量子场论讲义

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第 1 章　预备知识

1.1　量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由**粒子**组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon 方程**：

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\psi(\mathbf{x},t). \tag{1.1}$$

它给出的自由粒子能量为

$$E = \pm\sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}, \tag{1.2}$$

其中 $\mathbf{p}$ 为粒子的动量，m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2 \le E < \infty$；也可以为负，取值范围为 $-\infty < E \le mc^2$。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t}\psi\right), \tag{1.3}$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 协变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E < 0$ 的态都被填满而所有 $E > 0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E > 0$ 的电子跃迁到 $E < 0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的“反粒子 (anti-particle)”，称为**正电子** (positron)。正电子在 1932 年被 Carl Anderson 发现。

但是，Dirac 的空穴理论仍然面临一些困难，比如，为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场？另一方面，Dirac 方程一开始作为描述单个粒子波函数的方程提出来，但 Dirac 的解释却包含了无穷多个粒子。而且，像光子和 π 介子这些不满足 Pauli 不相容原理的粒子，空穴理论是不能成立的。此外，Dirac 方程只能描述自旋 1/2 的粒子，不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难，是否意味着处理这些问题的基础本身就不正确呢？确实是这样的。量子力学的一条基本原理是：观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而，**时间**显然是一个观测量，却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中，描述系统的量子态时可以让态依赖于一个时间参数 t，这是时间的概念进入量子力学的方式，但并没有假定这个参数是某个厄米算符的本征值。另一方面，粒子的空间位置 $\mathbf{x}$ 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见，在量子力学中，对时间和空间的处理方式是完全不同的。而在狭义相对论中，Lorentz 对称性将两者混合起来。因此，在结合量子力学与狭义相对论的过程中出现困难，也是正常的。

那么，如何在量子力学中平等地处理时间和空间呢？一种途径是将时间提升为一个厄米算符，但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数，不再由厄米算符描写。这样，我们可以在每个空间点 $\mathbf{x}$ 处定义一个算符 $\hat{\phi}(\mathbf{x})$，所有这些算符的集合称为**量子场**。在 Heisenberg 绘景中，量子场算符也依赖于时间 t：

$$\hat{\phi}(\mathbf{x},t) = e^{i\hat{H}t/\hbar}\hat{\phi}(\mathbf{x})e^{-i\hat{H}t/\hbar}. \tag{1.4}$$

如此，量子化的对象变成是由依赖于时空坐标的**场**组成的动力学系统，这就是**量子场论**。这里的量子算符用 ˆ 符号标记，为了简化记号，后面将省略 ˆ 符号。

在量子场论中，前面提到的困难都可以得到解决。现在，Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态，包含粒子的态则是激发态，激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子，由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的，不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x\,\rho$ 也可以用产生湮灭算符表达出来，虽然它不一定是正定的，但是它不再被解释为总概率，而是被解释为正粒子数与反粒子数之差，因而也不再出现负概率困难。

1.2　自然单位制

量子场论是结合量子力学和相对论的理论，因而时常出现约化 Planck 常量 $\hbar$ 和光速 c，这一点可以从上一节的几个公式中看出来。于是，为了简化表述，通常采用**自然单位制**，取

$$\hbar = c = 1. \tag{1.5}$$

从而，Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(\mathbf{x},t) = 0. \tag{1.6}$$

在自然单位制中，速度没有量纲 (dimension)；长度量纲与时间量纲相同，是能量量纲的倒数；能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22}\ \mathrm{MeV \cdot s}, \quad 1 = \hbar c = 1.973 \times 10^{-11}\ \mathrm{MeV \cdot cm}, \tag{1.7}$$

可得

$$1\ \mathrm{s}^{-1} = 6.582 \times 10^{-22}\ \mathrm{MeV}, \quad 1\ \mathrm{cm}^{-1} = 1.973 \times 10^{-11}\ \mathrm{MeV}. \tag{1.8}$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137.036} \tag{1.9}$$

是没有量纲的，它的数值在任何单位制下都应该相同。因此，自然单位制不可能将 $\hbar$、c、ε_0 和 e 这四个常数同时归一化。在量子场论中，通常再取真空介电常数

$$\varepsilon_0 = 1, \tag{1.10}$$

同时可得真空磁导率 $\mu_0 = 1/(\varepsilon_0 c^2) = 1$，这样做其实是取了 Heaviside-Lorentz 单位制。从而，不同于 Gauss 单位制，Maxwell 方程组中不会出现无理数 4π：

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{1.11}$$

此处的单位制称为**有理化**的自然单位制。现在，精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$，而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的；4π 因子会出现在 Coulomb 定律中，点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \tag{1.12}$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到**狭义相对论**，它的基本原理如下。

(1) 光速不变原理：在任意惯性参考系中，光速的大小不变。

(2) 狭义相对性原理：在任意惯性参考系中，物理定律具有相同的形式。

两个惯性参考系的直角坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动，则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \tag{1.13}$$

其中 Lorentz 因子 $\gamma \equiv (1-\beta^2)^{-1/2}$. 这种 Lorentz 变换称为沿 x 方向的**增速** (boost)。在此变换下，有

$$t'^2 - x'^2 - y'^2 - z'^2 = \gamma^2(t-\beta x)^2 - \gamma^2(x-\beta t)^2 - y^2 - z^2$$

$$= \frac{1}{1-\beta^2}(t^2+\beta^2x^2-2\beta xt-x^2-\beta^2t^2+2\beta xt)-y^2-z^2=t^2-x^2-y^2-z^2. \tag{1.14}$$

可见，$t^2-x^2-y^2-z^2$ 在 Lorentz 变换下不变，是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值，这是 Lorentz 变换对应的对称性，称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来，可以构成 Minkowski 时空，坐标记为

$$x^\mu=(x^0,x^1,x^2,x^3)=(t,x,y,z)=(x^0,\mathbf{x}),\quad 其中\ \mu=0,1,2,3. \tag{1.15}$$

上式中四种记法是等价的。x^μ 是一个*逆变* (contravariant) 的 Lorentz 四维矢量 (vector)，“逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发，可以定义 Lorentz 不变的*内积*[1]

$$x^2\equiv x\cdot x\equiv(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=(x^0)^2-|\mathbf{x}|^2. \tag{1.16}$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu}=g_{\nu\mu}=\begin{pmatrix}+1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}, \tag{1.17}$$

可以把内积 (1.16) 简洁地写成

$$x^2=g_{\mu\nu}x^\mu x^\nu. \tag{1.18}$$

这里采用了 **Einstein 求和约定**：*不写出求和符号，重复的指标即表示求和*。除非特别指出，后面都默认使用这个约定。在上式中，用同个字母表示的指标分别在上标和下标重复出现并求和，这称为**缩并** (contraction)，是 Lorentz 不变量的特点。

为了进一步简化记号，定义*协变* (covariant) 的 Lorentz 四维矢量

$$x_\mu=g_{\mu\nu}x^\nu=(x^0,-x^1,-x^2,-x^3)=(x^0,-\mathbf{x}). \tag{1.19}$$

“协变”指的是指标 μ 写在右下角。于是，内积 x^2 的表达式 (1.18) 可以简化为

$$x^2=x^\mu x_\mu. \tag{1.20}$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来，变成协变矢量 x_μ。从方阵的角度看，$g_{\mu\nu}$ 的逆为

$$g^{\mu\nu}=g^{\nu\mu}=\begin{pmatrix}+1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}, \tag{1.21}$$

[1]这里的记号有些不一致，第一个 x^2 是内积的记号，而第二个 x^2 是第 2 个空间坐标。

满足

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu, \tag{1.22}$$

其中 **Kronecker 符号** $\delta^\mu{}_\nu$ 定义为

$$\delta^\mu{}_\nu = \delta_\mu{}^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \tag{1.23}$$

对于 Minkowski 度规，$g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同，但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$，对 μ 求和，得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma{}_\nu x^\nu = x^\sigma, \tag{1.24}$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来，变成逆变矢量 x^σ。可见，逆变矢量与协变矢量是一一对应的，是对同一个 Lorentz 矢量的两种等价描述。

利用 Kronecker 符号的定义和 (1.22) 式，可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu{}_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \tag{1.25}$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho{}_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \tag{1.26}$$

这两条式子表明，度规也可以用来对度规自身的指标进行升降。

利用四维矢量的记号，可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.27}$$

其中

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.28}$$

注意：在将 $\Lambda^\mu{}_\nu$ 视作矩阵时，偏左的指标 μ 表示行的编号，偏右的指标 ν 表示列的编号。$\Lambda^\mu{}_\nu$ 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变，从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值。我们可以将 $\Lambda^\mu{}_\nu$ 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换，称为（齐次）**Lorentz 变换**，使下式成立：

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \tag{1.29}$$

可见，Lorentz 变换 $\Lambda^\mu{}_\nu$ 必须满足**保度规条件**

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}. \tag{1.30}$$

空间旋转变换保持 $|\mathbf{x}|^2$ 不变，由 (1.16) 式可知，这种变换也属于 Lorentz 变换。例如，绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \tag{1.31}$$

容易验证，它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并，可得

$$\Lambda_\nu{}^\gamma\Lambda^\nu{}_\beta = g^{\gamma\alpha}g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g^{\gamma\alpha}g_{\alpha\beta} = \delta^\gamma{}_\beta, \tag{1.32}$$

其中

$$\Lambda_\nu{}^\gamma \equiv g^{\gamma\alpha}g_{\mu\nu}\Lambda^\mu{}_\alpha \tag{1.33}$$

可以看作是用度规对 $\Lambda^\mu{}_\alpha$ 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu{}_\nu \equiv \Lambda_\nu{}^\mu, \tag{1.34}$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu{}_\rho\Lambda^\rho{}_\nu = \delta^\mu{}_\nu. \tag{1.35}$$

$\delta^\mu{}_\nu$ 也是一个 Lorentz 变换，它使得 $x'^\mu = \delta^\mu{}_\nu x^\nu = x^\mu$，即 x^μ 在这个变换下不变。可见，$\delta^\mu{}_\nu$ 是一个恒等变换。(1.35) 式表明，对时空坐标矢量先作 Λ 变换，再作 Λ^{-1} 变换，得到的矢量还是原来的矢量。也就是说，由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换，也是一个 Lorentz 变换。在这些记号下，协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu}x'^\nu = g_{\mu\nu}\Lambda^\nu{}_\rho x^\rho = g_{\mu\nu}\Lambda^\nu{}_\rho g^{\rho\sigma}x_\sigma = \Lambda_\mu{}^\sigma x_\sigma = x_\sigma(\Lambda^{-1})^\sigma{}_\mu. \tag{1.36}$$

Λ^{-1} 既然是一个 Lorentz 变换，必定满足保度规条件

$$g_{\mu\nu}(\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\nu{}_\beta = g_{\alpha\beta}, \tag{1.37}$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta}g^{\alpha\rho}g^{\beta\sigma} = g_{\mu\nu}(\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\nu{}_\beta g^{\alpha\rho}g^{\beta\sigma} = g^{\gamma\delta}g_{\gamma\mu}g_{\delta\nu}\Lambda_\alpha{}^\mu\Lambda_\beta{}^\nu g^{\alpha\rho}g^{\beta\sigma} \\ &= g^{\gamma\delta}(g^{\alpha\rho}g_{\gamma\mu}\Lambda_\alpha{}^\mu)(g^{\beta\sigma}g_{\delta\nu}\Lambda_\beta{}^\nu) = g^{\gamma\delta}\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta. \end{aligned} \tag{1.38}$$

这给出了保度规条件 (1.30) 的一个等价形式：

$$g^{\mu\nu}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = g^{\alpha\beta}. \tag{1.39}$$

将 $\Lambda^\mu{}_\nu$ 视作矩阵 Λ，则其转置矩阵 Λ^{T} 的分量满足 $(\Lambda^{\mathrm{T}})_\nu{}^\mu = \Lambda^\mu{}_\nu$，由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = (\Lambda^{\mathrm{T}})_\alpha{}^\mu g_{\mu\nu}\Lambda^\nu{}_\beta, \tag{1.40}$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda. \tag{1.41}$$

取行列式得 $\det\mathbf{g} = \det\Lambda^{\mathrm{T}}\cdot\det\mathbf{g}\cdot\det\Lambda = \det\mathbf{g}\cdot(\det\Lambda)^2$，因此，

$$(\det\Lambda)^2 = 1, \quad \det\Lambda = \pm 1. \tag{1.42}$$

Lorentz 坐标变换 $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ 的 Jacobi 行列式为

$$\mathcal{J} = \det\left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}\right] = \det\Lambda, \tag{1.43}$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}|d^4x = |\det\Lambda|d^4x = d^4x. \tag{1.44}$$

可见，Minkowski 时空的体积元是 Lorentz 不变的。

$\det\Lambda$ 的值可以用来为 Lorentz 变换分类：$\det\Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换，$\det\Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外，由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = (\Lambda^{0}{}_{0})^2 - (\Lambda^{i}{}_{0})^2, \tag{1.45}$$

则 $(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1$，故有 $\Lambda^0{}_0 \geq +1$ 或 $\Lambda^0{}_0 \leq -1$。$\Lambda^0{}_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换，$\Lambda^0{}_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上，对称性由群论描述。对称变换的集合称为**群**，群元素具有乘法，满足下列四个条件。

(1) 两个群元素的乘积即是两次对称变换相继作用，乘法满足*结合律*。

(2) 群中任意两个元素的乘积仍属于此群（*封闭性*）。

(3) 群中必有一个*恒元*（对应于恒等变换），它与任一元素的乘积仍为此元素。

(4) 任一元素都可以在群中找到一个*逆元*（对应于逆变换），两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数（如 β、θ 等）来描述，因而是一种*连续变换*，所以 Lorentz 群是一个*连续群*，参数的变化区域称为*群空间*。Lorentz 群的整个群空间不是连通的，它有四个连通分支，如图 1.1 所示，分别是固有保时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \geq +1$)、固有反时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \leq -1$)、非固有保时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \geq +1$) 和非固有反时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \leq -1$)，四个分支之间彼此不连通。恒元（即恒等变换）在固有保时向分支里，这个分支也称为**固有保时向 Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义*宇称* (parity) 变换为

$$\mathcal{P}^{\mu}{}_{\nu} = (\mathcal{P}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.46}$$

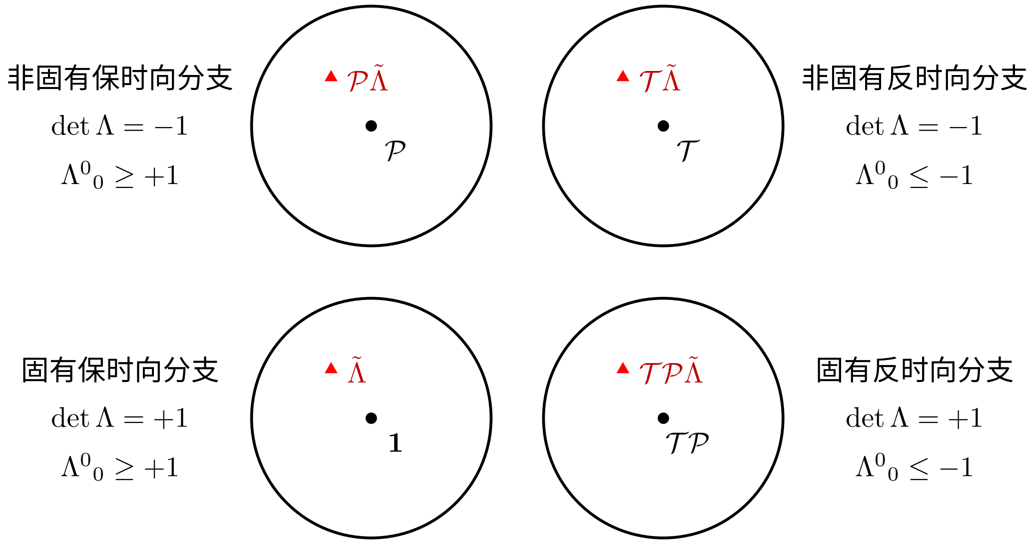


图 1.1: Lorentz 群的四个连通分支示意图。$\mathbf{1}$、$\mathcal{P}$ 和 $\mathcal{T}$ 分别代表恒等变换、宇称变换和时间反演变换，$\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的，亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$
{\mathcal{T}^\mu}_\nu = {(\mathcal{T}^{-1})^\mu}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \tag{1.47}
$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素，乘上宇称变换或（和）时间反演变换，就可以到达 Lorentz 群的其它分支。

1.4 Lorentz 矢量

如果一些 $m\times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同，就可以用这些矩阵来表示这个群，这些矩阵构成了这个群的一个 **m 维线性表示**。利用群的线性表示，可以将对称变换视作矩阵，将变换作用的态视作列矩阵。

在上一节中，我们已经用矩阵的形式表示过 Lorentz 变换 ${\Lambda^\mu}_\nu$，可见，${\Lambda^\mu}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**，因为 Lorentz 矢量 x^ν 可以看作是变换 ${\Lambda^\mu}_\nu$ 所作用的态。一般地，一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$
A'^\mu = {\Lambda^\mu}_\nu A^\nu. \tag{1.48}
$$

类似于 (1.36) 式，逆变矢量 A^μ 对应的协变矢量 $A_\mu = g_{\mu\nu}A^\nu$ 在 Lorentz 变换下满足

$$
A_\mu = A_\nu {(\Lambda^{-1})^\nu}_\mu. \tag{1.49}
$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \tag{1.50}$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量：

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \tag{1.51}$$

Lorentz 不变量也称为 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负，Lorentz 矢量 A^μ 的自我内积的符号不是确定的，可以分为三类。

(1) 若 $A^2 > 0$，则称 A^μ 为**类时**矢量。

(2) 若 $A^2 < 0$，则称 A^μ 为**类空**矢量。

(3) 若 $A^2 = 0$，则称 A^μ 为**类光**矢量。

由于 A^2 是 Lorentz 不变量，不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中，质点的能量 E、动量 $\mathbf{p}$ 和（静止）质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{1.52}$$

可以用 E 和 $\mathbf{p}$ 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \tag{1.53}$$

称为**四维动量**，它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \tag{1.54}$$

这是合理的，因为质量 m 在狭义相对论中是一个 Lorentz 不变量。p^μ 在 $m > 0$ 时是类时矢量，在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) = g^{\mu\nu} \partial_\nu, \tag{1.55}$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho{}^\nu = g^{\mu\nu}. \tag{1.56}$$

可见，这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ 时，时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \partial^\nu. \tag{1.57}$$

由上式、(1.56) 式和保度规条件 (1.39) 可得，

$$\partial'^\mu x'^\nu = \Lambda^\mu{}_\rho \partial^\rho (\Lambda^\nu{}_\sigma x^\sigma) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho x^\sigma = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} = g^{\mu\nu}, \tag{1.58}$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的，从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明，时空导数的 Lorentz 变换形式与 Lorentz 矢量相同，因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \tag{1.59}$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu}\partial'^\mu \partial'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma}\partial^\rho\partial^\sigma = \partial^2. \tag{1.60}$$

可见，∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \tag{1.61}$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 协变性。

1.5　Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p+q$ 阶的 (p,q) 型 **Lorentz 张量** $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ 具有 p 个逆变指标和 q 个协变指标，并满足如下 Lorentz 变换规则：

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_p}{}_{\rho_p} T^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\sigma_q}{}_{\nu_q}. \tag{1.62}$$

这里的逆变指标和协变指标统称为 *Lorentz* 指标。Lorentz 标量是 0 阶 Lorentz 张量，不具有 Lorentz 指标；Lorentz 矢量是 1 阶 Lorentz 张量，具有 1 个 Lorentz 指标。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0,2)$ 型 Lorentz 张量，不过它在任何惯性系中不变，Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62)，可以验证，如下表达式都是 Lorentz 标量（亦即 Lorentz 不变量）：

$$g_{\mu\nu}T^{\mu\nu}, \quad T^{\mu\nu}A_\mu B_\nu, \quad T^{\mu\nu}T_{\mu\nu}, \quad g_{\mu\sigma}T^{\mu\nu}{}_\rho T^{\sigma\rho}{}_\nu. \tag{1.63}$$

实际上,可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p,q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并，可以得到一个 $(p-1,q-1)$ 型 Lorentz 张量。例如，由

$$T'^{\mu\nu}{}_\mu = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma(\Lambda^{-1})^\gamma{}_\mu = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma \delta^\gamma{}_\alpha = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\alpha \tag{1.64}$$

可知，$T^{\mu\nu}{}_\mu$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita 符号**

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的偶次置换}, \\ -1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的奇次置换}, \\ 0, & \text{其它情况。} \end{cases} \tag{1.65}$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是全反对称的,即关于任意两个指标反对称,如 $\varepsilon^{\mu\nu\rho\sigma}=-\varepsilon^{\nu\mu\rho\sigma}=-\varepsilon^{\rho\nu\mu\sigma}=-\varepsilon^{\sigma\nu\rho\mu}$。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma}=g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.66}$$

$\varepsilon_{\mu\nu\rho\sigma}$ 也是全反对称的，如

$$\varepsilon_{\nu\mu\rho\sigma}=g_{\nu\alpha}g_{\mu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}=g_{\mu\beta}g_{\nu\alpha}g_{\rho\gamma}g_{\sigma\delta}(-\varepsilon^{\beta\alpha\gamma\delta})=-\varepsilon_{\mu\nu\rho\sigma}. \tag{1.67}$$

根据这些定义,

$$\varepsilon^{0123}=+1,\quad \varepsilon_{0123}=-1. \tag{1.68}$$

从而,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}=4!\,\varepsilon^{0123}\varepsilon_{0123}=-4!. \tag{1.69}$$

利用 Levi-Civita 符号可以把 $\det\Lambda$ 按照行列式定义写成

$$\det\Lambda=\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}=-\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.70}$$

对于固有 Lorentz 变换，$\det\Lambda=+1$，有

$$\varepsilon^{0123}=\varepsilon^{0123}\det\Lambda=\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.71}$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质，可得

$$\varepsilon^{1023}=-\varepsilon^{0123}=-\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}=-\Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}=\Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\beta\alpha\gamma\delta}. \tag{1.72}$$

依此类推，可以证明

$$\varepsilon^{\mu\nu\rho\sigma}=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.73}$$

可见，在固有 Lorentz 变换下，$\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量，不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中，ρ 是电荷密度，$\mathbf{J}$ 是电流密度，它们可以组成一个 Lorentz 矢量 $J^\mu=(\rho,\mathbf{J})$，从而，电流连续性方程

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0 \tag{1.74}$$

可以写成 Lorentz 协变的形式

$$\partial_\mu J^\mu=0. \tag{1.75}$$

此外，电场强度 $\mathbf{E}$ 和磁感应强度 $\mathbf{B}$ 可以用电势 Φ 和矢势 $\mathbf{A}$ 表达为

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t},\quad \mathbf{B}=\nabla\times\mathbf{A}. \tag{1.76}$$

这样，方程

$$\nabla\cdot\mathbf{B}=0 \tag{1.77}$$

是自动满足的。Φ 和 $\mathbf{A}$ 可以组成一个 Lorentz 矢量 $A^\mu=(\Phi,\mathbf{A})$，称为**四维矢势**，则 (1.76) 式的分量形式为

$$E^i=-\partial_i A^0-\partial_0 A^i,\quad B^k=\varepsilon^{kij}\partial_i A^j,\quad i,j,k=1,2,3. \tag{1.78}$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk}\equiv\varepsilon^{0ijk}, \tag{1.79}$$

因而 $\varepsilon^{123}=+1$。

引入电磁场的**场强张量** (field strength tensor)

$$F^{\mu\nu}=-F^{\nu\mu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu, \tag{1.80}$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序，从上述定义可得

$$\begin{aligned}\partial^\rho F^{\mu\nu}&=\partial^\rho(\partial^\mu A^\nu-\partial^\nu A^\mu)=\partial^\mu\partial^\rho A^\nu-\partial^\mu\partial^\nu A^\rho+\partial^\nu\partial^\mu A^\rho-\partial^\nu\partial^\rho A^\mu\\&=\partial^\mu F^{\rho\nu}+\partial^\nu F^{\mu\rho}=-\partial^\mu F^{\nu\rho}-\partial^\nu F^{\rho\mu},\end{aligned} \tag{1.81}$$

即

$$\partial^\rho F^{\mu\nu}+\partial^\mu F^{\nu\rho}+\partial^\nu F^{\rho\mu}=0. \tag{1.82}$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i}=\partial^0 A^i-\partial^i A^0=\partial_0 A^i+\partial_i A^0=-E^i, \tag{1.83}$$

可见，F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k}\varepsilon^{12k}=\varepsilon^{123}\varepsilon^{123}=1$ 和 $\varepsilon^{12k}\varepsilon^{21k}=\varepsilon^{123}\varepsilon^{213}=-1$，依此类推，可以归纳出如下求和关系：

$$\varepsilon^{ijk}\varepsilon^{kmn}=\varepsilon^{ijk}\varepsilon^{mnk}=\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm}, \tag{1.84}$$

利用这个关系，可得

$$\varepsilon^{ijk}B^k=\varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n=\delta^{im}\delta^{jn}\partial_m A^n-\delta^{in}\delta^{jm}\partial_m A^n=\partial_i A^j-\partial_j A^i, \tag{1.85}$$

从而，

$$F^{ij}=\partial^i A^j-\partial^j A^i=-\partial_i A^j+\partial_j A^i=-\varepsilon^{ijk}B^k, \tag{1.86}$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu}=\begin{pmatrix}0&-E^1&-E^2&-E^3\\E^1&0&-B^3&B^2\\E^2&B^3&0&-B^1\\E^3&-B^2&B^1&0\end{pmatrix}. \tag{1.87}$$

Gauss 定律对应的方程

$$\nabla\cdot\mathbf{E}=\rho \tag{1.88}$$

等价于

$$J^0 = \rho = \partial_i E^i = -\partial_i F^{0i} = \partial_i F^{i0} = \partial_i F^{i0} + \partial_0 F^{00} = \partial_\mu F^{\mu 0}, \tag{1.89}$$

而 Ampère 定律对应的方程

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{1.90}$$

等价于

$$J^i = \varepsilon^{ijk}\partial_j B^k - \partial_0 E^i = -\partial_j F^{ij} + \partial_0 F^{0i} = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \tag{1.91}$$

归纳起来，有

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.92}$$

这个方程完全是用 Lorentz 张量写出来的，它在不同惯性系中具有相同的形式，即具有 **Lorentz 协变性**，因而满足狭义相对性原理。

现在，Maxwell 方程组中还有一个方程没有讨论，它是 Maxwell–Faraday 方程

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.93}$$

将它写成分量的形式，得

$$\varepsilon^{kmn}\partial_m E^n = -\varepsilon^{kmn}\partial_m F^{0n} = \varepsilon^{kmn}\partial_m F^{n0} = -\partial_0 B^k, \tag{1.94}$$

从而

$$\partial_0 F^{ij} = -\varepsilon^{ijk}\partial_0 B^k = \varepsilon^{ijk}\varepsilon^{kmn}\partial_m F^{n0} = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\partial_m F^{n0} = \partial_i F^{j0} - \partial_j F^{i0}, \tag{1.95}$$

即

$$\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = 0. \tag{1.96}$$

这个方程与 Maxwell–Faraday 方程等价，不过，它只是前面得到的方程 (1.82) 取特定分量的形式。

利用四维 Levi-Civita 符号，可以定义电磁场的对偶场强张量 (duel field strength tensor)

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{1.97}$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk}\varepsilon^{1jk} = \varepsilon^{123}\varepsilon^{123} + \varepsilon^{132}\varepsilon^{132} = 2$ 和 $\varepsilon^{1jk}\varepsilon^{2jk} = \varepsilon^{123}\varepsilon^{223} + \varepsilon^{132}\varepsilon^{232} = 0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk}\varepsilon^{ljk} = 2\delta^{il}, \tag{1.98}$$

利用这个关系，可得

$$\begin{aligned}\tilde{F}^{0i} &= \frac{1}{2}\varepsilon^{0i\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\varepsilon^{0ijk}g_{j\mu}g_{k\nu}F^{\mu\nu} = \frac{1}{2}\varepsilon^{0ijk}g_{jm}g_{kn}F^{mn} = -\frac{1}{2}\varepsilon^{ijk}\delta^{jm}\delta^{kn}\varepsilon^{mnl}B^l \\ &= -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{jkl}B^l = -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}B^l = -\frac{1}{2}2\delta^{il}B^l = -B^i,\end{aligned} \tag{1.99}$$

故 $\tilde{F}^{0i}$ 对应于磁感应强度。另一方面，

$$\begin{aligned}\tilde{F}^{ij} &= \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(\varepsilon^{ij0k}F_{0k} + \varepsilon^{ijk0}F_{k0}) = \varepsilon^{0ijk}F_{0k} = \varepsilon^{0ijk}g_{0\mu}g_{k\nu}F^{\mu\nu} \\ &= \varepsilon^{ijk}g_{00}g_{kl}F^{0l} = -\varepsilon^{ijk}\delta^{kl}F^{0l} = -\varepsilon^{ijk}F^{0k} = \varepsilon^{ijk}E^k,\end{aligned} \tag{1.100}$$

说明 $\tilde{F}^{ij}$ 对应于电场强度。$\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \tag{1.101}$$

由 $\tilde{F}^{\mu\nu}$ 的定义，有

$$\begin{aligned}\partial_\mu\tilde{F}^{\mu\nu} &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\sigma\mu\rho}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\rho\sigma\mu}\partial_\mu F_{\rho\sigma}) \\ &= -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\mu\rho\sigma}\partial_\rho F_{\sigma\mu} + \varepsilon^{\nu\mu\rho\sigma}\partial_\sigma F_{\mu\rho}) = -\frac{1}{6}\varepsilon^{\nu\mu\rho\sigma}(\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho}),\end{aligned} \tag{1.102}$$

因此，方程 (1.82) 等价于

$$\partial_\mu\tilde{F}^{\mu\nu} = 0. \tag{1.103}$$

从这些讨论可以看到，用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的，而且方程的 Lorentz 协变性非常明确。

1.6　作用量原理

1.6.1　经典力学中的作用量原理

在经典力学中，质点力学系统可以用**拉格朗日量**（Lagrangian）描述。对于具有 n 个自由度的系统，可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i，它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i = dq_i/dt$。拉格朗日量是广义坐标和广义速度的函数 $L(q_i, \dot{q}_i)$。拉格朗日量的时间积分

$$S = \int_{t_1}^{t_2} dt\, L[q_i(t), \dot{q}_i(t)] \tag{1.104}$$

称为**作用量**。

作用量原理指出，作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零，则有

$$\delta\dot{q}_i = \delta\frac{dq_i}{dt} = \frac{d}{dt}\delta q_i, \tag{1.105}$$

即时间导数的变分等于变分的时间导数。从而可得

$$\delta S = \int_{t_1}^{t_2} dt\, \delta L[q_i(t), \dot{q}_i(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right)$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \right]$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\Big|_{t_1}^{t_2}, \tag{1.106}$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零，即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$，则上式最后一行第二项为零，而 $\delta S = 0$ 等价于

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \cdots, n. \tag{1.107}$$

这是 **Euler-Lagrange 方程**，它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \cdots, n. \tag{1.108}$$

反解上式表示的 n 个方程，则可以用 q_i 和 p_i 将 $\dot{q}_i$ 表达出来，然后用 Legendre 变换定义**哈密顿量** (Hamiltonian)

$$H(q_i, p_i) \equiv p_i\dot{q}_i - L, \tag{1.109}$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量，变分为

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt\, \delta(p_i\dot{q}_i - H) = \int_{t_1}^{t_2} dt \left(\dot{q}_i\delta p_i + p_i\delta\dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)\\
&= \int_{t_1}^{t_2} dt \left(\dot{q}_i\delta p_i + p_i\frac{d}{dt}\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)\\
&= \int_{t_1}^{t_2} dt \left[\dot{q}_i\delta p_i + \frac{d}{dt}(p_i\delta q_i) - \dot{p}_i\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right]\\
&= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right] + p_i\delta q_i\Big|_{t_1}^{t_2}.
\end{aligned} \tag{1.110}$$

根据前面的假设，上式最后一行第二项为零，于是，$\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \cdots, n. \tag{1.111}$$

这是 **Hamilton 正则运动方程**，相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.107).

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中，场 $\phi(\mathbf{x}, t)$ 是系统的广义坐标，每一个空间点 $\mathbf{x}$ 都是一个自由度，因此场论相当于具有无穷多自由度的质点力学。在局域场论中，拉格朗日量 $L = \int d^3x\, \mathcal{L}(x)$，其中 $\mathcal{L}(x)$ 称为**拉格朗日量密度**（下文将它简称为**拉氏量**）。$\mathcal{L}$ 是系统中 n 个场 $\phi_a(\mathbf{x}, t)$ $(a = 1, \cdots, n)$ 及其时空导数 $\partial_\mu\phi_a$ 的函数。现在，作用量可以表达为

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu\phi_a). \tag{1.112}$$

(1.44) 式告诉我们，时空体积元 d^4x 是 Lorentz 不变的，如果拉氏量 $\mathcal{L}$ 也是 Lorentz 不变的，则作用量 S 就是 Lorentz 不变的，从而，由作用量原理得到的运动方程满足狭义相对性原理。因此，构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 $\mathcal{L}$，即要求 $\mathcal{L}$ 是一个 *Lorentz 标量*。

类似于前面质点力学的处理方式，假设时空坐标的变分为零，则对场的时空导数的变分等于场变分的时空导数，即

$$\delta(\partial_\mu\phi_a) = \partial_\mu(\delta\phi_a). \tag{1.113}$$

于是，利用分部积分可得

$$\begin{aligned}
\delta S &= \int d^4x\,\delta\mathcal{L} = \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\right] = \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)\right]\\
&= \int d^4x\left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right] - \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a\right\}\\
&= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right].
\end{aligned} \tag{1.114}$$

上式最后一行第二项的积分项是关于时空坐标的全散度，利用 Stokes 定理，可以将它转化为积分区域边界面 $\mathcal{S}$ 上的积分：

$$\int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right] = \int_{\mathcal{S}} d\mathcal{S}_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a, \tag{1.115}$$

其中 $d\mathcal{S}_\mu$ 是 $\mathcal{S}$ 上的面元。进一步假设在边界面 $\mathcal{S}$ 上 $\delta\phi_a = 0$，则上式为零。我们通常讨论整个时空区域上的场，从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零，是很合理的。这样一来，$\delta S = 0$ 给出

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{1.116}$$

这就是*场的 Euler-Lagrange 方程*，它给出场的经典运动方程。

引入场的**共轭动量密度** (conjugate momentum density)

$$\pi_a(\mathbf{x},t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}, \tag{1.117}$$

则可以用 Legendre 变换将哈密顿量定义为

$$H \equiv \int d^3x\,\pi_a\dot{\phi}_a - L \equiv \int d^3x\,\mathcal{H}, \tag{1.118}$$

其中，**哈密顿量密度**

$$\mathcal{H}(\phi_a,\pi_a,\nabla\phi_a) = \pi_a\dot{\phi}_a - \mathcal{L}. \tag{1.119}$$

作用量变分为

$$\delta S = \int d^4x\,\delta\mathcal{L} = \int d^4x\,\delta(\pi_a\dot{\phi}_a - \mathcal{H})$$

$$
\begin{aligned}
&= \int d^4x \left[\dot{\phi}_a \delta\pi_a + \pi_a \delta\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)} \cdot \delta(\nabla\phi_a) \right] \\
&= \int d^4x \left[\dot{\phi}_a \delta\pi_a + \pi_a \frac{d}{dt}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)} \cdot \nabla(\delta\phi_a) \right] \\
&= \int d^4x \left\{ \dot{\phi}_a \delta\pi_a + \frac{d}{dt}(\pi_a\delta\phi_a) - \dot{\pi}_a\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a \right. \\
&\qquad\qquad \left. -\nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right] + \left[\nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\delta\phi_a \right\} \\
&= \int d^4x \left\{ \left(\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\pi_a} \right)\delta\pi_a - \left[\dot{\pi}_a + \frac{\partial\mathcal{H}}{\partial\phi_a} - \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)} \right]\delta\phi_a \right\} \\
&\quad + \int d^4x \, \frac{d}{dt}(\,\pi_a\delta\phi_a) - \int d^4x \, \nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right].
\end{aligned}
\tag{1.120}
$$

与前面一样，假设在时空区域边界面上 $\delta\phi_a = 0$，则上式最后一行的两项均为零，于是，$\delta S = 0$ 给出场的*正则运动方程*

$$
\dot{\phi}_a = \frac{\partial\mathcal{H}}{\partial\pi_a}, \quad \dot{\pi}_a = -\frac{\partial\mathcal{H}}{\partial\phi_a} + \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}. \tag{1.121}
$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述，则它是一种*连续变换*，连续变换对应的对称性称为连续对称性。**Noether 定理**指出，如果一个系统具有某种不显含时间的*连续对称性*，就必然存在一种对应的*守恒定律*。Noether 定理首先是在经典物理中给出的，但实际上它对所有物理行为由作用量原理决定的系统都成立。因此，可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$
S = \int_R d^4x \, \mathcal{L}(\phi_a, \partial_\mu\phi_a). \tag{1.122}
$$

考虑一个连续变换，使得

$$
\phi_a(x) \to \phi'_a(x'), \tag{1.123}
$$

其中已包含了坐标的变换

$$
x^\mu \to x'^\mu, \tag{1.124}
$$

它引起的拉氏量变换为

$$
\mathcal{L}(x) \to \mathcal{L}'(x'). \tag{1.125}
$$

记这个变换的无穷小变换形式为

$$
\phi'_a(x') = \phi_a(x) + \delta\phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}, \tag{1.126}
$$

如果在此变换下

$$\delta S = \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) = 0, \tag{1.127}$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4x' = |\mathcal{J}|d^4x, \quad \mathcal{J} = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) \simeq \det\left[\delta^\mu{}_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right], \tag{1.128}$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 $\mathbf{A}$ 满足 $\det(\mathbf{A}) \ll 1$，则有如下表达式：

$$\det(\mathbf{1}+\mathbf{A}) \simeq 1 + \mathrm{tr}(\mathbf{A}). \tag{1.129}$$

利用上式可以将 Jacobi 行列式 $\mathcal{J}$ 化为

$$\mathcal{J} \simeq 1 + \mathrm{tr}\left[\frac{\partial(\delta x^\mu)}{\partial x^\nu}\right] = 1 + \partial_\mu(\delta x^\mu), \tag{1.130}$$

从而，体积元的无穷小变换形式为

$$d^4x' \simeq [1 + \partial_\mu(\delta x^\mu)]d^4x. \tag{1.131}$$

作用量在此无穷小变换下的变分为

$$\begin{aligned}
\delta S &= \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) \\
&= \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}'(x') + \int_R d^4x\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) \\
&\simeq \int_R d^4x[1+\partial_\mu(\delta x^\mu)]\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}'(x') + \int_R d^4x\,\delta\mathcal{L} \\
&\simeq \int_R d^4x\,\mathcal{L}'(x')\partial_\mu(\delta x^\mu) + \int_R d^4x\,\delta\mathcal{L} \ \simeq \int_R d^4x\,[\delta\mathcal{L} + \mathcal{L}(x)\partial_\mu(\delta x^\mu)] \\
&= \int_R d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) + \ \mathcal{L}\partial_\mu(\delta x^\mu)\right].
\end{aligned} \tag{1.132}$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$，使得

$$\bar{\delta}\phi_a(x) = \phi'_a(x) - \phi_a(x). \tag{1.133}$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu\phi_a) = \partial_\mu(\bar{\delta}\phi_a), \tag{1.134}$$

δ 算符则不能。$\delta\phi_a$ 与 $\bar{\delta}\phi_a$ 的关系为

$$\begin{aligned}
\delta\phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta}\phi_a \\
&\simeq \bar{\delta}\phi_a + (\partial_\mu\phi'_a)\delta x^\mu \simeq \bar{\delta}\phi + (\partial_\mu\phi_a)\delta x^\mu,
\end{aligned} \tag{1.135}$$

即

$$\bar{\delta}\phi = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu. \tag{1.136}$$

同理,

$$\delta(\partial_\mu\phi_a) = \bar{\delta}(\partial_\mu\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu = \partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu. \tag{1.137}$$

将 (1.135) 和 (1.137) 式代入 (1.132) 式，得到

$$\begin{aligned}
\delta S =& \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}[\bar{\delta}\phi_a + (\partial_\mu\phi_a)\delta x^\mu] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}[\partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu] + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} \\
=& \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\bar{\delta}\phi_a + \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\bar{\delta}\phi_a \right. \\
& \left. + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu) \right\} \\
=& \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right] \right. \\
& \left. + \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu)\right] \right\} \\
=& \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] \right\}.
\end{aligned} \tag{1.138}$$

第二步用到分部积分，最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu) = \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu). \tag{1.139}$$

根据 Euler-Lagrange 方程 (1.116)，(1.138) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的，$\delta S = 0$ 等价于第二项为零，即

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] = 0. \tag{1.140}$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu, \tag{1.141}$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \tag{1.142}$$

方程 (1.142) 左边对整个三维空间积分，运用 Stokes 定理，得

$$\int d^3x\, \partial_\mu j^\mu = \int d^3x\, \partial_0 j^0 + \int d^3x\, \partial_i j^i = \frac{d}{dt}\int d^3x\, j^0 + \int_{\mathcal{S}} d\mathcal{S}_i\, j^i, \tag{1.143}$$

其中 $i = 1, 2, 3$。对于整个三维空间而言，边界面 $\mathcal{S}$ 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失，从而，在无穷远处 $j^i \to 0$，所以上式最后一项为零。定义**守恒荷** (conserved charge)

$$Q \equiv \int d^3x\, j^0, \tag{1.144}$$

则由 (1.143) 和 (1.142) 式可得

$$\frac{dQ}{dt}=\frac{d}{dt}\int d^3x\,j^0=\int d^3x\,\partial_\mu j^\mu=0. \tag{1.145}$$

可见，Q 不随时间变化，是守恒的。

综上，在场论中，如果一个系统具有某种连续对称性，则存在相应的守恒流 (1.141)，它满足守恒流方程 (1.142)，而守恒荷 (1.144) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2 时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^\mu=x^\mu-\varepsilon^\mu, \tag{1.146}$$

其中 ε^μ 是常数。要求场 ϕ_a 具有时空平移对称性，则

$$\phi'_a(x')=\phi'_a(x-\varepsilon)=\phi_a(x). \tag{1.147}$$

现在，$\delta x^\mu=-\varepsilon^\mu$，由 (1.136) 式可得

$$\bar{\delta}\phi_a=\delta\phi_a-(\partial_\mu\phi_a)\delta x^\mu=\phi'_a(x')-\phi_a(x)+\varepsilon^\mu\partial_\mu\phi_a=0+\varepsilon^\mu\partial_\mu\phi_a=\varepsilon^\rho\partial_\rho\phi_a, \tag{1.148}$$

代入到 Noether 守恒流表达式 (1.141)，得

$$j^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^\rho\partial_\rho\phi_a-\mathcal{L}\varepsilon^\mu=\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a-\delta^\mu{}_\rho\mathcal{L}\right]\varepsilon^\rho. \tag{1.149}$$

从而，$\partial_\mu j^\mu=0$ 给出

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a-\delta^\mu{}_\rho\mathcal{L}\right]=0, \tag{1.150}$$

各项乘以 $g^{\rho\nu}$，缩并，得

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a-g^{\mu\nu}\mathcal{L}\right]=0. \tag{1.151}$$

上式方括号部分是场的**能动张量** (energy-momentum tensor)

$$T^{\mu\nu}\equiv\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a-g^{\mu\nu}\mathcal{L}, \tag{1.152}$$

它满足

$$\partial_\mu T^{\mu\nu}=0. \tag{1.153}$$

因此，对 $T^{0\nu}$ $(\nu=0,1,2,3)$ 作全空间积分，就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00}=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^0\phi_a-\mathcal{L}, \tag{1.154}$$

与 (1.119) 和 (1.117) 式比较，可以看出 T^{00} 就是哈密顿量密度 $\mathcal{H}$。T^{00} 的全空间积分

$$H=\int d^3x\,T^{00}=\int d^3x\,\mathcal{H} \tag{1.155}$$

是场的哈密顿量，或者说**总能量**。$T^{\mu\nu}$ 的 $0i$ 分量

$$T^{0i}=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^i\phi_a=\pi_a\partial^i\phi_a \tag{1.156}$$

是场的动量密度，它的全空间积分

$$P^i=\int d^3x\,T^{0i}=\int d^3x\,\pi_a\partial^i\phi_a \tag{1.157}$$

是场的**总动量**。根据 (1.55) 式，上式也可以写成

$$\mathbf{P}=-\int d^3x\,\pi_a\nabla\phi_a. \tag{1.158}$$

H 和 P^i 都是守恒荷，可见，时间平移对称性对应于**能量守恒定律**，空间平移对称性对应于**动量守恒定律**。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu, \tag{1.159}$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30)，有

$$\begin{aligned}g_{\alpha\beta}&=g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta=g_{\mu\nu}(\delta^\mu{}_\alpha+\omega^\mu{}_\alpha)(\delta^\nu{}_\beta+\omega^\nu{}_\beta)\simeq g_{\mu\nu}\delta^\mu{}_\alpha\delta^\nu{}_\beta+g_{\mu\nu}\delta^\mu{}_\alpha\omega^\nu{}_\beta+g_{\mu\nu}\omega^\mu{}_\alpha\delta^\nu{}_\beta\\&=g_{\alpha\beta}+\omega_{\alpha\beta}+\omega_{\beta\alpha},\end{aligned} \tag{1.160}$$

可见，

$$\omega_{\mu\nu}\equiv g_{\mu\rho}\omega^\rho{}_\nu \tag{1.161}$$

关于两个指标反对称：

$$\omega_{\mu\nu}=-\omega_{\nu\mu}. \tag{1.162}$$

因此，$\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31)，利用三角函数展开式 $\cos\theta=1+\mathcal{O}(\theta^2)$ 和 $\sin\theta=\theta+\mathcal{O}(\theta^3)$，可得

$$\omega^\mu{}_\nu=\begin{pmatrix}0&&&\\&0&\theta&\\&-\theta&0&\\&&&0\end{pmatrix},\quad \omega_{\mu\nu}=g_{\mu\rho}\omega^\rho{}_\nu=\begin{pmatrix}0&&&\\&0&-\theta&\\&\theta&0&\\&&&0\end{pmatrix}. \tag{1.163}$$

对于沿 x 的增速变换 (1.28)，可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1}\beta, \tag{1.164}$$

再利用双曲函数公式 $\tanh\xi = \sinh\xi/\cosh\xi$ 和 $\cosh^2\xi - \sinh^2\xi = 1$ 得

$$\begin{aligned}\gamma &= (1-\beta^2)^{-1/2} = (1-\tanh^2\xi)^{-1/2} = \left(\frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi}\right)^{-1/2} = \cosh\xi,\\ \beta\gamma &= \tanh\xi\cosh\xi = \sinh\xi,\end{aligned} \tag{1.165}$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix}\cosh\xi & -\sinh\xi & & \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ & & & 1\end{pmatrix}. \tag{1.166}$$

根据双曲函数展开式 $\cosh\xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh\xi = \xi + \mathcal{O}(\xi^3)$，有

$$\omega^\mu{}_\nu = \begin{pmatrix}0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix}0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}. \tag{1.167}$$

在无穷小 Lorentz 变换 (1.159) 的作用下，一般地，场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right]\phi_b(x) = \phi_a(x) - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x), \tag{1.168}$$

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的**生成元** (generator)。由于 $\omega_{\mu\nu}$ 是反对称的，有

$$\omega_{\mu\nu}(I^{\mu\nu})_{ab} = \omega_{\nu\mu}(I^{\nu\mu})_{ab} = -\omega_{\mu\nu}(I^{\nu\mu})_{ab}, \tag{1.169}$$

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称：

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}. \tag{1.170}$$

现在，$\delta x^\mu = \omega^\mu{}_\nu x^\nu$，而

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu\phi_a)\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}(I^{\nu\rho})_{ab}\phi_b - (\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho, \tag{1.171}$$

故 Noether 流为

$$\begin{aligned}j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(I^{\nu\rho})_{ab}\phi_b - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho + \mathcal{L}\omega^\mu{}_\rho x^\rho\\ &= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a) - \delta^\mu{}_\nu\mathcal{L}\right]\omega^\nu{}_\rho x^\rho\end{aligned}$$

$$= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - T^\mu{}_\nu\omega^\nu{}_\rho x^\rho, \tag{1.172}$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho}g_{\rho\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L} \tag{1.173}$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作:

$$T^\mu{}_\nu\omega^\nu{}_\rho = T^\mu{}_\nu\delta^\nu{}_\sigma\omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha}g_{\alpha\sigma}\omega^\sigma{}_\rho = T^{\mu\alpha}\omega_{\alpha\rho} = T^{\mu\nu}\omega_{\nu\rho}, \tag{1.174}$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned} T^\mu{}_\nu\omega^\nu{}_\rho x^\rho &= T^{\mu\nu}\omega_{\nu\rho}x^\rho = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\nu}\omega_{\rho\nu}x^\rho) = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\rho}\omega_{\nu\rho}x^\nu) \\ &= \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu). \end{aligned} \tag{1.175}$$

于是，Noether 流 (1.172) 可化为

$$j^\mu = \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu) = \frac{1}{2}\mathbb{J}^{\mu\nu\rho}\omega_{\nu\rho} \tag{1.176}$$

其中

$$\mathbb{J}^{\mu\nu\rho} \equiv T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b. \tag{1.177}$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu\mathbb{J}^{\mu\nu\rho} = 0, \tag{1.178}$$

守恒荷为

$$\mathbb{J}^{\nu\rho} \equiv \int d^3x\, J^{0\nu\rho} = \int d^3x \left[T^{0\rho}x^\nu - T^{0\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b\right]. \tag{1.179}$$

易见 $\mathbb{J}^{\nu\rho} = -\mathbb{J}^{\rho\nu}$，因而一共有 6 个独立的守恒荷，满足 $d\mathbb{J}^{\nu\rho}/dt = 0$。

为明确物理含义，可将 $\mathbb{J}^{\nu\rho}$ 分解成两项:

$$\mathbb{J}^{\nu\rho} = \mathbb{L}^{\nu\rho} + \mathbb{S}^{\nu\rho}. \tag{1.180}$$

第一项为

$$\begin{aligned} \mathbb{L}^{\nu\rho} &\equiv \int d^3x\,(T^{0\rho}x^\nu - T^{0\nu}x^\rho) \\ &= \int d^3x \left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\rho\phi_a - g^{0\rho}\mathcal{L}\right)x^\nu - \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\nu\phi_a - g^{0\nu}\mathcal{L}\right)x^\rho\right] \\ &= \int d^3x\,[(\pi_a\partial^\rho\phi_a - g^{0\rho}\mathcal{L})x^\nu - (\pi_a\partial^\nu\phi_a - g^{0\nu}\mathcal{L})x^\rho] \\ &= \int d^3x\,[\pi_a(x^\nu\partial^\rho - x^\rho\partial^\nu)\phi_a + (g^{0\nu}x^\rho - g^{0\rho}x^\nu)\mathcal{L}]. \end{aligned} \tag{1.181}$$

它的纯空间分量 $\mathbb{L}^{jk}$ 中只有 3 个是独立的，可以等价地定义成

$$\mathbb{L}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathbb{L}^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\,\pi_a(x^j\partial^k - x^k\partial^j)\phi_a, \tag{1.182}$$

这是场的**轨道角动量**。第二项为

$$\mathbb{S}^{\nu\rho} \equiv \int d^3x\, \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b = \int d^3x\, \pi_a(-iI^{\nu\rho})_{ab}\phi_b, \tag{1.183}$$

同样，3 个独立的等价纯空间分量是

$$\mathbb{S}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathbb{S}^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\, \pi_a(-iI^{jk})_{ab}\phi_b, \tag{1.184}$$

这是场的**自旋角动量**。因此，$\mathbb{J}^{\nu\rho}$ 的纯空间分量等价于

$$\mathbb{J}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathbb{J}^{jk} = \mathbb{L}^i + \mathbb{S}^i, \tag{1.185}$$

这是场的**总角动量**。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 SO(3)，而空间旋转对称性对应于**角动量守恒定律**。

另一方面，$\mathbb{L}^{\nu\rho}$ 的 $i0$ 分量为

$$\mathbb{L}^{i0} = \int d^3x\,(T^{00}x^i - T^{0i}x^0) = \int d^3x\,(x^i\mathcal{H} - x^0\pi_a\partial^i\phi_a) = \int d^3x\, x^i\mathcal{H} - tP^i. \tag{1.186}$$

若 $d\mathbb{S}^{i0}/dt = 0$，则有 $d\mathbb{L}^{i0}/dt = 0$，从而

$$\mathbb{L}^{i0}(t) = \mathbb{L}^{i0}\big|_{t=0} = \int d^3x\, x^i\mathcal{H}(t=0), \tag{1.187}$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下，能量密度相当于质量密度，则 $\mathbb{L}^{i0}$ 是 $t=0$ 时刻的质心 (即质量中心，center of mass)。$\mathbb{L}^{i0}$ 的守恒在经典力学中对应于质心运动守恒定律：当没有外力存在时，质心的加速度为零，质心保持静止或作匀速直线运动。

1.7.4　U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu\phi^*)\partial_\mu\phi - m^2\phi^*\phi. \tag{1.188}$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta}\phi(x), \tag{1.189}$$

其中 θ 是不依赖于 x^μ 的连续变换实参数，q 是一个常数。这里不包含坐标的变换。$e^{iq\theta}$ 是个纯相位因子，可以看成是一个 1 维幺正 (unitary) 矩阵，形式为 $e^{iq\theta}$ 的所有变换组成的群称为 **U(1) 群**。**整体** (global) 指的是变换参数不依赖于时空坐标。相应地，ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta}\phi^*(x). \tag{1.190}$$

容易看出，由 (1.188) 式定义的 $\mathcal{L}$ 在这种变换下不变，即具有 U(1) 整体对称性。与前面叙述的两种对称性不同，这里的对称性出现在由场组成的抽象空间中，与时间和空间相对独立 ($\delta x^\mu = 0$)，因而是一种**内部对称性**。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \tag{1.191}$$

结合 $\delta x^\mu = 0$，有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \tag{1.192}$$

于是，Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\bar{\delta}\phi^* = \partial^\mu\phi^*(iq\theta\phi) + \partial^\mu\phi(-iq\theta\phi^*) \\ &= iq\theta[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*] = -q\theta\phi^* i\overleftrightarrow{\partial^\mu}\phi, \end{aligned} \tag{1.193}$$

其中，$\overleftrightarrow{\partial^\mu}$ 符号通过下式定义：

$$\phi^*\overleftrightarrow{\partial^\mu}\phi \equiv \phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi. \tag{1.194}$$

扔掉无穷小参数 $-\theta$，定义

$$J^\mu \equiv q\phi^* i\overleftrightarrow{\partial^\mu}\phi, \tag{1.195}$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$，相应的守恒荷为

$$Q = \int d^3x\, J^0 = q\int d^3x\, \phi^* i\overleftrightarrow{\partial^0}\phi. \tag{1.196}$$

在实际情况中，q 是由 ϕ 场描述的粒子所携带的某种荷，如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此，一种 U(1) 整体对称性对应于一条荷数守恒定律，比如，电磁 U(1) 整体对称性就对应于电荷守恒定律。

第 2 章　标量场

本章讲述标量场的**正则量子化** (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1　简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{2.1}$$

其中 m 是质量，ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 看作厄米算符，满足**正则对易关系**

$$[x, p] = xp - px = i. \tag{2.2}$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \tag{2.3}$$

a 称为**湮灭算符** (annihilation operator)，$a^\dagger$ 称为**产生算符** (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\ &= \frac{1}{2}(-i[x,p] + i[p,x]) = -i[x,p] = 1. \end{aligned} \tag{2.4}$$

根据 (2.3) 式，可以反过来用 a 和 $a^\dagger$ 表示 x 和 p：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \tag{2.5}$$

从而，哈密顿量表示成

$$\begin{aligned} H &= -\frac{1}{2m}\frac{m\omega}{2}(a - a^\dagger)^2 + \frac{1}{2}m\omega^2\frac{1}{2m\omega}(a + a^\dagger)^2 \\ &= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a). \end{aligned} \tag{2.6}$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$，于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right), \tag{2.7}$$

其中，$N \equiv a^\dagger a$ 是个厄米算符，称为**粒子数算符**。N 还是个**正定**算符，对于任意量子态 $|\psi\rangle$，N 的期待值 (expectation value) 非负：

$$\langle\psi| N |\psi\rangle = \langle\psi| a^\dagger a |\psi\rangle = \langle a\psi|a\psi\rangle \geq 0. \tag{2.8}$$

设 $|n\rangle$ 是 N 的本征态，归一化为 $\langle n|n\rangle = 1$。它满足本征方程

$$N|n\rangle = n|n\rangle. \tag{2.9}$$

由 $n = \langle n| n |n\rangle = \langle n| N |n\rangle \geq 0$ 可知，本征值 n 是个非负实数。利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B, \tag{2.10}$$

$$[A, BC] = ABC - BAC + BAC - BCA = [A,B]C + B[A,C], \tag{2.11}$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \tag{2.12}$$

从而，有

$$Na^\dagger|n\rangle = ([N, a^\dagger] + a^\dagger N)|n\rangle = (a^\dagger + a^\dagger n)|n\rangle = (n+1)a^\dagger|n\rangle, \tag{2.13}$$

$$Na|n\rangle = ([N, a] + aN)|n\rangle = (-a + an)|n\rangle = (n-1)a|n\rangle. \tag{2.14}$$

可见，$a^\dagger|n\rangle$ 和 $a|n\rangle$ 都是 N 的本征态，本征值分别为 $n+1$ 和 $n-1$，也就是说，

$$a^\dagger|n\rangle = c_1|n+1\rangle, \quad a|n\rangle = c_2|n-1\rangle, \tag{2.15}$$

其中 c_1 和 c_2 是两个归一化常数。$a^\dagger$ 将本征值为 n 的态变成本征值为 $n+1$ 的态，因而也称为升算符 (raising operator)；a 将本征值为 n 的态变成本征值为 $n-1$ 的态，因而也称为降算符 (lowering operator)。为确定归一化常数的值，可作如下计算：

$$n+1 = \langle n|(N+1)|n\rangle = \langle n|(a^\dagger a+1)|n\rangle = \langle n| aa^\dagger |n\rangle = |c_1|^2\langle n+1|n+1\rangle = |c_1|^2, \tag{2.16}$$

$$n = \langle n| N |n\rangle = \langle n| a^\dagger a |n\rangle = |c_2|^2\langle n-1|n-1\rangle = |c_2|^2. \tag{2.17}$$

将 c_1 和 c_2 都取为实数，则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$，故

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{2.18}$$

从 N 的某个本征态 $|n\rangle$ 出发，用降算符 a 逐步操作，可得本征值逐次减小的一系列本征态

$$a|n\rangle, \ a^2|n\rangle, \ a^3|n\rangle, \ \cdots, \tag{2.19}$$

本征值分别为

$$n-1,\ n-2,\ n-3,\ \cdots \tag{2.20}$$

由于 $n\geq 0$，必定存在一个最小本征值 n_0，它的本征态 $|n_0\rangle$ 满足

$$a\,|n_0\rangle = 0. \tag{2.21}$$

于是，有

$$N\,|n_0\rangle = a^\dagger a\,|n_0\rangle = 0 = 0\,|n_0\rangle\,, \tag{2.22}$$

可见，$n_0=0$，即

$$|n_0\rangle = |0\rangle\,. \tag{2.23}$$

反过来，从 $|0\rangle$ 出发，用升算符 $a^\dagger$ 逐步操作，可得本征值逐次增加的一系列本征态

$$a^\dagger\,|0\rangle\,,\ (a^\dagger)^2\,|0\rangle\,,\ (a^\dagger)^3\,|0\rangle\,,\ \cdots, \tag{2.24}$$

本征值分别为

$$1,\ 2,\ 3,\ \cdots \tag{2.25}$$

综上，本征值 n 的取值是非负整数，是量子化的；本征态 $|n\rangle$ 可以用 $a^\dagger$ 和 $|0\rangle$ 表示为

$$|n\rangle = c_3(a^\dagger)^n\,|0\rangle\,. \tag{2.26}$$

为确定归一化常数 c_3，可作如下运算：

$$\begin{aligned}\langle n|n\rangle &= |c_3|^2\,\langle 0|\,a^n(a^\dagger)^n\,|0\rangle = |c_3|^2\,\langle 1|\,a^{n-1}(a^\dagger)^{n-1}\,|1\rangle = 1\cdot 2\,|c_3|^2\,\langle 2|\,a^{n-2}(a^\dagger)^{n-2}\,|2\rangle = \cdots\\ &= (n-1)!\,|c_3|^2\,\langle n-1|\,aa^\dagger\,|n-1\rangle = n!\,|c_3|^2\,\langle n|n\rangle\,,\end{aligned} \tag{2.27}$$

故 $|c_3|^2=1/n!$。取 c_3 为实数，可得 $c_3=1/\sqrt{n!}$，于是

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle\,. \tag{2.28}$$

从 (2.7) 式容易看出，$|n\rangle$ 也是 H 的本征态：

$$H\,|n\rangle = \omega\left(N+\frac{1}{2}\right)|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle = E_n\,|n\rangle\,, \tag{2.29}$$

相应的能量本征值为

$$E_n = \omega\left(n+\frac{1}{2}\right). \tag{2.30}$$

基态 $|0\rangle$ 的能量本征值不是零，而是 $E_0=\omega/2$，称为零点能 (zero-point energy)，这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态，将 $n>0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态，每个声子具有一份能量 ω。这样一来，n 表示声子的数目，故粒子数算符 N 描述的是声子数。$a^\dagger$ 的作用是产生一个声子，从而增加一份能量；a 的作用是湮灭一个声子，从而减少一份能量。这是将 $a^\dagger$ 和 a 称为产生算符和湮灭算符的原因。

2.2　量子场论中的正则对易关系

在量子力学中，当系统的哈密顿量 H 不含时间时，Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法，它们之间可以通过含时的幺正变换联系起来。

在 **Schrödinger** 绘景中，态矢 $|\Psi(t)\rangle^{\mathrm{S}}$ 代表随时间演化的物理态，而算符 O^{S} 不依赖于时间。$|\Psi(t)\rangle^{\mathrm{S}}$ 与 $t=0$ 时刻态矢 $|\Psi(0)\rangle^{\mathrm{S}}$ 通过幺正变换 e^{-iHt} 联系起来：

$$|\Psi(t)\rangle^{\mathrm{S}} = e^{-iHt}|\Psi(0)\rangle^{\mathrm{S}}. \tag{2.31}$$

由于 H 不含时，有

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle^{\mathrm{S}} = i\frac{\partial e^{-iHt}}{\partial t}|\Psi(0)\rangle^{\mathrm{S}} = He^{-iHt}|\Psi(0)\rangle^{\mathrm{S}} = H|\Psi(t)\rangle^{\mathrm{S}}, \tag{2.32}$$

这就是 Schrödinger 方程。可见，(2.31) 式确实是 Schrödinger 方程的解。

在 **Heisenberg** 绘景中，态矢 $|\Psi\rangle^{\mathrm{H}}$ 定义为

$$|\Psi\rangle^{\mathrm{H}} = e^{iHt}|\Psi(t)\rangle^{\mathrm{S}} = |\Psi(0)\rangle^{\mathrm{S}}, \tag{2.33}$$

它不随时间演化：

$$i\frac{\partial}{\partial t}|\Psi\rangle^{\mathrm{H}} = 0. \tag{2.34}$$

而算符 $O^{\mathrm{H}}(t)$ 依赖于时间，通过一个含时的相似变换与 O^{S} 联系起来，

$$O^{\mathrm{H}}(t) = e^{iHt}O^{\mathrm{S}}e^{-iHt}. \tag{2.35}$$

从而，有

$${}^{\mathrm{H}}\langle\Psi|\,O^{\mathrm{H}}(t)|\Psi\rangle^{\mathrm{H}} = {}^{\mathrm{H}}\langle\Psi|\,e^{iHt}O^{\mathrm{S}}e^{-iHt}|\Psi\rangle^{\mathrm{H}} = {}^{\mathrm{S}}\langle\Psi(t)|\,O^{\mathrm{S}}|\Psi(t)\rangle^{\mathrm{S}}, \tag{2.36}$$

可见，在两种绘景中，力学量在态上的平均值相同。因此，两种绘景描述相同的物理。

上一节的量子化可以认为是在 Schrödinger 绘景中实现的，因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x^{\mathrm{S}},p^{\mathrm{S}}]=i$，它在 Heisenberg 绘景中的形式为

$$\begin{aligned}[x^{\mathrm{H}}(t),p^{\mathrm{H}}(t)] &= [e^{iHt}x^{\mathrm{S}}e^{-iHt},e^{iHt}p^{\mathrm{S}}e^{-iHt}] = e^{iHt}x^{\mathrm{S}}e^{-iHt}e^{iHt}p^{\mathrm{S}}e^{-iHt} - e^{iHt}p^{\mathrm{S}}e^{-iHt}e^{iHt}x^{\mathrm{S}}e^{-iHt}\\ &= e^{iHt}x^{\mathrm{S}}p^{\mathrm{S}}e^{-iHt} - e^{iHt}p^{\mathrm{S}}x^{\mathrm{S}}e^{-iHt} = e^{iHt}[x^{\mathrm{S}},p^{\mathrm{S}}]e^{-iHt} = e^{iHt}ie^{-iHt} = i.\end{aligned} \tag{2.37}$$

可见，正则对易关系的形式不依赖于绘景。(2.37) 式是在同一时刻 t 成立的，称为**等时** (equal time) 对易关系。

将讨论推广到自由度为 n 的系统，记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符，$p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响，这些算符需要满足如下等时对易关系：

$$[q_i(t),p_j(t)] = i\delta_{ij}, \quad [q_i(t),q_j(t)] = 0, \quad [p_i(t),p_j(t)] = 0. \tag{2.38}$$

1.1 节提到，在量子场论中，为了平等地处理时间和空间，空间坐标 $\mathbf{x}$ 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x},t)$ 的参数。由于这里量子场作为算符是依赖于时间的，使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行，省略绘景的标志性上标 H。

场论讨论的是无穷多自由度的系统，每一个空间点 $\mathbf{x}$ 上的 $\phi(\mathbf{x},t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度，我们可以先将空间离散化，划分成 n 个小体积元 V_i，然后再取 $V_i \to 0$ 的极限来得到 $n \to \infty$ 的结果。在体积元 V_i 中，定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i}\int_{V_i} d^3x\,\phi(\mathbf{x},t), \tag{2.39}$$

它是场 $\phi(\mathbf{x},t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi,\partial_\mu\phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i}\int_{V_i} d^3x\,\mathcal{L}(\phi,\partial_\mu\phi), \tag{2.40}$$

当体积元取得足够小时，它就成为 ϕ_i 和 $\partial_0\phi_i$ 的函数 $\mathcal{L}_i(\phi_i,\partial_0\phi_i)$。拉格朗日量可表达为

$$L = \int d^3x\,\mathcal{L} = \sum_i \int_{V_i} d^3x\,\mathcal{L} = \sum_i V_i \frac{1}{V_i}\int_{V_i} d^3x\,\mathcal{L} = \sum_i V_i\,\mathcal{L}_i(\phi_i,\partial_0\phi_i). \tag{2.41}$$

于是，由 (1.108) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j\delta_{ji}\frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} = V_i\pi_i(t), \tag{2.42}$$

其中，

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]}. \tag{2.43}$$

现在，等时对易关系变成

$$[\phi_i(t),\Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t),\phi_j(t)] = 0, \quad [\Pi_i(t),\Pi_j(t)] = 0. \tag{2.44}$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t),\pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t),\pi_j(t)] = 0. \tag{2.45}$$

对于任意连续函数 $f(x)$，**Dirac δ 函数** $\delta(x)$ 使下式成立：

$$f(x) = \int dy\,f(y)\delta(x-y). \tag{2.46}$$

函数 $\delta(x)$ 只在 $x=0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即

$$\delta(x) = \delta(-x), \tag{2.47}$$

而且满足

$$\int dx\,\delta(x) \,=\, 1, \tag{2.48}$$

$$f(x)\delta(x-y) = f(y)\delta(x-y), \tag{2.49}$$
$$x\delta(x) = 0. \tag{2.50}$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \tag{2.51}$$

则对于任意连续函数 $f(\mathbf{x})$，下式成立：

$$f(\mathbf{x}) = \int d^3y\, f(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{2.52}$$

类似地，函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x}=0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$，而且满足 $\int d^3x\, \delta^{(3)}(\mathbf{x}) = 1$。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值，则它会满足

$$f_i = \sum_j f_j\, \delta_{ij} = \sum_j V_j\, f_j\, \frac{\delta_{ij}}{V_j}. \tag{2.53}$$

(2.52) 式是 (2.53) 式在 $V_i \to 0$ 时的极限。可见，在 $V_i \to 0$ 极限下，

$$\frac{\delta_{ij}}{V_j} \to \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{2.54}$$

另一方面，在此极限下，$\phi_i(t) \to \phi(\mathbf{x},t)$，而 $\pi_i(t)$ 变成由 (1.117) 式定义的共轭动量密度：

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} \to \frac{\partial \mathcal{L}}{\partial[\partial_0\phi(\mathbf{x},t)]} = \pi(\mathbf{x},t). \tag{2.55}$$

因此，等时对易关系化为

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = 0, \quad [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0. \tag{2.56}$$

推广到包含若干个场 ϕ_a 的系统，假设不同的场不会相互影响，则有

$$[\phi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi_a(\mathbf{x},t),\phi_b(\mathbf{y},t)] = 0, \quad [\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)] = 0. \tag{2.57}$$

这就是**量子场论中的正则对易关系**。此时，$\phi_a(\mathbf{x},t)$ 和 $\pi_a(\mathbf{x},t)$ 都是算符。

2.3 实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量，就称它为**标量场**。在固有保时向 Lorentz 变换下，若时空坐标的变换为 $x' = \Lambda x$，则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \tag{2.58}$$

在本节中，我们讨论实标量场 $\phi(x)$，它满足自共轭 (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \tag{2.59}$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的自由实标量场，相应的 **Lorentz** 不变拉氏量可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{2.60}$$

注意到

$$\frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)\partial_\nu\phi = \frac{1}{2}[(\partial_0\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2], \tag{2.61}$$

可得

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi = \partial^0\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} = -\partial_i\phi = \partial^i\phi, \tag{2.62}$$

归纳起来，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi, \quad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi. \tag{2.63}$$

因此，Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\partial^\mu\phi + m^2\phi, \tag{2.64}$$

也就是说，$\phi(x)$ 满足 *Klein-Gordon 方程*

$$(\partial^2 + m^2)\phi(x) = 0. \tag{2.65}$$

2.3.1　平面波展开

设 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik\cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \tag{2.66}$$

则有

$$\partial^2\varphi = \partial^\mu\partial_\mu\varphi = \partial^\mu(-ik_\mu\varphi) = -ik_\mu\partial^\mu\varphi = (-i)^2k_\mu k^\mu\varphi = -k^2\varphi, \tag{2.67}$$

从而，

$$0 = (\partial^2 + m^2)\varphi = -(k^2 - m^2)\varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2]\varphi. \tag{2.68}$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$，即 $k^0 = \pm E_\mathbf{k}$，其中 $E_\mathbf{k} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$。因此，有两种平面波解。

(1) $k^0 = E_\mathbf{k}$ 对应于正能解

$$\varphi^{(+)}_\mathbf{k}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})]. \tag{2.69}$$

(2) $k^0 = -E_\mathbf{k}$ 对应于负能解

$$\varphi^{(-)}_\mathbf{k}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[i(E_\mathbf{k}t + \mathbf{k}\cdot\mathbf{x})]. \tag{2.70}$$

从而，场算符 $\phi(\mathbf{x},t)$ 的一般解可以写成如下形式：

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}\varphi_{\mathbf{k}}^{(+)}(x)+\tilde{a}_{\mathbf{k}}\varphi_{\mathbf{k}}^{(-)}(x)\right]\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_{\mathbf{k}}e^{i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}\right],\end{aligned} \tag{2.71}$$

其中 $a_{\mathbf{k}}$ 和 $\tilde{a}_{\mathbf{k}}$ 是两个只依赖于 $\mathbf{k}$ 的算符。这是一种 Fourier 变换，把 $\phi(\mathbf{x},t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭，得

$$\begin{aligned}\phi^\dagger(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a^\dagger_{\mathbf{k}}e^{i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}^\dagger_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}\right]\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a^\dagger_{-\mathbf{k}}e^{i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}+\tilde{a}^\dagger_{-\mathbf{k}}e^{-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.72}$$

第二步利用了如下性质：对整个三维动量空间进行积分时，将积分项中的 $\mathbf{k}$ 换成 $-\mathbf{k}$ 不会改变积分的结果。于是，由自共轭条件 $\phi^\dagger(\mathbf{x},t)=\phi(\mathbf{x},t)$ 可得

$$\tilde{a}_{\mathbf{k}} = a^\dagger_{-\mathbf{k}}. \tag{2.73}$$

（注意：由上式可以推出 $\tilde{a}^\dagger_{\mathbf{k}}=a_{-\mathbf{k}}$ 和 $\tilde{a}^\dagger_{-\mathbf{k}}=a_{\mathbf{k}}$。）因而，有

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+a^\dagger_{-\mathbf{k}}e^{i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}\right]\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+a^\dagger_{\mathbf{k}}e^{i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.74}$$

替换一下动量记号，可以把 $\phi(\mathbf{x},t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right), \tag{2.75}$$

其中，p^0 是正的，满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2+m^2}, \tag{2.76}$$

而 $a_{\mathbf{p}}$ 是湮灭算符，$a^\dagger_{\mathbf{p}}$ 是产生算符。$\phi(\mathbf{x},t)$ 对应的共轭动量密度算符为

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi = \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right). \tag{2.77}$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)]=0,\quad [\pi(\mathbf{x},t),\pi(\mathbf{y},t)]=0. \tag{2.78}$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} = \int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{p}), \tag{2.79}$$

可得

$$\begin{aligned}\int d^3x\, e^{iq\cdot x}\phi &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x \left[a_\mathbf{p}e^{-i(p-q)\cdot x} + a_\mathbf{p}^\dagger e^{i(p+q)\cdot x}\right] \\ &= \int d^3p\, \frac{1}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_\mathbf{p}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\ &= \frac{1}{\sqrt{2E_\mathbf{q}}}\left(a_\mathbf{q} + a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),\end{aligned} \tag{2.80}$$

以及

$$\begin{aligned}\int d^3x\, e^{iq\cdot x}\partial_0\phi &= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\int d^3x \left[a_\mathbf{p}e^{-i(p-q)\cdot x} - a_\mathbf{p}^\dagger e^{i(p+q)\cdot x}\right] \\ &= \int d^3p\, \frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_\mathbf{p}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\ &= \frac{-iq_0}{\sqrt{2E_\mathbf{q}}}\left(a_\mathbf{q} - a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right).\end{aligned} \tag{2.81}$$

从而，有

$$-i\sqrt{2E_\mathbf{q}}\,a_\mathbf{q} = \frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}a_\mathbf{q} = \int d^3x\, e^{iq\cdot x}\partial_0\phi - iq_0\int d^3x\, e^{iq\cdot x}\phi = \int d^3x\, e^{iq\cdot x}(\partial_0\phi - iq_0\phi), \tag{2.82}$$

亦即

$$a_\mathbf{p} = \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\left[\partial_0\phi(x) - ip_0\phi(x)\right]. \tag{2.83}$$

上式取厄米共轭，并使用自共轭条件 $\phi^\dagger = \phi$，得

$$a_\mathbf{p}^\dagger = \frac{-i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{-ip\cdot x}\left[\partial_0\phi(x) + ip_0\phi(x)\right]. \tag{2.84}$$

利用上面两个表达式和等时对易关系 (2.78)，可得

$$\begin{aligned}&[a_\mathbf{p}, a_\mathbf{q}^\dagger] \\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\}\right] \\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\left[\pi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\right] \\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right]\\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned}
\tag{2.85}
$$

在以上计算过程中，$x^0=y^0=t$。根据 δ 函数的性质 (2.49)，有

$$
\frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \frac{E_{\mathbf{p}}+E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}}}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}-\mathbf{q}).
\tag{2.86}
$$

故

$$
[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\tag{2.87}
$$

类似地,

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\}\right]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}[\pi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\left(-iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)]-ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right)\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})]\\
&= \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}).
\end{aligned}
\tag{2.88}
$$

根据 δ 函数的性质 (2.49)，有

$$
\frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) = \frac{E_{\mathbf{p}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}}}}e^{i(E_{\mathbf{p}}+E_{\mathbf{p}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0,
\tag{2.89}
$$

故

$$
[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0.
\tag{2.90}
$$

此外,

$$
[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}}a_{\mathbf{p}} - a_{\mathbf{p}}a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0.
\tag{2.91}
$$

综上，产生湮灭算符满足如下对易关系:

$$
[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0.
\tag{2.92}
$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.119)，实标量场的哈密顿量密度为

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = (\partial_0\phi)^2 - \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi + \frac{1}{2}m^2\phi^2 = \frac{1}{2}[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2]. \tag{2.93}$$

对全空间积分以得到哈密顿量:

$$\begin{aligned}
H =& \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\,[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2]\\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[\left(-ip_0a_{\mathbf{p}}e^{-ip\cdot x} + ip_0a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(-iq_0a_{\mathbf{q}}e^{-iq\cdot x} + iq_0a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\\
&+ \left(i\mathbf{p}\,a_{\mathbf{p}}e^{-ip\cdot x} - i\mathbf{p}\,a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\,a_{\mathbf{q}}e^{-iq\cdot x} - i\mathbf{q}\,a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\\
&+ m^2\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}}e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\Big]\\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p-q)\cdot x}\\
&+(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p+q)\cdot x} + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big]\\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right]\\
&+(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right]\Big\}\\
=& \frac{1}{2}\int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}\right]\\
&+\delta^{(3)}(\mathbf{p}+\mathbf{q})(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}\right]\Big\}\\
=& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right)\\
&+(-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\Big].
\end{aligned} \tag{2.94}$$

由 (2.76) 式可得 $-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 = 0$，故上式最后两行方括号中第二项没有贡献。从而，

$$\begin{aligned}
H =& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3\,2E_{\mathbf{p}}}2E_{\mathbf{p}}^2\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right)\\
=& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p})\right]\\
=& \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\,a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{0})\int \frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2},
\end{aligned} \tag{2.95}$$

其中第四步用到对易关系 (2.92)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。$a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 $\mathbf{p}$ 的模式对应的**粒子数密度算符**（动量空间中的密度），相应的能量是 $E_{\mathbf{p}}$。在 (2.95) 式最后一行中，第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.79) 式可得

$$(2\pi)^3\delta^{(3)}(\mathbf{0}) = \int d^3x = V, \tag{2.96}$$

其中 V 是进行积分的空间体积，对于全空间而言是无穷大的。因此，(2.95) 式最后一行的第二项是一个无穷大 c 数，是真空的零点能，是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明，一维简谐振子的零点能为 $E_0 = \omega/2$。这是自由度为 1 时的结果，推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象，这个零点能通常并不重要，因为实验上只能测量两个能量之差。经过正则量子化之后，实标量场的哈密顿量 H 是正定的，不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\, E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\, E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q}-\mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \tag{2.97}$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\, E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\, E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \tag{2.98}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则

$$H\,|E\rangle = E\,|E\rangle\,. \tag{2.99}$$

从而，有

$$H a_{\mathbf{p}}^\dagger\,|E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger)\,|E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger\,|E\rangle\,. \tag{2.100}$$

$$H a_{\mathbf{p}}\,|E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}})\,|E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p}}\,|E\rangle\,. \tag{2.101}$$

可见，当 $a_{\mathbf{p}}^\dagger\,|E\rangle \neq 0$ 时，产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{p}}$；当 $a_{\mathbf{p}}\,|E\rangle \neq 0$ 时，湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{p}}$。

根据 (1.158) 式，实标量场的总动量是

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi \nabla\phi = -\int d^3x (\partial_0\phi)\nabla\phi \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \left(-ip_0 a_{\mathbf{p}} e^{-ip\cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(i\mathbf{q}\, a_{\mathbf{q}} e^{-iq\cdot x} - i\mathbf{q}\, a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[-p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} - p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \\
&\qquad + p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x} \Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ -p_0\mathbf{q}\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right] \\
&\qquad + p_0\mathbf{q}\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right] \Big\} \\
&= -\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ -p_0\mathbf{q}\,\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t}\right] \\
&\qquad + p_0\mathbf{q}\,\delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p^0+q^0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p^0+q^0)t}\right] \Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (-E_{\mathbf{p}}\mathbf{p}) \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)
\end{aligned}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right). \tag{2.102}$$

先作 $\mathbf{p}\to-\mathbf{p}$ 的替换，再利用对易关系 (2.92)，可得

$$\begin{aligned}\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right) &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,(-\mathbf{p})\left(a_{-\mathbf{p}}a_{\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\\ &= -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right).\end{aligned} \tag{2.103}$$

可见，(2.102) 式最后一行圆括号中最后两项没有贡献。从而，

$$\begin{aligned}\mathbf{P} &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{0})\right]\\ &= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}\delta^{(3)}(\mathbf{0})\int d^3p\,\mathbf{p}.\end{aligned} \tag{2.104}$$

由于 $\int d^3p\,\mathbf{p} = \int d^3p\,(-\mathbf{p}) = -\int d^3p\,\mathbf{p}$，上式最后一行第二项没有贡献。于是，

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \tag{2.105}$$

即总动量是所有动量模式所有粒子贡献的动量之和。

$\mathbf{P}$ 与产生湮灭算符的对易子为

$$[\mathbf{P}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,a_{\mathbf{q}}^\dagger[a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\,\mathbf{q}\,a_{\mathbf{q}}^\dagger\delta^{(3)}(\mathbf{q}-\mathbf{p}) = \mathbf{p}\,a_{\mathbf{p}}^\dagger, \tag{2.106}$$

$$[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}]a_{\mathbf{q}} = -\int d^3q\,\mathbf{q}\,a_{\mathbf{q}}\delta^{(3)}(\mathbf{q}-\mathbf{p}) = -\mathbf{p}\,a_{\mathbf{p}}. \tag{2.107}$$

2.3.4 粒子态

真空态 $|0\rangle$ 是能量最低的态，对于任意动量 $\mathbf{p}$ 对应的湮灭算符 $a_{\mathbf{p}}$，满足

$$a_{\mathbf{p}}\,|0\rangle = 0, \tag{2.108}$$

归一化为

$$\langle 0|0\rangle = 1. \tag{2.109}$$

由哈密顿量的表达式 (2.95) 可得

$$H\,|0\rangle = E_{\text{vac}}\,|0\rangle\,, \quad E_{\text{vac}} = \delta^{(3)}(\mathbf{0})\int d^3p\,\frac{E_{\mathbf{p}}}{2}, \tag{2.110}$$

可见，这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外，由 (2.105) 式可知，$|0\rangle$ 的总动量本征值是零：

$$\mathbf{P}\,|0\rangle = \mathbf{0}\,|0\rangle\,, \tag{2.111}$$

即真空态不具有动量。

接着，定义动量为 $\mathbf{p}$ 的**单粒子态**为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, a_{\mathbf{p}}^{\dagger}\, |0\rangle \,. \tag{2.112}$$

从而，利用 (2.97) 和 (2.106) 式可得

$$H\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\, H a_{\mathbf{p}}^{\dagger}\, |0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\dagger} H + E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger})\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(E_{\text{vac}} + E_{\mathbf{p}})a_{\mathbf{p}}^{\dagger}\,|0\rangle = (E_{\text{vac}} + E_{\mathbf{p}})\,|\mathbf{p}\rangle \,, \tag{2.113}$$

$$\mathbf{P}\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{P}\, a_{\mathbf{p}}^{\dagger}\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^{\dagger}\,\mathbf{P} + \mathbf{p}\, a_{\mathbf{p}}^{\dagger})\,|0\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{p}\, a_{\mathbf{p}}^{\dagger}\,|0\rangle = \mathbf{p}\,|\mathbf{p}\rangle \,. \tag{2.114}$$

可以看出，相比于真空态 $|0\rangle$，单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$，也多了一份动量 $\mathbf{p}$。因此，$|\mathbf{p}\rangle$ 描述的是一个动量为 $\mathbf{p}$ 的粒子，这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$，满足狭义相对论中的能量—动量关系 (1.52)，而拉氏量 (2.60) 中的参数 m 就是粒子的**质量**。可以看出，产生算符 $a_{\mathbf{p}}^{\dagger}$ 的作用是产生一个动量为 $\mathbf{p}$ 的粒子。

此外，可作如下计算：

$$a_{\mathbf{p}}\,|\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}}\, a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger}\,|0\rangle = \sqrt{2E_{\mathbf{q}}}\,[a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle \,. \tag{2.115}$$

如果 $\mathbf{p} \neq \mathbf{q}$，则上式为零；如果 $\mathbf{p} = \mathbf{q}$，则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$。可见，湮灭算符 $a_{\mathbf{p}}$ 的作用是减少一个动量为 $\mathbf{p}$ 的粒子。

单粒子态的内积关系为

$$\begin{aligned}\langle\mathbf{q}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\, a_{\mathbf{q}} a_{\mathbf{p}}^{\dagger}\,|0\rangle = \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\,[a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle \\ &= 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{2.116}$$

上式是 *Lorentz* 不变的，这是 (2.112) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明　若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i，则如下等式成立：

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \tag{2.117}$$

引入**阶跃函数** (step function)

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{2.118}$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^μ 满足质壳条件 $p^2 - m^2 = 0$ 且能量为正 $(p^0 > 0)$ 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned}&\int d^4p\,\delta(p^2 - m^2)\theta(p^0)F(p) = \int d^3p\,dp^0\,\delta\left((p^0)^2 - |\mathbf{p}|^2 - m^2\right)\theta(p^0)F(p^0,\mathbf{p}) \\ &= \int d^3p\,\frac{1}{2\sqrt{|\mathbf{p}|^2 + m^2}}\,F\left(\sqrt{|\mathbf{p}|^2 + m^2},\mathbf{p}\right) = \int \frac{d^3p}{2E_{\mathbf{p}}}\,F\left(\sqrt{|\mathbf{p}|^2 + m^2},\mathbf{p}\right).\end{aligned} \tag{2.119}$$

这里第二步用到 (2.117) 式。可见,

$$\frac{d^3p}{2E_{\mathbf{p}}} \tag{2.120}$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$，按照 δ 函数定义，有

$$g(\mathbf{q}) = \int d^3p\, \delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p}) = \int \frac{d^3p}{2E_{\mathbf{p}}}\, 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p}). \tag{2.121}$$

由于上式最左边和最右边都是 Lorentz 不变的,

$$2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.122}$$

必定是 Lorentz 不变的。**证毕**。

进一步，可以定义动量分别为 $\mathbf{p}_1, \cdots, \mathbf{p}_n$ 的 n 个粒子对应的**多粒子态**为

$$|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle \equiv \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle\,. \tag{2.123}$$

H 对它的作用给出

$$\begin{aligned}
&H|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, H a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle \\
={}& \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, (a^\dagger_{\mathbf{p}_1} H + E_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_1}) \cdots a^\dagger_{\mathbf{p}_n} |\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle \\
={}& \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} H a^\dagger_{\mathbf{p}_2} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle + E_{\mathbf{p}_1}|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle \\
={}& \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} H \cdots a^\dagger_{\mathbf{p}_n} |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2})|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle \\
={}& \cdots = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} \cdots a^\dagger_{\mathbf{p}_n} H|0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n})|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle \\
={}& (E_{\text{vac}} + E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n})|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle\,,
\end{aligned} \tag{2.124}$$

同理，$\mathbf{P}$ 对它的作用给出

$$\mathbf{P}|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle = (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n)|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle\,. \tag{2.125}$$

也就是说，多粒子态 $|\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.92) 可得

$$\begin{aligned}
|\mathbf{p}_1, \cdots, \mathbf{p}_i, \cdots, \mathbf{p}_j, \cdots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_i} \cdots a^\dagger_{\mathbf{p}_j} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle \\
&= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}}\, a^\dagger_{\mathbf{p}_1} \cdots a^\dagger_{\mathbf{p}_j} \cdots a^\dagger_{\mathbf{p}_i} \cdots a^\dagger_{\mathbf{p}_n} |0\rangle \\
&= |\mathbf{p}_1, \cdots, \mathbf{p}_j, \cdots, \mathbf{p}_i, \cdots, \mathbf{p}_n\rangle\,.
\end{aligned} \tag{2.126}$$

可以看出，对调多粒子态中的任意两个粒子，得到的态相同，即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是**玻色子** (boson)，服从 Bose-Einstein 统计。得到这个结论的关键在于两个产生算符相互对易。

双粒子态的内积关系为

$$
\begin{aligned}
&\langle \mathbf{q}_1, \mathbf{q}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle = \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\, \langle 0| a_{\mathbf{q}_2} a_{\mathbf{q}_1} a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} |0\rangle \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\, \left[(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\langle 0| a_{\mathbf{q}_2} a^\dagger_{\mathbf{p}_2}|0\rangle + \langle 0| a_{\mathbf{q}_2} a^\dagger_{\mathbf{p}_1} a_{\mathbf{q}_1} a^\dagger_{\mathbf{p}_2}|0\rangle\right] \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\, \left[(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\langle 0| a_{\mathbf{q}_2} a^\dagger_{\mathbf{p}_2}|0\rangle + (2\pi)^3\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\langle 0| a_{\mathbf{q}_2} a^\dagger_{\mathbf{p}_1}|0\rangle\right] \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\, \left[(2\pi)^6\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2) + (2\pi)^6\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\right] \\
&= 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\left[\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2) + \delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\right]. \qquad (2.127)
\end{aligned}
$$

此外，还可以定义动量均为 $\mathbf{p}$ 的 n 个粒子对应的多粒子态为

$$
|n_{\mathbf{p}}\rangle \equiv (2E_{\mathbf{p}})^{n_{\mathbf{p}}/2}\left(a^\dagger_{\mathbf{p}}\right)^{n_{\mathbf{p}}}|0\rangle, \qquad (2.128)
$$

则粒子数密度算符

$$
N_{\mathbf{p}} \equiv a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \qquad (2.129)
$$

对它的作用为

$$
\begin{aligned}
N_{\mathbf{p}}|n_{\mathbf{q}}\rangle &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}}|0\rangle = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left[a^\dagger_{\mathbf{q}} a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle \\
&= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} a_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle \\
&= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^2 a_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-2}|0\rangle + 2(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle \\
&= \cdots = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}} a_{\mathbf{p}}|0\rangle + n_{\mathbf{q}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle \\
&= n_{\mathbf{q}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle. \qquad (2.130)
\end{aligned}
$$

在动量空间对粒子数密度算符进行积分，得到的是粒子数算符

$$
N \equiv \int \frac{d^3p}{(2\pi)^3} N_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \qquad (2.131)
$$

由 (2.130) 式，可得

$$
\begin{aligned}
N|n_{\mathbf{q}}\rangle &= \int \frac{d^3p}{(2\pi)^3} N_{\mathbf{p}}|n_{\mathbf{q}}\rangle = \int \frac{d^3p}{(2\pi)^3} n_{\mathbf{q}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}-1}|0\rangle \\
&= n_{\mathbf{q}}(2E_{\mathbf{q}})^{n_{\mathbf{q}}/2}\left(a^\dagger_{\mathbf{q}}\right)^{n_{\mathbf{q}}}|0\rangle = n_{\mathbf{q}}|n_{\mathbf{q}}\rangle. \qquad (2.132)
\end{aligned}
$$

因此，$|n_{\mathbf{q}}\rangle$ 是 N 的本征态，本征值为粒子数 $n_{\mathbf{q}}$。

更一般地，可以定义多粒子态

$$
|n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle \qquad (2.133)
$$

来描述动量为 $\mathbf{p}_1, \cdots, \mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}$ 个的情况。此时，有

$$
N|n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle
$$

$$
\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \prod_{i=1}^{m} (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^{m} (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right] a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^{m} (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right] \Big[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\
&\qquad + n_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}_1) a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}-1} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \Big] \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^{m} (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle\right] + n_{\mathbf{p}_1} |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \\
&= \cdots = \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^{m} (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} a_{\mathbf{p}} |0\rangle\right] \\
&\qquad + (n_{\mathbf{p}_1} + \cdots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \\
&= (n_{\mathbf{p}_1} + \cdots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle .
\end{aligned}
\tag{2.134}
$$

可见，N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中，我们讨论复标量场 $\phi(x)$，它不满足自共轭条件 (2.59)，即

$$\phi^\dagger(x) \neq \phi(x). \tag{2.135}$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.188) 式的形式。不过，由于 $\phi(x)$ 是量子场算符，需要把那里的复共轭记号 $*$ 改成厄米共轭记号 $\dagger$，故 **Lorentz** 不变拉氏量为

$$\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m^2 \phi^\dagger \phi. \tag{2.136}$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量，注意到

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^\dagger, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^\dagger, \tag{2.137}$$

则 Euler-Lagrange 方程 (1.116) 给出

$$(\partial^2 + m^2)\phi(x) = 0, \quad (\partial^2 + m^2)\phi^\dagger(x) = 0. \tag{2.138}$$

也就是说，$\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程.

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合：

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{2.139}$$

从而，拉氏量 (2.136) 化为

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[\partial^\mu(\phi_1 - i\phi_2)]\partial_\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2 - \frac{1}{2}m^2\phi_2^2.\end{aligned} \tag{2.140}$$

与 (2.60) 式比较可知，复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量。

2.4.1　平面波展开

对于复标量场，我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开，但不能够应用自共轭条件。因此，场算符 $\phi(\mathbf{x},t)$ 的一般解也具有 (2.71) 式的形式：

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_\mathbf{k}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_{-\mathbf{k}}e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.141}$$

由于不满足自共轭条件 (2.59)，算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_\mathbf{k}$ 没有什么关系，改记为

$$b_\mathbf{k}^\dagger = \tilde{a}_{-\mathbf{k}}, \tag{2.142}$$

则展开式变成

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + b_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right]. \tag{2.143}$$

替换一下动量记号，可以把 $\phi(\mathbf{x},t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip\cdot x} + b_\mathbf{p}^\dagger e^{ip\cdot x}\right), \tag{2.144}$$

其中，p^0 应该满足

$$p^0 = E_\mathbf{p} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \tag{2.145}$$

取厄米共轭，就得到 $\phi^\dagger(\mathbf{x},t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(b_\mathbf{p}e^{-ip\cdot x} + a_\mathbf{p}^\dagger e^{ip\cdot x}\right). \tag{2.146}$$

现在，$a_\mathbf{p}$ 和 $b_\mathbf{p}$ 是两个相互独立的湮灭算符，而 $a_\mathbf{p}^\dagger$ 和 $b_\mathbf{p}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}(-ip_0)\left(b_\mathbf{p}e^{-ip\cdot x} - a_\mathbf{p}^\dagger e^{ip\cdot x}\right), \tag{2.147}$$

$\phi^\dagger(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)} = \partial_0\phi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}(-ip_0)\left(a_\mathbf{p}e^{-ip\cdot x} - b_\mathbf{p}^\dagger e^{ip\cdot x}\right). \tag{2.148}$$

根据 (2.57) 式，等时对易关系为

$$
\begin{aligned}
[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0,\\
[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)] = [\pi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] = 0,\\
[\phi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= [\phi^\dagger(\mathbf{x},t),\pi(\mathbf{y},t)] = [\phi(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] = 0.
\end{aligned} \tag{2.149}
$$

2.4.2 产生湮灭算符的对易关系

由

$$
\begin{aligned}
\int d^3x\, e^{iq\cdot x}\phi &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\left[a_\mathbf{p}e^{-i(p-q)\cdot x} + b_\mathbf{p}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + b_\mathbf{p}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{1}{\sqrt{2E_\mathbf{q}}}\left(a_\mathbf{q} + b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right)
\end{aligned} \tag{2.150}
$$

和

$$
\begin{aligned}
\int d^3x\, e^{iq\cdot x}\partial_0\phi &= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\int d^3x\left[a_\mathbf{p}e^{-i(p-q)\cdot x} - b_\mathbf{p}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - b_\mathbf{p}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{-iq_0}{\sqrt{2E_\mathbf{q}}}\left(a_\mathbf{q} - b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),
\end{aligned} \tag{2.151}
$$

可得

$$
-i\sqrt{2E_\mathbf{q}}\,a_\mathbf{q} = \frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}a_\mathbf{q} = \int d^3x\, e^{iq\cdot x}\partial_0\phi - iq_0\int d^3x\, e^{iq\cdot x}\phi = \int d^3x\, e^{iq\cdot x}\left(\partial_0\phi - iq_0\phi\right). \tag{2.152}
$$

于是，

$$
a_\mathbf{p} = \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\left(\partial_0\phi - ip_0\phi\right), \quad a_\mathbf{p}^\dagger = \frac{-i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{-ip\cdot x}\left(\partial_0\phi^\dagger + ip_0\phi^\dagger\right). \tag{2.153}
$$

从而，有

$$
\begin{aligned}
&[a_\mathbf{p}, a_\mathbf{q}^\dagger]\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t) + iq_0\phi^\dagger(\mathbf{y},t)\}\right]\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\left[\pi^\dagger(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) + iq_0\phi^\dagger(\mathbf{y},t)\right]\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned}
\tag{2.154}
$$

以及

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\left[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\right]=0.
\end{aligned}
\tag{2.155}
$$

另一方面，由

$$
\begin{aligned}
\int d^3x\, e^{iq\cdot x}\phi^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}+a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})+a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}+a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right)
\end{aligned}
\tag{2.156}
$$

和

$$
\begin{aligned}
\int d^3x\, e^{iq\cdot x}\partial_0\phi^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}-a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})-a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}-a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),
\end{aligned}
\tag{2.157}
$$

可得

$$
-i\sqrt{2E_{\mathbf{q}}}\,b_{\mathbf{q}}=\frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}b_{\mathbf{q}}=\int d^3x\, e^{iq\cdot x}\partial_0\phi^\dagger-iq_0\int d^3x\, e^{iq\cdot x}\phi^\dagger=\int d^3x\, e^{iq\cdot x}\left(\partial_0\phi^\dagger-iq_0\phi^\dagger\right).
\tag{2.158}
$$

于是,

$$
b_{\mathbf{p}}=\frac{i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\left(\partial_0\phi^\dagger-ip_0\phi^\dagger\right),\quad b_{\mathbf{p}}^\dagger=\frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{-ip\cdot x}\left(\partial_0\phi+ip_0\phi\right).
\tag{2.159}
$$

从而，有

$$
\begin{aligned}
&[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\,[\pi(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)]\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)]-ip_0[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)]\right)\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\,[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})]\\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}=\frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned}\tag{2.160}
$$

以及

$$
\begin{aligned}
&[b_{\mathbf{p}},b_{\mathbf{q}}]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\}\right]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\,[\pi(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t),\ \pi(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)]=0.
\end{aligned}\tag{2.161}
$$

此外，还有

$$
\begin{aligned}
&[a_{\mathbf{p}},b^\dagger_{\mathbf{q}}]\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\}\right]\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\,[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)]=0,
\end{aligned}\tag{2.162}
$$

以及

$$
\begin{aligned}
&[a_{\mathbf{p}},b_{\mathbf{q}}]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\}\right]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\,[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)]\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\left(-iq_0[\pi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]-ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right)\\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\,[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})]\\
&= \frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}=\frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})=0.
\end{aligned}\tag{2.163}
$$

归纳起来，产生湮灭算符的对易关系如下:

$$
\begin{aligned}
[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad [a_{\mathbf{p}},a_{\mathbf{q}}]=[a^\dagger_{\mathbf{p}},a^\dagger_{\mathbf{q}}]=0,\\
[b_{\mathbf{p}},b^\dagger_{\mathbf{q}}] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad [b_{\mathbf{p}},b_{\mathbf{q}}]=[b^\dagger_{\mathbf{p}},b^\dagger_{\mathbf{q}}]=0,
\end{aligned}\tag{2.164}
$$

$$[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = [b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = [a_{\mathbf{p}}, b_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0.$$

这说明 $a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$ 与 $b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$ 是两套不同的产生湮灭算符，描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta}\phi(x), \quad [\phi^\dagger(x)]' = e^{-iq\theta}\phi^\dagger(x), \tag{2.165}$$

则拉氏量 (2.136) 不变。依照 1.7.4 小节的讨论，相应的守恒流为

$$J^\mu = q\phi^\dagger i\overleftrightarrow{\partial^\mu}\phi, \tag{2.166}$$

相应的守恒荷为

$$\begin{aligned}
Q &= q\int d^3x\,\phi^\dagger i\overleftrightarrow{\partial^0}\phi = iq\int d^3x\,[\phi^\dagger\partial^0\phi - (\partial^0\phi^\dagger)\phi] \\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[\left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\partial^0\left(a_{\mathbf{k}}e^{-ik\cdot x} + b_{\mathbf{k}}^\dagger e^{ik\cdot x}\right) \\
&\qquad - \partial^0\left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{k}}e^{-ik\cdot x} + b_{\mathbf{k}}^\dagger e^{ik\cdot x}\right)\Big] \\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[\left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)(-ik^0)\left(a_{\mathbf{k}}e^{-ik\cdot x} - b_{\mathbf{k}}^\dagger e^{ik\cdot x}\right) \\
&\qquad - (-ip^0)\left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{k}}e^{-ik\cdot x} + b_{\mathbf{k}}^\dagger e^{ik\cdot x}\right)\Big] \\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[(ik^0+ip^0)b_{\mathbf{p}}b_{\mathbf{k}}^\dagger e^{-i(p-k)\cdot x} + (-ik^0-ip^0)a_{\mathbf{p}}^\dagger a_{\mathbf{k}}e^{i(p-k)\cdot x} \\
&\qquad + (-ik^0+ip^0)b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + (ik^0-ip^0)a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(p+k)\cdot x}\Big] \\
&= q\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[-(E_{\mathbf{k}}+E_{\mathbf{p}})b_{\mathbf{p}}b_{\mathbf{k}}^\dagger e^{-i(p-k)\cdot x} + (E_{\mathbf{k}}+E_{\mathbf{p}})a_{\mathbf{p}}^\dagger a_{\mathbf{k}}e^{i(p-k)\cdot x} \\
&\qquad + (E_{\mathbf{k}}-E_{\mathbf{p}})b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + (-E_{\mathbf{k}}+E_{\mathbf{p}})a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(p+k)\cdot x}\Big] \\
&= q\int \frac{d^3p\,d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big\{(E_{\mathbf{k}}+E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{k})\left[-b_{\mathbf{p}}b_{\mathbf{k}}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{k}}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\right] \\
&\qquad + (E_{\mathbf{k}}-E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{k})\left[b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} - a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right]\Big\} \\
&= q\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}2E_{\mathbf{p}}\left(-b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = q\int\frac{d^3p}{(2\pi)^3}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}}b_{\mathbf{p}}^\dagger\right).
\end{aligned} \tag{2.167}$$

利用对易关系 (2.164)，可得

$$Q = \int\frac{d^3p}{(2\pi)^3}\left(q\,a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - q\,b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) - (2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}q. \tag{2.168}$$

上式第二项是零点荷。在第一项的圆括号中，粒子数密度算符 $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 的系数是 q，而粒子数密度算符 $b_{\mathbf{p}}^\dagger b_{\mathbf{p}}$ 的系数是 $-q$。可见，$a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q，习惯上称为**正粒子**；另一方面，

$b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$，习惯上称为**反粒子**。除去零点荷，总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式的全空间积分类似，但 Q/q 被解释为正粒子数与反粒子数之差，可正可负，因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的，并且来自于一种内部对称性，因而是一种**内部相加性量子数**。实际上，反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 U(1) 整体变换，则自共轭条件 (2.59) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^{\dagger} = [e^{iq\theta}\phi(x)]^{\dagger} = e^{-iq\theta}\phi^{\dagger}(x) = e^{-iq\theta}\phi(x). \tag{2.169}$$

上式要求 $q = 0$。因此，对实标量场不能进行非平庸的 U(1) 整体变换。实际上，自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数，也就是说，正粒子与反粒子是相同的，实标量场描述的是一种纯中性粒子。

2.4.4　哈密顿量和总动量

根据 (1.119) 式，复标量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi\partial_0\phi + \pi^{\dagger}\partial_0\phi^{\dagger} - \mathcal{L} = (\partial^0\phi^{\dagger})\partial_0\phi + (\partial^0\phi)\partial_0\phi^{\dagger} - (\partial^{\mu}\phi^{\dagger})\partial_{\mu}\phi + m^2\phi^{\dagger}\phi \\ &= (\partial^0\phi^{\dagger})\partial_0\phi + (\nabla\phi^{\dagger})\cdot\nabla\phi + m^2\phi^{\dagger}\phi.\end{aligned} \tag{2.170}$$

于是，哈密顿量可以写成

$$\begin{aligned}H &= \int d^3x\,\mathcal{H} = \int d^3x\,[(\partial^0\phi^{\dagger})\partial_0\phi + (\nabla\phi^{\dagger})\cdot\nabla\phi + m^2\phi^{\dagger}\phi] \\ &= \int d^3x\,[(\partial^0\phi^{\dagger})\partial_0\phi + \nabla\cdot(\phi^{\dagger}\nabla\phi) - \phi^{\dagger}\nabla^2\phi + m^2\phi^{\dagger}\phi] \\ &= \int d^3x\,[(\partial^0\phi^{\dagger})\partial_0\phi - \phi^{\dagger}\partial^0\partial_0\phi + \phi^{\dagger}(\partial^0\partial_0 - \nabla^2 + m^2)\phi] \\ &= \int d^3x\,[(\partial^0\phi^{\dagger})\partial_0\phi - \phi^{\dagger}\partial^0\partial_0\phi + \phi^{\dagger}(\partial^2 + m^2)\phi].\end{aligned} \tag{2.171}$$

上式第三步用了分部积分，第四步扔掉了一个全散度，最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.138) 消去。从而，得到

$$\begin{aligned}H &= \int d^3x\,[(\partial^0\phi^{\dagger})\partial_0\phi - \phi^{\dagger}\partial^0\partial_0\phi] \\ &= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[\partial^0\left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)\partial_0\left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^{\dagger}e^{iq\cdot x}\right) \\ &\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)\partial^0\partial_0\left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^{\dagger}e^{iq\cdot x}\right)\Big] \\ &= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{(-ip^0)\left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)(-iq_0)\left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^{\dagger}e^{iq\cdot x}\right) \\ &\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)\left[(-iq^0)(-iq_0)a_{\mathbf{q}}e^{-iq\cdot x} + iq^0iq_0b_{\mathbf{q}}^{\dagger}e^{iq\cdot x}\right]\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(p^0q_0 + q^0q_0)b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p^0q_0 + q^0q_0)a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \\
&\qquad\qquad + (-p^0q_0 + q^0q_0)b_{\mathbf{p}}a_{\mathbf{q}} e^{-i(p+q)\cdot x} + (-p^0q_0 + q^0q_0)a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, E_{\mathbf{q}} \Big\{(E_{\mathbf{p}} + E_{\mathbf{q}})\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right] \\
&\qquad\qquad + (E_{\mathbf{q}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[b_{\mathbf{p}}a_{\mathbf{q}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right]\Big\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\, 2E_{\mathbf{p}}^2 \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) + (2\pi)^3\delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}.
\end{aligned}
\tag{2.172}
$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 $\mathbf{p}$，正粒子与反粒子具有相同的能量 $E_{\mathbf{p}}$，因而它们具有相同的质量 m。

根据 (1.158) 式，复标量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\, (\pi\nabla\phi + \pi^\dagger\nabla\phi^\dagger) = -\int d^3x\, [(\partial_0\phi^\dagger)\nabla\phi + (\partial_0\phi)\nabla\phi^\dagger] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\partial_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \nabla \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad + \partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \nabla \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[-ip_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) i\mathbf{q} \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad - iq_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) i\mathbf{p} \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}})e^{-i(p-q)\cdot x} \\
&\qquad\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger)e^{i(p-q)\cdot x} \\
&\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}b_{\mathbf{p}})e^{-i(p+q)\cdot x} \\
&\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger)e^{i(p+q)\cdot x}\Big] \\
&= -\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}})e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \\
&\qquad\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger)e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big] \\
&\qquad\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[(E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}b_{\mathbf{p}})e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \\
&\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger)e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big]\Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \Big[-E_{\mathbf{p}}\mathbf{p}\, (b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}a_{\mathbf{p}}^\dagger) \\
&\qquad\qquad - E_{\mathbf{p}}\mathbf{p}\, (b_{\mathbf{p}}a_{-\mathbf{p}} - a_{-\mathbf{p}}b_{\mathbf{p}})e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}}\mathbf{p}\, (a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger - b_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger)e^{2iE_{\mathbf{p}}t}\Big]
\end{aligned}
$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} (b_\mathbf{p} b^\dagger_\mathbf{p} + b^\dagger_\mathbf{p} b_\mathbf{p} + a^\dagger_\mathbf{p} a_\mathbf{p} + a_\mathbf{p} a^\dagger_\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} (a^\dagger_\mathbf{p} a_\mathbf{p} + b^\dagger_\mathbf{p} b_\mathbf{p}) + \delta^{(3)}(\mathbf{0}) \int d^3p \, \mathbf{p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} (a^\dagger_\mathbf{p} a_\mathbf{p} + b^\dagger_\mathbf{p} b_\mathbf{p}). \tag{2.173}$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章　矢量场

3.1　量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换

$$|\Psi'\rangle = U(\Lambda)\,|\Psi\rangle\,, \tag{3.1}$$

其中 $U(\Lambda)$ 是一个线性幺正算符，描述**量子 Lorentz 变换**，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \tag{3.2}$$

先作 Lorentz 变换 Λ_1，再作 Lorentz 变换 Λ_2，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$，故以下同态 (homomorphic) 关系成立:

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \tag{3.3}$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(\mathbf{1}) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \tag{3.4}$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \tag{3.5}$$

将无穷小 Lorentz 变换 (1.159) 记为 $\Lambda_\omega = \mathbf{1} + \omega$，它诱导的无穷小幺正算符可表达为

$$U(\mathbf{1}+\omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \tag{3.6}$$

这里只展开到 ω 的一阶项。$J^{\mu\nu}$ 是量子 Lorentz 变换的**生成元算符**[1]。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的:

$$J^{\mu\nu} = -J^{\nu\mu}. \tag{3.7}$$

由 $U(\mathbf{1}+\omega)$ 的幺正性可得

$$1 = U^\dagger(\mathbf{1}+\omega)U(\mathbf{1}+\omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right]\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \tag{3.8}$$

[1]虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.179)。

最后一步忽略了 ω 的二阶项。可见，$J^{\mu\nu}$ 是厄米算符：

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \tag{3.9}$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda). \tag{3.10}$$

的左边和右边分别展开，得

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U^{-1}(\Lambda)\left(1-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)U(\Lambda) = 1-\frac{i}{2}U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda), \tag{3.11}$$

$$U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda) = U(\mathbf{1}+\Lambda^{-1}\omega\Lambda) = 1-\frac{i}{2}(\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu}. \tag{3.12}$$

因此，有

$$\begin{aligned} U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1}\omega\Lambda)^{\alpha}{}_{\nu}J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1})^{\alpha}{}_{\beta}\omega^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} \\ &= g_{\mu\alpha}\Lambda_{\beta}{}^{\alpha}\omega^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} = \Lambda^{\beta}{}_{\mu}\omega_{\beta\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} = \omega_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}, \end{aligned} \tag{3.13}$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立，于是，

$$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}. \tag{3.14}$$

因此，$J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值与它在 $|\Psi\rangle$ 中的期待值有如下关系：

$$\langle\Psi'|\, J^{\mu\nu}\,|\Psi'\rangle = \langle\Psi|\, U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)\,|\Psi\rangle = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\,\langle\Psi|\, J^{\rho\sigma}\,|\Psi\rangle\,. \tag{3.15}$$

也就是说，$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换：

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}. \tag{3.16}$$

可见，$J^{\mu\nu}$ 是一个 *2* 阶 *Lorentz* 张量。

接着，考虑 Λ 的无穷小形式 $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\nu}$，则

$$U(\Lambda) = 1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}. \tag{3.17}$$

忽略二阶小量，(3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) &= \left(1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}\right)J^{\mu\nu}\left(1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}\right) \\ &= J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\mu\nu}J^{\alpha\beta} + \frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \tag{3.18}$$

右边为

$$\begin{aligned} \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma} &= (\delta^{\mu}{}_{\rho}+\tilde{\omega}^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma}+\tilde{\omega}^{\nu}{}_{\sigma})J^{\rho\sigma} = \delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}J^{\rho\sigma} + \delta^{\mu}{}_{\rho}\tilde{\omega}^{\nu}{}_{\sigma}J^{\rho\sigma} + \tilde{\omega}^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^{\nu}{}_{\sigma}J^{\mu\sigma} + \tilde{\omega}^{\mu}{}_{\rho}J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho}g^{\mu\sigma}J^{\rho\nu} \end{aligned}$$

$$
\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma}J^{\mu\rho} + g^{\mu\rho}J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma}),
\end{aligned} \tag{3.19}
$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= i[g^{\nu\rho}J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma).
\end{aligned} \tag{3.20}
$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群是一类特殊的连续群，n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie 乘积*，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$
J^i \equiv \frac{1}{2}\varepsilon^{ijk}J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}
$$

即

$$
\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}
$$

J^i 与 J^j 的对易关系为

$$
\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij},
\end{aligned} \tag{3.23}
$$

第三、四步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.84) 式。由 (1.98) 式，有

$$
J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}
$$

从而推出

$$
[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}
$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$，写成分量的形式是 $L^i = \varepsilon^{ijk}x^j p^k$，从而，

$$
\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^l p^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^l p^m = x^i p^j - x^j p^i. \tag{3.26}
$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i, p^j] = i\delta^{ij}$ 可得

$$\begin{aligned}[L^i, L^j] &= \varepsilon^{ikl}\varepsilon^{jmn}[x^k p^l, x^m p^n] = \varepsilon^{ikl}\varepsilon^{jmn}\{x^k[p^l, x^m]p^n + x^m[x^k, p^n]p^l\} \\ &= \varepsilon^{ikl}\varepsilon^{jmn}(-i\delta^{lm}x^k p^n + i\delta^{kn}x^m p^l) = i(-\varepsilon^{ikl}\varepsilon^{jln}x^k p^n + \varepsilon^{ikl}\varepsilon^{jmk}x^m p^l) \\ &= i(\varepsilon^{ikl}\varepsilon^{jnl}x^k p^n - \varepsilon^{ilk}\varepsilon^{jmk}x^m p^l) = i[(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})x^k p^n - (\delta^{ij}\delta^{lm} - \delta^{im}\delta^{lj})x^m p^l] \\ &= i[\delta^{ij}x^k p^k - x^j p^i - \delta^{ij}x^l p^l + x^i p^j] = i(x^i p^j - x^j p^i) = i\varepsilon^{ijk}L^k. \end{aligned} \tag{3.27}$$

可见，$\mathbf{J}$ 与 $\mathbf{L}$ 具有相同的对易关系，$\mathbf{J}$ 也是一个角动量算符。实际上，$\mathbf{J}$ 描述**总角动量**，不止可以包含轨道角动量 $\mathbf{L}$，也可以包含自旋角动量。

满足

$$O^{\mathrm{T}}O = \mathbf{1} \tag{3.28}$$

的实方阵 O 称为实正交矩阵 (real orthogonal matrix)。对上式取行列式，得

$$1 = \det O^{\mathrm{T}} \cdot \det O = (\det O)^2. \tag{3.29}$$

可见，实正交矩阵 O 的行列式为 $\det O = \pm 1$。由行列式为 $+1$ 的 3 维实正交矩阵按照矩阵乘法构成的群，称为**空间旋转群 SO(3)**，描述三维空间中的旋转变换。1.7.3 小节提到，SO(3) 群是 Lorentz 群的子群，J^i 可以看作 SO(3) 群的生成元算符，而 (3.25) 式是 SO(3) 群的 Lie 代数关系。

另一方面，$\mathbf{K}$ 是增速算符。$\mathbf{J}$ 与 $\mathbf{K}$ 的对易关系为

$$\begin{aligned}[J^i, K^j] &= \frac{1}{2}\varepsilon^{ikl}[J^{kl}, J^{0j}] = \frac{i}{2}\varepsilon^{ikl}\{[g^{l0}J^{kj} - (k \leftrightarrow l)] - (0 \leftrightarrow j)\} \\ &= i\varepsilon^{ikl}[g^{l0}J^{kj} - (0 \leftrightarrow j)] = i\varepsilon^{ikl}(g^{l0}J^{kj} - g^{lj}J^{k0}) = -i\varepsilon^{ikl}g^{lj}J^{k0} = i\varepsilon^{ikl}\delta^{lj}J^{k0} \\ &= i\varepsilon^{ikj}J^{k0} = i\varepsilon^{ijk}J^{0k} = i\varepsilon^{ijk}K^k, \end{aligned} \tag{3.30}$$

而 $\mathbf{K}$ 自身的对易关系为

$$\begin{aligned}[K^i, K^j] &= [J^{0i}, J^{0j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) \\ &= -i(g^{00}J^{ij} + g^{ij}J^{00}) = -iJ^{ij} = -i\varepsilon^{ijk}J^k. \end{aligned} \tag{3.31}$$

归纳起来，有

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \quad [J^i, K^j] = i\varepsilon^{ijk}K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk}J^k. \tag{3.32}$$

3.2　量子矢量场的 Lorentz 变换

3.2.1　Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 $\omega^\alpha{}_\beta$ 可以转化为

$$\omega^\alpha{}_\beta = g^{\alpha\mu}\omega_{\mu\beta} = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\beta} - g^{\alpha\mu}\omega_{\beta\mu}) = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta - g^{\alpha\mu}\omega_{\nu\mu}\delta^\nu{}_\beta) = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta - g^{\alpha\nu}\omega_{\mu\nu}\delta^\mu{}_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}, \tag{3.33}$$

其中 $(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}$ 定义为

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} \equiv i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta}). \tag{3.34}$$

容易看出，$\mathcal{J}^{\mu\nu}$ 是反对称的:

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \tag{3.35}$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^{\gamma}{}_{\beta} = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\gamma}) = i(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}). \tag{3.36}$$

这样的话，可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}. \tag{3.37}$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned}
& [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^{\alpha}{}_{\beta} = (\mathcal{J}^{\mu\nu})^{\alpha}{}_{\gamma}(\mathcal{J}^{\rho\sigma})^{\gamma}{}_{\beta} - (\mathcal{J}^{\rho\sigma})^{\alpha}{}_{\gamma}(\mathcal{J}^{\mu\nu})^{\gamma}{}_{\beta} \\
= {} & i^2(g^{\mu\alpha}\delta^{\nu}{}_{\gamma} - \delta^{\mu}{}_{\gamma}g^{\nu\alpha})(g^{\rho\gamma}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
= {} & -g^{\mu\alpha}\delta^{\nu}{}_{\gamma}g^{\rho\gamma}\delta^{\sigma}{}_{\beta} + g^{\mu\alpha}\delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\beta}g^{\sigma\gamma} + \delta^{\mu}{}_{\gamma}g^{\nu\alpha}g^{\rho\gamma}\delta^{\sigma}{}_{\beta} - \delta^{\mu}{}_{\gamma}g^{\nu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
= {} & -g^{\mu\alpha}g^{\rho\nu}\delta^{\sigma}{}_{\beta} + g^{\mu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^{\sigma}{}_{\beta} - g^{\nu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
= {} & -g^{\nu\rho}g^{\mu\alpha}\delta^{\sigma}{}_{\beta} + g^{\mu\rho}g^{\nu\alpha}\delta^{\sigma}{}_{\beta} + g^{\nu\sigma}g^{\mu\alpha}\delta^{\rho}{}_{\beta} - g^{\mu\sigma}g^{\nu\alpha}\delta^{\rho}{}_{\beta} \\
& -[-g^{\sigma\mu}g^{\rho\alpha}\delta^{\nu}{}_{\beta} + g^{\rho\mu}g^{\sigma\alpha}\delta^{\nu}{}_{\beta} + g^{\sigma\nu}g^{\rho\alpha}\delta^{\mu}{}_{\beta} - g^{\rho\nu}g^{\sigma\alpha}\delta^{\mu}{}_{\beta}] \\
= {} & g^{\nu\rho}(g^{\sigma\alpha}\delta^{\mu}{}_{\beta} - g^{\mu\alpha}\delta^{\sigma}{}_{\beta}) + g^{\mu\rho}(g^{\nu\alpha}\delta^{\sigma}{}_{\beta} - g^{\sigma\alpha}\delta^{\nu}{}_{\beta}) + g^{\nu\sigma}(g^{\mu\alpha}\delta^{\rho}{}_{\beta} - g^{\rho\alpha}\delta^{\mu}{}_{\beta}) + g^{\mu\sigma}(g^{\rho\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\rho}{}_{\beta}) \\
= {} & -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^{\alpha}{}_{\beta} - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\beta} - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\beta} - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^{\alpha}{}_{\beta} \\
= {} & i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^{\alpha}{}_{\beta} - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\beta} - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\beta} + g^{\mu\sigma}(\mathcal{J}^{\nu\rho})^{\alpha}{}_{\beta}],
\end{aligned} \tag{3.38}$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}). \tag{3.39}$$

可见，$\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。$\Lambda^{\alpha}{}_{\beta}$ 属于 Lorentz 群的矢量表示，因而 $\mathcal{J}^{\mu\nu}$ 就是矢量表示的生成元。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \tag{3.40}$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^{\omega} = \sum_{n=0}^{\infty}\frac{\omega^n}{n!} \tag{3.41}$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 $\mathbf{g}$ 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}{}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}{}^{\delta}g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}{}_{\gamma}g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}{}_{\beta}, \tag{3.42}$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \tag{3.43}$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1}\left[\sum_{n=0}^{\infty}\frac{(\omega^{\mathrm{T}})^n}{n!}\right]\mathbf{g} = \sum_{n=0}^{\infty}\frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \tag{3.44}$$

若两个同阶方阵 A 和 B 相互对易，即 $[A,B]=0$，则二项式定理成立：

$$(A+B)^n = \sum_{j=0}^{n}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.45}$$

阶乘的定义可以推广到负整数：对于整数 $m<0$，定义

$$m! \to \infty, \qquad \frac{1}{m!} \to 0. \tag{3.46}$$

从而，对于 $j>n$，有 $[(n-j)!]^{-1}\to 0$。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A+B)^n = \sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.47}$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty}\frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j} = \sum_{j=0}^{\infty}\frac{A^j}{j!}\sum_{n=0}^{\infty}\frac{B^{n-j}}{(n-j)!} = e^Ae^B. \tag{3.48}$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^{\mathbf{0}} = \mathbf{1}. \tag{3.49}$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = \mathbf{g}, \tag{3.50}$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{3.51}$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的**有限变换**。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu}=0$，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

在正则量子化程序中，标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符，类似于 (3.16) 式，$\phi(x)$ 的固有保时向 Lorentz 变换关系 (2.58) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \tag{3.52}$$

上式表明，变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$，(3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{3.53}$$

相应地，$\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

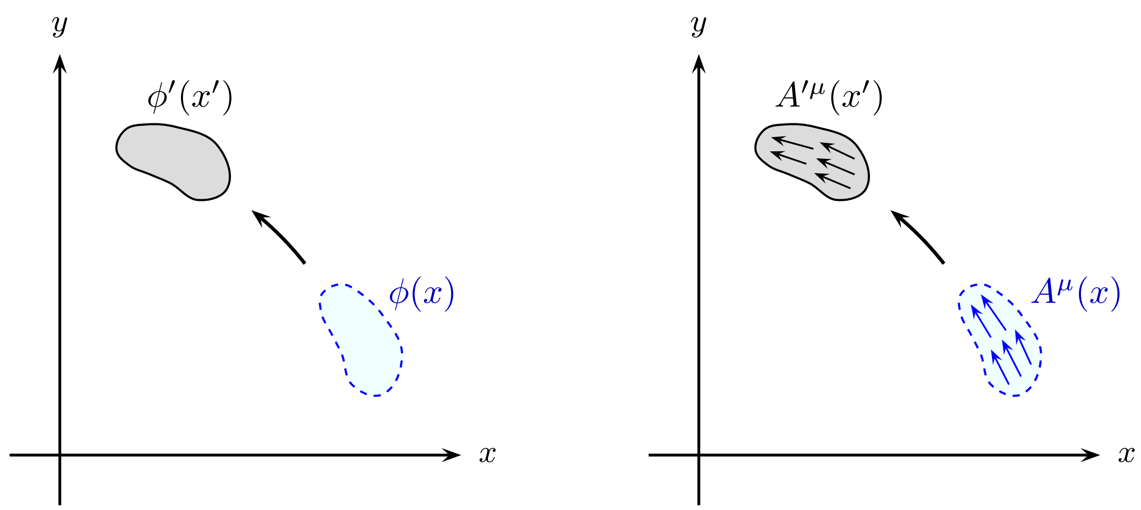
$$\langle\Psi'|\,\phi(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)\phi(x)U(\Lambda)\,|\Psi\rangle = \langle\Psi|\,\phi(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.54}$$

另一方面，由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的相应 Lorentz 变换形式为

$$\partial'^\mu\phi'(x') = U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda) = \partial'^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial'^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \tag{3.55}$$

于是，在固有保时向 Lorentz 变换下，自由实标量场的拉氏量 (2.60) 的变换形式为

$$\begin{aligned}
\mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial'^\mu\phi(x')\partial'_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda)\\
&= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial'^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\}\\
&= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)]\\
&= \mathcal{L}(x),
\end{aligned} \tag{3.56}$$



(a) 标量场 $\phi(x)$ (b) 矢量场 $A^\mu(x)$

图 3.1: 在绕 z 轴空间旋转变换下，标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而，

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(\Lambda^{-1}x). \tag{3.57}$$

可见，拉氏量 (2.60) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$，可得

$$\begin{aligned}(\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu = g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta = g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) = g_{\nu\beta}g^{\mu\beta} + g^{\mu\beta}\omega_{\nu\beta} = \delta^\mu{}_\nu - g^{\mu\beta}\omega_{\beta\nu}\\ &= \delta^\mu{}_\nu - \omega^\mu{}_\nu,\end{aligned} \tag{3.58}$$

从而，有

$$(\Lambda^{-1}x)^\mu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu. \tag{3.59}$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项，得

$$\begin{aligned}\phi(\Lambda^{-1}x) &= \phi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) = \phi(x) - \omega_{\mu\nu}x^\nu\partial^\mu\phi(x) = \phi(x) - \frac{1}{2}(\omega_{\mu\nu}x^\nu\partial^\mu + \omega_{\nu\mu}x^\mu\partial^\nu)\phi(x)\\ &= \phi(x) - \frac{1}{2}\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi(x) = \phi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x)\\ &= \phi(x) - \frac{i}{2}\omega_{\mu\nu}i(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x).\end{aligned} \tag{3.60}$$

根据 (3.6) 式，将 (3.53) 式左边展开到 ω 的一阶项，得

$$\begin{aligned}U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\\ &= \phi(x) - \frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}[\phi(x), J^{\mu\nu}].\end{aligned} \tag{3.61}$$

两相比较，给出

$$[\phi(x), J^{\mu\nu}] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x) = L^{\mu\nu}\phi(x), \tag{3.62}$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu} \equiv i(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{3.63}$$

对于空间分量 L^{ij}，可以等价地定义

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{i}{2}\varepsilon^{ijk}(x^j\partial^k - x^k\partial^j) = \frac{i}{2}(\varepsilon^{ijk}x^j\partial^k - \varepsilon^{ikj}x^j\partial^k) = i\varepsilon^{ijk}x^j\partial^k, \tag{3.64}$$

写成空间矢量的形式是

$$\mathbf{L} = -i\,\mathbf{x}\times\nabla. \tag{3.65}$$

可见，$\mathbf{L}$ 就是微分算符形式的轨道角动量算符。根据 (3.21) 式，(3.62) 式的纯空间分量部分可以改写为

$$[\phi(x), \mathbf{J}] = \mathbf{L}\,\phi(x). \tag{3.66}$$

上式表明，总角动量算符 $\mathbf{J}$ 生成了轨道角动量，但没有生成自旋角动量。这说明标量场没有自旋，对应于零自旋粒子。

3.2.3 量子矢量场的 Lorentz 变换形式

$\partial^\mu\phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有像 (3.55) 式那样的 Lorentz 变换形式，即

$$A'^\mu(x') = U^{-1}(\Lambda)A^\mu(x')U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \tag{3.67}$$

或者写成

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{3.68}$$

这就是**量子矢量场的 Lorentz 变换形式**。相应地，$A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\,A^\mu(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)A^\mu(x)U(\Lambda)\,|\Psi\rangle = \Lambda^\mu{}_\nu\,\langle\Psi|\,A^\nu(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.69}$$

对于固有保时向 Lorentz 变换，根据矢量表示中的无穷小形式 (3.40)，(3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]A^\nu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.70}$$

将上式与 (1.168) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出，在 Lorentz 变换下，除了矢量场的分布区域发生变化之外，矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式，在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned} A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta\partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta}x^\beta\partial^\alpha A^\nu(x) \\ &= A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x). \end{aligned} \tag{3.71}$$

从而，(3.68) 式右边可展开为

$$\begin{aligned} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\left[A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x)\right] \\ &= A^\mu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x) \\ &= A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)]. \end{aligned} \tag{3.72}$$

另一方面，(3.68) 式左边的无穷小展开式为

$$\begin{aligned} U^{-1}(\Lambda)A^\mu(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)A^\mu(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right) \\ &= A^\mu(x) - \frac{i}{2}\omega_{\alpha\beta}A^\mu(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}A^\mu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[A^\mu(x), J^{\rho\sigma}]. \end{aligned} \tag{3.73}$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.74}$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathcal{J}^{jk}, \quad \boldsymbol{\mathcal{J}} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \tag{3.75}$$

再根据 (3.21) 和 (3.64) 式，(3.74) 式的纯空间分量部分可以改写为

$$[A^\mu(x), \mathbf{J}] = \mathbf{L}\, A^\mu(x) + (\boldsymbol{\mathcal{J}})^\mu{}_\nu A^\nu(x). \tag{3.76}$$

上式表明，总角动量算符 $\mathbf{J}$ 不仅生成了轨道角动量，还生成了由 $\boldsymbol{\mathcal{J}}$ 描述的**自旋角动量**。$\mathcal{J}^i$ 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \tag{3.77}$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \tag{3.78}$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \tag{3.79}$$

只关注空间分量，可得

$$(\mathcal{J}^1\mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2\mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \tag{3.80}$$

因此，有

$$(\boldsymbol{\mathcal{J}}^2)^i{}_j = (\mathcal{J}^1\mathcal{J}^1 + \mathcal{J}^2\mathcal{J}^2 + \mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \tag{3.81}$$

根据量子力学的角动量理论，$\boldsymbol{\mathcal{J}}^2$ 的本征值为 $s(s+1)$，即 $(\boldsymbol{\mathcal{J}}^2)^i{}_j = s(s+1)\delta^i{}_j$，其中 s 为**自旋量子数**。可见，矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \tag{3.82}$$

经过量子化程序之后，矢量场 $A^\mu(x)$ 应当描述**自旋为 1** 的粒子。

3.3 有质量矢量场的正则量子化

类似于电磁场，对任意的矢量场 A^μ 可以定义反对称的场强张量

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3.83}$$

对于一个自由的**有质量**的实矢量场 A^μ，用场强张量可以将它的 **Lorentz** 不变拉氏量写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.84}$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned}
-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu] \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu.
\end{aligned} \tag{3.85}$$

从而，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \tag{3.86}$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \tag{3.87}$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{3.88}$$

上式称为 **Proca 方程**，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu\partial_\mu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = -\partial_\mu\partial_\nu F^{\nu\mu} = -\partial_\nu\partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu\partial_\mu F^{\mu\nu} = 0. \tag{3.89}$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu = m^2\partial_\nu A^\nu. \tag{3.90}$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz 条件**

$$\partial_\mu A^\mu = 0. \tag{3.91}$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu = \partial^2 A^\nu. \tag{3.92}$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon 方程*

$$(\partial^2 + m^2)A^\mu(x) = 0. \tag{3.93}$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \tag{3.94}$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \tag{3.95}$$

由于 $\pi_0 = 0$，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。实际上，由于 Lorenz 条件 (3.91) 的存在，A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于其它 3 个分量的量。因此，正则量子化程序要求独立的正则变量满足等时对易关系

$$[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = i\delta^i{}_j\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^i(\mathbf{x},t), A^j(\mathbf{y},t)] = [\pi_i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = 0. \tag{3.96}$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程，应该具有两个平面波解，即正能解 $\exp(-ip\cdot x)$ 和负能解 $\exp(ip\cdot x)$。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标，平面波展开式的系数也必须具有一个这样的指标。一般地，对于确定的动量 $\mathbf{p}$，矢量场的正能解模式具有如下形式：

$$A^\mu(x;\mathbf{p},\sigma) = e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \quad p^0 = E_\mathbf{p} = \sqrt{|\mathbf{p}|^2+m^2}. \tag{3.97}$$

这里的系数 $e^\mu(\mathbf{p},\sigma)$ 是 Lorentz 矢量，称为**极化矢量** (polarization vector)，它依赖于动量 p，而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而，可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点，一组极化矢量应当是线性独立且正交完备的。Lorentz 矢量空间是一个 4 维空间，因而这样的极化矢量应该有 4 个，包括 1 个类时的极化矢量 $e^\mu(\mathbf{p},0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$。

在没有额外约束的情况下，我们要求这 4 个极化矢量是**实**的，而且满足 Lorentz 矢量空间中的**正交归一关系**

$$e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma') = g_{\sigma\sigma'}. \tag{3.98}$$

进一步，要求这组极化矢量是完备的，也就是说，任意依赖于 $\mathbf{p}$ 的 Lorentz 矢量 $V_\mu(\mathbf{p})$ 能够以它们为基底展开成

$$V_\mu(\mathbf{p}) = \sum_{\sigma=0}^{3} v_\sigma(\mathbf{p})e_\mu(\mathbf{p},\sigma). \tag{3.99}$$

根据正交归一关系 (3.98)，可得

$$g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)V^\mu(\mathbf{p}) = g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)\sum_{\sigma=0}^{3} v_{\sigma'}(\mathbf{p})e^\mu(\mathbf{p},\sigma') = g_{\sigma\sigma}\sum_{\sigma=0}^{3} v_{\sigma'}(\mathbf{p})g_{\sigma\sigma'} = g_{\sigma\sigma}^2 v_\sigma(\mathbf{p}). \tag{3.100}$$

由于 $g_{\sigma\sigma}^2 = 1$，上式化为

$$v_\sigma(\mathbf{p}) = g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)V^\mu(\mathbf{p}). \tag{3.101}$$

这是展开系数 $v_\sigma(\mathbf{p})$ 的计算公式。将它代回 (3.99) 式，有

$$g_{\mu\nu}V^\nu(\mathbf{p}) = V_\mu(\mathbf{p}) = \sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\nu(\mathbf{p},\sigma)V^\nu(\mathbf{p})e_\mu(\mathbf{p},\sigma) = \left[\sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)\right]V^\nu(\mathbf{p}). \tag{3.102}$$

比较上式最左边和最右边，即得

$$\sum_{\sigma=0}^{3} g_{\sigma\sigma} e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{p},\sigma) = g_{\mu\nu}. \tag{3.103}$$

这就是**完备性关系**。正交归一关系 (3.98) 和完备性关系 (3.103) 都是 *Lorentz* 协变的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^μ 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空**横向极化矢量**

$$e^\mu(\mathbf{p},1) = (0,\mathbf{e}(\mathbf{p},1)), \quad e^\mu(\mathbf{p},2) = (0,\mathbf{e}(\mathbf{p},2)). \tag{3.104}$$

此处，

$$\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}(p^1p^3, p^2p^3, -|\mathbf{p}_\mathrm{T}|^2), \quad \mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^2, p^1, 0), \tag{3.105}$$

其中

$$|\mathbf{p}_\mathrm{T}| \equiv \sqrt{(p^1)^2 + (p^2)^2}. \tag{3.106}$$

“横向”指的是它们在三维空间中与 $\mathbf{p}$ 垂直，即

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}[(p^1)^2p^3 + (p^2)^2p^3 - p^3|\mathbf{p}_\mathrm{T}|^2] = 0, \tag{3.107}$$

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^1p^2 + p^2p^1) = 0. \tag{3.108}$$

此外，存在如下关系：

$$\begin{aligned}\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},1) &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}[(p^1)^2(p^3)^2 + (p^2)^2(p^3)^2 + |\mathbf{p}_\mathrm{T}|^4] \\ &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2[(p^3)^2 + |\mathbf{p}_\mathrm{T}|^2] = \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2|\mathbf{p}|^2 = 1,\end{aligned} \tag{3.109}$$

$$\mathbf{e}(\mathbf{p},2)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}[(p^2)^2 + (p^1)^2] = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2 = 1, \tag{3.110}$$

$$\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|^2}(-p^1p^3p^2 + p^2p^3p^1) = 0. \tag{3.111}$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j) = \delta_{ij}, \quad i,j = 1,2. \tag{3.112}$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_\mu e^\mu(\mathbf{p},1) = p_\mu e^\mu(\mathbf{p},2) = 0, \tag{3.113}$$

和正交归一关系

$$e_\mu(\mathbf{p},i)e^\mu(\mathbf{p},j) = -\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j) = -\delta_{ij} = g_{ij}. \tag{3.114}$$

接着，要求第 3 个类空极化矢量 $e^\mu(\mathbf{p},3)$ 是纵向的，即在三维空间中与 $\mathbf{p}$ 平行。这样还不能确定它的时间分量，为此，我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p},3)=0$，而正交归一关系 (3.98) 将决定它的归一化。于是，纵向极化矢量的形式为

$$e^\mu(\mathbf{p},3)=\left(\frac{|\mathbf{p}|}{m},\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}\right). \tag{3.115}$$

可以验证，它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p},3)=p^0\frac{|\mathbf{p}|}{m}-\mathbf{p}\cdot\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}=\frac{p^0|\mathbf{p}|}{m}-\frac{p^0|\mathbf{p}|}{m}=0, \tag{3.116}$$

和正交归一关系

$$\begin{aligned}
e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3) &= \frac{|\mathbf{p}|}{m}\frac{|\mathbf{p}|}{m}-\frac{(p^0)^2\,\mathbf{p}\cdot\mathbf{p}}{m^2|\mathbf{p}|^2}=\frac{|\mathbf{p}|^2}{m^2}-\frac{(p^0)^2}{m^2}=-\frac{(p^0)^2-|\mathbf{p}|^2}{m^2}=-1=g_{33}; &(3.117)\\
e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},i) &= -\frac{p^0}{m|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2. &(3.118)
\end{aligned}$$

最后，我们可以将类时极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p},0)=\frac{1}{m}\,p^\mu=\frac{1}{m}(p^0,\mathbf{p}). \tag{3.119}$$

它满足正交归一关系 (3.98)：

$$\begin{aligned}
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0) &= \frac{p^2}{m^2}=1=g_{00}; &(3.120)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},i) &= -\frac{1}{m}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2; &(3.121)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},3) &= \frac{1}{m^2}\,p^0|\mathbf{p}|-\frac{p^0}{m^2|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{p}=0. &(3.122)
\end{aligned}$$

不过，它不满足四维时空的横向条件：

$$p_\mu e^\mu(\mathbf{p},0)=\frac{p^2}{m}=m. \tag{3.123}$$

可以验证，由 (3.104)、(3.105)、(3.115) 和 (3.119) 式定义的这组极化矢量确实满足完备性关系 (3.103)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)\\
&= e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)-e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)-e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2)-e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\
&= \frac{1}{m^2}\begin{pmatrix} p^0p^0 & -p^0p^1 & -p^0p^2 & -p^0p^3\\ -p^1p^0 & p^1p^1 & p^1p^2 & p^1p^3\\ -p^2p^0 & p^2p^1 & p^2p^2 & p^2p^3\\ -p^3p^0 & p^3p^1 & p^3p^2 & p^3p^3\end{pmatrix}-\frac{1}{|\mathbf{p}|^2|\mathbf{p}_{\mathrm{T}}|^2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & p^1p^3p^1p^3 & p^1p^3p^2p^3 & -p^1p^3|\mathbf{p}_{\mathrm{T}}|^2\\ 0 & p^2p^3p^1p^3 & p^2p^3p^2p^3 & -p^2p^3|\mathbf{p}_{\mathrm{T}}|^2\\ 0 & -|\mathbf{p}_{\mathrm{T}}|^2p^1p^3 & -|\mathbf{p}_{\mathrm{T}}|^2p^2p^3 & |\mathbf{p}_{\mathrm{T}}|^4\end{pmatrix}
\end{aligned}$$

$$
-\frac{1}{|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix}0&0&0&0\\0&p^2p^2&-p^2p^1&0\\0&-p^1p^2&p^1p^1&0\\0&0&0&0\end{pmatrix}-\frac{1}{m^2}\begin{pmatrix}|\mathbf{p}|^2&-p^0p^1&-p^0p^2&-p^0p^3\\-p^0p^1&\frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^1&\frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^2&\frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^3\\-p^0p^2&\frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^1&\frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^2&\frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^3\\-p^0p^3&\frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^1&\frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^2&\frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^3\end{pmatrix}
$$

$$
=\begin{pmatrix}1&0&0&0\\
0&\frac{(p^1)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^1p^3)^2+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2}\\
0&\frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{(p^2)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^2p^3)^2+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2}\\
0&\frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2}&\frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2}&\frac{(p^3)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2}\end{pmatrix}
$$

$$
=\begin{pmatrix}1&0&0&0\\
0&-\frac{(p^1)^2|\mathbf{p}_\mathrm{T}|^2+(p^1)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&-\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2}&-\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2}\\
0&-\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2}&-\frac{(p^2)^2|\mathbf{p}_\mathrm{T}|^2+(p^2)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&-\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2}\\
0&-\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2}&-\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2}&-\frac{(p^3)^2+|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2}\end{pmatrix}
$$

$$
=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=g_{\mu\nu}. \tag{3.124}
$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91)，正能解模式 (3.97) 应满足

$$
0=\partial_\mu A^\mu(x;\mathbf{p},\sigma)=-ip_\mu e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \tag{3.125}
$$

即

$$
p_\mu e^\mu(\mathbf{p},\sigma)=0. \tag{3.126}
$$

也就是说，描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p},0)$ 不能用于描述有质量矢量场 A^μ。这说明 A^μ 只有 3 个物理的极化状态，由类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$ 描述。根据完备性关系 (3.103)，这 3 个物理的极化矢量满足

$$
-\sum_{\sigma=1}^{3}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=\sum_{\sigma=1}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=g_{\mu\nu}-g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)=g_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}, \tag{3.127}
$$

即具有求和关系

$$
\sum_{\sigma=1}^{3}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}. \tag{3.128}
$$

通过如下线性组合，我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p,\lambda)$，其中 $\lambda=+,0,-$：

$$
\varepsilon^\mu(\mathbf{p},\pm)\equiv\frac{1}{\sqrt{2}}\left[\mp e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)\right], \tag{3.129}
$$

$$\varepsilon^\mu(\mathbf{p},0) \equiv e^\mu(\mathbf{p},3). \tag{3.130}$$

这样定义的 $\varepsilon^\mu(p,\pm)$ 是复的，而 $\varepsilon^\mu(p,0)$ 是实的。它们都满足**四维横向条件**

$$p_\mu\varepsilon^\mu(\mathbf{p},\lambda) = 0. \tag{3.131}$$

它们还满足

$$\begin{aligned}
\varepsilon^*_\mu(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\pm) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][\mp e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)]\\
&= \frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1)+\frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2) = \frac{1}{2}(g_{11}+g_{22}) = -1, \qquad (3.132)\\
\varepsilon^*_\mu(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\mp) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][\pm e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)]\\
&= -\frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1)+\frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2) = \frac{1}{2}(-g_{11}+g_{22}) = 0, \qquad (3.133)\\
\varepsilon^*_\mu(\mathbf{p},0)\varepsilon^\mu(\mathbf{p},0) &= e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3) = -1, \qquad (3.134)\\
\varepsilon^*_\mu(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},0) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)]e^\mu(\mathbf{p},3) = 0, \qquad (3.135)
\end{aligned}$$

即具有**正交归一关系**

$$\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon^\mu(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'}. \tag{3.136}$$

极化矢量求和关系则是

$$\begin{aligned}
\sum_{\lambda=\pm,0}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) &= \frac{1}{2}[e_\mu(p,1)+ie_\mu(p,2)][e_\nu(p,1)-ie_\nu(p,2)]\\
&\quad+\frac{1}{2}[-e_\mu(p,1)+ie_\mu(p,2)][-e_\nu(p,1)-ie_\nu(p,2)]+e_\mu(p,3)e_\nu(p,3)\\
&= e_\mu(p,1)e_\nu(p,1)+e_\mu(p,2)e_\nu(p,2)+e_\mu(p,3)e_\nu(p,3)\\
&= \sum_{\sigma=1}^{3}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma), \qquad (3.137)
\end{aligned}$$

与 (3.128) 式左边相等，故

$$\sum_{\lambda=\pm,0}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}. \tag{3.138}$$

四维横向条件 (3.131) 在上式中体现为

$$p^\nu\sum_{\lambda=\pm,0}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -p_\mu+\frac{p_\mu p^2}{m^2} = -p_\mu+p_\mu = 0. \tag{3.139}$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 $\mathbf{p}$ 的方向由 $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$ 表征，于是，在 Lorentz 群矢量表示中，螺旋度矩阵定义为

$$\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}} = \frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\boldsymbol{\mathcal{J}} = \frac{1}{|\mathbf{p}|}\begin{pmatrix}0&&&\\&0&-ip^3&ip^2\\&ip^3&0&-ip^1\\&-ip^2&ip^1&0\end{pmatrix}. \tag{3.140}$$

这里已经使用了 $\boldsymbol{\mathcal{J}}$ 的矩阵表达式 (3.77)、(3.78) 和 (3.79)。将 (3.105) 和 (3.115) 式代入 (3.129) 和 (3.130) 式，得到 $\varepsilon^\mu(p,\lambda)$ 的列矢量形式为

$$
\varepsilon^\mu(p,0) = \frac{1}{m|\mathbf{p}|}\begin{pmatrix}|\mathbf{p}|^2\\ p^0p^1\\ p^0p^2\\ p^0p^3\end{pmatrix}, \quad \varepsilon^\mu(p,+) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ -p^1p^3+ip^2|\mathbf{p}|\\ -p^2p^3-ip^1|\mathbf{p}|\\ |\mathbf{p}_\mathrm{T}|^2\end{pmatrix},
$$

$$
\varepsilon^\mu(p,-) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ p^1p^3+ip^2|\mathbf{p}|\\ p^2p^3-ip^1|\mathbf{p}|\\ -|\mathbf{p}_\mathrm{T}|^2\end{pmatrix}. \tag{3.141}
$$

从而，可得

$$
(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,0) = \frac{1}{m|\mathbf{p}|^2}\begin{pmatrix}0\\ -ip^3p^0p^2+ip^2p^0p^3\\ ip^3p^0p^1-ip^1p^0p^3\\ -ip^2p^0p^1+ip^1p^0p^2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = 0\,\varepsilon^\mu(p,0), \tag{3.142}
$$

$$
\begin{aligned}
(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,+) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ ip^2(p^3)^2-p^1p^3|\mathbf{p}|+ip^2|\mathbf{p}_\mathrm{T}|^2\\ -ip^1(p^3)^2-p^2p^3|\mathbf{p}|-ip^1|\mathbf{p}_\mathrm{T}|^2\\ ip^1p^2p^3+(p^2)^2|\mathbf{p}|-ip^1p^2p^3+(p^1)^2|\mathbf{p}|\end{pmatrix}\\
&= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ -p^1p^3|\mathbf{p}|+ip^2|\mathbf{p}|^2\\ -p^2p^3|\mathbf{p}|-ip^1|\mathbf{p}|^2\\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}|\end{pmatrix} = +\varepsilon^\mu(p,+),
\end{aligned} \tag{3.143}
$$

$$
\begin{aligned}
(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,-) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ -ip^2(p^3)^2-p^1p^3|\mathbf{p}|-ip^2|\mathbf{p}_\mathrm{T}|^2\\ ip^1(p^3)^2-p^2p^3|\mathbf{p}|+ip^1|\mathbf{p}_\mathrm{T}|^2\\ -ip^1p^2p^3+(p^2)^2|\mathbf{p}|+ip^1p^2p^3+(p^1)^2|\mathbf{p}|\end{pmatrix}\\
&= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ -p^1p^3|\mathbf{p}|-ip^2|\mathbf{p}|^2\\ -p^2p^3|\mathbf{p}|+ip^1|\mathbf{p}|^2\\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}|\end{pmatrix} = -\varepsilon^\mu(p,-).
\end{aligned} \tag{3.144}
$$

归纳起来，有

$$
(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,\lambda) = \lambda\,\varepsilon^\mu(p,\lambda). \tag{3.145}
$$

上式说明极化矢量 $\varepsilon^\mu(p,\lambda)$ 是螺旋度的本征态，本征值为 λ。因此，$\varepsilon^\mu(p,\lambda)$ 描述动量为 $\mathbf{p}$、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda=\pm1$ 对应于两种**横向极化**，$\lambda=0$ 对应于**纵向极化**。

有质量的实矢量场算符 $A^\mu(\mathbf{x},t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态，形式为

$$A^\mu(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right],\tag{3.146}$$

其中 $p^0=E_\mathbf{p}=\sqrt{|\mathbf{p}|^2+m^2}$，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ。容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger=A^\mu(\mathbf{x},t).\tag{3.147}$$

根据 (3.95) 式，共轭动量密度为

$$\begin{aligned}\pi_i\ =\ -\partial_0A_i+\partial_iA_0=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\Big\{&[ip_0\varepsilon_i(\mathbf{p},\lambda)-ip_i\varepsilon_0(\mathbf{p},\lambda)]a_{\mathbf{p},\lambda}e^{-ip\cdot x}\\&+[-ip_0\varepsilon^*_i(\mathbf{p},\lambda)+ip_i\varepsilon^*_0(\mathbf{p},\lambda)]a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big\},\end{aligned}\tag{3.148}$$

引入

$$\tilde\varepsilon_i(\mathbf{p},\lambda)\equiv\varepsilon_i(\mathbf{p},\lambda)-\frac{p_i}{p_0}\varepsilon_0(\mathbf{p},\lambda),\tag{3.149}$$

则有

$$p_0\varepsilon_i(\mathbf{p},\lambda)-p_i\varepsilon_0(\mathbf{p},\lambda)=p_0\tilde\varepsilon_i(\mathbf{p},\lambda),\tag{3.150}$$

从而，可以将共轭动量密度的平面波展开式写得更加紧凑:

$$\pi_i(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\left[\tilde\varepsilon_i(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}-\tilde\varepsilon^*_i(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right].\tag{3.151}$$

易见，它也满足自共轭条件

$$[\pi_i(\mathbf{x},t)]^\dagger=\pi_i(\mathbf{x},t).\tag{3.152}$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned}&\int d^3x\,e^{iq\cdot x}A^\mu\\&=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p-q)\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p+q)\cdot x}\right]\\&=\int d^3p\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\&=\frac{1}{\sqrt{2E_\mathbf{q}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda}+\varepsilon^{\mu*}(-\mathbf{q},\lambda)a^\dagger_{-\mathbf{q},\lambda}e^{2iq^0t}\right]\end{aligned}\tag{3.153}$$

和

$$\int d^3x\,e^{iq\cdot x}\partial_0A^\mu\ =\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\int d^3x\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p-q)\cdot x}-\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p+q)\cdot x}\right]$$

$$
\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm,0} \Big[\varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&\qquad\qquad - \varepsilon^{\mu *}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big] \\
&= \frac{-iq_0}{\sqrt{2E_\mathbf{q}}} \sum_{\lambda=\pm,0} \Big[\varepsilon^\mu(\mathbf{q},\lambda) a_{\mathbf{q},\lambda} - \varepsilon^{\mu *}(-\mathbf{q},\lambda) a^\dagger_{-\mathbf{q},\lambda} e^{2iq^0 t}\Big],
\end{aligned} \tag{3.154}
$$

以及正交归一关系 (3.136)，可得

$$
\begin{aligned}
\varepsilon^*_\mu(\mathbf{q},\lambda') \int d^3x\, e^{iq\cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon^*_\mu(\mathbf{q},\lambda') \frac{-2iq_0}{\sqrt{2E_\mathbf{q}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{q},\lambda) a_{\mathbf{q},\lambda} \\
&= -i\sqrt{2E_\mathbf{q}} \sum_{\lambda=\pm,0} (-\delta_{\lambda'\lambda}) a_{\mathbf{q},\lambda} = i\sqrt{2E_\mathbf{q}}\, a_{\mathbf{q},\lambda'}.
\end{aligned} \tag{3.155}
$$

由 Lorenz 条件 (3.91) 可得

$$
\partial_0 A^0 = -\partial_i A^i, \tag{3.156}
$$

根据 (3.95) 式，有

$$
\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \tag{3.157}
$$

于是，湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$
\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_\mathbf{p}}} \varepsilon^*_\mu(\mathbf{p},\lambda) \int d^3x\, e^{ip\cdot x} (\partial_0 A^\mu - ip_0 A^\mu) \\
&= \frac{-i}{\sqrt{2E_\mathbf{p}}} \int d^3x\, e^{ip\cdot x} [\varepsilon^*_0(\mathbf{p},\lambda)\partial_0 A^0 + \varepsilon^*_i(\mathbf{p},\lambda)\partial_0 A^i - ip_0 \varepsilon^*_\mu(\mathbf{p},\lambda) A^\mu] \\
&= \frac{-i}{\sqrt{2E_\mathbf{p}}} \int d^3x\, e^{ip\cdot x} \big[-\varepsilon^*_0(\mathbf{p},\lambda)\partial_i A^i + \varepsilon^*_i(\mathbf{p},\lambda)\pi_i - \varepsilon^*_i(\mathbf{p},\lambda)\partial_i A^0 \\
&\qquad - ip_0 \varepsilon^*_0(\mathbf{p},\lambda) A^0 - ip_0 \varepsilon^*_i(\mathbf{p},\lambda) A^i \big].
\end{aligned} \tag{3.158}
$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$
\begin{aligned}
\int d^3x\, e^{ip\cdot x}[-\varepsilon^*_0(\mathbf{p},\lambda)\partial_i A^i - \varepsilon^*_i(\mathbf{p},\lambda)\partial_i A^0] &= \int d^3x\, [\varepsilon^*_0(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x}) A^i + \varepsilon^*_i(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x}) A^0] \\
&= \int d^3x\, [ip_i \varepsilon^*_0(\mathbf{p},\lambda) e^{ip\cdot x} A^i + ip_i \varepsilon^*_i(\mathbf{p},\lambda) e^{ip\cdot x} A^0] \\
&= \int d^3x\, e^{ip\cdot x}[i\varepsilon^*_0(\mathbf{p},\lambda) p_i A^i + ip_i \varepsilon^*_i(\mathbf{p},\lambda) A^0],
\end{aligned} \tag{3.159}
$$

从而，有

$$
\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_\mathbf{p}}} \int d^3x\, e^{ip\cdot x} \big[i\varepsilon^*_0(\mathbf{p},\lambda) p_i A^i + \varepsilon^*_i(\mathbf{p},\lambda)\pi_i + ip_i \varepsilon^*_i(\mathbf{p},\lambda) A^0 \\
&\qquad - ip_0 \varepsilon^*_0(\mathbf{p},\lambda) A^0 - ip_0 \varepsilon^*_i(\mathbf{p},\lambda) A^i \big] \\
&= \frac{-i}{\sqrt{2E_\mathbf{p}}} \int d^3x\, e^{ip\cdot x} \{\varepsilon^*_i(\mathbf{p},\lambda)\pi_i - ip^\mu \varepsilon^*_\mu(\mathbf{p},\lambda) A^0 - i[p_0 \varepsilon^*_i(\mathbf{p},\lambda) - p_i \varepsilon^*_0(\mathbf{p},\lambda)] A^i\}.
\end{aligned} \tag{3.160}
$$

再利用四维横向条件 (3.131) 和 (3.150) 式，得到

$$
\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\left[-\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(x) - ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(x)\right] \\
&= \frac{i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\left[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(x) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(x)\right].
\end{aligned}
\tag{3.161}
$$

对上式取厄米共轭，得

$$
a^\dagger_{\mathbf{p},\lambda} = \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{-ip\cdot x}\left[\varepsilon^{i}(\mathbf{p},\lambda)\pi_i(x) - ip_0\tilde{\varepsilon}_i(\mathbf{p},\lambda)A^i(x)\right]. \tag{3.162}
$$

利用等时对易关系 (3.96)，可得湮灭算符与产生算符的对易关系为

$$
\begin{aligned}
&[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\left[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(\mathbf{x},t),\right. \\
&\qquad\qquad \left.\varepsilon^{j}(\mathbf{q},\lambda')\pi_j(\mathbf{y},t) - iq_0\tilde{\varepsilon}_j(\mathbf{q},\lambda')A^j(\mathbf{y},t)\,\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\left\{-iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t), A^j(\mathbf{y},t)]\right. \\
&\qquad\qquad \left.+\, ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)]\,\right\} \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,[-q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')\delta^j{}_i - p_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')\delta^i{}_j] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\,[-E_{\mathbf{q}}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{q},\lambda') - E_{\mathbf{p}}\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')] \\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda')].
\end{aligned}
\tag{3.163}
$$

根据定义式 (3.149)、四维横向条件 (3.131) 和正交归一关系 (3.136)，有

$$
\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') - \frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') + \frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{\mu*}(\mathbf{p},\lambda)\varepsilon_\mu(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'},
\end{aligned}
\tag{3.164}
$$

取复共轭，可得

$$
\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'}. \tag{3.165}
$$

于是，

$$
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,(-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{3.166}
$$

另一方面，

$$
[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}]
$$

$$
\begin{aligned}
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\big[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t)+ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(\mathbf{x},t),\\
&\qquad\qquad \varepsilon^{j*}(\mathbf{q},\lambda')\pi_j(\mathbf{y},t)+iq_0\tilde{\varepsilon}_j^*(\mathbf{q},\lambda')A^j(\mathbf{y},t)\big]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\big\{iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j^*(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t),A^j(\mathbf{y},t)]\\
&\qquad\qquad +ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')[A^i(\mathbf{x},t),\pi_j(\mathbf{y},t)]\big\}\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,[q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j^*(\mathbf{q},\lambda')\delta^j{}_i-p_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')\delta^i{}_j]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, e^{i(p^0+q^0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\,[E_{\mathbf{q}}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(\mathbf{q},\lambda')-E_{\mathbf{p}}\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')]\\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{2iE_{\mathbf{p}}t}\,[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda')-\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')].
\end{aligned}\tag{3.167}
$$

对四维横向条件 (3.131) 取复共轭，得

$$p_\mu\varepsilon^{\mu*}(\mathbf{p},\lambda)=p_0\varepsilon^{0*}(\mathbf{p},\lambda)+p_i\varepsilon^{i*}(\mathbf{p},\lambda)=0.\tag{3.168}$$

将上式中的 $\mathbf{p}$ 替换成 $-\mathbf{p}$，得

$$p_0\varepsilon^{0*}(-\mathbf{p},\lambda)-p_i\varepsilon^{i*}(-\mathbf{p},\lambda)=0.\tag{3.169}$$

因此，有

$$p_i\varepsilon^{i*}(\mathbf{p},\lambda)=-p_0\varepsilon^{0*}(\mathbf{p},\lambda),\quad -p_i\varepsilon^{i*}(-\mathbf{p},\lambda)=-p_0\varepsilon^{0*}(-\mathbf{p},\lambda),\tag{3.170}$$

或者写成

$$\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)=p_0\varepsilon^{0*}(\mathbf{p},\lambda),\quad -\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda)=p_0\varepsilon^{0*}(-\mathbf{p},\lambda).\tag{3.171}$$

从而，可得

$$
\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\left[\varepsilon_i^*(-\mathbf{p},\lambda)+\frac{p_i}{p_0}\varepsilon_0^*(-\mathbf{p},\lambda)\right]\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda')+\frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda')\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda')-\frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda')\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda')-\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda'),
\end{aligned}\tag{3.172}
$$

$$
\begin{aligned}
\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') &= \left[\varepsilon_i^*(\mathbf{p},\lambda)-\frac{p_i}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)\right]\varepsilon^{i*}(-\mathbf{p},\lambda')\\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\frac{1}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)p_i\varepsilon^{i*}(-\mathbf{p},\lambda')\\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\frac{1}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)p_0\varepsilon^{0*}(-\mathbf{p},\lambda')\\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\varepsilon_0^*(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda').
\end{aligned}\tag{3.173}
$$

可见，$\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda')-\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')=0$，故

$$[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}]=0.\tag{3.174}$$

综上，产生湮灭算符的对易关系为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0. \tag{3.175}$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \tag{3.176}$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \tag{3.177}$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \tag{3.178}$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$，因此，

$$A^0 = -\frac{1}{m^2}\partial_\mu F^{\mu 0} = -\frac{1}{m^2}\partial_i F^{i0} = -\frac{1}{m^2}\partial_i \pi^i = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}. \tag{3.179}$$

从而，可得

$$-\boldsymbol{\pi}\cdot\dot{\mathbf{A}} = \boldsymbol{\pi}\cdot(\boldsymbol{\pi}+\nabla A_0) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) - A_0(\nabla\cdot\boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2. \tag{3.180}$$

另一方面，

$$\frac{1}{2}F_{0i}F^{0i} = \frac{1}{2}\pi_i\pi^i = -\frac{1}{2}\boldsymbol{\pi}^2. \tag{3.181}$$

利用 (1.84) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\partial^m A^n = \varepsilon^{ijk}\varepsilon^{kmn}\partial^m A^n = -\varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n, \tag{3.182}$$

从而，

$$\begin{aligned}\frac{1}{4}F_{ij}F^{ij} &= \frac{1}{4}F^{ij}F^{ij} = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{ijl}\varepsilon^{lpq}\partial_p A^q = \frac{1}{4}2\delta^{kl}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{lpq}\partial_p A^q \\ &= \frac{1}{2}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{kpq}\partial_p A^q = \frac{1}{2}(\nabla\times\mathbf{A})^2.\end{aligned} \tag{3.183}$$

于是，有

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2. \tag{3.184}$$

根据 (1.119) 式，有质量矢量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi_i\partial_0 A^i - \mathcal{L} = \pi_i\partial_0 A^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \\ &= -\boldsymbol{\pi}\cdot\dot{\mathbf{A}} - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2)\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2 - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}m^2\mathbf{A}^2\\
&= \frac{1}{2}\boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 + \frac{1}{2}m^2\mathbf{A}^2. \qquad (3.185)
\end{aligned}
$$

上式最后一行第二项是一个全散度，对全空间积分时它没有贡献。于是，哈密顿量为

$$
H = \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\left[\boldsymbol{\pi}^2 + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2 + (\nabla\times\mathbf{A})^2 + m^2\mathbf{A}^2\right]. \qquad (3.186)
$$

下面逐项进行计算。

哈密顿量的第一项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,\boldsymbol{\pi}^2\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,(ip_0)(iq_0)\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\cdot\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&= -\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\qquad-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} + \tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big]\\
&= -\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{-\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&\qquad+\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_{\mathbf{p}}}\,E_{\mathbf{p}}^2\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'} + \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&\qquad-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big]. \qquad (3.187)
\end{aligned}
$$

第二项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\frac{(ip_0)(iq_0)}{m^2}\left[i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\times\left[i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} + i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&= -\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}\,m^2}\Big\{-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\qquad-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} - [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - [\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big\}\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{q}}\,m^2}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\varepsilon}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\varepsilon}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad +[\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\varepsilon}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big)\\
&\qquad\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\varepsilon}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\varepsilon}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad +[\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big)\Big\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_\mathbf{p}}\frac{E^2_\mathbf{p}}{m^2}\Big\{[\mathbf{p}\cdot\tilde{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}\\
&\quad + [\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\varepsilon}(\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - [\mathbf{p}\cdot\tilde{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\varepsilon}(-\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\\
&\quad - [\mathbf{p}\cdot\tilde{\varepsilon}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\varepsilon}^*(-\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t}\Big\}.
\end{aligned}\tag{3.188}
$$

第三项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,(\nabla\times\mathbf{A})^2\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[i\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} - i\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big]\\
&\qquad\qquad \cdot\Big[i\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - i\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\Big]\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big\{[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} - [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\quad - [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big\}\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big)\\
&\qquad\qquad - \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big)\Big\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_\mathbf{p}}\Big\{[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}\\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} + [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t}\Big\}.
\end{aligned}\tag{3.189}
$$

第四项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,m^2\mathbf{A}^2\\
&= \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;m^2}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left[\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}+\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
=&\ \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\ m^2}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}+\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big]\\
=&\ \frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\ m^2}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&+\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
=&\ \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}m^2\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&+\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}\tag{3.190}
$$

综合起来，哈密顿量化为

$$
\begin{aligned}
H=&\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}\Big[f_1(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+f_1^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&+f_2(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}+f_2^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big],
\end{aligned}\tag{3.191}
$$

其中，

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda')\equiv&\ E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')+\frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda'),
\end{aligned}\tag{3.192}
$$

$$
\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda')\equiv&\ -E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')-\frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda').
\end{aligned}\tag{3.193}
$$

现在，我们计算 $f_1(\mathbf{p},\lambda,\lambda')$。由 (3.149)、(3.171) 和 (3.136) 式，可得

$$
\begin{aligned}
&\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')=\left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)-\frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\mathbf{p}}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]\\
=&\ \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)+\frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\ \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda')-\frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)+\frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\ -\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda')+\left(\frac{|\mathbf{p}|^2}{p_0^2}-1\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\ \delta_{\lambda\lambda'}-\frac{m^2}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}\tag{3.194}
$$

另一方面，

$$
\begin{aligned}
&[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]\\
=&\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]\\
=&[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]-\frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0^*(\mathbf{p},\lambda')\\
&-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]+\frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\,p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda')-\frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda')+\frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\left(p_0^2-2|\mathbf{p}|^2+\frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')=\left[p_0^2-|\mathbf{p}|^2+\frac{|\mathbf{p}|^2}{p_0^2}(|\mathbf{p}|^2-p_0^2)\right]\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
=&\left(m^2-m^2\frac{|\mathbf{p}|^2}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')=\frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}
\tag{3.195}
$$

对于任意空间矢量 $\mathbf{a}$ 和 $\mathbf{b}$，利用 (1.84) 式，有

$$
\begin{aligned}
(\mathbf{p}\times\mathbf{a})\cdot(\mathbf{p}\times\mathbf{b})&=\varepsilon^{ijk}p^ja^k\varepsilon^{imn}p^mb^n=(\delta^{jm}\delta^{kn}-\delta^{jn}\delta^{km})p^ja^kp^mb^n\\
&=p^ja^kp^jb^k-p^ja^kp^kb^j=|\mathbf{p}|^2\,\mathbf{a}\cdot\mathbf{b}-(\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b}),
\end{aligned}
\tag{3.196}
$$

从而，可得

$$
\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]&=|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]\\
&=|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda')\\
&=|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda').
\end{aligned}
\tag{3.197}
$$

于是，(3.192) 式化为

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda')&=E_{\mathbf{p}}^2\delta_{\lambda\lambda'}-m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')+m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&\quad+|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda')+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')\\
&=E_{\mathbf{p}}^2\delta_{\lambda\lambda'}+E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda')\\
&=E_{\mathbf{p}}^2\delta_{\lambda\lambda'}-E_{\mathbf{p}}^2\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda')=2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\end{aligned}
\tag{3.198}
$$

因此，

$$
f_1(\mathbf{p},\lambda,\lambda')=f_1^*(\mathbf{p},\lambda,\lambda')=2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\tag{3.199}
$$

接着，我们计算 $f_2(\mathbf{p},\lambda,\lambda')$。由 (3.149) 和 (3.171) 式，可得

$$
\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')=\left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)-\frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')+\frac{\mathbf{p}}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right]
$$

$$
\begin{aligned}
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^0(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{1}{E_{\mathbf{p}}^2}(2E_{\mathbf{p}}^2 - |\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda'). \qquad (3.200)
\end{aligned}
$$

另一方面,

$$
\begin{aligned}
& [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')] \\
&= \left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right] \\
&= [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] + \frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0(-\mathbf{p},\lambda') \\
&\quad -\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] - \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= -p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda') + \frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&\quad +\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= \left(-p_0^2 + 2|\mathbf{p}|^2 - \frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') = -\frac{1}{E_{\mathbf{p}}^2}(E_{\mathbf{p}}^2 - |\mathbf{p}|^2)^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= -\frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda'), \qquad (3.201)
\end{aligned}
$$

而

$$
\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] &= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] \\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(-\mathbf{p},\lambda') \\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda'). \qquad (3.202)
\end{aligned}
$$

于是，(3.193) 式化为

$$
\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda') &= -E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') + m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&\quad +|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda') + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') \\
&= (-2E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 + E_{\mathbf{p}}^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') = 0. \qquad (3.203)
\end{aligned}
$$

因此,

$$
f_2(\mathbf{p},\lambda,\lambda') = f_2^*(\mathbf{p},\lambda,\lambda') = 0. \qquad (3.204)
$$

将 (3.199) 和 (3.204) 式代入 (3.191) 式，再利用产生湮灭算符的对易关系 (3.175)，可得有质量矢量场的哈密顿量为

$$
H = \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'}\right) = \sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}\right)
$$

$$= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\, a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} \frac{3}{2} E_{\mathbf{p}}. \tag{3.205}$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和，第二项是零点能。

根据 (1.158) 式，有质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi_i \nabla A^i \\
&= -\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0) \left[\tilde{\varepsilon}_i(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}\right] \\
&\quad \times \left[i\mathbf{q}\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{q},\lambda'} e^{-iq\cdot x} - i\mathbf{q}\varepsilon^{i*}(\mathbf{q},\lambda') a^\dagger_{\mathbf{q},\lambda'} e^{iq\cdot x}\right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q\; p_0\mathbf{q}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \Big[-\tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{-i(p-q)\cdot x} \\
&\quad - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^i(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{i(p-q)\cdot x} + \tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} \\
&\quad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^{i*}(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{i(p+q)\cdot x} \Big] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p\, d^3q\; p_0\mathbf{q}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \Big\{ -\delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{-i(p_0-q_0)t} \\
&\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^i(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{i(p_0-q_0)t}\Big] \\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0+q_0)t} \\
&\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^{i*}(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{i(p_0+q_0)t}\Big] \Big\} \\
&= -\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^{i*}(\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda'} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^i(\mathbf{p},\lambda') a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'} \\
&\quad + \tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^i(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^{i*}(-\mathbf{p},\lambda') a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned} \tag{3.206}$$

由 (3.149) 和 (3.170) 式可得

$$\begin{aligned}
\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') &= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0} p_i \varepsilon^i(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda'),
\end{aligned} \tag{3.207}$$

从而，有

$$\begin{aligned}
&-\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p},\lambda) \varepsilon^i(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) \varepsilon^{i*}(-\mathbf{p},\lambda') a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t}\right] \\
= &\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big\{ [\varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\quad + [\varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \varepsilon_0^*(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda')] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t} \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} \Big\{ [\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda) - \varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)] a_{-\mathbf{p},\lambda'} a_{\mathbf{p},\lambda} e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda) - \varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)] a^\dagger_{-\mathbf{p},\lambda'} a^\dagger_{\mathbf{p},\lambda} e^{2iE_{\mathbf{p}}t} \Big\} \\
&= -\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big\{ [\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda) - \varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda) - \varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t} \Big\}.
\end{aligned} \tag{3.208}
$$

上式第二步进行了 $\mathbf{p} \to -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换，由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和，这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.175)。留意到第一步与第三步的结果互为相反数，可知上式为零。因此，(3.206) 式最后两行方括号中最后两项没有贡献。再利用 (3.165) 式，可得

$$
\begin{aligned}
\mathbf{P} &= -\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-\delta_{\lambda\lambda'} a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda'} - \delta_{\lambda\lambda'} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'} \right] = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda} + a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} \right] \\
&= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p}\, a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} + \frac{3}{2}\delta^{(3)}(\mathbf{0}) \int d^3p\, \mathbf{p} = \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p}\, a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}.
\end{aligned} \tag{3.209}
$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m = 0$ 时，由 (3.104) 和 (3.105) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 的形式不变，但 (3.115) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p},3)$ 的良好定义。实际上，在满足正确归一化的条件下，$m = 0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面，由于无质量矢量粒子的动量 p^μ 的内积为 $p^2 = 0$，也不能像 (3.119) 式那样将类时极化矢量 $e^\mu(\mathbf{p},0)$ 取为正比于 p^μ 的矢量，否则将出现 $e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0) = 0$ 而不能得到正确的归一化。因此，我们需要重新定义 $e^\mu(\mathbf{p},3)$ 和 $e^\mu(\mathbf{p},0)$。

在用 (3.104) 和 (3.105) 式定义 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 时，我们已经选取了一个特定的惯性参考系。在这个参考系中，可以定义一个类时单位矢量

$$n^\mu = (1,0,0,0), \tag{3.210}$$

它的 Lorentz 不变内积是

$$n^2 = 1. \tag{3.211}$$

然后，将类时极化矢量 $e^\mu(\mathbf{p},0)$ 在此参考系中的形式就取为 n^μ，即

$$e^\mu(\mathbf{p},0) = n^\mu. \tag{3.212}$$

$e^\mu(\mathbf{p},0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面，纵向极化矢量 $e^\mu(\mathbf{p},3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式：

$$e^\mu(\mathbf{p},3)=\frac{p^\mu-(p\cdot n)n^\mu}{p\cdot n}. \tag{3.213}$$

$p^2=(p^0)^2-|\mathbf{p}|^2=0$ 表明

$$p^0=|\mathbf{p}|, \tag{3.214}$$

从而，$e^\mu(\mathbf{p},3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p},3)=\frac{p^\mu-(p\cdot n)n^\mu}{p\cdot n}=\frac{p^\mu-p^0n^\mu}{p^0}=\left(0,\frac{\mathbf{p}}{|\mathbf{p}|}\right). \tag{3.215}$$

这样定义的 $e^\mu(\mathbf{p},0)$ 和 $e^\mu(\mathbf{p},3)$ 满足正交归一关系 (3.98)：

$$\begin{aligned}
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0) &= n^2=1,\quad e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3)=-\frac{\mathbf{p}\cdot\mathbf{p}}{|\mathbf{p}|^2}=-1; &(3.216)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},1) &= e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},2)=e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},3)=0; &(3.217)\\
e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},i) &= -\frac{1}{|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2. &(3.218)
\end{aligned}$$

此外，可以验证，由 (3.104)、(3.105)、(3.212) 和 (3.213) 式定义的这组极化矢量确实满足完备性关系 (3.103)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)\\
={}& e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)-e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)-e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2)-e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\
={}& \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}-\frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix}0&0&0&0\\0&p^1p^3p^1p^3&p^1p^3p^2p^3&-p^1p^3|\mathbf{p}_\mathrm{T}|^2\\0&p^2p^3p^1p^3&p^2p^3p^2p^3&-p^2p^3|\mathbf{p}_\mathrm{T}|^2\\0&-|\mathbf{p}_\mathrm{T}|^2p^1p^3&-|\mathbf{p}_\mathrm{T}|^2p^2p^3&|\mathbf{p}_\mathrm{T}|^4\end{pmatrix}\\
&-\frac{1}{|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix}0&0&0&0\\0&p^2p^2&-p^2p^1&0\\0&-p^1p^2&p^1p^1&0\\0&0&0&0\end{pmatrix}-\frac{1}{|\mathbf{p}|^2}\begin{pmatrix}0&0&0&0\\0&p^1p^1&p^1p^2&p^1p^3\\0&p^2p^1&p^2p^2&p^2p^3\\0&p^3p^1&p^3p^2&p^3p^3\end{pmatrix}\\
={}& \begin{pmatrix}1&0&0&0\\0&-\frac{(p^1)^2(p^3)^2+(p^2)^2|\mathbf{p}|^2+(p^1)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{p^1p^3}{|\mathbf{p}|^2}-\frac{p^1p^3}{|\mathbf{p}|^2}\\0&-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&-\frac{(p^2)^2(p^3)^2+(p^1)^2|\mathbf{p}|^2+(p^2)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}&\frac{p^2p^3}{|\mathbf{p}|^2}-\frac{p^2p^3}{|\mathbf{p}|^2}\\0&\frac{p^1p^3}{|\mathbf{p}|^2}-\frac{p^1p^3}{|\mathbf{p}|^2}&\frac{p^2p^3}{|\mathbf{p}|^2}-\frac{p^2p^3}{|\mathbf{p}|^2}&-\frac{|\mathbf{p}_\mathrm{T}|^2+(p^3)^2}{|\mathbf{p}|^2}\end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \tag{3.219}
$$

不过，$e^\mu(\mathbf{p},0)$ 和 $e^\mu(\mathbf{p},3)$ 都不满足四维横向条件：

$$
p_\mu e^\mu(\mathbf{p},0) = p\cdot n = p^0 = |\mathbf{p}|, \quad p_\mu e^\mu(\mathbf{p},3) = -\frac{\mathbf{p}\cdot\mathbf{p}}{|\mathbf{p}|} = -|\mathbf{p}| = -p\cdot n. \tag{3.220}
$$

横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 具有求和关系

$$
\begin{aligned}
-\sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) &= \sum_{\sigma=1}^{2} g_{\sigma\sigma} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu} - g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0) - g_{33}e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3) \\
&= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu-(p\cdot n)n_\mu}{p\cdot n}\frac{p_\nu-(p\cdot n)n_\nu}{p\cdot n} \\
&= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu p_\nu-(p\cdot n)p_\mu n_\nu-(p\cdot n)p_\nu n_\mu+(p\cdot n)^2 n_\mu n_\nu}{(p\cdot n)^2} \\
&= g_{\mu\nu} + \frac{p_\mu p_\nu}{(p\cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n},
\end{aligned} \tag{3.221}
$$

即

$$
\sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}. \tag{3.222}
$$

根据 (3.129) 式，作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 满足

$$
\begin{aligned}
\sum_{\lambda=\pm} \varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) &= \frac{1}{2}[e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)] \\
&\quad +\frac{1}{2}[-e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][-e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)] \\
&= e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)+e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2) = \sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma),
\end{aligned} \tag{3.223}
$$

因而具有**求和关系**

$$
\sum_{\lambda=\pm} \varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}. \tag{3.224}
$$

四维横向条件 $p_\mu \varepsilon^\mu(\mathbf{p},\pm) = 0$ 在上式中体现为

$$
p^\nu \sum_{\lambda=\pm} \varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -p_\mu - \frac{p_\mu p^2}{(p\cdot n)^2} + \frac{p_\mu(p\cdot n)+p^2 n_\mu}{p\cdot n} = -p_\mu + p_\mu = 0. \tag{3.225}
$$

3.4.2　无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中，令参数 $m=0$，就得到自由无质量实矢量场 $A^\mu(x)$ 的拉氏量

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.226}$$

其中 $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$。同理，令 Proca 方程中 $m=0$，就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu} = 0. \tag{3.227}$$

根据 1.5 节的讨论，这个方程就是*无源的* **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子，**光子**是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作**规范变换** (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu\chi(x), \tag{3.228}$$

其中，作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数，依赖于时空坐标，因而这样的变换是**局域** (local) 变换。在此规范变换下，场强张量不变：

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu[A^\nu(x)+\partial^\nu\chi(x)] - \partial^\nu[A^\mu(x)+\partial^\mu\chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu\partial^\nu\chi(x) - \partial^\nu\partial^\mu\chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \tag{3.229}$$

因而，拉氏量 (3.226) 和无源 Maxwell 方程 (3.227) 都不会改变，这称为**规范对称性** (gauge symmetry)。

在经典电动力学中，这种对称性广为人知，它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定，因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$，也就是说，不依赖于**规范**的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算，常常将规范固定下来，使得计算过程依赖于选取的规范，不过，最后得出的可观测量必须是*规范不变* (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**，规范条件为

$$\partial_\mu A^\mu = 0. \tag{3.230}$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同，但是，在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件，而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$，令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2\chi(x) = -\partial_\mu A^\mu(x), \tag{3.231}$$

那么，作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件：

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2\chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \tag{3.232}$$

但是，经过这种变换之后，矢量场仍然没有被唯一地确定：对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$，取满足齐次波动方程

$$\partial^2\tilde{\chi}(x) = 0 \tag{3.233}$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换，都能得到满足 Lorenz 规范条件的另一个矢量场 $\tilde{A}'^\mu(x)$。可见，存在无穷多个规范等价的矢量场，它们描述相同的物理，而且全都满足 Lorenz 规范条件 (3.230)。

矢量场 $A^\mu(x)$ 有 4 个分量，因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是，上述规范等价性表明，$A^\mu(x)$ 并没有 3 个独立的自由度，否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上，无质量矢量场 $A^\mu(x)$ 只具有 *2 个独立的自由度*，也就是说，有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论：电磁波具有 2 种独立的极化态，以螺旋度 λ 来表征的话，就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有*质量*矢量场 $A^\mu(x)$ 的量子化程序时，由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度，因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难，因为 Proca 方程要求 $A^0(x)$ 不是独立变量，而是由 (3.179) 式决定的：

$$A^0 = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}. \tag{3.234}$$

于是，以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的，自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时，(3.234) 式显然不能成立。因此，对于无质量矢量场，最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度，可以在拉氏量中增加一个不会影响最终物理结果的项：

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.235}$$

其中 ξ 是一个可以自由选取的实参数。可以看出，在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.230) 的情况下，由 (3.235) 式定义的 $\mathcal{L}_1$ 等价于由 (3.226) 式定义的 $\mathcal{L}$。新增的项 $-\frac{1}{2}\xi(\partial_\mu A^\mu)^2$ 破坏了规范对称性，相当于把规范固定下来，因而称为**规范固定项** (gauge-fixing term)。可以将 $\mathcal{L}_1$ 展开为

$$\mathcal{L}_1 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.236}$$

从而，A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}_1}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi(\partial_\nu A^\nu)\frac{\partial(\partial_\sigma A^\sigma)}{\partial(\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0}\partial_\nu A^\nu, \tag{3.237}$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi\partial_\mu A^\mu. \tag{3.238}$$

因此，$\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0。

现在，正则量子化程序要求算符 A^μ 和 π_μ 满足如下等时对易关系：

$$[A^\mu(\mathbf{x},t),\pi_\nu(\mathbf{y},t)]=i\delta^\mu{}_\nu\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [A^\mu(\mathbf{x},t),A^\nu(\mathbf{y},t)]=[\pi_\mu(\mathbf{x},t),\pi_\nu(\mathbf{y},t)]=0. \tag{3.239}$$

但是，这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子，利用 (3.238) 式，可得

$$[A^0(\mathbf{x},t),\partial_\mu A^\mu(\mathbf{y},t)]=-\frac{1}{\xi}[A^0(\mathbf{x},t),\pi_0(\mathbf{y},t)]=-\frac{i}{\xi}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.240}$$

上式在 $\mathbf{x}=\mathbf{y}$ 处非零，因而必有 $\partial_\mu A^\mu\neq 0$。所以，A^μ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.230)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$，但对于量子场 $A^\mu(x)$ 来说限制太强了，下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu A_\nu)}=-\partial^\mu A^\nu+\partial^\nu A^\mu-\xi g^{\mu\nu}(\partial_\rho A^\rho),\quad \frac{\partial\mathcal{L}_1}{\partial A_\nu}=0, \tag{3.241}$$

可得，与 $\mathcal{L}_1$ 对应的 Euler-Lagrange 方程为

$$0=\partial_\mu\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu A_\nu)}-\frac{\partial\mathcal{L}_1}{\partial A_\nu}=-\partial^2A^\nu+\partial^\nu\partial_\mu A^\mu-\xi g^{\mu\nu}\partial_\mu(\partial_\rho A^\rho)=-\partial^2A^\nu+(1-\xi)\partial^\nu(\partial_\rho A^\rho), \tag{3.242}$$

即

$$\partial^2A^\mu-(1-\xi)\partial^\mu(\partial_\nu A^\nu)=0. \tag{3.243}$$

若取 $\xi=1$，则上式化为 **d'Alembert 方程**

$$\partial^2A^\mu(x)=0, \tag{3.244}$$

可以看作无质量情况下的 Klein-Gordon 方程。可见，将规范固定参数取为

$$\xi=1 \tag{3.245}$$

将有利于简化计算，这种取法称为 **Feynman 规范**，本节后续计算采用这个规范。在 Feynman 规范下，拉氏量化为

$$\begin{aligned}\mathcal{L}_1&=-\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu+\frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu-\frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu)\\&=-\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu+\frac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu)-\frac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu-\frac{1}{2}\partial^\mu(A_\mu\partial_\nu A^\nu)+\frac{1}{2}A_\mu\partial^\mu\partial_\nu A^\nu\\&=-\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu+\frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu-A^\mu\partial_\nu A^\nu).\end{aligned} \tag{3.246}$$

上式最后一行第二项是一个全散度，它不会影响作用量和运动方程，可以舍弃。因此，可以采用更加简化的拉氏量

$$\mathcal{L}_2=-\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \tag{3.247}$$

此时，共轭动量密度为

$$\pi_\mu=\frac{\partial\mathcal{L}_2}{\partial(\partial^0A^\mu)}=-\partial_0A_\mu. \tag{3.248}$$

对于 d'Alembert 方程 (3.244)，平面波解的正能解和负能解分别正比于 $\exp(-ip\cdot x)$ 和 $\exp(ip\cdot x)$，其中

$$p^0 = E_{\mathbf{p}} = |\mathbf{p}|. \tag{3.249}$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p},\sigma)$，可以对无质量矢量场 $A^\mu(\mathbf{x},t)$ 作如下平面波展开：

$$A^\mu(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right). \tag{3.250}$$

容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t). \tag{3.251}$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x},t) = -\partial_0 A_\mu = \int \frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e_\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right), \tag{3.252}$$

它也满足自共轭条件

$$[\pi_\mu(\mathbf{x},t)]^\dagger = \pi_\mu(\mathbf{x},t). \tag{3.253}$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}A^\mu &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\, \frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right]
\end{aligned} \tag{3.254}$$

和

$$\begin{aligned}
&\int d^3x\, e^{iq\cdot x}\partial_0 A^\mu \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right],
\end{aligned} \tag{3.255}$$

以及正交归一关系 (3.98)，可得

$$
\begin{aligned}
e_\mu(\mathbf{q},\sigma')\int d^3x\,e^{iq\cdot x}\,(\partial_0 A^\mu - iq_0 A^\mu) &= e_\mu(\mathbf{q},\sigma')\,\frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma}\\
&= -i\sqrt{2E_\mathbf{q}}\sum_{\sigma=0}^{3} g_{\sigma'\sigma}a_{\mathbf{q};\sigma'} = -i\sqrt{2E_\mathbf{q}}\,g_{\sigma'\sigma'}a_{\mathbf{q};\sigma'}. \qquad (3.256)
\end{aligned}
$$

注意，虽然上式出现了重复的指标 σ'，但此处不需要对 σ' 求和。于是，有

$$
a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_\mathbf{p}}}\,g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)\int d^3x\,e^{ip\cdot x}\,(\partial_0 A^\mu - ip_0 A^\mu). \qquad (3.257)
$$

对上式取厄米共轭，得

$$
a^\dagger_{\mathbf{p};\sigma} = \frac{-i}{\sqrt{2E_\mathbf{p}}}\,g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)\int d^3x\,e^{-ip\cdot x}\,(\partial_0 A^\mu + ip_0 A^\mu). \qquad (3.258)
$$

根据等时对易关系 (3.239)，湮灭算符与产生算符的对易关系为

$$
\begin{aligned}
[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] &= \frac{g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\\
&\qquad\times[\partial_0A^\mu(\mathbf{x},t)-ip_0A^\mu(\mathbf{x},t),\ \partial_0A^\nu(\mathbf{y},t)+iq_0A^\nu(\mathbf{y},t)]\\
&= \frac{g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\\
&\qquad\times[-\pi^\mu(\mathbf{x},t)-ip_0A^\mu(\mathbf{x},t),\ -\pi^\nu(\mathbf{y},t)+iq_0A^\nu(\mathbf{y},t)]\\
&= \frac{g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\\
&\qquad\times\{-iq_0\,[\pi^\mu(\mathbf{x},t),A^\nu(\mathbf{y},t)]+ip_0\,[A^\mu(\mathbf{x},t),\pi^\nu(\mathbf{y},t)]\}\\
&= \frac{g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\,[-(p_0+q_0)g^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})]\\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,g_{\sigma\sigma}g_{\sigma'\sigma'}e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,g_{\sigma\sigma}g_{\sigma'\sigma'}e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')e^{i(E_\mathbf{p}-E_\mathbf{q})t}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= -(2\pi)^3 g_{\sigma\sigma}g_{\sigma'\sigma'}e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= -(2\pi)^3 g_{\sigma\sigma}g_{\sigma'\sigma'}g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}). \qquad (3.259)
\end{aligned}
$$

倒数第二步用到正交归一关系 (3.98)。另一方面，两个湮灭算符之间的对易关系为

$$
\begin{aligned}
[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\\
&\qquad\times[\partial_0A^\mu(\mathbf{x},t)-ip_0A^\mu(\mathbf{x},t),\ \partial_0A^\nu(\mathbf{y},t)-iq_0A^\nu(\mathbf{y},t)]\\
&= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma')\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\\
&\qquad\times[-\pi^\mu(\mathbf{x},t)-ip_0A^\mu(\mathbf{x},t),\ -\pi^\nu(\mathbf{y},t)-iq_0A^\nu(\mathbf{y},t)]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)} \\
&\qquad \times \{iq_0\,[\pi^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] + ip_0\,[A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)]\} \\
&= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\left[(q_0-p_0)g^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, g_{\sigma\sigma}g_{\sigma'\sigma'}e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \int d^3x\, e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, g_{\sigma\sigma}g_{\sigma'\sigma'}e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0.
\end{aligned}
\tag{3.260}
$$

归纳起来，产生湮灭算符的对易关系为

$$
[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a^\dagger_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = 0. \tag{3.261}
$$

3.4.4　哈密顿量和总动量

根据 (1.119)、(3.248) 和 (3.247) 式，无质量矢量场的哈密顿量密度是

$$
\begin{aligned}
\mathcal{H} &= \pi_\mu\partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu \\
&= -\frac{1}{2}(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_i A_\mu)\partial^i A^\mu = -\frac{1}{2}\left[\pi_\mu\pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right].
\end{aligned}
\tag{3.262}
$$

于是，哈密顿量表达为

$$
\begin{aligned}
H &= \int d^3x\, \mathcal{H} = -\frac{1}{2}\int d^3x\, [\pi_\mu\pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \Big[(ip_0)(iq_0)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\left(a_{\mathbf{q},\sigma'}e^{-iq\cdot x} - a^\dagger_{\mathbf{q},\sigma'}e^{iq\cdot x}\right) \\
&\qquad\quad + \left(i\mathbf{p}\, a_{\mathbf{p};\sigma}e^{-ip\cdot x} - i\mathbf{p}\, a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\, a_{\mathbf{q};\sigma'}e^{-iq\cdot x} - i\mathbf{q}\, a^\dagger_{\mathbf{q};\sigma'}e^{iq\cdot x}\right)\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')\Big[(p_0q_0+\mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p-q)\cdot x} \\
&\qquad + (p_0q_0+\mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p-q)\cdot x} + (p_0q_0-\mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p+q)\cdot x} \\
&\qquad + (-p_0q_0-\mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p+q)\cdot x}\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')(p_0q_0+\mathbf{p}\cdot\mathbf{q}) \\
&\qquad \times\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p_0-q_0)t}\right] \\
&\qquad\quad - \delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p_0+q_0)t}\right]\Big\} \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E^2_{\mathbf{p}}+|\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E_\mathbf{p}^2-|\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_\mathbf{p}t}+a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_\mathbf{p}t}\right)\Big]\\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\,e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)\\
&= -\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\,g_{\sigma\sigma'}\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)=\int\frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})\,\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)\\
&= \int\frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma})\,+(2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2\\
&= \int\frac{d^3p}{(2\pi)^3}\,E_\mathbf{p}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}+\sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)\,+(2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}\,2E_\mathbf{p}.
\end{aligned}\tag{3.263}
$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负，与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规，时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论，将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态，满足

$$
a_{\mathbf{p};\sigma}\,|0\rangle=0,\quad \langle 0|0\rangle=1,\quad H\,|0\rangle=E_{\text{vac}}\,|0\rangle\,,\quad E_{\text{vac}}=2\delta^{(3)}(\mathbf{0})\int d^3p\,E_\mathbf{p}.\tag{3.264}
$$

动量为 $\mathbf{p}$、极化态为 σ 的单粒子态定义为

$$
|\mathbf{p};\sigma\rangle\equiv\sqrt{2E_\mathbf{p}}\,a^\dagger_{\mathbf{p};\sigma}\,|0\rangle\,.\tag{3.265}
$$

从而，由

$$
\begin{aligned}
[H,a^\dagger_{\mathbf{p};\sigma}] &= \int\frac{d^3q}{(2\pi)^3}\,E_\mathbf{q}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})[a^\dagger_{\mathbf{q};\sigma'}a_{\mathbf{q};\sigma'},a^\dagger_{\mathbf{p};\sigma}]=\int\frac{d^3q}{(2\pi)^3}\,E_\mathbf{q}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})a^\dagger_{\mathbf{q};\sigma'}[a_{\mathbf{q};\sigma'},a^\dagger_{\mathbf{p};\sigma}]\\
&= \int\frac{d^3q}{(2\pi)^3}\,E_\mathbf{q}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})a^\dagger_{\mathbf{q};\sigma'}(2\pi)^3(-g_{\sigma'\sigma})\delta^{(3)}(\mathbf{q}-\mathbf{p})\\
&= E_\mathbf{p}\sum_{\sigma'=0}^{3}g_{\sigma'\sigma'}g_{\sigma'\sigma}a^\dagger_{\mathbf{p};\sigma'}=E_\mathbf{p}a^\dagger_{\mathbf{p};\sigma}
\end{aligned}\tag{3.266}
$$

可得

$$
\begin{aligned}
H\,|\mathbf{p};\sigma\rangle &= \sqrt{2E_\mathbf{p}}\,Ha^\dagger_{\mathbf{p};\sigma}\,|0\rangle=\sqrt{2E_\mathbf{p}}\,(E_\mathbf{p}a^\dagger_{\mathbf{p};\sigma}+a^\dagger_{\mathbf{p};\sigma}H)\,|0\rangle\\
&= \sqrt{2E_\mathbf{p}}\,(E_\mathbf{p}+E_{\text{vac}})a^\dagger_{\mathbf{p};\sigma}\,|0\rangle=(E_\mathbf{p}+E_{\text{vac}})\,|\mathbf{p};\sigma\rangle\,.
\end{aligned}\tag{3.267}
$$

这似乎是一个正常的结果，说明单粒子态 $|\mathbf{p};\sigma\rangle$ 比真空多了一份能量 $E_\mathbf{p}$。

利用产生湮灭算符的对易关系 (3.261)，可以计算单粒子态的内积：

$$
\begin{aligned}
\langle\mathbf{q};\sigma'|\mathbf{p};\sigma\rangle &= \sqrt{2E_\mathbf{q}2E_\mathbf{p}}\,\langle 0|\,a_{\mathbf{q};\sigma'}a^\dagger_{\mathbf{p};\sigma}\,|0\rangle=\sqrt{2E_\mathbf{q}2E_\mathbf{p}}\,\langle 0|\left[a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}-(2\pi)^3g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]|0\rangle\\
&= -2E_\mathbf{p}(2\pi)^3g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned}\tag{3.268}
$$

于是，有

$$\langle \mathbf{p};0|\mathbf{p};0\rangle = -2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0}), \qquad \langle \mathbf{p};i|\mathbf{p};i\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0}), \quad i=1,2,3. \tag{3.269}$$

上式表明，单粒子态 $|\mathbf{p};0\rangle$ 的自我内积是负的，从而导致它的能量期待值也是负的：

$$\langle \mathbf{p};0|\, H\, |\mathbf{p};0\rangle = (E_{\mathbf{p}}+E_{\mathrm{vac}})\,\langle \mathbf{p};0|\mathbf{p};0\rangle = -2E_{\mathbf{p}}(E_{\mathbf{p}}+E_{\mathrm{vac}})(2\pi)^3\delta^{(3)}(\mathbf{0}) < 0. \tag{3.270}$$

这个负能量结果在物理上看起来是不可接受的，它的根源在于不定度规。

不过，如前所述，无质量矢量场只有 2 种独立的极化态，对应于 2 种横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$，纵向极化和类时极化都应该是非物理的。选取一定的规范条件，应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.230) 与正则量子化程序不相容，我们不能直接使用这个条件，而需要将它转换到物理 Hilbert 空间中的态的期待值上，要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle\Psi|\,\partial_\mu A^\mu(x)\,|\Psi\rangle = 0. \tag{3.271}$$

上式称为**弱 Lorenz 规范条件**。

$A^\mu(x)$ 的平面波展开式 (3.250) 可以分解成正能解和负能解两个部分：

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x). \tag{3.272}$$

其中，正能解部分为

$$A^{\mu(+)}(\mathbf{x},t) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\, a_{\mathbf{p};\sigma}e^{-ip\cdot x}, \tag{3.273}$$

上式的厄米共轭即是负能解部分

$$A^{\mu(-)}(\mathbf{x},t) \equiv [A^{\mu(+)}(\mathbf{x},t)]^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\, a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}. \tag{3.274}$$

如果要求

$$\partial_\mu A^{\mu(+)}(x)\,|\Psi\rangle = 0 \tag{3.275}$$

对任意物理态 $|\Psi\rangle$ 成立，则伴随有

$$\langle\Psi|\,\partial_\mu A^{\mu(-)}(x) = \langle\Psi|\,[\partial_\mu A^{\mu(+)}(x)]^\dagger = 0, \tag{3.276}$$

从而，弱 Lorenz 规范条件 (3.271) 得到满足：

$$\langle\Psi|\,\partial_\mu A^\mu(x)\,|\Psi\rangle = \langle\Psi|\,\partial_\mu A^{\mu(+)}(x)\,|\Psi\rangle + \langle\Psi|\,\partial_\mu A^{\mu(-)}(x)\,|\Psi\rangle = 0. \tag{3.277}$$

利用 (3.113) 和 (3.220) 式，规范条件 (3.275) 可化为

$$0 = \partial_\mu A^{\mu(+)}(x)\,|\Psi\rangle$$

$$
\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip\cdot x}}{\sqrt{2E_{\mathbf{p}}}} \left[p_\mu e^\mu(\mathbf{p},0)a_{\mathbf{p};0} + p_\mu e^\mu(\mathbf{p},1)a_{\mathbf{p};1} + p_\mu e^\mu(\mathbf{p},2)a_{\mathbf{p};2} + p_\mu e^\mu(\mathbf{p},3)a_{\mathbf{p};3}\right] |\Psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip\cdot x}}{\sqrt{2E_{\mathbf{p}}}}\, p\cdot n\,(a_{\mathbf{p};0} - a_{\mathbf{p};3})\,|\Psi\rangle\,. \qquad (3.278)
\end{aligned}
$$

这意味着

$$
(a_{\mathbf{p};0} - a_{\mathbf{p};3})\,|\Psi\rangle = 0 \qquad (3.279)
$$

对任意物理态 $|\Psi\rangle$ 和任意动量 $\mathbf{p}$ 成立。从而，也有

$$
\langle\Psi|\,(a^\dagger_{\mathbf{p};0} - a^\dagger_{\mathbf{p};3}) = 0. \qquad (3.280)
$$

于是，

$$
\langle\Psi|\, a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}\,|\Psi\rangle = \langle\Psi|\, a^\dagger_{\mathbf{p};3}a_{\mathbf{p};3}\,|\Psi\rangle\,. \qquad (3.281)
$$

这样一来，根据 (3.263) 式计算，$|\Psi\rangle$ 的能量期待值为

$$
\begin{aligned}
\langle\Psi|\,H\,|\Psi\rangle &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\Psi| \left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3} a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) |\Psi\rangle + E_{\mathrm{vac}}\,\langle\Psi|\Psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=1}^{2} \langle\Psi|\, a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\,|\Psi\rangle + E_{\mathrm{vac}}\,\langle\Psi|\Psi\rangle\,. \qquad (3.282)
\end{aligned}
$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.158) 式可得无质量矢量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\,\pi_\mu \nabla A^\mu \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\qquad \times (ip_0)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\left(i\mathbf{q}\,a_{\mathbf{q};\sigma'}e^{-iq\cdot x} - i\mathbf{q}\,a^\dagger_{\mathbf{q};\sigma'}e^{iq\cdot x}\right) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, p_0\mathbf{q}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \left[-a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p-q)\cdot x} + a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p+q)\cdot x} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p+q)\cdot x}\right] \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, p_0\mathbf{q}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \Big\{ -\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p_0-q_0)t}\right] \\
&\qquad\quad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p_0+q_0)t}\right] \Big\} \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big[-e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)
\end{aligned}
$$

$$- e_\mu(\mathbf{p},\sigma)e^\mu(-\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_\mathbf{p}t}+a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_\mathbf{p}t}\right)\Big]. \tag{3.283}$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p}\to-\mathbf{p}$ 的替换和 $\sigma\leftrightarrow\sigma'$ 的互换，可得

$$\begin{aligned}
&-\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\,e_\mu(\mathbf{p},\sigma)e^\mu(-\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_\mathbf{p}t}+a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_\mathbf{p}t}\right)\\
=&-\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{-\mathbf{p}}{2}\,e_\mu(-\mathbf{p},\sigma')e^\mu(\mathbf{p},\sigma)\left(a_{-\mathbf{p};\sigma'}a_{\mathbf{p};\sigma}e^{-2iE_\mathbf{p}t}+a^\dagger_{-\mathbf{p};\sigma'}a^\dagger_{\mathbf{p};\sigma}e^{2iE_\mathbf{p}t}\right)\\
=&\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\,e_\mu(-\mathbf{p},\sigma')e^\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_\mathbf{p}t}+a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_\mathbf{p}t}\right).
\end{aligned}\tag{3.284}$$

可以看出，上式为零。于是，总动量化为

$$\begin{aligned}
\mathbf{P}\;&=-\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\,e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)\\
&=-\sum_{\sigma\sigma'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\,g_{\sigma\sigma'}\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right)=\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma}+a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)\\
&=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma})+\delta^{(3)}(\mathbf{0})\int d^3p\,\frac{\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2\\
&=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}+\sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right).
\end{aligned}\tag{3.285}$$

根据 (3.281) 式，物理态 $|\Psi\rangle$ 的动量期待值为

$$\langle\Psi|\,\mathbf{P}\,|\Psi\rangle=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}\,\langle\Psi|\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}+\sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)|\Psi\rangle=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}\sum_{\sigma=1}^{2}\langle\Psi|\,a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\,|\Psi\rangle. \tag{3.286}$$

同样，只有两种物理的横向极化才对动量有净贡献。

通过线性组合，可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$a_{\mathbf{p},\pm}\equiv\frac{1}{\sqrt{2}}(\mp a_{\mathbf{p};1}+ia_{\mathbf{p};2}), \tag{3.287}$$

相应的产生算符可以通过取厄米共轭得到。反过来，有

$$a_{\mathbf{p};1}=-\frac{1}{\sqrt{2}}(a_{\mathbf{p},+}-a_{\mathbf{p},-}),\quad a_{\mathbf{p};2}=-\frac{i}{\sqrt{2}}(a_{\mathbf{p},+}+a_{\mathbf{p},-}). \tag{3.288}$$

利用对易关系 (3.261)，可得

$$\begin{aligned}
[a_{\mathbf{p},\pm},a^\dagger_{\mathbf{q},\pm}]\;&=\frac{1}{2}[\mp a_{\mathbf{p};1}+ia_{\mathbf{p};2},\,\mp a^\dagger_{\mathbf{q};1}-ia^\dagger_{\mathbf{q};2}]=\frac{1}{2}[a_{\mathbf{p};1},\,a^\dagger_{\mathbf{q};1}]+\frac{1}{2}[a_{\mathbf{p};2},\,a^\dagger_{\mathbf{q};2}]=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),\\
[a_{\mathbf{p},\pm},a^\dagger_{\mathbf{q},\mp}]\;&=\frac{1}{2}[\mp a_{\mathbf{p};1}+ia_{\mathbf{p};2},\,\pm a^\dagger_{\mathbf{q};1}-ia^\dagger_{\mathbf{q};2}]=-\frac{1}{2}[a_{\mathbf{p};1},\,a^\dagger_{\mathbf{q};1}]+\frac{1}{2}[a_{\mathbf{p};2},\,a^\dagger_{\mathbf{q};2}]=0,\\
[a_{\mathbf{p},\pm},a_{\mathbf{q},\pm}]\;&=\frac{1}{2}[\mp a_{\mathbf{p};1}+ia_{\mathbf{p};2},\,\mp a_{\mathbf{q};1}+ia_{\mathbf{q};2}]=0,
\end{aligned}$$

$$[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] = \frac{1}{2}[\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2}, \pm a_{\mathbf{q};1} + ia_{\mathbf{q};2}] = 0. \tag{3.289}$$

于是，这组产生湮灭算符的对易关系可以整理为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0, \quad \lambda, \lambda' = \pm. \tag{3.290}$$

根据 (3.129) 式，可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 表示 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$：

$$e^\mu(\mathbf{p},1) = -\frac{1}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)], \quad e^\mu(\mathbf{p},2) = \frac{i}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)]. \tag{3.291}$$

从而，有

$$\begin{aligned}\sum_{\sigma=1}^{2} e^\mu(\mathbf{p},\sigma)a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p},1)a_{\mathbf{p};1} + e^\mu(\mathbf{p},2)a_{\mathbf{p};2} \\ &= \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= \varepsilon^\mu(\mathbf{p},+)a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p},-)a_{\mathbf{p},-} = \sum_{\lambda=\pm}\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},\end{aligned} \tag{3.292}$$

取厄米共轭，得

$$\sum_{\sigma=1}^{2} e^\mu(\mathbf{p},\sigma)a^\dagger_{\mathbf{p};\sigma} = \sum_{\lambda=\pm}\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}. \tag{3.293}$$

于是，可以把 $A^\mu(x)$ 的平面波展开式 (3.250) 改写成

$$\begin{aligned}A^\mu(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0,3} e^\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right) \\ &\quad + \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right],\end{aligned} \tag{3.294}$$

第一行对应于非物理极化态，第二行对应于两种物理的螺旋度本征极化态。可见，(3.287) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda = \pm$ 对应的湮灭算符。

此外，由 (3.288) 式可得

$$\begin{aligned}\sum_{\sigma=1}^{2} a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma} &= a^\dagger_{\mathbf{p};1}a_{\mathbf{p};1} + a^\dagger_{\mathbf{p};2}a_{\mathbf{p};2} = \frac{1}{2}(a^\dagger_{\mathbf{p},+} - a^\dagger_{\mathbf{p},-})(a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}(a^\dagger_{\mathbf{p},+} + a^\dagger_{\mathbf{p},-})(a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= a^\dagger_{\mathbf{p},+}a_{\mathbf{p},+} + a^\dagger_{\mathbf{p},-}a_{\mathbf{p},-} = \sum_{\lambda=\pm}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda},\end{aligned} \tag{3.295}$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle\Psi| H |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_\mathbf{p}\sum_{\lambda=\pm}\langle\Psi| a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}|\Psi\rangle + E_{\text{vac}}\langle\Psi|\Psi\rangle, \tag{3.296}$$

$$\langle\Psi| \mathbf{P} |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\sum_{\lambda=\pm}\langle\Psi| a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}|\Psi\rangle. \tag{3.297}$$

第 4 章　旋量场

4.1　Lorentz 群的旋量表示

旋量表示 (spinor representation) 是 Lorentz 群的一个线性表示，它在物理上扮演着非常重要的角色，Dirac 在 1928 年首次将它引入到描述电子的理论中。3.1 节提到，Lorentz 群的线性表示可以通过构造满足 Lorentz 代数关系 (3.20) 的生成元矩阵来得到，下面我们就用这样的方式来建立旋量表示。

首先，我们假设能够找到一组满足如下**反对易关系**的 $N\times N$ 矩阵 γ^μ ($\mu=0,1,2,3$)：

$$\{\gamma^\mu,\gamma^\nu\}\equiv\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}\mathbf{1}=2g^{\mu\nu}. \tag{4.1}$$

最后一步是一种简写，省略了 $N\times N$ 单位矩阵 $\mathbf{1}$。这样的 γ^μ 称为 **Dirac 矩阵**。当 $\mu\neq\nu$ 时，γ^μ 与 γ^ν 是反对易的，即

$$\gamma^\mu\gamma^\nu=-\gamma^\nu\gamma^\mu,\quad \mu\neq\nu. \tag{4.2}$$

当 $\mu=\nu$ 时，有

$$(\gamma^0)^2=\frac{1}{2}\{\gamma^0,\gamma^0\}=g^{00}=\mathbf{1},\quad (\gamma^i)^2=\frac{1}{2}\{\gamma^i,\gamma^i\}=g^{ii}=-\mathbf{1}. \tag{4.3}$$

我们约定 γ^0 是厄米矩阵，γ^i 是反厄米矩阵，即

$$(\gamma^0)^\dagger=\gamma^0,\quad (\gamma^i)^\dagger=-\gamma^i, \tag{4.4}$$

则可得

$$(\gamma^0)^\dagger\gamma^0=(\gamma^0)^2=\mathbf{1},\quad (\gamma^i)^\dagger\gamma^i=-(\gamma^i)^2=\mathbf{1}. \tag{4.5}$$

可见，在此约定下，γ^0 和 γ^i 都是幺正矩阵。

然后，以 Dirac 矩阵的对易子定义另一组 $N\times N$ 矩阵

$$S^{\mu\nu}\equiv\frac{i}{4}[\gamma^\mu,\gamma^\nu]. \tag{4.6}$$

显然，$S^{\mu\nu}$ 关于 μ 和 ν 反对称：

$$S^{\mu\nu}=-S^{\nu\mu}. \tag{4.7}$$

因而 $S^{\mu\nu}$ 的独立分量有 6 个。

利用对易子公式

$$[AB,C]=ABC+ACB-ACB-CAB=A\{B,C\}-\{A,C\}B, \tag{4.8}$$

可得

$$\begin{aligned}[S^{\mu\nu},\gamma^\rho] &= \frac{i}{4}[\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu,\gamma^\rho]=\frac{i}{4}[\gamma^\mu\gamma^\nu-(2g^{\nu\mu}-\gamma^\mu\gamma^\nu),\gamma^\rho]=\frac{i}{2}[\gamma^\mu\gamma^\nu,\gamma^\rho]-\frac{i}{2}[g^{\nu\mu},\gamma^\rho]\\ &= \frac{i}{2}[\gamma^\mu\gamma^\nu,\gamma^\rho]=\frac{i}{2}(\gamma^\mu\{\gamma^\nu,\gamma^\rho\}-\{\gamma^\mu,\gamma^\rho\}\gamma^\nu)=i(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho}).\end{aligned} \tag{4.9}$$

从而，根据对易子公式 (2.11)，有

$$\begin{aligned}[S^{\mu\nu},S^{\rho\sigma}] &= \frac{i}{4}[S^{\mu\nu},\gamma^\rho\gamma^\sigma-\gamma^\sigma\gamma^\rho]=\frac{i}{4}([S^{\mu\nu},\gamma^\rho\gamma^\sigma]-[S^{\mu\nu},\gamma^\sigma\gamma^\rho])\\ &= \frac{i}{4}([S^{\mu\nu},\gamma^\rho]\gamma^\sigma+\gamma^\rho[S^{\mu\nu},\gamma^\sigma]-[S^{\mu\nu},\gamma^\sigma]\gamma^\rho-\gamma^\sigma[S^{\mu\nu},\gamma^\rho])\\ &= \frac{i}{4}[i(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})\gamma^\sigma+i\gamma^\rho(\gamma^\mu g^{\nu\sigma}-\gamma^\nu g^{\mu\sigma})\\ &\qquad -i(\gamma^\mu g^{\nu\sigma}-\gamma^\nu g^{\mu\sigma})\gamma^\rho-i\gamma^\sigma(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})]\\ &= \frac{i^2}{4}(\gamma^\mu\gamma^\sigma g^{\nu\rho}-\gamma^\nu\gamma^\sigma g^{\mu\rho}+\gamma^\rho\gamma^\mu g^{\nu\sigma}-\gamma^\rho\gamma^\nu g^{\mu\sigma}\\ &\qquad -\gamma^\mu\gamma^\rho g^{\nu\sigma}+\gamma^\nu\gamma^\rho g^{\mu\sigma}-\gamma^\sigma\gamma^\mu g^{\nu\rho}+\gamma^\sigma\gamma^\nu g^{\mu\rho})\\ &= \frac{i^2}{4}[g^{\nu\rho}(\gamma^\mu\gamma^\sigma-\gamma^\sigma\gamma^\mu)-g^{\mu\rho}(\gamma^\nu\gamma^\sigma-\gamma^\sigma\gamma^\nu)-g^{\nu\sigma}(\gamma^\mu\gamma^\rho-\gamma^\rho\gamma^\mu)+g^{\mu\sigma}(\gamma^\nu\gamma^\rho-\gamma^\rho\gamma^\nu)]\\ &= i(g^{\nu\rho}S^{\mu\sigma}-g^{\mu\rho}S^{\nu\sigma}-g^{\nu\sigma}S^{\mu\rho}+g^{\mu\sigma}S^{\nu\rho}).\end{aligned} \tag{4.10}$$

可见，$S^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)，因而是 Lorentz 群某个线性表示的生成元。以 $S^{\mu\nu}$ 生成的线性表示就是**旋量表示**。

根据 (3.2.1) 小节的讨论，一组变换参数 $\omega_{\mu\nu}$ 在 Lorentz 群的矢量表示中可以生成固有保时向的有限变换 (3.51)：

$$\Lambda=\exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)=e^X,\quad X\equiv-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \tag{4.11}$$

类似地，这组参数在旋量表示中生成了固有保时向的有限变换

$$D(\Lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n=\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)=e^Y,\quad Y\equiv-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}. \tag{4.12}$$

这样定义的 $D(\Lambda)$ 是旋量表示中的 Lorentz 变换矩阵，对于任意的 Lorentz 变换 Λ_1 和 Λ_2，满足同态关系

$$D(\Lambda_2\Lambda_1)=D(\Lambda_2)D(\Lambda_1). \tag{4.13}$$

由 (3.48) 式可得

$$e^{-Y}e^{Y}=e^{-Y+Y}=e^{\mathbf{0}}=\mathbf{1}, \tag{4.14}$$

故 $D(\Lambda)$ 的逆矩阵为

$$D(\Lambda^{-1}) = D^{-1}(\Lambda) = e^{-Y} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{4.15}$$

这里先来介绍一些将会用到的对易子公式。以如下方式定义 B 与 A 的多重对易子 $[B, A^{(n)}]$:

$$\begin{aligned} [B, A^{(0)}] &= B, \quad [B, A^{(1)}] = [B, A] = [[B, A^{(0)}], A] \\ [B, A^{(2)}] &= [[B, A], A] = [[B, A^{(1)}], A], \quad \cdots, \quad [B, A^{(n)}] = [[B, A^{(n-1)}], A]. \end{aligned} \tag{4.16}$$

于是，下式成立:

$$BA^k = \sum_{n=0}^{k} \frac{k!}{(k-n)!n!} A^{k-n}[B, A^{(n)}]. \tag{4.17}$$

下面用数学归纳法证明这个等式。

证明　当 $k=0$ 和 $k=1$ 时，(4.17) 式明显成立:

$$BA^0 = B = [B, A^{(0)}] = \frac{0!}{(0-0)!0!}A^{0-0}[B, A^{(0)}], \tag{4.18}$$

$$BA^1 = BA = AB + [B, A] = \frac{1!}{(1-0)!0!}A^{1-0}[B, A^{(0)}] + \frac{1!}{(1-1)!1!}A^{1-1}[B, A^{(1)}]. \tag{4.19}$$

假设 $k=m$ 时 (4.17) 式成立，则有

$$\begin{aligned} BA^{m+1} &= \sum_{n=0}^{m} \frac{m!}{(m-n)!n!}A^{m-n}[B, A^{(n)}]A = \sum_{n=0}^{m} \frac{m!}{(m-n)!n!}A^{m-n}(A[B, A^{(n)}] + [[B, A^{(n)}], A]) \\ &= \sum_{n=0}^{m} \frac{m!}{(m-n)!n!}A^{m+1-n}[B, A^{(n)}] + \sum_{n=0}^{m} \frac{m!}{(m-n)!n!}A^{m-n}[B, A^{(n+1)}] \\ &= \sum_{n=0}^{m} \frac{m!}{(m-n)!n!}A^{m+1-n}[B, A^{(n)}] + \sum_{j=1}^{m+1} \frac{m!}{(m-j+1)!(j-1)!}A^{m-j+1}[B, A^{(j)}] \\ &= \frac{m!}{(m-0)!0!}A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m}\left[\frac{m!}{(m-n)!n!} + \frac{m!}{(m-n+1)!(n-1)!}\right]A^{m+1-n}[B, A^{(n)}] \\ &\quad + \frac{m!}{[m-(m+1)+1]![(m+1)-1]!}A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m}\left[\frac{m!}{(m-n)!n!} + \frac{n}{m-n+1}\frac{m!}{(m-n)!n!}\right]A^{m+1-n}[B, A^{(n)}] \\ &\quad + A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= \frac{(m+1)!}{[(m+1)-0]!0!}A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m} \frac{(m+1)!}{(m-n+1)!n!}A^{m+1-n}[B, A^{(n)}] \\ &\quad + \frac{(m+1)!}{[(m+1)-(m+1)]!(m+1)!}A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= \sum_{n=0}^{m+1} \frac{(m+1)!}{[(m+1)-n]!n!}A^{(m+1)-n}[B, A^{(n)}], \end{aligned} \tag{4.20}$$

即 $k=m+1$ 时 (4.17) 式也成立。于是，(4.17) 式对任意非负整数 k 成立。**证毕。**

根据推广的阶乘定义 (3.46) 可以将 (4.17) 式右边的级数化成无穷级数：

$$BA^k = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} A^{k-n}[B, A^{(n)}]. \tag{4.21}$$

利用上式，可得

$$\begin{aligned}
e^{-A}Be^{A} &= e^{-A}\sum_{k=0}^{\infty}\frac{1}{k!}BA^k = e^{-A}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{\infty}\frac{k!}{(k-n)!n!}A^{k-n}[B, A^{(n)}] \\
&= e^{-A}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{(k-n)!}A^{k-n}[B, A^{(n)}] = e^{-A}\sum_{n=0}^{\infty}\frac{1}{n!}e^{A}[B, A^{(n)}] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}[B, A^{(n)}].
\end{aligned} \tag{4.22}$$

现在，我们继续讨论 Lorentz 群的旋量表示。由 (4.9) 和 (3.34) 式可得

$$[\gamma^\mu, S^{\rho\sigma}] = -[S^{\rho\sigma}, \gamma^\mu] = [S^{\sigma\rho}, \gamma^\mu] = i(\gamma^\sigma g^{\rho\mu} - \gamma^\rho g^{\sigma\mu}) = i(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu)\gamma^\nu = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu. \tag{4.23}$$

从而，有

$$\begin{aligned}
[\gamma^\mu, Y^{(1)}] &= [\gamma^\mu, Y] = -\frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] = -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu = X^\mu{}_\nu\gamma^\nu, \\
[\gamma^\mu, Y^{(2)}] &= [[\gamma^\mu, Y^{(1)}], Y] = X^\mu{}_\nu[\gamma^\nu, Y] = X^\mu{}_\nu X^\nu{}_\rho\gamma^\rho = (X^2)^\mu{}_\nu\gamma^\nu, \\
&\cdots \\
[\gamma^\mu, Y^{(n)}] &= (X^n)^\mu{}_\nu\gamma^\nu.
\end{aligned} \tag{4.24}$$

于是，利用 (4.22) 式可以推出

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = e^{-Y}\gamma^\mu e^{Y} = \sum_{n=0}^{\infty}\frac{1}{n!}[\gamma^\mu, Y^{(n)}] = \sum_{n=0}^{\infty}\frac{1}{n!}(X^n)^\mu{}_\nu\gamma^\nu = (e^X)^\mu{}_\nu\gamma^\nu, \tag{4.25}$$

即

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{4.26}$$

上式是 γ^μ 在旋量表示中的 Lorentz 变换关系，它说明 γ^μ 是一个 Lorentz 矢量。相应的协变矢量为

$$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu, \tag{4.27}$$

从而，

$$\gamma_0 = \gamma^0, \quad \gamma_i = -\gamma^i, \quad i = 1, 2, 3. \tag{4.28}$$

$N \times N$ 单位矩阵 $\mathbf{1}$ 满足

$$D^{-1}(\Lambda)\mathbf{1}D(\Lambda) = \mathbf{1}, \tag{4.29}$$

因而 $\mathbf{1}$ 是一个 Lorentz 标量。生成元 $S^{\mu\nu}$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)S^{\mu\nu}D(\Lambda)=\frac{i}{4}[D^{-1}(\Lambda)\gamma^\mu D(\Lambda),D^{-1}(\Lambda)\gamma^\nu D(\Lambda)]=\frac{i}{4}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma[\gamma^\rho,\gamma^\sigma]=\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma S^{\rho\sigma},\quad(4.30)$$

可见，$S^{\mu\nu}$ 是一个 2 阶反对称 Lorentz 张量。

$S^{\mu\nu}$ 是用 2 个 Dirac 矩阵的乘积构造出来的反对称张量，类似地，我们也可以用 3 个 Dirac 矩阵的乘积来构造一个 3 阶全反对称张量

$$\Gamma^{\mu\nu\rho}\equiv\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}\equiv\frac{1}{3!}(\gamma^\mu\gamma^\nu\gamma^\rho+\gamma^\rho\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\rho\gamma^\mu-\gamma^\mu\gamma^\rho\gamma^\nu-\gamma^\rho\gamma^\nu\gamma^\mu-\gamma^\nu\gamma^\mu\gamma^\rho).\qquad(4.31)$$

上式第二步中的中括号表示对 μ、ν、ρ 三个指标作全反对称操作：在偶次置换前面加上正号，奇次置换前面加上负号，然后对所有置换求和并除以置换方式的数目。$\Gamma^{\mu\nu\rho}$ 的 Lorentz 变换形式是

$$D^{-1}(\Lambda)\Gamma^{\mu\nu\rho}D(\Lambda)=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Gamma^{\alpha\beta\gamma}.\qquad(4.32)$$

由全反对称性可知，$\Gamma^{\mu\nu\rho}$ 的独立分量只有 4 个，可取为 Γ^{012}、Γ^{023}、Γ^{013} 和 Γ^{123}。根据 (4.2) 式和定义式 (4.31)，可得

$$\Gamma^{012}=\gamma^0\gamma^1\gamma^2,\quad\Gamma^{023}=\gamma^0\gamma^2\gamma^3,\quad\Gamma^{013}=\gamma^0\gamma^1\gamma^3,\quad\Gamma^{123}=\gamma^1\gamma^2\gamma^3.\qquad(4.33)$$

更进一步，可以用 4 个 Dirac 矩阵的乘积来构造一个 4 阶全反对称张量

$$\begin{aligned}\Gamma^{\mu\nu\rho\sigma}&\equiv\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}\\&\equiv\frac{1}{4!}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma+\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho+\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu-\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho-\gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu-\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\\&\quad-\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho-\gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu-\gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu+\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu+\gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho\\&\quad+\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu+\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu+\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma-\gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu-\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma-\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\\&\quad-\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu-\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma-\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho+\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma+\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho+\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu).\end{aligned}\quad(4.34)$$

从而，$\Gamma^{\mu\nu\rho\sigma}$ 具有如下性质：

$$\Gamma^{\mu\nu\rho\sigma}=\begin{cases}+\Gamma^{0123},&(\mu,\nu,\rho,\sigma)\text{ 是 }(0,1,2,3)\text{ 的偶次置换},\\-\Gamma^{0123},&(\mu,\nu,\rho,\sigma)\text{ 是 }(0,1,2,3)\text{ 的奇次置换},\\0,&\text{其它情况。}\end{cases}\qquad(4.35)$$

可见，它只有 1 个独立分量，可取为

$$\Gamma^{0123}=\gamma^0\gamma^1\gamma^2\gamma^3.\qquad(4.36)$$

结合四维 Levi-Civita 符号的定义 (1.65)，可得

$$\Gamma^{\mu\nu\rho\sigma}=\varepsilon^{\mu\nu\rho\sigma}\Gamma^{0123}=\varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3\qquad(4.37)$$

受到四维时空的维度限制，我们不能以同样的方式定义高于 4 阶的全反对称张量。现在，我们拥有一组矩阵

$$\{\mathbf{1}, \gamma^\mu, S^{\mu\nu}, \Gamma^{\mu\nu\rho}, \Gamma^{\mu\nu\rho\sigma}\}, \tag{4.38}$$

它们各自的独立分量个数之和为 $1+4+6+4+1=16$。利用反对易关系 (4.1)，可以将任意多个 Dirac 矩阵的乘积转化为集合 (4.38) 中的矩阵与度规张量乘积的线性组合。例如，

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\gamma^\nu\gamma^\mu + g^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + g^{\mu\nu} = -2iS^{\mu\nu} + g^{\mu\nu}. \tag{4.39}$$

又如，

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma^\rho &= \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
&= \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{4}\gamma^\nu\gamma^\mu\gamma^\rho + \frac{1}{2}g^{\mu\nu}\gamma^\rho - \frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu + \frac{1}{4}\gamma^\rho\gamma^\mu\gamma^\nu - \frac{1}{2}g^{\mu\rho}\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
&= \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{8}\gamma^\nu\gamma^\mu\gamma^\rho + \frac{1}{8}\gamma^\nu\gamma^\rho\gamma^\mu - \frac{1}{4}g^{\rho\mu}\gamma^\nu + \frac{1}{2}g^{\mu\nu}\gamma^\rho - \frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu \\
&\quad + \frac{1}{8}\gamma^\rho\gamma^\mu\gamma^\nu - \frac{1}{8}\gamma^\rho\gamma^\nu\gamma^\mu + \frac{1}{4}g^{\mu\nu}\gamma^\rho - \frac{1}{2}g^{\mu\rho}\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
&= \frac{3!}{8}\Gamma^{\mu\nu\rho} + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{8}\gamma^\mu\gamma^\rho\gamma^\nu - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu \\
&= \frac{3}{4}\Gamma^{\mu\nu\rho} + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{4}g^{\rho\nu}\gamma^\mu - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu \\
&= \frac{3}{4}\Gamma^{\mu\nu\rho} + \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + \frac{3}{4}g^{\rho\nu}\gamma^\mu,
\end{aligned} \tag{4.40}$$

故

$$\gamma^\mu\gamma^\nu\gamma^\rho = \Gamma^{\mu\nu\rho} - g^{\rho\mu}\gamma^\nu + g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu. \tag{4.41}$$

因此，对于由 Dirac 矩阵乘积的线性组合构造的矩阵，集合 (4.38) 构成一组完备的基底。

这里引入一个新的矩阵

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.42}$$

从 (4.2) 式可得

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{cases} +\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的偶次置换,} \\ -\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的奇次置换。} \end{cases} \tag{4.43}$$

这种置换性质与四维 Levi-Civita 符号 (1.65) 相同,因而置换操作带来的符号在 $\varepsilon_{\mu\nu\rho\sigma}$ 与 $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ 的乘积中相互抵消，如

$$\varepsilon_{1023}\gamma^1\gamma^0\gamma^2\gamma^3 = -\varepsilon_{0123}(-\gamma^0\gamma^1\gamma^2\gamma^3) = \varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.44}$$

由此可得

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{4.45}$$

对于固有保时向 Lorentz 变换 (4.11)，用度规对 (1.73) 式升降指标，有

$$\varepsilon_{\mu\nu\rho\sigma} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \Lambda_\rho{}^\gamma \Lambda_\sigma{}^\delta \varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta} (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu (\Lambda^{-1})^\gamma{}_\rho (\Lambda^{-1})^\delta{}_\sigma. \tag{4.46}$$

于是，γ^5 的 Lorentz 变换形式为

$$\begin{aligned} D^{-1}(\Lambda)\gamma^5 D(\Lambda) &= -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\ &= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}(\Lambda^{-1})^\kappa{}_\mu (\Lambda^{-1})^\lambda{}_\nu (\Lambda^{-1})^\tau{}_\rho (\Lambda^{-1})^\varepsilon{}_\sigma \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\ &= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}\delta^\kappa{}_\alpha \delta^\lambda{}_\beta \delta^\tau{}_\gamma \delta^\varepsilon{}_\delta \gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = -\frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = \gamma^5. \end{aligned} \tag{4.47}$$

可见，γ^5 是一个 Lorentz 标量。γ^5 的平方为

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0 = -(-\mathbf{1})^3 = \mathbf{1}. \tag{4.48}$$

根据约定 (4.4)，γ^5 是厄米矩阵：

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{4.49}$$

γ^5 与 γ^μ 反对易：

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu - \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu) = 0. \tag{4.50}$$

由 (4.37) 式可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon^{\mu\nu\rho\sigma}\gamma^5. \tag{4.51}$$

可见，$\Gamma^{\mu\nu\rho\sigma}$ 正比于 γ^5。此外，由 (4.33) 式有

$$\Gamma^{012} = \gamma^0\gamma^1\gamma^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^3\gamma^5 = i\gamma_3\gamma^5 = i\varepsilon^{0123}\gamma_3\gamma^5, \tag{4.52}$$

$$\Gamma^{023} = \gamma^0\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^5 = i\gamma_1\gamma^5 = i\varepsilon^{0231}\gamma_1\gamma^5, \tag{4.53}$$

$$\Gamma^{013} = \gamma^0\gamma^1\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = -\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^2\gamma^5 = -i\gamma_2\gamma^5 = i\varepsilon^{0132}\gamma_2\gamma^5, \tag{4.54}$$

$$\Gamma^{123} = \gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5 = -i\gamma_0\gamma^5 = i\varepsilon^{1230}\gamma_0\gamma^5. \tag{4.55}$$

综合起来，得

$$\Gamma^{\mu\nu\rho} = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5. \tag{4.56}$$

根据上式，$\Gamma^{\mu\nu\rho}$ 可以写成 $\gamma^\mu\gamma^5$ 的 4 个独立分量的线性组合。$\gamma^\mu\gamma^5$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)\gamma^\mu\gamma^5 D(\Lambda) = D^{-1}(\Lambda)\gamma^\mu D(\Lambda)D^{-1}(\Lambda)\gamma^5 D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu\gamma^5, \tag{4.57}$$

因而它是一个 Lorentz 矢量。再引入

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2S^{\mu\nu}, \tag{4.58}$$

它正比于 $S^{\mu\nu}$，所以也是一个 2 阶反对称 Lorentz 张量：

$$D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda) = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\sigma^{\mu\nu}. \tag{4.59}$$

于是，我们可以用 γ^5、$\gamma^\mu\gamma^5$ 和 $\sigma^{\mu\nu}$ 分别代替集合 (4.38) 中的 $\Gamma^{\mu\nu\rho\sigma}$、$\Gamma^{\mu\nu\rho}$ 和 $S^{\mu\nu}$ 作为基底，从而得到另一组完备的矩阵基底

$$\{\mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}, \tag{4.60}$$

它们各自的独立分量个数之和仍是 16。

依照约定 (4.4)，γ^0 即是厄米的又是幺正的，我们可以用它定义一个幺正变换矩阵 β：

$$\beta^{-1} = \beta^\dagger = \beta \equiv \gamma^0. \tag{4.61}$$

从而，有

$$\beta^{-1}\gamma^0\beta = \gamma^0\gamma^0\gamma^0 = +\gamma^0, \quad \beta^{-1}\gamma^i\beta = \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i. \tag{4.62}$$

根据宇称变换 $\mathcal{P}$ 的定义 (1.46)，可以将这两个式子合写为

$$\beta^{-1}\gamma^\mu\beta = \mathcal{P}^\mu{}_\nu\gamma^\nu. \tag{4.63}$$

这表明 β 相当于旋量表示中的宇称变换矩阵 $D(\mathcal{P})$，它是非固有保时向的，上式就是 γ^μ 的宇称变换形式。(4.62) 式说明 γ^0 是宇称本征态，本征值为 $+$，即具有**偶宇称**；γ^i 也是宇称本征态，本征值为 $-$，即具有**奇宇称**。虽然单位矩阵 $\mathbf{1}$ 与 γ_5 都是 Lorentz 标量，但它们的宇称是不同的：

$$\beta^{-1}\mathbf{1}\beta = +\mathbf{1}, \quad \beta^{-1}\gamma^5\beta = \gamma^0\gamma^5\gamma^0 = -\gamma^5\gamma^0\gamma^0 = -\gamma^5. \tag{4.64}$$

像 γ^5 这样具有奇宇称的 Lorentz 标量，称为**赝标量** (pseudoscalar)。此外，$\gamma^\mu\gamma^5$ 的宇称变换形式是

$$\beta^{-1}\gamma^\mu\gamma^5\beta = \beta^{-1}\gamma^\mu\beta\beta^{-1}\gamma^5\beta = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \tag{4.65}$$

即

$$\beta^{-1}\gamma^0\gamma^5\beta = -\gamma^0\gamma^5, \quad \beta^{-1}\gamma^i\gamma^5\beta = +\gamma^i\gamma^5. \tag{4.66}$$

可以看出，虽然 $\gamma^\mu\gamma^5$ 也是 Lorentz 矢量，但它的分量的宇称性质与 γ^μ 相反。宇称变换性质像 $\gamma^\mu\gamma^5$ 这样的 Lorentz 矢量称为**轴矢量** (axial vector)。最后，$\sigma^{\mu\nu}$ 的宇称变换形式为

$$\beta^{-1}\sigma^{\mu\nu}\beta = \frac{i}{2}[\beta^{-1}\gamma^\mu\beta, \beta^{-1}\gamma^\nu\beta] = \frac{i}{2}\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta[\gamma^\alpha, \gamma^\beta] = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\sigma^{\alpha\beta}, \tag{4.67}$$

即

$$\beta^{-1}\sigma^{0i}\beta = \mathcal{P}^0{}_\alpha\mathcal{P}^i{}_\beta\sigma^{\alpha\beta} = -\sigma^{0i}, \quad \beta^{-1}\sigma^{ij}\beta = \mathcal{P}^i{}_\alpha\mathcal{P}^j{}_\beta\sigma^{\alpha\beta} = +\sigma^{ij}. \tag{4.68}$$

可见，基底集合 (4.60) 是由标量 $\mathbf{1}$、赝标量 γ^5、矢量 γ^μ、轴矢量 $\gamma^\mu\gamma^5$ 和 2 阶反对称张量 $\sigma^{\mu\nu}$ 组成的，综合考虑固有保时向 Lorentz 变换和宇称变换，则这些基底的变换性质各不相同，因而它们彼此之间是相互独立的，总共有 16 个独立而完备的基底。由于独立的 $N\times N$ 矩阵最多有 N^2 个，为了得到 16 个这样的基底，需要 $N\geq 4$。我们考虑最简单的情况，将 Dirac 矩阵取为 4×4 矩阵。

4.2 Dirac 旋量场

在 Lorentz 群的旋量表示中，被变换矩阵 $D(\Lambda)$ 作用的态称为 **Dirac 旋量** (spinor)。由于 $D(\Lambda)$ 是 4×4 矩阵，一个 Dirac 旋量 ψ_a 应当具有 4 个分量 $(a=1,2,3,4)$，相应的 Lorentz 变换形式为

$$\psi'_a = D_{ab}(\Lambda)\psi_b. \tag{4.69}$$

隐去旋量指标 a 和 b，上式化为

$$\psi' = D(\Lambda)\psi. \tag{4.70}$$

我们可以将 ψ 和 ψ' 看作列矢量，而上式右边的乘积就是线性代数中矩阵与列矢量的乘积。

进一步，如果 ψ_a 依赖于时空坐标 x^μ，它就成为 **Dirac 旋量场** $\psi_a(x)$。类似于 (3.67) 式，量子 Dirac 旋量场的 Lorentz 变换形式是

$$\psi'_a(x') = U^{-1}(\Lambda)\psi_a(x')U(\Lambda) = D_{ab}(\Lambda)\psi_b(x). \tag{4.71}$$

对于固有保时向 Lorentz 变换，由 (4.12) 式可得 $D_{ab}(\Lambda)$ 的无穷小形式为

$$D_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}, \tag{4.72}$$

于是，(4.71) 式的无穷小形式是

$$\psi'_a(x') = \psi_a(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}\psi_b(x). \tag{4.73}$$

将上式与 (1.168) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在旋量表示中对应于 $S^{\mu\nu}$。隐去旋量指标，则 (4.71) 式化为

$$\psi'(x') = U^{-1}(\Lambda)\psi(x')U(\Lambda) = D(\Lambda)\psi(x), \tag{4.74}$$

也可以写成

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x). \tag{4.75}$$

对于无穷小变换，根据 (3.59) 式，将 $\psi(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned}\psi(\Lambda^{-1}x) &= \psi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi(x) = \psi(x) - \omega_{\mu\nu}x^\nu\partial^\mu\psi(x) = \psi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi(x)\\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu}i(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x),\end{aligned} \tag{4.76}$$

其中 $L^{\mu\nu}$ 是 (3.63) 式定义的微分算符。从而，(4.75) 式右边展开到 ω 一阶项的形式为

$$D(\Lambda)\psi(\Lambda^{-1}x) = \left(\mathbf{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left[\psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x)\right] = \psi(x) - \frac{i}{2}\omega_{\mu\nu}(L^{\mu\nu}+S^{\mu\nu})\psi(x). \tag{4.77}$$

另一方面，根据 (3.6) 式可以将 (4.75) 式左边展开为

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = \left(1 + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\psi(x)\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)$$

$$= \psi(x) - \frac{i}{2}\omega_{\mu\nu}\psi(x)J^{\mu\nu} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu}[\psi(x), J^{\mu\nu}]. \tag{4.78}$$

两相比较，得到

$$[\psi(x), J^{\mu\nu}] = (L^{\mu\nu} + S^{\mu\nu})\psi(x). \tag{4.79}$$

$S^{\mu\nu}$ 的空间分量等价于三维矢量

$$S^i \equiv \frac{1}{2}\varepsilon^{ijk}S^{jk}, \quad \mathbf{S} = (S^{23}, S^{31}, S^{12}). \tag{4.80}$$

再根据 (3.21) 和 (3.64) 式，(4.79) 式的纯空间分量部分可以改写为

$$[\psi(x), \mathbf{J}] = (\mathbf{L} + \mathbf{S})\psi(x). \tag{4.81}$$

上式表明，除了轨道角动量 $\mathbf{L}$，总角动量算符 $\mathbf{J}$ 还生成了由 $\mathbf{S}$ 描述的自旋角动量。

描述半整数自旋经常用到 3 个 2×2 的 **Pauli 矩阵**

$$\sigma^1 \equiv \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{4.82}$$

它们都是既厄米又幺正的：

$$(\sigma^i)^{-1} = (\sigma^i)^\dagger = \sigma^i. \tag{4.83}$$

Pauli 矩阵的两两乘积为

$$\begin{aligned}
(\sigma^1)^2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad (\sigma^2)^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\
(\sigma^3)^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \sigma^1\sigma^2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\sigma^3, \\
\sigma^2\sigma^1 &= \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} -i & \\ & i \end{pmatrix} = -i\sigma^3, \quad \sigma^2\sigma^3 = \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma^1, \\
\sigma^3\sigma^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} & -i \\ -i & \end{pmatrix} = -i\sigma^1, \quad \sigma^3\sigma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma^2, \\
\sigma^1\sigma^3 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i\sigma^2.
\end{aligned} \tag{4.84}$$

归纳起来，有

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k. \tag{4.85}$$

从而可得

$$[\sigma^i, \sigma^j] = i\varepsilon^{ijk}\sigma^k - i\varepsilon^{jik}\sigma^k = 2i\varepsilon^{ijk}\sigma^k, \tag{4.86}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} + i\varepsilon^{ijk}\sigma^k + i\varepsilon^{jik}\sigma^k = 2\delta^{ij}. \tag{4.87}$$

利用 Pauli 矩阵可以将 Dirac 矩阵表示成 2×2 分块形式:

$$\gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}, \tag{4.88}$$

其中 $\mathbf{1}$ 表示 2×2 单位矩阵。容易验证，这样表示的 Dirac 矩阵符合约定 (4.4)，而且满足反对易关系 (4.1):

$$\{\gamma^0,\gamma^0\} = 2\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} = 2\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{00}, \tag{4.89}$$

$$\{\gamma^0,\gamma^i\} = \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} = 0 = 2g^{0i}, \tag{4.90}$$

$$\{\gamma^i,\gamma^j\} = \begin{pmatrix} -\sigma^i\sigma^j - \sigma^j\sigma^i & \\ & -\sigma^i\sigma^j - \sigma^j\sigma^i \end{pmatrix} = -2\delta^{ij}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{ij}. \tag{4.91}$$

实际上，Dirac 矩阵有多种表示方式，(4.88) 式这种表示方式称为 **Weyl 表象**，也称为**手征表象** (chiral representation)。Dirac 矩阵的所有表示方式都是等价的，彼此可以通过相似变换联系起来。

在 Weyl 表象中，由 (4.86) 式可得 $S^{\mu\nu}$ 的空间分量为

$$\begin{aligned} S^{ij} &= \frac{i}{4}[\gamma^i,\gamma^j] = \frac{i}{4}\begin{pmatrix} -\sigma^i\sigma^j + \sigma^j\sigma^i & \\ & -\sigma^i\sigma^j + \sigma^j\sigma^i \end{pmatrix} \\ &= \frac{i}{4}\begin{pmatrix} -2i\varepsilon^{ijk}\sigma^k & \\ & -2i\varepsilon^{ijk}\sigma^k \end{pmatrix} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}, \end{aligned} \tag{4.92}$$

从 Pauli 矩阵的厄米性可知，S^{ij} 是厄米矩阵:

$$(S^{ij})^\dagger = S^{ij}. \tag{4.93}$$

由 (1.98) 式可得

$$S^i = \frac{1}{2}\varepsilon^{ijk}S^{jk} = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{jkl}\begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{4}2\delta^{il}\begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}. \tag{4.94}$$

于是，自旋角动量矩阵的平方为

$$\mathbf{S}^2 = S^iS^i = \frac{1}{4}\begin{pmatrix} \sigma^i\sigma^i & \\ & \sigma^i\sigma^i \end{pmatrix} = \frac{3}{4}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right) = s(s+1). \tag{4.95}$$

上式最后两步省略了 4×4 单位矩阵。可见，Dirac 旋量场 $\psi(x)$ 的自旋量子数是

$$s = \frac{1}{2}. \tag{4.96}$$

经过量子化程序之后，$\psi(x)$ 应当描述**自旋为 1/2** 的粒子。

4.3 Dirac 方程

为了写下 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量，我们需要结合两个旋量场来得到 Lorentz 标量。在 Weyl 表象中，$S^{\mu\nu}$ 的 $0i$ 分量为

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{i}{2}\gamma^0\gamma^i = \frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}. \tag{4.97}$$

由 Pauli 矩阵的厄米性可得

$$(S^{0i})^\dagger = -\frac{i}{2}\begin{pmatrix} -(\sigma^i)^\dagger & \\ & (\sigma^i)^\dagger \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} = -S^{0i}. \tag{4.98}$$

可见，S^{0i} 不是厄米矩阵。于是，当 $\omega_{0i} \neq 0$ 时，

$$D^\dagger(\Lambda) = \left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\right]^\dagger = \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right] \neq \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = D^{-1}(\Lambda), \tag{4.99}$$

即 $D(\Lambda)$ 不是幺正矩阵。因此，一般地，$\psi^\dagger(x)\psi(x)$ 不是 Lorentz 标量：

$$\psi'^\dagger(x')\psi'(x') = \psi^\dagger(x)D^\dagger(\Lambda)D(\Lambda)\psi(x) \neq \psi^\dagger(x)\psi(x). \tag{4.100}$$

根据约定 (4.4)，可得

$$(\gamma^0)^\dagger\gamma^0 = \gamma^0\gamma^0, \quad (\gamma^i)^\dagger\gamma^0 = -\gamma^i\gamma^0 = \gamma^0\gamma^i. \tag{4.101}$$

这两条式子可以合起来写成

$$(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu. \tag{4.102}$$

从而，有

$$(S^{\mu\nu})^\dagger\gamma^0 = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger\gamma^0 = -\frac{i}{4}[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0 = -\frac{i}{4}\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) = \gamma^0 S^{\mu\nu}. \tag{4.103}$$

于是，可得

$$\begin{aligned} D^\dagger(\Lambda)\gamma^0 &= \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\gamma^0 = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]^n\gamma^0 = \gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n \\ &= \gamma^0\exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \gamma^0 D^{-1}(\Lambda). \end{aligned} \tag{4.104}$$

根据上式，定义

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0, \tag{4.105}$$

则它的 Lorentz 变换形式为

$$\bar{\psi}'(x') = \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)D^\dagger(\Lambda)\gamma^0 = \psi^\dagger(x)\gamma^0 D^{-1}(\Lambda) = \bar{\psi}(x)D^{-1}(\Lambda). \tag{4.106}$$

这样一来，$\bar{\psi}(x)\psi(x)$ 就是一个 Lorentz 标量：

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)D(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x). \tag{4.107}$$

$\bar{\psi}(x)\psi(x)$ 这种形式的量属于旋量双线性型 (spinor bilinear)，我们可以使用 $\bar{\psi}(x)$ 构造一些 Lorentz 协变的其它旋量双线性型。$\bar{\psi}(x)i\gamma^5\psi(x)$ 是一个 Lorentz 标量：

$$\bar{\psi}'(x')i\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)i\gamma^5 D(\Lambda)\psi(x) = \bar{\psi}(x)i\gamma^5\psi(x). \tag{4.108}$$

$\bar{\psi}(x)\gamma^\mu\psi(x)$ 和 $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ 都是 Lorentz 矢量：

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x), \tag{4.109}$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu\gamma^5 D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x). \tag{4.110}$$

$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ 是一个 2 阶反对称 Lorentz 张量：

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda)\psi(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x) \tag{4.111}$$

如果将 $\psi(x)$ 看作旋量空间中的列矢量，则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 都是行矢量，因而这些旋量双线性型都只是旋量空间中的 1×1 矩阵，也就是数。由 γ^0 和 γ^5 的厄米性及 (4.102) 式可知，这些旋量双线性型都是厄米的，或者说，都是实数：

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \tag{4.112}$$

$$(\bar{\psi}i\gamma^5\psi)^\dagger = -i\psi^\dagger\gamma^5\gamma^0\psi = i\psi^\dagger\gamma^0\gamma^5\psi = \bar{\psi}i\gamma^5\psi, \tag{4.113}$$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = \psi^\dagger(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \tag{4.114}$$

$$(\bar{\psi}\gamma^\mu\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \tag{4.115}$$

$$(\bar{\psi}\sigma^{\mu\nu}\psi)^\dagger = -\frac{i}{2}\psi^\dagger[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0\psi = -\frac{i}{2}\psi^\dagger\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\psi = \bar{\psi}\sigma^{\mu\nu}\psi. \tag{4.116}$$

此外，包含时空导数的旋量双线性型 $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ 是 Lorentz 标量：

$$\begin{aligned}\bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x') &= \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) = \bar{\psi}(x)\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x)\\ &= \bar{\psi}(x)\delta^\nu{}_\rho\gamma^\rho\partial_\nu\psi(x) = \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x).\end{aligned} \tag{4.117}$$

利用 $\bar{\psi}\gamma^\mu\partial_\mu\psi$ 和 $\bar{\psi}\psi$ 可以写下自由 Dirac 旋量场 $\psi(x)$ 的 **Lorentz 不变拉氏量**

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{4.118}$$

于是，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}. \tag{4.119}$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi}. \tag{4.120}$$

对上式取厄米共轭，得到

$$0 = -i(\gamma^\mu)^\dagger\partial_\mu(\psi^\dagger\gamma^0)^\dagger + m(\psi^\dagger\gamma^0)^\dagger = -i(\gamma^\mu)^\dagger\gamma^0\partial_\mu\psi + m\gamma^0\psi = -\gamma^0(i\gamma^\mu\partial_\mu - m)\psi, \tag{4.121}$$

即

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \tag{4.122}$$

上式就是 **Dirac 方程**，标明旋量指标的形式为

$$[i(\gamma^\mu)_{ab}\partial_\mu - m\delta_{ab}]\psi_b(x) = 0. \tag{4.123}$$

可以验证，Dirac 方程具有 Lorentz 协变性：

$$\begin{aligned}(i\gamma^\mu\partial'_\mu - m)\psi'(x') &= [i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]D(\Lambda)\psi(x) = D(\Lambda)[iD^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x)\\ &= D(\Lambda)[i\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) = D(\Lambda)(i\delta^\nu{}_\rho\gamma^\rho\partial_\nu - m)\psi(x)\\ &= D(\Lambda)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0.\end{aligned} \tag{4.124}$$

对 Dirac 方程 (4.122) 左边乘以 $(-i\gamma^\mu\partial_\mu - m)$，利用反对易关系 (4.1)，可得

$$\begin{aligned}0 &= (-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = \left[\frac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu) + m^2\right]\psi\\ &= \left[\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu + m^2\right]\psi = (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = (\partial^2 + m^2)\psi.\end{aligned} \tag{4.125}$$

也就是说，自由的 Dirac 旋量场 $\psi(x)$ 满足 *Klein-Gordon 方程*

$$(\partial^2 + m^2)\psi(x) = 0. \tag{4.126}$$

由 (4.92) 和 (4.97) 式可以看出，旋量表示的生成元在 Weyl 表象中都是分块对角的，因而它可以分解为两个 2 维表示的直和。相应地，可以把具有 4 个分量的 Dirac 旋量场 ψ 分解为两个二分量旋量 φ_L 和 φ_R：

$$\psi = \begin{pmatrix}\varphi_\mathrm{L}\\ \varphi_\mathrm{R}\end{pmatrix}. \tag{4.127}$$

这样的二分量旋量称为 **Weyl 旋量**，其中，φ_L 称为**左手** (left-handed) Weyl 旋量，φ_R 称为**右手** (right-handed) Weyl 旋量。

用 2×2 单位矩阵和 Pauli 矩阵定义

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma}), \tag{4.128}$$

那么，Weyl 表象中的 Dirac 矩阵 (4.88) 可以简洁地表示成

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu\\ \bar{\sigma}^\mu & \end{pmatrix}. \tag{4.129}$$

从而，Dirac 方程 (4.122) 化为

$$0=(i\gamma^\mu\partial_\mu-m)\psi=\begin{pmatrix}-m & i\sigma^\mu\partial_\mu\\ i\bar{\sigma}^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\varphi_{\mathrm{L}}\\ \varphi_{\mathrm{R}}\end{pmatrix}=\begin{pmatrix}i\sigma^\mu\partial_\mu\varphi_{\mathrm{R}}-m\varphi_{\mathrm{L}}\\ i\bar{\sigma}^\mu\partial_\mu\varphi_{\mathrm{L}}-m\varphi_{\mathrm{R}}\end{pmatrix},\tag{4.130}$$

即

$$\begin{cases}i\bar{\sigma}^\mu\partial_\mu\varphi_{\mathrm{L}}-m\varphi_{\mathrm{R}}=0,\\ i\sigma^\mu\partial_\mu\varphi_{\mathrm{R}}-m\varphi_{\mathrm{L}}=0.\end{cases}\tag{4.131}$$

这是一组相互耦合的方程。如果 $m=0$，方程组中的两个方程就变得相互独立了：

$$i\bar{\sigma}^\mu\partial_\mu\varphi_{\mathrm{L}}=0,\quad i\sigma^\mu\partial_\mu\varphi_{\mathrm{R}}=0.\tag{4.132}$$

这两个独立的方程称为 **Weyl 方程**。可见，非零质量 m 的存在将左手和右手 Weyl 旋量耦合起来。

4.4 Dirac 旋量场的平面波展开

4.4.1 平面波解的一般形式

本小节讨论与表象选取无关。

对于确定的动量 $\mathbf{p}$，我们假设 Dirac 方程具有如下形式的平面波解：

$$\psi_a(x;\mathbf{k})=w_a(k^0,\mathbf{k})e^{-ik\cdot x}.\tag{4.133}$$

其中，系数 $w_a(k^0,\mathbf{k})$ 是四分量旋量，带着一个旋量指标 a。隐去旋量指标，将这个平面波解代入到 Dirac 方程 (4.122) 中，可得

$$0=(i\gamma^\mu\partial_\mu-m)\psi(x;\mathbf{k})=(\gamma^\mu k_\mu-m)w(k^0,\mathbf{k})e^{-ik\cdot x}=(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)w(k^0,\mathbf{k})e^{-ik\cdot x}.\tag{4.134}$$

因此，有

$$(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)w(k^0,\mathbf{k})=0.\tag{4.135}$$

对上式左乘 γ^0，可得

$$[k^0-\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})-m\gamma^0]w(k^0,\mathbf{k})=0.\tag{4.136}$$

通过移项，上式化为

$$[\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})+m\gamma^0]w(k^0,\mathbf{k})=k^0w(k^0,\mathbf{k}).\tag{4.137}$$

这是一个本征值方程，它具有非平庸解的条件是特征多项式 $\det[k^0-\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})-m\gamma^0]$ 为零，即

$$\det[k^0-\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})-m\gamma^0]=0.\tag{4.138}$$

这个方程的根给出 k^0 的本征值，相应的非平庸解是本征矢量。

方程 (4.138) 可化为

$$0=\det[k^0-\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})-m\gamma^0]=\det[\gamma^0(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det(\gamma^0)\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m). \quad (4.139)$$

由 (4.3) 式可得 $[\det(\gamma^0)]^2=\det(\gamma^0\gamma^0)=\det(\mathbf{1})=1$，故 $\det(\gamma^0)\neq 0$。因而方程 (4.138) 等价于

$$\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=0. \quad (4.140)$$

利用 (4.48) 式，上式左边可化为

$$\begin{aligned}\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m) &= \det[(\gamma^5)^2(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det[\gamma^5(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)\gamma^5]\\ &= \det[(\gamma^5)^2(-k^0\gamma^0+\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det[-(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m]. \quad (4.141)\end{aligned}$$

这里第二步用到行列式性质

$$\det(AB)=\det(BA), \quad (4.142)$$

第三步用到 γ^5 与 γ^μ 反对易的性质 (4.50)。由反对易关系 (4.1) 有

$$(k_\mu\gamma^\mu)^2=k_\mu k_\nu\gamma^\mu\gamma^\nu=\frac{1}{2}k_\mu k_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)=k_\mu k_\nu g^{\mu\nu}\mathbf{1}=k^2\mathbf{1}=[(k^0)^2-|\mathbf{k}|^2]\mathbf{1}. \quad (4.143)$$

从而，可得

$$\begin{aligned}[\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]^2 &= \det[(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m]\det[-(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m]\\ &= \det(k_\mu\gamma^\mu-m)\det(-k_\mu\gamma^\mu-m)=\det[(k_\mu\gamma^\mu-m)(-k_\mu\gamma^\mu-m)]\\ &= \det[-(k_\mu\gamma^\mu)^2+m^2]=\det\{[-(k^0)^2+|\mathbf{k}|^2+m^2]\mathbf{1}\}\\ &= [-(k^0)^2+|\mathbf{k}|^2+m^2]^4=[E_{\mathbf{k}}^2-(k^0)^2]^4, \quad (4.144)\end{aligned}$$

其中 $E_{\mathbf{k}}\equiv\sqrt{|\mathbf{k}|^2+m^2}$。于是，方程 (4.140) 化为

$$0=\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=[E_{\mathbf{k}}^2-(k^0)^2]^2=(E_{\mathbf{k}}+k^0)^2(E_{\mathbf{k}}-k^0)^2. \quad (4.145)$$

这个方程有 2 个根 $k^0=\pm E_{\mathbf{k}}$；这 2 个根都是 2 重根，各自对应于 2 个独立的本征矢量，共有 4 个线性无关的本征矢量。

(1) $k^0=E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma),\quad \sigma=1,2. \quad (4.146)$$

因而平面波解中有 2 个正能解，形式为

$$w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma)\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})],\quad \sigma=1,2. \quad (4.147)$$

(2) $k^0=-E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma),\quad \sigma=1,2. \quad (4.148)$$

因而平面波解中有 2 个负能解，形式为

$$w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma)\exp[i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})],\quad \sigma=1,2. \quad (4.149)$$

可以将这 4 个本征矢量的正交归一关系取为

$$\begin{aligned} w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma') &= 2E_{\mathbf{k}}\delta_{\sigma\sigma'}, \quad w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma') = 2E_{\mathbf{k}}\delta_{\sigma\sigma'}, \\ w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma') = 0. \end{aligned} \tag{4.150}$$

按如下定义引入四分量旋量 $u(\mathbf{k};\sigma)$ 和 $v(\mathbf{k};\sigma)$：

$$u(\mathbf{k};\sigma) \equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma), \quad v(-\mathbf{k};\sigma) \equiv w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma), \quad \sigma = 1,2. \tag{4.151}$$

第二个定义式等价于

$$v(\mathbf{k};\sigma) = w^{(-)}(-E_{\mathbf{k}},-\mathbf{k};\sigma). \tag{4.152}$$

于是，Dirac 方程的正能解和负能解可以分别写作

$$\psi^{(+)}(x;\mathbf{k};\sigma) \equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma)\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = u(\mathbf{k};\sigma)\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})], \tag{4.153}$$

$$\psi^{(-)}(x;\mathbf{k};\sigma) \equiv w^{(-)}(-E_{\mathbf{k}},-\mathbf{k};\sigma)\exp[i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = v(\mathbf{k};\sigma)\exp[i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]. \tag{4.154}$$

替换一下动量记号，可得

$$\psi^{(+)}(x;\mathbf{p};\sigma) = u(\mathbf{p};\sigma)e^{-ip\cdot x}, \quad \psi^{(-)}(x;\mathbf{p};\sigma) = v(\mathbf{p};\sigma)e^{ip\cdot x}, \quad p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2+m^2}. \tag{4.155}$$

从而，Dirac 旋量场算符 $\psi(\mathbf{x},t)$ 的平面波展开式可写作

$$\begin{aligned} \psi(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=1}^{2}\left[\psi^{(+)}(x;\mathbf{p};\sigma)a_{\mathbf{p};\sigma} + \psi^{(-)}(x;\mathbf{p};\sigma)b^\dagger_{\mathbf{p};\sigma}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=1}^{2}\left[u(\mathbf{p};\sigma)a_{\mathbf{p};\sigma}e^{-ip\cdot x} + v(\mathbf{p};\sigma)b^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right]. \end{aligned} \tag{4.156}$$

其中，$a_{\mathbf{p};\sigma}$ 是湮灭算符，$b^\dagger_{\mathbf{p};\sigma}$ 是产生算符。一般地，$a_{\mathbf{p};\sigma} \neq b_{\mathbf{p};\sigma}$。

旋量系数 $u(\mathbf{p};\sigma)$ 和 $v(\mathbf{p};\sigma)$ 的正交归一关系为

$$u^\dagger(\mathbf{p};\sigma)u(\mathbf{p};\sigma') = w^{(+)\dagger}(E_{\mathbf{p}},\mathbf{p};\sigma)\,w^{(+)}(E_{\mathbf{p}},\mathbf{p};\sigma') = 2E_{\mathbf{p}}\delta_{\sigma\sigma'}, \tag{4.157}$$

$$v^\dagger(\mathbf{p};\sigma)v(\mathbf{p};\sigma') = w^{(-)\dagger}(-E_{\mathbf{p}},-\mathbf{p};\sigma)\,w^{(-)}(-E_{\mathbf{p}},-\mathbf{p};\sigma') = 2E_{\mathbf{p}}\delta_{\sigma\sigma'}, \tag{4.158}$$

$$u^\dagger(\mathbf{p};\sigma)v(-\mathbf{p};\sigma') = w^{(+)\dagger}(E_{\mathbf{p}},\mathbf{p};\sigma)\,w^{(-)}(-E_{\mathbf{p}},\mathbf{p};\sigma') = 0. \tag{4.159}$$

4.4.2 Weyl 表象中的平面波解

本小节在 *Weyl* 表象中讨论 Dirac 方程的平面波解。

Dirac 旋量场描述自旋为 1/2 的粒子，因而粒子的自旋在动量方向上的投影有 2 种取值，+1/2 和 −1/2，归一化后对应于 2 种螺旋度 $\lambda = \pm$。类似于矢量场的情况，Dirac 旋量场所描述的粒子的状态可以用螺旋度本征值 λ 来表征。因此，无论是平面波解的正能解还是负能解，都能够以 2 种螺旋度本征态作为 2 个独立的本征矢量。

按照这个思路，可以把正能解的 2 个本征矢量记作

$$\psi^{(+)}(x;\mathbf{p},\lambda) = u(\mathbf{p},\lambda)e^{-ip\cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{4.160}$$

根据 Dirac 方程 (4.122)，有

$$0 = (i\gamma^\mu\partial_\mu - m)\psi^{(+)}(x;\mathbf{p},\lambda) = (p_\mu\gamma^\mu - m)u(\mathbf{p},\lambda)e^{-ip\cdot x}, \tag{4.161}$$

即

$$(\not{p} - m)u(\mathbf{p},\lambda) = 0, \tag{4.162}$$

其中，$\not{p}$ 的定义为

$$\not{p} \equiv p_\mu\gamma^\mu. \tag{4.163}$$

这种斜线记号称为 **Dirac 斜线** (slash)，是 R. Feynman 引进的。

将四分量旋量 $u(\mathbf{p},\lambda)$ 分解为两个二分量旋量 $f_\lambda(\mathbf{p})$ 和 $g_\lambda(\mathbf{p})$，

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.164}$$

那么，根据 Weyl 表象中的 Dirac 矩阵表达式 (4.129)，方程 (4.162) 化为

$$0 = (\not{p} - m)u(\mathbf{p},\lambda) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix}\begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu\sigma^\mu g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) \\ p_\mu\bar{\sigma}^\mu f_\lambda(\mathbf{p}) - mg_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.165}$$

即

$$(p\cdot\sigma)g_\lambda(\mathbf{p}) - mf_\lambda(\mathbf{p}) = 0, \tag{4.166}$$

$$(p\cdot\bar{\sigma})f_\lambda(\mathbf{p}) - mg_\lambda(\mathbf{p}) = 0. \tag{4.167}$$

将 (4.129) 式代入反对易关系 (4.1)，可得

$$2g^{\mu\nu}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}, \tag{4.168}$$

故

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}, \tag{4.169}$$

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu}. \tag{4.170}$$

因而，有

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu\sigma^\mu\bar{\sigma}^\nu = \frac{1}{2}p_\mu p_\nu(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu) = \frac{1}{2}p_\mu p_\nu 2g^{\mu\nu} = p^2. \tag{4.171}$$

由方程 (4.167) 可得

$$g_\lambda(\mathbf{p}) = \frac{p\cdot\bar{\sigma}}{m}f_\lambda(\mathbf{p}). \tag{4.172}$$

将上式代入到由方程 (4.166) 得出的关系中，有

$$f_\lambda(\mathbf{p}) = \frac{p\cdot\sigma}{m} g_\lambda(\mathbf{p}) = \frac{1}{m^2}(p\cdot\sigma)(p\cdot\bar{\sigma}) f_\lambda(\mathbf{p}) = \frac{p^2}{m^2} f_\lambda(\mathbf{p}) = f_\lambda(\mathbf{p}). \tag{4.173}$$

可见，关系式 (4.172) 是自洽的。这样的话，只要选取合适的 $f_\lambda(\mathbf{p})$，然后由 (4.164) 和 (4.172) 式得到

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.174}$$

就可以满足方程 (4.162)。

在 Weyl 表象中，根据 (4.94) 式，自旋角动量矩阵 $\mathbf{S}$ 在动量 $\mathbf{p}$ 方向上的投影为

$$\hat{\mathbf{p}}\cdot\mathbf{S} = \frac{1}{2}\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{4.175}$$

归一化后，得到**螺旋度**矩阵

$$2\,\hat{\mathbf{p}}\cdot\mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{4.176}$$

上式的两个分块相同，因此，左手和右手 Weyl 旋量对应的螺旋度矩阵是相同的，都是

$$\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}. \tag{4.177}$$

引入作为螺旋度本征态的二分量旋量 $\xi_\lambda(\mathbf{p})$，满足

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda\,\xi_\lambda(\mathbf{p}), \quad \lambda = \pm. \tag{4.178}$$

我们要求 $\xi_\lambda(\mathbf{p})$ 具有正交归一关系

$$\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'} \tag{4.179}$$

和完备性关系

$$\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}. \tag{4.180}$$

此外，由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 可得

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda|\mathbf{p}|\xi_\lambda(\mathbf{p}) \tag{4.181}$$

我们将 $\xi_\lambda(\mathbf{p})$ 称为**螺旋态**。在实际应用中，可以把螺旋态 $\xi_\lambda(\mathbf{p})$ 取为如下形式：

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1 + ip^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix}. \tag{4.182}$$

可以验证，它们确实是 $\lambda = \pm$ 的本征态：

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_+(\mathbf{p}) = \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1 + ip^2 \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3(|\mathbf{p}|+p^3)+(p^1-ip^2)(p^1+ip^2) \\ (p^1+ip^2)(|\mathbf{p}|+p^3)-p^3(p^1+ip^2)\end{pmatrix}\\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3|\mathbf{p}|+|\mathbf{p}|^2 \\ (p^1+ip^2)|\mathbf{p}|\end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3+|\mathbf{p}| \\ p^1+ip^2\end{pmatrix}\\
&= +\xi_+(\mathbf{p}),
\end{aligned}
\tag{4.183}
$$

$$
\begin{aligned}
(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_-(\mathbf{p}) &= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3\end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3\end{pmatrix}\\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^3(p^1-ip^2)+(p^1-ip^2)(|\mathbf{p}|+p^3) \\ (p^1+ip^2)(-p^1+ip^2)-p^3(|\mathbf{p}|+p^3)\end{pmatrix}\\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} (p^1-ip^2)|\mathbf{p}| \\ -|\mathbf{p}|^2-p^3|\mathbf{p}|\end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^1-ip^2 \\ -|\mathbf{p}|-p^3\end{pmatrix}\\
&= -\xi_-(\mathbf{p}).
\end{aligned}
\tag{4.184}
$$

而且，满足正交归一关系：

$$
\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_+(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2\end{pmatrix}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+ip^2\end{pmatrix}\\
&= \frac{(|\mathbf{p}|+p^3)^2+|p^1+ip^2|^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2+2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1,
\end{aligned}
\tag{4.185}
$$

$$
\begin{aligned}
\xi_-^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} -p^1-ip^2 & |\mathbf{p}|+p^3\end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3\end{pmatrix}\\
&= \frac{|-p^1+ip^2|^2+(|\mathbf{p}|+p^3)^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2+2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1,
\end{aligned}
\tag{4.186}
$$

$$
\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2\end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3\end{pmatrix}\\
&= \frac{-(|\mathbf{p}|+p^3)(p^1-ip^2)+(|\mathbf{p}|+p^3)(p^1-ip^2)}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 0.
\end{aligned}
\tag{4.187}
$$

也满足完备性关系：

$$
\begin{aligned}
&\sum_{\lambda=\pm}\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \xi_+(\mathbf{p})\xi_+^\dagger(\mathbf{p})+\xi_-(\mathbf{p})\xi_-^\dagger(\mathbf{p})\\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} (|\mathbf{p}|+p^3)^2+|-p^1+ip^2|^2 & (|\mathbf{p}|+p^3)(p^1-ip^2)+(|\mathbf{p}|+p^3)(-p^1+ip^2) \\ (|\mathbf{p}|+p^3)(p^1+ip^2)+(|\mathbf{p}|+p^3)(-p^1-ip^2) & |p^1+ip^2|^2+(|\mathbf{p}|+p^3)^2\end{pmatrix}\\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} 2|\mathbf{p}|^2+2p^3|\mathbf{p}| & \\ & 2|\mathbf{p}|^2+2p^3|\mathbf{p}|\end{pmatrix} = \begin{pmatrix} 1 & \\ & 1\end{pmatrix} = \mathbf{1}.
\end{aligned}
\tag{4.188}
$$

当 $p^3=-|\mathbf{p}|$ 时，(4.182) 式失去良好的定义，此时我们可以将螺旋态取成

$$
\xi_+(\mathbf{p}) = \begin{pmatrix} 0 \\ 1\end{pmatrix}, \quad \xi_-(\mathbf{p}) = \begin{pmatrix} -1 \\ 0\end{pmatrix}.
\tag{4.189}
$$

现在，将 $f_\lambda(\mathbf{p})$ 取为

$$f_\lambda(\mathbf{p}) = C_\lambda\, \xi_\lambda(\mathbf{p}), \tag{4.190}$$

其中 C_λ 是常数。从而，利用 (4.181) 式，(4.174) 式可化为

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{E_{\mathbf{p}}+\mathbf{p}\cdot\boldsymbol{\sigma}}{m}\xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{E_{\mathbf{p}}+\lambda|\mathbf{p}|}{m}\xi_\lambda(\mathbf{p}) \end{pmatrix}. \tag{4.191}$$

再取

$$C_\lambda = \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \tag{4.192}$$

则由

$$\sqrt{(E_{\mathbf{p}}+\lambda|\mathbf{p}|)(E_{\mathbf{p}}-\lambda|\mathbf{p}|)} = \sqrt{E_{\mathbf{p}}^2 - \lambda^2|\mathbf{p}|^2} = \sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2} = m, \tag{4.193}$$

有

$$\begin{aligned} C_\lambda \frac{E_{\mathbf{p}}+\lambda|\mathbf{p}|}{m} &= \sqrt{E_{\mathbf{p}}-\lambda|\mathbf{p}|}\,\frac{E_{\mathbf{p}}+\lambda|\mathbf{p}|}{m} = \frac{\sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}}{m}\sqrt{(E_{\mathbf{p}}+\lambda|\mathbf{p}|)(E_{\mathbf{p}}-\lambda|\mathbf{p}|)} \\ &= \sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}. \end{aligned} \tag{4.194}$$

于是，得到 $u(\mathbf{p},\lambda)$ 的螺旋态表达式

$$u(\mathbf{p},\lambda) = \begin{pmatrix} \sqrt{E_{\mathbf{p}}-\lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.195}$$

其中，$\omega_\lambda(\mathbf{p})$ 定义为

$$\omega_\lambda(\mathbf{p}) \equiv \sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}, \tag{4.196}$$

它是关于 $\mathbf{p}$ 的偶函数:

$$\omega_\lambda(-\mathbf{p}) = \omega_\lambda(\mathbf{p}). \tag{4.197}$$

这样的话，根据 (4.176) 式，$u(\mathbf{p},\lambda)$ 是螺旋度本征态，本征值为 λ：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{S})u(\mathbf{p},\lambda) = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda\, u(\mathbf{p},\lambda). \tag{4.198}$$

另一方面，可以把负能解的 2 个本征矢量记作

$$\psi^{(-)}(x;\mathbf{p},\lambda) = v(\mathbf{p},\lambda)e^{ip\cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}. \tag{4.199}$$

根据 Dirac 方程 (4.122)，有

$$0 = (i\gamma^\mu\partial_\mu - m)\psi^{(-)}(x;\mathbf{p},\lambda) = (-p_\mu\gamma^\mu - m)v(\mathbf{p},\lambda)e^{ip\cdot x}, \tag{4.200}$$

即

$$(\not{p} + m)v(\mathbf{p},\lambda) = 0. \tag{4.201}$$

同样，将四分量旋量 $v(\mathbf{p},\lambda)$ 分解为两个二分量旋量 $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$，

$$v(\mathbf{p},\lambda)=\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix}, \tag{4.202}$$

则有

$$0=(\not{p}+m)v(\mathbf{p},\lambda)=\begin{pmatrix}m & \sigma^\mu p_\mu\\ \bar{\sigma}^\mu p_\mu & m\end{pmatrix}\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix}=\begin{pmatrix}p_\mu\sigma^\mu\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})\\ p_\mu\bar{\sigma}^\mu\tilde{f}_\lambda(\mathbf{p})+m\tilde{g}_\lambda(\mathbf{p})\end{pmatrix}, \tag{4.203}$$

即

$$(p\cdot\sigma)\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=0, \tag{4.204}$$

$$(p\cdot\bar{\sigma})\tilde{f}_\lambda(\mathbf{p})+m\tilde{g}_\lambda(\mathbf{p})=0. \tag{4.205}$$

由方程 (4.205) 可得

$$\tilde{g}_\lambda(\mathbf{p})=-\frac{p\cdot\bar{\sigma}}{m}\tilde{f}_\lambda(\mathbf{p}). \tag{4.206}$$

将上式代入到由方程 (4.204) 得出的关系中，根据 (4.171) 式，有

$$\tilde{f}_\lambda(\mathbf{p})=-\frac{p\cdot\sigma}{m}\tilde{g}_\lambda(\mathbf{p})=\frac{1}{m^2}(p\cdot\sigma)(p\cdot\bar{\sigma})\tilde{f}_\lambda(\mathbf{p})=\frac{p^2}{m^2}\tilde{f}_\lambda(\mathbf{p})=\tilde{f}_\lambda(\mathbf{p}). \tag{4.207}$$

可见，关系式 (4.206) 是自洽的。

现在，将 $\tilde{f}_\lambda(\mathbf{p})$ 取为

$$\tilde{f}_\lambda(\mathbf{p})=\tilde{C}_\lambda\,\xi_{-\lambda}(\mathbf{p}), \tag{4.208}$$

其中 $\tilde{C}_\lambda$ 是常数。在这里，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，而非 $\xi_\lambda(\mathbf{p})$。这种取法的原因将在 4.5.4 小节中说明，现在姑且接受这种选择。从而，有

$$v(\mathbf{p},\lambda)=\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ -\dfrac{p\cdot\bar{\sigma}}{m}\tilde{f}_\lambda(\mathbf{p})\end{pmatrix}=\tilde{C}_\lambda\begin{pmatrix}\xi_{-\lambda}(\mathbf{p})\\ -\dfrac{E_{\mathbf{p}}+\mathbf{p}\cdot\boldsymbol{\sigma}}{m}\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=\tilde{C}_\lambda\begin{pmatrix}\xi_{-\lambda}(\mathbf{p})\\ -\dfrac{E_{\mathbf{p}}-\lambda|\mathbf{p}|}{m}\xi_{-\lambda}(\mathbf{p})\end{pmatrix}. \tag{4.209}$$

再取

$$\tilde{C}_\lambda=-\lambda\sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}, \tag{4.210}$$

则由

$$\begin{aligned}-\tilde{C}_\lambda\frac{E_{\mathbf{p}}-\lambda|\mathbf{p}|}{m}&=\lambda\sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}\,\frac{E_{\mathbf{p}}-\lambda|\mathbf{p}|}{m}=\lambda\frac{\sqrt{E_{\mathbf{p}}-\lambda|\mathbf{p}|}}{m}\sqrt{(E_{\mathbf{p}}+\lambda|\mathbf{p}|)(E_{\mathbf{p}}-\lambda|\mathbf{p}|)}\\&=\lambda\sqrt{E_{\mathbf{p}}-\lambda|\mathbf{p}|},\end{aligned} \tag{4.211}$$

可得 $v(\mathbf{p},\lambda)$ 的螺旋态表达式

$$v(\mathbf{p},\lambda)=\begin{pmatrix}-\lambda\sqrt{E_{\mathbf{p}}+\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\\ \lambda\sqrt{E_{\mathbf{p}}-\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}. \tag{4.212}$$

这样一来，$v(\mathbf{p},\lambda)$ 是螺旋度本征态，本征值为 $-\lambda$：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{S})v(\mathbf{p},\lambda)=\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=-\lambda\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=-\lambda\,v(\mathbf{p},\lambda). \tag{4.213}$$

根据 $\xi_\lambda(\mathbf{p})$ 的正交归一关系 (4.179)，可以验证，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足 (4.157) 和 (4.158) 式表示的正交归一关系：

$$\begin{aligned}
u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')&=\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix}\\
&=[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})+\omega_\lambda(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p})=[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})\delta_{\lambda\lambda'}+\omega_\lambda(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'}\\
&=[\omega_{-\lambda}^2(\mathbf{p})+\omega_\lambda^2(\mathbf{p})]\delta_{\lambda\lambda'}=[(E_\mathbf{p}-\lambda|\mathbf{p}|)+(E_\mathbf{p}+\lambda|\mathbf{p}|)]\delta_{\lambda\lambda'}=2E_\mathbf{p}\delta_{\lambda\lambda'},
\end{aligned}\tag{4.214}$$

$$\begin{aligned}
v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')&=\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix}\\
&=\lambda\lambda'[\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})+\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})=\lambda\lambda'[\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})+\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'}\\
&=\lambda^2[\omega_\lambda^2(\mathbf{p})+\omega_{-\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'}=[(E_\mathbf{p}+\lambda|\mathbf{p}|)+(E_\mathbf{p}-\lambda|\mathbf{p}|)]\delta_{\lambda\lambda'}=2E_\mathbf{p}\delta_{\lambda\lambda'},
\end{aligned}\tag{4.215}$$

依照螺旋态的本征值方程 (4.178)，可得

$$(-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p})=-\lambda\,\xi_{-\lambda}(-\mathbf{p}), \tag{4.216}$$

从而，有

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p})=\lambda\,\xi_{-\lambda}(-\mathbf{p}). \tag{4.217}$$

可见，$\xi_{-\lambda}(-\mathbf{p})$ 与 $\xi_\lambda(\mathbf{p})$ 服从相同的本征值方程，这意味着 $\xi_{-\lambda}(-\mathbf{p})\propto\xi_\lambda(\mathbf{p})$，故

$$\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p})\propto\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p})=\delta_{\lambda\lambda'}. \tag{4.218}$$

于是，(4.159) 式表示的正交关系也成立：

$$\begin{aligned}
u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')&=\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p})\end{pmatrix}\\
&=\lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(-\mathbf{p})+\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(-\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p})\\
&\propto\lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})+\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'}\\
&\propto\lambda[-\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})+\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{\lambda\lambda'}=0.
\end{aligned}\tag{4.219}$$

整理一下，旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足如下**正交归一关系**：

$$u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')=v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')=2E_\mathbf{p}\delta_{\lambda\lambda'},\quad u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')=v^\dagger(-\mathbf{p},\lambda)u(\mathbf{p},\lambda')=0. \tag{4.220}$$

此外，由 (4.193) 式有

$$\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})=\sqrt{(E_\mathbf{p}+\lambda|\mathbf{p}|)(E_\mathbf{p}-\lambda|\mathbf{p}|)}=m. \tag{4.221}$$

从而，利用

$$\begin{aligned}
\bar{u}(\mathbf{p},\lambda) &= u^\dagger(\mathbf{p},\lambda)\gamma^0 = \begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\\
&= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}, \qquad (4.222)\\
\bar{v}(\mathbf{p},\lambda) &= v^\dagger(\mathbf{p},\lambda)\gamma^0 = \begin{pmatrix}-\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\\
&= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}, \qquad (4.223)
\end{aligned}$$

可得

$$\begin{aligned}
\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix}\\
&= [\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = 2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} = 2m\delta_{\lambda\lambda'}, \quad (4.224)\\
\bar{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix}\\
&= -\lambda\lambda'[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = -2\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'}\\
&= -2m\delta_{\lambda\lambda'}, \qquad (4.225)\\
\bar{u}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix}\\
&= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\,\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\
&= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\,\delta_{\lambda,-\lambda'}\\
&= -\lambda[-\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p})\,]\,\delta_{\lambda,-\lambda'} = 0, \qquad (4.226)\\
\bar{v}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix}\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\,\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{-\lambda,\lambda'}\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p}) - \omega_{\lambda}(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\,\delta_{-\lambda,\lambda'} = 0. \qquad (4.227)
\end{aligned}$$

整理一下，有

$$\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 2m\delta_{\lambda\lambda'}, \quad \bar{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = -2m\delta_{\lambda\lambda'}, \quad \bar{u}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = \bar{v}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 0. \qquad (4.228)$$

另一方面，利用等式

$$\begin{aligned}
(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p} + \mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p} + \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \qquad (4.229)\\
(p\cdot\sigma)\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p} - \mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p} - \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \qquad (4.230)
\end{aligned}$$

以及 (4.221) 式和 $\xi_\lambda(\mathbf{p})$ 的完备性关系 (4.180)，可得

$$
\begin{aligned}
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm}\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}m\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\\ (p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & m\xi_\lambda^\dagger(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar{\sigma} & m\end{pmatrix} = p_\mu\gamma^\mu + m.
\end{aligned}\tag{4.231}
$$

通过等式

$$
(p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}+\mathbf{p}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}-\lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \tag{4.232}
$$

$$
(p\cdot\sigma)\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}-\mathbf{p}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}+\lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p}), \tag{4.233}
$$

则可以得到

$$
\begin{aligned}
\sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm}\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}\begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\,\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}-\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda^2\omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ \lambda^2\omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda^2\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}-m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ (p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar{\sigma} & -m\end{pmatrix} = p_\mu\gamma^\mu - m.
\end{aligned}\tag{4.234}
$$

整理一下，有如下螺旋度求和关系，或者说，**自旋求和关系**：

$$
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) = \not{p}+m, \quad \sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) = \not{p}-m. \tag{4.235}
$$

用 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 可以把 Dirac 旋量场算符 $\psi(\mathbf{x},t)$ 的平面波展开式写作

$$
\begin{aligned}
\psi(\mathbf{x},t) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\Big[\psi^{(+)}(x;\mathbf{p},\lambda)a_{\mathbf{p},\lambda} + \psi^{(-)}(x;\mathbf{p},\lambda)b_{\mathbf{p},\lambda}^\dagger\Big]\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\Big[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + v(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x}\Big].
\end{aligned}\tag{4.236}
$$

从而，有

$$
\psi^\dagger(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\Big[u^\dagger(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\Big], \tag{4.237}
$$

$$
\bar{\psi}(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\Big[\bar{u}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x} + \bar{v}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\Big]. \tag{4.238}
$$

4.4.3　哈密顿量和产生湮灭算符

根据 (4.119) 式，$\psi(x)$ 对应的共轭动量密度是

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \tag{4.239}$$

它的平面波展开式为

$$\pi(\mathbf{x},t) = i\psi^\dagger(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{i}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]. \tag{4.240}$$

自由运动的旋量场 $\psi(x)$ 满足 Dirac 方程 (4.122)，相应地，拉氏量 (4.118) 化为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{4.241}$$

于是，根据 (1.119) 式，自由 Dirac 旋量场的哈密顿量密度为

$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L} = \pi\partial_0\psi = i\psi^\dagger\partial_0\psi. \tag{4.242}$$

从而，哈密顿量为

$$\begin{aligned}
H \;&= \int d^3x\,\mathcal{H} = \int d^3x\,\psi^\dagger i\partial_0\psi\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]\\
&\qquad\qquad\times\left[q_0u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - q_0v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,q_0\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}\right.\\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} - u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\\
&\qquad\left. + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,E_\mathbf{q}\left\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(E_\mathbf{p}-E_\mathbf{q})t}\right.\right.\\
&\qquad\qquad\left. - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_\mathbf{p}-E_\mathbf{q})t}\right]\\
&\qquad\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\left[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(E_\mathbf{p}+E_\mathbf{q})t}\right.\\
&\qquad\qquad\left.\left. + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(E_\mathbf{p}+E_\mathbf{q})t}\right]\right\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_\mathbf{p}}\,E_\mathbf{p}\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right.\\
&\qquad\left. - u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t} + v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_\mathbf{p}}\,E_\mathbf{p}\left(2E_\mathbf{p}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - 2E_\mathbf{p}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right)
\end{aligned}$$

$$= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b^\dagger_{\mathbf{p},\lambda} \right). \tag{4.243}$$

倒数第二步用到正交归一关系 (4.220)。

另一方面，利用正交归一关系 (4.220)，可得

$$\begin{aligned}
&\int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})\cdot t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&\qquad\qquad + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a_{\mathbf{p},\lambda'} + u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t} \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}}\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda'}) = \sqrt{2E_{\mathbf{p}}}\, a_{\mathbf{p},\lambda}.
\end{aligned} \tag{4.244}$$

从而，湮灭算符 $a_{\mathbf{p},\lambda}$ 和产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 可以表示为

$$a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad a^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \psi^\dagger(\mathbf{x},t)u(\mathbf{p},\lambda). \tag{4.245}$$

同理，可以推出

$$\begin{aligned}
&\int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \Big[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&\qquad\qquad + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b^\dagger_{\mathbf{p},\lambda'} \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}b^\dagger_{\mathbf{p},\lambda'}\right) = \sqrt{2E_{\mathbf{p}}}\, b^\dagger_{\mathbf{p},\lambda}.
\end{aligned} \tag{4.246}$$

于是，产生算符 $b^\dagger_{\mathbf{p},\lambda}$ 和湮灭算符 $b_{\mathbf{p},\lambda}$ 可以表示为

$$b^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \psi^\dagger(\mathbf{x},t)v(\mathbf{p},\lambda). \tag{4.247}$$

4.5 Dirac 旋量场的正则量子化

4.5.1 用等时对易关系量子化 Dirac 旋量场的困难

在标量场和矢量场的正则量子化程序中，我们先假设场算符与其共轭动量密度算符满足等时对易关系 (2.57)，然后推导出产生湮灭算符的对易关系，再通过计算给出正定的哈密顿量（对于无质量矢量场，需要用弱 Lorenz 规范条件来得到正定的哈密顿量期待值），从而说明在量子场论中使用正则量子化方法是合理的。在本小节中，我们将尝试用类似的等时对易关系对 Dirac 旋量场进行量子化，不过，我们会发现这种方法并不能给出正定的哈密顿量，因而是不可行的。

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=0. \tag{4.248}$$

这里已经将旋量指标明显地写出来。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=0. \tag{4.249}$$

接下来，我们计算产生湮灭算符的对易关系。由 (4.245) 式和正交归一关系 (4.220)，可得

$$\begin{aligned}
[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]u_b(\mathbf{q},\lambda')\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')\\
&= \frac{1}{2E_\mathbf{p}}u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{4.250}$$

另外，有

$$[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}]=\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b^\dagger(\mathbf{q},\lambda')[\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=0. \tag{4.251}$$

由 (4.247) 式和正交归一关系 (4.220)，可得

$$\begin{aligned}
[b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')[\psi_a^\dagger(\mathbf{x},t),\psi_b(\mathbf{y},t)]v_a(\mathbf{p},\lambda)\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')v_a(\mathbf{p},\lambda)(-\delta_{ba})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&= -\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}v^\dagger(\mathbf{q},\lambda')v(\mathbf{p},\lambda)\\
&= -\frac{1}{2E_\mathbf{p}}v^\dagger(\mathbf{p},\lambda')v(\mathbf{p},\lambda)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=-(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{4.252}$$

注意，这个结果非同寻常地多了一个负号。此外，还有

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}[\psi_a^\dagger(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)]v_a(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda') = 0, \tag{4.253}$$

$$[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b^\dagger(\mathbf{q},\lambda')[\psi_a(\mathbf{x},t), \psi_b(\mathbf{y},t)] = 0, \tag{4.254}$$

以及

$$\begin{aligned}
[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)[\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)]v_b(\mathbf{q},\lambda') \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\, e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda') \\
&= \frac{1}{2E_{\mathbf{p}}}e^{2iE_{\mathbf{p}}t}u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0.
\end{aligned} \tag{4.255}$$

上式最后一步用到正交归一关系 (4.220)。

整理起来，通过等时对易关系 (4.248) 得到的产生湮灭算符对易关系为

$$\begin{aligned}
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0, \\
[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] &= -(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0, \\
[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] &= [b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0.
\end{aligned} \tag{4.256}$$

利用这样的对易关系，可以把哈密顿量 (4.243) 化为

$$\begin{aligned}
H &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right) \\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right) + (2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{d^3p}{(2\pi)^3}\,2E_{\mathbf{p}}.
\end{aligned} \tag{4.257}$$

上式最后一行第二项是零点能。在第一项中由 $a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为正，但由 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为负。从而，粒子数密度 $b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}$ 越大，场的总能量越少，这显然是非物理的。因此，用正则对易关系 (4.248) 对 Dirac 旋量场进行量子化是不可行的。

4.5.2 用等时反对易关系量子化 Dirac 旋量场

从 (4.257) 式的计算过程可以看出，如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 位置的同时可以改变圆括号中第二项的符号，就可以得到正定的哈密顿量。这意味着我们需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的对易关系，而是反对易关系。为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的反对易关系，则需要舍弃等时对易关系 (4.248)，代之以等时反对易关系

$$\{\psi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)\} = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad \{\psi_a(\mathbf{x},t), \psi_b(\mathbf{y},t)\} = \{\pi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)\} = 0. \tag{4.258}$$

采用反对易关系进行量子化的方法称为 **Jordan-Wigner 量子化**。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时反对易关系

$$\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad \{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=\{\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}=0.\tag{4.259}$$

接下来，我们计算产生湮灭算符的反对易关系。计算过程与上一小节类似，只是我们要将 (4.250) 至 (4.255) 式中表示对易的方括号改成表示反对易的花括号。因此，可得

$$\begin{aligned}\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}u_b(\mathbf{q},\lambda')\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\end{aligned}\tag{4.260}$$

和

$$\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b^\dagger(\mathbf{q},\lambda')\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=0.\tag{4.261}$$

另外，有

$$\begin{aligned}\{b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')\{\psi_a^\dagger(\mathbf{x},t),\psi_b(\mathbf{y},t)\}v_a(\mathbf{p},\lambda)\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')v_a(\mathbf{p},\lambda)\delta_{ba}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}v^\dagger(\mathbf{q},\lambda')v(\mathbf{p},\lambda)\\ &= \frac{1}{2E_\mathbf{p}}v^\dagger(\mathbf{p},\lambda')v(\mathbf{p},\lambda)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned}\tag{4.262}$$

与 (4.252) 式不同，上式的结果具有正常的符号。此外，还有

$$\{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\} = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\{\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}v_a(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')=0,\tag{4.263}$$

$$\{a_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\} = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b^\dagger(\mathbf{q},\lambda')\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=0,\tag{4.264}$$

以及

$$\begin{aligned}\{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}v_b(\mathbf{q},\lambda')\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})=0.\end{aligned}\tag{4.265}$$

整理起来，通过等时反对易关系 (4.258) 得到的产生湮灭算符反对易关系为

$$\begin{aligned}\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\} &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a^\dagger_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=0,\\ \{b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\} &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\}=\{b^\dagger_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\}=0,\\ \{a_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\} &= \{b_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=\{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\}=\{a^\dagger_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\}=0.\end{aligned}\tag{4.266}$$

$a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 各自描述一种粒子。利用这样的反对易关系，可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b^\dagger_{\mathbf{p},\lambda} \right) \\ &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda} \right) - (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \end{aligned} \tag{4.267}$$

上式最后一行第二项是零点能。第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和，它是正定的。可见，用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的。

利用 (4.8) 式和反对易关系 (4.266)，可得哈密顿量 H 与产生湮灭算符的对易子为

$$\begin{aligned} [H, a^\dagger_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q},\lambda'} a_{\mathbf{q},\lambda'} + b^\dagger_{\mathbf{q},\lambda'} b_{\mathbf{q},\lambda'},\, a^\dagger_{\mathbf{p},\lambda} \right] \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \Big(a^\dagger_{\mathbf{q},\lambda'} \{a_{\mathbf{q},\lambda'}, a^\dagger_{\mathbf{p},\lambda}\} - \{a^\dagger_{\mathbf{q},\lambda'}, a^\dagger_{\mathbf{p},\lambda}\} a_{\mathbf{q},\lambda'} \\ &\qquad + b^\dagger_{\mathbf{q},\lambda'} \{b_{\mathbf{q},\lambda'}, a^\dagger_{\mathbf{p},\lambda}\} - \{b^\dagger_{\mathbf{q},\lambda'}, a^\dagger_{\mathbf{p},\lambda}\} b_{\mathbf{q},\lambda'} \Big) \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a^\dagger_{\mathbf{q},\lambda'} \{a_{\mathbf{q},\lambda'}, a^\dagger_{\mathbf{p},\lambda}\} \\ &= \sum_{\lambda'} \int d^3q\, E_{\mathbf{q}} a^\dagger_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = E_{\mathbf{p}} a^\dagger_{\mathbf{p},\lambda}, \end{aligned} \tag{4.268}$$

$$\begin{aligned} [H, a_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q},\lambda'} a_{\mathbf{q},\lambda'} + b^\dagger_{\mathbf{q},\lambda'} b_{\mathbf{q},\lambda'},\, a_{\mathbf{p},\lambda} \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(-\{a^\dagger_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}\} a_{\mathbf{q},\lambda'} \right) \\ &= -\sum_{\lambda'} \int d^3q\, E_{\mathbf{q}} a_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p},\lambda}, \end{aligned} \tag{4.269}$$

$$\begin{aligned} [H, b^\dagger_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q},\lambda'} a_{\mathbf{q},\lambda'} + b^\dagger_{\mathbf{q},\lambda'} b_{\mathbf{q},\lambda'},\, b^\dagger_{\mathbf{p},\lambda} \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} b^\dagger_{\mathbf{q},\lambda'} \{b_{\mathbf{q},\lambda'}, b^\dagger_{\mathbf{p},\lambda}\} \\ &= \sum_{\lambda'} \int d^3q\, E_{\mathbf{q}} b^\dagger_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = E_{\mathbf{p}} b^\dagger_{\mathbf{p},\lambda}, \end{aligned} \tag{4.270}$$

$$\begin{aligned} [H, b_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a^\dagger_{\mathbf{q},\lambda'} a_{\mathbf{q},\lambda'} + b^\dagger_{\mathbf{q},\lambda'} b_{\mathbf{q},\lambda'}, b_{\mathbf{p},\lambda} \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(-\{b^\dagger_{\mathbf{q},\lambda'}, b_{\mathbf{p},\lambda}\} b_{\mathbf{q},\lambda'} \right) \\ &= -\sum_{\lambda'} \int d^3q\, E_{\mathbf{q}} b_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -E_{\mathbf{p}} b_{\mathbf{p},\lambda}. \end{aligned} \tag{4.271}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则

$$H\,|E\rangle = E\,|E\rangle\,. \tag{4.272}$$

从而，可得

$$\begin{aligned} Ha^\dagger_{\mathbf{p},\lambda}\,|E\rangle &= (a^\dagger_{\mathbf{p},\lambda} H + E_{\mathbf{p}} a^\dagger_{\mathbf{p},\lambda})\,|E\rangle = (E + E_{\mathbf{p}}) a^\dagger_{\mathbf{p},\lambda}\,|E\rangle\,, \\ Ha_{\mathbf{p},\lambda}\,|E\rangle &= (a_{\mathbf{p},\lambda} H - E_{\mathbf{p}} a_{\mathbf{p},\lambda})\,|E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p},\lambda}\,|E\rangle\,, \end{aligned}$$

$$
\begin{aligned}
Hb^\dagger_{\mathbf{p},\lambda}\left|E\right> &= (b^\dagger_{\mathbf{p},\lambda}H + E_{\mathbf{p}}b^\dagger_{\mathbf{p},\lambda})\left|E\right> = (E+E_{\mathbf{p}})b^\dagger_{\mathbf{p},\lambda}\left|E\right>,\\
Hb_{\mathbf{p},\lambda}\left|E\right> &= (b_{\mathbf{p},\lambda}H - E_{\mathbf{p}}b_{\mathbf{p},\lambda})\left|E\right> = (E-E_{\mathbf{p}})b_{\mathbf{p},\lambda}\left|E\right>. \qquad (4.273)
\end{aligned}
$$

可见，当 $a^\dagger_{\mathbf{p},\lambda}\left|E\right>$ 和 $b^\dagger_{\mathbf{p},\lambda}\left|E\right>$ 不为零时，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 的作用都是使能量本征值增加 $E_{\mathbf{p}}$；当 $a_{\mathbf{p},\lambda}\left|E\right>$ 和 $b_{\mathbf{p},\lambda}\left|E\right>$ 不为零时，湮灭算符 $a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的作用都是使能量本征值减少 $E_{\mathbf{p}}$。

根据 (1.158) 式，Dirac 旋量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\,\pi\nabla\psi = \int d^3x\,\psi^\dagger(-i\nabla)\psi\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}+v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]\\
&\qquad\times\left[\mathbf{q}\,u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}-\mathbf{q}\,v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\mathbf{q}\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}\right.\\
&\qquad -v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\\
&\qquad \left.+v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\mathbf{q}\left\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right.\right.\\
&\qquad\qquad \left.-v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right]\\
&\qquad\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{q})\left[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right.\\
&\qquad\qquad \left.\left.+v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right]\right\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left[\mathbf{p}\,u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}-\mathbf{p}\,v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right.\\
&\qquad \left.+\mathbf{p}\,u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}-\mathbf{p}\,v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\,\mathbf{p}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}-2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right)-2\delta^{(3)}(\mathbf{0})\int d^3p\,\mathbf{p}\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right). \qquad (4.274)
\end{aligned}
$$

倒数第四步用到正交归一关系 (4.220)，倒数第二步用到反对易关系 (4.266)。总动量是所有动量模式所有螺旋度所有粒子贡献的动量之和。

4.5.3　U(1) 整体对称性

类似于复标量场，Dirac 旋量场也具有 U(1) 整体对称性。对 Dirac 旋量场 $\psi(x)$ 作 U(1) 整体变换

$$\psi'(x) = e^{iq\theta}\psi(x), \tag{4.275}$$

则 $\psi^\dagger(x)$ 和 $\bar\psi(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x)e^{-iq\theta}, \quad [\bar\psi(x)]' = \bar\psi'(x) = [\psi'(x)]^\dagger\gamma^0 = \bar\psi(x)e^{-iq\theta}. \tag{4.276}$$

在此变换下，拉氏量 (4.118) 不变：

$$\begin{aligned}\mathcal{L}'(x) &= \bar\psi'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar\psi(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi'(x)\\ &= \bar\psi(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x).\end{aligned} \tag{4.277}$$

容易验证，4.3 节中列举的旋量双线性型都在这种 U(1) 整体变换下不变。因此，用这些旋量双线性型构造的拉氏量都具有 U(1) 整体对称性。

U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + iq\theta\psi(x). \tag{4.278}$$

由于 $\delta x^\mu = 0$，根据 (1.136) 式可得

$$\bar\delta\psi = \delta\psi = iq\theta\psi. \tag{4.279}$$

按照 (1.141) 式，相应的 Noether 守恒流为

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar\delta\psi = i\bar\psi\gamma^\mu(iq\theta\psi) = -q\theta\bar\psi\gamma^\mu\psi. \tag{4.280}$$

扔掉无穷小参数 $-\theta$，定义

$$J^\mu \equiv q\bar\psi\gamma^\mu\psi, \tag{4.281}$$

则 Noether 定理给出

$$\partial_\mu J^\mu = 0. \tag{4.282}$$

相应的守恒荷为

$$\begin{aligned}Q &= \int d^3x\, J^0 = q\int d^3x\,\bar\psi\gamma^0\psi = q\int d^3x\,\psi^\dagger\psi\\ &= q\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]\\ &\qquad\times\left[u(\mathbf{k},\lambda')a_{\mathbf{k},\lambda'}e^{-ik\cdot x} + v(\mathbf{k},\lambda')b^\dagger_{\mathbf{k},\lambda'}e^{ik\cdot x}\right]\\ &= q\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{i(p-k)\cdot x}\right.\end{aligned}$$

$$
\begin{aligned}
&\quad +v^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{-i(p-k)\cdot x}+u^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{i(p+k)\cdot x}\\
&\quad +v^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(p+k)\cdot x}\Big]\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{k})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\\
&\qquad\qquad +v^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\Big]\\
&\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{k})\Big[u^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\\
&\qquad\qquad +v^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\Big]\Big\}\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\\
&\quad +u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}+v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big]\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right)\\
&= q\sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(q\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-q\,b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right)+2\delta^{(3)}(\mathbf{0})\int d^3p\,q.
\end{aligned}
\tag{4.283}
$$

上式第二项是零点荷。从第一项的形式可以看出，由 $a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 描述的粒子是**正粒子**，具有的荷为 q；由 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 描述的粒子是**反粒子**，具有的荷为 $-q$。除去零点荷，总荷是所有动量模式所有螺旋度所有正反粒子贡献的荷之和。

4.5.4 粒子态

对于自由的 Dirac 旋量场，真空态定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$
a_{\mathbf{p},\lambda}|0\rangle=b_{\mathbf{p},\lambda}|0\rangle=0,\tag{4.284}
$$

满足

$$
\langle 0|0\rangle=1,\quad H|0\rangle=E_{\text{vac}}|0\rangle,\quad E_{\text{vac}}=-2\delta^{(3)}(\mathbf{0})\int d^3p\,E_{\mathbf{p}}.\tag{4.285}
$$

动量为 $\mathbf{p}$、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$
|\mathbf{p},\lambda,+\rangle\equiv\sqrt{2E_{\mathbf{p}}}\,a^\dagger_{\mathbf{p},\lambda}|0\rangle,\quad |\mathbf{p},\lambda,-\rangle\equiv\sqrt{2E_{\mathbf{p}}}\,b^\dagger_{\mathbf{p},\lambda}|0\rangle.\tag{4.286}
$$

根据 (4.268) 和 (4.270) 式，有

$$
\begin{aligned}
H|\mathbf{p},\lambda,+\rangle &= \sqrt{2E_{\mathbf{p}}}\,Ha^\dagger_{\mathbf{p},\lambda}|0\rangle=\sqrt{2E_{\mathbf{p}}}\,(a^\dagger_{\mathbf{p},\lambda}H+E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda})|0\rangle\\
&= \sqrt{2E_{\mathbf{p}}}\,(E_{\text{vac}}+E_{\mathbf{p}})a^\dagger_{\mathbf{p},\lambda}|0\rangle=(E_{\text{vac}}+E_{\mathbf{p}})|\mathbf{p},\lambda,+\rangle,\\
H|\mathbf{p},\lambda,-\rangle &= \sqrt{2E_{\mathbf{p}}}\,Hb^\dagger_{\mathbf{p},\lambda}|0\rangle=\sqrt{2E_{\mathbf{p}}}\,(b^\dagger_{\mathbf{p},\lambda}H+E_{\mathbf{p}}b^\dagger_{\mathbf{p},\lambda})|0\rangle
\end{aligned}
\tag{4.287}
$$

$$= \sqrt{2E_{\mathbf{p}}}\,(E_{\mathrm{vac}}+E_{\mathbf{p}})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle = (E_{\mathrm{vac}}+E_{\mathbf{p}})\,|\mathbf{p},\lambda,-\rangle\,. \tag{4.288}$$

可见，$|\mathbf{p},\lambda,+\rangle$ 和 $|\mathbf{p},\lambda,-\rangle$ 都比真空态多了一份能量 $E_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}$。

将 $\psi(x)$ 的平面波解 (4.236) 代入 (4.81) 式左边，得

$$[\psi(x),\mathbf{J}] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left\{u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}]e^{-ip\cdot x}+v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}]e^{ip\cdot x}\right\}, \tag{4.289}$$

代入右边，得

$$(\mathbf{L}+\mathbf{S})\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}(-i\mathbf{x}\times\nabla+\mathbf{S})\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[(\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+(-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]. \tag{4.290}$$

可见，对于动量模式 $\mathbf{p}$ 和螺旋度 λ，有

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}] = (\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},\quad v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}] = (-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}. \tag{4.291}$$

根据 (4.198) 和 (4.213) 式，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态，因而

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = 2\,\hat{\mathbf{p}}\cdot(\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = (2\,\hat{\mathbf{p}}\cdot\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = \lambda\,u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}, \tag{4.292}$$

$$v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = 2\,\hat{\mathbf{p}}\cdot(-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda} = (2\,\hat{\mathbf{p}}\cdot\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda} = -\lambda\,v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}. \tag{4.293}$$

故

$$[a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = \lambda\,a_{\mathbf{p},\lambda},\quad [b^\dagger_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = -\lambda\,b^\dagger_{\mathbf{p},\lambda}. \tag{4.294}$$

由于 $\mathbf{J}$ 是厄米算符，对第一式取厄米共轭可得

$$\lambda\,a^\dagger_{\mathbf{p},\lambda} = [a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}]^\dagger = (2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda} - a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) = [2\,\hat{\mathbf{p}}\cdot\mathbf{J},a^\dagger_{\mathbf{p},\lambda}]. \tag{4.295}$$

于是，有

$$[2\,\hat{\mathbf{p}}\cdot\mathbf{J},a^\dagger_{\mathbf{p},\lambda}] = \lambda\,a^\dagger_{\mathbf{p},\lambda},\quad [2\,\hat{\mathbf{p}}\cdot\mathbf{J},b^\dagger_{\mathbf{p},\lambda}] = \lambda\,b^\dagger_{\mathbf{p},\lambda}. \tag{4.296}$$

$\mathbf{J}$ 是总角动量算符，真空态 $|0\rangle$ 不具有角动量，所以满足

$$\mathbf{J}\,|0\rangle = \mathbf{0}. \tag{4.297}$$

由此，可得

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J})+\lambda\,a^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\,a^\dagger_{\mathbf{p},\lambda}\,|0\rangle\,, \tag{4.298}$$

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [b^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J})+\lambda\,b^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\,b^\dagger_{\mathbf{p},\lambda}\,|0\rangle\,. \tag{4.299}$$

在没有轨道角动量的情况下，$2\,\hat{\mathbf{p}}\cdot\mathbf{J}$ 是螺旋度算符。因此，上面两式说明 $|\mathbf{p},\lambda,+\rangle$ 和 $|\mathbf{p},\lambda,-\rangle$ 都是螺旋度本征态，本征值为 λ：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})\,|\mathbf{p},\lambda,\pm\rangle = \lambda\,|\mathbf{p},\lambda,\pm\rangle\,. \tag{4.300}$$

这正是我们所期望的。

以上讨论表明，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的正粒子，另一个产生算符 $b^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的反粒子。正粒子和反粒子具有相同的质量 m。

在 (4.208) 式中，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，使得 $v(\mathbf{p},\lambda)$ 的螺旋度本征值为 $-\lambda$，从而得到 $b^\dagger_{\mathbf{p},\lambda}|0\rangle$ 的螺旋度本征值为 λ 的结果。如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$，依照上述推导，$b^\dagger_{\mathbf{p},\lambda}|0\rangle$ 的螺旋度本征值就会变成 $-\lambda$；也就是说，$b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 将描述螺旋度为 $-\lambda$ 的反粒子。这不符合我们的记号，因此，我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 (4.208) 式的形式。

由反对易关系 (4.266)，可得

$$\begin{aligned}
a_{\mathbf{p},\lambda}|\mathbf{q},\lambda',+\rangle &= \sqrt{2E_\mathbf{q}}\,a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{p},\lambda}]|0\rangle\\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})|0\rangle, \qquad (4.301)\\
b_{\mathbf{p},\lambda}|\mathbf{q},\lambda',-\rangle &= \sqrt{2E_\mathbf{q}}\,b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{p},\lambda}]|0\rangle\\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})|0\rangle. \qquad (4.302)
\end{aligned}$$

可以看出，湮灭算符 $a_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 $\mathbf{p}$、螺旋度为 λ 的正粒子，湮灭算符 $b_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 $\mathbf{p}$、螺旋度为 λ 的反粒子。

将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle \equiv \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{p}_3}E_{\mathbf{p}_4}}\,a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle. \qquad (4.303)$$

根据反对易关系 (4.266)，有

$$\begin{aligned}
a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle &= -a^\dagger_{\mathbf{p}_2,\lambda_2}a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle = -a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle\\
&= -a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_4,\lambda_4}b^\dagger_{\mathbf{p}_3,\lambda_3}|0\rangle = -b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle\\
&= -a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_4,\lambda_4}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle = -b^\dagger_{\mathbf{p}_4,\lambda_4}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_1,\lambda_1}|0\rangle. \qquad (4.304)
\end{aligned}$$

从而，可得

$$\begin{aligned}
|\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle,\\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle,\\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_3,\lambda_3,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle,\\
|\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle,\\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle,\\
|\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_1,\lambda_1,+\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle. \qquad (4.305)
\end{aligned}$$

也就是说，交换任意两个粒子，得到的态相差一个负号，故多粒子态对于全同粒子交换是反对称的。这说明旋量场描述的粒子是**费米子** (fermion)，服从 Fermi-Dirac 统计。得到这个结论的

关键在于两个产生算符相互反对易。对于两个相同的产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 或 $b^\dagger_{\mathbf{p},\lambda}$，反对易关系导致

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}|0\rangle = -a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}|0\rangle, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}|0\rangle = -b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}|0\rangle, \tag{4.306}$$

故

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}|0\rangle = 0, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}|0\rangle = 0. \tag{4.307}$$

这说明在没有其它自由度的情况下，不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态，这就是 **Pauli 不相容原理**。

在第 2 章和第 3 章中，我们分别讨论了自旋为 0 的标量场和自旋为 1 的矢量场，合适的处理方式是通过对易关系对它们进行量子化，因而它们都描述玻色子。另一方面，在本章中，我们需要采用反对易关系才能对自旋为 1/2 的旋量场进行合适的量子化，因而旋量场描述的粒子是费米子。实际上，这样的状况是普遍的，存在**自旋–统计定理**：整数自旋的物理场必须用对易关系进行量子化，对应的粒子是玻色子；半整数自旋的物理场必须用反对易关系进行量子化，对应的粒子是费米子。

将两个正费米子组成的双粒子态记为

$$|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+\rangle \equiv \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\,a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle, \tag{4.308}$$

则双粒子态的内积关系是

$$\begin{aligned}
&\langle \mathbf{q}_1,\lambda_1',+;\,\mathbf{q}_2,\lambda_2',+\,|\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+\rangle\\
=\ &\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle\\
=\ &\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle\\
&\qquad -\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle\Big]\\
=\ &\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle\\
&\qquad -(2\pi)^3\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}|0\rangle\Big]\\
=\ &\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\big[(2\pi)^6\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\\
&\qquad -(2\pi)^6\delta_{\lambda_2\lambda_1'}\delta_{\lambda_1\lambda_2'}\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\big]\\
=\ &4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\,\big[\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\\
&\qquad -\delta_{\lambda_1\lambda_2'}\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\big].
\end{aligned} \tag{4.309}$$

上式最后两行方括号中第二项前面有一个负号，由产生湮灭算符的反对易关系引起。这是双费米子态内积关系与双玻色子态内积关系 (2.127) 在形式上的不同之处。

第 5 章　量子场的相互作用

第 2、3、4 章分别讨论了标量场、矢量场、旋量场的正则量子化。不过，这些讨论只涉及自由量子场的拉氏量，没有考虑到量子场的相互作用。像 (2.60)、(3.84) 和 (4.118) 式这样的自由场拉氏量包含着动能项和质量项，它们都是双线性的，即每一项均包含 2 个场算符。如果我们更进一步，考虑拉氏量包含多于 2 个场算符的项，则这些项将描述场的**相互作用** (interaction)。在局域场论中，拉氏量 $\mathcal{L}(x)$ 中的相互作用项只能包含同一个时空点处的几个场，例如 $[\phi(x)]^3$；不能包含处于不同时空点上的场，例如 $[\phi(x)]^2\phi(y)$。这样可以保持理论的*因果性* (causality)。

相互作用项可以只包含同一种场，从而描述场的*自相互作用* (self-interaction)。例如，对于实标量场 $\phi(x)$，可以构造如下拉氏量：

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5.1}$$

前两项与 (2.60) 式相同，第三项描述四个实标量场的自相互作用，其中，λ 是一个**耦合常数** (coupling constant)，它的大小决定耦合的强度。$\mathcal{L}_{\phi^4}$ 描述的理论称为实标量场的 **ϕ^4 理论**。

在自然单位制中，时空坐标 x^μ 的量纲是能量量纲的倒数，即 $[x^\mu]=[E]^{-1}$，故时空导数的量纲是 $[\partial_\mu]=[E]$，时空体积元的量纲则是 $[d^4x]=[E]^{-4}$。由于作用量 $S=\int d^4x\,\mathcal{L}$ 没有量纲，拉氏量的量纲是

$$[\mathcal{L}] = [E]^4. \tag{5.2}$$

于是，从拉氏量 (5.1) 的第一项可以看出，标量场的量纲是

$$[\phi] = [E]. \tag{5.3}$$

从而，$[\phi^4]=[E]^4$，故 $[\lambda]=1$，即耦合常数 λ 是无量纲的。

相互作用项也可以涉及不同类型的场。例如，用实标量场 $\phi(x)$ 和 Dirac 旋量场 $\psi(x)$ 可以构造拉氏量

$$\mathcal{L}_{\mathrm{Yukawa}} = \mathcal{L}_{\mathrm{S}} + \mathcal{L}_{\mathrm{D}} + \mathcal{L}_{\mathrm{Y}}, \tag{5.4}$$

其中，

$$\mathcal{L}_{\mathrm{S}} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 \tag{5.5}$$

包含 ϕ 的动能项和质量项，

$$\mathcal{L}_{\mathrm{D}} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_\psi\bar{\psi}\psi \tag{5.6}$$

包含 ψ 的动能项和质量项，而相互作用项

$$\mathcal{L}_{\mathrm{Y}} = -y\phi\bar{\psi}\psi \tag{5.7}$$

描述标量场 ϕ 与旋量场 ψ 之间的 **Yukawa 相互作用**，这里 y 是耦合常数。由拉氏量 (5.6) 的第一项可以看出，旋量场的量纲是 $[E]^{3/2}$，故

$$[\psi] = [\bar{\psi}] = [E]^{3/2}. \tag{5.8}$$

因此，$[\phi\bar{\psi}\psi] = [E]^4$，于是 Yukawa 耦合常数 y 没有量纲。这类相互作用最先由汤川秀树 (Hideki Yukawa) 于 1935 年提出，当时引入 π 介子 (对应于 ϕ) 来传递核子 (对应于 ψ) 之间的强相互作用。

存在相互作用时，场的经典运动方程是非线性的。例如，由 Euler-Lagrange 方程 (1.116) 可得，ϕ^4 理论的场方程为

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3. \tag{5.9}$$

如果像 Yukawa 相互作用那样，相互作用项包含不同类型的场，则会得到多个相互耦合的场方程。这样的场方程在经典场论中不容易求解，在量子场论中就更加困难了。所幸的是，当耦合常数（如 λ、y）比较小时，在**微扰论** (perturbation theory) 中利用微扰级数展开可以得到比较可靠的近似解。本章主要介绍用微扰论处理量子场相互作用的思路。

如果拉氏量中的相互作用项 $\mathcal{L}_{\mathrm{int}}$ 不包含场 $\phi_a(x)$ 的时空导数 $\partial_\mu\phi_a$，则 $\partial\mathcal{L}_{\mathrm{int}}/\partial\dot{\phi}_a = 0$。上面两个例子都属于这种情况。按照定义式 (1.117)，此时场的共轭动量密度 $\pi_a(x)$ 不会受到 $\mathcal{L}_{\mathrm{int}}(\phi_a)$ 的影响，因而与没有相互作用时的量相同。这样的话，等时对易关系 (2.57) 或等时反对易关系 (4.258) 不会受到影响，我们可以继续采用这些关系。将哈密顿量密度 $\mathcal{H}$ 分解成自由部分 $\mathcal{H}_{\mathrm{free}}$（与没有相互作用时的哈密顿量密度相同）和相互作用部分 $\mathcal{H}_{\mathrm{int}}$，

$$\mathcal{H} = \mathcal{H}_{\mathrm{free}} + \mathcal{H}_{\mathrm{int}}, \tag{5.10}$$

则根据定义式 (1.119) 有

$$\mathcal{H}_{\mathrm{int}}(\phi_a) = -\mathcal{L}_{\mathrm{int}}(\phi_a). \tag{5.11}$$

从而，哈密顿量中描述相互作用的项是

$$H_{\mathrm{int}} = \int d^3x\, \mathcal{H}_{\mathrm{int}}(\phi_a) = -\int d^3x\, \mathcal{L}_{\mathrm{int}}(\phi_a). \tag{5.12}$$

如果 $\mathcal{L}_{\mathrm{int}}$ 包含场的时空导数 $\partial_\mu\phi_a$，则共轭动量密度 $\pi_a(x)$ 将与没有相互作用的情况不同。此时，正则量子化方法并不方便，更容易的处理方法是采用路径积分量子化。因此，本章讨论仅局限于 $\mathcal{L}_{\mathrm{int}}$ 不包含 $\partial_\mu\phi_a$ 的情况，其余情况留待路径积分量子化方法处理。

5.1 相互作用绘景

在 2.2 节中已经介绍过，当系统的哈密顿量 H 不含时间（这对于封闭系统是成立的）时，可以建立 Heisenberg 绘景。Heisenberg 绘景中的不含时态矢 $|\Psi\rangle^{\mathrm{H}}$ 和含时算符 $O^{\mathrm{H}}(t)$（场算符或描述物理量的算符）与 Schrödinger 绘景中的含时态矢 $|\Psi(t)\rangle^{\mathrm{S}}$ 和不含时算符 O^{S} 之间的关系为

$$|\Psi\rangle^{\mathrm{H}} = e^{iHt}|\Psi(t)\rangle^{\mathrm{S}}, \quad O^{\mathrm{H}}(t) = e^{iHt}O^{\mathrm{S}}e^{-iHt}. \tag{5.13}$$

由 $[H,H]=0$，有

$$e^{iHt}He^{-iHt} = He^{iHt}e^{-iHt} = H. \tag{5.14}$$

可见，哈密顿量 H 在这两种绘景中是相同的:

$$H^{\mathrm{H}} = H^{\mathrm{S}} = H. \tag{5.15}$$

此外，可以得到

$$\begin{aligned} i\partial_0 O^{\mathrm{H}}(t) &= (i\partial_0 e^{iHt})O^{\mathrm{S}}e^{-iHt} + e^{iHt}O^{\mathrm{S}}(i\partial_0 e^{-iHt}) = -He^{iHt}O^{\mathrm{S}}e^{-iHt} + e^{iHt}O^{\mathrm{S}}e^{-iHt}H \\ &= [e^{iHt}O^{\mathrm{S}}e^{-iHt}, H], \end{aligned} \tag{5.16}$$

即 Heisenberg 绘景中的含时算符 $O^{\mathrm{H}}(t)$ 满足 **Heisenberg 运动方程**

$$i\frac{\partial}{\partial t}O^{\mathrm{H}}(t) = [O^{\mathrm{H}}(t), H]. \tag{5.17}$$

由于 Heisenberg 绘景能够明确地处理场算符的时间依赖性，前面章节中自由场的正则量子化程序都是在这个绘景中进行的。为便于讨论，接下来以实标量场为例进行表述。自由实标量场 $\phi(x)$ 的哈密顿量可以用产生湮灭算符表达成 (2.95) 式的形式:

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{5.18}$$

这里我们省略了零点能，因为零点能是一个 c 数，只决定总能量的零点，不会影响下面的讨论。湮灭算符 $a_{\mathbf{p}}$ 和产生算符 $a^\dagger_{\mathbf{p}}$ 不依赖于时间 t，它们实际上是 *Schrödinger* 绘景中的算符。由 (2.98) 式，可得

$$\begin{aligned} [a_{\mathbf{p}}, (-iHt)^{(1)}] &= [a_{\mathbf{p}}, -iHt] = -it[a_{\mathbf{p}}, H] = -iE_{\mathbf{p}}ta_{\mathbf{p}}, \\ [a_{\mathbf{p}}, (-iHt)^{(2)}] &= [[a_{\mathbf{p}}, -iH^{(1)}t], -iHt] = -iE_{\mathbf{p}}t[a_{\mathbf{p}}, H] = (-iE_{\mathbf{p}}t)^2 a_{\mathbf{p}}, \\ &\cdots \\ [a_{\mathbf{p}}, (-iHt)^{(n)}] &= (-iE_{\mathbf{p}}t)^n a_{\mathbf{p}}. \end{aligned} \tag{5.19}$$

从而，由 (4.22) 式可以推出 Heisenberg 绘景中的湮灭算符为

$$a^{\mathrm{H}}_{\mathbf{p}}(t) = e^{iHt}a_{\mathbf{p}}e^{-iHt} = \sum_{n=0}^{\infty}\frac{1}{n!}[a_{\mathbf{p}}, (-iHt)^{(n)}] = \sum_{n=0}^{\infty}\frac{1}{n!}(-iE_{\mathbf{p}}t)^n a_{\mathbf{p}} = e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}}, \tag{5.20}$$

而相应的产生算符 $a_{\mathbf{p}}^{\mathrm{H}\dagger}(t)$ 满足

$$e^{iHt}a_{\mathbf{p}}^{\dagger}e^{-iHt}=a_{\mathbf{p}}^{\mathrm{H}\dagger}(t)=e^{iE_{\mathbf{p}}t}a_{\mathbf{p}}^{\dagger}. \tag{5.21}$$

根据这两条关系，可以把自由实标量场的平面波展开式 (2.75) 表示成

$$\phi^{\mathrm{H}}(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}^{\mathrm{H}}(t)e^{i\mathbf{p}\cdot\mathbf{x}}+a_{\mathbf{p}}^{\mathrm{H}\dagger}(t)e^{-i\mathbf{p}\cdot\mathbf{x}}\right]. \tag{5.22}$$

在最右边的表达式中，场算符的时间依赖性只包含在 Heisenberg 绘景中的产生湮灭算符里面。反过来，在 Schrödinger 绘景中，自由实标量场的平面波展开式为

$$\begin{aligned}\phi^{\mathrm{S}}(\mathbf{x})&=e^{-iHt}\phi^{\mathrm{H}}(\mathbf{x},t)e^{iHt}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[e^{-iHt}a_{\mathbf{p}}^{\mathrm{H}}(t)e^{iHt}e^{i\mathbf{p}\cdot\mathbf{x}}+e^{-iHt}a_{\mathbf{p}}^{\mathrm{H}\dagger}(t)e^{iHt}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\\&=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}+a_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).\end{aligned} \tag{5.23}$$

可见，场算符在 Schrödinger 绘景中确实不依赖于时间。同样，将共轭动量密度的展开式 (2.77) 变换到 Schrödinger 绘景中，则共轭动量密度也不依赖于时间：

$$\begin{aligned}\pi^{\mathrm{S}}(\mathbf{x})&=e^{-iHt}\pi^{\mathrm{H}}(\mathbf{x},t)e^{iHt}=\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}e^{-iHt}\left[a_{\mathbf{p}}^{\mathrm{H}}(t)e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}^{\mathrm{H}\dagger}(t)e^{-i\mathbf{p}\cdot\mathbf{x}}\right]e^{iHt}\\&=\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).\end{aligned} \tag{5.24}$$

我们在 2.2 节中提到，正则对易关系的形式与绘景无关。这一点很容易验证，比如，实标量场的等时对易关系 (2.78) 在 Schrödinger 绘景中化为

$$[\phi^{\mathrm{S}}(\mathbf{x}),\pi^{\mathrm{S}}(\mathbf{y})]=i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad[\phi^{\mathrm{S}}(\mathbf{x}),\phi^{\mathrm{S}}(\mathbf{y})]=[\pi^{\mathrm{S}}(\mathbf{x}),\pi^{\mathrm{S}}(\mathbf{y})]=0. \tag{5.25}$$

如果从这些正则对易关系和展开式 (5.23)、(5.24) 出发，可以推出产生湮灭算符的对易关系，结果必定与在 Heisenberg 绘景中导出的 (2.92) 式相同。于是，可以进一步导出哈密顿量的表达式 (5.18)。这说明在 Schrödinger 绘景中进行计算也会得到自洽结果。

现在，考虑存在相互作用的情况，假设系统的哈密顿量 $H=H^{\mathrm{S}}=H^{\mathrm{H}}$ 在 Schrödinger 绘景中分解为两个部分

$$H=H_0^{\mathrm{S}}+H_1^{\mathrm{S}}, \tag{5.26}$$

其中，主要部分 H_0^{S} 是自由（没有相互作用）的哈密顿量，微扰部分 H_1^{S} 描述相互作用，只给出较小的影响。此时，可以建立**相互作用绘景** (interaction picture)，它也称为 Dirac 绘景。建立方式是把主要部分 H_0^{S} 的影响塞进态矢里面，将态矢定义为

$$|\Psi(t)\rangle^{\mathrm{I}}=e^{iH_0^{\mathrm{S}}t}|\Psi(t)\rangle^{\mathrm{S}}, \tag{5.27}$$

算符定义为

$$O^{\mathrm{I}}(t)=e^{iH_0^{\mathrm{S}}t}O^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t}. \tag{5.28}$$

这样一来，相互作用绘景中哈密顿量的自由部分与 Schrödinger 绘景相同，

$$H_0^{\mathrm{I}} = e^{iH_0^{\mathrm{S}}t} H_0^{\mathrm{S}} e^{-iH_0^{\mathrm{S}}t} = H_0^{\mathrm{S}}; \tag{5.29}$$

但总哈密顿量不同，

$$H^{\mathrm{I}} = e^{iH_0^{\mathrm{S}}t} H e^{-iH_0^{\mathrm{S}}t}; \tag{5.30}$$

微扰部分则满足

$$H_1^{\mathrm{I}} = e^{iH_0^{\mathrm{S}}t} H_1^{\mathrm{S}} e^{-iH_0^{\mathrm{S}}t} = e^{iH_0^{\mathrm{S}}t}(H - H_0^{\mathrm{S}}) e^{-iH_0^{\mathrm{S}}t} = H^{\mathrm{I}} - H_0^{\mathrm{S}} = H^{\mathrm{I}} - H_0^{\mathrm{I}}. \tag{5.31}$$

此外，由 (5.13) 式有

$$|\Psi(t)\rangle^{\mathrm{S}} = e^{-iHt}|\Psi\rangle^{\mathrm{H}}, \quad O^{\mathrm{S}} = e^{-iHt} O^{\mathrm{H}}(t) e^{iHt}, \tag{5.32}$$

故相互作用绘景与 Heisenberg 绘景之间的关系为

$$|\Psi(t)\rangle^{\mathrm{I}} = e^{iH_0^{\mathrm{S}}t} e^{-iHt}|\Psi\rangle^{\mathrm{H}}, \quad O^{\mathrm{I}}(t) = e^{iH_0^{\mathrm{S}}t} e^{-iHt} O^{\mathrm{H}}(t) e^{iHt} e^{-iH_0^{\mathrm{S}}t}. \tag{5.33}$$

于是，等时对易关系的形式不变，如

$$\begin{aligned}
[\phi^{\mathrm{I}}(\mathbf{x}, y), \pi^{\mathrm{I}}(\mathbf{y}, t)] &= [e^{iH_0^{\mathrm{S}}t} e^{-iHt} \phi^{\mathrm{H}}(\mathbf{x}, y) e^{iHt} e^{-iH_0^{\mathrm{S}}t}, e^{iH_0^{\mathrm{S}}t} e^{-iHt} \pi^{\mathrm{H}}(\mathbf{y}, t) e^{iHt} e^{-iH_0^{\mathrm{S}}t}] \\
&= e^{iH_0^{\mathrm{S}}t} e^{-iHt} [\phi^{\mathrm{H}}(\mathbf{x}, y), \pi^{\mathrm{H}}(\mathbf{y}, t)] e^{iHt} e^{-iH_0^{\mathrm{S}}t} = e^{iH_0^{\mathrm{S}}t} e^{-iHt} i\delta^{(3)}(\mathbf{x} - \mathbf{y}) e^{iHt} e^{-iH_0^{\mathrm{S}}t} \\
&= i\delta^{(3)}(\mathbf{x} - \mathbf{y}).
\end{aligned} \tag{5.34}$$

当 $t = 0$ 时，三种绘景是一致的，

$$|\Psi(0)\rangle^{\mathrm{I}} = |\Psi(0)\rangle^{\mathrm{S}} = |\Psi\rangle^{\mathrm{H}}, \quad O^{\mathrm{I}}(0) = O^{\mathrm{S}} = O^{\mathrm{H}}(0). \tag{5.35}$$

在任意 t 时刻，均有

$${}^{\mathrm{I}}\langle\Psi(t)| O^{\mathrm{I}}(t) |\Psi(t)\rangle^{\mathrm{I}} = {}^{\mathrm{S}}\langle\Psi(t)| O^{\mathrm{S}} |\Psi(t)\rangle^{\mathrm{S}} = {}^{\mathrm{H}}\langle\Psi| O^{\mathrm{H}}(t) |\Psi\rangle^{\mathrm{H}}, \tag{5.36}$$

因而三种绘景描述相同的物理。如果没有相互作用，$H = H_0^{\mathrm{S}}$，则相互作用绘景与 Heisenberg 绘景相同。

在 Schrödinger 绘景中，**Schrödinger 方程**是

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle^{\mathrm{S}} = H|\Psi(t)\rangle^{\mathrm{S}}. \tag{5.37}$$

由此可得

$$\begin{aligned}
i\partial_0|\Psi(t)\rangle^{\mathrm{I}} &= \left(i\partial_0 e^{iH_0^{\mathrm{S}}t}\right)|\Psi(t)\rangle^{\mathrm{S}} + e^{iH_0^{\mathrm{S}}t} i\partial_0|\Psi(t)\rangle^{\mathrm{S}} = \left(-H_0^{\mathrm{S}} e^{iH_0^{\mathrm{S}}t} + e^{iH_0^{\mathrm{S}}t} H\right)|\Psi(t)\rangle^{\mathrm{S}} \\
&= \left(-H_0^{\mathrm{S}} + e^{iH_0^{\mathrm{S}}t} H e^{-iH_0^{\mathrm{S}}t}\right) e^{iH_0^{\mathrm{S}}t}|\Psi(t)\rangle^{\mathrm{S}} = \left(-H_0^{\mathrm{I}} + H^{\mathrm{I}}\right) e^{iH_0^{\mathrm{S}}t}|\Psi(t)\rangle^{\mathrm{S}},
\end{aligned} \tag{5.38}$$

即

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle^{\mathrm{I}} = H_1^{\mathrm{I}}|\Psi(t)\rangle^{\mathrm{I}}. \tag{5.39}$$

这是态矢 $|\Psi(t)\rangle^{\mathrm{I}}$ 的演化方程。可见，在相互作用绘景中，态矢的演化只由相互作用哈密顿量 H_1^{I} 决定。另一方面，有

$$\begin{aligned} i\partial_0 O^{\mathrm{I}}(t) &= (i\partial_0 e^{iH_0^{\mathrm{S}}t})O^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t} + e^{iH_0^{\mathrm{S}}t}O^{\mathrm{S}}(i\partial_0 e^{-iH_0^{\mathrm{S}}t}) \\ &= -H_0^{\mathrm{S}}e^{iH_0^{\mathrm{S}}t}O^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t} + e^{iH_0^{\mathrm{S}}t}O^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t}H_0^{\mathrm{S}} = [e^{iH_0^{\mathrm{S}}t}O^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t}, H_0^{\mathrm{S}}], \end{aligned} \tag{5.40}$$

即

$$i\frac{\partial}{\partial t}O^{\mathrm{I}}(t) = [O^{\mathrm{I}}(t), H_0^{\mathrm{S}}]. \tag{5.41}$$

这个方程表明相互作用绘景中算符的演化只由自由哈密顿量 $H_0^{\mathrm{S}} = H_0^{\mathrm{I}}$ 决定。

综上，在相互作用绘景中，态矢的演化规律与 Schrödinger 绘景中的运动方程 (5.37) 相同，但必须将那里的总哈密顿量 H 换成相互作用哈密顿量 H_1^{I}, 这部分演化属于**动力学 (dynamics) 演化**；算符的演化规律与 Heisenberg 绘景中的运动方程 (5.17) 相同，但必须将那里的总哈密顿量 H 换成自由哈密顿量 H_0^{I}, 这部分演化属于**运动学 (kinematics) 演化**。在 Heisenberg 绘景中，对未加微扰的系统求出各个算符之间的关系之后，加入微扰一般会让这些关系发生改变。幸运的是，加入微扰之后各个算符在相互作用绘景中的关系仍然与加入微扰之前它们在 Heisenberg 绘景中的关系相同，可以直接套用原来的公式。这就是相互作用绘景的好处。

因此，在相互作用绘景中，具有相互作用的场算符的平面波展开式将与没有相互作用的场算符在 Heisenberg 绘景中的展开式相同。于是，在存在相互作用的情况下，我们仍然可以沿用第 2、3、4 章中导出的许多自由场关系式，比如产生湮灭算符的对易或反对易关系。

5.1.1　例：实标量场

下面以实标量场为例讨论相互作用绘景。假设 $t = 0$ 时，实标量场 $\phi(x)$ 的平面波展开式与自由场展开式 (5.23) 和 (5.24) 一样，

$$\phi^{\mathrm{I}}(\mathbf{x},0) = \phi^{\mathrm{H}}(\mathbf{x},0) = \phi^{\mathrm{S}}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{5.42}$$

$$\pi^{\mathrm{I}}(\mathbf{x},0) = \pi^{\mathrm{H}}(\mathbf{x},0) = \pi^{\mathrm{S}}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \tag{5.43}$$

其中，产生湮灭算符 $a_{\mathbf{p}}^\dagger$ 和 $a_{\mathbf{p}}$ 满足对易关系 (2.92)。哈密顿量的自由部分 H_0^{S} 具有 (5.18) 式的形式：

$$H_0^{\mathrm{S}} = \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{5.44}$$

类似于 (5.19) 式，我们有

$$[a_{\mathbf{p}}, (-iH_0^{\mathrm{S}}t)^{(n)}] = (-iE_{\mathbf{p}}t)^n a_{\mathbf{p}}. \tag{5.45}$$

从而由 (4.22) 式可得

$$a^{\mathrm{I}}_{\mathbf{p}}(t)=e^{iH^{\mathrm{S}}_0t}a_{\mathbf{p}}e^{-iH^{\mathrm{S}}_0t}=e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}},\quad a^{\mathrm{I}\dagger}_{\mathbf{p}}(t)=e^{iH^{\mathrm{S}}_0t}a^{\dagger}_{\mathbf{p}}e^{-iH^{\mathrm{S}}_0t}=e^{iE_{\mathbf{p}}t}a^{\dagger}_{\mathbf{p}}. \tag{5.46}$$

于是，相互作用绘景中任意 t 时刻的场算符展开式为

$$\begin{aligned}\phi^{\mathrm{I}}(\mathbf{x},t) &= e^{iH^{\mathrm{S}}_0t}\phi^{\mathrm{S}}(\mathbf{x})e^{-iH^{\mathrm{S}}_0t}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a^{\mathrm{I}}_{\mathbf{p}}(t)e^{i\mathbf{p}\cdot\mathbf{x}}+a^{\mathrm{I}\dagger}_{\mathbf{p}}(t)e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}e^{i\mathbf{p}\cdot\mathbf{x}}+a^{\dagger}_{\mathbf{p}}e^{iE_{\mathbf{p}}t}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right),\end{aligned} \tag{5.47}$$

共轭动量密度的展开式为

$$\pi^{\mathrm{I}}(\mathbf{x},t)=e^{iH^{\mathrm{S}}_0t}\pi^{\mathrm{S}}(\mathbf{x})e^{-iH^{\mathrm{S}}_0t}=\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right). \tag{5.48}$$

正如所期望的，这两个式子与自由实标量场在 Heisenberg 绘景中的展开式 (2.75) 和 (2.77) 一致。

因此，根据产生湮灭算符的对易关系 (2.92)，可以证明 $\phi^{\mathrm{I}}(x)$ 和 $\pi^{\mathrm{I}}(x)$ 满足与 (2.78) 形式相同的等时对易关系

$$[\phi^{\mathrm{I}}(\mathbf{x},t),\pi^{\mathrm{I}}(\mathbf{y},t)]=i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\phi^{\mathrm{I}}(\mathbf{x},t),\phi^{\mathrm{I}}(\mathbf{y},t)]=[\pi^{\mathrm{I}}(\mathbf{x},t),\pi^{\mathrm{I}}(\mathbf{y},t)]=0. \tag{5.49}$$

可以验证，场算符展开式符合演化方程 (5.41)：类似于 (2.97) 式和 (2.98) 式，可以推出

$$[a_{\mathbf{p}},H^{\mathrm{S}}_0]=E_{\mathbf{p}}a_{\mathbf{p}},\quad [a^{\dagger}_{\mathbf{p}},H^{\mathrm{S}}_0]=-E_{\mathbf{p}}a^{\dagger}_{\mathbf{p}}, \tag{5.50}$$

从而，有

$$\begin{aligned}i\frac{\partial}{\partial t}\phi^{\mathrm{I}}(\mathbf{x},t) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(E_{\mathbf{p}}a_{\mathbf{p}}e^{-ip\cdot x}-E_{\mathbf{p}}a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right)\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left([a_{\mathbf{p}},H^{\mathrm{S}}_0]e^{-ip\cdot x}+[a^{\dagger}_{\mathbf{p}},H^{\mathrm{S}}_0]e^{ip\cdot x}\right)=[\phi^{\mathrm{I}}(\mathbf{x},t),H^{\mathrm{S}}_0].\end{aligned} \tag{5.51}$$

5.1.2　例：有质量矢量场

不难将上述讨论推广到复标量场、无质量矢量场和 Dirac 旋量场。但是，推广到有质量矢量场 $A^\mu(x)$ 却会得到不同寻常的结果，原因在于 $A^0(x)$ 不是一个独立的场分量，不具备相应的共轭动量密度和正则对易关系，因而在绘景变换中具有特殊的性质。

假设参与相互作用的有质量矢量场具有拉氏量

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1, \tag{5.52}$$

其中，自由场的拉氏量为

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \tag{5.53}$$

相互作用项为

$$\mathcal{L}_1 = J_\mu A^\mu. \tag{5.54}$$

此处，$J_\mu(x)$ 是由其它的场组成的流，如 $g\bar{\psi}(x)\gamma_\mu\psi(x)$。根据 (1.116) 式及

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu + J^\nu, \tag{5.55}$$

可得有质量矢量场的 Euler-Lagrange 方程为

$$\partial_\mu F^{\mathrm{H},\mu\nu} + m^2 A^{\mathrm{H},\nu} = -J^{\mathrm{H},\nu}. \tag{5.56}$$

这里我们将 Heisenberg 绘景的标记明确写出来。由于 $J_\mu(x)$ 不包含 A^μ 的时间导数，正则动量密度与自由情况形式相同：

$$\pi_i^{\mathrm{H}} = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^{\mathrm{H},i})} = -F_{0i}^{\mathrm{H}}, \quad \pi^{\mathrm{H},i} = F^{\mathrm{H},i0} = -\partial^0 A^{\mathrm{H},i} + \partial^i A^{\mathrm{H},0}. \tag{5.57}$$

写成空间矢量的形式，得

$$\boldsymbol{\pi}^{\mathrm{H}} = -\dot{\mathbf{A}}^{\mathrm{H}} - \nabla A^{\mathrm{H},0}, \quad \dot{\mathbf{A}}^{\mathrm{H}} = -\boldsymbol{\pi}^{\mathrm{H}} - \nabla A^{\mathrm{H},0}. \tag{5.58}$$

当 $\nu = 0$ 时，运动方程变成

$$\partial_i F^{\mathrm{H},i0} + m^2 A^{\mathrm{H},0} = -J^{\mathrm{H},0}, \tag{5.59}$$

故

$$A^{\mathrm{H},0} = -\frac{1}{m^2}(\partial_i F^{\mathrm{H},i0} + J^{\mathrm{H},0}) = -\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}} + J^{\mathrm{H},0}). \tag{5.60}$$

与自由情况 (3.179) 不同，此处 $A^{\mathrm{H},0}$ 还依赖于 $J^{\mathrm{H},0}$。

现在，哈密顿量密度是

$$\begin{aligned}\mathcal{H}^{\mathrm{H}} &= \pi_i^{\mathrm{H}}\partial_0 A^{\mathrm{H},i} - \mathcal{L} = -\boldsymbol{\pi}^{\mathrm{H}}\cdot\dot{\mathbf{A}}^{\mathrm{H}} - \mathcal{L}\\ &= -\boldsymbol{\pi}^{\mathrm{H}}\cdot\dot{\mathbf{A}}^{\mathrm{H}} - \frac{1}{2}(\boldsymbol{\pi}^{\mathrm{H}})^2 + \frac{1}{2}(\nabla\times\mathbf{A}^{\mathrm{H}})^2 - \frac{1}{2}m^2[(A^{\mathrm{H},0})^2 - (\mathbf{A}^{\mathrm{H}})^2] - J^{\mathrm{H},0}A^{\mathrm{H},0} + \mathbf{J}^{\mathrm{H}}\cdot\mathbf{A}^{\mathrm{H}}.\end{aligned} \tag{5.61}$$

我们需要知道它比自由哈密顿量密度 (3.185) 多了什么。(5.61) 式第一项可化为

$$\begin{aligned}-\boldsymbol{\pi}^{\mathrm{H}}\cdot\dot{\mathbf{A}}^{\mathrm{H}} &= \boldsymbol{\pi}^{\mathrm{H}}\cdot(\boldsymbol{\pi}^{\mathrm{H}} + \nabla A^{\mathrm{H},0}) = (\boldsymbol{\pi}^{\mathrm{H}})^2 + \nabla\cdot(A^{\mathrm{H},0}\boldsymbol{\pi}^{\mathrm{H}}) - A^{\mathrm{H},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}}\\ &= (\boldsymbol{\pi}^{\mathrm{H}})^2 + \nabla\cdot(A^{\mathrm{H},0}\boldsymbol{\pi}^{\mathrm{H}}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}})^2 + \frac{1}{m^2}J^{\mathrm{H},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}}.\end{aligned} \tag{5.62}$$

最后一行第二项是全散度，不会影响哈密顿量。(5.61) 式第四项中包括

$$-\frac{1}{2}m^2(A^{\mathrm{H},0})^2 = -\frac{1}{2}m^2\frac{1}{m^4}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}} + J^{\mathrm{H},0})^2$$

$$= -\frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}})^2 - \frac{1}{2m^2}(J^{\mathrm{H},0})^2 - \frac{1}{m^2}J^{\mathrm{H},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}}, \tag{5.63}$$

而第五项为

$$-J^{\mathrm{H},0}(A^{\mathrm{H}})^0 = \frac{1}{m^2}J^{\mathrm{H},0}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}} + J^{\mathrm{H},0}) = \frac{1}{m^2}J^{\mathrm{H},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}} + \frac{1}{m^2}(J^{\mathrm{H},0})^2. \tag{5.64}$$

这里包含 J^μ 的项都是自由场不具备的，应该归为相互作用项。于是，我们可以将哈密顿量分解为

$$H^{\mathrm{H}} = \int d^3x\,\mathcal{H}^{\mathrm{H}} = H_0^{\mathrm{H}} + H_1^{\mathrm{H}}, \tag{5.65}$$

其中，

$$H_0^{\mathrm{H}} = \frac{1}{2}\int d^3x\left[(\boldsymbol{\pi}^{\mathrm{H}})^2 + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}})^2 + (\nabla\times\mathbf{A}^{\mathrm{H}})^2 + m^2(\mathbf{A}^{\mathrm{H}})^2\right] \tag{5.66}$$

与自由哈密顿量密度 (3.186) 形式相同，而

$$H_1^{\mathrm{H}} = \int d^3x\left[\mathbf{J}^{\mathrm{H}}\cdot\mathbf{A}^{\mathrm{H}} + \frac{1}{m^2}J^{\mathrm{H},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}} + \frac{1}{2m^2}(J^{\mathrm{H},0})^2\right] \tag{5.67}$$

描述相互作用。

根据等时对易关系 (3.96)，有

$$\begin{aligned}[A^{\mathrm{H},i}(x),(\boldsymbol{\pi}^{\mathrm{H}}(y))^2] &= [A^{\mathrm{H},i}(x),\pi_j^{\mathrm{H}}(y)]\pi_j^{\mathrm{H}}(y) + \pi_j^{\mathrm{H}}(y)[A^{\mathrm{H},i}(x),\pi_j^{\mathrm{H}}(y)]\\ &= 2i\delta^i{}_j\delta^{(3)}(\mathbf{x}-\mathbf{y})\pi_j^{\mathrm{H}}(y) = -2i\delta^{(3)}(\mathbf{x}-\mathbf{y})\pi^{\mathrm{H},i}(y),\end{aligned} \tag{5.68}$$

写成空间矢量的形式是

$$[\mathbf{A}^{\mathrm{H}}(x),(\boldsymbol{\pi}^{\mathrm{H}}(y))^2] = -2i\delta^{(3)}(\mathbf{x}-\mathbf{y})\pi^{\mathrm{H}}(y). \tag{5.69}$$

另一方面，用 ∇_y 表示对空间矢量 $\mathbf{y}$ 的梯度算符，可得

$$[A^{\mathrm{H},i}(x),\nabla_y\cdot\boldsymbol{\pi}^{\mathrm{H}}(y)] = -\frac{\partial}{\partial y^j}[A^{\mathrm{H},i}(x),\pi_j^{\mathrm{H}}(y)] = -i\delta^i{}_j\frac{\partial}{\partial y^j}\delta^{(3)}(\mathbf{x}-\mathbf{y}) = -i\frac{\partial}{\partial y^i}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{5.70}$$

即

$$[\mathbf{A}^{\mathrm{H}}(x),\nabla_y\cdot\boldsymbol{\pi}^{\mathrm{H}}(y)] = -i\nabla_y\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{5.71}$$

从而，我们能够导出

$$\begin{aligned}[\mathbf{A}^{\mathrm{H}}(x),H_0^{\mathrm{H}}] &= \frac{1}{2}\int d^3y\left\{[\mathbf{A}^{\mathrm{H}}(x),(\boldsymbol{\pi}^{\mathrm{H}}(y))^2] + \frac{1}{m^2}[\mathbf{A}^{\mathrm{H}}(x),(\nabla_y\cdot\boldsymbol{\pi}^{\mathrm{H}}(y))^2]\right\}\\ &= \int d^3y\left\{-i\delta^{(3)}(\mathbf{x}-\mathbf{y})\boldsymbol{\pi}^{\mathrm{H}}(y) - \frac{i}{m^2}(\nabla_y\cdot\boldsymbol{\pi}^{\mathrm{H}}(y))\nabla_y\delta^{(3)}(\mathbf{x}-\mathbf{y})\right\}\\ &= -i\boldsymbol{\pi}^{\mathrm{H}}(x) + \frac{i}{m^2}\int d^3y\,\{\delta^{(3)}(\mathbf{x}-\mathbf{y})\nabla_y(\nabla_y\cdot\boldsymbol{\pi}^{\mathrm{H}}(y))\}\\ &= -i\boldsymbol{\pi}^{\mathrm{H}}(x) + \frac{i}{m^2}\nabla_x(\nabla_x\cdot\boldsymbol{\pi}^{\mathrm{H}}(x))\end{aligned} \tag{5.72}$$

接下来，我们转换到相互作用绘景，

$$\mathbf{A}^{\mathrm{I}}=e^{iH_0^{\mathrm{S}}t}e^{-iHt}\mathbf{A}^{\mathrm{H}}e^{iHt}e^{-iH_0^{\mathrm{S}}t},\quad \boldsymbol{\pi}^{\mathrm{I}}=e^{iH_0^{\mathrm{S}}t}e^{-iHt}\boldsymbol{\pi}^{\mathrm{H}}e^{iHt}e^{-iH_0^{\mathrm{S}}t},\tag{5.73}$$

则有

$$\begin{aligned}H_0^{\mathrm{S}}&=H_0^{\mathrm{I}}=e^{iH_0^{\mathrm{S}}t}e^{-iHt}H_0^{\mathrm{H}}e^{iHt}e^{-iH_0^{\mathrm{S}}t}\\&=\frac{1}{2}\int d^3x\left[(\boldsymbol{\pi}^{\mathrm{I}})^2+\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi}^{\mathrm{I}})^2+(\nabla\times\mathbf{A}^{\mathrm{I}})^2+m^2(\mathbf{A}^{\mathrm{I}})^2\right].\end{aligned}\tag{5.74}$$

将演化方程 (5.41) 应用到 $\mathbf{A}^{\mathrm{I}}$ 上，利用 (5.72) 式，可得

$$\begin{aligned}i\dot{\mathbf{A}}^{\mathrm{I}}&=[\mathbf{A}^{\mathrm{I}},H_0^{\mathrm{S}}]=e^{iH_0^{\mathrm{S}}t}e^{-iHt}[\mathbf{A}^{\mathrm{H}},H_0^{\mathrm{H}}]e^{iHt}e^{-iH_0^{\mathrm{S}}t}\\&=e^{iH_0^{\mathrm{S}}t}e^{-iHt}\left[-i\boldsymbol{\pi}^{\mathrm{H}}+\frac{i}{m^2}\nabla(\nabla\cdot\boldsymbol{\pi}^{\mathrm{H}})\right]e^{iHt}e^{-iH_0^{\mathrm{S}}t}=-i\boldsymbol{\pi}^{\mathrm{I}}+\frac{i}{m^2}\nabla(\nabla\cdot\boldsymbol{\pi}^{\mathrm{I}}),\end{aligned}\tag{5.75}$$

即

$$\boldsymbol{\pi}^{\mathrm{I}}=-\dot{\mathbf{A}}^{\mathrm{I}}+\frac{1}{m^2}\nabla(\nabla\cdot\boldsymbol{\pi}^{\mathrm{I}}).\tag{5.76}$$

与 (3.177) 式和 (3.179) 式比较，可以看出，这个等式与自由场情况形式相同。

现在，假设 $t=0$ 时 $A^\mu(x)$ 和 $\pi_i(x)$ 的平面波展开式与 $t=0$ 时的自由场展开式 (3.146) 和 (3.151) 相同，

$$\begin{aligned}A^{\mathrm{I},\mu}(\mathbf{x},0)&=A^{\mathrm{H},\mu}(\mathbf{x},0)=A^{\mathrm{S},\mu}(\mathbf{x})\\&=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{i\mathbf{p}\cdot\mathbf{x}}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],\end{aligned}\tag{5.77}$$

$$\begin{aligned}\pi_i^{\mathrm{I}}(\mathbf{x},0)&=\pi_i^{\mathrm{H}}(\mathbf{x},0)=\pi_i^{\mathrm{S}}(\mathbf{x})\\&=\int\frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\left[\tilde{\varepsilon}_i(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{i\mathbf{p}\cdot\mathbf{x}}-\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],\end{aligned}\tag{5.78}$$

其中，产生湮灭算符 $a^\dagger_{\mathbf{p},\lambda}$ 和 $a_{\mathbf{p},\lambda}$ 满足对易关系 (3.175)。哈密顿量的自由部分 H_0^{S} 具有 (3.205) 式的形式（略去零点能）：

$$H_0^{\mathrm{S}}=\sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}.\tag{5.79}$$

从而，有

$$[H_0^{\mathrm{S}},a^\dagger_{\mathbf{p},\lambda}]=\sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}E_{\mathbf{q}}[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}]=\sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}a^\dagger_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda},\tag{5.80}$$

$$[H_0^{\mathrm{S}},a_{\mathbf{p},\lambda}]=\sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}E_{\mathbf{q}}[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'},a_{\mathbf{p},\lambda}]=-\sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}a_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=-E_{\mathbf{p}}a_{\mathbf{p},\lambda}.\tag{5.81}$$

于是，我们能够得到与 (5.19) 形式相同的式子

$$[a_{\mathbf{p},\lambda},(-iH_0^{\mathrm{S}}t)^{(n)}]=(-iE_{\mathbf{p}}t)^{(n)}a_{\mathbf{p},\lambda},\tag{5.82}$$

再根据 (4.22) 式，可以导出

$$a^{\mathrm{I}}_{\mathbf{p},\lambda}(t)=e^{iH_0^{\mathrm{S}}t}a_{\mathbf{p},\lambda}e^{-iH_0^{\mathrm{S}}t}=e^{-iE_{\mathbf{p}}t}a_{\mathbf{p},\lambda},\quad a^{\mathrm{I}\dagger}_{\mathbf{p},\lambda}(t)=e^{iH_0^{\mathrm{S}}t}a^{\dagger}_{\mathbf{p},\lambda}e^{-iH_0^{\mathrm{S}}t}=e^{iE_{\mathbf{p}}t}a^{\dagger}_{\mathbf{p},\lambda}. \tag{5.83}$$

更进一步，推出

$$A^{\mathrm{I},\mu}(\mathbf{x},t)=e^{iH_0^{\mathrm{S}}t}A^{\mathrm{S},\mu}(\mathbf{x})e^{-iH_0^{\mathrm{S}}t}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\left[\varepsilon^{\mu}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}e^{ip\cdot x}\right], \tag{5.84}$$

$$\pi^{\mathrm{I}}_i(\mathbf{x},t)=e^{iH_0^{\mathrm{S}}t}\pi^{\mathrm{S}}_i(\mathbf{x})e^{-iH_0^{\mathrm{S}}t}=\int\frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\left[\tilde{\varepsilon}_i(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}-\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}e^{ip\cdot x}\right]. \tag{5.85}$$

也就是说，对于任意 t 时刻，$A^{\mathrm{I},\mu}(x)$ 和 $\pi^{\mathrm{I}}_i(x)$ 的展开式与 Heisenberg 绘景中的自由场展开式 (3.146) 和 (3.151) 一致。这是我们期望的结果。

因此，$\pi^{\mathrm{I}}_i(x)$ 和 $A^{\mathrm{I},\mu}(x)$ 的关系也与自由场情况 (3.95) 式一样:

$$\pi^{\mathrm{I}}_i=-\partial_0A^{\mathrm{I}}_i+\partial_iA^{\mathrm{I}}_0, \tag{5.86}$$

即

$$\boldsymbol{\pi}^{\mathrm{I}}=-\dot{\mathbf{A}}^{\mathrm{I}}-\nabla A^{\mathrm{I},0}. \tag{5.87}$$

与 (5.76) 式比较，就得到

$$A^{\mathrm{I},0}=-\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}^{\mathrm{I}}. \tag{5.88}$$

这个式子不同于 Heisenberg 绘景中的关系式 (5.60)，反而与自由场中的关系式 (3.179) 一致。实际上，由于 $A^{\mathrm{H},0}$ 不是独立的场分量，我们在 Heisenberg 绘景中可以利用场的 Euler-Lagrange 方程导出关系式 (5.60) 来确定它，但我们无法保证这个关系式在相互作用绘景中成立，因而不能通过相似变换定义 $A^{\mathrm{H},0}$ 在相互作用绘景中对应的量。

根据 (5.88) 式，相互作用哈密顿量 (5.67) 在相互作用绘景中将变成

$$\begin{aligned}H^{\mathrm{I}}_1&=e^{iH_0^{\mathrm{S}}t}e^{-iHt}H^{\mathrm{H}}_1e^{iHt}e^{-iH_0^{\mathrm{S}}t}=\int d^3x\left[\mathbf{J}^{\mathrm{I}}\cdot\mathbf{A}^{\mathrm{I}}+\frac{1}{m^2}J^{\mathrm{I},0}\nabla\cdot\boldsymbol{\pi}^{\mathrm{I}}+\frac{1}{2m^2}(J^{\mathrm{I},0})^2\right]\\&=\int d^3x\left[\mathbf{J}^{\mathrm{I}}\cdot\mathbf{A}^{\mathrm{I}}-J^{\mathrm{I},0}A^{\mathrm{I},0}+\frac{1}{2m^2}(J^{\mathrm{I},0})^2\right]=\int d^3x\left[-J^{\mathrm{I}}_{\mu}A^{\mathrm{I},\mu}+\frac{1}{2m^2}(J^{\mathrm{I},0})^2\right]\\&=\int d^3x\left[-\mathcal{L}^{\mathrm{I}}_1+\frac{1}{2m^2}(J^{\mathrm{I},0})^2\right].\end{aligned} \tag{5.89}$$

最后一行方括号中第一项 $-\mathcal{L}^{\mathrm{I}}_1=-J^{\mathrm{I}}_{\mu}A^{\mathrm{I},\mu}$ 是我们期望得到的，它是 Lorentz 不变的。但第二项异乎寻常，不具有 Lorentz 不变性，我们将它记为

$$\mathcal{H}_{J^0}=\frac{1}{2m^2}(J^{\mathrm{I},0})^2. \tag{5.90}$$

在这里，$\mathcal{H}_{J^0}$ 看起来会破坏理论的 Lorentz 协变性，不过，在后续微扰论分析中，我们将看到它的贡献恰好抵消了矢量场传播子中的非协变项。最终，理论仍然是 Lorentz 协变的。

5.2 时间演化算符和 S 矩阵

如前所述，在相互作用绘景中，态矢 $|\Psi(t)\rangle^{\mathrm{I}}$ 承载着动力学演化，它的演化方程 (5.39) 是微扰论处理量子场相互作用的一个出发点。引入**时间演化算符** (time-evolution operator) $U(t,t_0)$，用于联系 t_0 和 t 两个时刻的态矢：

$$|\Psi(t)\rangle^{\mathrm{I}} = U(t,t_0)|\Psi(t_0)\rangle^{\mathrm{I}}. \tag{5.91}$$

由 (5.33) 式，有

$$|\Psi(t)\rangle^{\mathrm{I}} = e^{iH_0^{\mathrm{S}}t}e^{-iHt}|\Psi\rangle^{\mathrm{H}} = e^{iH_0^{\mathrm{S}}t}e^{-iH(t-t_0)}e^{-iH_0^{\mathrm{S}}t_0}|\Psi(t_0)\rangle^{\mathrm{I}}. \tag{5.92}$$

因此，时间演化算符可以表示为

$$U(t,t_0) = e^{iH_0^{\mathrm{S}}t}e^{-iH(t-t_0)}e^{-iH_0^{\mathrm{S}}t_0}. \tag{5.93}$$

容易看出，时间演化算符满足

$$U(t_0,t_0) = 1. \tag{5.94}$$

两个时间演化算符相继作用得出的乘法规则为

$$\begin{aligned} U(t_2,t_1)U(t_1,t_0) &= e^{iH_0^{\mathrm{S}}t_2}e^{-iH(t_2-t_1)}e^{-iH_0^{\mathrm{S}}t_1}e^{iH_0^{\mathrm{S}}t_1}e^{-iH(t_1-t_0)}e^{-iH_0^{\mathrm{S}}t_0} = e^{iH_0^{\mathrm{S}}t_2}e^{-iH(t_2-t_0)}e^{-iH_0^{\mathrm{S}}t_0} \\ &= U(t_2,t_0). \end{aligned} \tag{5.95}$$

上式取 $t_2 = t_0$，即得

$$U(t_0,t_1)U(t_1,t_0) = U(t_0,t_0) = 1, \tag{5.96}$$

故时间演化算符的逆算符满足

$$U^{-1}(t,t_0) = U(t_0,t). \tag{5.97}$$

再由 H 和 H_0^{S} 的厄米性，可得

$$U^\dagger(t,t_0) = e^{iH_0^{\mathrm{S}}t_0}e^{iH(t-t_0)}e^{-iH_0^{\mathrm{S}}t} = e^{iH_0^{\mathrm{S}}t_0}e^{-iH(t_0-t)}e^{-iH_0^{\mathrm{S}}t} = U(t_0,t) = U^{-1}(t_0,t), \tag{5.98}$$

也就是说，时间演化算符是幺正算符。取 $t_0 = 0$，有

$$U(t,0) = e^{iH_0^{\mathrm{S}}t}e^{-iHt}, \quad U^{-1}(t,0) = e^{iHt}e^{-iH_0^{\mathrm{S}}t}, \tag{5.99}$$

因而根据 (5.33) 式和 (5.35) 式可得

$$|\Psi(t)\rangle^{\mathrm{I}} = U(t,0)|\Psi\rangle^{\mathrm{H}}, \quad O^{\mathrm{I}}(t) = U(t,0)O^{\mathrm{H}}(t)U^{-1}(t,0). \tag{5.100}$$

可见，$U(t,0)$ 就是联系 Heisenberg 绘景和相互作用绘景的幺正变换算符。

从态矢的演化方程 (5.39) 可以得出

$$i\frac{\partial}{\partial t}U(t,t_0)|\Psi(t_0)\rangle^{\mathrm{I}} = i\frac{\partial}{\partial t}|\Psi(t)\rangle^{\mathrm{I}} = H_1^{\mathrm{I}}(t)|\Psi(t)\rangle^{\mathrm{I}} = H_1^{\mathrm{I}}(t)U(t,t_0)|\Psi(t_0)\rangle^{\mathrm{I}}, \tag{5.101}$$

即

$$i\frac{\partial}{\partial t}U(t,t_0) = H_1^{\mathrm{I}}(t)U(t,t_0). \tag{5.102}$$

这是时间演化算符需要满足的微分方程，结合边值条件 (5.94)，可以将方程的解表达为

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\, H_1^{\mathrm{I}}(t_1)U(t_1,t_0). \tag{5.103}$$

上式左右两边均包含时间演化算符，可以进行重复迭代，从而得到级数

$$\begin{aligned}U(t,t_0) \;=\; & 1 + (-i)\int_{t_0}^{t} dt_1\, H_1^{\mathrm{I}}(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\, H_1^{\mathrm{I}}(t_1)H_1^{\mathrm{I}}(t_2) \\ & + \cdots + \left[(-i)^n\int_{t_0}^{t} dt_1\cdots\int_{t_0}^{t_{n-1}} dt_n\, H_1^{\mathrm{I}}(t_1)\cdots H_1^{\mathrm{I}}(t_n)\right] + \cdots\end{aligned} \tag{5.104}$$

这个级数用起来不够方便，需要进一步化简。

从现在开始，我们将**省略**表示相互作用绘景的上标 I，因为本章余下内容均在相互作用绘景中讨论。

在级数 (5.104) 中，作为积分上限的时刻是降序排列的，即 $t \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq \cdots \geq t_0$。由于积分上限相互依赖，这样的多重积分是很难处理的。为了将级数中每个积分的上限都扩展到 t 时刻，需要引入**时序乘积** (time-ordered product) 的概念。时序乘积使若干个含时算符的乘积强行按照它们相应的时刻降序排列。以 n 个 $H_1(t)$ 算符为例，用 T 表示这种时序操作，有

$$\mathsf{T}[H_1(t_1)H_1(t_2)\cdots H_1(t_n)] = H_1(t_{i_1})H_1(t_{i_2})\cdots H_1(t_{i_n}), \quad t_{i_1} \geq t_{i_2} \geq \cdots \geq t_{i_n}. \tag{5.105}$$

这里 $t_{i_1}, t_{i_2}, \cdots, t_{i_n}$ 是由 $t_1, t_2, \cdots, t_n$ 降序排列得到的：

$$t_{i_1} \geq t_{i_2} \geq \cdots \geq t_{i_n}. \tag{5.106}$$

又如，两个标量场算符 $\phi(x)$ 和 $\phi(y)$ 的时序乘积可以用阶跃函数表示为

$$\mathsf{T}[\phi(x)\phi(y)] = \phi(x)\phi(y)\theta(x^0 - y^0) + \phi(y)\phi(x)\theta(y^0 - x^0). \tag{5.107}$$

对于费米子算符，需要顾及到它们的反对易性质，因此，如果时序操作使费米子算符之间交换了奇数次，则应该额外加上一个负号。比如，两个旋量场算符 $\psi_a(x)$ 和 $\bar{\psi}_b(y)$ 的时序乘积是

$$\mathsf{T}[\psi_a(x)\bar{\psi}_b(y)] = \psi_a(x)\bar{\psi}_b(y)\theta(x^0 - y^0) - \bar{\psi}_b(y)\psi_a(x)\theta(y^0 - x^0). \tag{5.108}$$

现在考虑级数 (5.104) 的第三项，它包含一个关于 t_1 和 t_2 的二重积分，积分区域如图 5.1(a) 所示，先对 t_2 积分，再对 t_1 积分。这个二重积分可以重新表达为

$$\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\, H_1(t_1)H_1(t_2) = \int_{t_0}^{t} dt_2\int_{t_2}^{t} dt_1\, H_1(t_1)H_1(t_2) = \int_{t_0}^{t} dt_1\int_{t_1}^{t} dt_2\, H_1(t_2)H_1(t_1). \tag{5.109}$$

在上式第一步中，我们改成先对 t_1 积分，再对 t_2 积分，积分区域不变，如图 5.1(b) 所示。第二步，我们交换了积分变量 t_1 和 t_2，对应的积分区域如图 5.1(c) 所示。由此，可得

$$\begin{aligned}2!\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\,H_1(t_1)H_1(t_2) &= \int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\,H_1(t_1)H_1(t_2)+\int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2\,H_1(t_2)H_1(t_1)\\ &= \int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\,\mathsf{T}[H_1(t_1)H_1(t_2)]. \end{aligned}\tag{5.110}$$

这里利用时序乘积将 t_1 和 t_2 的积分范围都扩展到整个 $[t_0, t_1]$ 区间，因为图 5.1(a) 中的积分区域与图 5.1(c) 中的积分区域恰好拼成一个正方形。在上式第一步第一项中，t_1 是 t_2 的积分上限，显然有 $t_1 \geq t_2$，因而 $H_1(t_1)H_1(t_2)$ 是正确的时序乘积；在第二项中，t_1 是 t_2 的积分下限，故 $t_2 \geq t_1$，此时 $H_1(t_2)H_1(t_1)$ 才是正确的时序乘积；两项相加，就得到第二步的结果。

将上述讨论推广到级数 (5.104) 中的第 n 项，可得

$$n!\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t_{n-1}}dt_n\,H_1(t_1)\cdots H_1(t_n)=\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t}dt_n\,\mathsf{T}[H_1(t_1)\cdots H_1(t_n)].\tag{5.111}$$

上式出现 $n!$ 是因为此时对 n 个时间积分变量有 $n!$ 种排列方式。于是，级数 (5.104) 可以用时序乘积表达为

$$\begin{aligned}U(t,t_0) &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t}dt_n\,\mathsf{T}[H_1(t_1)\cdots H_1(t_n)]\\ &\equiv \mathsf{T}\exp\left[-i\int_{t_0}^{t}dt'\,H_1(t')\right].\end{aligned}\tag{5.112}$$

由于这个级数具有指数函数的级数展开形式，这里进一步用指数记号来表示。

像 (5.12) 式一样，在局域场论中 $H_1(t)$ 是相应哈密顿量密度 $\mathcal{H}_1(x)$ 的空间积分

$$H_1(t)=\int d^3x\,\mathcal{H}_1(x).\tag{5.113}$$

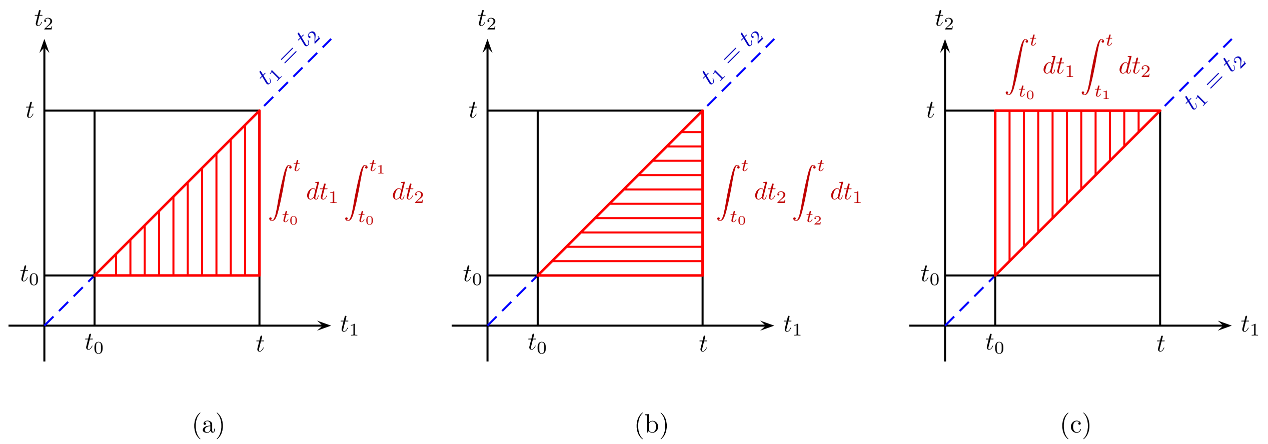


图 5.1: $t_1 - t_2$ 平面上的积分区域。

因此，时间演化算符满足

$$U(t,t_0) = \mathsf{T}\exp\left[-i\int_{t_0}^{t} dt' \int d^3x'\, \mathcal{H}_1(x)\right]. \tag{5.114}$$

S **矩阵**，或者称为**散射矩阵** (scattering matrix)，是量子散射理论的核心概念，它描述系统从初态跃迁到末态的概率振幅。在相互作用绘景中，S 矩阵可以用时间演化算符来构造。

假设系统的初态 $|i\rangle$ 和末态 $|f\rangle$ 均处于自由状态，而相互作用只发生在很短的时间间隔里，则初始时刻处于遥远过去，而终末时刻处于遥远未来。若将 t 时刻处描述系统的态矢记为 $|\Psi(t)\rangle$，它从遥远过去 $(t\to-\infty)$ 的初态 $|i\rangle$ 演化而来，因而可以用时间演化算符表达为

$$|\Psi(t)\rangle = \lim_{t_0\to-\infty} U(t,t_0)\,|i\rangle\,. \tag{5.115}$$

此过程相应的 S 矩阵元 S_{fi} 定义为态矢 $|\Psi(t)\rangle$ 演化到遥远未来 $(t\to+\infty)$ 处与末态 $|f\rangle$ 的内积，即

$$S_{fi} = \lim_{t\to+\infty}\langle f|\Psi(t)\rangle = \lim_{t\to+\infty}\lim_{t_0\to-\infty}\langle f|U(t,t_0)|i\rangle\,. \tag{5.116}$$

引入 S **算符**，它在初态与末态之间的期待值就是 S 矩阵元 S_{fi}：

$$S_{fi} = \langle f|S|i\rangle\,. \tag{5.117}$$

那么，我们可以得出

$$S = U(+\infty,-\infty). \tag{5.118}$$

从而，S 算符可以表达为相互作用哈密顿量的积分级数

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1\cdots\int_{-\infty}^{+\infty}dt_n\,\mathsf{T}[H_1(t_1)\cdots H_1(t_n)] \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\,\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]. \end{aligned} \tag{5.119}$$

由时间演化算符的幺正性可知，S 算符也是幺正的，

$$S^\dagger S = 1. \tag{5.120}$$

5.3 散射截面和衰变宽度

在没有相互作用的理论中，S 算符就是单位算符 1，因而 S 矩阵为 $S_{fi}=\langle f|i\rangle$。对于存在相互作用的理论，上一节的计算告诉我们，$S$ 算符可以展开为级数 (5.119)。这个级数的 $n=0$ 项也是单位算符，因此我们可以将 S 算符分解为

$$S = 1 + iT, \tag{5.121}$$

其中 iT 包含所有 $n \geq 1$ 的项。从而，S 矩阵可以分解为

$$S_{fi} = \langle f|i\rangle + \langle f|\, iT\, |i\rangle\,. \tag{5.122}$$

右边第一项意味着，即使理论中存在相互作用，初态也有一定概率自由地演化，也就是说，初态中的粒子仍然有一定概率不发生任何相互作用。由此可见，S 矩阵中真正描述相互作用的项是 $\langle f|\, iT\, |i\rangle$。由于能动量守恒定律，初态中所有粒子的四维动量之和 p_i^μ 必定等于末态中所有粒子的四维动量之和 p_f^μ。因此，$\langle f|\, iT\, |i\rangle$ 具有如下形式：

$$\langle f|\, iT\, |i\rangle = (2\pi)^4\delta^{(4)}(p_i - p_f)\, i\mathcal{M}_{fi}. \tag{5.123}$$

上式右边的四维 δ 函数体现了能动量守恒定律，而 $\mathcal{M}_{fi}$ 是 Lorentz 不变的，称为**不变矩阵元** (invariant matrix element)，或者**不变散射振幅** (invariant scattering amplitude)，它是初态和末态动量的函数。

5.3.1　跃迁概率

在发生相互作用时，$i \to f$ 的跃迁概率可以表示成

$$P_{fi} = \frac{|\langle f|\, iT\, |i\rangle|^2}{\langle i|i\rangle\, \langle f|f\rangle}, \tag{5.124}$$

其中，$\langle i|i\rangle$ 和 $\langle f|f\rangle$ 分别是初态 $|i\rangle$ 和末态 $|f\rangle$ 的归一化因子。上式右边的分子为

$$|\langle f|\, iT\, |i\rangle|^2 = [(2\pi)^4\delta^{(4)}(p_i - p_f)]^2|\mathcal{M}_{fi}|^2 = (2\pi)^{(4)}\delta^{(4)}(0)\cdot(2\pi)^4\delta^{(4)}(p_i - p_f)|\mathcal{M}_{fi}|^2. \tag{5.125}$$

根据 Fourier 变换公式

$$\int d^4x\, e^{\pm ip\cdot x} = (2\pi)^4\delta^{(4)}(p), \tag{5.126}$$

有

$$(2\pi)^4\delta^{(4)}(0) = \int d^4x = \tilde{V}\tilde{T}. \tag{5.127}$$

其中，$\tilde{V}$ 是空间积分区域的体积，$\tilde{T}$ 是时间积分范围的长度，对于全空间全时间积分，它们趋于无穷大。于是，(5.125) 式可以写作

$$|\langle f|\, iT\, |i\rangle|^2 = \tilde{V}\tilde{T}(2\pi)^4\delta^{(4)}(p_i - p_f)|\mathcal{M}_{fi}|^2. \tag{5.128}$$

现在，讨论 2 体初态到 n 体末态的跃迁过程，即初态包含 2 个粒子 $\mathcal{A}$ 和 $\mathcal{B}$，它们通过相互作用发生散射，从而产生包含 n 个粒子的末态。设初态中两个粒子的动量分别为 $\mathbf{p}_\mathcal{A}$ 和 $\mathbf{p}_\mathcal{B}$，则 $|i\rangle$ 可以用相应的产生算符表达为

$$|i\rangle = \sqrt{2E_\mathcal{A}2E_\mathcal{B}}\, a^\dagger_{\mathbf{p}_\mathcal{A}} a^\dagger_{\mathbf{p}_\mathcal{B}}\, |0\rangle\,, \quad E_{\mathcal{A},\mathcal{B}} = p^0_{\mathcal{A},\mathcal{B}} = \sqrt{|\mathbf{p}_{\mathcal{A},\mathcal{B}}|^2 + m^2_{\mathcal{A},\mathcal{B}}}\,. \tag{5.129}$$

此处，我们省略了产生算符的螺旋度指标（或者说，自旋指标）。$|0\rangle$ 是理论的真空态，理论中任意湮灭算符作用到它身上都将得到零。类似地，末态 $|f\rangle$ 可以写成

$$|f\rangle = \left(\prod_{j=1}^{n} \sqrt{2E_j}\, a^\dagger_{\mathbf{p}_j}\right)|0\rangle, \quad E_j = p_j^0 = \sqrt{|\mathbf{p}_j|^2 + m_j^2}. \tag{5.130}$$

其中，$\mathbf{p}_j$ $(j = 1, \cdots, n)$ 是 n 个末态粒子的动量。此时，初态和末态的四维总动量分别是

$$p_i^\mu = p_{\mathcal{A}}^\mu + p_{\mathcal{B}}^\mu, \quad p_f^\mu = \sum_{j=1}^{n} p_j^\mu. \tag{5.131}$$

我们可以把初态 $|i\rangle$ 改写为直积，

$$|i\rangle = \sqrt{2E_{\mathcal{A}}}\, a^\dagger_{\mathbf{p}_{\mathcal{A}}}|0\rangle_{\mathcal{A}} \otimes \sqrt{2E_{\mathcal{B}}}\, a^\dagger_{\mathbf{p}_{\mathcal{B}}}|0\rangle_{\mathcal{B}} = |\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}} \otimes |\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}}. \tag{5.132}$$

这里 $|0\rangle_{\mathcal{A}}$ 和 $|0\rangle_{\mathcal{B}}$ 分别是描述 $\mathcal{A}$ 和 $\mathcal{B}$ 的两个量子场所对应的真空态。如同 (2.116) 式，单粒子态 $|\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}}$ 和 $|\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}}$ 的自我内积分别是

$$\langle \mathbf{p}_{\mathcal{A}}|\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}} = 2E_{\mathcal{A}}(2\pi)^3\delta^{(3)}(\mathbf{0}) = 2E_{\mathcal{A}}\tilde{V}, \quad \langle \mathbf{p}_{\mathcal{B}}|\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}} = 2E_{\mathcal{B}}(2\pi)^3\delta^{(3)}(\mathbf{0}) = 2E_{\mathcal{B}}\tilde{V}. \tag{5.133}$$

于是，我们得到

$$\langle i|i\rangle = \langle \mathbf{p}_{\mathcal{A}}|\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}}\langle \mathbf{p}_{\mathcal{B}}|\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}} = 4E_{\mathcal{A}}E_{\mathcal{B}}\tilde{V}^2. \tag{5.134}$$

同理可得

$$\langle f|f\rangle = \prod_{j=1}^{n}(2E_j\tilde{V}). \tag{5.135}$$

从而，跃迁概率化为

$$P_{fi} = \frac{|\langle f|\, iT\, |i\rangle|^2}{\langle i|i\rangle\, \langle f|f\rangle} = \frac{\tilde{V}\tilde{T}(2\pi)^4\delta^{(4)}(p_i - p_f)|\mathcal{M}_{fi}|^2}{4E_{\mathcal{A}}E_{\mathcal{B}}\tilde{V}^2\prod_{j=1}^{n}(2E_j\tilde{V})} = \frac{\tilde{T}(2\pi)^4\delta^{(4)}(p_i - p_f)|\mathcal{M}_{fi}|^2}{4E_{\mathcal{A}}E_{\mathcal{B}}\tilde{V}\prod_{j=1}^{n}(2E_j\tilde{V})}. \tag{5.136}$$

对于一组特定的动量 $\{p_j\}$，单位时间内的跃迁概率为

$$R_{\{p_j\}} = \frac{P_{fi}}{\tilde{T}} = \frac{1}{4E_{\mathcal{A}}E_{\mathcal{B}}\tilde{V}\prod_{j=1}^{n}(2E_j\tilde{V})}(2\pi)^4\delta^{(4)}\Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j\Big)|\mathcal{M}_{fi}|^2. \tag{5.137}$$

此处四维 δ 函数可以分解为

$$\delta^{(4)}\Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j\Big) = \delta^{(3)}\Big(\mathbf{p}_{\mathcal{A}} + \mathbf{p}_{\mathcal{B}} - \sum_{j=1}^{n} \mathbf{p}_j\Big)\delta\Big(E_{\mathcal{A}} + E_{\mathcal{B}} - \sum_{j=1}^{n} E_j\Big). \tag{5.138}$$

在这样一个 $2 \to n$ 散射过程中，末态中 n 个粒子的动量可以取任意满足运动学要求的值，而运动学条件

$$p_{\mathcal{A}}^\mu + p_{\mathcal{B}}^\mu - \sum_{j=1}^{n} p_j^\mu = 0 \tag{5.139}$$

已经体现在 (5.137) 式的四维 δ 函数中。为了计算总的跃迁率，我们需要将 $\{p_j\}$ 的所有可能取值包含起来，也就是说，需要对末态的动量相空间积分。

接下来，我们讨论如何包含末态粒子所有可能的动量取值。考察一维情况，先假定粒子局限在 $x \in [-L/2, L/2]$ 范围内运动，最后让 $L \to \infty$。为了确保动量算符 $p_x = -i\partial/\partial x$ 在区间 $[-L/2, L/2]$ 上是厄米算符，必须要求描述粒子的波函数 $\varphi(x)$ 满足周期性边界条件

$$\varphi\left(-\frac{L}{2}\right) = \varphi\left(\frac{L}{2}\right). \tag{5.140}$$

作为动量本征态的波函数是平面波解 $\varphi_p(x) \propto \exp(ipx)$，结合周期性边界条件，有

$$\exp\left(-\frac{i}{2}pL\right) = \exp\left(\frac{i}{2}pL\right), \tag{5.141}$$

故

$$\exp(ipL) = 1, \quad \sin(pL) = 0, \quad \cos(pL) = 1. \tag{5.142}$$

上式成立意味着

$$pL = 2k\pi, \quad k = 0, \pm 1, \pm 2, \cdots \tag{5.143}$$

因此，动量本征值是

$$p_k = \frac{2\pi}{L}k, \quad k \in \mathbb{Z}. \tag{5.144}$$

当 $L \to \infty$ 时，相邻动量本征值之差变成动量的微分：

$$\Delta p_k = p_{k+1} - p_k = \frac{2\pi}{L} \to dp. \tag{5.145}$$

从而可得

$$\sum_{k=-\infty}^{+\infty} \Delta p_k = \frac{2\pi}{L} \sum_{k=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} dp, \tag{5.146}$$

即

$$\sum_{k=-\infty}^{+\infty} \to \frac{L}{2\pi} \int_{-\infty}^{+\infty} dp. \tag{5.147}$$

推广到三维情况，先假定粒子局限在体积为 $\tilde{V} = L^3$ 的立方体中运动，周期性边界条件相当于将立方体表面上任意一点视作与位于相对的面上的对应点等同。满足此条件的动量本征值为

$$\mathbf{p} = \frac{2\pi}{L}(k_1, k_2, k_3), \quad k_1, k_2, k_3 \in \mathbb{Z}. \tag{5.148}$$

当 $L \to \infty$ 时，我们得到

$$\sum_{k_1 k_2 k_3} \to \frac{L^3}{(2\pi)^3} \int d^3p = \frac{\tilde{V}}{(2\pi)^3} \int d^3p\,. \tag{5.149}$$

上式最左边代表对所有动量取值求和，当动量可取连续数值时，这种求和就化作最右边的动量相空间积分。将 n 个末态粒子的所有动量取值都考虑进来，要对 (5.137) 式积分，从而得到单位时间内 $2 \to n$ 散射过程的跃迁概率为

$$\begin{aligned}R &= \left(\prod_{j=1}^{n} \frac{\tilde{V}}{(2\pi)^3} \int d^3 p_j\right) R_{\{p_j\}} \\ &= \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}} \tilde{V}} \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j}\right) (2\pi)^4 \delta^{(4)}\Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j\Big) |\mathcal{M}_{fi}|^2. \end{aligned} \tag{5.150}$$

根据 2.3.4 小节的讨论，我们知道体积元 (2.120) 是 Lorentz 不变的，因而上式中相空间体积元

$$\frac{d^3 p_j}{(2\pi)^3 2E_j} \tag{5.151}$$

也是 Lorentz 不变的。

5.3.2 散射截面

现在，我们讨论*束流打靶实验*，靶 (target) 由 $\mathcal{A}$ 粒子组成，束流 (beam) 由 $\mathcal{B}$ 粒子组成。设束流中每个 $\mathcal{B}$ 粒子的运动速度相同，记为 $\mathbf{v}_{\mathcal{B}}$，按照狭义相对论，有 $\mathbf{v}_{\mathcal{B}} \equiv \mathbf{p}_{\mathcal{B}}/E_{\mathcal{B}}$。记束流的横截面积为 A，则 t 时间内束流的一个横截面经过的体积为 $V = A|\mathbf{v}_{\mathcal{B}}|t$。再设束流中 $\mathcal{B}$ 粒子的数密度为 $n_{\mathcal{B}}$，从而，体积 V 中的粒子数为 $N_{\mathcal{B}} = n_{\mathcal{B}} V = n_{\mathcal{B}} A|\mathbf{v}_{\mathcal{B}}|t$。在单位时间内穿过单位面积的 $\mathcal{B}$ 粒子数称为*流密度*，记作 $j_{\mathcal{B}}$，可以通过下式计算，

$$j_{\mathcal{B}} = \frac{N_{\mathcal{B}}}{At} = \frac{n_{\mathcal{B}} A|\mathbf{v}_{\mathcal{B}}|t}{At} = n_{\mathcal{B}}|\mathbf{v}_{\mathcal{B}}|. \tag{5.152}$$

考虑流密度为 $j_{\mathcal{B}}$ 的束流打到由 $N_{\mathcal{A}}$ 个 $\mathcal{A}$ 粒子组成的靶上，则 t 时间内散射发生的次数可以表示为

$$N = N_{\mathcal{A}}\, j_{\mathcal{B}}\, \sigma\, t \tag{5.153}$$

这里引入了物理量 σ，由量纲分析知道它具有面积量纲，称为**散射截面** (scattering cross section)，或简称为**截面** (cross section)。散射截面表征散射过程的*强度*，由 $\mathcal{A}$ 粒子与 $\mathcal{B}$ 粒子的相互作用性质决定，常用单位是靶 (barn)，记作 b，

$$1\ \text{b} = 10^{-28}\ \text{m}^2 = 2.568 \times 10^3\ \text{GeV}^{-2}. \tag{5.154}$$

于是，单位时间单位体积内散射发生的次数为

$$\mathcal{R} = \frac{N}{Vt} = \frac{N_{\mathcal{A}} j_{\mathcal{B}} \sigma}{V} = \frac{N_{\mathcal{A}} n_{\mathcal{B}} |\mathbf{v}_{\mathcal{B}}| \sigma}{V} = n_{\mathcal{A}} n_{\mathcal{B}} \sigma |\mathbf{v}_{\mathcal{B}}|, \tag{5.155}$$

其中 $n_{\mathcal{A}} = N_{\mathcal{A}}/V$ 相当于 $\mathcal{A}$ 粒子在体积 V 中的密度。

如果只考虑一个 $\mathcal{B}$ 粒子打到一个 $\mathcal{A}$ 粒子上，那么，可以看作在体积 $\tilde{V}$ 中仅有这两个粒子，因而 $n_{\mathcal{A}} = n_{\mathcal{B}} = 1/\tilde{V}$，此时 $\mathcal{R}$ 可以用单位时间内的跃迁概率 R 表示为 $\mathcal{R} = R/\tilde{V}$。于是，根据 (5.150) 式，我们得到

$$\begin{aligned}\sigma &= \frac{\mathcal{R}}{n_{\mathcal{A}} n_{\mathcal{B}} |\mathbf{v}_{\mathcal{B}}|} = \frac{R}{\tilde{V}} \frac{\tilde{V}^2}{|\mathbf{v}_{\mathcal{B}}|} = \frac{R\tilde{V}}{|\mathbf{v}_{\mathcal{B}}|} \\ &= \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}} |\mathbf{v}_{\mathcal{B}}|} \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)} \Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j \Big) |\mathcal{M}_{fi}|^2. \end{aligned} \tag{5.156}$$

上式对 $\mathcal{A}$ 粒子静止的参考系成立。我们想把它推广到任意惯性系，从而可以处理 $\mathcal{A}$ 粒子和 $\mathcal{B}$ 粒子处于任意运动状态的情况。为此，把散射截面 σ 定义为 Lorentz 不变量会比较方便。(5.156) 式最后一行中，除了第一个因子 $(4E_{\mathcal{A}} E_{\mathcal{B}} |\mathbf{v}_{\mathcal{B}}|)^{-1}$ 之外，其余部分是 Lorentz 不变的。在 $\mathcal{A}$ 粒子静止的参考系中，$|\mathbf{v}_{\mathcal{B}}|$ 就是 $\mathcal{B}$ 粒子与 $\mathcal{A}$ 粒子之间的相对速度。相对速度可以定义为

$$v_{\text{rel}} \equiv |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|, \tag{5.157}$$

其中 $\mathbf{v}_{\mathcal{A}} \equiv \mathbf{p}_{\mathcal{A}}/E_{\mathcal{A}}$ 是 $\mathcal{A}$ 粒子的运行速度。不过，$E_{\mathcal{A}} E_{\mathcal{B}}\, v_{\text{rel}}$ 并不是 Lorentz 不变量。我们要做的是将相对速度替换成另一个物理量 *Møller 速度*，定义是

$$v_{\text{Møl}} \equiv \frac{1}{E_{\mathcal{A}} E_{\mathcal{B}}} \sqrt{(p_{\mathcal{A}} \cdot p_{\mathcal{B}})^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2}\,. \tag{5.158}$$

容易看出，$E_{\mathcal{A}} E_{\mathcal{B}}\, v_{\text{Møl}}$ 是 Lorentz 不变量。现在，我们将散射截面定义为

$$\sigma = \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}}\, v_{\text{Møl}}} \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)} \Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j \Big) |\mathcal{M}_{fi}|^2. \tag{5.159}$$

它是 Lorentz 不变的，而且 $\mathcal{R} = n_{\mathcal{A}} n_{\mathcal{B}} \sigma v_{\text{Møl}}$ 也是 Lorentz 不变的。当 $\mathcal{A}$ 粒子静止时，$E_{\mathcal{A}} = m_{\mathcal{A}}$，$\mathbf{p}_{\mathcal{A}} = \mathbf{0}$，故

$$v_{\text{Møl}} = \frac{1}{m_{\mathcal{A}} E_{\mathcal{B}}} \sqrt{m_{\mathcal{A}}^2 E_{\mathcal{B}}^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2} = \frac{\sqrt{E_{\mathcal{B}}^2 - m_{\mathcal{B}}^2}}{E_{\mathcal{B}}} = \frac{|\mathbf{p}_{\mathcal{B}}|}{E_{\mathcal{B}}} = |\mathbf{v}_{\mathcal{B}}|, \tag{5.160}$$

此时截面定义式 (5.159) 可以回复到 (5.156) 式。

在 (5.159) 式右边，不变振幅模方 $|\mathcal{M}_{fi}|^2$ 是*动力学因素*，而其它部分都属于*运动学因素*。在运动学因素中，对末态动量的积分具有如下形式：

$$\int d\Pi_n = \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)} \Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j \Big). \tag{5.161}$$

这个积分称为 **n 体不变相空间**。利用这个记号，可以把 (5.159) 式写得简洁一些：

$$\sigma = \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}}\, v_{\text{Møl}}} \int d\Pi_n\, |\mathcal{M}_{fi}|^2. \tag{5.162}$$

如果 (5.159) 式右边不作积分，则对应于**微分散射截面**

$$d\sigma = \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}}\, v_{\text{Møl}}} \left(\prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)} \Big(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^{n} p_j \Big) |\mathcal{M}_{fi}|^2. \tag{5.163}$$

下面进一步考察 Møller 速度 $v_{\text{Møl}}$ 的性质。设 $\mathcal{A}$ 粒子与 $\mathcal{B}$ 粒子运动方向之间的夹角为 α，则有

$$\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B}=|\mathbf{v}_\mathcal{A}||\mathbf{v}_\mathcal{B}|\cos\alpha,\quad |\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|=|\mathbf{v}_\mathcal{A}||\mathbf{v}_\mathcal{B}|\sin\alpha, \tag{5.164}$$

故

$$(\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B})^2=|\mathbf{v}_\mathcal{A}|^2|\mathbf{v}_\mathcal{B}|^2\cos^2\alpha=|\mathbf{v}_\mathcal{A}|^2|\mathbf{v}_\mathcal{B}|^2(1-\sin^2\alpha)=|\mathbf{v}_\mathcal{A}|^2|\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2. \tag{5.165}$$

从而，可以推出

$$\begin{aligned}(1-\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B})^2 &= 1-2\,\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B}+(\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B})^2=1-2\,\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B}+|\mathbf{v}_\mathcal{A}|^2|\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2\\ &= 1+|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}|^2-|\mathbf{v}_\mathcal{B}|^2+|\mathbf{v}_\mathcal{A}|^2|\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2\\ &= |\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2+(1-|\mathbf{v}_\mathcal{A}|^2)(1-|\mathbf{v}_\mathcal{B}|^2).\end{aligned} \tag{5.166}$$

将 $\mathcal{A}$ 和 $\mathcal{B}$ 的四维动量分解为时间分量和空间分量，得

$$p_\mathcal{A}^\mu=(E_\mathcal{A},\mathbf{p}_\mathcal{A})=E_\mathcal{A}(1,\mathbf{v}_\mathcal{A}),\quad p_\mathcal{B}^\mu=(E_\mathcal{B},\mathbf{p}_\mathcal{B})=E_\mathcal{B}(1,\mathbf{v}_\mathcal{B}). \tag{5.167}$$

这两个四维动量的内积为

$$p_\mathcal{A}\cdot p_\mathcal{B}=E_\mathcal{A}E_\mathcal{B}-\mathbf{p}_\mathcal{A}\cdot\mathbf{p}_\mathcal{B}=E_\mathcal{A}E_\mathcal{B}(1-\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B}). \tag{5.168}$$

于是，可以导出

$$\begin{aligned}&(p_\mathcal{A}\cdot p_\mathcal{B})^2-m_\mathcal{A}^2m_\mathcal{B}^2=E_\mathcal{A}^2E_\mathcal{B}^2(1-\mathbf{v}_\mathcal{A}\cdot\mathbf{v}_\mathcal{B})^2-m_\mathcal{A}^2m_\mathcal{B}^2\\ =\;& E_\mathcal{A}^2E_\mathcal{B}^2(|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2)+E_\mathcal{A}^2E_\mathcal{B}^2(1-|\mathbf{v}_\mathcal{A}|^2)(1-|\mathbf{v}_\mathcal{B}|^2)-m_\mathcal{A}^2m_\mathcal{B}^2\\ =\;& E_\mathcal{A}^2E_\mathcal{B}^2(|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2)+(E_\mathcal{A}^2-|\mathbf{p}_\mathcal{A}|^2)(E_\mathcal{B}^2-|\mathbf{p}_\mathcal{B}|^2)-m_\mathcal{A}^2m_\mathcal{B}^2\\ =\;& E_\mathcal{A}^2E_\mathcal{B}^2(|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2)+m_\mathcal{A}^2m_\mathcal{B}^2-m_\mathcal{A}^2m_\mathcal{B}^2\\ =\;& E_\mathcal{A}^2E_\mathcal{B}^2(|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2).\end{aligned} \tag{5.169}$$

这样的话，由 Møller 速度的定义 (5.158) 可得

$$\begin{aligned}v_{\text{Møl}} &= \frac{1}{E_\mathcal{A}E_\mathcal{B}}\sqrt{(p_\mathcal{A}\cdot p_\mathcal{B})^2-m_\mathcal{A}^2m_\mathcal{B}^2}=\frac{1}{E_\mathcal{A}E_\mathcal{B}}\sqrt{E_\mathcal{A}^2E_\mathcal{B}^2(|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2)}\\ &= \sqrt{|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|^2-|\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}|^2}.\end{aligned} \tag{5.170}$$

如果 $\mathcal{A}$ 粒子与 $\mathcal{B}$ 粒子的运动方向相同或相反，则 $\mathbf{v}_\mathcal{A}\times\mathbf{v}_\mathcal{B}=\mathbf{0}$，因而

$$v_{\text{Møl}}=|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|=v_{\text{rel}}, \tag{5.171}$$

即 Møller 速度与相对速度相同。这种情况在对撞机 (collider) 实验中经常遇到，因为在束流迎头对撞时，两股束流中的粒子具有相反的运动方向。此时，散射截面 (5.159) 化为

$$\sigma=\frac{1}{4E_\mathcal{A}E_\mathcal{B}|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|}\left(\prod_{j=1}^{n}\int\frac{d^3p_j}{(2\pi)^3 2E_j}\right)(2\pi)^4\delta^{(4)}\Big(p_\mathcal{A}+p_\mathcal{B}-\sum_{j=1}^{n}p_j\Big)|\mathcal{M}_{fi}|^2. \tag{5.172}$$

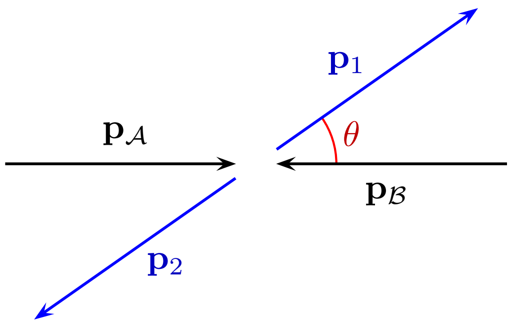


图 5.2: 质心系中 $2 \to 2$ 散射过程的动量示意图。

在非相对论极限下，$v_{\mathrm{rel}} = |\mathbf{v}_\mathcal{A} - \mathbf{v}_\mathcal{B}|$ 确实是 A 与 B 的相对速度，但是，对于相对论极限下的束流对撞，$|\mathbf{v}_\mathcal{A}| = |\mathbf{v}_\mathcal{B}| = 1$ 且 $\mathbf{v}_\mathcal{B} = -\mathbf{v}_\mathcal{A}$，故 $v_{\mathrm{rel}} = |\mathbf{v}_\mathcal{A} - \mathbf{v}_\mathcal{B}| = 2$，它是真空光速的 2 倍，显然不是真正意义的相对速度。

接下来讨论 $2 \to 2$ 散射，即 $n = 2$ 的情况，此时末态包含 2 个粒子。在系统的质量中心参考系（简称**质心系**，center-of-mass system）中，总动量为零，即

$$\mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}, \tag{5.173}$$

因而

$$|\mathbf{p}_\mathcal{A}| = |\mathbf{p}_\mathcal{B}|, \quad |\mathbf{p}_1| = |\mathbf{p}_2|. \tag{5.174}$$

可见，初态中 $\mathbf{p}_\mathcal{A}$ 与 $\mathbf{p}_\mathcal{B}$ 大小相等，方向相反，故 $v_{\mathrm{Møl}} = v_{\mathrm{rel}}$；末态中 $\mathbf{p}_1$ 与 $\mathbf{p}_2$ 也是大小相等，方向相反。这些动量在质心系中的关系如图 5.2 所示，其中，散射角 θ 是 $\mathbf{p}_1$ 与 $\mathbf{p}_\mathcal{A}$ 之间的夹角。质心系中系统的总能量称为**质心能** (center-of-mass energy) E_{CM}，满足

$$E_{\mathrm{CM}} = E_\mathcal{A} + E_\mathcal{B} = E_1 + E_2. \tag{5.175}$$

由

$$(p_\mathcal{A} + p_\mathcal{B})^2 = (E_\mathcal{A} + E_\mathcal{B})^2 - (\mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B})^2 = (E_\mathcal{A} + E_\mathcal{B})^2 = E_{\mathrm{CM}}^2 \tag{5.176}$$

可知，质心能 E_{CM} 是 Lorentz 不变量.

根据 (5.172) 和 (5.161) 式，质心系中 $2 \to 2$ 散射截面可以写成

$$\sigma = \frac{1}{4E_\mathcal{A}E_\mathcal{B}\,|\mathbf{v}_\mathcal{A} - \mathbf{v}_\mathcal{B}|} \int d\Pi_2\, |\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to p_1, p_2)|^2. \tag{5.177}$$

其中，不变散射振幅 $\mathcal{M}$ 的动量依赖性已经明显表示出来。计算 2 体不变相空间中的积分，可得

$$\begin{aligned}
\int d\Pi_2 &= \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_\mathcal{A} + p_\mathcal{B} - p_1 - p_2) \\
&= \int \frac{d^3p_1}{(2\pi)^2 4E_1E_2}\, \delta(E_{\mathrm{CM}} - E_1 - E_2) \\
&= \int d\Omega\, d|\mathbf{p}_1|\, \frac{|\mathbf{p}_1|^2}{16\pi^2 E_1E_2}\, \delta\Big(E_{\mathrm{CM}} - \sqrt{|\mathbf{p}_1|^2 + m_1^2} - \sqrt{|\mathbf{p}_1|^2 + m_2^2}\Big).
\end{aligned} \tag{5.178}$$

第二步结合三维 δ 函数 $\delta^{(3)}(\mathbf{p}_\mathcal{A}+\mathbf{p}_\mathcal{B}-\mathbf{p}_1-\mathbf{p}_2)$ 作出 $\mathbf{p}_2$ 的三维积分。这样积分看起来没有效果，但实际上是要求 $\mathbf{p}_2$ 满足动量守恒条件 $\mathbf{p}_\mathcal{A}+\mathbf{p}_\mathcal{B}-\mathbf{p}_1-\mathbf{p}_2=\mathbf{0}$，因此后续计算中出现的 $\mathbf{p}_2$ 应该满足这个条件，在质心系中则体现为 $\mathbf{p}_2=-\mathbf{p}_1$，故 $E_2=\sqrt{|\mathbf{p}_2|^2+m_2^2}=\sqrt{|\mathbf{p}_1|^2+m_2^2}$。第三步利用球坐标将 $\mathbf{p}_1$ 动量空间的体积元分解为 $d^3p_1=|\mathbf{p}_1|^2\,d|\mathbf{p}_1|\,d\Omega$，而立体角的微分可以用散射角 θ 表示为

$$d\Omega=\sin\theta\,d\theta\,d\phi,\tag{5.179}$$

其中方位角 ϕ 在垂直于 $\mathbf{p}_\mathcal{A}$ 方向的平面上定义。现在，δ 函数的宗量是关于 $|\mathbf{p}_1|$ 的函数，利用 (2.117) 式，可得作出 $|\mathbf{p}_1|$ 的积分，得到

$$\begin{aligned}&\int d|\mathbf{p}_1|\,\delta\Big(E_\mathrm{CM}-\sqrt{|\mathbf{p}_1|^2+m_1^2}-\sqrt{|\mathbf{p}_1|^2+m_2^2}\Big)\\
&=\left|\frac{d}{d|\mathbf{p}_1|}\Big(E_\mathrm{CM}-\sqrt{|\mathbf{p}_1|^2+m_1^2}-\sqrt{|\mathbf{p}_1|^2+m_2^2}\Big)\right|^{-1}=\left(\frac{2|\mathbf{p}_1|}{2\sqrt{|\mathbf{p}_1|^2+m_1^2}}+\frac{2|\mathbf{p}_1|}{2\sqrt{|\mathbf{p}_1|^2+m_2^2}}\right)^{-1}\\
&=\left[|\mathbf{p}_1|\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\right]^{-1}=\frac{E_1E_2}{|\mathbf{p}_1|(E_1+E_2)}=\frac{E_1E_2}{|\mathbf{p}_1|E_\mathrm{CM}}.\end{aligned}\tag{5.180}$$

于是，(5.178) 式化为

$$\int d\Pi_2=\int d\Omega\,\frac{|\mathbf{p}_1|^2}{16\pi^2E_1E_2}\frac{E_1E_2}{|\mathbf{p}_1|E_\mathrm{CM}}=\int d\Omega\,\frac{|\mathbf{p}_1|}{16\pi^2E_\mathrm{CM}}.\tag{5.181}$$

将上式代入散射截面表达式 (5.177)，得

$$\sigma=\frac{1}{4E_\mathcal{A}E_\mathcal{B}\,|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|}\int d\Omega\,\frac{|\mathbf{p}_1|}{16\pi^2E_\mathrm{CM}}\,|\mathcal{M}(p_\mathcal{A},p_\mathcal{B}\to p_1,p_2)|^2.\tag{5.182}$$

于是，质心系中关于立体角的微分散射截面是

$$\left(\frac{d\sigma}{d\Omega}\right)_\mathrm{CM}=\frac{1}{64\pi^2}\frac{1}{E_\mathcal{A}E_\mathcal{B}\,|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|}\frac{|\mathbf{p}_1|}{E_\mathrm{CM}}\,|\mathcal{M}(p_\mathcal{A},p_\mathcal{B}\to p_1,p_2)|^2.\tag{5.183}$$

利用末态粒子在质心系中的动量关系 $|\mathbf{p}_1|=|\mathbf{p}_2|$，可得

$$E_\mathrm{CM}=E_1+E_2=E_1+\sqrt{|\mathbf{p}_1|^2+m_2^2}=E_1+\sqrt{E_1^2-m_1^2+m_2^2}\,,\tag{5.184}$$

故

$$E_1^2-m_1^2+m_2^2=(E_\mathrm{CM}-E_1)^2=E_\mathrm{CM}^2-2E_\mathrm{CM}E_1+E_1^2,\tag{5.185}$$

即

$$2E_\mathrm{CM}E_1=E_\mathrm{CM}^2+m_1^2-m_2^2,\tag{5.186}$$

从而，E_1 可以表示为

$$E_1=\frac{1}{2E_\mathrm{CM}}\,(E_\mathrm{CM}^2+m_1^2-m_2^2).\tag{5.187}$$

同理，E_2 可以表示为

$$E_2=\frac{1}{2E_\mathrm{CM}}\,(E_\mathrm{CM}^2+m_2^2-m_1^2).\tag{5.188}$$

根据动量与能量的关系，有

$$
\begin{aligned}
|\mathbf{p}_1|^2 &= E_1^2 - m_1^2 = \frac{1}{4E_{\mathrm{CM}}^2}(E_{\mathrm{CM}}^2 + m_1^2 - m_2^2)^2 - m_1^2 \\
&= \frac{1}{4E_{\mathrm{CM}}^2}[E_{\mathrm{CM}}^4 + m_1^4 + m_2^4 + 2E_{\mathrm{CM}}^2 m_1^2 - 2E_{\mathrm{CM}}^2 m_2^2 - 2m_1^2 m_2^2 - 4E_{\mathrm{CM}}^2 m_1^2] \\
&= \frac{1}{4E_{\mathrm{CM}}^2}(E_{\mathrm{CM}}^4 + m_1^4 + m_2^4 - 2E_{\mathrm{CM}}^2 m_1^2 - 2E_{\mathrm{CM}}^2 m_2^2 - 4m_1^2 m_2^2) \\
&= \frac{1}{4E_{\mathrm{CM}}^2}\lambda(E_{\mathrm{CM}}^2, m_1^2, m_2^2).
\end{aligned}
\tag{5.189}
$$

其中，λ 函数定义为

$$
\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz, \tag{5.190}
$$

它关于 x, y, z 对称。可见，末态粒子的动量满足

$$
|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{1}{2E_{\mathrm{CM}}}\lambda^{1/2}(E_{\mathrm{CM}}^2, m_1^2, m_2^2) = \frac{E_{\mathrm{CM}}}{2}\lambda^{1/2}\left(1, \frac{m_1^2}{E_{\mathrm{CM}}^2}, \frac{m_2^2}{E_{\mathrm{CM}}^2}\right). \tag{5.191}
$$

于是，(5.183) 式可以改写成

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{128\pi^2 E_{\mathcal{A}} E_{\mathcal{B}}\,|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|}\lambda^{1/2}\left(1, \frac{m_1^2}{E_{\mathrm{CM}}^2}, \frac{m_2^2}{E_{\mathrm{CM}}^2}\right)|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2)|^2. \tag{5.192}
$$

下面讨论几种特殊情况。

(1) 如果散射过程关于对撞轴 ($\mathbf{p}_{\mathcal{A}}$ 对应的直线) 对称，则不变振幅 $\mathcal{M}$ 与 ϕ 无关，是 θ 的函数，从而，

$$
\int d\Omega\,|\mathcal{M}(\theta)|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\,|\mathcal{M}(\theta)|^2 = 2\pi\int_0^{\pi} d\theta\,\sin\theta\,|\mathcal{M}(\theta)|^2. \tag{5.193}
$$

此时散射截面为

$$
\sigma = \int d\Omega\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{64\pi E_{\mathcal{A}} E_{\mathcal{B}}\,|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|}\lambda^{1/2}\left(1, \frac{m_1^2}{E_{\mathrm{CM}}^2}, \frac{m_2^2}{E_{\mathrm{CM}}^2}\right)\int_0^{\pi} d\theta\,\sin\theta\,|\mathcal{M}(\theta)|^2. \tag{5.194}
$$

(2) 如果末态 2 个粒子质量相同，$m_1 = m_2 = m$，则由

$$
\lambda(x,y,y) = x^2 + 2y^2 - 4xy - 2y^2 = x(x-4y) \tag{5.195}
$$

可得

$$
\lambda^{1/2}\left(1, \frac{m_1^2}{E_{\mathrm{CM}}^2}, \frac{m_2^2}{E_{\mathrm{CM}}^2}\right) = \lambda^{1/2}\left(1, \frac{m^2}{E_{\mathrm{CM}}^2}, \frac{m^2}{E_{\mathrm{CM}}^2}\right) = \sqrt{1 - \frac{4m^2}{E_{\mathrm{CM}}^2}}. \tag{5.196}
$$

(3) 如果初末态 4 个粒子的质量相同，即 $m_{\mathcal{A}} = m_{\mathcal{B}} = m_1 = m_2$，则有

$$
E_{\mathcal{A}} = E_{\mathcal{B}} = \frac{E_{\mathrm{CM}}}{2} = E_1 = E_2, \quad |\mathbf{p}_{\mathcal{A}}| = |\mathbf{p}_{\mathcal{B}}| = |\mathbf{p}_1| = |\mathbf{p}_2|. \tag{5.197}
$$

从而，可得

$$|\mathbf{v}_\mathcal{A}-\mathbf{v}_\mathcal{B}|=\left|\frac{\mathbf{p}_\mathcal{A}}{E_\mathcal{A}}-\frac{\mathbf{p}_\mathcal{B}}{E_\mathcal{B}}\right|=\frac{2|\mathbf{p}_\mathcal{A}|}{E_\mathcal{A}}=\frac{4|\mathbf{p}_1|}{E_\mathrm{CM}}. \tag{5.198}$$

于是，(5.183) 式化为

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_\mathrm{CM}&=\frac{1}{64\pi^2}\frac{4}{E_\mathrm{CM}^2}\frac{E_\mathrm{CM}}{4|\mathbf{p}_1|}\frac{|\mathbf{p}_1|}{E_\mathrm{CM}}|\mathcal{M}(p_\mathcal{A},p_\mathcal{B}\to p_1,p_2)|^2\\&=\frac{1}{64\pi^2E_\mathrm{CM}^2}|\mathcal{M}(p_\mathcal{A},p_\mathcal{B}\to p_1,p_2)|^2.\end{aligned} \tag{5.199}$$

5.3.3 衰变宽度

即使没有与其它粒子散射，一个粒子也不一定是稳定的。不稳定粒子 $\mathcal{A}$ 自身可以通过相互作用**衰变** (decay) 成其它粒子。在 $\mathcal{A}$ 粒子的静止参考系中，它在衰变之前存活的时间 t 服从指数分布，概率密度为

$$P(t)=\frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)=\Gamma\exp(-\Gamma t). \tag{5.200}$$

其中，τ 是常数，称为粒子的**寿命** (lifetime)，由

$$\langle t\rangle=\frac{1}{\tau}\int_0^\infty te^{-t/\tau}dt=-\int_0^\infty t\,de^{-t/\tau}=-te^{-t/\tau}\Big|_0^\infty+\int_0^\infty e^{-t/\tau}dt=-\,\tau e^{-t/\tau}\Big|_0^\infty=\tau \tag{5.201}$$

可知，寿命是粒子存活的平均时间。因此，

$$\Gamma\equiv\frac{1}{\tau} \tag{5.202}$$

是 $\mathcal{A}$ 粒子在静止系中发生衰变的平均速率，它在自然单位制中具有质量的量纲，称为**衰变宽度** (decay width)，简称**宽度**。

$\mathcal{A}$ 粒子可能有多种衰变过程。在一次衰变中，某个衰变过程 $i\to f$ 发生的概率称为此过程的**分支比** (branching ratio)，记作 $\mathrm{BR}(f)$。衰变过程 $i\to f$ 的**分宽度** (partial decay width) 定义为

$$\Gamma_f=\Gamma\cdot\mathrm{BR}(f), \tag{5.203}$$

它是 $\mathcal{A}$ 粒子静止系中衰变过程 $i\to f$ 发生的平均速率。所有衰变过程的分支比之和应该是归一的，故

$$\sum_f\mathrm{BR}(f)=\frac{1}{\Gamma}\sum_f\Gamma_f=1,\quad \Gamma=\sum_f\Gamma_f. \tag{5.204}$$

我们可以通过跃迁概率计算衰变过程 $i\to f$ 的分宽度。现在，初态 $|i\rangle$ 只包含 1 个粒子 $\mathcal{A}$，末态 $|f\rangle$ 则包含 $n\geq 2$ 个粒子。因此，$|i\rangle$ 的自我内积为

$$\langle i|i\rangle=2E_\mathcal{A}\tilde{V}, \tag{5.205}$$

跃迁概率是

$$P_{fi}=\frac{|\langle f|\,iT\,|i\rangle|^2}{\langle i|i\rangle\,\langle f|f\rangle}=\frac{\tilde{V}\tilde{T}(2\pi)^4\delta^{(4)}(p_\mathcal{A}-p_f)|\mathcal{M}_{fi}|^2}{2E_\mathcal{A}\tilde{V}\prod\limits_{j=1}^n(2E_jV)}=\frac{\tilde{T}(2\pi)^4\delta^{(4)}(p_\mathcal{A}-p_f)|\mathcal{M}_{fi}|^2}{2E_\mathcal{A}\prod\limits_{j=1}^n(2E_j\tilde{V})}. \tag{5.206}$$

对于一组特定的末态动量 $\{p_j\}$，单位时间内的跃迁概率为

$$R_{\{p_j\}} = \frac{P_{fi}}{\tilde{T}} = \frac{1}{2E_{\mathcal{A}} \prod_{j=1}^{n} (2E_j \tilde{V})} (2\pi)^4 \delta^{(4)}\Big(p_{\mathcal{A}} - \sum_{j=1}^{n} p_j\Big) |\mathcal{M}_{fi}|^2. \tag{5.207}$$

将末态动量的所有取值考虑进来，可得单位时间内衰变过程 $i \to f$ 的发生概率为

$$R_f = \left(\prod_{j=1}^{n} \frac{\tilde{V}}{(2\pi)^3} \int d^3 p_j\right) R_{\{p_j\}} = \frac{1}{2E_{\mathcal{A}}} \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j}\right) (2\pi)^4 \delta^{(4)} \left(p_{\mathcal{A}} - \sum_{j=1}^{n} p_j\right) |\mathcal{M}_{fi}|^2. \tag{5.208}$$

在 $\mathcal{A}$ 粒子静止系中，$E_{\mathcal{A}} = m_{\mathcal{A}}$，而 R_f 的值就是分宽度 Γ_f，故

$$\Gamma_f = \frac{1}{2m_{\mathcal{A}}} \left(\prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 2E_j}\right) (2\pi)^4 \delta^{(4)}\Big(p_{\mathcal{A}} - \sum_{j=1}^{n} p_j\Big) |\mathcal{M}_{fi}|^2. \tag{5.209}$$

若 $\mathcal{A}$ 粒子是标量粒子，自旋为 0，则 $\mathcal{A}$ 粒子静止系没有特殊的方向，于是，任一末态粒子在动量方向上呈球对称分布。若 $\mathcal{A}$ 粒子具有非零自旋，则自旋方向是 $\mathcal{A}$ 粒子静止系的特殊方向，于是，末态粒子在动量方向上呈轴对称分布，以 $\mathcal{A}$ 粒子自旋方向为轴；在实际情况中，初态中 $\mathcal{A}$ 粒子自旋的取向往往是不确定的，而且它取不同方向具有相同的概率，那么，我们可以对 $\mathcal{A}$ 粒子的自旋方向取平均，从而，末态粒子在动量方向上也呈球对称分布。

下面分别讨论二体衰变和三体衰变。

(1) 对于二体衰变，$n = 2$，末态两个粒子的质心系就是 $\mathcal{A}$ 粒子的静止系，故 $E_{\text{CM}} = m_{\mathcal{A}}$。于是，(5.187) 和 (5.188) 式化为

$$E_1 = \frac{1}{2m_{\mathcal{A}}} (E_{\text{CM}}^2 + m_1^2 - m_2^2), \quad E_2 = \frac{1}{2m_{\mathcal{A}}} (E_{\text{CM}}^2 + m_2^2 - m_1^2). \tag{5.210}$$

而 (5.191) 式化为

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{m_{\mathcal{A}}}{2} \lambda^{1/2} \left(1, \frac{m_1^2}{m_{\mathcal{A}}^2}, \frac{m_2^2}{m_{\mathcal{A}}^2}\right). \tag{5.211}$$

2 体不变相空间 (5.181) 变成

$$\int d\Pi_2 = \int d\Omega \, \frac{|\mathbf{p}_1|}{16\pi^2 m_{\mathcal{A}}}. \tag{5.212}$$

此处，$d\Omega = \sin\theta \, d\theta \, d\phi$ 中的 θ 和 ϕ 分别是 $\mathbf{p}_1$ 在某个球坐标系中的极角 (polar angle) 和方位角 (azimuthal angle)。于是，衰变过程 $i \to f$ 的分宽度可以表达为

$$\begin{aligned}
\Gamma_f &= \frac{1}{2m_{\mathcal{A}}} \int d\Pi_2 \, |\mathcal{M}(p_{\mathcal{A}} \to p_1, p_2)|^2 = \frac{|\mathbf{p}_1|}{32\pi^2 m_{\mathcal{A}}^2} \int d\Omega \, |\mathcal{M}(p_{\mathcal{A}} \to p_1, p_2)|^2 \\
&= \frac{1}{64\pi^2 m_{\mathcal{A}}} \lambda^{1/2} \left(1, \frac{m_1^2}{m_{\mathcal{A}}^2}, \frac{m_2^2}{m_{\mathcal{A}}^2}\right) \int d\Omega \, |\mathcal{M}(p_{\mathcal{A}} \to p_1, p_2)|^2.
\end{aligned} \tag{5.213}$$

如果 $\mathcal{A}$ 粒子的自旋为 0，或者对它的自旋方向取平均，按照前述讨论，末态粒子在动量方向上呈球对称分布。此时，不变振幅模方 $|\mathcal{M}|^2$ 与 θ、ϕ 无关，对立体角积分只给出一个 4π 因子，故分宽度为

$$\Gamma_f = \frac{|\mathbf{p}_1|}{8\pi m_{\mathcal{A}}^2} |\mathcal{M}|^2 = \frac{|\mathcal{M}|^2}{16\pi m_{\mathcal{A}}} \lambda^{1/2} \left(1, \frac{m_1^2}{m_{\mathcal{A}}^2}, \frac{m_2^2}{m_{\mathcal{A}}^2}\right). \tag{5.214}$$

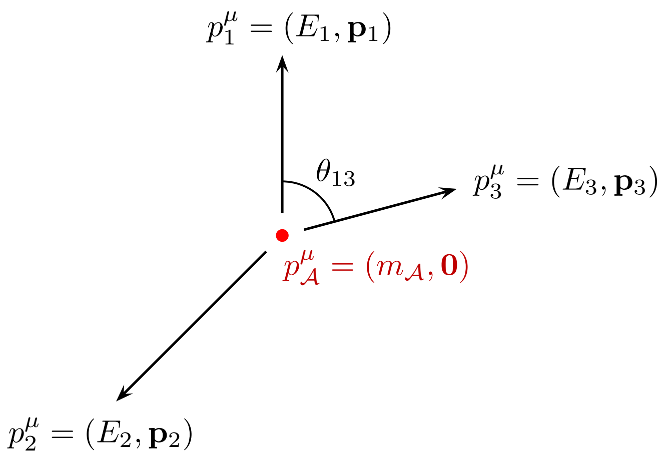


图 5.3: $\mathcal{A}$ 粒子静止系中三体衰变过程的动量示意图。

(2) 对于三体衰变，$n = 3$，衰变过程 $i \to f$ 的分宽度可以表示成

$$\Gamma_f = \frac{1}{2m_{\mathcal{A}}} \int d\Pi_3 \, |\mathcal{M}(p_{\mathcal{A}} \to p_1, p_2, p_3)|^2, \tag{5.215}$$

其中，3 体不变相空间为

$$\int d\Pi_3 = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} - p_1 - p_2 - p_3). \tag{5.216}$$

这里，我们只在 $\mathcal{A}$ 粒子的静止系中讨论它没有自旋或者对它的自旋方向取平均的情况，如前所述，此时末态粒子在动量方向上呈球对称分布，不变振幅模方 $|\mathcal{M}|^2$ 与末态粒子的运动方向无关。根据动量守恒定律，$\mathbf{0} = \mathbf{p}_{\mathcal{A}} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$，即末态 3 个粒子的三维动量之和为零，因而这 3 个三维动量矢量处在同一个平面内，如图 5.3 所示。对于确定的 $\mathbf{p}_1$ 和 $\mathbf{p}_3$，第 2 个粒子的三维动量 $\mathbf{p}_2 = -\mathbf{p}_1 - \mathbf{p}_3$ 由动量守恒定律决定。对 $\mathbf{p}_2$ 积分，可消去代表动量守恒定律的 $\delta^{(3)}(\mathbf{p}_{\mathcal{A}} - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)$，得到

$$\begin{aligned}\int d\Pi_3 &= \frac{1}{8(2\pi)^5} \int \frac{d^3p_1\, d^3p_3}{E_1 E_2 E_3} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3) \\ &= \frac{1}{8(2\pi)^5} \int d\Omega_1\, d|\mathbf{p}_1|\, d\Omega_3\, d|\mathbf{p}_3| \frac{|\mathbf{p}_1|^2 |\mathbf{p}_3|^2}{E_1 E_2 E_3} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3).\end{aligned} \tag{5.217}$$

其中，Ω_1 和 Ω_3 分别是 $\mathbf{p}_1$ 和 $\mathbf{p}_3$ 相对应的立方角。

对粒子 1 的质壳条件 $|\mathbf{p}_1|^2 + m_1^2 = E_1^2$ 两边求微分，得 $2|\mathbf{p}_1| d|\mathbf{p}_1| = 2E_1 dE_1$，对粒子 3 也可以得到类似的式子，故

$$|\mathbf{p}_1| d|\mathbf{p}_1| = E_1 dE_1, \quad |\mathbf{p}_3| d|\mathbf{p}_3| = E_3 dE_3. \tag{5.218}$$

从而，有

$$\int d\Pi_3 = \frac{1}{8(2\pi)^5} \int d\Omega_1\, d\Omega_3\, dE_1\, dE_3 \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3)$$

$$= \frac{1}{4(2\pi)^4} \int d\Omega_3 \, dE_1 \, dE_3 \, \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \, \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3). \tag{5.219}$$

第二步对 Ω_1 作了积分，由于粒子 1 在动量方向上呈球对称分布，此积分只给出一个 4π 因子。

将 $\mathbf{p}_1$ 与 $\mathbf{p}_3$ 方向之间的夹角记为 θ_{13}，则粒子 3 的立体角微分可以表示为

$$d\Omega_3 = \sin\theta_{13} \, d\theta_{13} \, d\phi_3 = d\cos\theta_{13} \, d\phi_3, \tag{5.220}$$

其中 ϕ_3 是粒子 3 的方位角。这样的话，对 Ω_3 积分不是平庸的，这是因为 E_2 依赖于 $\cos\theta_{13}$，

$$E_2 = \sqrt{m_2^2 + |\mathbf{p}_2|^2} = \sqrt{m_2^2 + |\mathbf{p}_1 + \mathbf{p}_3|^2} = \sqrt{m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13}}\,, \tag{5.221}$$

导致 $\delta(m_{\mathcal{A}} - E_1 - E_2 - E_3)$ 也依赖于 $\cos\theta_{13}$。由

$$\frac{\partial E_2}{\partial \cos\theta_{13}} = \frac{2|\mathbf{p}_1||\mathbf{p}_3|}{2\sqrt{m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta_{13}}} = \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \tag{5.222}$$

有

$$\left| \frac{\partial(m_{\mathcal{A}} - E_1 - E_2 - E_3)}{\partial \cos\theta_{13}} \right| = \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2}. \tag{5.223}$$

再利用 (2.117) 式，作出关于 Ω_3 的积分，得

$$\begin{aligned} \int d\Pi_3 &= \frac{1}{4(2\pi)^4} \int_0^{2\pi} d\phi_3 \int_{-1}^{1} d\cos\theta_{13} \int dE_1 \, dE_3 \, \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \, \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3) \\ &= \frac{1}{4(2\pi)^4} \int_0^{2\pi} d\phi_3 \int dE_1 \, dE_3 \, \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \left| \frac{\partial(m_{\mathcal{A}} - E_1 - E_2 - E_3)}{\partial \cos\theta_{13}} \right|^{-1} \\ &= \frac{1}{4(2\pi)^3} \int dE_1 \, dE_3. \end{aligned} \tag{5.224}$$

从而，分宽度 (5.215) 化为

$$\Gamma_f = \frac{1}{(2\pi)^3} \frac{1}{8m_{\mathcal{A}}} \int_{E_1^{\min}}^{E_1^{\max}} dE_1 \int_{E_3^{\min}}^{E_3^{\max}} dE_3 \, |\mathcal{M}(E_1, E_2)|^2. \tag{5.225}$$

注意，使用上式计算时需要把不变振幅 $\mathcal{M}$ 表达为 E_1 和 E_2 的函数，而且要仔细考虑 E_1 和 E_2 的积分上下限。

在实践中，把 E_1 和 E_2 当作积分变量并不方便，我们可以将它们替换成更加便利的变量。引入两个 Lorentz 不变量

$$s_{12} \equiv (p_1 + p_2)^2 = (p_{\mathcal{A}} - p_3)^2 = m_{\mathcal{A}}^2 + m_3^2 - 2m_{\mathcal{A}}E_3, \tag{5.226}$$

$$s_{23} \equiv (p_2 + p_3)^2 = (p_{\mathcal{A}} - p_1)^2 = m_{\mathcal{A}}^2 + m_1^2 - 2m_{\mathcal{A}}E_1, \tag{5.227}$$

它们在不同参考系中分别具有相同的值。我们可以把粒子 1 和 2 组成的系统看成一个等效粒子，四维动量为 $p_{12}^\mu = p_1^\mu + p_2^\mu$。由于 $p_{12}^2 = (p_1 + p_2)^2 = s_{12}$，$\sqrt{s_{12}}$ 相当于这个等效粒子的质量，称

为粒子 1 和 2 的**不变质量 (invariant mass)**，它也是粒子 1 和 2 组成的系统的质心能。类似地，$\sqrt{s_{23}}$ 是粒子 2 和 3 的不变质量。s_{12} 和 s_{23} 的微分分别正比于 E_1 和 E_2 的微分，

$$ds_{12} = -2m_{\mathcal{A}}dE_3, \quad ds_{23} = -2m_{\mathcal{A}}dE_1. \tag{5.228}$$

于是，分宽度的积分式 (5.225) 可以改写为

$$\Gamma_f = \frac{1}{(2\pi)^3}\frac{1}{32m_{\mathcal{A}}^3}\int_{s_{12}^{\min}}^{s_{12}^{\max}} ds_{12}\int_{s_{23}^{\min}}^{s_{23}^{\max}} ds_{23}\,|\mathcal{M}(s_{12},s_{23})|^2. \tag{5.229}$$

使用上式计算时，需要把不变振幅 $\mathcal{M}$ 表达为 s_{12} 和 s_{23} 的函数。接下来，我们讨论 s_{12} 和 s_{23} 的积分上下限。注意，对 s_{23} 的积分位于内层，积分上下限会依赖于 s_{12}。

在粒子 1 和 2 的质心系中，$\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 = \mathbf{0}$，质心能 $\tilde{E}_{\mathrm{CM}} = \sqrt{s_{12}}$。这里我们用波浪线标记此参考系中的物理量。根据 (5.188) 式，粒子 2 的能量为

$$\tilde{E}_2 = \frac{1}{2\sqrt{s_{12}}}(s_{12} - m_1^2 + m_2^2). \tag{5.230}$$

动量守恒定律给出 $\tilde{\mathbf{p}}_3 = \tilde{\mathbf{p}}_{\mathcal{A}} - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2 = \tilde{\mathbf{p}}_{\mathcal{A}}$，由 s_{12} 的 Lorentz 不变性有

$$\begin{aligned} s_{12} &= (p_1+p_2)^2 = (\tilde{p}_1+\tilde{p}_2)^2 = (\tilde{p}_{\mathcal{A}}-\tilde{p}_3)^2 = p_{\mathcal{A}}^2 + p_3^2 - 2\,p_{\mathcal{A}}\cdot p_3 \\ &= m_{\mathcal{A}}^2 + m_3^2 - 2\tilde{E}_{\mathcal{A}}\tilde{E}_3 + 2\,\tilde{\mathbf{p}}_{\mathcal{A}}\cdot\tilde{\mathbf{p}}_3 = m_{\mathcal{A}}^2 + m_3^2 - 2\sqrt{|\tilde{\mathbf{p}}_3|^2 + m_{\mathcal{A}}^2}\,\tilde{E}_3 + 2|\tilde{\mathbf{p}}_3|^2 \\ &= m_{\mathcal{A}}^2 + m_3^2 - 2\sqrt{\tilde{E}_3^2 - m_3^2 + m_{\mathcal{A}}^2}\,\tilde{E}_3 + 2\tilde{E}_3^2 - 2m_3^2 \\ &= m_{\mathcal{A}}^2 - 2\sqrt{\tilde{E}_3^2 - m_3^2 + m_{\mathcal{A}}^2}\,\tilde{E}_3 + 2\tilde{E}_3^2 - m_3^2. \end{aligned} \tag{5.231}$$

整理，得 $2\sqrt{\tilde{E}_3^2 - m_3^2 + m_{\mathcal{A}}^2}\,\tilde{E}_3 = m_{\mathcal{A}}^2 - s_{12} + 2\tilde{E}_3^2 - m_3^2$，两边平方，得

$$\begin{aligned} 4(\tilde{E}_3^2 - m_3^2 + m_{\mathcal{A}}^2)\tilde{E}_3^2 &= (m_{\mathcal{A}}^2 - s_{12} + 2\tilde{E}_3^2 - m_3^2)^2 \\ &= (m_{\mathcal{A}}^2 - s_{12} - m_3^2)^2 + 4\tilde{E}_3^4 + 4(m_{\mathcal{A}}^2 - s_{12} - m_3^2)\tilde{E}_3^2. \end{aligned} \tag{5.232}$$

再整理，得 $4s_{12}\tilde{E}_3^2 = (m_{\mathcal{A}}^2 - s_{12} - m_3^2)^2$，故粒子 3 的能量为

$$\tilde{E}_3 = \frac{1}{2\sqrt{s_{12}}}(m_{\mathcal{A}}^2 - s_{12} - m_3^2). \tag{5.233}$$

(5.230) 式和 (5.233) 式右边是 Lorentz 不变的，而且，对于确定的 s_{12}，$\tilde{E}_2$ 和 $\tilde{E}_3$ 是确定的。

另一方面，由 s_{23} 的 Lorentz 不变性有

$$s_{23} = (p_2+p_3)^2 = (\tilde{p}_2+\tilde{p}_3)^2 = (\tilde{E}_2+\tilde{E}_3)^2 - |\tilde{\mathbf{p}}_2+\tilde{\mathbf{p}}_3|^2, \tag{5.234}$$

这里，

$$|\tilde{\mathbf{p}}_2+\tilde{\mathbf{p}}_3|^2 = |\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|^2 + 2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3|\cos\tilde{\theta}_{23} \tag{5.235}$$

其中 $\tilde{\theta}_{23}$ 是 $\tilde{\mathbf{p}}_2$ 与 $\tilde{\mathbf{p}}_3$ 方向之间的夹角。当 $\cos\tilde{\theta}_{23}=1$ 时，$|\tilde{\mathbf{p}}_2+\tilde{\mathbf{p}}_3|^2=|\tilde{\mathbf{p}}_2|^2+|\tilde{\mathbf{p}}_3|^2+2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3|=(|\tilde{\mathbf{p}}_2|^2+|\tilde{\mathbf{p}}_3|)^2$，而 s_{23} 取得最小值

$$s_{23}^{\min}=(\tilde{E}_2+\tilde{E}_3)^2-(|\tilde{\mathbf{p}}_2|^2+|\tilde{\mathbf{p}}_3|)^2=(\tilde{E}_2+\tilde{E}_3)^2-\left(\sqrt{\tilde{E}_2^2-m_2^2}+\sqrt{\tilde{E}_3^2-m_3^2}\right)^2. \tag{5.236}$$

当 $\cos\tilde{\theta}_{23}=-1$ 时，$|\tilde{\mathbf{p}}_2+\tilde{\mathbf{p}}_3|^2=|\tilde{\mathbf{p}}_2|^2+|\tilde{\mathbf{p}}_3|^2-2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3|=(|\tilde{\mathbf{p}}_2|^2-|\tilde{\mathbf{p}}_3|)^2$，而 s_{23} 取得最大值

$$s_{23}^{\max}=(\tilde{E}_2+\tilde{E}_3)^2-(|\tilde{\mathbf{p}}_2|^2-|\tilde{\mathbf{p}}_3|)^2=(\tilde{E}_2+\tilde{E}_3)^2-\left(\sqrt{\tilde{E}_2^2-m_2^2}-\sqrt{\tilde{E}_3^2-m_3^2}\right)^2. \tag{5.237}$$

对于确定的 s_{12}，(5.236) 式和 (5.237) 式分别给出 s_{23} 的积分下限和上限。注意，它们是 Lorentz 不变的。

在粒子 1 和 2 的质心系中，

$$s_{12}=(\tilde{p}_1+\tilde{p}_2)^2=(\tilde{E}_1+\tilde{E}_2)^2-|\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2|^2=(\tilde{E}_1+\tilde{E}_2)^2. \tag{5.238}$$

可见，当 $\tilde{E}_1=m_1$ 且 $\tilde{E}_2=m_2$ 时，s_{12} 取得最小值

$$s_{12}^{\min}=(m_1+m_2)^2. \tag{5.239}$$

在 $\mathcal{A}$ 粒子的静止系中，根据 (5.226) 式，当 $E_3=m_3$ 时，s_{12} 取得最大值

$$s_{12}^{\max}=m_{\mathcal{A}}^2+m_3^2-2m_{\mathcal{A}}m_3=(m_{\mathcal{A}}-m_3)^2. \tag{5.240}$$

注意 s_{12} 的积分下限 (5.239) 和积分上限 (5.240) 也是 Lorentz 不变的。

5.4 Wick 定理

5.4.1 正规乘积和 Wick 定理

在 5.2 节中，借助时序乘积，我们把 S 算符写成了一个紧凑的级数形式 (5.119)。不过，如何适当地处理级数每一项中的时序乘积 $\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]$ 呢？在量子场论中，相互作用哈密顿量密度 $\mathcal{H}_1(x)$ 是由若干个场算符构成的，因而我们需要处理的是多个场算符的时序乘积。这看来不是一个简单的问题，幸好接下来将要介绍的 Wick 定理为我们提供了一个简便的方法。

在相互作用绘景中，实标量场 $\phi(x)$ 的平面波展开式 (5.47) 可以分解成两个部分：

$$\phi(x)=\phi^{(+)}(x)+\phi^{(-)}(x), \tag{5.241}$$

其中正能解部分为

$$\phi^{(+)}(x)\equiv\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,a_{\mathbf{p}}e^{-ip\cdot x}, \tag{5.242}$$

负能解部分为

$$\phi^{(-)}(x)\equiv\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\,a_{\mathbf{p}}^\dagger e^{ip\cdot x}. \tag{5.243}$$

根据 (5.84) 式，我们同样可以把有质量矢量场 $A^\mu(x)$ 分为正能解和负能解两部分:

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x), \tag{5.244}$$

其中,

$$A^{\mu(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x}, \tag{5.245}$$

$$A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}. \tag{5.246}$$

前面提到，Dirac 旋量场 $\psi_a(x)$ 在相互作用绘景中的平面波展开式也具有 Heisenberg 绘景中自由场展开式 (4.236) 的形式，即

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x), \tag{5.247}$$

其中,

$$\psi_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} u_a(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x}, \tag{5.248}$$

$$\psi_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} v_a(\mathbf{p},\lambda) b^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}. \tag{5.249}$$

可以看到，正能解部分只包含湮灭算符，而负能解部分只包含产生算符。

引入**正规乘积** (normal product) 的概念，以 N 为记号，它的作用是将乘积中的所有湮灭算符移动到所有产生算符的右边，形成*正规次序* (normal order)；考虑到费米子算符的反对易性，移动过程中若涉及奇数次费米子算符之间的交换，则应额外增加一个负号。例如，对于标量场的产生湮灭算符，有

$$\mathsf{N}(a_{\mathbf{p}} a^\dagger_{\mathbf{q}} a_{\mathbf{k}} a^\dagger_{\mathbf{l}}) = a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{l}} a_{\mathbf{p}} a_{\mathbf{k}} = a^\dagger_{\mathbf{l}} a^\dagger_{\mathbf{q}} a_{\mathbf{p}} a_{\mathbf{k}} = a^\dagger_{\mathbf{l}} a^\dagger_{\mathbf{q}} a_{\mathbf{k}} a_{\mathbf{p}}; \tag{5.250}$$

对于旋量场的产生湮灭算符，则有

$$\mathsf{N}(b_{\mathbf{p},\lambda_1} a^\dagger_{\mathbf{q},\lambda_2} a_{\mathbf{k},\lambda_3} b^\dagger_{\mathbf{l},\lambda_4}) = -a^\dagger_{\mathbf{q},\lambda_2} b^\dagger_{\mathbf{l},\lambda_4} b_{\mathbf{p},\lambda_1} a_{\mathbf{k},\lambda_3} = b^\dagger_{\mathbf{l},\lambda_4} a^\dagger_{\mathbf{q},\lambda_2} b_{\mathbf{p},\lambda_1} a_{\mathbf{k},\lambda_3} = -b^\dagger_{\mathbf{l},\lambda_4} a^\dagger_{\mathbf{q},\lambda_2} a_{\mathbf{k},\lambda_3} b_{\mathbf{p},\lambda_1}. \tag{5.251}$$

于是，可以得到两个标量场的正规乘积为

$$\mathsf{N}[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x), \tag{5.252}$$

最后一项中 $\phi^{(+)}(x)$ 被正规操作移动到 $\phi^{(-)}(y)$ 的右边。而两个旋量场的正规乘积为

$$\mathsf{N}[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y) + \psi_a^{(-)}(x)\psi_b^{(+)}(y) + \psi_a^{(+)}(x)\psi_b^{(+)}(y) - \psi_b^{(-)}(y)\psi_a^{(+)}(x), \tag{5.253}$$

最后一项中 $\psi_a^{(+)}(x)$ 被正规操作移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号。湮灭算符对真空态 $|0\rangle$ 的作用为零，如 $a_{\mathbf{p}}|0\rangle = 0$，$\langle 0| a^\dagger_{\mathbf{p}} = 0$，因此，对一组产生湮灭算符的任意乘积取正规次序之后，真空期待值为零:

$$\langle 0| \mathsf{N}\,(\text{产生湮灭算符的乘积})\,|0\rangle = 0. \tag{5.254}$$

用统一的记号 $\Phi_a(x)$ 代表一般的场算符，它可以是标量场 $\phi(x)$ 或 $\phi^\dagger(x)$，也可以是矢量场 $A^\mu(x)$ 的一个分量，还可以是旋量场 $\psi_a(x)$、$\psi_a^\dagger(x)$ 或 $\bar\psi_a(x)$ 的一个分量。比如，$\Phi_a(x)\Phi_b(x)\Phi_c(x)$ 可以表示 $\phi(x)\phi(x)\phi(x)$，也可以表示 $A_\mu(x)\bar\psi_a(x)\psi_b(x)$。后者不是 Lorentz 不变的，但利用 Dirac 矩阵可以线性地组合出 Lorentz 不变量 $A_\mu(x)\bar\psi_a(x)(\gamma^\mu)_{ab}\psi_b(x) = A_\mu(x)\bar\psi(x)\gamma^\mu\psi(x)$。将 $\Phi_a(x)$ 分解为正能解部分 $\Phi_a^{(+)}(x)$ 和负能解部分 $\Phi_a^{(-)}(x)$，

$$\Phi_a(x) = \Phi_a^{(+)}(x) + \Phi_a^{(-)}(x), \tag{5.255}$$

则可得

$$\Phi_a(x)\Phi_b(y) = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y). \tag{5.256}$$

由于正能解部分和负能解部分分别只包含湮灭算符和产生算符，我们有

$$\Phi_a^{(+)}(x)\left|0\right\rangle = 0, \quad \left\langle 0\right|\Phi_a^{(-)}(x) = 0, \tag{5.257}$$

从而，可以推出

$$\left\langle 0\right|\Phi_a(x)\Phi_b(y)\left|0\right\rangle = \left\langle 0\right|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\left|0\right\rangle. \tag{5.258}$$

现在，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的正规乘积可以表达为

$$\mathsf{N}[\Phi_a(x)\Phi_b(y)] = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x), \tag{5.259}$$

其中，因子 $\epsilon_{ab} = \pm 1$ 来自费米子算符的反对易性。若 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符，则 $\epsilon_{ab} = -1$；其余情况 $\epsilon_{ab} = +1$。利用 ϵ_{ab}，我们可以交换 (5.259) 式右边第一项和第三项各自的两个场算符，得到

$$\begin{aligned}\mathsf{N}[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)\\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x)],\end{aligned} \tag{5.260}$$

即

$$\mathsf{N}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)]. \tag{5.261}$$

也就是说，两个场算符的位置交换后，正规乘积只相差一个由费米子算符的反对易性导致的符号。另一方面，$\Phi_a(x)\Phi_b(y)$ 的时序乘积可以写作

$$\begin{aligned}\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0-y^0) + \epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0-x^0)\\ &= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0-y^0) + \Phi_b(y)\Phi_a(x)\theta(y^0-x^0)],\end{aligned} \tag{5.262}$$

因此，两个场算符的位置交换后，时序乘积也只相差一个由费米子算符的反对易性导致的符号：

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]. \tag{5.263}$$

当 $x^0 \geq y^0$ 时，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积为

$$\begin{aligned}\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\\ &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y). \end{aligned}\tag{5.264}$$

最后一项可以改写成

$$\begin{aligned}\Phi_a^{(+)}(x)\Phi_b^{(-)}(y) &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_a^{(+)}(x), \Phi_b^{(-)}(y) - \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)\\ &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}. \end{aligned}\tag{5.265}$$

这里 $[\cdot,\cdot]_- = [\cdot,\cdot]$ 代表对易子，$[\cdot,\cdot]_+ = \{\cdot,\cdot\}$ 代表反对易子。$\mp$ 号仅当 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符时取负号，其余情况取正号。于是，由 (5.259) 式可以得到

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}. \tag{5.266}$$

注意，$[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}$ 必定是一个 c 数，因为 $\Phi_a^{(+)}(x)$ 中的湮灭算符与 $\Phi_b^{(-)}(y)$ 中的产生算符的对易子或反对易子并不是算符，而是 c 数。从而，根据 (5.257) 式和 (5.258) 式可得

$$\begin{aligned}[][\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} &= \langle 0|\,[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}\,|0\rangle = \langle 0|\,\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\,|0\rangle = \langle 0|\,\Phi_a(x)\Phi_b(y)\,|0\rangle\\ &= \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle\,. \end{aligned}\tag{5.267}$$

当 $x^0 \leq y^0$ 时，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积变成

$$\begin{aligned}\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b(y)\Phi_a(x)\\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(-)}(x)]\\ &= \epsilon_{ab}\{\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x)\\ &\qquad + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}\}\\ &= \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}\\ &= \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}. \end{aligned}\tag{5.268}$$

最后一步用到 (5.261) 式。根据 (5.263) 式，有

$$\begin{aligned}\epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} &= \epsilon_{ab}\,\langle 0|\,[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}\,|0\rangle = \epsilon_{ab}\,\langle 0|\,\Phi_b^{(+)}(y)\Phi_a^{(-)}(x)\,|0\rangle\\ &= \epsilon_{ab}\,\langle 0|\,\Phi_b(y)\Phi_a(x)\,|0\rangle = \epsilon_{ab}\,\langle 0|\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]\,|0\rangle = \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle\,. \end{aligned}\tag{5.269}$$

综合这两种情况，我们发现 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积可以统一地表达为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle\,. \tag{5.270}$$

引入场算符的**缩并** (contraction) 概念，将两个场算符 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

$$\overline{\Phi_a(x)\Phi_b(y)} \equiv \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}, & x^0 \geq y^0,\\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}, & x^0 < y^0. \end{cases}\tag{5.271}$$

上式仅当 $\Phi_a^{(+)}(x)$ 中的湮灭算符与 $\Phi_b^{(-)}(y)$ 中的产生算符属于同一套产生湮灭算符时非零，因而不同类型的场算符的缩并为零。两个场算符的缩并是一个 c 数，不会受到正规操作 N 的影响。在正规乘积中出现缩并记号时，参与缩并的一对场算符可以不相邻。为了使它们相邻，需要适当地交换场算符，交换时应计入费米子算符的反对易性引起的符号差异，我们约定这样得到的式子与原先的式子相等。例如，

$$\mathsf{N}(\wick{\Phi_a \c1 \Phi_b \c2 \Phi_c \c1 \Phi_d \Phi_e \c2 \Phi_f}) = \epsilon_{cd}\epsilon_{ef}\mathsf{N}(\wick{\Phi_a \c1 \Phi_b \c1 \Phi_d \c2 \Phi_c \c2 \Phi_f \Phi_e}) = \epsilon_{cd}\epsilon_{ef}\wick{\c1 \Phi_b \c1 \Phi_d \c2 \Phi_c \c2 \Phi_f}\mathsf{N}(\Phi_a\Phi_e) \tag{5.272}$$

于是，(5.270) 式可以改记为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y) + \wick{\c1 \Phi_a(x) \c1 \Phi_b(y)}]. \tag{5.273}$$

上式表明，两个场算符的时序乘积等于它们的正规乘积加上它们的缩并。

这个结论可以推广成 **Wick 定理**：*一组场算符的时序乘积可以分解为它们的正规乘积与所有可能缩并的正规乘积之和*，也就是说，

$$\begin{aligned}\mathsf{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)] \ = \ & \mathsf{N}\big[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) \\ & + \big(\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\text{的所有可能缩并}\big)\big].\end{aligned} \tag{5.274}$$

例如，对于四个场算符的情况，有

$$\begin{aligned}\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d) \ = \ & \mathsf{N}\big(\Phi_a\Phi_b\Phi_c\Phi_d + \wick{\c1 \Phi_a \c1 \Phi_b \Phi_c \Phi_d} + \wick{\c1 \Phi_a \Phi_b \c1 \Phi_c \Phi_d} + \wick{\c1 \Phi_a \Phi_b \Phi_c \c1 \Phi_d} \\ & + \wick{\Phi_a \c1 \Phi_b \c1 \Phi_c \Phi_d} + \wick{\Phi_a \c1 \Phi_b \Phi_c \c1 \Phi_d} + \wick{\Phi_a \Phi_b \c1 \Phi_c \c1 \Phi_d} \\ & + \wick{\c1 \Phi_a \c1 \Phi_b \c2 \Phi_c \c2 \Phi_d} + \wick{\c1 \Phi_a \c2 \Phi_b \c1 \Phi_c \c2 \Phi_d} + \wick{\c1 \Phi_a \c2 \Phi_b \c2 \Phi_c \c1 \Phi_d}\big).\end{aligned} \tag{5.275}$$

根据正规乘积的性质 (5.254)，上式的真空期待值为

$$\begin{aligned}\langle 0|\,\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d)\,|0\rangle \ = \ & \wick{\c1 \Phi_a \c1 \Phi_b \c2 \Phi_c \c2 \Phi_d} + \wick{\c1 \Phi_a \c2 \Phi_b \c1 \Phi_c \c2 \Phi_d} + \wick{\c1 \Phi_a \c2 \Phi_b \c2 \Phi_c \c1 \Phi_d} \\ = \ & \wick{\c1 \Phi_a \c1 \Phi_b \c2 \Phi_c \c2 \Phi_d} + \epsilon_{bc}\wick{\c1 \Phi_a \c1 \Phi_c \c2 \Phi_b \c2 \Phi_d} + \epsilon_{cd}\epsilon_{bd}\wick{\c1 \Phi_a \c1 \Phi_d \c2 \Phi_b \c2 \Phi_c} \\ = \ & \langle 0|\,\mathsf{T}(\Phi_a\Phi_b)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_c\Phi_d)\,|0\rangle + \epsilon_{bc}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_c)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_d)\,|0\rangle \\ & +\epsilon_{cd}\epsilon_{bd}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_d)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_c)\,|0\rangle\,.\end{aligned} \tag{5.276}$$

5.4.2　Wick 定理的证明

为了证明 Wick 定理，我们需要先证明如下引理。

引理　如果场算符 $\Phi_b(x_b)$ 的时间坐标比 n 个场算符 $\Phi_{a_1}(x_1),\cdots,\Phi_{a_n}(x_n)$ 的时间坐标都小，即 $x_b^0 \le x_1^0,\cdots,x_n^0$，那么，以下等式成立：

$$\begin{aligned}\mathsf{N}[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)]\Phi_b(x_b) = \ & \mathsf{N}\big[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)\Phi_b(x_b) + \wick{\c1 \Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n) \c1 \Phi_b(x_b)} \\ & +\Phi_{a_1}(x_1)\wick{\c1 \Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) \c1 \Phi_b(x_b)} + \cdots\cdots + \Phi_{a_1}(x_1)\cdots\wick{\c1 \Phi_{a_n}(x_n) \c1 \Phi_b(x_b)}\big].\end{aligned} \tag{5.277}$$

如果 $\Phi_{a_1},\cdots,\Phi_{a_n}$ 中有些算符已经先彼此缩并了，也存在与 (5.277) 形式相同的等式，如

$$
\begin{aligned}
\contraction{\mathsf{N}(\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
\mathsf{N}(\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n})\Phi_b = {}&
\contraction{\mathsf{N}\big(\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
\mathsf{N}\big(\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}\Phi_b \\
&\contraction[2ex]{+}{\Phi}{_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}}{\Phi}
\contraction{+\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
+\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}\Phi_b
\contraction{+\Phi_{a_1}\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
\contraction[2ex]{+\Phi_{a_1}\Phi_{a_1}\Phi_{a_2}}{\Phi}{_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}\Phi_b \\
&\contraction{+\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
\contraction{+\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}}{\Phi}{_{a_5}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}\Phi_b+\cdots\cdots
\contraction{+\Phi_{a_1}}{\Phi}{_{a_2}\Phi_{a_3}}{\Phi}
\contraction{+\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots}{\Phi}{_{a_n}}{\Phi}
+\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Phi_{a_5}\cdots\Phi_{a_n}\Phi_b\big).
\end{aligned}
\tag{5.278}
$$

证明 我们分四步来证明。

(1) 将 Φ_b 分解为正能解部分和负能解部分，$\Phi_b=\Phi_b^{(+)}+\Phi_b^{(-)}$，则可以证明正能解部分 $\Phi_b^{(+)}$ 满足

$$
\begin{aligned}
\mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_n})\Phi_b^{(+)} = {}& \mathsf{N}\big(\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(+)}
\contraction{+}{\Phi}{_{a_1}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(+)}
\contraction{+\Phi_{a_1}}{\Phi}{_{a_2}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\Phi_b^{(+)} \\
&\contraction{+\cdots\cdots+\Phi_{a_1}\cdots}{\Phi}{_{a_n}}{\Phi}
+\cdots\cdots+\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(+)}\big).
\end{aligned}
\tag{5.279}
$$

由于 $x_b^0\le x_1^0,\cdots,x_n^0$，$\Phi_{a_1}(x_i)$ $(i=1,\cdots,n)$ 与 $\Phi_b^{(+)}$ 的缩并为零：

$$
\contraction{}{\Phi}{_{a_i}(x_i)}{\Phi}
\Phi_{a_i}(x_i)\Phi_b^{(+)}(x_b)=\langle 0|\,\mathsf{T}\big[\Phi_{a_i}(x_i)\Phi_b^{(+)}(x_b)\big]\,|0\rangle=\langle 0|\,\Phi_{a_i}(x_i)\Phi_b^{(+)}(x_b)\,|0\rangle=0.
\tag{5.280}
$$

因此，(5.279) 式右边除第一项外的其它项均为零。另一方面，(5.279) 式左边和右边第一项已经按正规次序排列了，故 (5.279) 式成立。现在，只需要证明负能解部分 $\Phi_b^{(-)}$ 满足

$$
\begin{aligned}
\mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_n})\Phi_b^{(-)} = {}& \mathsf{N}\big(\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(-)}
\contraction{+}{\Phi}{_{a_1}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(-)}
\contraction{+\Phi_{a_1}}{\Phi}{_{a_2}\cdots\Phi_{a_n}}{\Phi}
+\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\Phi_b^{(-)} \\
&\contraction{+\cdots\cdots+\Phi_{a_1}\cdots}{\Phi}{_{a_n}}{\Phi}
+\cdots\cdots+\Phi_{a_1}\cdots\Phi_{a_n}\Phi_b^{(-)}\big).
\end{aligned}
\tag{5.281}
$$

将 $\Phi_{a_1},\cdots,\Phi_{a_n}$ 都分解为正能解部分和负能解部分，则 $\mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_n})$ 将包含 2^n 项，每一项是 j 个负能解部分 $(j=0,\cdots,n)$ 与 $n-j$ 个正能解部分之积

$$
\Phi_{a_1}^{(-)}\cdots\Phi_{a_j}^{(-)}\Phi_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)},
\tag{5.282}
$$

负能解部分都处于正能解部分的左边。

(2) 可以证明，通项 (5.282) 中右边正能解部分之积 $\Phi_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}$ 满足

$$
\begin{aligned}
\mathsf{N}\big(\Phi_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)} = {}& \mathsf{N}\big(\Phi_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}
\contraction{+}{\Phi}{_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}}{\Phi}
+\Phi_{a_{j+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}
\contraction{+\Phi_{a_{j+1}}^{(+)}}{\Phi}{_{a_{j+2}}^{(+)}\cdots\Phi_{a_n}^{(+)}}{\Phi}
+\Phi_{a_{j+1}}^{(+)}\Phi_{a_{j+2}}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)} \\
&\contraction{+\cdots\cdots+\Phi_{a_1}^{(+)}\cdots}{\Phi}{_{a_n}^{(+)}}{\Phi}
+\cdots\cdots+\Phi_{a_1}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}\big).
\end{aligned}
\tag{5.283}
$$

下面用数学归纳法证明 (5.283) 式。

对于 $\mathsf{N}\big(\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)}$，存在与 (5.283) 形式相同的等式，这是因为由 (5.273) 式可以得到

$$
\contraction{\mathsf{N}\big(\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)}=\Phi_{a_n}^{(+)}\Phi_b^{(-)}=\mathsf{T}\big(\Phi_{a_n}^{(+)}\Phi_b^{(-)}\big)=\mathsf{N}\big(\Phi_{a_n}^{(+)}\Phi_b^{(-)}+}{\Phi}{_{a_n}^{(+)}}{\Phi}
\mathsf{N}\big(\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)}=\Phi_{a_n}^{(+)}\Phi_b^{(-)}=\mathsf{T}\big(\Phi_{a_n}^{(+)}\Phi_b^{(-)}\big)=\mathsf{N}\big(\Phi_{a_n}^{(+)}\Phi_b^{(-)}+\Phi_{a_n}^{(+)}\Phi_b^{(-)}\big).
\tag{5.284}
$$

这样的话，需要证明的是可以从上式递推地导出 (5.283) 式。

假设 $\mathsf{N}\big(\Phi_{a_k}^{(+)}\cdots\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)}$ $(j+2\le k\le n)$ 满足与 (5.283) 形式相同的等式

$$
\mathsf{N}\big(\Phi_{a_k}^{(+)}\cdots\Phi_{a_n}^{(+)}\big)\Phi_b^{(-)} = \mathsf{N}\big(\Phi_{a_k}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}
\contraction{+}{\Phi}{_{a_k}^{(+)}\cdots\Phi_{a_n}^{(+)}}{\Phi}
+\Phi_{a_k}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}
\contraction{+\Phi_{a_k}^{(+)}}{\Phi}{_{a_{k+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}}{\Phi}
+\Phi_{a_k}^{(+)}\Phi_{a_{k+1}}^{(+)}\cdots\Phi_{a_n}^{(+)}\Phi_b^{(-)}
$$

$$+\cdots\cdots+\Phi^{(+)}_{a_k}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\Big),\tag{5.285}$$

那么，可以得到

$$\begin{aligned}&\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b=\Phi^{(+)}_{a_{k-1}}\mathsf{N}\big(\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b\\
=\;&\Phi^{(+)}_{a_{k-1}}\mathsf{N}\big(\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b\big)+\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\overbracket{\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}+\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\overbracket{\Phi^{(+)}_{a_{k+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\
&\qquad+\cdots\cdots+\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big).\end{aligned}\tag{5.286}$$

接着，我们进一步整理上式第二步的第一项，

$$\begin{aligned}&\Phi^{(+)}_{a_{k-1}}\mathsf{N}\big(\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b\big)\\
=\;&\Phi^{(+)}_{a_{k-1}}\epsilon_1\,\mathsf{N}\big(\Phi^{(-)}_b\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)=\epsilon_1\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\\
=\;&\epsilon_1\mathsf{T}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b\big)\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}=\epsilon_1\,\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b+\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b}\big)\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\\
=\;&\epsilon_1\,\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b\big)\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}+\epsilon_1\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\\
=\;&\epsilon_1\epsilon_{a_{k-1}b}\,\mathsf{N}\big(\Phi^{(-)}_b\Phi^{(+)}_{a_{k-1}}\big)\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}+\epsilon_1\,\mathsf{N}\big(\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(-)}_b}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)\\
=\;&\epsilon_1\epsilon_{a_{k-1}b}\,\mathsf{N}\big(\Phi^{(-)}_b\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)+\mathsf{N}\big(\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big)\\
=\;&\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b\big)+\mathsf{N}\big(\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big).\end{aligned}\tag{5.287}$$

第一步重复利用 (5.261) 式，将 $\Phi^{(-)}_b$ 从正规乘积中的最左边移动到最右边，因而出现因子

$$\epsilon_1=\epsilon_{a_nb}\epsilon_{a_{n-1}b}\cdots\epsilon_{a_{k+1}b}\epsilon_{a_kb}.\tag{5.288}$$

第三步利用到 $x^0_b\leq x^0_{k-1}$ 的条件。第四步使用了 (5.273) 式。第六至八步再多次利用 (5.261) 式。将 (5.287) 式代入 (5.286) 式，立即得到

$$\begin{aligned}\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b&=\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b+\overbracket{\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\
&\quad+\Phi^{(+)}_{a_{k-1}}\overbracket{\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}+\cdots\cdots+\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big).\end{aligned}\tag{5.289}$$

因此，$\mathsf{N}\big(\Phi^{(+)}_{a_{k-1}}\Phi^{(+)}_{a_k}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b$ 也满足与 (5.283) 形式相同的等式。结合 (5.284) 式，可知 (5.283) 式成立。

(3) 根据 (5.283) 式，通项 (5.282) 满足

$$\begin{aligned}&\mathsf{N}\big(\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b=\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\mathsf{N}\big(\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b\\
=\;&\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\mathsf{N}\big(\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b+\overbracket{\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\
&\qquad+\Phi^{(+)}_{a_{j+1}}\overbracket{\Phi^{(+)}_{a_{j+2}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}+\cdots\cdots+\Phi^{(+)}_{a_1}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big)\\
=\;&\mathsf{N}\big(\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b+\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\overbracket{\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\
&\quad+\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\overbracket{\Phi^{(+)}_{a_{j+2}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}+\cdots\cdots+\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_1}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big).\end{aligned}\tag{5.290}$$

由

$$\overbracket{\Phi^{(-)}_{a_i}(x_i)\Phi^{(-)}_b}(x_b) = \langle 0|\,\mathsf{T}\big[\Phi^{(-)}_{a_i}(x_i)\Phi^{(-)}_b(x_b)\big]\,|0\rangle = 0, \tag{5.291}$$

可得

$$\begin{aligned}&\mathsf{N}\big(\overbracket{\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b} + \Phi^{(-)}_{a_1}\overbracket{\Phi^{(-)}_{a_2}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\ &\quad + \cdots\cdots + \Phi^{(-)}_{a_1}\cdots\overbracket{\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big) = 0.\end{aligned} \tag{5.292}$$

因此，将上式左边添加到 (5.290) 式右边，等式仍然成立：

$$\begin{aligned}&\mathsf{N}\big(\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\big)\Phi^{(-)}_b\\ &= \mathsf{N}\big(\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b + \overbracket{\Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\ &\quad + \Phi^{(-)}_{a_1}\overbracket{\Phi^{(-)}_{a_2}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b} + \cdots\cdots + \Phi^{(-)}_{a_1}\cdots\overbracket{\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\ &\quad + \Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\overbracket{\Phi^{(+)}_{a_{j+1}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b} + \Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_{j+1}}\overbracket{\Phi^{(+)}_{a_{j+2}}\cdots\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\\ &\quad + \cdots\cdots + \Phi^{(-)}_{a_1}\cdots\Phi^{(-)}_{a_j}\Phi^{(+)}_{a_1}\cdots\overbracket{\Phi^{(+)}_{a_n}\Phi^{(-)}_b}\big).\end{aligned} \tag{5.293}$$

也就是说，$\mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_n})$ 分解后每一项都满足与 (5.281) 形式相同的等式，故 (5.281) 式成立。结合第 (1) 步结论，(5.277) 式成立。

(4) 如果 $\Phi_{a_1},\cdots,\Phi_{a_n}$ 中有些算符已经先彼此缩并了，可以按照第 (1)、(2)、(3) 步的方法进行类似的证明。因此，像 (5.278) 这样的等式也成立。引理证毕。

现在，我们可以利用这个引理来证明 Wick 定理。

证明 用数学归纳法证明。

当 $n=2$ 时，(5.274) 式变成

$$\mathsf{T}[\Phi_{a_1}(x)\Phi_{a_2}(y)] = \mathsf{N}[\Phi_{a_1}(x)\Phi_{a_2}(y) + \overbracket{\Phi_{a_1}(x)\Phi_{a_2}}(y)]. \tag{5.294}$$

这是成立的，因为它的形式与 (5.273) 式相同。

假设当 $n=k$ 时，(5.274) 式成立，即

$$\mathsf{T}[\Phi_{a_1}(x_1)\cdots\Phi_{a_k}(x_k)] = \mathsf{N}\left[\Phi_{a_1}(x_1)\cdots\Phi_{a_k}(x_k) + \big(\Phi_{a_1}\cdots\Phi_{a_k}\text{的所有可能缩并}\big)\right]. \tag{5.295}$$

如果 $x^0_{k+1}\leq x^0_1,\cdots,x^0_k$，我们就可以得到

$$\begin{aligned}&\mathsf{T}[\Phi_{a_1}(x_1)\cdots\Phi_{a_k}(x_k)\Phi_{a_{k+1}}(x_{k+1})] = \mathsf{T}[\Phi_{a_1}(x_1)\cdots\Phi_{a_k}(x_k)]\Phi_{a_{k+1}}(x_{k+1})\\ &= \mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_k})\Phi_{a_{k+1}} + \mathsf{N}\big(\Phi_{a_1}\cdots\Phi_{a_k}\text{的所有可能缩并}\big)\Phi_{a_{k+1}}.\end{aligned} \tag{5.296}$$

根据上述引理中的 (5.277) 式，(5.296) 式第二行第一项为

$$\begin{aligned}\mathsf{N}(\Phi_{a_1}\cdots\Phi_{a_k})\Phi_{a_{k+1}} &= \mathsf{N}\big(\Phi_{a_1}\cdots\Phi_{a_k}\Phi_{a_{k+1}} + \overbracket{\Phi_{a_1}\cdots\Phi_{a_k}\Phi_{a_{k+1}}} + \Phi_{a_1}\overbracket{\Phi_{a_2}\cdots\Phi_{a_k}\Phi_{a_{k+1}}}\\ &\quad + \cdots + \Phi_{a_1}\cdots\overbracket{\Phi_{a_k}\Phi_{a_{k+1}}}\big),\end{aligned} \tag{5.297}$$

上式右边的缩并项穷尽了只有一次缩并时与 $\Phi_{a_{k+1}}$ 有关的缩并。另一方面，上述引理中有些算符已经先彼此缩并的情况可以应用到 (5.296) 式第二行的其它项上，得到的项都包含缩并，在这些项里面，只包含一次缩并的项中的缩并必定与 $\Phi_{a_{k+1}}$ 无关，余下的项则穷尽了 $\Phi_{a_1}\cdots\Phi_{a_k+1}$ 的包含一次以上缩并的所有情况。因此，(5.296) 式已经包含了 $\Phi_{a_1}\cdots\Phi_{a_k+1}$ 的所有可能缩并，故

$$\mathsf{T}[\Phi_{a_1}(x_1)\cdots\Phi_{a_{k+1}}(x_{k+1})] = \mathsf{N}\left[\Phi_{a_1}(x_1)\cdots\Phi_{a_{k+1}}(x_{k+1}) + \left(\Phi_{a_1}\cdots\Phi_{a_{k+1}}\text{的所有可能缩并}\right)\right]. \tag{5.298}$$

因此，对于 $x^0_{k+1}\leq x^0_1,\cdots,x^0_k$ 的情形，当 $n=k+1$ 时 (5.274) 式也成立。结合 (5.294) 式，我们就证明了 (5.274) 式对 $x^0_1\geq x^0_2\geq\cdots\geq x^0_n$ 成立。

当 $x^0_1\geq x^0_2\geq\cdots\geq x^0_n$ 这个条件不成立时，我们可以交换 $\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)$ 中各个算符的位置，得到符合时序的乘积

$$\Phi'_{a_1}(x'_1)\Phi'_{a_2}(x'_2)\cdots\Phi'_{a_n}(x'_n),$$

其中时间坐标已经按降序排列，$x'^0_1\geq x'^0_2\geq\cdots\geq x'^0_n$。从而，等式

$$\mathsf{T}[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n)] = \mathsf{N}\left[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n) + \left(\Phi'_{a_1}\cdots\Phi'_{a_n}\text{的所有可能缩并}\right)\right] \tag{5.299}$$

成立。(5.263) 式和 (5.261) 式表明，时序乘积与正规乘积关于算符交换的性质是相同的。因此，如果我们分别在时序乘积和正规乘积中通过交换算符将 $\Phi'_{a_1}(x'_1)\Phi'_{a_2}(x'_2)\cdots\Phi'_{a_n}(x'_n)$ 调回到原来的形式 $\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)$，将出现一个共同的因子 $\epsilon_2=\pm1$，它由费米子算符的反对易性所致。也就是说，我们得到了

$$\mathsf{T}[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n)] = \epsilon_2\mathsf{T}[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)], \tag{5.300}$$

和

$$\begin{aligned}&\mathsf{N}\left[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n) + \left(\Phi'_{a_1}\cdots\Phi'_{a_n}\text{的所有可能缩并}\right)\right]\\&\quad=\epsilon_2\,\mathsf{N}\left[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n) + \left(\Phi_{a_1}\cdots\Phi_{a_n}\text{的所有可能缩并}\right)\right].\end{aligned} \tag{5.301}$$

将以上两式分别代入到 (5.299) 式的左右两边，消去 ϵ_2，我们就证明了 (5.274) 式对 $x^0_1,x^0_2,\cdots,x^0_n$ 的任意次序成立。证毕。

5.5　Feynman 传播子

在应用 Wick 定理时，两个场算符的缩并是一种基本要素。在上一节中我们已经指出，仅当参与缩并的场算符中含有同一套产生湮灭算符时，缩并的结果才不为零。这些非零缩并就是 **Feynman 传播子**，在本节中，我们将导出它们的显式结果。

5.5.1 实标量场的 Feynman 传播子

实标量场 $\phi(x)$ 的 Feynman 传播子 $D_\mathrm{F}(x-y)$ 定义为

$$D_\mathrm{F}(x-y) \equiv \overbrace{\phi(x)\phi(y)} = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle\,. \tag{5.302}$$

根据展开式 (5.242) 和 (5.243)，当 $x^0 > y^0$ 时，有

$$\begin{aligned}
&\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \langle 0|\,\phi(x)\phi(y)\,|0\rangle = \langle 0|\,\phi^{(+)}(x)\phi^{(-)}(y)\,|0\rangle \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,\langle 0|\,a_\mathbf{p}e^{-ip\cdot x}a_\mathbf{q}^\dagger e^{iq\cdot y}\,|0\rangle = \int \frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,\langle 0|\,([a_\mathbf{p},a_\mathbf{q}^\dagger]+a_\mathbf{q}^\dagger a_\mathbf{p})\,|0\rangle \\
&= \int \frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{-ip\cdot(x-y)}}{2E_\mathbf{p}} \\
&= \int \frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}.
\end{aligned} \tag{5.303}$$

第四步用到产生湮灭算符的对易关系 (2.92)。借助复变函数的知识，可以将上式最后一行中的因子 $e^{-iE_\mathbf{p}(x^0-y^0)}/(2E_\mathbf{p})$ 化为一维积分的结果。

将 p^0 视作复变量，在 p^0 的复平面上考虑函数

$$\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})} \tag{5.304}$$

的曲线积分。这个函数具有两个一阶极点，$p^0 = \pm E_\mathbf{p}$，均位于实轴上。图 5.4(a) 中画出了 p^0 复平面上的几条积分路径。路径 Γ_F 在两个极点处分别通过一个半径无穷小的半圆绕过极点，当

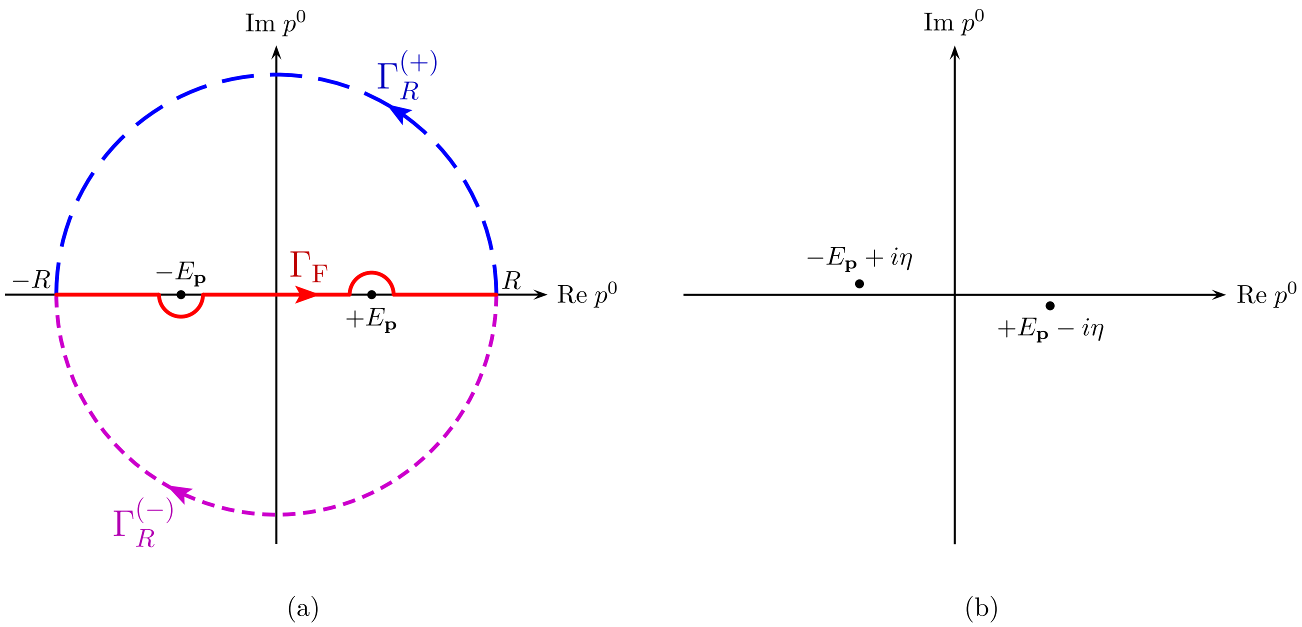


图 5.4: Feynman 传播子的极点和积分路径。

$R\to\infty$ 时，Γ_F 将从 $p^0=-\infty$ 一直延伸到 $p^0=+\infty$。将 Γ_{F} 与下半平面上的半圆弧 $\Gamma_{\text{R}}^{(-)}$ 组成一条围线 $C_{\text{F}}^{(-)}=\Gamma_{\text{F}}+\Gamma_{\text{R}}^{(-)}$，方向为顺时针方向，即反方向。由于 $x^0-y^0>0$，根据复变函数的 Jordan 引理，可得

$$\lim_{R\to\infty}\int_{\Gamma_R^{(-)}}dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}=0. \tag{5.305}$$

从而，当 $R\to\infty$ 时，由留数定理可以计算相应的积分主值，

$$\begin{aligned}\int_{\Gamma_{\text{F}}}dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}&=\int_{C_{\text{F}}^{(-)}}dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}\\&=-2\pi i\,\text{Res}\left[\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})},+E_{\mathbf{p}}\right]=-2\pi i\,\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}.\end{aligned} \tag{5.306}$$

利用

$$(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})=(p^0)^2-E_{\mathbf{p}}^2=(p^0)^2-|\mathbf{p}|^2-m^2=p^2-m^2, \tag{5.307}$$

我们进一步得到

$$\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=-\frac{1}{2\pi i}\int_{\Gamma_{\text{F}}}dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}=\int_{\Gamma_{\text{F}}}\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{p^2-m^2}. \tag{5.308}$$

如图 5.4(b) 所示，如果我们将左边极点向正虚轴方向移动一个无穷小量 $\eta>0$，右边极点向负虚轴方向同样移动无穷小量 η，则沿正实轴积分将等价于原来沿 Γ_{F} 积分。此时，极点位置为 $p^0=\pm(E_{\mathbf{p}}-i\eta)$，积分项中的分母应改为

$$[p^0-(E_{\mathbf{p}}-i\eta)][p^0+(E_{\mathbf{p}}-i\eta)]=(p^0)^2-(E_{\mathbf{p}}-i\eta)^2=(p^0)^2-E_{\mathbf{p}}^2+2i\eta E_{\mathbf{p}}+\eta^2\simeq p^2-m^2+i\epsilon. \tag{5.309}$$

最后一步忽略了二阶小量，而 $\epsilon=2\eta E_{\mathbf{p}}>0$ 也是一个无穷小量。于是，我们可以得到

$$\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{[p^0-(E_{\mathbf{p}}-i\eta)][p^0+(E_{\mathbf{p}}-i\eta)]}=\int\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}. \tag{5.310}$$

将上式代入到 (5.303) 式，立即推出

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle=\int\frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi}\frac{ie^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}=\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \tag{5.311}$$

当 $x^0<y^0$ 时，时序操作将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

$$\begin{aligned}\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle&=\langle 0|\,\phi(y)\phi(x)\,|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(y-x)}=\int\frac{d^3p}{(2\pi)^3}\frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}\\&=\int\frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}.\end{aligned} \tag{5.312}$$

最后一步将积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$。将 Γ_{F} 与上半平面上的半圆弧 $\Gamma_{\text{R}}^{(+)}$ 组成一条围线 $C_{\text{F}}^{(+)}=\Gamma_{\text{F}}+\Gamma_{\text{R}}^{(+)}$，方向为逆时针方向，即正方向。由于 $x^0-y^0<0$，根据 Jordan 引理，可得

$$\lim_{R\to\infty}\int_{\Gamma_R^{(+)}}dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}=0. \tag{5.313}$$

从而，当 $R\to\infty$ 时，可以推出

$$\begin{aligned}\int_{\Gamma_\mathrm{F}} dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})} &= \int_{C_\mathrm{F}^{(+)}} dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}\\ &= 2\pi i\,\mathrm{Res}\left[\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})},-E_\mathbf{p}\right] = -2\pi i\,\frac{e^{iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}.\end{aligned} \tag{5.314}$$

故

$$\frac{e^{iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}} = -\frac{1}{2\pi i}\int_{\Gamma_\mathrm{F}} dp^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})} = \int\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}, \tag{5.315}$$

代入到 (5.312) 式，即得

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int\frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi}\frac{ie^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon} = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \tag{5.316}$$

(5.311) 式和 (5.316) 式是一样的。因此，无论 x^0 和 y^0 孰大孰小，Feynman 传播子都可以表达为

$$D_\mathrm{F}(x-y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}. \tag{5.317}$$

它是 Lorentz 不变的，而且是一个偶函数：

$$D_\mathrm{F}(y-x) = D_\mathrm{F}(x-y). \tag{5.318}$$

5.5.2 复标量场的 Feynman 传播子

在相互作用绘景中，复标量场 $\phi(x)$ 的平面波展开式仍然具有 (2.144) 的形式。将 $\phi(x)$ 和 $\phi^\dagger(x)$ 分解为正能解和负能解两部分，得

$$\phi(x) = \phi^{(+)}(x)+\phi^{(-)}(x),\quad \phi^\dagger(x) = \phi^{\dagger(+)}(x)+\phi^{\dagger(-)}(x), \tag{5.319}$$

其中，

$$\phi^{(+)}(x) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\,a_\mathbf{p}e^{-ip\cdot x},\quad \phi^{(-)}(x) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\,b_\mathbf{p}^\dagger e^{ip\cdot x}, \tag{5.320}$$

$$\phi^{\dagger(+)}(x) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\,b_\mathbf{p}e^{-ip\cdot x},\quad \phi^{\dagger(-)}(x) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\,a_\mathbf{p}^\dagger.e^{ip\cdot x}. \tag{5.321}$$

容易看出，

$$\overbracket{\phi(x)\phi}(y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = 0,\quad \overbracket{\phi^\dagger(x)\phi}^\dagger(y) = \langle 0|\,\mathsf{T}[\phi^\dagger(x)\phi^\dagger(y)]\,|0\rangle = 0. \tag{5.322}$$

复标量场的 Feynman 传播子定义为

$$D_\mathrm{F}(x-y) \equiv \overbracket{\phi(x)\phi}^\dagger(y) = \langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle\,. \tag{5.323}$$

类似于上一小节的计算，利用产生湮灭算符的对易关系 (2.164)，可以得到

$$\begin{aligned}&\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle = \langle 0|\,\phi^{(+)}(x)\phi^{\dagger(-)}(y)\,|0\rangle \\ &= \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,\langle 0|\,a_\mathbf{p}e^{-ip\cdot x}a_\mathbf{q}^\dagger e^{iq\cdot y}\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2E_\mathbf{p}},\end{aligned} \tag{5.324}$$

以及

$$\begin{aligned}&\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle = \langle 0|\,\phi^{\dagger(+)}(y)\phi^{(-)}(x)\,|0\rangle \\ &= \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,\langle 0|\,b_\mathbf{p}e^{-ip\cdot y}b_\mathbf{q}^\dagger e^{iq\cdot x}\,|0\rangle = \int \frac{d^3p\,d^3q\,e^{-i(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,\langle 0|\,([b_\mathbf{p},b_\mathbf{q}^\dagger]+b_\mathbf{q}^\dagger b_\mathbf{p})\,|0\rangle \\ &= \int \frac{d^3p\,d^3q\,e^{-i(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{d^3p}{(2\pi)^3}\frac{e^{ip\cdot(x-y)}}{2E_\mathbf{p}}.\end{aligned} \tag{5.325}$$

归纳上一小节的计算过程，可得

$$\theta(x^0-y^0)\frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}} + \theta(y^0-x^0)\frac{e^{iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}} = \int\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}, \tag{5.326}$$

其中 $\epsilon>0$ 是一个无穷小量。从而，有

$$\begin{aligned}&\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\ &= \int\frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\theta(x^0-y^0)\frac{e^{-iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}+\theta(y^0-x^0)\frac{e^{iE_\mathbf{p}(x^0-y^0)}}{2E_\mathbf{p}}\right] \\ &= \int\frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int\frac{dp^0}{2\pi}\frac{ie^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon} = \int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon},\end{aligned} \tag{5.327}$$

于是，复标量场的 Feynman 传播子能够表达为

$$\begin{aligned}D_\mathrm{F}(x-y) &= \langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}.\end{aligned} \tag{5.328}$$

可以看出，复标量场与实标量场具有相同形式的 Feynman 传播子。此外，由 (5.263) 式有

$$\overbracket{\phi^\dagger(x)\phi}(y) = \langle 0|\,\mathsf{T}[\phi^\dagger(x)\phi(y)]\,|0\rangle = \langle 0|\,\mathsf{T}[\phi(y)\phi^\dagger(x)]\,|0\rangle = D_\mathrm{F}(y-x) = D_\mathrm{F}(x-y). \tag{5.329}$$

也就是说，$\overbracket{\phi^\dagger(x)\phi}(y)$ 与 $\overbracket{\phi(x)\phi^\dagger}(y)$ 相等。

5.5.3　有质量矢量场的 Feynman 传播子

有质量矢量场 $A^\mu(x)$ 的 Feynman 传播子 $\Delta_\mathrm{F}(x-y)$ 定义为

$$\Delta_\mathrm{F}^{\mu\nu}(x-y) \equiv \contraction{}{A}{^\mu(x)}{A}A^\mu(x)A^\nu(y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle\,. \tag{5.330}$$

根据展开式 (5.245) 和 (5.246)，及产生湮灭算符的对易关系 (3.175)，可得

$$\begin{aligned}
&\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle = \langle 0|\,A^{\mu(+)}(x)A^{\nu(-)}(y)\,|0\rangle \\
={}& \int\frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\sum_{\lambda\lambda'}\langle 0|\,\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}\varepsilon^{\nu*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot y}\,|0\rangle \\
={}& \int\frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\sum_{\lambda\lambda'}\varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')\,\langle 0|\,([a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}]+a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{p},\lambda})\,|0\rangle \\
={}& \int\frac{d^3p\,d^3q\,e^{-i(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\sum_{\lambda\lambda'}\varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
={}& \int\frac{d^3p\,e^{-ip\cdot(x-y)}}{(2\pi)^3 2E_\mathbf{p}}\sum_{\lambda}\varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{p},\lambda) = \int\frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{e^{-ip\cdot(x-y)}}{2E_\mathbf{p}},
\end{aligned} \tag{5.331}$$

以及

$$\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle = \int\frac{d^3p}{(2\pi)^3}\left(-g^{\nu\mu}+\frac{p^\nu p^\mu}{m^2}\right)\frac{e^{-ip\cdot(y-x)}}{2E_\mathbf{p}} = \int\frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{e^{ip\cdot(x-y)}}{2E_\mathbf{p}}. \tag{5.332}$$

从而，有

$$\begin{aligned}
\Delta_\mathrm{F}^{\mu\nu}(x-y) &= \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle \\
&= \int\frac{d^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{1}{2E_\mathbf{p}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}].
\end{aligned} \tag{5.333}$$

最后一行圆括号中的项 $p^\mu p^\nu/m^2$ 与 p^0 有关，因此直接应用 (5.326) 式不能得到适当的结果。

为了得到简洁的表达式，我们需要将 $p^\mu p^\nu/m^2$ 转换为时空导数。记 $\partial_x^\mu\equiv\partial/\partial x_\mu$，利用阶跃函数与 δ 函数的关系

$$\theta'(x) = \delta(x), \tag{5.334}$$

可以推出

$$\begin{aligned}
&\partial_x^\mu\partial_x^\nu[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
={}& \partial_x^\mu[-ip^\nu\theta(x^0-y^0)e^{-ip\cdot(x-y)}+g^{\nu0}\delta(x^0-y^0)e^{-ip\cdot(x-y)}+ip^\nu\theta(y^0-x^0)e^{ip\cdot(x-y)} \\
&\quad -g^{\nu0}\delta(y^0-x^0)e^{ip\cdot(x-y)}] \\
={}& -p^\mu p^\nu\theta(x^0-y^0)e^{-ip\cdot(x-y)}-ig^{\mu0}p^\nu\delta(x^0-y^0)e^{-ip\cdot(x-y)}-ip^\mu g^{\nu0}\delta(x^0-y^0)e^{-ip\cdot(x-y)} \\
&+g^{\mu0}g^{\nu0}\partial_x^0\delta(x^0-y^0)e^{-ip\cdot(x-y)}-p^\mu p^\nu\theta(y^0-x^0)e^{ip\cdot(x-y)}-ig^{\mu0}p^\nu\delta(y^0-x^0)e^{ip\cdot(x-y)}
\end{aligned}$$

$$
\begin{aligned}
&- ip^\mu g^{\nu 0}\delta(y^0-x^0)e^{ip\cdot(x-y)} + g^{\mu 0}g^{\nu 0}\partial_x^0\delta(y^0-x^0)e^{ip\cdot(x-y)} \\
=& - p^\mu p^\nu[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&- i(g^{\mu 0}p^\nu + g^{\nu 0}p^\mu)\delta(x^0-y^0)[e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}] \\
&+ g^{\mu 0}g^{\nu 0}\partial_x^0\delta(x^0-y^0)[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}],
\end{aligned}
\tag{5.335}
$$

故

$$
\begin{aligned}
&\frac{p^\mu p^\nu}{m^2}[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
=& - \frac{\partial_x^\mu \partial_x^\nu}{m^2}[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&- \frac{i}{m^2}(g^{\mu 0}p^\nu + g^{\nu 0}p^\mu)\delta(x^0-y^0)[e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}] \\
&+ \frac{g^{\mu 0}g^{\nu 0}}{m^2}\partial_x^0\delta(x^0-y^0)[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}].
\end{aligned}
\tag{5.336}
$$

因此，$\Delta_{\mathrm{F}}^{\mu\nu}(x-y)$ 可以分解成三个部分，

$$
\Delta_{\mathrm{F}}^{\mu\nu}(x-y) = f_1^{\mu\nu}(x,y) + f_2^{\mu\nu}(x,y) + f_3^{\mu\nu}(x,y),
\tag{5.337}
$$

分别为

$$
f_1^{\mu\nu}(x,y) \equiv -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}],
\tag{5.338}
$$

$$
f_2^{\mu\nu}(x,y) \equiv -\frac{i}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(g^{\mu 0}p^\nu + g^{\nu 0}p^\mu)\delta(x^0-y^0)[e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}],
\tag{5.339}
$$

$$
f_3^{\mu\nu}(x,y) \equiv \frac{g^{\mu 0}g^{\nu 0}}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\partial_x^0\delta(x^0-y^0)[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}].
\tag{5.340}
$$

根据 (5.327) 式，$f_1^{\mu\nu}(x,y)$ 化为

$$
f_1^{\mu\nu}(x,y) = -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right)\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\frac{-i(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}.
\tag{5.341}
$$

$\delta(x^0-y^0)$ 只在 $x^0-y^0=0$ 处非零，此时有 $e^{-iE_{\mathbf{p}}(x^0-y^0)} = e^{iE_{\mathbf{p}}(x^0-y^0)} = 1$，故

$$
f_2^{i0}(x,y) = f_2^{0i}(x,y) = -\frac{i}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{p^i}{2E_{\mathbf{p}}}\delta(x^0-y^0)[e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] = 0.
\tag{5.342}
$$

上式中积分项是关于 $\mathbf{p}$ 的奇函数，因而对整个三维动量空间积分为零。此外，利用 Fourier 变换公式

$$
\int \frac{d^3p}{(2\pi)^3}\,e^{i\mathbf{p}\cdot\mathbf{x}} = \int \frac{d^3p}{(2\pi)^3}\,e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^{(3)}(\mathbf{x}),
\tag{5.343}
$$

可以导出

$$
f_2^{00}(x,y) = -\frac{i}{m^2}\int \frac{d^3p}{(2\pi)^3}\frac{2p^0}{2E_{\mathbf{p}}}\,\delta(x^0-y^0)[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}]
$$

$$= -\frac{2i}{m^2}\delta(x^0-y^0)\delta^{(3)}(\mathbf{x}-\mathbf{y}) = -\frac{2i}{m^2}\delta^{(4)}(x-y). \tag{5.344}$$

归纳起来，得到

$$f_2^{\mu\nu}(x,y) = -\frac{2i}{m^2}g^{\mu 0}g^{\nu 0}\delta^{(4)}(x-y). \tag{5.345}$$

另一方面，根据 δ 函数的导数的定义，有

$$\int dx\, f(x)\delta'(x-a) = -f'(a) = -\int dx\, f'(x)\delta(x-a), \tag{5.346}$$

因而对 (5.340) 式中的积分项可作替换

$$\partial_x^0\delta(x^0-y^0)[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}] \to -\delta(x^0-y^0)\partial_x^0[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}], \tag{5.347}$$

则

$$\begin{aligned} f_3^{\mu\nu}(x,y) &= -\frac{g^{\mu 0}g^{\nu 0}}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\delta(x^0-y^0)\partial_x^0[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}] \\ &= -\frac{g^{\mu 0}g^{\nu 0}}{m^2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\delta(x^0-y^0)[-ip^0e^{-ip\cdot(x-y)} - ip^0e^{ip\cdot(x-y)}] \\ &= \frac{i}{m^2}g^{\mu 0}g^{\nu 0}\int\frac{d^3p}{(2\pi)^3}\delta(x^0-y^0)[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] = \frac{i}{m^2}g^{\mu 0}g^{\nu 0}\delta^{(4)}(x-y). \end{aligned} \tag{5.348}$$

综合起来，有质量矢量场 Feynman 传播子的表达式为

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) = \int\frac{d^4p}{(2\pi)^4}\frac{-i(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)} - \frac{i}{m^2}g^{\mu 0}g^{\nu 0}\delta^{(4)}(x-y). \tag{5.349}$$

第一项是 Lorentz 协变的，但第二项是非协变的。幸好，这个非协变项在微扰论中的贡献被相互作用哈密顿量密度中的非协变项 (5.90) 精确抵消，从而理论是 Lorentz 协变的。因此，在实际计算中可以只保留协变项：

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) \to \int\frac{d^4p}{(2\pi)^4}\frac{-i(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}. \tag{5.350}$$

5.5.4 无质量矢量场的 Feynman 传播子

无质量矢量场的 Feynman 传播子依赖于规范的选择，这里我们取 Feynman 规范 ($\xi = 1$)。在相互作用绘景中，无质量矢量场 $A^\mu(x)$ 的平面波展开式仍然具有 (3.250) 的形式，把它分解为正能解和负能解两部分，得

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x), \tag{5.351}$$

其中，

$$A^{\mu(+)}(x) \equiv \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3}e^\mu(\mathbf{p},\sigma)a_{\mathbf{p};\sigma}e^{-ip\cdot x}, \tag{5.352}$$

$$A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^{3} e^{\mu}(\mathbf{p},\sigma) a^{\dagger}_{\mathbf{p};\sigma} e^{ip\cdot x}. \tag{5.353}$$

相应的 Feynman 传播子定义为

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) \equiv \overbracket{A^{\mu}(x)A^{\nu}}(y) = \langle 0|\,\mathsf{T}[A^{\mu}(x)A^{\nu}(y)]\,|0\rangle\,. \tag{5.354}$$

根据产生湮灭算符的对易关系 (3.261) 和极化矢量的完备性关系 (3.103)，可以得到

$$\begin{aligned}
&\langle 0|\, A^{\mu}(x)A^{\nu}(y)\,|0\rangle = \langle 0|\, A^{\mu(+)}(x)A^{\nu(-)}(y)\,|0\rangle \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} \langle 0|\, e^{\mu}(\mathbf{p},\sigma) a_{\mathbf{p};\sigma} e^{-ip\cdot x} e^{\nu}(\mathbf{q},\sigma') a^{\dagger}_{\mathbf{q};\sigma'} e^{iq\cdot y}\,|0\rangle \\
&= \int \frac{d^3p\,d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^{\mu}(\mathbf{p},\sigma) e^{\nu}(\mathbf{q},\sigma') \langle 0|\,([a_{\mathbf{p};\sigma}, a^{\dagger}_{\mathbf{q};\sigma'}] + a^{\dagger}_{\mathbf{q};\sigma'} a_{\mathbf{p};\sigma})\,|0\rangle \\
&= -\int \frac{d^3p\,d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^{\mu}(\mathbf{p},\sigma) e^{\nu}(\mathbf{q},\sigma') (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -\int \frac{d^3p\, e^{-ip\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\sigma} g_{\sigma\sigma} e^{\mu}(\mathbf{p},\lambda) e^{\nu}(\mathbf{p},\lambda) = -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}},
\end{aligned} \tag{5.355}$$

以及

$$\langle 0|\, A^{\nu}(y)A^{\mu}(x)\,|0\rangle = -g^{\nu\mu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(y-x)}}{2E_{\mathbf{p}}} = -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}}. \tag{5.356}$$

当质量 $m=0$ 时，(5.327) 式化为

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}] = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2+i\epsilon}. \tag{5.357}$$

于是，*Feynman* 规范下无质量矢量场的 Feynman 传播子可以表达为

$$\begin{aligned}
\Delta_{\mathrm{F}}^{\mu\nu}(x-y) &= \langle 0|\,\mathsf{T}[A^{\mu}(x)A^{\nu}(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\, A^{\mu}(x)A^{\nu}(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\, A^{\nu}(y)A^{\mu}(x)\,|0\rangle \\
&= -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0-y^0)e^{-ip\cdot(x-y)} + \theta(y^0-x^0)e^{ip\cdot(x-y)}] \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2+i\epsilon} e^{-ip\cdot(x-y)}.
\end{aligned} \tag{5.358}$$

5.5.5 Dirac 旋量场的 Feynman 传播子

Dirac 旋量场 $\psi_a(x)$ 的 Feynman 传播子 $S_{\mathrm{F},ab}(x-y)$ 定义为

$$S_{\mathrm{F},ab}(x-y) \equiv \overbracket{\psi_a(x)\bar{\psi}_b}(y) = \langle 0|\,\mathsf{T}[\psi_a(x)\bar{\psi}_b(y)]\,|0\rangle\,. \tag{5.359}$$

在相互作用绘景中，$\bar{\psi}_a(x)$ 的平面波展开式仍然具有 (4.238) 的形式，将它分解为正能解和负能解两个部分，有

$$\bar{\psi}_a(x) = \bar{\psi}_a^{(+)}(x) + \bar{\psi}_a^{(-)}(x), \tag{5.360}$$

其中，

$$\bar{\psi}_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \bar{u}_a(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x}, \tag{5.361}$$

$$\bar{\psi}_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \bar{v}_a(\mathbf{p},\lambda) b_{\mathbf{p},\lambda} e^{-ip\cdot x}. \tag{5.362}$$

再利用 $\psi_a^{(\pm)}(x)$ 的展开式 (5.248) 和 (5.249)、产生湮灭算符的反对易关系 (4.266)、自旋求和关系 (4.235)，可得

$$\begin{aligned}
\langle 0|\, \psi_a(x)\bar{\psi}_b(y)\, |0\rangle &= \langle 0|\, \psi_a^{(+)}(x)\bar{\psi}_b^{(-)}(y)\, |0\rangle \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0|\, u_a(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} \bar{u}_b(\mathbf{q},\lambda') a^\dagger_{\mathbf{q},\lambda'}{}^{iq\cdot y}\, |0\rangle \\
&= \int \frac{d^3p\, d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')\, \langle 0|\, (\{a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}\} - a^\dagger_{\mathbf{q},\lambda'} a_{\mathbf{p},\lambda})\, |0\rangle \\
&= \int \frac{d^3p\, d^3q\, e^{-i(p\cdot x - q\cdot y)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{d^3p\, e^{-ip\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{p},\lambda) = \int \frac{d^3p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} \frac{e^{-ip\cdot(x-y)}}{2E_{\mathbf{p}}} \\
&= \int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} e^{-ip\cdot(x-y)} \delta(p^2 - m^2)\theta(p^0),
\end{aligned} \tag{5.363}$$

最后一步逆向利用 (2.119) 式的推导过程将 d^3p 积分化为 d^4p 积分。类似地，还可以导出

$$\begin{aligned}
\langle 0|\, \bar{\psi}_b(y)\psi_a(x)\, |0\rangle &= \langle 0|\, \bar{\psi}_b^{(+)}(y)\psi_a^{(-)}(x)\, |0\rangle \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0|\, \bar{v}_b(\mathbf{p},\lambda) b_{\mathbf{p},\lambda} e^{-ip\cdot y} v_a(\mathbf{q},\lambda') b^\dagger_{\mathbf{q},\lambda'}{}^{iq\cdot x}\, |0\rangle \\
&= \int \frac{d^3p\, d^3q\, e^{-i(p\cdot y - q\cdot x)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} v_a(\mathbf{q},\lambda')\bar{v}_b(\mathbf{p},\lambda)\, \langle 0|\, (\{b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} - b^\dagger_{\mathbf{q},\lambda'} b_{\mathbf{p},\lambda})\, |0\rangle \\
&= \int \frac{d^3p\, d^3q\, e^{-i(p\cdot y - q\cdot x)}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} v_a(\mathbf{q},\lambda')\bar{v}_b(\mathbf{p},\lambda)(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{d^3p\, e^{-ip\cdot(y-x)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} v_a(\mathbf{p},\lambda)\bar{v}_b(\mathbf{p},\lambda) = \int \frac{d^3p}{(2\pi)^3} (\gamma^\mu p_\mu - m)_{ab} \frac{e^{ip\cdot(x-y)}}{2E_{\mathbf{p}}} \\
&= \int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu - m)_{ab} e^{ip\cdot(x-y)} \delta(p^2 - m^2)\theta(p^0) \\
&= -\int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} e^{-ip\cdot(x-y)} \delta(p^2 - m^2)\theta(-p^0).
\end{aligned} \tag{5.364}$$

最后一步作了变量替换 $p^\mu \to -p^\mu$。于是，Feynman 传播子为

$$S_{\mathrm{F},ab}(x-y) = \langle 0|\, \mathsf{T}[\psi_a(x)\bar{\psi}_b(y)]\, |0\rangle$$

$$
\begin{aligned}
&= \theta(x^0-y^0)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle \\
&= \int \frac{d^4p}{(2\pi)^3}(\gamma_\mu p^\mu+m)_{ab}e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2). \quad (5.365)
\end{aligned}
$$

现在要想办法将 (5.365) 式转化为简洁的表达式。由

$$
\begin{aligned}
&\partial_x^\mu\{e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2)\} \\
&\quad = -ip^\mu e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2) \\
&\qquad +g^{\mu 0}e^{-ip\cdot(x-y)}[\delta(x^0-y^0)\theta(p^0)-\delta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2), \quad (5.366)
\end{aligned}
$$

可得

$$
\begin{aligned}
&p^\mu e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2) \\
&\quad = i\partial_x^\mu\{e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2)\} \\
&\qquad -ig^{\mu 0}e^{-ip\cdot(x-y)}[\theta(p^0)-\theta(-p^0)]\delta(x^0-y^0)\delta(p^2-m^2). \quad (5.367)
\end{aligned}
$$

将上式代入 (5.365) 式，得到

$$
\begin{aligned}
&S_{\mathrm{F},ab}(x-y) \\
&= \int \frac{d^4p}{(2\pi)^3}\Big[(i\gamma_\mu\partial_x^\mu+m)_{ab}\{e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2)\} \\
&\qquad -i(\gamma_\mu)_{ab}g^{\mu 0}e^{-ip\cdot(x-y)}[\theta(p^0)-\theta(-p^0)]\delta(x^0-y^0)\delta(p^2-m^2)\Big] \\
&= (i\gamma_\mu\partial_x^\mu+m)_{ab}\int \frac{d^4p}{(2\pi)^3}e^{-ip\cdot(x-y)}\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2) \\
&\quad -i(\gamma^0)_{ab}\int \frac{d^4p}{(2\pi)^3}e^{-ip\cdot(x-y)}[\theta(p^0)-\theta(-p^0)]\delta(x^0-y^0)\delta(p^2-m^2). \quad (5.368)
\end{aligned}
$$

先计算 (5.368) 式最后一行。利用 δ 函数的性质 (2.49)，有

$$
e^{-ip^0(x^0-y^0)}\delta(x^0-y^0) = e^{-ip^0(x^0-x^0)}\delta(x^0-y^0) = \delta(x^0-y^0), \quad (5.369)
$$

由此可得

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^3}e^{-ip\cdot(x-y)}\theta(p^0)\delta(x^0-y^0)\delta(p^2-m^2) \\
&= \int \frac{d^4p}{(2\pi)^3}e^{-ip^0(x^0-y^0)}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\theta(p^0)\delta(x^0-y^0)\delta(p^2-m^2) \\
&= \int \frac{d^4p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\theta(p^0)\delta(x^0-y^0)\delta(p^2-m^2), \quad (5.370)
\end{aligned}
$$

以及

$$
\int \frac{d^4p}{(2\pi)^3}e^{-ip\cdot(x-y)}\theta(-p^0)\delta(x^0-y^0)\delta(p^2-m^2)
$$

$$
\begin{aligned}
&= \int \frac{d^4p}{(2\pi)^3}\, e^{-ip^0(x^0-y^0)} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(-p^0)\delta(x^0-y^0)\delta(p^2-m^2) \\
&= \int \frac{d^4p}{(2\pi)^3}\, e^{ip^0(x^0-y^0)} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(p^0)\delta(x^0-y^0)\delta(p^2-m^2) \\
&= \int \frac{d^4p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(p^0)\delta(x^0-y^0)\delta(p^2-m^2). \qquad (5.371)
\end{aligned}
$$

第二步作了变量替换 $p^0 \to -p^0$。结合以上两式，有

$$
\int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot(x-y)}[\theta(p^0)-\theta(-p^0)]\delta(x^0-y^0)\delta(p^2-m^2) = 0. \qquad (5.372)
$$

故 (5.368) 式最后一行为零。另一方面，(5.368) 式倒数第二行中积分可化为

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot(x-y)}[\theta(x^0-y^0)\theta(p^0)+\theta(y^0-x^0)\theta(-p^0)]\delta(p^2-m^2) \\
&= \int \frac{d^4p}{(2\pi)^3}\, [e^{-ip\cdot(x-y)}\theta(x^0-y^0)+e^{ip\cdot(x-y)}\theta(y^0-x^0)]\theta(p^0)\delta(p^2-m^2) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)e^{-ip\cdot(x-y)}+\theta(y^0-x^0)e^{ip\cdot(x-y)}] = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}. \qquad (5.373)
\end{aligned}
$$

第一步作了变量替换 $p^\mu \to -p^\mu$，第二步利用 (2.119) 式的推导过程将 d^4p 积分化为 d^3p 积分，第三步用到 (5.327) 式。将上式代入 (5.368) 式，则 Dirac 旋量场的 Feynman 传播子可以表达为

$$
S_{\mathrm{F},ab}(x-y) = (i\gamma_\mu\partial_x^\mu+m)_{ab}\int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}. \qquad (5.374)
$$

写成旋量空间矩阵的形式是

$$
S_{\mathrm{F}}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}. \qquad (5.375)
$$

根据 Dirac 矩阵的反对易关系 (4.1)，有

$$
\not{p}\not{p} = p_\mu p_\nu \gamma^\mu\gamma^\nu = \frac{1}{2}p_\mu p_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu) = p_\mu p_\nu g^{\mu\nu} = p^2, \qquad (5.376)
$$

从而，可得

$$
(\not{p}+m)(\not{p}-m) = \not{p}\not{p}-m^2 = p^2-m^2, \qquad (5.377)
$$

故

$$
(\not{p}+m)(\not{p}-m+i\epsilon) = p^2-m^2+i\epsilon(\not{p}+m). \qquad (5.378)
$$

$i\epsilon(\not{p}+m)$ 是一个无穷小量，因而上式右边与 (5.375) 式右边分式中的分母等价。于是，(5.375) 式也可以表示成

$$
S_{\mathrm{F}}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{(\not{p}+m)(\not{p}-m+i\epsilon)}\, e^{-ip\cdot(x-y)} = \int \frac{d^4p}{(2\pi)^4}\frac{i}{\not{p}-m+i\epsilon}\, e^{-ip\cdot(x-y)}. \qquad (5.379)
$$

上式最右边在表达方式上更为简洁，但在矩阵的意义上不好理解，应将它转化回到 (5.375) 式来理解。

附录 A　英汉对照

Annihilation operator: 湮灭算符
Antichronous: 反时向
Anti-particle: 反粒子
Axial vector: 轴矢量
Azimuthal angle: 方位角
Beam: 束流
Boost: 增速
Boson: 玻色子
Branching ratio: 分支比
Canonical quantization: 正则量子化
Causality: 因果性
Center-of-mass energy: 质心能
Center-of-mass system: 质心系
Chiral representation: 手征表象
Collider: 对撞机
Conjugate momentum density: 共轭动量密度
Conserved charge: 守恒荷
Conserved current: 守恒流
Contraction: 缩并
Contravariant vector: 逆变矢量
Coupling constant: 耦合常数
Covariant vector: 协变矢量
Creation operator: 产生算符
Cross section: 截面
Decay: 衰变
Decay width: 衰变宽度
Dirac slash: Dirac 斜线
Dynamics: 动力学
Electron: 电子
Energy-momentum tensor: 能动张量
Expectation value: 期待值
Fermion: 费米子
Field strength tensor: 场强张量
Gauge-fixing term: 规范固定项
Gauge invariant: 规范不变量
Gauge symmetry: 规范对称性
Gauge transformation: 规范变换
Generalized coordinate: 广义坐标
Generator: 生成元
Global: 整体
Hamiltonian: 哈密顿量
Helicity: 螺旋度
Hermitian conjugate: 厄米共轭
Hermitian operator: 厄米算符
Homomorphic: 同态
Improper: 非固有
Interaction: 相互作用
Interaction picture: 相互作用绘景
Invariant mass: 不变质量
Invariant matrix element: 不变矩阵元
Invariant scattering amplitude: 不变散射振幅
Kinematics: 运动学
Lagrangian: 拉格朗日量
Left-handed: 左手
Lifetime: 寿命
Local: 局域
Lowering operator: 降算符
Metric: 度规
Mode: 模式
Normal order: 正规次序
Normal product: 正规乘积
Orthochronous: 保时向

Parity: 宇称

Partial decay width: 分宽度

Perturbation theory: 微扰论

Phonon: 声子

Picture: 绘景

Plane-wave solution: 平面波解

Polar angle: 极角

Polarization vector: 极化矢量

Positron: 正电子

Proper: 固有

Pseudoscalar: 赝标量

Raising operator: 升算符

Right-handed: 右手

Real orthogonal matrix: 实正交矩阵

Scalar: 标量

Scattering cross section: 散射截面

Scattering matrix: 散射矩阵

Self-conjugate: 自共轭

Self-interaction: 自相互作用

Simple harmonic oscillator: 简谐振子

Space inversion: 空间反射

Spinor: 旋量

Spinor bilinear: 旋量双线性型

Spinor representation: 旋量表示

Step function: 阶跃函数

Target: 靶

Tensor: 张量

Time-evolution operator: 时间演化算符

Time-ordered product: 时序乘积

Time reversal: 时间反演

Unitary: 幺正

Vacuum: 真空

Vector: 矢量

Zero-point energy: 零点能