi(\mathcal{T}x), \end{aligned} \quad (117)$$

而在 C 变换下,

$$C^{-1}\bar{\psi}i\gamma^\mu\partial_\mu\psi C = \psi^T\mathcal{C}i\gamma^\mu\mathcal{C}\partial_\mu\bar{\psi}^T = -i\partial_\mu\bar{\psi}\mathcal{C}^T(\gamma^\mu)^T\mathcal{C}^T\psi = -i\partial_\mu\bar{\psi}\gamma^\mu\psi = -i\partial_\mu(\bar{\psi}\gamma^\mu\psi) + \bar{\psi}i\gamma^\mu\partial_\mu\psi, \quad (118)$$

丢掉全散度项, 化为

$$C^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)C = +\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x). \quad (119)$$

可见动能项 $\bar{\psi}i\gamma^\mu\partial_\mu\psi$ 在 P, T, C 变换下分别保持不变.

对于带有向旋量场求导操作的 2 阶张量 (twist-2) 算符

$$\begin{aligned} \bar{\psi}\gamma^\mu\partial^\nu\psi + \text{h.c.} &= \bar{\psi}\gamma^\mu\partial^\nu\psi + (\partial^\nu\bar{\psi})\gamma^\mu\psi, & i\bar{\psi}\gamma^\mu\partial^\nu\psi + \text{h.c.} &= i\bar{\psi}\gamma^\mu\partial^\nu\psi - i(\partial^\nu\bar{\psi})\gamma^\mu\psi, \\ \bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi + \text{h.c.} &= \bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi + (\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi, & i\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi + \text{h.c.} &= i\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi - i(\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi, \end{aligned}$$

$$\begin{aligned}\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.} &= \bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + (\partial^\nu\bar{\psi}_{L/R})\gamma^\mu\psi_{L/R}, \\ i\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.} &= i\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} - i(\partial^\nu\bar{\psi}_{L/R})\gamma^\mu\psi_{L/R},\end{aligned}$$

P 和 T 的变换性质为

$$P^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P = +\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \quad (120)$$

$$P^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]P = -\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \quad (121)$$

$$P^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]P = +\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}_{R/L}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi_{R/L}(\mathcal{P}x) + \text{h.c.}], \quad (122)$$

$$T^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]T = -\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[\bar{\psi}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \quad (123)$$

$$T^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]T = -\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[\bar{\psi}(\mathcal{T}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \quad (124)$$

$$T^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]T = -\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[\bar{\psi}_{L/R}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{T}x) + \text{h.c.}], \quad (125)$$

由于 T 变换的反么正性,

$$T^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]T = +\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[i\bar{\psi}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \quad (126)$$

$$T^{-1}[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]T = +\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[i\bar{\psi}(\mathcal{T}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \quad (127)$$

$$T^{-1}[i\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]T = +\mathcal{T}^\mu_\rho\mathcal{T}^\nu_\sigma[i\bar{\psi}_{L/R}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{T}x) + \text{h.c.}]. \quad (128)$$

另一方面, C 变换会联系互为厄米共轭的两项,

$$C^{-1}\bar{\psi}\gamma^\mu\partial^\nu\psi C = \psi^\text{T}\mathcal{C}\gamma^\mu\mathcal{C}\partial^\nu\bar{\psi}^\text{T} = -\partial^\nu\bar{\psi}\mathcal{C}^\text{T}(\gamma^\mu)^\text{T}\mathcal{C}^\text{T}\psi = -(\partial^\nu\bar{\psi})\gamma^\mu\psi, \quad (129)$$

$$C^{-1}(\partial^\nu\bar{\psi})\gamma^\mu\psi C = (\partial^\nu\psi^\text{T})\mathcal{C}\gamma^\mu\mathcal{C}\bar{\psi}^\text{T} = -\bar{\psi}\mathcal{C}^\text{T}(\gamma^\mu)^\text{T}\mathcal{C}^\text{T}\partial^\nu\psi = -\bar{\psi}\gamma^\mu\partial^\nu\psi, \quad (130)$$

$$C^{-1}\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi C = \psi^\text{T}\mathcal{C}\gamma^\mu\gamma_5\mathcal{C}\partial^\nu\bar{\psi}^\text{T} = -\partial^\nu\bar{\psi}\mathcal{C}^\text{T}(\gamma^\mu\gamma_5)^\text{T}\mathcal{C}^\text{T}\psi = (\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi, \quad (131)$$

$$C^{-1}(\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi C = (\partial^\nu\psi^\text{T})\mathcal{C}\gamma^\mu\gamma_5\mathcal{C}\bar{\psi}^\text{T} = -\bar{\psi}\mathcal{C}^\text{T}(\gamma^\mu\gamma_5)^\text{T}\mathcal{C}^\text{T}\partial^\nu\psi = \bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi. \quad (132)$$

故

$$C^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]C = -[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}], \quad (133)$$

$$C^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]C = +[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}], \quad (134)$$

$$C^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]C = -[\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}], \quad (135)$$

$$C^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]C = +[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}], \quad (136)$$

$$C^{-1}[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]C = -[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}], \quad (137)$$

$$C^{-1}[i\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]C = +[i\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}]. \quad (138)$$

于是, CP 变换性质为

$$\begin{aligned}(CP)^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]CP &= -P^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}],\end{aligned} \quad (139)$$

$$\begin{aligned}(CP)^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]CP &= +P^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}],\end{aligned} \quad (140)$$

$$\begin{aligned}(CP)^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]CP &= -P^{-1}[\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma[\bar{\psi}_{L/R}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{P}x) + \text{h.c.}],\end{aligned} \quad (141)$$

$$(CP)^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]CP = +P^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P$$

$$= +\mathcal{P}^\mu_\rho \mathcal{P}^\nu_\sigma [i\bar{\psi}(\mathcal{P}x)\gamma^\rho \partial^\sigma \psi(\mathcal{P}x) + \text{h.c.}], \quad (142)$$

$$\begin{aligned} (CP)^{-1}[i\bar{\psi}(x)\gamma^\mu \gamma_5 \partial^\nu \psi(x) + \text{h.c.}]CP &= -P^{-1}[i\bar{\psi}(x)\gamma^\mu \gamma_5 \partial^\nu \psi(x) + \text{h.c.}]P \\ &= +\mathcal{P}^\mu_\rho \mathcal{P}^\nu_\sigma [i\bar{\psi}(\mathcal{P}x)\gamma^\rho \gamma_5 \partial^\sigma \psi(\mathcal{P}x) + \text{h.c.}], \end{aligned} \quad (143)$$

$$\begin{aligned} (CP)^{-1}[i\bar{\psi}_{L/R}(x)\gamma^\mu \partial^\nu \psi_{L/R}(x) + \text{h.c.}]CP &= +P^{-1}[i\bar{\psi}_{R/L}(x)\gamma^\mu \partial^\nu \psi_{R/L}(x) + \text{h.c.}]P \\ &= +\mathcal{P}^\mu_\rho \mathcal{P}^\nu_\sigma [i\bar{\psi}_{L/R}(\mathcal{P}x)\gamma^\rho \partial^\sigma \psi_{L/R}(\mathcal{P}x) + \text{h.c.}]. \end{aligned} \quad (144)$$

值得注意的是, 有无虚数 i 对应的两组算符的 CP 变换性质相反.

3 电磁场

电磁场 A^μ 的 P, T, C 变换性质可以通过分析有源 Maxwell 方程得到 [4]. 在经典电动力学中, Maxwell 方程在 P, T 和 C 变换下分别保持不变. 根据对应原理, 量子电动力学中在 Lorenz 规范下的电磁场运动方程

$$\partial^2 A^\mu = ej^\mu \quad (145)$$

也应在这些变换下保持不变, 其中 $j^\mu = \bar{\psi}\gamma^\mu\psi$. 从而, 由

$$P^{-1}j^\mu(x)P = \mathcal{P}^\mu_\nu j^\nu(\mathcal{P}x), \quad T^{-1}j^\mu(x)T = -\mathcal{T}^\mu_\nu j^\nu(\mathcal{T}x), \quad C^{-1}j^\mu(x)C = -j^\mu(x), \quad (146)$$

和

$$P^{-1}\partial^\mu P = \mathcal{P}^\mu_\nu \partial^\nu, \quad T^{-1}\partial^\mu T = \mathcal{T}^\mu_\nu \partial^\nu, \quad C^{-1}\partial^\mu C = \partial^\mu, \quad (147)$$

可以推出

$$P^{-1}A^\mu(x)P = \mathcal{P}^\mu_\nu A^\nu(\mathcal{P}x), \quad T^{-1}A^\mu(x)T = -\mathcal{T}^\mu_\nu A^\nu(\mathcal{T}x), \quad C^{-1}A^\mu(x)C = -A^\mu(x). \quad (148)$$

场强张量 $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ 的变换性质如下,

$$P^{-1}F^{\mu\nu}(x)P = \mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \partial^\alpha A^\beta(\mathcal{P}x) - \mathcal{P}^\nu_\beta \mathcal{P}^\mu_\alpha \partial^\beta A^\alpha(\mathcal{P}x) = \mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta F^{\alpha\beta}(\mathcal{P}x), \quad (149)$$

$$T^{-1}F^{\mu\nu}(x)T = -\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta \partial^\alpha A^\beta(\mathcal{T}x) + \mathcal{T}^\nu_\beta \mathcal{T}^\mu_\alpha \partial^\beta A^\alpha(\mathcal{T}x) = -\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta F^{\alpha\beta}(\mathcal{T}x), \quad (150)$$

$$C^{-1}F^{\mu\nu}(x)C = -\partial^\mu A^\nu(x) + \partial^\nu A^\mu(x) = -F^{\mu\nu}(x). \quad (151)$$

若 λ_1 和 λ_2 分别取 $\{0, 1, 2, 3\}$ 中的两个数字, 而 λ_3 和 λ_4 分别取剩余的另外两个数字, 则有 $\mathcal{P}_{\lambda_3}^{\lambda_3} \mathcal{P}_{\lambda_4}^{\lambda_4} = -\mathcal{P}_{\lambda_1}^{\lambda_1} \mathcal{P}_{\lambda_2}^{\lambda_2}$, 从而

$$\varepsilon^{\lambda_1 \lambda_2 \alpha \beta} \mathcal{P}_\alpha^{\lambda_3} \mathcal{P}_\beta^{\lambda_4} = \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \mathcal{P}_{\lambda_3}^{\lambda_3} \mathcal{P}_{\lambda_4}^{\lambda_4} = -\mathcal{P}_{\lambda_1}^{\lambda_1} \mathcal{P}_{\lambda_2}^{\lambda_2} \varepsilon^{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = -\mathcal{P}_{\lambda_1}^{\lambda_1} \mathcal{P}_{\lambda_2}^{\lambda_2} \varepsilon^{\alpha \beta \lambda_3 \lambda_4}, \quad (152)$$

于是, 下式成立:

$$\varepsilon^{\mu\nu\alpha\beta} \mathcal{P}_\alpha^\rho \mathcal{P}_\beta^\sigma = -\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \varepsilon^{\alpha\beta\rho\sigma}. \quad (153)$$

同理, 可以推出

$$\varepsilon^{\mu\nu\alpha\beta} \mathcal{T}_\alpha^\rho \mathcal{T}_\beta^\sigma = -\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta \varepsilon^{\alpha\beta\rho\sigma}. \quad (154)$$

是故, 对偶场强张量 $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ 的变换性质如下,

$$\begin{aligned} P^{-1}\tilde{F}^{\mu\nu}(x)P &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} P^{-1}F_{\alpha\beta}(x)P = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} \mathcal{P}_\alpha^\rho \mathcal{P}_\beta^\sigma F_{\rho\sigma}(\mathcal{P}x) \\ &= -\frac{1}{2}\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \varepsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma}(\mathcal{P}x) = -\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \tilde{F}^{\alpha\beta}(\mathcal{P}x), \end{aligned} \quad (155)$$

$$\begin{aligned}
T^{-1}\tilde{F}^{\mu\nu}(x)T &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}T^{-1}F_{\alpha\beta}(x)T = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{T}_\alpha{}^\rho\mathcal{T}_\beta{}^\sigma F_{\rho\sigma}(\mathcal{T}x) \\
&= +\frac{1}{2}\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}(\mathcal{T}x) = +\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\tilde{F}^{\alpha\beta}(\mathcal{T}x), \tag{156}
\end{aligned}$$

$$C^{-1}\tilde{F}^{\mu\nu}(x)C = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}C^{-1}F_{\rho\sigma}(x)C = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}(x) = -\tilde{F}^{\mu\nu}(x). \tag{157}$$

由此, $F_{\mu\nu}(x)F^{\mu\nu}(x)$ 和 $F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)$ 的变换性质分别为:

$$P^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)P = +\mathcal{P}_\mu{}^\alpha\mathcal{P}_\nu{}^\beta\mathcal{P}^\mu{}_\gamma\mathcal{P}^\nu{}_\delta F_{\alpha\beta}(\mathcal{P}x)F^{\gamma\delta}(\mathcal{P}x) = +F_{\mu\nu}(\mathcal{P}x)F^{\mu\nu}(\mathcal{P}x), \tag{158}$$

$$T^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)T = +\mathcal{T}_\mu{}^\alpha\mathcal{T}_\nu{}^\beta\mathcal{T}^\mu{}_\gamma\mathcal{T}^\nu{}_\delta F_{\alpha\beta}(\mathcal{T}x)F^{\gamma\delta}(\mathcal{T}x) = +F_{\mu\nu}(\mathcal{T}x)F^{\mu\nu}(\mathcal{T}x), \tag{159}$$

$$C^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)C = +F_{\mu\nu}(x)F^{\mu\nu}(x); \tag{160}$$

$$P^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)P = -\mathcal{P}_\mu{}^\alpha\mathcal{P}_\nu{}^\beta\mathcal{P}^\mu{}_\gamma\mathcal{P}^\nu{}_\delta F_{\alpha\beta}(\mathcal{P}x)\tilde{F}^{\gamma\delta}(\mathcal{P}x) = -F_{\mu\nu}(\mathcal{P}x)\tilde{F}^{\mu\nu}(\mathcal{P}x), \tag{161}$$

$$T^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)T = -\mathcal{T}_\mu{}^\alpha\mathcal{T}_\nu{}^\beta\mathcal{T}^\mu{}_\gamma\mathcal{T}^\nu{}_\delta F_{\alpha\beta}(\mathcal{T}x)\tilde{F}^{\gamma\delta}(\mathcal{T}x) = -F_{\mu\nu}(\mathcal{T}x)\tilde{F}^{\mu\nu}(\mathcal{T}x), \tag{162}$$

$$C^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)C = +F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x). \tag{163}$$

4 电弱规范场

电弱理论由 $SU(2)_L \times U(1)_Y$ 规范群描述, 它同时违反 P 和 C 对称性. 然而, 在大多数弱作用过程里, CP 联合对称性依然得以保持. CP 破坏只出现在少数稀有过程里, 被认为是由 CKM 矩阵里的 CP 相角引起的.

电弱理论的协变导数可表达为

$$D_\mu = \partial_\mu - ig_1 B_\mu Y - ig_2 W_\mu^a T^a = \partial_\mu - ig_1 B_\mu Y - ig_2 W_\mu, \tag{164}$$

其中 $W_\mu \equiv W_\mu^a T^a$. 对于左手费米子二重态 ψ_L (规范本征态) 而言, $T^a = \frac{\sigma^a}{2}$, 而

$$\bar{\psi}_L i\gamma^\mu D_\mu \psi_L = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + g_2 \bar{\psi}_L \gamma^\mu W_\mu^a T^a \psi_L. \tag{165}$$

CP 守恒体现在相互作用项 $g_2 \bar{\psi}_L \gamma^\mu W_\mu^a T^a \psi_L$ 上, 应有

$$(CP)^{-1} \bar{\psi}_L(x) \gamma^\mu W_\mu(x) \psi_L(x) CP = \bar{\psi}_L(\mathcal{P}x) \gamma^\mu W_\mu(\mathcal{P}x) \psi_L(\mathcal{P}x). \tag{166}$$

利用

$$P^{-1} \psi_L(x) P = \frac{1}{2}(1 - \gamma_5) i\beta \psi(\mathcal{P}x) = i\beta \frac{1}{2}(1 + \gamma_5) \psi(\mathcal{P}x) = i\beta \psi_R(\mathcal{P}x), \tag{167}$$

$$\begin{aligned}
P^{-1} \bar{\psi}_L(x) P &= P^{-1} \psi^\dagger(x) \frac{1}{2}(1 - \gamma_5) \beta P = \left[P^{-1} \beta \frac{1}{2}(1 - \gamma_5) \psi(x) P \right]^\dagger \\
&= \left[\beta \frac{1}{2}(1 - \gamma_5) i\beta \psi(\mathcal{P}x) \right]^\dagger = -i \left[\frac{1}{2}(1 + \gamma_5) \psi(\mathcal{P}x) \right]^\dagger = -i \bar{\psi}_R(\mathcal{P}x) \beta, \tag{168}
\end{aligned}$$

$$(CP)^{-1} \psi_L(x) CP = i\beta \frac{1}{2}(1 + \gamma_5) C^{-1} \psi(\mathcal{P}x) C = i\beta \frac{1}{2}(1 + \gamma_5) C \bar{\psi}^T(\mathcal{P}x), \tag{169}$$

$$\begin{aligned}
(CP)^{-1} \bar{\psi}_L(x) CP &= -i C^{-1} \left[\frac{1}{2}(1 + \gamma_5) \psi(\mathcal{P}x) \right]^\dagger C = -i \left[C^{-1} \frac{1}{2}(1 + \gamma_5) \psi(\mathcal{P}x) C \right]^\dagger \\
&= -i \left[\frac{1}{2}(1 + \gamma_5) C \bar{\psi}^T(\mathcal{P}x) \right]^\dagger = -i \left\{ \left[\psi^\dagger(\mathcal{P}x) \beta C^T \frac{1}{2}(1 + \gamma_5^T) \right]^\dagger \right\}^T
\end{aligned}$$

$$\begin{aligned}
&= -i \left[\frac{1}{2} (1 + \gamma_5^T) \mathcal{C}^* \beta \psi(\mathcal{P}x) \right]^T = -i \psi^T(\mathcal{P}x) \beta^T \mathcal{C}^\dagger \frac{1}{2} (1 + \gamma_5) \\
&= -i \psi^T(\mathcal{P}x) \beta^T \mathcal{C}^{-1} \frac{1}{2} (1 + \gamma_5) = i \psi^T(\mathcal{P}x) \mathcal{C}^{-1} \beta \frac{1}{2} (1 + \gamma_5) \\
&= -i \psi^T(\mathcal{P}x) \mathcal{C} \beta \frac{1}{2} (1 + \gamma_5) = -i \psi^T(\mathcal{P}x) \mathcal{C} \frac{1}{2} (1 - \gamma_5) \beta,
\end{aligned} \tag{170}$$

可得

$$\begin{aligned}
& (CP)^{-1} \bar{\psi}_L(x) \gamma^\mu W_\mu(x) \psi_L(x) CP \\
&= (CP)^{-1} \bar{\psi}_L(x) \gamma^\mu CP (CP)^{-1} W_\mu(x) CP (CP)^{-1} \psi_L(x) CP \\
&= -i \psi^T(\mathcal{P}x) \mathcal{C} \frac{1}{2} (1 - \gamma_5) \beta \gamma^\mu (CP)^{-1} W_\mu(x) CP i \beta \frac{1}{2} (1 + \gamma_5) \mathcal{C} \bar{\psi}^T(\mathcal{P}x) \\
&= \psi^T(\mathcal{P}x) \mathcal{C} \frac{1}{2} (1 - \gamma_5) \beta \gamma^\mu (CP)^{-1} W_\mu(x) CP \beta \frac{1}{2} (1 + \gamma_5) \mathcal{C} \bar{\psi}^T(\mathcal{P}x) \\
&= -\bar{\psi}(\mathcal{P}x) \mathcal{C}^T \frac{1}{2} (1 + \gamma_5^T) \beta^T [(CP)^{-1} W_\mu(x) CP]^T \gamma^{\mu T} \beta^T \frac{1}{2} (1 - \gamma_5^T) \mathcal{C}^T \psi(\mathcal{P}x) \\
&= \bar{\psi}(\mathcal{P}x) \mathcal{C}^{-1} \frac{1}{2} (1 + \gamma_5^T) \beta^T [(CP)^{-1} W_\mu(x) CP]^T \gamma^{\mu T} \beta^T \frac{1}{2} (1 - \gamma_5^T) \mathcal{C} \psi(\mathcal{P}x) \\
&= -\bar{\psi}(\mathcal{P}x) \frac{1}{2} (1 + \gamma_5) \beta \gamma^\mu \beta [(CP)^{-1} W_\mu(x) CP]^T \frac{1}{2} (1 - \gamma_5) \psi(\mathcal{P}x) \\
&= -\bar{\psi}_L(\mathcal{P}x) \mathcal{P}^\mu{}_\nu \gamma^\nu [(CP)^{-1} W_\mu(x) CP]^T \psi_L(\mathcal{P}x).
\end{aligned} \tag{171}$$

可见,

$$-\mathcal{P}^\nu{}_\mu \gamma^\mu [(CP)^{-1} W_\nu(x) CP]^T = \gamma^\mu W_\mu(\mathcal{P}x), \tag{172}$$

即 $-\mathcal{P}^\nu{}_\mu [(CP)^{-1} W_\nu(x) CP]^T = W_\mu(\mathcal{P}x)$. 因此, 规范场 W^μ 的 CP 变换性质为

$$(CP)^{-1} W^\mu(x) CP = -\mathcal{P}^\mu{}_\nu W^{\nu T}(\mathcal{P}x), \quad (CP)^{-1} W^{a\mu}(x) T^a CP = -\mathcal{P}^\mu{}_\nu W^{a\nu}(\mathcal{P}x) (T^a)^*. \tag{173}$$

令

$$T^\pm \equiv T^1 \pm iT^2 = \frac{1}{2} (\sigma^1 \pm i\sigma^2) = \sigma^\pm, \quad W_\mu^\pm \equiv \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2), \tag{174}$$

则有

$$W_\mu^a T^a = W_\mu^1 T^1 + W_\mu^2 T^2 + W_\mu^3 T^3 = \frac{1}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) + W_\mu^3 T^3. \tag{175}$$

由 $(\sigma^1)^* = \sigma^1$, $(\sigma^2)^* = -\sigma^2$ 和 $(\sigma^3)^* = \sigma^3$, 得

$$W_\mu^a (T^a)^* = W_\mu^1 T^1 - W_\mu^2 T^2 + W_\mu^3 T^3 = \frac{1}{\sqrt{2}} (W_\mu^- T^+ + W_\mu^+ T^-) + W_\mu^3 T^3, \tag{176}$$

故 W^\pm 和 W^3 的 CP 变换性质为

$$(CP)^{-1} W^{\pm\mu}(x) CP = -\mathcal{P}^\mu{}_\nu W^{\mp\nu}(\mathcal{P}x), \quad (CP)^{-1} W^{3\mu}(x) CP = -\mathcal{P}^\mu{}_\nu W^{3\nu}(\mathcal{P}x). \tag{177}$$

同理, 可以得到

$$(CP)^{-1} B^\mu(x) CP = -\mathcal{P}^\mu{}_\nu B^\nu(\mathcal{P}x). \tag{178}$$

于是, $Z^\mu \equiv W^{3\mu} \cos \theta_W - B^\mu \sin \theta_W$ 和 $A^\mu \equiv W^{3\mu} \sin \theta_W + B^\mu \cos \theta_W$ 的 CP 变换性质为

$$(CP)^{-1} Z^\mu(x) CP = -\mathcal{P}^\mu{}_\nu Z^\nu(\mathcal{P}x), \quad (CP)^{-1} A^\mu(x) CP = -\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x). \tag{179}$$

另一方面, T 对称性要求

$$T^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)T = \bar{\psi}_L(\mathcal{T}x)\gamma^\mu W_\mu(\mathcal{T}x)\psi_L(\mathcal{T}x). \quad (180)$$

利用

$$T^{-1}\psi_L(x)T = \frac{1}{2}(1 - \gamma_5)\mathcal{C}\gamma_5\psi(\mathcal{T}x) = \mathcal{C}\gamma_5\frac{1}{2}(1 - \gamma_5)\psi(\mathcal{T}x) = \mathcal{C}\gamma_5\psi_L(x), \quad (181)$$

$$\begin{aligned} T^{-1}\bar{\psi}_L(x)T &= T^{-1}\psi^\dagger(x)\frac{1}{2}(1 - \gamma_5)\beta T = [\mathcal{C}\gamma_5\psi(\mathcal{T}x)]^\dagger\frac{1}{2}(1 - \gamma_5)\beta \\ &= \psi^\dagger(\mathcal{T}x)\gamma_5\mathcal{C}^\dagger\frac{1}{2}(1 - \gamma_5)\beta = \psi^\dagger(\mathcal{T}x)\frac{1}{2}(1 - \gamma_5)\beta\gamma_5\mathcal{C}^{-1} = \bar{\psi}_L(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}, \end{aligned} \quad (182)$$

可得

$$\begin{aligned} T^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)T &= T^{-1}\bar{\psi}_L(x)TT^{-1}\gamma^\mu TT^{-1}W_\mu(x)TT^{-1}\psi_L(x)T \\ &= \bar{\psi}_L(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*T^{-1}W_\mu(x)T\mathcal{C}\gamma_5\psi_L(\mathcal{T}x) \\ &= -\bar{\psi}_L(\mathcal{T}x)\mathcal{T}^\mu{}_\nu\gamma^\nu T^{-1}W_\mu(x)T\psi_L(\mathcal{T}x), \end{aligned} \quad (183)$$

可见,

$$-\mathcal{T}^\nu{}_\mu\gamma^\mu T^{-1}W_\nu(x)T = \gamma^\mu W_\mu(\mathcal{T}x). \quad (184)$$

注意到时间反演算符 T 的反么正性, 规范场 W^μ 的 T 变换性质为

$$T^{-1}W^\mu(x)T = -\mathcal{T}^\mu{}_\nu W^\nu(\mathcal{T}x), \quad T^{-1}W^{a\mu}(x)T^a = -\mathcal{T}^\mu{}_\nu W^{a\nu}(\mathcal{T}x)T^a. \quad (185)$$

由 $(T^\pm)^* = T^\pm$ 和 $(T^3)^* = T^3$, 得

$$\begin{aligned} &\frac{1}{\sqrt{2}}[T^{-1}W^{+\mu}(x)TT^+ + T^{-1}W^{-\mu}(x)TT^-] + T^{-1}W^{3\mu}(x)TT^3 \\ &= -\mathcal{T}^\mu{}_\nu \left\{ \frac{1}{\sqrt{2}}[W^{+\nu}(\mathcal{T}x)T^+ + W^{-\nu}(\mathcal{T}x)T^-] + W^{3\nu}(\mathcal{T}x)T^3 \right\}, \end{aligned} \quad (186)$$

故 W^\pm 和 W^3 的 T 变换性质为

$$T^{-1}W^{\pm\mu}(x)T = -\mathcal{T}^\mu{}_\nu W^{\pm\nu}(\mathcal{T}x), \quad T^{-1}W^{3\mu}(x)T = -\mathcal{T}^\mu{}_\nu W^{3\nu}(\mathcal{T}x). \quad (187)$$

同理, 可以得到

$$T^{-1}B^\mu(x)T = -\mathcal{T}^\mu{}_\nu B^\nu(\mathcal{T}x). \quad (188)$$

于是, Z^μ 和 A^μ 的 T 变换性质为

$$T^{-1}Z^\mu(x)T = -\mathcal{T}^\mu{}_\nu Z^\nu(\mathcal{T}x), \quad T^{-1}A^\mu(x)T = -\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x). \quad (189)$$

Z^μ 和 B^μ 的 CP 和 T 变换性质均与电磁场 A^μ 的变换性质相同, $W^{\pm\mu}$ 也类似, 唯一的不同的是 CP 变换后会改变电荷符号. 因此, $Z^{\mu\nu} \equiv \partial^\mu Z^\nu - \partial^\nu Z^\mu$ 和 $B^{\mu\nu} \equiv \partial^\mu B^\nu - \partial^\nu B^\mu$ 均与 $F^{\mu\nu}$ 具有相同的 CP 和 T 变换性质, 而 $W^{\pm\mu\nu} \equiv \partial^\mu W^{\pm\nu} - \partial^\nu W^{\pm\mu}$ 在 CP 变换时则有电荷符号上的差异, 即

$$(CP)^{-1}W^{\pm\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma W^{\mp\rho\sigma}(\mathcal{P}x), \quad T^{-1}W^{\pm\mu\nu}(x)T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma W^{\pm\rho\sigma}(\mathcal{T}x), \quad (190)$$

$$(CP)^{-1}Z^{\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma Z^{\rho\sigma}(\mathcal{P}x), \quad T^{-1}Z^{\mu\nu}(x)T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma Z^{\rho\sigma}(\mathcal{T}x), \quad (191)$$

$$(CP)^{-1}B^{\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma B^{\rho\sigma}(\mathcal{P}x), \quad T^{-1}B^{\mu\nu}(x)T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma B^{\rho\sigma}(\mathcal{T}x). \quad (192)$$

进而, $\tilde{W}^{\pm\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}W_{\rho\sigma}^{\pm}$, $\tilde{Z}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}Z_{\rho\sigma}$ 和 $\tilde{B}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}B_{\rho\sigma}$ 的 CP 和 T 变换性质为

$$(CP)^{-1}\tilde{W}^{\pm\mu\nu}(x)CP = \mathcal{P}^{\mu}_{\alpha}\mathcal{P}^{\nu}_{\beta}\tilde{W}^{\mp\alpha\beta}(\mathcal{P}x), \quad T^{-1}\tilde{W}^{\pm\mu\nu}(x)T = \mathcal{T}^{\mu}_{\alpha}\mathcal{T}^{\nu}_{\beta}\tilde{W}^{\pm\alpha\beta}(\mathcal{T}x), \quad (193)$$

$$(CP)^{-1}\tilde{Z}^{\mu\nu}(x)CP = \mathcal{P}^{\mu}_{\alpha}\mathcal{P}^{\nu}_{\beta}\tilde{Z}^{\alpha\beta}(\mathcal{P}x), \quad T^{-1}\tilde{Z}^{\mu\nu}(x)T = \mathcal{T}^{\mu}_{\alpha}\mathcal{T}^{\nu}_{\beta}\tilde{Z}^{\alpha\beta}(\mathcal{T}x), \quad (194)$$

$$(CP)^{-1}\tilde{B}^{\mu\nu}(x)CP = \mathcal{P}^{\mu}_{\alpha}\mathcal{P}^{\nu}_{\beta}\tilde{B}^{\alpha\beta}(\mathcal{P}x), \quad T^{-1}\tilde{B}^{\mu\nu}(x)T = \mathcal{T}^{\mu}_{\alpha}\mathcal{T}^{\nu}_{\beta}\tilde{B}^{\alpha\beta}(\mathcal{T}x). \quad (195)$$

李群的生成元 t^a 满足

$$[t^a, t^b] = if^{abc}t^c, \quad [t^{a*}, t^{b*}] = -if^{abc}t^{c*}, \quad \text{tr}(t^a t^b) = \text{tr}(t^{a*} t^{b*}) = \frac{1}{2}\delta^{ab}. \quad (196)$$

对于 $SU(2)_L$ 规范场, 场强张量

$$W^{\mu\nu} = W^{a\mu\nu}T^a = \partial^{\mu}W^{a\nu}T^a - \partial^{\nu}W^{a\mu}T^a - ig_2[W^{a\mu}T^a, W^{b\nu}T^b], \quad (197)$$

其中

$$W^{a\mu\nu} = \partial^{\mu}W^{a\nu} - \partial^{\nu}W^{a\mu} + g_2\varepsilon^{abc}W^{b\mu}W^{c\nu}. \quad (198)$$

场强张量 $W^{\mu\nu}$ 的变换性质为

$$\begin{aligned} & (CP)^{-1}W^{a\mu\nu}(x)T^aCP \\ &= (CP)^{-1}(\partial^{\mu}W^{a\nu}(x)T^a - \partial^{\nu}W^{a\mu}(x)T^a - ig_2[W^{a\mu}(x)T^a, W^{b\nu}(x)T^b])CP \\ &= -\mathcal{P}^{\mu}_{\rho}\mathcal{P}^{\nu}_{\sigma}\partial^{\rho}W^{a\sigma}(\mathcal{P}x)T^{a*} + \mathcal{P}^{\nu}_{\rho}\mathcal{P}^{\mu}_{\sigma}\partial^{\rho}W^{a\sigma}(\mathcal{P}x)T^{a*} - ig_2[\mathcal{P}^{\mu}_{\rho}W^{a\rho}(\mathcal{P}x)T^{a*}, \mathcal{P}^{\nu}_{\sigma}W^{b\sigma}(\mathcal{P}x)T^{b*}] \\ &= -\mathcal{P}^{\mu}_{\rho}\mathcal{P}^{\nu}_{\sigma}\{\partial^{\rho}W^{a\sigma}(\mathcal{P}x)T^{a*} - \partial^{\sigma}W^{a\rho}(\mathcal{P}x)T^{a*} + ig_2[W^{a\rho}(\mathcal{P}x)T^{a*}, W^{b\sigma}(\mathcal{P}x)T^{b*}]\} \\ &= -\mathcal{P}^{\mu}_{\rho}\mathcal{P}^{\nu}_{\sigma}\{\partial^{\rho}W^{a\sigma}(\mathcal{P}x)T^{a*} - \partial^{\sigma}W^{a\rho}(\mathcal{P}x)T^{a*} + g_2\varepsilon^{abc}W^{a\rho}(\mathcal{P}x)W^{b\sigma}(\mathcal{P}x)T^{c*}\} \\ &= -\mathcal{P}^{\mu}_{\rho}\mathcal{P}^{\nu}_{\sigma}W^{a\rho\sigma}(\mathcal{P}x)T^{a*}, \\ & T^{-1}W^{a\mu\nu}(x)T^aT \\ &= T^{-1}(\partial^{\mu}W^{a\nu}(x)T^a - \partial^{\nu}W^{a\mu}(x)T^a - ig_2[W^{a\mu}(x)T^a, W^{b\nu}(x)T^b])T \\ &= -\mathcal{T}^{\mu}_{\rho}\mathcal{T}^{\nu}_{\sigma}\partial^{\rho}W^{a\sigma}(\mathcal{T}x)T^a + \mathcal{T}^{\nu}_{\rho}\mathcal{T}^{\mu}_{\sigma}\partial^{\rho}W^{a\sigma}(\mathcal{T}x)T^a + ig_2[\mathcal{T}^{\mu}_{\rho}W^{a\rho}(\mathcal{T}x)T^a, \mathcal{T}^{\nu}_{\sigma}W^{b\sigma}(\mathcal{T}x)T^b] \\ &= -\mathcal{T}^{\mu}_{\rho}\mathcal{T}^{\nu}_{\sigma}\{\partial^{\rho}W^{a\sigma}(\mathcal{T}x)T^a - \partial^{\sigma}W^{a\rho}(\mathcal{T}x)T^a - ig_2[W^{a\rho}(\mathcal{T}x)T^a, W^{b\sigma}(\mathcal{T}x)T^b]\} \\ &= -\mathcal{T}^{\mu}_{\rho}\mathcal{T}^{\nu}_{\sigma}\{\partial^{\rho}W^{a\sigma}(\mathcal{T}x)T^a - \partial^{\sigma}W^{a\rho}(\mathcal{T}x)T^a + g_2\varepsilon^{abc}W^{a\rho}(\mathcal{T}x)W^{b\sigma}(\mathcal{T}x)T^c\} \\ &= -\mathcal{T}^{\mu}_{\rho}\mathcal{T}^{\nu}_{\sigma}W^{a\rho\sigma}(\mathcal{T}x)T^a. \end{aligned} \quad (199)$$

注意到 (153) 和 (154) 式, 对偶场强张量 $\tilde{W}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}W_{\alpha\beta}$ 的变换性质为

$$\begin{aligned} (CP)^{-1}\tilde{W}^{a\mu\nu}(x)T^aCP &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(CP)^{-1}W_{\alpha\beta}^a(x)T^aCP = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{P}_{\alpha}^{\rho}\mathcal{P}_{\beta}^{\sigma}W_{\rho\sigma}^a(\mathcal{P}x)T^{a*} \\ &= \frac{1}{2}\mathcal{P}^{\mu}_{\alpha}\mathcal{P}^{\nu}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}W_{\rho\sigma}^a(\mathcal{P}x)T^{a*} = \mathcal{P}^{\mu}_{\alpha}\mathcal{P}^{\nu}_{\beta}\tilde{W}^{a\alpha\beta}(\mathcal{P}x)T^{a*}, \end{aligned} \quad (201)$$

$$\begin{aligned} T^{-1}\tilde{W}^{a\mu\nu}(x)T^aT &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}T^{-1}W_{\alpha\beta}^a(x)T^aT = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{T}_{\alpha}^{\rho}\mathcal{T}_{\beta}^{\sigma}W_{\rho\sigma}^a(\mathcal{T}x)T^a \\ &= \frac{1}{2}\mathcal{T}^{\mu}_{\alpha}\mathcal{T}^{\nu}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}W_{\rho\sigma}^a(\mathcal{T}x)T^a = \mathcal{T}^{\mu}_{\alpha}\mathcal{T}^{\nu}_{\beta}\tilde{W}^{a\alpha\beta}(\mathcal{T}x)T^a. \end{aligned} \quad (202)$$

于是, $W^{a\mu\nu}W_{\mu\nu}^a$ 和 $W^{a\mu\nu}\tilde{W}_{\mu\nu}^a$ 的变换性质为

$$(CP)^{-1}W^{a\mu\nu}(x)W_{\mu\nu}^a(x)CP$$

$$\begin{aligned}
&= (CP)^{-1} 2\text{tr}[W^{a\mu\nu}(x)T^a W_{\mu\nu}^b(x)T^b]CP = 2\text{tr}[(CP)^{-1}W^{a\mu\nu}(x)T^a CP(CP)^{-1}W_{\mu\nu}^b(x)T^b CP] \\
&= 2\text{tr}[\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta W^{a\alpha\beta}(\mathcal{P}x)T^{a*} \mathcal{P}_\mu^\gamma \mathcal{P}_\nu^\delta W_{\gamma\delta}^b(\mathcal{P}x)T^{b*}] = 2\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \mathcal{P}_\mu^\gamma \mathcal{P}_\nu^\delta W^{a\alpha\beta}(\mathcal{P}x)W_{\gamma\delta}^b(\mathcal{P}x)\text{tr}(T^{a*}T^{b*}) \\
&= \mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \mathcal{P}_\mu^\gamma \mathcal{P}_\nu^\delta W^{a\alpha\beta}(\mathcal{P}x)W_{\gamma\delta}^a(\mathcal{P}x) = +W^{a\mu\nu}(\mathcal{P}x)W_{\mu\nu}^a(\mathcal{P}x), \tag{203}
\end{aligned}$$

$$\begin{aligned}
&T^{-1}W^{a\mu\nu}(x)W_{\mu\nu}^a(x)T \\
&= T^{-1}2\text{tr}[W^{a\mu\nu}(x)T^a W_{\mu\nu}^b(x)T^b]T = 2\text{tr}[T^{-1}W^{a\mu\nu}(x)T^a TT^{-1}W_{\mu\nu}^b(x)T^b T] \\
&= 2\text{tr}[\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta W^{a\rho\sigma}(\mathcal{T}x)T^a \mathcal{T}_\mu^\gamma \mathcal{T}_\nu^\delta W_{\gamma\delta}^b(\mathcal{T}x)T^b] = 2\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta \mathcal{T}_\mu^\gamma \mathcal{T}_\nu^\delta W^{a\rho\sigma}(\mathcal{T}x)W_{\gamma\delta}^b(\mathcal{T}x)\text{tr}(T^a T^b) \\
&= \mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta \mathcal{T}_\mu^\gamma \mathcal{T}_\nu^\delta W^{a\rho\sigma}(\mathcal{T}x)W_{\gamma\delta}^a(\mathcal{T}x) = +W^{a\mu\nu}(\mathcal{T}x)W_{\mu\nu}^a(\mathcal{T}x), \tag{204}
\end{aligned}$$

$$\begin{aligned}
&(CP)^{-1}W^{a\mu\nu}(x)\tilde{W}_{\mu\nu}^a(x)CP = 2\text{tr}[(CP)^{-1}W^{a\mu\nu}(x)T^a CP(CP)^{-1}\tilde{W}_{\mu\nu}^b(x)T^b CP] \\
&= -2\mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \mathcal{P}_\mu^\gamma \mathcal{P}_\nu^\delta W^{a\alpha\beta}(\mathcal{P}x)\tilde{W}_{\gamma\delta}^b(\mathcal{P}x)\text{tr}(T^{a*}T^{b*}) \\
&= -W^{a\mu\nu}(\mathcal{P}x)\tilde{W}_{\mu\nu}^a(\mathcal{P}x), \tag{205}
\end{aligned}$$

$$\begin{aligned}
&T^{-1}W^{a\mu\nu}(x)\tilde{W}_{\mu\nu}^a(x)T = 2\text{tr}[T^{-1}W^{a\mu\nu}(x)T^a TT^{-1}\tilde{W}_{\mu\nu}^b(x)T^b T] \\
&= -2\mathcal{T}^\mu_\alpha \mathcal{T}^\nu_\beta \mathcal{T}_\mu^\gamma \mathcal{T}_\nu^\delta W^{a\rho\sigma}(\mathcal{T}x)\tilde{W}_{\gamma\delta}^a(\mathcal{T}x)\text{tr}(T^a T^b) \\
&= -W^{a\mu\nu}(\mathcal{T}x)\tilde{W}_{\mu\nu}^a(\mathcal{T}x). \tag{206}
\end{aligned}$$

下面讨论算符 $\varepsilon^{abc}W_{\mu\rho}^a W_{\nu}^b W^{c\rho} F^{\mu\nu}$ [5]. 仅当 $SU(2)$ 指标 a, b, c 彼此不同时, 这个算符才不为零, 故必有一个 $SU(2)$ 指标等于 2. 在生成元 T^1, T^2, T^3 中, 只有 T^2 的复共轭不等于自身而表达成 $(T^2)^* = -T^2$. 参考 (173), (185), (199) 和 (200) 式可知, $\varepsilon^{abc}W_{\mu\rho}^a W_{\nu}^b W^{c\rho}$ 的 CP 和 T 变换都会有一个额外的负号, 来自生成元 T^2 的非自复共轭性质. 于是,

$$\begin{aligned}
&(CP)^{-1}\varepsilon^{abc}W_{\mu\rho}^a(x)W_{\nu}^b(x)W^{c\rho}(x)F^{\mu\nu}(x)CP \\
&= -\varepsilon^{abc}(-\mathcal{P}_\mu^\alpha \mathcal{P}_\rho^\beta)W_{\alpha\beta}^a(\mathcal{P}x)(-\mathcal{P}_\nu^\gamma)W_{\gamma}^b(\mathcal{P}x)(-\mathcal{P}^\rho_\delta)W^{c\delta}(\mathcal{P}x)(-\mathcal{P}^\mu_\lambda \mathcal{P}^\nu_\tau)F^{\lambda\tau}(\mathcal{P}x) \\
&= -\varepsilon^{abc}\mathcal{P}_\mu^\alpha \mathcal{P}_\rho^\beta \mathcal{P}_\nu^\gamma \mathcal{P}^\rho_\delta \mathcal{P}^\mu_\lambda \mathcal{P}^\nu_\tau W_{\alpha\beta}^a(\mathcal{P}x)W_{\gamma}^b(\mathcal{P}x)W^{c\delta}(\mathcal{P}x)F^{\lambda\tau}(\mathcal{P}x) \\
&= -\varepsilon^{abc}W_{\mu\rho}^a(\mathcal{P}x)W_{\nu}^b(\mathcal{P}x)W^{c\rho}(\mathcal{P}x)F^{\mu\nu}(\mathcal{P}x), \tag{207}
\end{aligned}$$

$$\begin{aligned}
&T^{-1}\varepsilon^{abc}W_{\mu\rho}^a(x)W_{\nu}^b(x)W^{c\rho}(x)F^{\mu\nu}(x)T \\
&= -\varepsilon^{abc}(-\mathcal{T}_\mu^\alpha \mathcal{T}_\rho^\beta)W_{\alpha\beta}^a(\mathcal{T}x)(-\mathcal{T}_\nu^\gamma)W_{\gamma}^b(\mathcal{T}x)(-\mathcal{T}^\rho_\delta)W^{c\delta}(\mathcal{T}x)(-\mathcal{T}^\mu_\lambda \mathcal{T}^\nu_\tau)F^{\lambda\tau}(\mathcal{T}x) \\
&= -\varepsilon^{abc}\mathcal{T}_\mu^\alpha \mathcal{T}_\rho^\beta \mathcal{T}_\nu^\gamma \mathcal{T}^\rho_\delta \mathcal{T}^\mu_\lambda \mathcal{T}^\nu_\tau W_{\alpha\beta}^a(\mathcal{T}x)W_{\gamma}^b(\mathcal{T}x)W^{c\delta}(\mathcal{T}x)F^{\lambda\tau}(\mathcal{T}x) \\
&= -\varepsilon^{abc}W_{\mu\rho}^a(\mathcal{T}x)W_{\nu}^b(\mathcal{T}x)W^{c\rho}(\mathcal{T}x)F^{\mu\nu}(\mathcal{T}x). \tag{208}
\end{aligned}$$

5 总结

一些算符的 P, T, C 变换性质如 Tab. 1 所示.

可以看到, 带有 n 个矢量指标的旋量场双线性型在 CPT 变换下的奇偶性与数字 n 的奇偶性相同. 由于 ∂_μ 在 CPT 变换下是奇的, 这一结论也可以推广到含有时空导数的情况. 对于标量场和矢量场, 总可以选择它们在 C, P, T 变换下的相位因子, 使得它们满足: 由场和时空导数组成的厄米算符在 CPT 变换下的奇偶性与未收缩矢量指标个数的奇偶性相同. 于是, 由任意场和时空导数组成的厄米 Lorentz 标量在 CPT 变换下是偶的. 由于拉氏量必须由这样的标量构成, 在 CPT 变换下有 $\mathcal{L}(x) \rightarrow \mathcal{L}(-x)$, 从而作用量 $S = \int d^4x \mathcal{L}(x)$ 是 CPT 不变量. 这就是 CPT 定理.

参考文献

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Table 1: 一些算符的 P, T, C 变换性质. 对于 $\mu = 0$, $[-]^\mu \equiv 1$; 对于 $\mu = 1, 2, 3$, $[-]^\mu \equiv -1$.

Operator	P	T	C	CP	CPT
i	+	−	+	+	−
∂^μ	$[-]^\mu$	$-[-]^\mu$	+	$[-]^\mu$	−
$\phi^\dagger \phi$	+	+	+	+	+
$\phi^\dagger i \overleftrightarrow{\partial}^\mu \phi$	$[-]^\mu$	$[-]^\mu$	−	$-[-]^\mu$	−
$\partial^\mu \phi^\dagger \partial_\mu \phi$	+	+	+	+	+
$\bar{\psi} \psi$	+	+	+	+	+
$\bar{\psi} i \gamma_5 \psi$	−	−	+	−	+
$\bar{\psi} \gamma^\mu \psi$	$[-]^\mu$	$[-]^\mu$	−	$-[-]^\mu$	−
$\bar{\psi} \gamma^\mu \gamma_5 \psi$	$-[-]^\mu$	$[-]^\mu$	+	$-[-]^\mu$	−
$\bar{\psi} \sigma^{\mu\nu} \psi$	$[-]^\mu [-]^\nu$	$-[-]^\mu [-]^\nu$	−	$-[-]^\mu [-]^\nu$	+
$\bar{\psi}_L \gamma^\mu \psi_L$	$[-]^\mu \bar{\psi}_R \gamma^\mu \psi_R$	$[-]^\mu$	$-\bar{\psi}_R \gamma^\mu \psi_R$	$-[-]^\mu$	−
$\bar{\psi}_R \gamma^\mu \psi_R$	$[-]^\mu \bar{\psi}_L \gamma^\mu \psi_L$	$[-]^\mu$	$-\bar{\psi}_L \gamma^\mu \psi_L$	$-[-]^\mu$	−
$\bar{\psi} i \gamma^\mu \partial_\mu \psi$	+	+	+	+	+
$\bar{\psi} \gamma^\mu \partial^\nu \psi + \text{h.c.}$	$[-]^\mu [-]^\nu$	$-[-]^\mu [-]^\nu$	−	$-[-]^\mu [-]^\nu$	+
$\bar{\psi} \gamma^\mu \gamma_5 \partial^\nu \psi + \text{h.c.}$	$-[-]^\mu [-]^\nu$	$-[-]^\mu [-]^\nu$	+	$-[-]^\mu [-]^\nu$	+
$\bar{\psi}_{L/R} \gamma^\mu \partial^\nu \psi_{L/R} + \text{h.c.}$	$[-]^\mu [-]^\nu (L \leftrightarrow R)$	$-[-]^\mu [-]^\nu$	$-(L \leftrightarrow R)$	$-[-]^\mu [-]^\nu$	+
$i \bar{\psi} \gamma^\mu \partial^\nu \psi + \text{h.c.}$	$[-]^\mu [-]^\nu$	$[-]^\mu [-]^\nu$	+	$[-]^\mu [-]^\nu$	+
$i \bar{\psi} \gamma^\mu \gamma_5 \partial^\nu \psi + \text{h.c.}$	$-[-]^\mu [-]^\nu$	$[-]^\mu [-]^\nu$	−	$[-]^\mu [-]^\nu$	+
$i \bar{\psi}_{L/R} \gamma^\mu \partial^\nu \psi_{L/R} + \text{h.c.}$	$[-]^\mu [-]^\nu (L \leftrightarrow R)$	$[-]^\mu [-]^\nu$	$+(L \leftrightarrow R)$	$[-]^\mu [-]^\nu$	+
A^μ	$[-]^\mu$	$[-]^\mu$	−	$-[-]^\mu$	−
$F^{\mu\nu}$	$[-]^\mu [-]^\nu$	$-[-]^\mu [-]^\nu$	−	$-[-]^\mu [-]^\nu$	+
$\tilde{F}^{\mu\nu}$	$-[-]^\mu [-]^\nu$	$[-]^\mu [-]^\nu$	−	$[-]^\mu [-]^\nu$	+
$F^{\mu\nu} F_{\mu\nu}$	+	+	+	+	+
$F^{\mu\nu} \tilde{F}_{\mu\nu}$	−	−	+	−	+
Z^μ		$[-]^\mu$		$-[-]^\mu$	−
$Z^{\mu\nu}$		$-[-]^\mu [-]^\nu$		$-[-]^\mu [-]^\nu$	+
$\tilde{Z}^{\mu\nu}$		$[-]^\mu [-]^\nu$		$[-]^\mu [-]^\nu$	+
$W^{\pm\mu}$		$[-]^\mu$		$-[-]^\mu W^{\mp\mu}$	$-W^{\mp\mu}$
$W^{\pm\mu\nu}$		$-[-]^\mu [-]^\nu$		$-[-]^\mu [-]^\nu W^{\mp\mu\nu}$	$+W^{\mp\mu\nu}$
$\tilde{W}^{\pm\mu\nu}$		$[-]^\mu [-]^\nu$		$[-]^\mu [-]^\nu \tilde{W}^{\mp\mu\nu}$	$+\tilde{W}^{\mp\mu\nu}$
$W^{a\mu\nu} W_{\mu\nu}^a, B^{\mu\nu} B_{\mu\nu}$		+		+	+
$W^{a\mu\nu} \tilde{W}_{\mu\nu}^a, B^{\mu\nu} \tilde{B}_{\mu\nu}$		−		−	+
$\varepsilon^{abc} W_{\mu\rho}^a W_\nu^b W^{c\rho} F^{\mu\nu}$		−		−	+

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