量子场论讲义

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第 1 章　预备知识

1.1　量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由**粒子**组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon 方程**：

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=(-\hbar^2c^2\nabla^2+m^2c^4)\psi(\mathbf{x},t). \tag{1.1}$$

它给出的自由粒子能量为

$$E=\pm\sqrt{|\mathbf{p}|^2c^2+m^2c^4}, \tag{1.2}$$

其中 $\mathbf{p}$ 为粒子的动量，m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2\leq E<\infty$；也可以为负，取值范围为 $-\infty<E\leq mc^2$。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho=\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right), \tag{1.3}$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 协变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E<0$ 的态都被填满而所有 $E>0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E>0$ 的电子跃迁到 $E<0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的"反粒子 (anti-particle)"，称为**正电子** (positron)。正电子在 1932 年被 Carl Anderson 发现。

但是，Dirac 的空穴理论仍然面临一些困难，比如，为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场？另一方面，Dirac 方程一开始作为描述单个粒子波函数的方程提出来，但 Dirac 的解释却包含了无穷多个粒子。而且，像光子和 π 介子这些不满足 Pauli 不相容原理的粒子，空穴理论是不能成立的。此外，Dirac 方程只能描述自旋 1/2 的粒子，不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难，是否意味着处理这些问题的基础本身就不正确呢？确实是这样的。量子力学的一条基本原理是：观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而，**时间**显然是一个观测量，却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中，描述系统的量子态时可以让态依赖于一个时间参数 t，这是时间的概念进入量子力学的方式，但并没有假定这个参数是某个厄米算符的本征值。另一方面，粒子的空间位置 $\mathbf{x}$ 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见，在量子力学中，对时间和空间的处理方式是完全不同的。而在狭义相对论中，Lorentz 对称性将两者混合起来。因此，在结合量子力学与狭义相对论的过程中出现困难，也是正常的。

那么，如何在量子力学中平等地处理时间和空间呢？一种途径是将时间提升为一个厄米算符，但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数，不再由厄米算符描写。这样，我们可以在每个空间点 $\mathbf{x}$ 处定义一个算符 $\hat{\phi}(\mathbf{x})$，所有这些算符的集合称为**量子场**。在 Heisenberg 绘景中，量子场算符也依赖于时间 t：

$$\hat{\phi}(\mathbf{x},t) = e^{i\hat{H}t/\hbar}\hat{\phi}(\mathbf{x})e^{-i\hat{H}t/\hbar}. \tag{1.4}$$

如此，量子化的对象变成是由依赖于时空坐标的**场**组成的动力学系统，这就是**量子场论**。这里的量子算符用 ˆ 符号标记，为了简化记号，后面将省略 ˆ 符号。

在量子场论中，前面提到的困难都可以得到解决。现在，Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态，包含粒子的态则是激发态，激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子，由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的，不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x\,\rho$ 也可以用产生湮灭算符表达出来，虽然它不一定是正定的，但是它不再被解释为总概率，而是被解释为正粒子数与反粒子数之差，因而也不再出现负概率困难。

1.2 自然单位制

量子场论是结合量子力学和相对论的理论，因而时常出现约化 Planck 常量 $\hbar$ 和光速 c，这一点可以从上一节的几个公式中看出来。于是，为了简化表述，通常采用**自然单位制**，取

$$\hbar = c = 1. \tag{1.5}$$

从而，Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(\mathbf{x},t) = 0. \tag{1.6}$$

在自然单位制中，速度没有量纲 (dimension)；长度量纲与时间量纲相同，是能量量纲的倒数；能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22}\ \mathrm{MeV \cdot s}, \quad 1 = \hbar c = 1.973 \times 10^{-11}\ \mathrm{MeV \cdot cm}, \tag{1.7}$$

可得

$$1\ \mathrm{s}^{-1} = 6.582 \times 10^{-22}\ \mathrm{MeV}, \quad 1\ \mathrm{cm}^{-1} = 1.973 \times 10^{-11}\ \mathrm{MeV}. \tag{1.8}$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.036} \tag{1.9}$$

是没有量纲的，它的数值在任何单位制下都应该相同。因此，自然单位制不可能将 $\hbar$、c、ε_0 和 e 这四个常数同时归一化。在量子场论中，通常再取真空介电常数

$$\varepsilon_0 = 1, \tag{1.10}$$

同时可得真空磁导率 $\mu_0 = 1/(\varepsilon_0 c^2) = 1$，这样做其实是取了 Heaviside-Lorentz 单位制。从而，不同于 Gauss 单位制，Maxwell 方程组中不会出现无理数 4π：

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{1.11}$$

此处的单位制称为**有理化**的自然单位制。现在，精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$，而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的；4π 因子会出现在 Coulomb 定律中，点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \tag{1.12}$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到**狭义相对论**，它的基本原理如下。

(1) 光速不变原理：在任意惯性参考系中，光速的大小不变。

(2) 狭义相对性原理：在任意惯性参考系中，物理定律具有相同的形式。

两个惯性参考系的 Descartes 坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动，则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \tag{1.13}$$

其中 Lorentz 因子 $\gamma \equiv (1 - \beta^2)^{-1/2}$. 这种 Lorentz 变换称为沿 x 方向的**增速** (boost)。在此变换下，有

$$t'^2 - x'^2 - y'^2 - z'^2 = \gamma^2(t - \beta x)^2 - \gamma^2(x - \beta t)^2 - y^2 - z^2$$

$$= \frac{1}{1-\beta^2}(t^2+\beta^2x^2-2\beta xt-x^2-\beta^2t^2+2\beta xt)-y^2-z^2 = t^2-x^2-y^2-z^2. \quad (1.14)$$

可见，$t^2-x^2-y^2-z^2$ 在 Lorentz 变换下不变，是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值，这是 Lorentz 变换对应的对称性，称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来，可以构成 Minkowski 时空，坐标记为

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \mathbf{x}), \quad \text{其中 } \mu = 0, 1, 2, 3. \quad (1.15)$$

上式中四种记法是等价的。x^μ 是一个*逆变* (contravariant) 的 Lorentz 四维矢量 (vector)，“逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发，可以定义 Lorentz 不变的*内积*[1]

$$x^2 \equiv x\cdot x \equiv (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2 = (x^0)^2-|\mathbf{x}|^2. \quad (1.16)$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu} = g_{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.17)$$

可以把内积 (1.16) 简洁地写成

$$x^2 = g_{\mu\nu}x^\mu x^\nu. \quad (1.18)$$

这里采用了 **Einstein 求和约定**：*不写出求和符号，重复的指标即表示求和*。除非特别指出，后面都默认使用这个约定。在上式中，用同个字母表示的指标分别在上标和下标重复出现并求和，这称为**缩并** (contraction)，是 Lorentz 不变量的特点。

为了进一步简化记号，定义*协变* (covariant) 的 Lorentz 四维矢量

$$x_\mu = g_{\mu\nu}x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}). \quad (1.19)$$

“协变”指的是指标 μ 写在右下角。于是，内积 x^2 的表达式 (1.18) 可以简化为

$$x^2 = x^\mu x_\mu. \quad (1.20)$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来，变成协变矢量 x_μ。从方阵的角度看，$g_{\mu\nu}$ 的逆为

$$g^{\mu\nu} = g^{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.21)$$

[1]这里的记号有些不一致，第一个 x^2 是内积的记号，而第二个 x^2 是第 2 个空间坐标。

满足

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu, \tag{1.22}$$

其中 **Kronecker 符号** $\delta^\mu{}_\nu$ 定义为

$$\delta^\mu{}_\nu = \delta_\mu{}^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \tag{1.23}$$

对于 Minkowski 度规，$g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同，但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$，对 μ 求和，得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma{}_\nu x^\nu = x^\sigma, \tag{1.24}$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来，变成逆变矢量 x^σ。可见，逆变矢量与协变矢量是一一对应的，是对同一个 Lorentz 矢量的两种等价描述。

利用 Kronecker 符号的定义和 (1.22) 式，可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu{}_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \tag{1.25}$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho{}_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \tag{1.26}$$

这两条式子表明，度规也可以用来对度规自身的指标进行升降。

利用四维矢量的记号，可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.27}$$

其中

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.28}$$

注意：在将 $\Lambda^\mu{}_\nu$ 视作矩阵时，偏左的指标 μ 表示行的编号，偏右的指标 ν 表示列的编号。$\Lambda^\mu{}_\nu$ 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变，从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值。我们可以将 $\Lambda^\mu{}_\nu$ 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换，称为（齐次）**Lorentz 变换**，使下式成立：

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \tag{1.29}$$

可见，Lorentz 变换 $\Lambda^\mu{}_\nu$ 必须满足**保度规条件**

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}. \tag{1.30}$$

空间旋转变换保持 $|\mathbf{x}|^2$ 不变，由 (1.16) 式可知，这种变换也属于 Lorentz 变换。例如，绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \tag{1.31}$$

容易验证，它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并，可得

$$\Lambda_\nu{}^\gamma \Lambda^\nu{}_\beta = g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g^{\gamma\alpha} g_{\alpha\beta} = \delta^\gamma{}_\beta, \tag{1.32}$$

其中

$$\Lambda_\nu{}^\gamma \equiv g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu{}_\alpha \tag{1.33}$$

可以看作是用度规对 $\Lambda^\mu{}_\alpha$ 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu{}_\nu \equiv \Lambda_\nu{}^\mu, \tag{1.34}$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu{}_\rho \Lambda^\rho{}_\nu = \delta^\mu{}_\nu. \tag{1.35}$$

$\delta^\mu{}_\nu$ 也是一个 Lorentz 变换，它使得 $x'^\mu = \delta^\mu{}_\nu x^\nu = x^\mu$，即 x^μ 在这个变换下不变。可见，$\delta^\mu{}_\nu$ 是一个恒等变换。(1.35) 式表明，对时空坐标矢量先作 Λ 变换，再作 Λ^{-1} 变换，得到的矢量还是原来的矢量。也就是说，由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换，也是一个 Lorentz 变换。在这些记号下，协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} x_\sigma = \Lambda_\mu{}^\sigma x_\sigma = x_\sigma (\Lambda^{-1})^\sigma{}_\mu. \tag{1.36}$$

Λ^{-1} 既然是一个 Lorentz 变换，必定满足保度规条件

$$g_{\mu\nu} (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta = g_{\alpha\beta}, \tag{1.37}$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma} = g_{\mu\nu} (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta g^{\alpha\rho} g^{\beta\sigma} = g^{\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu g^{\alpha\rho} g^{\beta\sigma} \\ &= g^{\gamma\delta} (g^{\alpha\rho} g_{\gamma\mu} \Lambda_\alpha{}^\mu)(g^{\beta\sigma} g_{\delta\nu} \Lambda_\beta{}^\nu) = g^{\gamma\delta} \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta. \end{aligned} \tag{1.38}$$

这给出了保度规条件 (1.30) 的一个等价形式：

$$g^{\mu\nu} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = g^{\alpha\beta}. \tag{1.39}$$

将 $\Lambda^\mu{}_\nu$ 视作矩阵 Λ，则其转置矩阵 Λ^{T} 的分量满足 $(\Lambda^{\mathrm{T}})_\nu{}^\mu = \Lambda^\mu{}_\nu$，由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = (\Lambda^{\mathrm{T}})_\alpha{}^\mu g_{\mu\nu} \Lambda^\nu{}_\beta, \tag{1.40}$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^{\mathrm{T}} \mathbf{g}\, \Lambda. \tag{1.41}$$

取行列式得 $\det \mathbf{g} = \det \Lambda^{\mathrm{T}} \cdot \det \mathbf{g} \cdot \det \Lambda = \det \mathbf{g} \cdot (\det \Lambda)^2$，因此，

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \tag{1.42}$$

Lorentz 坐标变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ 的 Jacobi 行列式为

$$\mathcal{J} = \det\left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}\right] = \det\Lambda, \tag{1.43}$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}|d^4x = |\det\Lambda|d^4x = d^4x. \tag{1.44}$$

可见，Minkowski 时空的体积元是 Lorentz 不变的。

$\det\Lambda$ 的值可以用来为 Lorentz 变换分类：$\det\Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换，$\det\Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外，由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2, \tag{1.45}$$

则 $(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1$，故有 $\Lambda^0{}_0 \geq +1$ 或 $\Lambda^0{}_0 \leq -1$。$\Lambda^0{}_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换，$\Lambda^0{}_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上，对称性由群论描述。对称变换的集合称为**群**，群元素具有乘法，满足下列四个条件。

(1) 两个群元素的乘积即是两次对称变换相继作用，乘法满足结合律。

(2) 群中任意两个元素的乘积仍属于此群（封闭性）。

(3) 群中必有一个恒元（对应于恒等变换），它与任一元素的乘积仍为此元素。

(4) 任一元素都可以在群中找到一个逆元（对应于逆变换），两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数（如 β、θ 等）来描述，因而是一种连续变换，所以 Lorentz 群是一个连续群，参数的变化区域称为群空间。Lorentz 群的整个群空间不是连通的，它有四个连通分支，如图 1.1 所示，分别是固有保时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \geq +1$)、固有反时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \leq -1$)、非固有保时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \geq +1$) 和非固有反时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \leq -1$)，四个分支之间彼此不连通。恒元（即恒等变换）在固有保时向分支里，这个分支也称为**固有保时向 Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义宇称 (parity) 变换为

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.46}$$

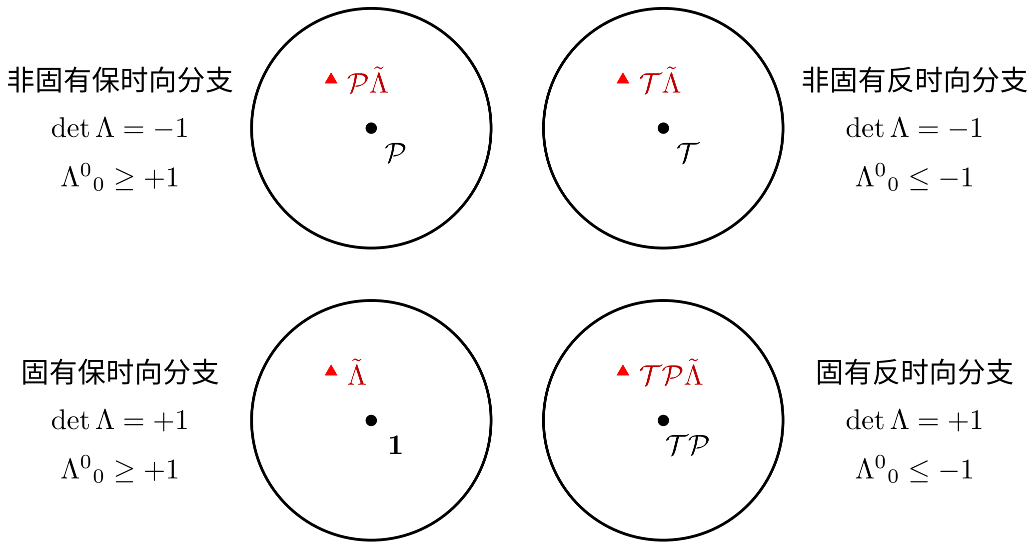


图 1.1: Lorentz 群的四个连通分支示意图。**1**、$\mathcal{P}$ 和 $\mathcal{T}$ 分别代表恒等变换、宇称变换和时间反演变换，$\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的，亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$\mathcal{T}^\mu{}_\nu=(\mathcal{T}^{-1})^\mu{}_\nu=\begin{pmatrix}-1&&&\\&+1&&\\&&+1&\\&&&+1\end{pmatrix},\tag{1.47}$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素，乘上宇称变换或（和）时间反演变换，就可以到达 Lorentz 群的其它分支。

1.4　Lorentz 矢量

如果一些 $m\times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同，就可以用这些矩阵来表示这个群，这些矩阵构成了这个群的一个 m 维**线性表示**。利用群的线性表示，可以将对称变换视作矩阵，将变换作用的态视作列矩阵。

在上一节中，我们已经用矩阵的形式表示过 Lorentz 变换 $\Lambda^\mu{}_\nu$，可见，$\Lambda^\mu{}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**，因为 Lorentz 矢量 x^ν 可以看作是变换 $\Lambda^\mu{}_\nu$ 所作用的态。一般地，一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$A'^\mu=\Lambda^\mu{}_\nu A^\nu.\tag{1.48}$$

类似于 (1.36) 式，逆变矢量 A^μ 对应的协变矢量 $A_\mu=g_{\mu\nu}A^\nu$ 在 Lorentz 变换下满足

$$A_\mu=A_\nu(\Lambda^{-1})^\nu{}_\mu.\tag{1.49}$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \tag{1.50}$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量：

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \tag{1.51}$$

Lorentz 不变量也称为 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负，Lorentz 矢量 A^μ 的自我内积的符号不是确定的，可以分为三类。

(1) 若 $A^2 > 0$，则称 A^μ 为**类时**矢量。

(2) 若 $A^2 < 0$，则称 A^μ 为**类空**矢量。

(3) 若 $A^2 = 0$，则称 A^μ 为**类光**矢量。

由于 A^2 是 Lorentz 不变量，不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中，质点的能量 E、动量 $\mathbf{p}$ 和（静止）质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{1.52}$$

可以用 E 和 $\mathbf{p}$ 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \tag{1.53}$$

称为**四维动量**，它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \tag{1.54}$$

这是合理的，因为质量 m 在狭义相对论中是一个 Lorentz 不变量。p^μ 在 $m > 0$ 时是类时矢量，在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) = g^{\mu\nu} \partial_\nu, \tag{1.55}$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho{}^\nu = g^{\mu\nu}. \tag{1.56}$$

可见，这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ 时，时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \partial^\nu. \tag{1.57}$$

由上式、(1.56) 式和保度规条件 (1.39) 可得，

$$\partial'^\mu x'^\nu = \Lambda^\mu{}_\rho \partial^\rho (\Lambda^\nu{}_\sigma x^\sigma) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho x^\sigma = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} = g^{\mu\nu}, \tag{1.58}$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的，从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明，时空导数的 Lorentz 变换形式与 Lorentz 矢量相同，因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \tag{1.59}$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu}\partial'^\mu \partial'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma}\partial^\rho \partial^\sigma = \partial^2. \tag{1.60}$$

可见，∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \tag{1.61}$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 协变性。

1.5　Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p+q$ 阶的 (p,q) 型 **Lorentz 张量** $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ 具有 p 个逆变指标和 q 个协变指标，并满足如下 Lorentz 变换规则：

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_p}{}_{\rho_p} T^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\sigma_q}{}_{\nu_q}. \tag{1.62}$$

这里的逆变指标和协变指标统称为 *Lorentz* 指标。Lorentz 标量是 0 阶 Lorentz 张量，不具有 Lorentz 指标；Lorentz 矢量是 1 阶 Lorentz 张量，具有 1 个 Lorentz 指标。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0,2)$ 型 Lorentz 张量，不过它在任何惯性系中不变，Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62)，可以验证，如下表达式都是 Lorentz 标量（亦即 Lorentz 不变量）：

$$g_{\mu\nu}T^{\mu\nu}, \quad T^{\mu\nu}A_\mu B_\nu, \quad T^{\mu\nu}T_{\mu\nu}, \quad g_{\mu\sigma}T^{\mu\nu}{}_\rho T^{\sigma\rho}{}_\nu. \tag{1.63}$$

实际上，可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p,q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并，可以得到一个 $(p-1,q-1)$ 型 Lorentz 张量。例如，由

$$T'^{\mu\nu}{}_\mu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma (\Lambda^{-1})^\gamma{}_\mu = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma \delta^\gamma{}_\alpha = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\alpha \tag{1.64}$$

可知，$T^{\mu\nu}{}_\mu$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita 符号**

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的偶次置换，} \\ -1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的奇次置换，} \\ 0, & \text{其它情况。} \end{cases} \tag{1.65}$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是**全反对称**的，即关于任意两个指标反对称，如 $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\rho\nu\mu\sigma} = -\varepsilon^{\sigma\nu\rho\mu}$。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.66}$$

$\varepsilon_{\mu\nu\rho\sigma}$ 也是全反对称的，如

$$\varepsilon_{\nu\mu\rho\sigma} = g_{\nu\alpha}g_{\mu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta} = g_{\mu\beta}g_{\nu\alpha}g_{\rho\gamma}g_{\sigma\delta}(-\varepsilon^{\beta\alpha\gamma\delta}) = -\varepsilon_{\mu\nu\rho\sigma}. \tag{1.67}$$

根据这些定义，

$$\varepsilon^{0123} = +1, \quad \varepsilon_{0123} = -1. \tag{1.68}$$

从而，

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 4!\,\varepsilon^{0123}\varepsilon_{0123} = -4!. \tag{1.69}$$

利用 Levi-Civita 符号可以把 $\det\Lambda$ 按照行列式定义写成

$$\det\Lambda = \Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.70}$$

对于固有 Lorentz 变换，$\det\Lambda = +1$，有

$$\varepsilon^{0123} = \varepsilon^{0123}\det\Lambda = \Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.71}$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质，可得

$$\varepsilon^{1023} = -\varepsilon^{0123} = -\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = -\Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = \Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\beta\alpha\gamma\delta}. \tag{1.72}$$

依此类推，可以证明

$$\varepsilon^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.73}$$

可见，在固有 Lorentz 变换下，$\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量，不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中，ρ 是电荷密度，$\mathbf{J}$ 是电流密度，它们可以组成一个 Lorentz 矢量 $J^\mu = (\rho, \mathbf{J})$，从而，电流连续性方程

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{1.74}$$

可以写成 Lorentz 协变的形式

$$\partial_\mu J^\mu = 0. \tag{1.75}$$

此外，电场强度 $\mathbf{E}$ 和磁感应强度 $\mathbf{B}$ 可以用电势 Φ 和矢势 $\mathbf{A}$ 表达为

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1.76}$$

这样，方程

$$\nabla \cdot \mathbf{B} = 0 \tag{1.77}$$

是自动满足的。Φ 和 $\mathbf{A}$ 可以组成一个 Lorentz 矢量 $A^\mu = (\Phi, \mathbf{A})$，称为**四维矢势**，则 (1.76) 式的分量形式为

$$E^i = -\partial_i A^0 - \partial_0 A^i, \quad B^k = \varepsilon^{kij}\partial_i A^j, \quad i,j,k = 1,2,3. \tag{1.78}$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk} \equiv \varepsilon^{0ijk}, \tag{1.79}$$

因而 $\varepsilon^{123} = +1$。

引入电磁场的**场强张量** (field strength tensor)

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{1.80}$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序，从上述定义可得

$$\begin{aligned}\partial^\rho F^{\mu\nu} &= \partial^\rho(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu\partial^\rho A^\nu - \partial^\mu\partial^\nu A^\rho + \partial^\nu\partial^\mu A^\rho - \partial^\nu\partial^\rho A^\mu \\ &= \partial^\mu F^{\rho\nu} + \partial^\nu F^{\mu\rho} = -\partial^\mu F^{\nu\rho} - \partial^\nu F^{\rho\mu},\end{aligned} \tag{1.81}$$

即

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \tag{1.82}$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = -E^i, \tag{1.83}$$

可见，F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k}\varepsilon^{12k} = \varepsilon^{123}\varepsilon^{123} = 1$ 和 $\varepsilon^{12k}\varepsilon^{21k} = \varepsilon^{123}\varepsilon^{213} = -1$，依此类推，可以归纳出如下求和关系：

$$\varepsilon^{ijk}\varepsilon^{kmn} = \varepsilon^{ijk}\varepsilon^{mnk} = \delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}, \tag{1.84}$$

利用这个关系，可得

$$\varepsilon^{ijk}B^k = \varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n = \delta^{im}\delta^{jn}\partial_m A^n - \delta^{in}\delta^{jm}\partial_m A^n = \partial_i A^j - \partial_j A^i, \tag{1.85}$$

从而，

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\varepsilon^{ijk}B^k, \tag{1.86}$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu}=\begin{pmatrix}0 & -E^1 & -E^2 & -E^3\\ E^1 & 0 & -B^3 & B^2\\ E^2 & B^3 & 0 & -B^1\\ E^3 & -B^2 & B^1 & 0\end{pmatrix}. \tag{1.87}$$

Gauss 定律对应的方程

$$\nabla\cdot\mathbf{E}=\rho \tag{1.88}$$

等价于

$$J^0=\rho=\partial_i E^i=-\partial_i F^{0i}=\partial_i F^{i0}=\partial_i F^{i0}+\partial_0 F^{00}=\partial_\mu F^{\mu 0}, \tag{1.89}$$

而 Ampère 定律对应的方程

$$\nabla\times\mathbf{B}=\mathbf{J}+\frac{\partial\mathbf{E}}{\partial t} \tag{1.90}$$

等价于

$$J^i=\varepsilon^{ijk}\partial_j B^k-\partial_0 E^i=-\partial_j F^{ij}+\partial_0 F^{0i}=\partial_j F^{ji}+\partial_0 F^{0i}=\partial_\mu F^{\mu i}. \tag{1.91}$$

归纳起来，有

$$\partial_\mu F^{\mu\nu}=J^\nu. \tag{1.92}$$

这个方程完全是用 Lorentz 张量写出来的，它在不同惯性系中具有相同的形式，即具有 **Lorentz 协变性**，因而满足狭义相对性原理。

现在，Maxwell 方程组中还有一个方程没有讨论，它是 Maxwell–Faraday 方程

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{1.93}$$

将它写成分量的形式，得

$$\varepsilon^{kmn}\partial_m E^n=-\varepsilon^{kmn}\partial_m F^{0n}=\varepsilon^{kmn}\partial_m F^{n0}=-\partial_0 B^k, \tag{1.94}$$

从而

$$\partial_0 F^{ij}=-\varepsilon^{ijk}\partial_0 B^k=\varepsilon^{ijk}\varepsilon^{kmn}\partial_m F^{n0}=(\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm})\partial_m F^{n0}=\partial_i F^{j0}-\partial_j F^{i0}, \tag{1.95}$$

即

$$\partial^0 F^{ij}+\partial^i F^{j0}+\partial^j F^{0i}=0. \tag{1.96}$$

这个方程与 Maxwell–Faraday 方程等价，不过，它只是前面得到的方程 (1.82) 取特定分量的形式。

利用四维 Levi-Civita 符号，可以定义电磁场的**对偶场强张量** (duel field strength tensor)

$$\tilde{F}^{\mu\nu}=-\tilde{F}^{\nu\mu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{1.97}$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk}\varepsilon^{1jk}=\varepsilon^{123}\varepsilon^{123}+\varepsilon^{132}\varepsilon^{132}=2$ 和 $\varepsilon^{1jk}\varepsilon^{2jk}=\varepsilon^{123}\varepsilon^{223}+\varepsilon^{132}\varepsilon^{232}=0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk}\varepsilon^{ljk}=2\delta^{il},\tag{1.98}$$

利用这个关系，可得

$$\begin{aligned}\tilde{F}^{0i}&=\frac{1}{2}\varepsilon^{0i\rho\sigma}F_{\rho\sigma}=\frac{1}{2}\varepsilon^{0ijk}F_{jk}=\frac{1}{2}\varepsilon^{0ijk}g_{j\mu}g_{k\nu}F^{\mu\nu}=\frac{1}{2}\varepsilon^{0ijk}g_{jm}g_{kn}F^{mn}=-\frac{1}{2}\varepsilon^{ijk}\delta^{jm}\delta^{kn}\varepsilon^{mnl}B^l\\&=-\frac{1}{2}\varepsilon^{ijk}\varepsilon^{jkl}B^l=-\frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}B^l=-\frac{1}{2}2\delta^{il}B^l=-B^i,\end{aligned}\tag{1.99}$$

故 $\tilde{F}^{0i}$ 对应于磁感应强度。另一方面，

$$\begin{aligned}\tilde{F}^{ij}&=\frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma}=\frac{1}{2}(\varepsilon^{ij0k}F_{0k}+\varepsilon^{ijk0}F_{k0})=\varepsilon^{0ijk}F_{0k}=\varepsilon^{0ijk}g_{0\mu}g_{k\nu}F^{\mu\nu}\\&=\varepsilon^{ijk}g_{00}g_{kl}F^{0l}=-\varepsilon^{ijk}\delta^{kl}F^{0l}=-\varepsilon^{ijk}F^{0k}=\varepsilon^{ijk}E^k,\end{aligned}\tag{1.100}$$

说明 $\tilde{F}^{ij}$ 对应于电场强度。$\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu}=\begin{pmatrix}0&-B^1&-B^2&-B^3\\B^1&0&E^3&-E^2\\B^2&-E^3&0&E^1\\B^3&E^2&-E^1&0\end{pmatrix}.\tag{1.101}$$

由 $\tilde{F}^{\mu\nu}$ 的定义，有

$$\begin{aligned}\partial_\mu\tilde{F}^{\mu\nu}&=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma}=-\frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma}=-\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma}+\varepsilon^{\nu\sigma\mu\rho}\partial_\mu F_{\rho\sigma}+\varepsilon^{\nu\rho\sigma\mu}\partial_\mu F_{\rho\sigma})\\&=-\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma}+\varepsilon^{\nu\mu\rho\sigma}\partial_\rho F_{\sigma\mu}+\varepsilon^{\nu\mu\rho\sigma}\partial_\sigma F_{\mu\rho})=-\frac{1}{6}\varepsilon^{\nu\mu\rho\sigma}(\partial_\mu F_{\rho\sigma}+\partial_\rho F_{\sigma\mu}+\partial_\sigma F_{\mu\rho}),\end{aligned}\tag{1.102}$$

因此，方程 (1.82) 等价于

$$\partial_\mu\tilde{F}^{\mu\nu}=0.\tag{1.103}$$

从这些讨论可以看到，用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的，而且方程的 Lorentz 协变性非常明确。

1.6 作用量原理

1.6.1 经典力学中的作用量原理

在经典力学中，质点力学系统可以用**拉格朗日量**（Lagrangian）描述。对于具有 n 个自由度的系统，可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i，它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i=dq_i/dt$。拉格朗日量是广义坐标和广义速度的函数 $L(q_i,\dot{q}_i)$。拉格朗日量的时间积分

$$S=\int_{t_1}^{t_2}dt\,L[q_i(t),\dot{q}_i(t)]\tag{1.104}$$

称为**作用量**。

作用量原理指出，作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零，则有

$$\delta\dot{q}_i = \delta\frac{dq_i}{dt} = \frac{d}{dt}\delta q_i, \tag{1.105}$$

即时间导数的变分等于变分的时间导数。从而可得

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt\,\delta L[q_i(t),\dot{q}_i(t)] = \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) = \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right)\\ &= \int_{t_1}^{t_2} dt\left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right]\\ &= \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \left.\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right|_{t_1}^{t_2},\end{aligned} \tag{1.106}$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零，即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$，则上式最后一行第二项为零，而 $\delta S = 0$ 等价于

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,\cdots,n. \tag{1.107}$$

这是 **Euler-Lagrange 方程**，它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1,\cdots,n. \tag{1.108}$$

反解上式表示的 n 个方程，则可以用 q_i 和 p_i 将 $\dot{q}_i$ 表达出来，然后用 Legendre 变换定义**哈密顿量** (Hamiltonian)

$$H(q_i,p_i) \equiv p_i\dot{q}_i - L, \tag{1.109}$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量，变分为

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} dt\,\delta L = \int_{t_1}^{t_2} dt\,\delta(p_i\dot{q}_i - H) = \int_{t_1}^{t_2} dt\left(\dot{q}_i\delta p_i + p_i\delta\dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)\\ &= \int_{t_1}^{t_2} dt\left(\dot{q}_i\delta p_i + p_i\frac{d}{dt}\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)\\ &= \int_{t_1}^{t_2} dt\left[\dot{q}_i\delta p_i + \frac{d}{dt}(p_i\delta q_i) - \dot{p}_i\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right]\\ &= \int_{t_1}^{t_2} dt\left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right] + \left.p_i\delta q_i\right|_{t_1}^{t_2}.\end{aligned} \tag{1.110}$$

根据前面的假设，上式最后一行第二项为零，于是，$\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1,\cdots,n. \tag{1.111}$$

这是 **Hamilton 正则运动方程**，相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.107).

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中，场 $\phi(\mathbf{x},t)$ 是系统的广义坐标，每一个空间点 $\mathbf{x}$ 都是一个自由度，因此场论相当于具有无穷多自由度的质点力学。在局域场论中，拉格朗日量 $L=\int d^3x\,\mathcal{L}(x)$，其中 $\mathcal{L}(x)$ 称为**拉格朗日量密度**（下文将它简称为**拉氏量**）。$\mathcal{L}$ 是系统中 n 个场 $\phi_a(\mathbf{x},t)$ $(a=1,\cdots,n)$ 及其时空导数 $\partial_\mu\phi_a$ 的函数。现在，作用量可以表达为

$$S=\int dt\,L=\int d^4x\,\mathcal{L}(\phi_a,\partial_\mu\phi_a). \tag{1.112}$$

(1.44) 式告诉我们，时空体积元 d^4x 是 Lorentz 不变的，如果拉氏量 $\mathcal{L}$ 也是 Lorentz 不变的，则作用量 S 就是 Lorentz 不变的，从而，由作用量原理得到的运动方程满足狭义相对性原理。因此，构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 $\mathcal{L}$，即要求 $\mathcal{L}$ 是一个 *Lorentz* 标量。

类似于前面质点力学的处理方式，假设时空坐标的变分为零，则对场的时空导数的变分等于场变分的时空导数，即

$$\delta(\partial_\mu\phi_a)=\partial_\mu(\delta\phi_a). \tag{1.113}$$

于是，利用分部积分可得

$$\begin{aligned}\delta S&=\int d^4x\,\delta\mathcal{L}=\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\right]=\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)\right]\\&=\int d^4x\left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right]-\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a\right\}\\&=\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a+\int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right].\end{aligned} \tag{1.114}$$

上式最后一行第二项的积分项是关于时空坐标的全散度，利用 Stokes 定理，可以将它转化为积分区域边界面 $\mathcal{S}$ 上的积分：

$$\int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right]=\int_{\mathcal{S}}d\mathcal{S}_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a, \tag{1.115}$$

其中 $d\mathcal{S}_\mu$ 是 $\mathcal{S}$ 上的面元。进一步假设在边界面 $\mathcal{S}$ 上 $\delta\phi_a=0$，则上式为零。我们通常讨论整个时空区域上的场，从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零，是很合理的。这样一来，$\delta S=0$ 给出

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}-\frac{\partial\mathcal{L}}{\partial\phi_a}=0. \tag{1.116}$$

这就是*场的 Euler-Lagrange 方程*，它给出场的经典运动方程。

引入场的**共轭动量密度** (conjugate momentum density)

$$\pi_a(\mathbf{x},t)\equiv\frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}, \tag{1.117}$$

则可以用 Legendre 变换将哈密顿量定义为

$$H\equiv\int d^3x\,\pi_a\dot{\phi}_a-L\equiv\int d^3x\,\mathcal{H}, \tag{1.118}$$

其中，**哈密顿量密度**

$$\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a\dot{\phi}_a - \mathcal{L}. \tag{1.119}$$

作用量变分为

$$\begin{aligned}
\delta S &= \int d^4x\,\delta\mathcal{L} = \int d^4x\,\delta(\pi_a\dot{\phi}_a - \mathcal{H}) \\
&= \int d^4x\left[\dot{\phi}_a\delta\pi_a + \pi_a\delta\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\cdot\delta(\nabla\phi_a)\right] \\
&= \int d^4x\left[\dot{\phi}_a\delta\pi_a + \pi_a\frac{d}{dt}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\cdot\nabla(\delta\phi_a)\right] \\
&= \int d^4x\left\{\dot{\phi}_a\delta\pi_a + \frac{d}{dt}(\pi_a\delta\phi_a) - \dot{\pi}_a\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a\right. \\
&\qquad\qquad \left. -\nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right] + \left[\nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\delta\phi_a\right\} \\
&= \int d^4x\left\{\left(\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\right)\delta\pi_a - \left[\dot{\pi}_a + \frac{\partial\mathcal{H}}{\partial\phi_a} - \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\delta\phi_a\right\} \\
&\quad + \int d^4x\,\frac{d}{dt}(\,\pi_a\delta\phi_a) - \int d^4x\,\nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right].
\end{aligned} \tag{1.120}$$

与前面一样，假设在时空区域边界面上 $\delta\phi_a = 0$，则上式最后一行的两项均为零，于是，$\delta S = 0$ 给出场的正则运动方程

$$\dot{\phi}_a = \frac{\partial\mathcal{H}}{\partial\pi_a}, \quad \dot{\pi}_a = -\frac{\partial\mathcal{H}}{\partial\phi_a} + \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}. \tag{1.121}$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述，则它是一种连续变换，连续变换对应的对称性称为连续对称性。**Noether 定理**指出，如果一个系统具有某种不显含时间的连续对称性，就必然存在一种对应的守恒定律。Noether 定理首先是在经典物理中给出的，但实际上它对所有物理行为由作用量原理决定的系统都成立。因此，可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$S = \int_R d^4x\,\mathcal{L}(\phi_a, \partial_\mu\phi_a). \tag{1.122}$$

考虑一个连续变换，使得

$$\phi_a(x) \to \phi'_a(x'), \tag{1.123}$$

其中已包含了坐标的变换

$$x^\mu \to x'^\mu, \tag{1.124}$$

它引起的拉氏量变换为

$$\mathcal{L}(x) \to \mathcal{L}'(x'). \tag{1.125}$$

记这个变换的无穷小变换形式为

$$\phi'_a(x') = \phi_a(x) + \delta\phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}, \tag{1.126}$$

如果在此变换下

$$\delta S = \int_{R'} d^4x'\, \mathcal{L}'(x') - \int_R d^4x\, \mathcal{L}(x) = 0, \tag{1.127}$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4x' = |\mathcal{J}| d^4x, \quad \mathcal{J} = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) \simeq \det\left[\delta^\mu{}_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right], \tag{1.128}$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 $\mathbf{A}$ 满足 $\det(\mathbf{A}) \ll 1$，则有如下表达式：

$$\det(\mathbf{1} + \mathbf{A}) \simeq 1 + \mathrm{tr}(\mathbf{A}). \tag{1.129}$$

利用上式可以将 Jacobi 行列式 $\mathcal{J}$ 化为

$$\mathcal{J} \simeq 1 + \mathrm{tr}\left[\frac{\partial(\delta x^\mu)}{\partial x^\nu}\right] = 1 + \partial_\mu(\delta x^\mu), \tag{1.130}$$

从而，体积元的无穷小变换形式为

$$d^4x' \simeq [1 + \partial_\mu(\delta x^\mu)] d^4x. \tag{1.131}$$

作用量在此无穷小变换下的变分为

$$\begin{aligned}
\delta S &= \int_{R'} d^4x'\, \mathcal{L}'(x') - \int_R d^4x\, \mathcal{L}(x) \\
&= \int_{R'} d^4x'\, \mathcal{L}'(x') - \int_R d^4x\, \mathcal{L}'(x') + \int_R d^4x\, \mathcal{L}'(x') - \int_R d^4x\, \mathcal{L}(x) \\
&\simeq \int_R d^4x [1 + \partial_\mu(\delta x^\mu)]\, \mathcal{L}'(x') - \int_R d^4x\, \mathcal{L}'(x') + \int_R d^4x\, \delta\mathcal{L} \\
&\simeq \int_R d^4x\, \mathcal{L}'(x') \partial_\mu(\delta x^\mu) + \int_R d^4x\, \delta\mathcal{L} \;\simeq \int_R d^4x\, [\delta\mathcal{L} + \mathcal{L}(x)\partial_\mu(\delta x^\mu)] \\
&= \int_R d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) + \mathcal{L}\partial_\mu(\delta x^\mu)\right].
\end{aligned} \tag{1.132}$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$，使得

$$\bar{\delta}\phi_a(x) = \phi'_a(x) - \phi_a(x). \tag{1.133}$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu\phi_a) = \partial_\mu(\bar{\delta}\phi_a), \tag{1.134}$$

δ 算符则不能。$\delta\phi_a$ 与 $\bar{\delta}\phi_a$ 的关系为

$$\begin{aligned}\delta\phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta}\phi_a \\ &\simeq \bar{\delta}\phi_a + (\partial_\mu\phi'_a)\delta x^\mu \simeq \bar{\delta}\phi + (\partial_\mu\phi_a)\delta x^\mu,\end{aligned} \tag{1.135}$$

即

$$\bar{\delta}\phi = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu. \tag{1.136}$$

同理，

$$\delta(\partial_\mu\phi_a) = \bar{\delta}(\partial_\mu\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu = \partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu. \tag{1.137}$$

将 (1.135) 和 (1.137) 式代入 (1.132) 式，得到

$$\begin{aligned}\delta S &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}[\bar{\delta}\phi_a + (\partial_\mu\phi_a)\delta x^\mu] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}[\partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu] + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\bar{\delta}\phi_a + \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\bar{\delta}\phi_a \right. \\ &\qquad \left. + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right] \right. \\ &\qquad \left. + \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu)\right] \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] \right\}.\end{aligned} \tag{1.138}$$

第二步用到分部积分，最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu) = \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu). \tag{1.139}$$

根据 Euler-Lagrange 方程 (1.116)，(1.138) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的，$\delta S = 0$ 等价于第二项为零，即

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] = 0. \tag{1.140}$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu, \tag{1.141}$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \tag{1.142}$$

方程 (1.142) 左边对整个三维空间积分，运用 Stokes 定理，得

$$\int d^3x\, \partial_\mu j^\mu = \int d^3x\, \partial_0 j^0 + \int d^3x\, \partial_i j^i = \frac{d}{dt}\int d^3x\, j^0 + \int_{\mathcal{S}} d\mathcal{S}_i\, j^i, \tag{1.143}$$

其中 $i=1,2,3$。对于整个三维空间而言，边界面 $\mathcal{S}$ 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失，从而，在无穷远处 $j^i\to 0$，所以上式最后一项为零。定义**守恒荷** (conserved charge)

$$Q\equiv\int d^3x\,j^0,\tag{1.144}$$

则由 (1.143) 和 (1.142) 式可得

$$\frac{dQ}{dt}=\frac{d}{dt}\int d^3x\,j^0=\int d^3x\,\partial_\mu j^\mu=0.\tag{1.145}$$

可见，Q 不随时间变化，是守恒的。

综上，在场论中，如果一个系统具有某种连续对称性，则存在相应的守恒流 (1.141)，它满足守恒流方程 (1.142)，而守恒荷 (1.144) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2　时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^\mu=x^\mu-\varepsilon^\mu,\tag{1.146}$$

其中 ε^μ 是常数。要求场 ϕ_a 具有时空平移对称性，则

$$\phi'_a(x')=\phi'_a(x-\varepsilon)=\phi_a(x).\tag{1.147}$$

现在，$\delta x^\mu=-\varepsilon^\mu$，由 (1.136) 式可得

$$\bar{\delta}\phi_a=\delta\phi_a-(\partial_\mu\phi_a)\delta x^\mu=\phi'_a(x')-\phi_a(x)+\varepsilon^\mu\partial_\mu\phi_a=0+\varepsilon^\mu\partial_\mu\phi_a=\varepsilon^\rho\partial_\rho\phi_a,\tag{1.148}$$

代入到 Noether 守恒流表达式 (1.141)，得

$$j^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^\rho\partial_\rho\phi_a-\mathcal{L}\varepsilon^\mu=\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a-\delta^\mu{}_\rho\mathcal{L}\right]\varepsilon^\rho.\tag{1.149}$$

从而，$\partial_\mu j^\mu=0$ 给出

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a-\delta^\mu{}_\rho\mathcal{L}\right]=0,\tag{1.150}$$

各项乘以 $g^{\rho\nu}$，缩并，得

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a-g^{\mu\nu}\mathcal{L}\right]=0.\tag{1.151}$$

上式方括号部分是场的**能动张量** (energy-momentum tensor)

$$T^{\mu\nu}\equiv\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a-g^{\mu\nu}\mathcal{L},\tag{1.152}$$

它满足

$$\partial_\mu T^{\mu\nu}=0.\tag{1.153}$$

因此，对 $T^{0\nu}$ $(\nu = 0, 1, 2, 3)$ 作全空间积分，就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^0 \phi_a - \mathcal{L}, \tag{1.154}$$

与 (1.119) 和 (1.117) 式比较，可以看出 T^{00} 就是哈密顿量密度 $\mathcal{H}$。T^{00} 的全空间积分

$$H = \int d^3x\, T^{00} = \int d^3x\, \mathcal{H} \tag{1.155}$$

是场的哈密顿量，或者说**总能量**。$T^{\mu\nu}$ 的 $0i$ 分量

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^i \phi_a = \pi_a \partial^i \phi_a \tag{1.156}$$

是场的动量密度，它的全空间积分

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \pi_a \partial^i \phi_a \tag{1.157}$$

是场的**总动量**。根据 (1.55) 式，上式也可以写成

$$\mathbf{P} = -\int d^3x\, \pi_a \nabla \phi_a. \tag{1.158}$$

H 和 P^i 都是守恒荷，可见，时间平移对称性对应于**能量守恒定律**，空间平移对称性对应于**动量守恒定律**。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{1.159}$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30)，有

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\mu\nu}(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha)(\delta^\nu{}_\beta + \omega^\nu{}_\beta) \simeq g_{\mu\nu}\delta^\mu{}_\alpha \delta^\nu{}_\beta + g_{\mu\nu}\delta^\mu{}_\alpha \omega^\nu{}_\beta + g_{\mu\nu}\omega^\mu{}_\alpha \delta^\nu{}_\beta \\ &= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{aligned} \tag{1.160}$$

可见，

$$\omega_{\mu\nu} \equiv g_{\mu\rho}\omega^\rho{}_\nu \tag{1.161}$$

关于两个指标反对称：

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{1.162}$$

因此，$\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31)，利用三角函数展开式 $\cos\theta = 1 + \mathcal{O}(\theta^2)$ 和 $\sin\theta = \theta + \mathcal{O}(\theta^3)$，可得

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}. \tag{1.163}$$

对于沿 x 的增速变换 (1.28)，可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1}\beta, \tag{1.164}$$

再利用双曲函数公式 $\tanh\xi = \sinh\xi/\cosh\xi$ 和 $\cosh^2\xi - \sinh^2\xi = 1$ 得

$$\begin{aligned} \gamma &= (1-\beta^2)^{-1/2} = (1-\tanh^2\xi)^{-1/2} = \left(\frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi}\right)^{-1/2} = \cosh\xi, \\ \beta\gamma &= \tanh\xi\cosh\xi = \sinh\xi, \end{aligned} \tag{1.165}$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\xi & -\sinh\xi & & \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.166}$$

根据双曲函数展开式 $\cosh\xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh\xi = \xi + \mathcal{O}(\xi^3)$，有

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \tag{1.167}$$

在无穷小 Lorentz 变换 (1.159) 的作用下，一般地，场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right]\phi_b(x) = \phi_a(x) - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x), \tag{1.168}$$

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的**生成元** (generator)。由于 $\omega_{\mu\nu}$ 是反对称的，有

$$\omega_{\mu\nu}(I^{\mu\nu})_{ab} = \omega_{\nu\mu}(I^{\nu\mu})_{ab} = -\omega_{\mu\nu}(I^{\nu\mu})_{ab}, \tag{1.169}$$

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称：

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}. \tag{1.170}$$

现在，$\delta x^\mu = \omega^\mu{}_\nu x^\nu$，而

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu\phi_a)\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}(I^{\nu\rho})_{ab}\phi_b - (\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho, \tag{1.171}$$

故 Noether 流为

$$\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(I^{\nu\rho})_{ab}\phi_b - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho + \mathcal{L}\omega^\mu{}_\rho x^\rho \\
&= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a) - \delta^\mu{}_\nu\mathcal{L}\right]\omega^\nu{}_\rho x^\rho \\
&= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - T^\mu{}_\nu\omega^\nu{}_\rho x^\rho, \qquad (1.172)
\end{aligned}$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho}g_{\rho\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L} \qquad (1.173)$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作：

$$T^\mu{}_\nu\omega^\nu{}_\rho = T^\mu{}_\nu\delta^\nu{}_\sigma\omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha}g_{\alpha\sigma}\omega^\sigma{}_\rho = T^{\mu\alpha}\omega_{\alpha\rho} = T^{\mu\nu}\omega_{\nu\rho}, \qquad (1.174)$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned}
T^\mu{}_\nu\omega^\nu{}_\rho x^\rho &= T^{\mu\nu}\omega_{\nu\rho}x^\rho = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\nu}\omega_{\rho\nu}x^\rho) = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\rho}\omega_{\nu\rho}x^\nu) \\
&= \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu). \qquad (1.175)
\end{aligned}$$

于是，Noether 流 (1.172) 可化为

$$j^\mu = \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu) = \frac{1}{2}J^{\mu\nu\rho}\omega_{\nu\rho} \qquad (1.176)$$

其中

$$J^{\mu\nu\rho} \equiv T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b. \qquad (1.177)$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu J^{\mu\nu\rho} = 0, \qquad (1.178)$$

守恒荷为

$$J^{\nu\rho} \equiv \int d^3x\, J^{0\nu\rho} = \int d^3x \left[T^{0\rho}x^\nu - T^{0\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b\right]. \qquad (1.179)$$

易见 $J^{\nu\rho} = -J^{\rho\nu}$，因而一共有 6 个独立的守恒荷，满足 $dJ^{\nu\rho}/dt = 0$。

为明确物理含义，可将 $J^{\nu\rho}$ 分解成两项：

$$J^{\nu\rho} = L^{\nu\rho} + S^{\nu\rho}. \qquad (1.180)$$

第一项为

$$\begin{aligned}
L^{\nu\rho} &\equiv \int d^3x\,(T^{0\rho}x^\nu - T^{0\nu}x^\rho) \\
&= \int d^3x \left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\rho\phi_a - g^{0\rho}\mathcal{L}\right)x^\nu - \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\nu\phi_a - g^{0\nu}\mathcal{L}\right)x^\rho\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int d^3x\,[(\pi_a\partial^\rho\phi_a - g^{0\rho}\mathcal{L})x^\nu - (\pi_a\partial^\nu\phi_a - g^{0\nu}\mathcal{L})x^\rho] \\
&= \int d^3x\,[\pi_a(x^\nu\partial^\rho - x^\rho\partial^\nu)\phi_a + (g^{0\nu}x^\rho - g^{0\rho}x^\nu)\mathcal{L}]. \qquad (1.181)
\end{aligned}
$$

它的纯空间分量 L^{jk} 中只有 3 个是独立的，可以等价地定义成

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\,\pi_a(x^j\partial^k - x^k\partial^j)\phi_a, \tag{1.182}$$

这是场的**轨道角动量**。第二项为

$$S^{\nu\rho} \equiv \int d^3x\,\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b = \int d^3x\,\pi_a(-iI^{\nu\rho})_{ab}\phi_b, \tag{1.183}$$

同样，3 个独立的等价纯空间分量是

$$S^i \equiv \frac{1}{2}\varepsilon^{ijk}S^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\,\pi_a(-iI^{jk})_{ab}\phi_b, \tag{1.184}$$

这是场的**自旋角动量**。因此，$J^{\nu\rho}$ 的纯空间分量等价于

$$J^i \equiv \frac{1}{2}\varepsilon^{ijk}J^{jk} = L^i + S^i, \tag{1.185}$$

这是场的**总角动量**。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 SO(3)，而空间旋转对称性对应于**角动量守恒定律**。

另一方面，$L^{\nu\rho}$ 的 $i0$ 分量为

$$L^{i0} = \int d^3x\,(T^{00}x^i - T^{0i}x^0) = \int d^3x\,(x^i\mathcal{H} - x^0\pi_a\partial^i\phi_a) = \int d^3x\,x^i\mathcal{H} - tP^i. \tag{1.186}$$

若 $dS^{i0}/dt = 0$，则有 $dL^{i0}/dt = 0$，从而

$$L^{i0}(t) = L^{i0}\big|_{t=0} = \int d^3x\,x^i\mathcal{H}(t=0), \tag{1.187}$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下，能量密度相当于质量密度，则 L^{i0} 是 $t=0$ 时刻的质心 (即质量中心，center of mass)。L^{i0} 的守恒在经典力学中对应于质心运动守恒定律：当没有外力存在时，质心的加速度为零，质心保持静止或作匀速直线运动。

1.7.4 U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu\phi^*)\partial_\mu\phi - m^2\phi^*\phi. \tag{1.188}$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta}\phi(x), \tag{1.189}$$

其中 θ 是不依赖于 x^μ 的连续变换实参数，q 是一个常数。这里不包含坐标的变换。$e^{iq\theta}$ 是个纯相位因子，可以看成是一个 1 维幺正 (unitary) 矩阵，形式为 $e^{iq\theta}$ 的所有变换组成的群称为 **U(1) 群**。**整体** (global) 指的是变换参数不依赖于时空坐标。相应地，ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta}\phi^*(x). \tag{1.190}$$

容易看出，由 (1.188) 式定义的 $\mathcal{L}$ 在这种变换下不变，即具有 U(1) 整体对称性。与前面叙述的两种对称性不同，这里的对称性出现在由场组成的抽象空间中，与时间和空间相对独立 ($\delta x^\mu = 0$)，因而是一种**内部对称性**。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \tag{1.191}$$

结合 $\delta x^\mu = 0$，有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \tag{1.192}$$

于是，Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\bar{\delta}\phi^* = \partial^\mu\phi^*(iq\theta\phi) + \partial^\mu\phi(-iq\theta\phi^*) \\ &= iq\theta[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*] = -q\theta\phi^* i\overleftrightarrow{\partial^\mu}\phi, \end{aligned} \tag{1.193}$$

其中，$\overleftrightarrow{\partial^\mu}$ 符号通过下式定义：

$$\phi^*\overleftrightarrow{\partial^\mu}\phi \equiv \phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi. \tag{1.194}$$

扔掉无穷小参数 $-\theta$，定义

$$J^\mu \equiv q\phi^* i\overleftrightarrow{\partial^\mu}\phi, \tag{1.195}$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$，相应的守恒荷为

$$Q = \int d^3x\, J^0 = q\int d^3x\,\phi^* i\overleftrightarrow{\partial^0}\phi. \tag{1.196}$$

在实际情况中，q 是由 ϕ 场描述的粒子所携带的某种荷，如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此，一种 U(1) 整体对称性对应于一条荷数守恒定律，比如，电磁 U(1) 整体对称性就对应于**电荷守恒定律**。

第 2 章　标量场

本章讲述标量场的**正则量子化** (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1　简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{2.1}$$

其中 m 是质量，ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 看作厄米算符，满足**正则对易关系**

$$[x, p] = xp - px = i. \tag{2.2}$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \tag{2.3}$$

a 称为**湮灭算符** (annihilation operator)，$a^\dagger$ 称为**产生算符** (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\ &= \frac{1}{2}(-i[x,p] + i[p,x]) = -i[x,p] = 1. \end{aligned} \tag{2.4}$$

根据 (2.3) 式，可以反过来用 a 和 $a^\dagger$ 表示 x 和 p：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \tag{2.5}$$

从而，哈密顿量表示成

$$\begin{aligned} H &= -\frac{1}{2m}\frac{m\omega}{2}(a - a^\dagger)^2 + \frac{1}{2}m\omega^2\frac{1}{2m\omega}(a + a^\dagger)^2 \\ &= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a). \end{aligned} \tag{2.6}$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$，于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right), \tag{2.7}$$

其中，$N \equiv a^\dagger a$ 是个厄米算符，称为**粒子数算符**。N 还是个**正定**算符，对于任意量子态 $|\psi\rangle$，N 的期待值 (expectation value) 非负：

$$\langle\psi| N |\psi\rangle = \langle\psi| a^\dagger a |\psi\rangle = \langle a\psi|a\psi\rangle \geq 0. \tag{2.8}$$

设 $|n\rangle$ 是 N 的本征态，归一化为 $\langle n|n\rangle = 1$。它满足本征方程

$$N|n\rangle = n|n\rangle. \tag{2.9}$$

由 $n = \langle n| n |n\rangle = \langle n| N |n\rangle \geq 0$ 可知，本征值 n 是个非负实数。利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B, \tag{2.10}$$
$$[A, BC] = ABC - BAC + BAC - BCA = [A, B]C + B[A, C], \tag{2.11}$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \tag{2.12}$$

从而，有

$$Na^\dagger|n\rangle = ([N, a^\dagger] + a^\dagger N)|n\rangle = (a^\dagger + a^\dagger n)|n\rangle = (n+1)a^\dagger|n\rangle, \tag{2.13}$$
$$Na|n\rangle = ([N, a] + aN)|n\rangle = (-a + an)|n\rangle = (n-1)a|n\rangle. \tag{2.14}$$

可见，$a^\dagger|n\rangle$ 和 $a|n\rangle$ 都是 N 的本征态，本征值分别为 $n+1$ 和 $n-1$，也就是说，

$$a^\dagger|n\rangle = c_1|n+1\rangle, \quad a|n\rangle = c_2|n-1\rangle, \tag{2.15}$$

其中 c_1 和 c_2 是两个归一化常数。$a^\dagger$ 将本征值为 n 的态变成本征值为 $n+1$ 的态，因而也称为升算符 (raising operator)；a 将本征值为 n 的态变成本征值为 $n-1$ 的态，因而也称为降算符 (lowering operator)。为确定归一化常数的值，可作如下计算：

$$n+1 = \langle n|(N+1)|n\rangle = \langle n|(a^\dagger a+1)|n\rangle = \langle n|aa^\dagger|n\rangle = |c_1|^2\langle n+1|n+1\rangle = |c_1|^2, \tag{2.16}$$
$$n = \langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = |c_2|^2\langle n-1|n-1\rangle = |c_2|^2. \tag{2.17}$$

将 c_1 和 c_2 都取为实数，则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$，故

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle. \tag{2.18}$$

从 N 的某个本征态 $|n\rangle$ 出发，用降算符 a 逐步操作，可得本征值逐次减小的一系列本征态

$$a|n\rangle, \ a^2|n\rangle, \ a^3|n\rangle, \ \cdots, \tag{2.19}$$

本征值分别为

$$n-1,\ n-2,\ n-3,\ \cdots \tag{2.20}$$

由于 $n\geq 0$，必定存在一个最小本征值 n_0，它的本征态 $|n_0\rangle$ 满足

$$a\,|n_0\rangle = 0. \tag{2.21}$$

于是，有

$$N\,|n_0\rangle = a^\dagger a\,|n_0\rangle = 0 = 0\,|n_0\rangle\,, \tag{2.22}$$

可见，$n_0 = 0$，即

$$|n_0\rangle = |0\rangle\,. \tag{2.23}$$

反过来，从 $|0\rangle$ 出发，用升算符 $a^\dagger$ 逐步操作，可得本征值逐次增加的一系列本征态

$$a^\dagger\,|0\rangle\,,\ (a^\dagger)^2\,|0\rangle\,,\ (a^\dagger)^3\,|0\rangle\,,\ \cdots, \tag{2.24}$$

本征值分别为

$$1,\ 2,\ 3,\ \cdots \tag{2.25}$$

综上，本征值 n 的取值是非负整数，是量子化的；本征态 $|n\rangle$ 可以用 $a^\dagger$ 和 $|0\rangle$ 表示为

$$|n\rangle = c_3(a^\dagger)^n\,|0\rangle\,. \tag{2.26}$$

为确定归一化常数 c_3，可作如下运算：

$$\begin{aligned}\langle n|n\rangle &= |c_3|^2\,\langle 0|\,a^n(a^\dagger)^n\,|0\rangle = |c_3|^2\,\langle 1|\,a^{n-1}(a^\dagger)^{n-1}\,|1\rangle = 1\cdot 2\;|c_3|^2\,\langle 2|\,a^{n-2}(a^\dagger)^{n-2}\,|2\rangle = \cdots\\ &= (n-1)!\;|c_3|^2\,\langle n-1|\,aa^\dagger\,|n-1\rangle = n!\;|c_3|^2\,\langle n|n\rangle\,,\end{aligned} \tag{2.27}$$

故 $|c_3|^2 = 1/n!$。取 c_3 为实数，可得 $c_3 = 1/\sqrt{n!}$，于是

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle\,. \tag{2.28}$$

从 (2.7) 式容易看出，$|n\rangle$ 也是 H 的本征态：

$$H\,|n\rangle = \omega\left(N+\frac{1}{2}\right)|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle = E_n\,|n\rangle\,, \tag{2.29}$$

相应的能量本征值为

$$E_n = \omega\left(n+\frac{1}{2}\right). \tag{2.30}$$

基态 $|0\rangle$ 的能量本征值不是零，而是 $E_0 = \omega/2$，称为零点能 (zero-point energy)，这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态，将 $n>0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态，每个声子具有一份能量 ω。这样一来，n 表示声子的数目，故粒子数算符 N 描述的是声子数。$a^\dagger$ 的作用是产生一个声子，从而增加一份能量；a 的作用是湮灭一个声子，从而减少一份能量。这是将 $a^\dagger$ 和 a 称为产生算符和湮灭算符的原因。

2.2　量子场论中的正则对易关系

在量子力学中，Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法，它们之间可以通过一个幺正变换联系起来。在 Schrödinger 绘景中，波函数 $\psi_{\mathrm{S}}(t)$ 代表随时间演化的物理态，而任意算符 O_{S} 不依赖于时间。在 Heisenberg 绘景中，波函数 ψ_{H} 代表不随时间演化的物理态，它与 $\psi_{\mathrm{S}}(t)$ 的关系为

$$\psi_{\mathrm{H}} = e^{iHt}\psi_{\mathrm{S}}(t), \tag{2.31}$$

其中 H 是哈密顿量；而算符 $O_{\mathrm{H}}(t)$ 依赖于时间，通过下式与 O_{S} 联系起来：

$$O_{\mathrm{H}}(t) = e^{iHt}O_{\mathrm{S}}e^{-iHt}. \tag{2.32}$$

上一节的量子化可以认为是在 Schrödinger 绘景中实现的，因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x_{\mathrm{S}}, p_{\mathrm{S}}] = i$，它在 Heisenberg 绘景中的形式为

$$\begin{aligned}[x_{\mathrm{H}}(t), p_{\mathrm{H}}(t)] &= [e^{iHt}x_{\mathrm{S}}e^{-iHt}, e^{iHt}p_{\mathrm{S}}e^{-iHt}] = e^{iHt}x_{\mathrm{S}}e^{-iHt}e^{iHt}p_{\mathrm{S}}e^{-iHt} - e^{iHt}p_{\mathrm{S}}e^{-iHt}e^{iHt}x_{\mathrm{S}}e^{-iHt}\\ &= e^{iHt}x_{\mathrm{S}}p_{\mathrm{S}}e^{-iHt} - e^{iHt}p_{\mathrm{S}}x_{\mathrm{S}}e^{-iHt} = e^{iHt}[x_{\mathrm{S}}, p_{\mathrm{S}}]e^{-iHt} = e^{iHt}ie^{-iHt} = i.\end{aligned} \tag{2.33}$$

可见，正则对易关系的形式不依赖于绘景。(2.33) 式是在同一时刻 t 成立的，称为**等时** (equal time) 对易关系。

将讨论推广到自由度为 n 的系统，记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符，$p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响，这些算符需要满足如下等时对易关系：

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [q_i(t), q_j(t)] = 0, \quad [p_i(t), p_j(t)] = 0. \tag{2.34}$$

1.1 节提到，在量子场论中，为了平等地处理时间和空间，空间坐标 $\mathbf{x}$ 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x}, t)$ 的参数。由于这里量子场作为算符是依赖于时间的，使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行，省略绘景的标志性下标 H。

场论讨论的是无穷多自由度的系统，每一个空间点 $\mathbf{x}$ 上的 $\phi(\mathbf{x}, t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度，我们可以先将空间离散化，划分成 n 个小体积元 V_i，然后再取 $V_i \to 0$ 的极限来得到 $n \to \infty$ 的结果。在体积元 V_i 中，定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i}\int_{V_i} d^3x\, \phi(\mathbf{x}, t), \tag{2.35}$$

它是场 $\phi(\mathbf{x}, t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi, \partial_\mu\phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i}\int_{V_i} d^3x\, \mathcal{L}(\phi, \partial_\mu\phi), \tag{2.36}$$

当体积元取得足够小时，它就成为 ϕ_i 和 $\partial_0\phi_i$ 的函数 $\mathcal{L}_i(\phi_i, \partial_0\phi_i)$。拉格朗日量可表达为

$$L = \int d^3x\, \mathcal{L} = \sum_i \int_{V_i} d^3x\, \mathcal{L} = \sum_i V_i \frac{1}{V_i}\int_{V_i} d^3x\, \mathcal{L} = \sum_i V_i\, \mathcal{L}_i(\phi_i, \partial_0\phi_i). \tag{2.37}$$

于是，由 (1.108) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \delta_{ji} \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} = V_i \pi_i(t), \quad (2.38)$$

其中，

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]}. \quad (2.39)$$

现在，等时对易关系变成

$$[\phi_i(t), \Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t), \phi_j(t)] = 0, \quad [\Pi_i(t), \Pi_j(t)] = 0. \quad (2.40)$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t), \pi_j(t)] = 0. \quad (2.41)$$

对于任意连续函数 $f(x)$，**Dirac δ 函数** $\delta(x)$ 使下式成立：

$$f(x) = \int dy\, f(y)\delta(x-y). \quad (2.42)$$

函数 $\delta(x)$ 只在 $x = 0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即

$$\delta(x) = \delta(-x), \quad (2.43)$$

而且满足

$$\int dx\, \delta(x) = 1. \quad (2.44)$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \quad (2.45)$$

则对于任意连续函数 $f(\mathbf{x})$，下式成立：

$$f(\mathbf{x}) = \int d^3y\, f(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.46)$$

类似地，函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x} = 0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$，而且满足 $\int d^3x\, \delta^{(3)}(\mathbf{x}) = 1$。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值，则它会满足

$$f_i = \sum_j f_j\, \delta_{ij} = \sum_j V_j\, f_j\, \frac{\delta_{ij}}{V_j}. \quad (2.47)$$

(2.46) 式是 (2.47) 式在 $V_i \to 0$ 时的极限。可见，在 $V_i \to 0$ 极限下，

$$\frac{\delta_{ij}}{V_j} \to \delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.48)$$

另一方面，在此极限下，$\phi_i(t) \to \phi(\mathbf{x}, t)$，而 $\pi_i(t)$ 变成由 (1.117) 式定义的共轭动量密度：

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial[\partial_0 \phi_i(t)]} \to \frac{\partial \mathcal{L}}{\partial[\partial_0 \phi(\mathbf{x}, t)]} = \pi(\mathbf{x}, t). \tag{2.49}$$

因此，等时对易关系化为

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \tag{2.50}$$

推广到包含若干个场 ϕ_a 的系统，假设不同的场不会相互影响，则有

$$[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0, \quad [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \tag{2.51}$$

这就是**量子场论中的正则对易关系**。此时，$\phi_a(\mathbf{x}, t)$ 和 $\pi_a(\mathbf{x}, t)$ 都是算符。

2.3　实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量，就称它为**标量场**。在固有保时向 Lorentz 变换下，若时空坐标的变换为 $x' = \Lambda x$，则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \tag{2.52}$$

在本节中，我们讨论**实**标量场 $\phi(x)$，它满足*自共轭* (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \tag{2.53}$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的*自由*实标量场，相应的 **Lorentz 不变拉氏量**可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m^2\phi^2. \tag{2.54}$$

注意到

$$\frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu \phi)\partial_\nu \phi = \frac{1}{2}[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2], \tag{2.55}$$

可得

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} = -\partial_i \phi = \partial^i \phi, \tag{2.56}$$

归纳起来，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi. \tag{2.57}$$

因此，Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi, \tag{2.58}$$

也就是说，$\phi(x)$ 满足 *Klein-Gordon 方程*

$$(\partial^2 + m^2)\phi(x) = 0. \tag{2.59}$$

2.3.1 平面波展开

设 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik \cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \tag{2.60}$$

则有

$$\partial^2\varphi = \partial^\mu\partial_\mu\varphi = \partial^\mu(-ik_\mu\varphi) = -ik_\mu\partial^\mu\varphi = (-i)^2 k_\mu k^\mu\varphi = -k^2\varphi, \tag{2.61}$$

从而，

$$0 = (\partial^2 + m^2)\varphi = -(k^2 - m^2)\varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2]\varphi. \tag{2.62}$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$，即 $k^0 = \pm E_\mathbf{k}$，其中 $E_\mathbf{k} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$。因此，有两种平面波解。

(1) $k^0 = E_\mathbf{k}$ 对应于正能解

$$\varphi_\mathbf{k}^{(+)}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})]. \tag{2.63}$$

(2) $k^0 = -E_\mathbf{k}$ 对应于负能解

$$\varphi_\mathbf{k}^{(-)}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[i(E_\mathbf{k}t + \mathbf{k}\cdot\mathbf{x})]. \tag{2.64}$$

从而，场算符 $\phi(\mathbf{x}, t)$ 的通解可以写成如下形式：

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}\varphi_\mathbf{k}^{(+)}(x) + \tilde{a}_\mathbf{k}\varphi_\mathbf{k}^{(-)}(x)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_\mathbf{k}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right],\end{aligned} \tag{2.65}$$

其中 $a_\mathbf{k}$ 和 $\tilde{a}_\mathbf{k}$ 是两个只依赖于 $\mathbf{k}$ 的算符。这是一种 Fourier 变换，把 $\phi(\mathbf{x}, t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭，得

$$\begin{aligned}\phi^\dagger(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_\mathbf{k}^\dagger e^{-i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_{-\mathbf{k}}^\dagger e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.66}$$

第二步利用了如下性质：对整个三维动量空间进行积分时，将积分项中的 $\mathbf{k}$ 换成 $-\mathbf{k}$ 不会改变积分的结果。于是，由自共轭条件 $\phi^\dagger(\mathbf{x}, t) = \phi(\mathbf{x}, t)$ 可得

$$\tilde{a}_\mathbf{k} = a_{-\mathbf{k}}^\dagger. \tag{2.67}$$

（注意：由上式可以推出 $\tilde{a}_\mathbf{k}^\dagger = a_{-\mathbf{k}}$ 和 $\tilde{a}_{-\mathbf{k}}^\dagger = a_\mathbf{k}$。）因而，有

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + a_{-\mathbf{k}}^\dagger e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \right]. \tag{2.68}$$

替换一下动量记号，可以把 $\phi(\mathbf{x},t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right), \tag{2.69}$$

其中，p^0 是正的，满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}, \tag{2.70}$$

而 $a_{\mathbf{p}}$ 是湮灭算符，$a_{\mathbf{p}}^\dagger$ 是产生算符。$\phi(\mathbf{x},t)$ 对应的共轭动量密度算符为

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right). \tag{2.71}$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = 0, \quad [\pi(\mathbf{x},t), \pi(\mathbf{y},t)] = 0. \tag{2.72}$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} = \int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{p}), \tag{2.73}$$

可得

$$\begin{aligned} \int d^3x\, e^{iq\cdot x}\phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \left[a_{\mathbf{p}} e^{-i(p-q)\cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x} \right] \\ &= \int d^3p\, \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0t} \right), \end{aligned} \tag{2.74}$$

以及

$$\begin{aligned} \int d^3x\, e^{iq\cdot x}\partial_0\phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \left[a_{\mathbf{p}} e^{-i(p-q)\cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x} \right] \\ &= \int d^3p\, \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0t} \right). \end{aligned} \tag{2.75}$$

从而，有

$$-i\sqrt{2E_{\mathbf{q}}}\, a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x\, e^{iq\cdot x}\partial_0\phi - iq_0 \int d^3x\, e^{iq\cdot x}\phi = \int d^3x\, e^{iq\cdot x}(\partial_0\phi - iq_0\phi), \tag{2.76}$$

亦即

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \left[\partial_0\phi(x) - ip_0\phi(x)\right]. \tag{2.77}$$

上式取厄米共轭，并使用自共轭条件 $\phi^\dagger = \phi$，得

$$a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \left[\partial_0\phi(x) + ip_0\phi(x)\right]. \tag{2.78}$$

利用上面两个表达式和等时对易关系 (2.72)，可得

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left(iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{2.79}$$

在以上计算过程中，$x^0 = y^0 = t$。上式最后一行中的 $\delta^{(3)}(\mathbf{p}-\mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = \mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$，则 $E_{\mathbf{p}} + E_{\mathbf{q}} = 2E_{\mathbf{p}} = \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}$，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{2.80}$$

类似地，

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t) - iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) - iq_0\phi(\mathbf{y},t)\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left(-iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right) \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} (2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}).
\end{aligned} \tag{2.81}$$

最后一行中的 $\delta^{(3)}(\mathbf{p}+\mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = -\mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \tag{2.82}$$

此外，

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}}a_{\mathbf{p}} - a_{\mathbf{p}}a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \tag{2.83}$$

综上，产生湮灭算符满足如下对易关系：

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \tag{2.84}$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.119)，实标量场的哈密顿量密度为

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = (\partial_0\phi)^2 - \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi + \frac{1}{2}m^2\phi^2 = \frac{1}{2}[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2]. \tag{2.85}$$

对全空间积分以得到哈密顿量：

$$\begin{aligned}
H =& \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\,[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2] \\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\left(-ip_0a_{\mathbf{p}}e^{-ip\cdot x} + ip_0a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(-iq_0a_{\mathbf{q}}e^{-iq\cdot x} + iq_0a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&+ \left(i\mathbf{p}\,a_{\mathbf{p}}e^{-ip\cdot x} - i\mathbf{p}\,a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\,a_{\mathbf{q}}e^{-iq\cdot x} - i\mathbf{q}\,a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&+ m^2\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}}e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\Big] \\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p-q)\cdot x} \\
&+(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p+q)\cdot x} + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
=& \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right] \\
&+(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right]\Big\} \\
=& \frac{1}{2}\int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}\right] \\
&+\delta^{(3)}(\mathbf{p}+\mathbf{q})(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}\right]\Big\} \\
=& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \Big[(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&+(-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\Big].
\end{aligned} \tag{2.86}$$

由 (2.70) 式可得 $-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 = 0$，故上式最后两行方括号中第二项没有贡献。从而，

$$\begin{aligned}
H =& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2)\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3\,2E_{\mathbf{p}}}2E_{\mathbf{p}}^2\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
=& \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p})\right]
\end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2}, \tag{2.87}$$

其中第四步用到对易关系 (2.84)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。$a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 $\mathbf{p}$ 的模式对应的**粒子数密度算符**（动量空间中的密度），相应的能量是 $E_{\mathbf{p}}$。在 (2.87) 式最后一行中，第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.73) 式可得

$$(2\pi)^3\delta^{(3)}(0) = \int d^3x = V, \tag{2.88}$$

其中 V 是进行积分的空间体积，对于全空间而言是无穷大的。因此，(2.87) 式最后一行的第二项是一个无穷大 c 数，是真空的零点能，是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明，一维简谐振子的零点能为 $E_0 = \omega/2$。这是自由度为 1 时的结果，推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象，这个零点能通常并不重要，因为实验上只能测量两个能量之差。经过正则量子化之后，实标量场的哈密顿量 H 是正定的，不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q}-\mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \tag{2.89}$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \tag{2.90}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则

$$H\,|E\rangle = E\,|E\rangle\,. \tag{2.91}$$

从而，有

$$Ha_{\mathbf{p}}^\dagger\,|E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger)\,|E\rangle = (E+E_{\mathbf{p}})a_{\mathbf{p}}^\dagger\,|E\rangle\,. \tag{2.92}$$

$$Ha_{\mathbf{p}}\,|E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}})\,|E\rangle = (E-E_{\mathbf{p}})a_{\mathbf{p}}\,|E\rangle\,. \tag{2.93}$$

可见，当 $a_{\mathbf{p}}^\dagger\,|E\rangle \neq 0$ 时，产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{p}}$；当 $a_{\mathbf{p}}\,|E\rangle \neq 0$ 时，湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{p}}$。

根据 (1.158) 式，实标量场的总动量是

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi\nabla\phi = -\int d^3x (\partial_0\phi)\nabla\phi \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \left(-ip_0 a_{\mathbf{p}} e^{-ip\cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(i\mathbf{q}\, a_{\mathbf{q}} e^{-iq\cdot x} - i\mathbf{q}\, a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[-p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} - p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ - p_0\mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right] \\
&\qquad\qquad + p_0\mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right] \Big\} \\
&= -\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ - p_0\mathbf{q}\, \delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t}\right] \\
&\qquad\qquad + p_0\mathbf{q}\, \delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p^0+q^0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p^0+q^0)t}\right] \Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (-E_{\mathbf{p}}\mathbf{p}) \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p} \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right).
\end{aligned} \tag{2.94}
$$

先作 $\mathbf{p} \to -\mathbf{p}$ 的替换，再利用对易关系 (2.84)，可得

$$
\begin{aligned}
&\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} (-\mathbf{p}) \left(a_{-\mathbf{p}} a_{\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right) \\
&\quad = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right).
\end{aligned} \tag{2.95}
$$

可见，(2.94) 式最后一行圆括号中最后两项没有贡献。从而，

$$
\begin{aligned}
\mathbf{P} &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p} \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \mathbf{p} \left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0)\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\, \mathbf{p}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}\delta^{(3)}(0) \int d^3p\, \mathbf{p}.
\end{aligned} \tag{2.96}
$$

由于 $\int d^3p\, \mathbf{p} = \int d^3p\, (-\mathbf{p}) = -\int d^3p\, \mathbf{p}$，上式最后一行第二项没有贡献。于是，

$$
\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\, \mathbf{p}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \tag{2.97}
$$

即总动量是所有动量模式所有粒子贡献的动量之和。

$\mathbf{P}$ 与产生湮灭算符的对易子为

$$
[\mathbf{P}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\, \mathbf{q}\, [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\, \mathbf{q}\, a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\, \mathbf{q}\, a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q}-\mathbf{p}) = \mathbf{p}\, a_{\mathbf{p}}^\dagger, \tag{2.98}
$$

$$
[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\, \mathbf{q}\, [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\, \mathbf{q}\, [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int d^3q\, \mathbf{q}\, a_{\mathbf{q}} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -\mathbf{p}\, a_{\mathbf{p}}. \tag{2.99}
$$

2.3.4　粒子态

真空态 $|0\rangle$ 是能量最低的态，对于任意动量 $\mathbf{p}$ 对应的湮灭算符 $a_{\mathbf{p}}$，满足

$$
a_{\mathbf{p}}\, |0\rangle = 0, \tag{2.100}
$$

归一化为

$$\langle 0|0\rangle = 1. \tag{2.101}$$

由哈密顿量的表达式 (2.87)可得

$$H\,|0\rangle = E_{\text{vac}}\,|0\rangle\,, \quad E_{\text{vac}} = \delta^{(3)}(0)\int d^3p\,\frac{E_{\mathbf{p}}}{2}, \tag{2.102}$$

可见，这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外，由 (2.97) 式可知，$|0\rangle$ 的总动量本征值是零：

$$\mathbf{P}\,|0\rangle = 0\,|0\rangle\,, \tag{2.103}$$

即真空态不具有动量。

接着，定义动量为 $\mathbf{p}$ 的**单粒子态**为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,a_{\mathbf{p}}^\dagger\,|0\rangle\,. \tag{2.104}$$

从而，利用 (2.89) 和 (2.98) 式可得

$$H\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,Ha_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}}a_{\mathbf{p}}^\dagger)\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(E_{\text{vac}} + E_{\mathbf{p}})a_{\mathbf{p}}^\dagger\,|0\rangle = (E_{\text{vac}} + E_{\mathbf{p}})\,|\mathbf{p}\rangle\,, \tag{2.105}$$

$$\mathbf{P}\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{P}\,a_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^\dagger\,\mathbf{P} + \mathbf{p}\,a_{\mathbf{p}}^\dagger)\,|0\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{p}\,a_{\mathbf{p}}^\dagger\,|0\rangle = \mathbf{p}\,|\mathbf{p}\rangle\,. \tag{2.106}$$

可以看出，相比于真空态 $|0\rangle$，单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$，也多了一份动量 $\mathbf{p}$。因此，$|\mathbf{p}\rangle$ 描述的是一个动量为 $\mathbf{p}$ 的粒子，这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$，满足狭义相对论中的能量–动量关系 (1.52)，而拉氏量 (2.54) 中的参数 m 就是粒子的**质量**。可以看出，产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是产生一个动量为 $\mathbf{p}$ 的粒子。

此外，可作如下计算：

$$a_{\mathbf{p}}\,|\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}}\,a_{\mathbf{p}}a_{\mathbf{q}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{q}}}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle\,. \tag{2.107}$$

如果 $\mathbf{p} \neq \mathbf{q}$，则上式为零；如果 $\mathbf{p} = \mathbf{q}$，则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$。可见，湮灭算符 $a_{\mathbf{p}}$ 的作用是减少一个动量为 $\mathbf{p}$ 的粒子。

单粒子态的内积关系为

$$\begin{aligned}\langle \mathbf{q}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\,a_{\mathbf{q}}a_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\,[a_{\mathbf{p}}^\dagger a_{\mathbf{q}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle \\ &= 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{2.108}$$

上式是 *Lorentz* 不变的，这是 (2.104) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明 若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i，则如下等式成立：

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \tag{2.109}$$

引入**阶跃函数** (step function)

$$\theta(x)=\begin{cases}1, & x\geq 0,\\ 0, & x<0,\end{cases}\tag{2.110}$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^μ 满足质壳条件 $p^2-m^2=0$ 且能量为正 $(p^0>0)$ 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned}&\int d^4p\,\delta(p^2-m^2)\theta(p^0)F(p)=\int d^3p\,dp^0\,\delta\left((p^0)^2-|\mathbf{p}|^2-m^2\right)\theta(p^0)F(p^0,\mathbf{p})\\ =&\int d^3p\,\frac{1}{2\sqrt{|\mathbf{p}|^2+m^2}}\,F\left(\sqrt{|\mathbf{p}|^2+m^2},\mathbf{p}\right)=\int\frac{d^3p}{2E_{\mathbf{p}}}\,F\left(\sqrt{|\mathbf{p}|^2+m^2},\mathbf{p}\right).\end{aligned}\tag{2.111}$$

这里第二步用到 (2.109) 式。可见，

$$\frac{d^3p}{2E_{\mathbf{p}}}\tag{2.112}$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$，按照 δ 函数定义，有

$$g(\mathbf{q})=\int d^3p\,\delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p})=\int\frac{d^3p}{2E_{\mathbf{p}}}\,2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p}).\tag{2.113}$$

由于上式最左边和最右边都是 Lorentz 不变的，

$$2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})\tag{2.114}$$

必定是 Lorentz 不变的。**证毕**。

进一步，可以定义动量分别为 $\mathbf{p}_1,\cdots,\mathbf{p}_n$ 的 n 个粒子对应的**多粒子态**为

$$|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\equiv\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle\,.\tag{2.115}$$

H 对它的作用给出

$$\begin{aligned}&H\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,Ha^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle\\ =&\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,(a^\dagger_{\mathbf{p}_1}H+E_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_1})\cdots a^\dagger_{\mathbf{p}_n}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ =&\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}Ha^\dagger_{\mathbf{p}_2}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle+E_{\mathbf{p}_1}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ =&\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}H\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle+(E_{\mathbf{p}_1}+E_{\mathbf{p}_2})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ =&\cdots=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}\cdots a^\dagger_{\mathbf{p}_n}H\,|0\rangle+(E_{\mathbf{p}_1}+E_{\mathbf{p}_2}+\cdots+E_{\mathbf{p}_n})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ =&(E_{\text{vac}}+E_{\mathbf{p}_1}+E_{\mathbf{p}_2}+\cdots+E_{\mathbf{p}_n})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,,\end{aligned}\tag{2.116}$$

同理，$\mathbf{P}$ 对它的作用给出

$$\mathbf{P}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=(\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n)\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,.\tag{2.117}$$

也就是说，多粒子态 $|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.84) 可得

$$\begin{aligned}|\mathbf{p}_1,\cdots,\mathbf{p}_i,\cdots,\mathbf{p}_j,\cdots,\mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_i}\cdots a^\dagger_{\mathbf{p}_j}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle\\ &= \sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_j}\cdots a^\dagger_{\mathbf{p}_i}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle\\ &= |\mathbf{p}_1,\cdots,\mathbf{p}_j,\cdots,\mathbf{p}_i,\cdots,\mathbf{p}_n\rangle\,. \end{aligned}\tag{2.118}$$

可以看出，对调多粒子态中的任意两个粒子，得到的态相同，即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是**玻色子** (boson)，服从 Bose-Einstein 统计。得到这个结论的关键在于两个产生算符相互对易。

双粒子态的内积关系为

$$\begin{aligned}&\langle\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2\rangle = \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\langle 0|\,a_{\mathbf{q}_2}a_{\mathbf{q}_1}a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}|0\rangle\\ =&\ \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_2}|0\rangle+\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_1}a_{\mathbf{q}_1}a^\dagger_{\mathbf{p}_2}|0\rangle\right]\\ =&\ \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[(2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_2}|0\rangle+(2\pi)^3\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_1}|0\rangle\right]\\ =&\ \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[(2\pi)^6\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)+(2\pi)^6\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\right]\\ =&\ 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\left[\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)+\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\right].\end{aligned}\tag{2.119}$$

此外，还可以定义动量均为 $\mathbf{p}$ 的 n 个粒子对应的多粒子态为

$$|n_\mathbf{p}\rangle \equiv (2E_\mathbf{p})^{n_\mathbf{p}/2}\left(a^\dagger_\mathbf{p}\right)^{n_\mathbf{p}}|0\rangle\,,\tag{2.120}$$

则粒子数密度算符

$$N_\mathbf{p}\equiv a^\dagger_\mathbf{p}a_\mathbf{p}\tag{2.121}$$

对它的作用为

$$\begin{aligned}N_\mathbf{p}|n_\mathbf{q}\rangle &= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}|0\rangle = (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left[a^\dagger_\mathbf{q}a_\mathbf{p}+(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\\ &= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}a^\dagger_\mathbf{q}a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle+(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\\ &= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^2a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-2}|0\rangle+2(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\\ &= \cdots = (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}a_\mathbf{p}|0\rangle+n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\\ &= n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\,.\end{aligned}\tag{2.122}$$

在动量空间对粒子数密度算符进行积分，得到的是**粒子数算符**

$$N\equiv\int\frac{d^3p}{(2\pi)^3}N_\mathbf{p}=\int\frac{d^3p}{(2\pi)^3}a^\dagger_\mathbf{p}a_\mathbf{p}.\tag{2.123}$$

由 (2.122) 式，可得

$$\begin{aligned}N|n_\mathbf{q}\rangle &= \int\frac{d^3p}{(2\pi)^3}N_\mathbf{p}|n_\mathbf{q}\rangle=\int\frac{d^3p}{(2\pi)^3}n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\\ &= n_\mathbf{q}(2E_\mathbf{q})^{n_\mathbf{q}/2}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}|0\rangle=n_\mathbf{q}|n_\mathbf{q}\rangle\,.\end{aligned}\tag{2.124}$$

因此，$|n_{\mathbf{q}}\rangle$ 是 N 的本征态，本征值为粒子数 $n_{\mathbf{q}}$。

更一般地，可以定义多粒子态

$$|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle \tag{2.125}$$

来描述动量为 $\mathbf{p}_1,\cdots,\mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}$ 个的情况。此时，有

$$\begin{aligned}
&N|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\,a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_1}\right)^{n_{\mathbf{p}_1}}\cdots\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]\Big[a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_1}\right)^{n_{\mathbf{p}1}}a_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_2}\right)^{n_{\mathbf{p}2}}\cdots\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle\\
&\qquad+n_{\mathbf{p}_1}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}_1)a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_1}\right)^{n_{\mathbf{p}_1}-1}\left(a^\dagger_{\mathbf{p}_2}\right)^{n_{\mathbf{p}2}}\cdots\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle\Big]\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]\left[a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_1}\right)^{n_{\mathbf{p}1}}a_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_2}\right)^{n_{\mathbf{p}2}}\cdots\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}|0\rangle\right]+n_{\mathbf{p}_1}|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&\cdots=\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]\left[a^\dagger_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}_1}\right)^{n_{\mathbf{p}_1}}\cdots\left(a^\dagger_{\mathbf{p}_i}\right)^{n_{\mathbf{p}_i}}a_{\mathbf{p}}|0\rangle\right]\\
&\quad+(n_{\mathbf{p}_1}+\cdots+n_{\mathbf{p}_m})|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&(n_{\mathbf{p}_1}+\cdots+n_{\mathbf{p}_m})|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle.
\end{aligned}\tag{2.126}$$

可见，N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中，我们讨论复标量场 $\phi(x)$，它不满足自共轭条件 (2.53)，即

$$\phi^\dagger(x)\neq\phi(x). \tag{2.127}$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.188) 式的形式。不过，由于 $\phi(x)$ 是量子场算符，需要把那里的复共轭记号 $*$ 改成厄米共轭记号 $\dagger$，故 **Lorentz** 不变拉氏量为

$$\mathcal{L}=(\partial^\mu\phi^\dagger)\partial_\mu\phi-m^2\phi^\dagger\phi. \tag{2.128}$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量，注意到

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}=\partial^\mu\phi,\quad \frac{\partial\mathcal{L}}{\partial\phi^\dagger}=-m^2\phi,\quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=\partial^\mu\phi^\dagger,\quad \frac{\partial\mathcal{L}}{\partial\phi}=-m^2\phi^\dagger, \tag{2.129}$$

则 Euler-Lagrange 方程 (1.116) 给出

$$(\partial^2 + m^2)\phi(x) = 0, \quad (\partial^2 + m^2)\phi^\dagger(x) = 0. \tag{2.130}$$

也就是说，$\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程.

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合：

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{2.131}$$

从而，拉氏量 (2.128) 化为

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[\partial^\mu(\phi_1 - i\phi_2)]\partial_\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2 - \frac{1}{2}m^2\phi_2^2.\end{aligned} \tag{2.132}$$

与 (2.54) 式比较可知，复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量。

2.4.1 平面波展开

对于复标量场，我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开，但不能够应用自共轭条件。因此，场算符 $\phi(\mathbf{x}, t)$ 的通解也具有的 (2.65) 式的形式：

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})} + \tilde{a}_\mathbf{k}e^{i(E_\mathbf{k}t + \mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})} + \tilde{a}_{-\mathbf{k}}e^{i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.133}$$

由于不满足自共轭条件 (2.53)，算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_\mathbf{k}$ 没有什么关系，改记为

$$b_\mathbf{k}^\dagger = \tilde{a}_{-\mathbf{k}}, \tag{2.134}$$

则展开式变成

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})} + b_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})}\right]. \tag{2.135}$$

替换一下动量记号，可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip\cdot x} + b_\mathbf{p}^\dagger e^{ip\cdot x}\right), \tag{2.136}$$

其中，p^0 应该满足

$$p^0 = E_\mathbf{p} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \tag{2.137}$$

取厄米共轭，就得到 $\phi^\dagger(\mathbf{x}, t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(b_\mathbf{p}e^{-ip\cdot x} + a_\mathbf{p}^\dagger e^{ip\cdot x}\right). \tag{2.138}$$

现在，$a_{\mathbf{p}}$ 和 $b_{\mathbf{p}}$ 是两个相互独立的湮灭算符，而 $a_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^\dagger = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(-ip_0)\left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right), \tag{2.139}$$

$\phi^\dagger(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)} = \partial_0\phi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(-ip_0)\left(a_{\mathbf{p}}e^{-ip\cdot x} - b_{\mathbf{p}}^\dagger e^{ip\cdot x}\right). \tag{2.140}$$

根据 (2.51) 式，等时对易关系为

$$\begin{aligned}
[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0,\\
[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)] = [\pi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] = 0,\\
[\phi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= [\phi^\dagger(\mathbf{x},t),\pi(\mathbf{y},t)] = [\phi(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] = 0.
\end{aligned} \tag{2.141}$$

2.4.2 产生湮灭算符的对易关系

由

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}\phi &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[a_{\mathbf{p}}e^{-i(p-q)\cdot x} + b_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right)
\end{aligned} \tag{2.142}$$

和

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}\partial_0\phi &= \int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[a_{\mathbf{p}}e^{-i(p-q)\cdot x} - b_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}} - b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),
\end{aligned} \tag{2.143}$$

可得

$$-i\sqrt{2E_{\mathbf{q}}}\,a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}a_{\mathbf{q}} = \int d^3x\,e^{iq\cdot x}\partial_0\phi - iq_0\int d^3x\,e^{iq\cdot x}\phi = \int d^3x\,e^{iq\cdot x}\left(\partial_0\phi - iq_0\phi\right). \tag{2.144}$$

于是，

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\,e^{ip\cdot x}\left(\partial_0\phi - ip_0\phi\right), \quad a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\,e^{-ip\cdot x}\left(\partial_0\phi^\dagger + ip_0\phi^\dagger\right). \tag{2.145}$$

从而，有

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi^{\dagger}(\mathbf{y},t)+iq_0\phi^{\dagger}(\mathbf{y},t)\}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\left[\pi^{\dagger}(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t)+iq_0\phi^{\dagger}(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi^{\dagger}(\mathbf{x},t),\phi^{\dagger}(\mathbf{y},t)]-ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned}
\tag{2.146}
$$

以及

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\left[\pi^{\dagger}(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi^{\dagger}(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\right]=0.
\end{aligned}
\tag{2.147}
$$

另一方面，由

$$
\begin{aligned}
\int d^3x\,e^{iq\cdot x}\phi^{\dagger} &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}+a_{\mathbf{p}}^{\dagger}e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})+a_{\mathbf{p}}^{\dagger}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}+a_{-\mathbf{q}}^{\dagger}e^{2iq^0t}\right)
\end{aligned}
\tag{2.148}
$$

和

$$
\begin{aligned}
\int d^3x\,e^{iq\cdot x}\partial_0\phi^{\dagger} &= \int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}-a_{\mathbf{p}}^{\dagger}e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})-a_{\mathbf{p}}^{\dagger}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}-a_{-\mathbf{q}}^{\dagger}e^{2iq^0t}\right),
\end{aligned}
\tag{2.149}
$$

可得

$$
-i\sqrt{2E_{\mathbf{q}}}\,b_{\mathbf{q}}=\frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}b_{\mathbf{q}}=\int d^3x\,e^{iq\cdot x}\partial_0\phi^{\dagger}-iq_0\int d^3x\,e^{iq\cdot x}\phi^{\dagger}=\int d^3x\,e^{iq\cdot x}\left(\partial_0\phi^{\dagger}-iq_0\phi^{\dagger}\right).
\tag{2.150}
$$

于是，

$$b_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \left(\partial_0\phi^\dagger - ip_0\phi^\dagger\right), \quad b^\dagger_{\mathbf{p}} = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \left(\partial_0\phi + ip_0\phi\right). \tag{2.151}$$

从而，有

$$\begin{aligned}
&[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y \left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t) - ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi^\dagger(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \left(iq_0[\pi(\mathbf{x},t), \phi(\mathbf{y},t)] - ip_0[\phi^\dagger(\mathbf{x},t), \pi^\dagger(\mathbf{y},t)]\right) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} \left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned} \tag{2.152}$$

以及

$$\begin{aligned}
&[b_{\mathbf{p}}, b_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y \left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t) - ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t) - iq_0\phi^\dagger(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi^\dagger(\mathbf{x},t),\ \pi(\mathbf{y},t) - iq_0\phi^\dagger(\mathbf{y},t)\right] = 0.
\end{aligned} \tag{2.153}$$

此外，还有

$$\begin{aligned}
&[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left[\pi^\dagger(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\right] = 0,
\end{aligned} \tag{2.154}$$

以及

$$\begin{aligned}
&[a_{\mathbf{p}}, b_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t) - iq_0\phi^\dagger(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \left[\pi^\dagger(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) - iq_0\phi^\dagger(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \left(-iq_0[\pi^\dagger(\mathbf{x},t), \phi^\dagger(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t), \pi(\mathbf{y},t)]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\,[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})]\\
&= \frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}-E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0. \qquad (2.155)
\end{aligned}
$$

归纳起来，产生湮灭算符的对易关系如下：

$$
\begin{aligned}
[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = 0,\\
[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p}}, b_{\mathbf{q}}] = [b^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = 0, \qquad (2.156)\\
[a_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] &= [b_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = [a_{\mathbf{p}}, b_{\mathbf{q}}] = [a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = 0.
\end{aligned}
$$

这说明 $a^\dagger_{\mathbf{p}}, a_{\mathbf{p}}$ 与 $b^\dagger_{\mathbf{p}}, b_{\mathbf{p}}$ 是两套不同的产生湮灭算符，描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$
\phi'(x) = e^{iq\theta}\phi(x), \quad [\phi^\dagger(x)]' = e^{-iq\theta}\phi^\dagger(x), \qquad (2.157)
$$

则拉氏量 (2.128) 不变。依照 1.7.4 小节的讨论，相应的守恒流为

$$
J^\mu = q\phi^\dagger i\overleftrightarrow{\partial^\mu}\phi, \qquad (2.158)
$$

相应的守恒荷为

$$
\begin{aligned}
Q &= q\int d^3x\,\phi^\dagger i\overleftrightarrow{\partial^0}\phi = iq\int d^3x\,[\phi^\dagger\partial^0\phi - (\partial^0\phi^\dagger)\phi]\\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[\left(b_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\partial^0\left(a_{\mathbf{k}}e^{-ik\cdot x} + b^\dagger_{\mathbf{k}}e^{ik\cdot x}\right)\\
&\qquad - \partial^0\left(b_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\left(a_{\mathbf{k}}e^{-ik\cdot x} + b^\dagger_{\mathbf{k}}e^{ik\cdot x}\right)\Big]\\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[\left(b_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)(-ik^0)\left(a_{\mathbf{k}}e^{-ik\cdot x} - b^\dagger_{\mathbf{k}}e^{ik\cdot x}\right)\\
&\qquad - (-ip^0)\left(b_{\mathbf{p}}e^{-ip\cdot x} - a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\left(a_{\mathbf{k}}e^{-ik\cdot x} + b^\dagger_{\mathbf{k}}e^{ik\cdot x}\right)\Big]\\
&= iq\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[(ik^0+ip^0)b_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{-i(p-k)\cdot x} + (-ik^0-ip^0)a^\dagger_{\mathbf{p}}a_{\mathbf{k}}e^{i(p-k)\cdot x}\\
&\qquad + (-ik^0+ip^0)b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + (ik^0-ip^0)a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{i(p+k)\cdot x}\Big]\\
&= q\int \frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big[-(E_{\mathbf{k}}+E_{\mathbf{p}})b_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{-i(p-k)\cdot x} + (E_{\mathbf{k}}+E_{\mathbf{p}})a^\dagger_{\mathbf{p}}a_{\mathbf{k}}e^{i(p-k)\cdot x}\\
&\qquad + (E_{\mathbf{k}}-E_{\mathbf{p}})b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(p+k)\cdot x} + (-E_{\mathbf{k}}+E_{\mathbf{p}})a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{i(p+k)\cdot x}\Big]\\
&= q\int \frac{d^3p\,d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big\{(E_{\mathbf{k}}+E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{k})\left[-b_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} + a^\dagger_{\mathbf{p}}a_{\mathbf{k}}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\right]
\end{aligned}
$$

$$+ (E_{\mathbf{k}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{k})\left[b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} - a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right]\Bigg\}$$
$$= q\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\, 2E_{\mathbf{p}}\left(-b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right) = q\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right). \tag{2.159}$$

利用对易关系 (2.156)，可得

$$Q = \int \frac{d^3p}{(2\pi)^3}\left(q\, a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - q\, b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) - (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\, q. \tag{2.160}$$

上式第二项是零点荷。在第一项的圆括号中，粒子数密度算符 $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ 的系数是 q，而粒子数密度算符 $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$ 的系数是 $-q$。可见，$a^\dagger_{\mathbf{p}}, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q，习惯上称为**正粒子**；另一方面，$b^\dagger_{\mathbf{p}}, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$，习惯上称为**反粒子**。除去零点荷，总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式类似，但 Q/q 被解释为正粒子数与反粒子数之差，因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的，并且来自于一种内部对称性，因而是一种**内部相加性量子数**。实际上，反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 U(1) 整体变换，则自共轭条件 (2.53) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^\dagger = [e^{iq\theta}\phi(x)]^\dagger = e^{-iq\theta}\phi^\dagger(x) = e^{-iq\theta}\phi(x). \tag{2.161}$$

上式要求 $q = 0$。因此，对实标量场不能进行非平庸的 U(1) 整体变换。实际上，自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数，也就是说，正粒子与反粒子是相同的，实标量场描述的是一种纯中性粒子。

2.4.4　哈密顿量和总动量

根据 (1.119) 式，复标量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L} = (\partial^0\phi^\dagger)\partial_0\phi + (\partial^0\phi)\partial_0\phi^\dagger - (\partial^\mu\phi^\dagger)\partial_\mu\phi + m^2\phi^\dagger\phi\\ &= (\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger)\cdot\nabla\phi + m^2\phi^\dagger\phi.\end{aligned} \tag{2.162}$$

于是，哈密顿量可以写成

$$\begin{aligned}H &= \int d^3x\,\mathcal{H} = \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger)\cdot\nabla\phi + m^2\phi^\dagger\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi + \nabla\cdot(\phi^\dagger\nabla\phi) - \phi^\dagger\nabla^2\phi + m^2\phi^\dagger\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^0\partial_0 - \nabla^2 + m^2)\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^2 + m^2)\phi].\end{aligned} \tag{2.163}$$

上式第三步用了分部积分，第四步扔掉了一个全散度，最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.130) 消去。从而，得到

$$H = \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi]$$

$$
\begin{aligned}
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\partial^0 \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \partial^0\partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\Big] \\
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{(-ip^0) \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) (-iq_0) \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \left[(-iq^0)(-iq_0)a_{\mathbf{q}}e^{-iq\cdot x} + iq^0 iq_0 b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right]\Big\} \\
&= \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(p^0q_0 + q^0q_0)b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p^0q_0 + q^0q_0)a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \\
&\qquad\qquad + (-p^0q_0 + q^0q_0)b_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p+q)\cdot x} + (-p^0q_0 + q^0q_0)a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} E_{\mathbf{q}} \Big\{(E_{\mathbf{p}} + E_{\mathbf{q}})\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right] \\
&\qquad\qquad + (E_{\mathbf{q}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[b_{\mathbf{p}}a_{\mathbf{q}}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right]\Big\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) + (2\pi)^3\delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}.
\end{aligned} \tag{2.164}
$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 $\mathbf{p}$，正粒子与反粒子具有相同的能量 $E_{\mathbf{p}}$，因而它们具有相同的质量 m。

根据 (1.158) 式，复标量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x \left(\pi\nabla\phi + \pi^\dagger\nabla\phi^\dagger\right) = -\int d^3x \left[(\partial_0\phi^\dagger)\nabla\phi + (\partial_0\phi)\nabla\phi^\dagger\right] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\partial_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \nabla \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad + \partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \nabla \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[- ip_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) i\mathbf{q} \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad\qquad - iq_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) i\mathbf{p} \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\Big] \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}})e^{-i(p-q)\cdot x} \\
&\qquad\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger)e^{i(p-q)\cdot x} \\
&\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}b_{\mathbf{p}})e^{-i(p+q)\cdot x} \\
&\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger)e^{i(p+q)\cdot x}\Big] \\
&= -\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}})e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \\
&\qquad\qquad\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger)e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[(E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}} b_{\mathbf{p}})e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \\
&\qquad + (E_{\mathbf{p}}\mathbf{q}\, a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, b^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{p}})e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big]\Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[- E_{\mathbf{p}}\mathbf{p}\,(b_{\mathbf{p}} b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}) \\
&\qquad - E_{\mathbf{p}}\mathbf{p}\,(b_{\mathbf{p}} a_{-\mathbf{p}} - a_{-\mathbf{p}} b_{\mathbf{p}})e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}}\mathbf{p}\,(a^\dagger_{\mathbf{p}} b^\dagger_{-\mathbf{p}} - b^\dagger_{-\mathbf{p}} a^\dagger_{\mathbf{p}})e^{2iE_{\mathbf{p}}t}\Big] \\
&= \int \frac{d^3p}{(2\pi)^3}\,\frac{\mathbf{p}}{2}\,(b_{\mathbf{p}} b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) + \delta^{(3)}(0)\int d^3p\,\mathbf{p} \\
&= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}). \qquad (2.165)
\end{aligned}
$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章　矢量场

3.1　量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换

$$|\Psi'\rangle = U(\Lambda)\,|\Psi\rangle\,, \tag{3.1}$$

其中 $U(\Lambda)$ 是一个线性幺正算符，描述**量子 Lorentz 变换**，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \tag{3.2}$$

先作 Lorentz 变换 Λ_1，再作 Lorentz 变换 Λ_2，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$，故以下同态 (homomorphic) 关系成立：

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \tag{3.3}$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(1) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \tag{3.4}$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \tag{3.5}$$

将无穷小 Lorentz 变换 (1.159) 记为 $\Lambda_\omega = \mathbf{1} + \omega$，它诱导的无穷小幺正算符可表达为

$$U(\mathbf{1} + \omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \tag{3.6}$$

这里只展开到 ω 的一阶项。$J^{\mu\nu}$ 是量子 Lorentz 变换的**生成元算符**[1]。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的：

$$J^{\mu\nu} = -J^{\nu\mu}. \tag{3.7}$$

由 $U(\mathbf{1} + \omega)$ 的幺正性可得

$$1 = U^\dagger(\mathbf{1} + \omega)U(\mathbf{1} + \omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right]\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \tag{3.8}$$

[1]虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.179)。

最后一步忽略了 ω 的二阶项。可见，$J^{\mu\nu}$ 是厄米算符：

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \tag{3.9}$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda). \tag{3.10}$$

的左边和右边分别展开，得

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U^{-1}(\Lambda)\left(1-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)U(\Lambda) = 1-\frac{i}{2}U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda), \tag{3.11}$$

$$U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda) = U(\mathbf{1}+\Lambda^{-1}\omega\Lambda) = 1-\frac{i}{2}(\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu}. \tag{3.12}$$

因此，有

$$\begin{aligned} U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1}\omega\Lambda)^{\alpha}{}_{\nu}J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1})^{\alpha}{}_{\beta}\omega^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} \\ &= g_{\mu\alpha}\Lambda_{\beta}{}^{\alpha}\omega^{\beta}{}_{\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} = \Lambda^{\beta}{}_{\mu}\omega_{\beta\gamma}\Lambda^{\gamma}{}_{\nu}J^{\mu\nu} = \omega_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}, \end{aligned} \tag{3.13}$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立，于是，

$$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}. \tag{3.14}$$

因此，$J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值与它在 $|\Psi\rangle$ 中的期待值有如下关系：

$$\langle\Psi'|\,J^{\mu\nu}\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)\,|\Psi\rangle = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\,\langle\Psi|\,J^{\rho\sigma}\,|\Psi\rangle\,. \tag{3.15}$$

也就是说，$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换：

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma}. \tag{3.16}$$

可见，$J^{\mu\nu}$ 是一个 *2* 阶 *Lorentz* 张量。

接着，考虑 Λ 的无穷小形式 $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \tilde{\omega}^{\mu}{}_{\nu}$，则

$$U(\Lambda) = 1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}. \tag{3.17}$$

忽略二阶小量，(3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) &= \left(1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}\right)J^{\mu\nu}\left(1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}\right) \\ &= J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\mu\nu}J^{\alpha\beta} + \frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \tag{3.18}$$

右边为

$$\begin{aligned} \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}J^{\rho\sigma} &= (\delta^{\mu}{}_{\rho}+\tilde{\omega}^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma}+\tilde{\omega}^{\nu}{}_{\sigma})J^{\rho\sigma} = \delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}J^{\rho\sigma} + \delta^{\mu}{}_{\rho}\tilde{\omega}^{\nu}{}_{\sigma}J^{\rho\sigma} + \tilde{\omega}^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma}J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^{\nu}{}_{\sigma}J^{\mu\sigma} + \tilde{\omega}^{\mu}{}_{\rho}J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho}g^{\mu\sigma}J^{\rho\nu} \end{aligned}$$

$$
\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma}J^{\mu\rho} + g^{\mu\rho}J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma}),
\end{aligned}
\tag{3.19}
$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= i[g^{\nu\rho}J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma).
\end{aligned}
\tag{3.20}
$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群是一类特殊的连续群，n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie 乘积*，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$
J^i \equiv \frac{1}{2}\varepsilon^{ijk}J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}
$$

即

$$
\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}
$$

J^i 与 J^j 的对易关系为

$$
\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij},
\end{aligned}
\tag{3.23}
$$

第二、三步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.84) 式。由 (1.98) 式，有

$$
J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}
$$

从而推出

$$
[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}
$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$，写成分量的形式是 $L^i = \varepsilon^{ijk}x^j p^k$，从而，

$$
\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^l p^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^l p^m = x^i p^j - x^j p^i. \tag{3.26}
$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i, p^j] = i\delta^{ij}$ 可得

$$\begin{aligned}[L^i, L^j] &= \varepsilon^{ikl}\varepsilon^{jmn}[x^k p^l, x^m p^n] = \varepsilon^{ikl}\varepsilon^{jmn}\{x^k[p^l, x^m]p^n + x^m[x^k, p^n]p^l\} \\ &= \varepsilon^{ikl}\varepsilon^{jmn}(-i\delta^{lm}x^k p^n + i\delta^{kn}x^m p^l) = i(-\varepsilon^{ikl}\varepsilon^{jln}x^k p^n + \varepsilon^{ikl}\varepsilon^{jmk}x^m p^l) \\ &= i(\varepsilon^{ikl}\varepsilon^{jnl}x^k p^n - \varepsilon^{ilk}\varepsilon^{jmk}x^m p^l) = i[(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})x^k p^n - (\delta^{ij}\delta^{lm} - \delta^{im}\delta^{lj})x^m p^l] \\ &= i[\delta^{ij}x^k p^k - x^j p^i - \delta^{ij}x^l p^l + x^i p^j] = i(x^i p^j - x^j p^i) = i\varepsilon^{ijk}L^k. \end{aligned} \tag{3.27}$$

可见，$\mathbf{J}$ 与 $\mathbf{L}$ 具有相同的对易关系，$\mathbf{J}$ 也是一个角动量算符。实际上，$\mathbf{J}$ 描述**总角动量**，不止可以包含轨道角动量 $\mathbf{L}$，也可以包含*自旋角动量*。

满足

$$O^{\mathrm{T}}O = \mathbf{1} \tag{3.28}$$

的实方阵 O 称为*实正交矩阵* (real orthogonal matrix)。对上式取行列式，得

$$1 = \det O^{\mathrm{T}} \cdot \det O = (\det O)^2. \tag{3.29}$$

可见，实正交矩阵 O 的行列式为 $\det O = \pm 1$。由行列式为 $+1$ 的 3 维实正交矩阵按照矩阵乘法构成的群，称为**空间旋转群 SO(3)**，描述三维空间中的旋转变换。1.7.3 小节提到，SO(3) 群是 Lorentz 群的*子群*，J^i 可以看作 SO(3) 群的生成元算符，而 (3.25) 式是 SO(3) 群的 Lie 代数关系。

另一方面，$\mathbf{K}$ 是**增速算符**。$\mathbf{J}$ 与 $\mathbf{K}$ 的对易关系为

$$\begin{aligned}[J^i, K^j] &= \frac{1}{2}\varepsilon^{ikl}[J^{kl}, J^{0j}] = \frac{i}{2}\varepsilon^{ikl}\{[g^{l0}J^{kj} - (k \leftrightarrow l)] - (0 \leftrightarrow j)\} \\ &= i\varepsilon^{ikl}[g^{l0}J^{kj} - (0 \leftrightarrow j)] = i\varepsilon^{ikl}(g^{l0}J^{kj} - g^{lj}J^{k0}) = -i\varepsilon^{ikl}g^{lj}J^{k0} = i\varepsilon^{ikl}\delta^{lj}J^{k0} \\ &= i\varepsilon^{ikj}J^{k0} = i\varepsilon^{ijk}J^{0k} = i\varepsilon^{ijk}K^k, \end{aligned} \tag{3.30}$$

而 $\mathbf{K}$ 自身的对易关系为

$$\begin{aligned}[K^i, K^j] &= [J^{0i}, J^{0j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) \\ &= -i(g^{00}J^{ij} + g^{ij}J^{00}) = -iJ^{ij} = -i\varepsilon^{ijk}J^k. \end{aligned} \tag{3.31}$$

归纳起来，有

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \quad [J^i, K^j] = i\varepsilon^{ijk}K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk}J^k. \tag{3.32}$$

3.2　量子矢量场的 Lorentz 变换

3.2.1　Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 $\omega^\alpha{}_\beta$ 可以转化为

$$\omega^\alpha{}_\beta = g^{\alpha\mu}\omega_{\mu\beta} = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\beta} - g^{\alpha\mu}\omega_{\beta\mu}) = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta - g^{\alpha\mu}\omega_{\nu\mu}\delta^\nu{}_\beta) = \frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta - g^{\alpha\nu}\omega_{\mu\nu}\delta^\mu{}_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}, \tag{3.33}$$

其中 $(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}$ 定义为

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} \equiv i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta}). \tag{3.34}$$

容易看出，$\mathcal{J}^{\mu\nu}$ 是反对称的：

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \tag{3.35}$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^{\gamma}{}_{\beta} = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}g^{\nu\gamma}) = i(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}). \tag{3.36}$$

这样的话，可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}. \tag{3.37}$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned}
&[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^{\alpha}{}_{\beta} = (\mathcal{J}^{\mu\nu})^{\alpha}{}_{\gamma}(\mathcal{J}^{\rho\sigma})^{\gamma}{}_{\beta} - (\mathcal{J}^{\rho\sigma})^{\alpha}{}_{\gamma}(\mathcal{J}^{\mu\nu})^{\gamma}{}_{\beta}\\
=\ & i^2(g^{\mu\alpha}\delta^{\nu}{}_{\gamma} - \delta^{\mu}{}_{\gamma}g^{\nu\alpha})(g^{\rho\gamma}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma)\\
=\ & -g^{\mu\alpha}\delta^{\nu}{}_{\gamma}g^{\rho\gamma}\delta^{\sigma}{}_{\beta} + g^{\mu\alpha}\delta^{\nu}{}_{\gamma}\delta^{\rho}{}_{\beta}g^{\sigma\gamma} + \delta^{\mu}{}_{\gamma}g^{\nu\alpha}g^{\rho\gamma}\delta^{\sigma}{}_{\beta} - \delta^{\mu}{}_{\gamma}g^{\nu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma)\\
=\ & -g^{\mu\alpha}g^{\rho\nu}\delta^{\sigma}{}_{\beta} + g^{\mu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^{\sigma}{}_{\beta} - g^{\nu\alpha}\delta^{\rho}{}_{\beta}g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma)\\
=\ & -g^{\nu\rho}g^{\mu\alpha}\delta^{\sigma}{}_{\beta} + g^{\mu\rho}g^{\nu\alpha}\delta^{\sigma}{}_{\beta} + g^{\nu\sigma}g^{\mu\alpha}\delta^{\rho}{}_{\beta} - g^{\mu\sigma}g^{\nu\alpha}\delta^{\rho}{}_{\beta}\\
&-[-g^{\sigma\mu}g^{\rho\alpha}\delta^{\nu}{}_{\beta} + g^{\rho\mu}g^{\sigma\alpha}\delta^{\nu}{}_{\beta} + g^{\sigma\nu}g^{\rho\alpha}\delta^{\mu}{}_{\beta} - g^{\rho\nu}g^{\sigma\alpha}\delta^{\mu}{}_{\beta}]\\
=\ & g^{\nu\rho}(g^{\sigma\alpha}\delta^{\mu}{}_{\beta} - g^{\mu\alpha}\delta^{\sigma}{}_{\beta}) + g^{\mu\rho}(g^{\nu\alpha}\delta^{\sigma}{}_{\beta} - g^{\sigma\alpha}\delta^{\nu}{}_{\beta}) + g^{\nu\sigma}(g^{\mu\alpha}\delta^{\rho}{}_{\beta} - g^{\rho\alpha}\delta^{\mu}{}_{\beta}) + g^{\mu\sigma}(g^{\rho\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\rho}{}_{\beta})\\
=\ & -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^{\alpha}{}_{\beta} - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\beta} - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\beta} - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^{\alpha}{}_{\beta}\\
=\ & i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^{\alpha}{}_{\beta} - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^{\alpha}{}_{\beta} - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^{\alpha}{}_{\beta} + g^{\mu\sigma}(\mathcal{J}^{\nu\rho})^{\alpha}{}_{\beta}],
\end{aligned} \tag{3.38}$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}). \tag{3.39}$$

可见，$\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。$\Lambda^{\alpha}{}_{\beta}$ 属于 Lorentz 群的矢量表示，因而 $\mathcal{J}^{\mu\nu}$ 就是**矢量表示的生成元**。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \tag{3.40}$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^{\omega} = \sum_{n=0}^{\infty}\frac{\omega^n}{n!} \tag{3.41}$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 $\mathbf{g}$ 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}{}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}{}^{\delta}g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}{}_{\gamma}g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}{}_{\beta}, \tag{3.42}$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \tag{3.43}$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1}\left[\sum_{n=0}^{\infty}\frac{(\omega^{\mathrm{T}})^n}{n!}\right]\mathbf{g} = \sum_{n=0}^{\infty}\frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \tag{3.44}$$

若两个同阶方阵 A 和 B 相互对易，即 $[A,B]=0$，则二项式定理成立：

$$(A+B)^n = \sum_{j=0}^{n}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.45}$$

阶乘的定义可以推广到负整数：对于整数 $m<0$，定义

$$m! \to \infty, \qquad \frac{1}{m!} \to 0. \tag{3.46}$$

从而，对于 $j>n$，有 $[(n-j)!]^{-1} \to 0$。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A+B)^n = \sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.47}$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty}\frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j} = \sum_{j=0}^{\infty}\frac{A^j}{j!}\sum_{n=0}^{\infty}\frac{B^{n-j}}{(n-j)!} = e^Ae^B. \tag{3.48}$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^{\mathbf{0}} = \mathbf{1}. \tag{3.49}$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = \mathbf{g}, \tag{3.50}$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{3.51}$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的**有限变换**。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu}=0$，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

在正则量子化程序中，标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符，类似于 (3.16) 式，$\phi(x)$ 的固有保时向 Lorentz 变换关系 (2.52) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \tag{3.52}$$

上式表明，变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$，(3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{3.53}$$

相应地，$\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

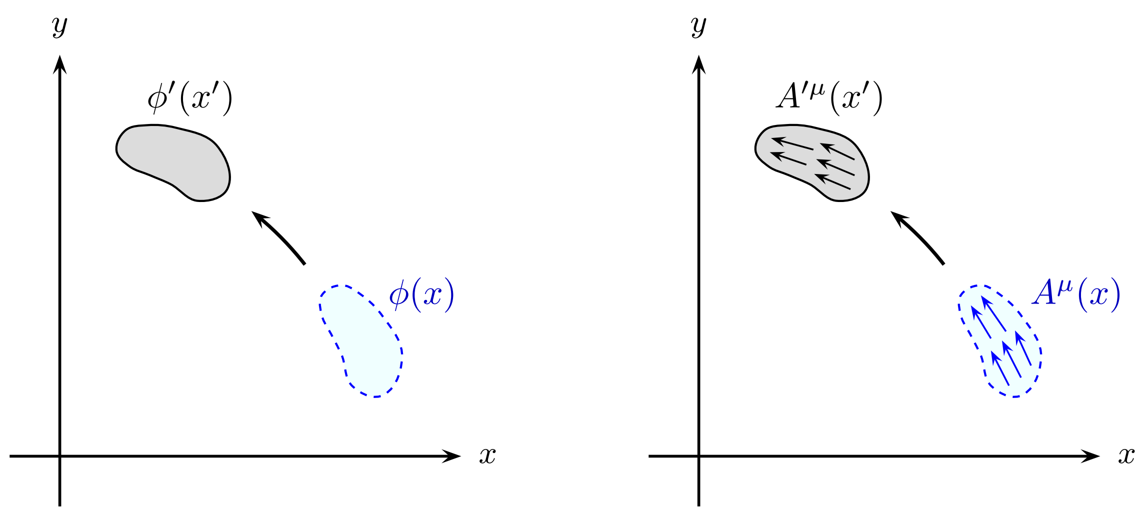
$$\langle\Psi'|\,\phi(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)\phi(x)U(\Lambda)\,|\Psi\rangle = \langle\Psi|\,\phi(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.54}$$

另一方面，由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的相应 Lorentz 变换形式为

$$\partial'^\mu\phi'(x') = U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda) = \partial'^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial'^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \tag{3.55}$$

于是，在固有保时向 Lorentz 变换下，自由实标量场的拉氏量 (2.54) 的变换形式为

$$\begin{aligned}\mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial'^\mu\phi(x')\partial'_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda)\\ &= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial'^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\}\\ &= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)]\\ &= \mathcal{L}(x),\end{aligned} \tag{3.56}$$



(a) 标量场 $\phi(x)$　　(b) 矢量场 $A^\mu(x)$

图 3.1: 在绕 z 轴空间旋转变换下，标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而，

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(\Lambda^{-1}x). \tag{3.57}$$

可见，拉氏量 (2.54) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$，可得

$$\begin{aligned}(\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu = g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta = g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) = g_{\nu\beta}g^{\mu\beta} + g^{\mu\beta}\omega_{\nu\beta} = \delta^\mu{}_\nu - g^{\mu\beta}\omega_{\beta\nu}\\ &= \delta^\mu{}_\nu - \omega^\mu{}_\nu,\end{aligned} \tag{3.58}$$

从而，有

$$(\Lambda^{-1}x)^\mu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu. \tag{3.59}$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项，得

$$\begin{aligned}\phi(\Lambda^{-1}x) &= \phi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\phi(x) = \phi(x) - \omega_{\mu\nu}x^\nu\partial^\mu\phi(x) = \phi(x) - \frac{1}{2}(\omega_{\mu\nu}x^\nu\partial^\mu + \omega_{\nu\mu}x^\mu\partial^\nu)\phi(x)\\ &= \phi(x) - \frac{1}{2}\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi(x) = \phi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x)\\ &= \phi(x) - \frac{i}{2}\omega_{\mu\nu}i(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x).\end{aligned} \tag{3.60}$$

根据 (3.6) 式，将 (3.53) 式左边展开到 ω 的一阶项，得

$$\begin{aligned}U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\\ &= \phi(x) - \frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}[\phi(x), J^{\mu\nu}].\end{aligned} \tag{3.61}$$

两相比较，给出

$$[\phi(x), J^{\mu\nu}] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x) = L^{\mu\nu}\phi(x), \tag{3.62}$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu} \equiv i(x^\mu\partial^\nu - x^\nu\partial^\mu). \tag{3.63}$$

对于空间分量 L^{ij}，可以等价地定义

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{i}{2}\varepsilon^{ijk}(x^j\partial^k - x^k\partial^j) = \frac{i}{2}(\varepsilon^{ijk}x^j\partial^k - \varepsilon^{ikj}x^j\partial^k) = i\varepsilon^{ijk}x^j\partial^k, \tag{3.64}$$

写成空间矢量的形式是

$$\mathbf{L} = -i\,\mathbf{x}\times\nabla. \tag{3.65}$$

可见，$\mathbf{L}$ 就是微分算符形式的轨道角动量算符。根据 (3.21) 式，(3.62) 式的纯空间分量部分可以改写为

$$[\phi(x), \mathbf{J}] = \mathbf{L}\,\phi(x). \tag{3.66}$$

上式表明，总角动量算符 $\mathbf{J}$ 生成了轨道角动量，但没有生成自旋角动量。这说明标量场没有自旋，对应于**零自旋**粒子。

3.2.3　量子矢量场的 Lorentz 变换形式

$\partial^\mu\phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有像 (3.55) 式那样的 Lorentz 变换形式，即

$$A'^\mu(x') = U^{-1}(\Lambda)A^\mu(x')U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \tag{3.67}$$

或者写成

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{3.68}$$

这就是**量子矢量场的 Lorentz 变换形式**。相应地，$A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\,A^\mu(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)A^\mu(x)U(\Lambda)\,|\Psi\rangle = \Lambda^\mu{}_\nu\,\langle\Psi|\,A^\nu(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.69}$$

对于固有保时向 Lorentz 变换，根据矢量表示中的无穷小形式 (3.40)，(3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]A^\nu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.70}$$

将上式与 (1.168) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出，在 Lorentz 变换下，除了矢量场的分布区域发生变化之外，矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式，在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned}A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta\partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta}x^\beta\partial^\alpha A^\nu(x)\\ &= A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x).\end{aligned} \tag{3.71}$$

从而，(3.68) 式右边可展开为

$$\begin{aligned}\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\left[A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x)\right]\\ &= A^\mu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)\\ &= A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)].\end{aligned} \tag{3.72}$$

另一方面，(3.68) 式左边的无穷小展开式为

$$\begin{aligned}&U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)A^\mu(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\\ &= A^\mu(x) - \frac{i}{2}\omega_{\alpha\beta}A^\mu(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}A^\mu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[A^\mu(x), J^{\rho\sigma}].\end{aligned} \tag{3.73}$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.74}$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathcal{J}^{jk}, \quad \boldsymbol{\mathcal{J}} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \tag{3.75}$$

再根据 (3.21) 和 (3.64) 式，(3.74) 式的纯空间分量部分可以改写为

$$[A^\mu(x),\mathbf{J}] = \mathbf{L}\,A^\mu(x) + (\boldsymbol{\mathcal{J}})^\mu{}_\nu A^\nu(x). \tag{3.76}$$

上式表明，总角动量算符 $\mathbf{J}$ 不仅生成了轨道角动量，还生成了由 $\boldsymbol{\mathcal{J}}$ 描述的**自旋角动量**。$\mathcal{J}^i$ 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \tag{3.77}$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \tag{3.78}$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \tag{3.79}$$

只关注空间分量，可得

$$(\mathcal{J}^1\mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2\mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \tag{3.80}$$

因此，有

$$(\boldsymbol{\mathcal{J}}^2)^i{}_j = (\mathcal{J}^1\mathcal{J}^1 + \mathcal{J}^2\mathcal{J}^2 + \mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \tag{3.81}$$

根据量子力学的角动量理论，$\boldsymbol{\mathcal{J}}^2$ 的本征值为 $s(s+1)$，即 $(\boldsymbol{\mathcal{J}}^2)^i{}_j = s(s+1)\delta^i{}_j$，其中 s 为**自旋量子数**。可见，矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \tag{3.82}$$

经过量子化程序之后，矢量场 $A^\mu(x)$ 应当描述**自旋为 1** 的粒子。

3.3　有质量矢量场的正则量子化

类似于电磁场，对任意的矢量场 A^μ 可以定义反对称的**场强张量**

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3.83}$$

对于一个自由的**有质量**的实矢量场 A^μ，用场强张量可以将它的 **Lorentz 不变拉氏量**写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.84}$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu]\\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu.\end{aligned} \tag{3.85}$$

从而，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \tag{3.86}$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \tag{3.87}$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{3.88}$$

上式称为 **Proca 方程**，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu\partial_\mu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = -\partial_\mu\partial_\nu F^{\nu\mu} = -\partial_\nu\partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu\partial_\mu F^{\mu\nu} = 0. \tag{3.89}$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu = m^2\partial_\nu A^\nu. \tag{3.90}$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz 条件**

$$\partial_\mu A^\mu = 0. \tag{3.91}$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu = \partial^2 A^\nu. \tag{3.92}$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon 方程*

$$(\partial^2 + m^2)A^\mu(x) = 0. \tag{3.93}$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \tag{3.94}$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \tag{3.95}$$

由于 $\pi_0 = 0$，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。实际上，由于 Lorenz 条件 (3.91) 的存在，A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于其它 3 个分量的量。因此，正则量子化程序要求独立的正则变量满足等时对易关系

$$[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = i\delta^i{}_j\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^i(\mathbf{x},t), A^j(\mathbf{y},t)] = [\pi_i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = 0. \tag{3.96}$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程，应该具有两个平面波解，即正能解 $\exp(-ip\cdot x)$ 和负能解 $\exp(ip\cdot x)$。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标，平面波展开式的系数也必须具有一个这样的指标。一般地，对于确定的动量 $\mathbf{p}$，矢量场的正能解模式具有如下形式：

$$A^\mu(x;\mathbf{p},\sigma) = e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}. \tag{3.97}$$

这里的系数 $e^\mu(\mathbf{p},\sigma)$ 是 Lorentz 矢量，称为**极化矢量** (polarization vector)，它依赖于动量 p，而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而，可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点，一组极化矢量应当是*线性独立且正交完备*的。Lorentz 矢量空间是一个 4 维空间，因而这样的极化矢量应该有 4 个，包括 1 个类时的极化矢量 $e^\mu(\mathbf{p},0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$。在没有额外约束的情况下，我们要求这 4 个极化矢量是**实**的，而且满足 Lorentz 矢量空间中的**正交归一关系**

$$e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma') = g_{\sigma\sigma'} \tag{3.98}$$

和**完备性关系**

$$\sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu}. \tag{3.99}$$

上面这两个关系都是 *Lorentz 协变*的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^μ 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空**横向**极化矢量

$$e^\mu(\mathbf{p},1) = (0,\mathbf{e}(\mathbf{p},1)), \quad e^\mu(\mathbf{p},2) = (0,\mathbf{e}(\mathbf{p},2)). \tag{3.100}$$

此处，

$$\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_{\mathrm{T}}|}(p^1p^3, p^2p^3, -|\mathbf{p}_{\mathrm{T}}|^2), \quad \mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_{\mathrm{T}}|}(-p^2, p^1, 0), \tag{3.101}$$

其中

$$|\mathbf{p}_{\mathrm{T}}| \equiv \sqrt{(p^1)^2+(p^2)^2}. \tag{3.102}$$

“横向”指的是它们在三维空间中与 $\mathbf{p}$ 垂直，即

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}[(p^1)^2p^3+(p^2)^2p^3-p^3|\mathbf{p}_\mathrm{T}|^2]=0, \tag{3.103}$$

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^1p^2+p^2p^1)=0. \tag{3.104}$$

此外，存在如下关系：

$$\begin{aligned}\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},1) &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}[(p^1)^2(p^3)^2+(p^2)^2(p^3)^2+|\mathbf{p}_\mathrm{T}|^4]\\ &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2[(p^3)^2+|\mathbf{p}_\mathrm{T}|^2] = \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2|\mathbf{p}|^2=1,\end{aligned} \tag{3.105}$$

$$\mathbf{e}(\mathbf{p},2)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}[(p^2)^2+(p^1)^2] = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2=1, \tag{3.106}$$

$$\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|^2}(-p^1p^3p^2+p^2p^3p^1)=0. \tag{3.107}$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j)=\delta_{ij},\quad i,j=1,2. \tag{3.108}$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_\mu e^\mu(\mathbf{p},1)=p_\mu e^\mu(\mathbf{p},2)=0, \tag{3.109}$$

和正交归一关系

$$e_\mu(\mathbf{p},i)e^\mu(\mathbf{p},j)=-\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j)=-\delta_{ij}=g_{ij}. \tag{3.110}$$

接着，要求第 3 个类空极化矢量 $e^\mu(\mathbf{p},3)$ 是**纵向**的，即在三维空间中与 $\mathbf{p}$ 平行。这样还不能确定它的时间分量，为此，我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p},3)=0$，而正交归一关系 (3.98) 将决定它的归一化。于是，纵向极化矢量的形式为

$$e^\mu(\mathbf{p},3)=\left(\frac{|\mathbf{p}|}{m},\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}\right). \tag{3.111}$$

可以验证，它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p},3)=p^0\frac{|\mathbf{p}|}{m}-\mathbf{p}\cdot\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}=\frac{p^0|\mathbf{p}|}{m}-\frac{p^0|\mathbf{p}|}{m}=0, \tag{3.112}$$

和正交归一关系

$$e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3) = \frac{|\mathbf{p}|}{m}\frac{|\mathbf{p}|}{m}-\frac{(p^0)^2\,\mathbf{p}\cdot\mathbf{p}}{m^2|\mathbf{p}|^2}=\frac{|\mathbf{p}|^2}{m^2}-\frac{(p^0)^2}{m^2}=-\frac{(p^0)^2-|\mathbf{p}|^2}{m^2}=-1=g_{33}; \tag{3.113}$$

$$e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},i) = -\frac{p^0}{m|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2. \tag{3.114}$$

最后，我们可以将**类时**极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p},0)=\frac{1}{m}\,p^\mu=\frac{1}{m}(p^0,\mathbf{p}).\tag{3.115}$$

它满足正交归一关系 (3.98)：

$$\begin{aligned}
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0) &= \frac{p^2}{m^2}=1=g_{00};\qquad(3.116)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},i) &= -\frac{1}{m}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2;\qquad(3.117)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},3) &= \frac{1}{m^2}\,p^0|\mathbf{p}|-\frac{p^0}{m^2|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{p}=0.\qquad(3.118)
\end{aligned}$$

不过，它不满足四维时空的横向条件：

$$p_\mu e^\mu(\mathbf{p},0)=\frac{p^2}{m}=m.\tag{3.119}$$

可以验证，由 (3.100)、(3.101)、(3.111) 和 (3.115) 式定义的这组极化矢量确实满足完备性关系 (3.99)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)\\
&=e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)-e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)-e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2)-e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\
&=\frac{1}{m^2}\begin{pmatrix}p^0p^0 & -p^0p^1 & -p^0p^2 & -p^0p^3\\ -p^1p^0 & p^1p^1 & p^1p^2 & p^1p^3\\ -p^2p^0 & p^2p^1 & p^2p^2 & p^2p^3\\ -p^3p^0 & p^3p^1 & p^3p^2 & p^3p^3\end{pmatrix}-\frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & p^1p^3p^1p^3 & p^1p^3p^2p^3 & -p^1p^3|\mathbf{p}_\mathrm{T}|^2\\ 0 & p^2p^3p^1p^3 & p^2p^3p^2p^3 & -p^2p^3|\mathbf{p}_\mathrm{T}|^2\\ 0 & -|\mathbf{p}_\mathrm{T}|^2p^1p^3 & -|\mathbf{p}_\mathrm{T}|^2p^2p^3 & |\mathbf{p}_\mathrm{T}|^4\end{pmatrix}\\
&\quad-\frac{1}{|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & p^2p^2 & -p^2p^1 & 0\\ 0 & -p^1p^2 & p^1p^1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}-\frac{1}{m^2}\begin{pmatrix}|\mathbf{p}|^2 & -p^0p^1 & -p^0p^2 & -p^0p^3\\ -p^0p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^3\\ -p^0p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^3\\ -p^0p^3 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^3\end{pmatrix}\\
&=\begin{pmatrix}1 & 0 & 0 & 0\\
0 & \frac{(p^1)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^1p^3)^2+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2}\\
0 & \frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{(p^2)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^2p^3)^2+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2}\\
0 & \frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2} & \frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2} & \frac{(p^3)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2}
\end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2|\mathbf{p}_\mathrm{T}|^2+(p^1)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{(p^2)^2|\mathbf{p}_\mathrm{T}|^2+(p^2)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{(p^3)^2+|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2} \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \tag{3.120}
$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91)，正能解模式 (3.97) 应满足

$$
0 = \partial_\mu A^\mu(x;\mathbf{p},\sigma) = -ip_\mu e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \tag{3.121}
$$

即

$$
p_\mu e^\mu(\mathbf{p},\sigma) = 0. \tag{3.122}
$$

也就是说，描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p},0)$ 不能用于描述有质量矢量场 A^μ。这说明 A^μ 只有 3 个物理的极化状态，由类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$ 描述。根据完备性关系 (3.99)，这 3 个物理的极化矢量满足

$$
-\sum_{\sigma=1}^{3} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = \sum_{\sigma=1}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu}-g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0) = g_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}, \tag{3.123}
$$

即具有求和关系

$$
\sum_{\sigma=1}^{3} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{3.124}
$$

通过如下线性组合，我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p,\lambda)$，其中 $\lambda = +,0,-$：

$$
\varepsilon^\mu(\mathbf{p},\pm) \equiv \frac{1}{\sqrt{2}}\left[\mp e^\mu(\mathbf{p},1) - ie^\mu(\mathbf{p},2)\right], \tag{3.125}
$$

$$
\varepsilon^\mu(\mathbf{p},0) \equiv e^\mu(\mathbf{p},3). \tag{3.126}
$$

这样定义的 $\varepsilon^\mu(p,\pm)$ 是**复**的，而 $\varepsilon^\mu(p,0)$ 是实的。它们都满足**四维横向条件**

$$
p_\mu \varepsilon^\mu(\mathbf{p},\lambda) = 0. \tag{3.127}
$$

它们还满足

$$
\begin{aligned}
\varepsilon^*_\mu(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\pm) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1) + ie_\mu(\mathbf{p},2)][\mp e^\mu(\mathbf{p},1) - ie^\mu(\mathbf{p},2)] \\
&= \frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1) + \frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2) = \frac{1}{2}(g_{11}+g_{22}) = -1,
\end{aligned} \tag{3.128}
$$

$$\begin{aligned}\varepsilon_\mu^*(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\mp) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][\pm e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)]\\ &= -\frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1)+\frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2)=\frac{1}{2}(-g_{11}+g_{22})=0, \quad (3.129)\\ \varepsilon_\mu^*(\mathbf{p},0)\varepsilon^\mu(\mathbf{p},0) &= e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3)=-1, \quad (3.130)\\ \varepsilon_\mu^*(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},0) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)]e^\mu(\mathbf{p},3)=0, \quad (3.131)\end{aligned}$$

即具有**正交归一关系**

$$\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon^\mu(\mathbf{p},\lambda')=-\delta_{\lambda\lambda'}. \quad (3.132)$$

极化矢量求和关系则是

$$\begin{aligned}\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) &= \frac{1}{2}[e_\mu(p,1)+ie_\mu(p,2)][e_\nu(p,1)-ie_\nu(p,2)]\\ &\quad +\frac{1}{2}[-e_\mu(p,1)+ie_\mu(p,2)][-e_\nu(p,1)-ie_\nu(p,2)]+e_\mu(p,3)e_\nu(p,3)\\ &= e_\mu(p,1)e_\nu(p,1)+e_\mu(p,2)e_\nu(p,2)+e_\mu(p,3)e_\nu(p,3)\\ &= \sum_{\sigma=1}^{3}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma), \quad (3.133)\end{aligned}$$

与 (3.124) 式左边相等，故

$$\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}. \quad (3.134)$$

四维横向条件 (3.127) 在上式中体现为

$$p^\nu\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-p_\mu+\frac{p_\mu p^2}{m^2}=-p_\mu+p_\mu=0. \quad (3.135)$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 $\mathbf{p}$ 的方向由 $\hat{\mathbf{p}}\equiv\mathbf{p}/|\mathbf{p}|$ 表征，于是，在 Lorentz 群矢量表示中，螺旋度矩阵定义为

$$\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}}=\frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\boldsymbol{\mathcal{J}}=\frac{1}{|\mathbf{p}|}\begin{pmatrix}0 & & & \\ & 0 & -ip^3 & ip^2\\ & ip^3 & 0 & -ip^1\\ & -ip^2 & ip^1 & 0\end{pmatrix}. \quad (3.136)$$

将 (3.101) 和 (3.111) 式代入 (3.125) 和 (3.126) 式，得到 $\varepsilon^\mu(p,\lambda)$ 的列矢量形式为

$$\varepsilon^\mu(p,0)=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}|\mathbf{p}|^2\\ p^0p^1\\ p^0p^2\\ p^0p^3\end{pmatrix},\quad \varepsilon^\mu(p,+)=\frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\ -p^1p^3+ip^2|\mathbf{p}|\\ -p^2p^3-ip^1|\mathbf{p}|\\ |\mathbf{p}_\mathrm{T}|^2\end{pmatrix},$$

$$\varepsilon^\mu(p,-) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ p^1p^3 + ip^2|\mathbf{p}| \\ p^2p^3 - ip^1|\mathbf{p}| \\ -|\mathbf{p}_\mathrm{T}|^2 \end{pmatrix}. \tag{3.137}$$

从而，可得

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,0) = \frac{1}{m|\mathbf{p}|^2}\begin{pmatrix} 0 \\ -ip^3p^0p^2 + ip^2p^0p^3 \\ ip^3p^0p^1 - ip^1p^0p^3 \\ -ip^2p^0p^1 + ip^1p^0p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\,\varepsilon^\mu(p,0), \tag{3.138}$$

$$\begin{aligned}(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,+) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ ip^2(p^3)^2 - p^1p^3|\mathbf{p}| + ip^2|\mathbf{p}_\mathrm{T}|^2 \\ -ip^1(p^3)^2 - p^2p^3|\mathbf{p}| - ip^1|\mathbf{p}_\mathrm{T}|^2 \\ ip^1p^2p^3 + (p^2)^2|\mathbf{p}| - ip^1p^2p^3 + (p^1)^2|\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -p^1p^3|\mathbf{p}| + ip^2|\mathbf{p}|^2 \\ -p^2p^3|\mathbf{p}| - ip^1|\mathbf{p}|^2 \\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}| \end{pmatrix} = +\varepsilon^\mu(p,+), \end{aligned} \tag{3.139}$$

$$\begin{aligned}(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,-) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -ip^2(p^3)^2 - p^1p^3|\mathbf{p}| - ip^2|\mathbf{p}_\mathrm{T}|^2 \\ ip^1(p^3)^2 - p^2p^3|\mathbf{p}| + ip^1|\mathbf{p}_\mathrm{T}|^2 \\ -ip^1p^2p^3 + (p^2)^2|\mathbf{p}| + ip^1p^2p^3 + (p^1)^2|\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -p^1p^3|\mathbf{p}| - ip^2|\mathbf{p}|^2 \\ -p^2p^3|\mathbf{p}| + ip^1|\mathbf{p}|^2 \\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}| \end{pmatrix} = -\varepsilon^\mu(p,-). \end{aligned} \tag{3.140}$$

归纳起来，有

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,\lambda) = \lambda\,\varepsilon^\mu(p,\lambda). \tag{3.141}$$

上式说明极化矢量 $\varepsilon^\mu(p,\lambda)$ 是螺旋度的本征态，本征值为 λ。因此，$\varepsilon^\mu(p,\lambda)$ 描述动量为 $\mathbf{p}$ 、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda=\pm1$ 对应于两种**横向极化**，$\lambda=0$ 对应于**纵向极化**。

有质量的实矢量场算符 $A^\mu(\mathbf{x},t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态，形式为

$$A^\mu(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big], \tag{3.142}$$

其中 $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ。容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t). \tag{3.143}$$

根据 (3.95) 式，共轭动量密度为

$$\begin{aligned}\pi_i &= -\partial_0 A_i + \partial_i A_0 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \Big\{ [ip_0\varepsilon_i(\mathbf{p},\lambda) - ip_i\varepsilon_0(\mathbf{p},\lambda)] a_{\mathbf{p},\lambda} e^{-ip\cdot x} \\ &\qquad + [-ip_0\varepsilon_i^*(\mathbf{p},\lambda) + ip_i\varepsilon_0^*(\mathbf{p},\lambda)] a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x} \Big\},\end{aligned} \tag{3.144}$$

引入

$$\tilde{\varepsilon}_i(\mathbf{p},\lambda) \equiv \varepsilon_i(\mathbf{p},\lambda) - \frac{p_i}{p_0}\varepsilon_0(\mathbf{p},\lambda), \tag{3.145}$$

则有

$$p_0\varepsilon_i(\mathbf{p},\lambda) - p_i\varepsilon_0(\mathbf{p},\lambda) = p_0\tilde{\varepsilon}_i(\mathbf{p},\lambda), \tag{3.146}$$

从而，可以将共轭动量密度的平面波展开式写得更加紧凑：

$$\pi_i(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x} \right]. \tag{3.147}$$

易见，它也满足自共轭条件

$$[\pi_i(\mathbf{x},t)]^\dagger = \pi_i(\mathbf{x},t). \tag{3.148}$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned}&\int d^3x\, e^{iq\cdot x} A^\mu \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-i(p-q)\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{i(p+q)\cdot x} \right] \\ &= \int d^3p\, \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q},\lambda) a_{\mathbf{q},\lambda} + \varepsilon^{\mu*}(-\mathbf{q},\lambda) a^\dagger_{-\mathbf{q},\lambda} e^{2iq^0t} \right]\end{aligned} \tag{3.149}$$

和

$$\begin{aligned}\int d^3x\, e^{iq\cdot x} \partial_0 A^\mu &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-i(p-q)\cdot x} - \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{i(p+q)\cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \Big[\varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ &\qquad - \varepsilon^{\mu*}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big]\end{aligned}$$

$$= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda} - \varepsilon^{\mu*}(-\mathbf{q},\lambda)a^\dagger_{-\mathbf{q},\lambda}e^{2iq^0t}\right], \tag{3.150}$$

以及正交归一关系 (3.132)，可得

$$\begin{aligned}\varepsilon^*_\mu(\mathbf{q},\lambda')\int d^3x\, e^{iq\cdot x}\,(\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon^*_\mu(\mathbf{q},\lambda')\frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}\sum_{\lambda=\pm,0}\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda}\\ &= -i\sqrt{2E_{\mathbf{q}}}\sum_{\lambda=\pm,0}(-\delta_{\lambda'\lambda})a_{\mathbf{q},\lambda} = i\sqrt{2E_{\mathbf{q}}}\,a_{\mathbf{q},\lambda'}.\end{aligned} \tag{3.151}$$

由 Lorenz 条件 (3.91) 可得

$$\partial_0 A^0 = -\partial_i A^i, \tag{3.152}$$

根据 (3.95) 式，有

$$\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \tag{3.153}$$

于是，湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$\begin{aligned} a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\varepsilon^*_\mu(\mathbf{p},\lambda)\int d^3x\, e^{ip\cdot x}\,(\partial_0 A^\mu - ip_0 A^\mu)\\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\,[\varepsilon^*_0(\mathbf{p},\lambda)\partial_0 A^0 + \varepsilon^*_i(\mathbf{p},\lambda)\partial_0 A^i - ip_0\varepsilon^*_\mu(\mathbf{p},\lambda)A^\mu]\\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\,\big[-\varepsilon^*_0(\mathbf{p},\lambda)\partial_i A^i + \varepsilon^*_i(\mathbf{p},\lambda)\pi_i - \varepsilon^*_i(\mathbf{p},\lambda)\partial_i A^0\\ &\qquad - ip_0\varepsilon^*_0(\mathbf{p},\lambda)A^0 - ip_0\varepsilon^*_i(\mathbf{p},\lambda)A^i\big]. \end{aligned} \tag{3.154}$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$\begin{aligned}\int d^3x\, e^{ip\cdot x}[-\varepsilon^*_0(\mathbf{p},\lambda)\partial_i A^i - \varepsilon^*_i(\mathbf{p},\lambda)\partial_i A^0] &= \int d^3x\,[\varepsilon^*_0(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x})A^i + \varepsilon^*_i(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x})A^0]\\ &= \int d^3x\,[ip_i\varepsilon^*_0(\mathbf{p},\lambda)e^{ip\cdot x}A^i + ip_i\varepsilon^*_i(\mathbf{p},\lambda)e^{ip\cdot x}A^0]\\ &= \int d^3x\, e^{ip\cdot x}[i\varepsilon^*_0(\mathbf{p},\lambda)p_i A^i + ip_i\varepsilon^*_i(\mathbf{p},\lambda)A^0],\end{aligned} \tag{3.155}$$

从而，有

$$\begin{aligned} a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\,\big[i\varepsilon^*_0(\mathbf{p},\lambda)p_i A^i + \varepsilon^*_i(\mathbf{p},\lambda)\pi_i + ip_i\varepsilon^*_i(\mathbf{p},\lambda)A^0\\ &\qquad - ip_0\varepsilon^*_0(\mathbf{p},\lambda)A^0 - ip_0\varepsilon^*_i(\mathbf{p},\lambda)A^i\big]\\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\,\{\varepsilon^*_i(\mathbf{p},\lambda)\pi_i - ip^\mu\varepsilon^*_\mu(\mathbf{p},\lambda)A^0 - i[p_0\varepsilon^*_i(\mathbf{p},\lambda) - p_i\varepsilon^*_0(\mathbf{p},\lambda)]A^i\}. \end{aligned} \tag{3.156}$$

再利用四维横向条件 (3.127) 和 (3.146) 式，得到

$$a_{\mathbf{p},\lambda} = \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\left[-\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(x) - ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)A^i(x)\right]$$

$$= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \left[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(x) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(x)\right]. \tag{3.157}$$

对上式取厄米共轭，得

$$a_{\mathbf{p},\lambda}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \left[\varepsilon^{i}(\mathbf{p},\lambda)\pi_i(x) - ip_0\tilde{\varepsilon}_i(\mathbf{p},\lambda)A^i(x)\right]. \tag{3.158}$$

利用等时对易关系 (3.96)，可得湮灭算符与产生算符的对易关系为

$$\begin{aligned}
&[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(\mathbf{x},t),\right. \\
&\qquad\qquad \left.\varepsilon^{j}(\mathbf{q},\lambda')\pi_j(\mathbf{y},t) - iq_0\tilde{\varepsilon}_j(\mathbf{q},\lambda')A^j(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left\{ -iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t), A^j(\mathbf{y},t)]\right. \\
&\qquad\qquad \left. + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)] \right\} \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \left[-q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')\delta^j{}_i - p_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')\delta^i{}_j\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \left[-E_{\mathbf{q}}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{q},\lambda') - E_{\mathbf{p}}\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')\right] \\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda')\right].
\end{aligned} \tag{3.159}$$

根据定义式 (3.145)、四维横向条件 (3.127) 和正交归一关系 (3.132)，有

$$\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') - \frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') + \frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{\mu*}(\mathbf{p},\lambda)\varepsilon_\mu(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'},
\end{aligned} \tag{3.160}$$

取复共轭，可得

$$\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'}. \tag{3.161}$$

于是，

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,(-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{3.162}$$

另一方面，

$$\begin{aligned}
&[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \left[\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(\mathbf{x},t),\right. \\
&\qquad\qquad \left.\varepsilon^{j*}(\mathbf{q},\lambda')\pi_j(\mathbf{y},t) + iq_0\tilde{\varepsilon}_j^*(\mathbf{q},\lambda')A^j(\mathbf{y},t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\left\{iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_j(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t),A^j(\mathbf{y},t)]\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.+\,ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')[A^i(\mathbf{x},t),\pi_j(\mathbf{y},t)]\right\}\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,[q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_j(\mathbf{q},\lambda')\delta^j{}_i-p_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')\delta^i{}_j]\\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0+q^0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\,[E_{\mathbf{q}}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_i(\mathbf{q},\lambda')-E_{\mathbf{p}}\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')]\\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{2iE_{\mathbf{p}}t}\,[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_i(-\mathbf{p},\lambda')-\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')].
\end{aligned}
\tag{3.163}
$$

对四维横向条件 (3.127) 取复共轭，得

$$
p_\mu\varepsilon^{\mu*}(\mathbf{p},\lambda)=p_0\varepsilon^{0*}(\mathbf{p},\lambda)+p_i\varepsilon^{i*}(\mathbf{p},\lambda)=0. \tag{3.164}
$$

将上式中的 $\mathbf{p}$ 替换成 $-\mathbf{p}$，得

$$
p_0\varepsilon^{0*}(-\mathbf{p},\lambda)-p_i\varepsilon^{i*}(-\mathbf{p},\lambda)=0. \tag{3.165}
$$

因此，有

$$
p_i\varepsilon^{i*}(\mathbf{p},\lambda)=-p_0\varepsilon^{0*}(\mathbf{p},\lambda),\quad -p_i\varepsilon^{i*}(-\mathbf{p},\lambda)=-p_0\varepsilon^{0*}(-\mathbf{p},\lambda), \tag{3.166}
$$

或者写成

$$
\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)=p_0\varepsilon^{0*}(\mathbf{p},\lambda),\quad -\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda)=p_0\varepsilon^{0*}(-\mathbf{p},\lambda). \tag{3.167}
$$

从而，可得

$$
\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_i(-\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\left[\varepsilon^*_i(-\mathbf{p},\lambda)+\frac{p_i}{p_0}\varepsilon^*_0(-\mathbf{p},\lambda)\right]\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon^*_i(-\mathbf{p},\lambda')+\frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon^*_0(-\mathbf{p},\lambda')\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon^*_i(-\mathbf{p},\lambda')-\frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon^*_0(-\mathbf{p},\lambda')\\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon^*_i(-\mathbf{p},\lambda')-\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon^*_0(-\mathbf{p},\lambda'),
\end{aligned}
\tag{3.168}
$$

$$
\begin{aligned}
\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') &= \left[\varepsilon^*_i(\mathbf{p},\lambda)-\frac{p_i}{p_0}\varepsilon^*_0(\mathbf{p},\lambda)\right]\varepsilon^{i*}(-\mathbf{p},\lambda')\\
&= \varepsilon^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\frac{1}{p_0}\varepsilon^*_0(\mathbf{p},\lambda)p_i\varepsilon^{i*}(-\mathbf{p},\lambda')\\
&= \varepsilon^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\frac{1}{p_0}\varepsilon^*_0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(-\mathbf{p},\lambda')\\
&= \varepsilon^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')-\varepsilon^*_0(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda').
\end{aligned}
\tag{3.169}
$$

可见，$\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_i(-\mathbf{p},\lambda')-\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')=0$，故

$$
[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}]=0. \tag{3.170}
$$

综上，产生湮灭算符的对易关系为

$$
[a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}]=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad [a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}]=[a^\dagger_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}]=0. \tag{3.171}
$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \tag{3.172}$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \tag{3.173}$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \tag{3.174}$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$，因此，

$$A^0 = -\frac{1}{m^2}\partial_\mu F^{\mu 0} = -\frac{1}{m^2}\partial_i F^{i0} = -\frac{1}{m^2}\partial_i \pi^i = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}. \tag{3.175}$$

从而，可得

$$-\boldsymbol{\pi}\cdot\dot{\mathbf{A}} = \boldsymbol{\pi}\cdot(\boldsymbol{\pi}+\nabla A_0) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) - A_0(\nabla\cdot\boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2. \tag{3.176}$$

另一方面，

$$\frac{1}{2}F_{0i}F^{0i} = \frac{1}{2}\pi_i\pi^i = -\frac{1}{2}\boldsymbol{\pi}^2. \tag{3.177}$$

利用 (1.84) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\partial^m A^n = \varepsilon^{ijk}\varepsilon^{kmn}\partial^m A^n = -\varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n, \tag{3.178}$$

从而，

$$\begin{aligned}\frac{1}{4}F_{ij}F^{ij} &= \frac{1}{4}F^{ij}F^{ij} = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{ijl}\varepsilon^{lpq}\partial_p A^q = \frac{1}{4}2\delta^{kl}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{lpq}\partial_p A^q \\ &= \frac{1}{2}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{kpq}\partial_p A^q = \frac{1}{2}(\nabla\times\mathbf{A})^2.\end{aligned} \tag{3.179}$$

于是，有

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2. \tag{3.180}$$

根据 (1.119) 式，有质量矢量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi_i\partial_0 A^i - \mathcal{L} = \pi_i\partial_0 A^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \\ &= -\boldsymbol{\pi}\cdot\dot{\mathbf{A}} - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2) \\ &= \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2 - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}m^2\mathbf{A}^2 \\ &= \frac{1}{2}\boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 + \frac{1}{2}m^2\mathbf{A}^2.\end{aligned} \tag{3.181}$$

上式最后一行第二项是一个全散度，对全空间积分时它没有贡献。于是，哈密顿量为

$$H=\int d^3x\,\mathcal{H}=\frac{1}{2}\int d^3x\left[\boldsymbol{\pi}^2+\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2+(\nabla\times\mathbf{A})^2+m^2\mathbf{A}^2\right]. \tag{3.182}$$

下面逐项进行计算。

哈密顿量的第一项是

$$\begin{aligned}
&\frac{1}{2}\int d^3x\,\boldsymbol{\pi}^2\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,(ip_0)(iq_0)\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\cdot\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\qquad-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}+\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{-\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&\qquad+\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
&=\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_{\mathbf{p}}}\,E_{\mathbf{p}}^2\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&\qquad-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}\tag{3.183}$$

第二项是

$$\begin{aligned}
&\frac{1}{2}\int d^3x\,\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\frac{(ip_0)(iq_0)}{m^2}\left[i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\times\left[i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}+i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}\,m^2}\Big\{-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\qquad-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\qquad-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big\}\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}\,m^2}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big)\\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad +[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big)\Big\}\\
= &\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_\mathbf{p}}\frac{E_\mathbf{p}^2}{m^2}\Big\{[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}\\
&+[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\\
&-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t}\Big\}.
\end{aligned}
\tag{3.184}
$$

第三项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,(\nabla\times\mathbf{A})^2\\
=&\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[i\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}-i\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big]\\
&\qquad\cdot\Big[i\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}-i\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\Big]\\
=&\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big\{[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}-[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&-[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big\}\\
=&\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad+[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big)\\
&\qquad-\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad+[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big)\Big\}\\
=&\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{4E_\mathbf{p}}\Big\{[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t}\Big\}.
\end{aligned}
\tag{3.185}
$$

第四项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,m^2\mathbf{A}^2\\
=&\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;m^2}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big]\\
&\qquad\cdot\Big[\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}+\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\Big]\\
=&\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;m^2}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}+\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\quad +\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big]\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;m^2}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad +\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad +\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
&=\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}m^2\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&\quad +\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}+\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}\tag{3.186}
$$

综合起来，哈密顿量化为

$$
\begin{aligned}
H=\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}\Big[&f_1(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+f_1^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&+f_2(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}+f_2^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big],
\end{aligned}\tag{3.187}
$$

其中，

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda')\equiv\;&E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')+\frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda'),
\end{aligned}\tag{3.188}
$$

$$
\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda')\equiv\;&-E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')-\frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda').
\end{aligned}\tag{3.189}
$$

现在，我们计算 $f_1(\mathbf{p},\lambda,\lambda')$。由 (3.145)、(3.167) 和 (3.132) 式，可得

$$
\begin{aligned}
&\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')=\left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)-\frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\mathbf{p}}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]\\
&=\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)+\frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&=\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')-\frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda')-\frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)+\frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&=-\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda')+\left(\frac{|\mathbf{p}|^2}{p_0^2}-1\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&=\delta_{\lambda\lambda'}-\frac{m^2}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}\tag{3.190}
$$

另一方面，

$$
[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]
$$

$$
\begin{aligned}
&= \left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]\\
&= [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] - \frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0^*(\mathbf{p},\lambda')\\
&\quad - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] + \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&\quad - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda') + \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= \left(p_0^2 - 2|\mathbf{p}|^2 + \frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') = \left[p_0^2 - |\mathbf{p}|^2 + \frac{|\mathbf{p}|^2}{p_0^2}(|\mathbf{p}|^2 - p_0^2)\right]\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= \left(m^2 - m^2\frac{|\mathbf{p}|^2}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') = \frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}
\tag{3.191}
$$

对于任意空间矢量 $\mathbf{a}$ 和 $\mathbf{b}$，利用 (1.84) 式，有

$$
\begin{aligned}
(\mathbf{p}\times\mathbf{a})\cdot(\mathbf{p}\times\mathbf{b}) &= \varepsilon^{ijk}p^j a^k \varepsilon^{imn}p^m b^n = (\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km})p^j a^k p^m b^n\\
&= p^j a^k p^j b^k - p^j a^k p^k b^j = |\mathbf{p}|^2\,\mathbf{a}\cdot\mathbf{b} - (\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b}),
\end{aligned}
\tag{3.192}
$$

从而，可得

$$
\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] &= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')]\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda')\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda').
\end{aligned}
\tag{3.193}
$$

于是，(3.188) 式化为

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda') &= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} - m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') + m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&\quad + |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda') + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')\\
&= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} + E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda')\\
&= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} - E_{\mathbf{p}}^2\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda') = 2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\end{aligned}
\tag{3.194}
$$

因此，

$$
f_1(\mathbf{p},\lambda,\lambda') = f_1^*(\mathbf{p},\lambda,\lambda') = 2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\tag{3.195}
$$

接着，我们计算 $f_2(\mathbf{p},\lambda,\lambda')$。由 (3.145) 和 (3.167) 式，可得

$$
\begin{aligned}
\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda') &= \left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\mathbf{p}}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right]\\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^0(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')
\end{aligned}
$$

$$= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{1}{E_{\mathbf{p}}^2}(2E_{\mathbf{p}}^2 - |\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda'). \tag{3.196}$$

另一方面，

$$\begin{aligned}
&[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]\\
=&\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right]\\
=&[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] + \frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0(-\mathbf{p},\lambda')\\
&-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] - \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&-p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda') + \frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
&+\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&\left(-p_0^2 + 2|\mathbf{p}|^2 - \frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') = -\frac{1}{E_{\mathbf{p}}^2}(E_{\mathbf{p}}^2 - |\mathbf{p}|^2)^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&-\frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda'),
\end{aligned} \tag{3.197}$$

而

$$\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] &= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(-\mathbf{p},\lambda')\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda').
\end{aligned} \tag{3.198}$$

于是，(3.189) 式化为

$$\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda') =& -E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') + m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
&+|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda') + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')\\
=& (-2E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 + E_{\mathbf{p}}^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') = 0.
\end{aligned} \tag{3.199}$$

因此，

$$f_2(\mathbf{p},\lambda,\lambda') = f_2^*(\mathbf{p},\lambda,\lambda') = 0. \tag{3.200}$$

将 (3.195) 和 (3.200) 式代入 (3.187) 式，再利用产生湮灭算符的对易关系 (3.171)，可得有质量矢量场的哈密顿量为

$$\begin{aligned}
H =& \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'}\right) = \sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}\right)\\
=& \sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\,a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + (2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}\frac{3}{2}E_{\mathbf{p}}.
\end{aligned} \tag{3.201}$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和，第二项是零点能。

根据 (1.158) 式，有质量矢量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi_i \nabla A^i \\
&= -\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0) \left[\tilde{\varepsilon}_i(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x}\right] \\
&\qquad \times \left[i\mathbf{q}\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{q},\lambda'} e^{-iq\cdot x} - i\mathbf{q}\varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq\cdot x}\right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q\; p_0 \mathbf{q}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \Big[-\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q)\cdot x} \\
&\qquad - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q)\cdot x} + \tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} \\
&\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p\, d^3q\; p_0 \mathbf{q}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \Big\{ -\delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0-q_0)t} \\
&\qquad\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0-q_0)t}\Big] \\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0+q_0)t} \\
&\qquad\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0+q_0)t}\Big]\Big\} \\
&= -\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \\
&\qquad + \tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}
\tag{3.202}
$$

由 (3.145) 和 (3.166) 式可得

$$
\begin{aligned}
\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') &= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0} p_i \varepsilon^i(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda'),
\end{aligned}
\tag{3.203}
$$

从而，有

$$
\begin{aligned}
&-\sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t}\right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big\{[\varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \varepsilon_0^*(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda')] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t}\Big\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} \Big\{[\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda) - \varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)] a_{-\mathbf{p},\lambda'} a_{\mathbf{p},\lambda} e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda) - \varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)] a_{-\mathbf{p},\lambda'}^\dagger a_{\mathbf{p},\lambda}^\dagger e^{2iE_{\mathbf{p}}t}\Big\}
\end{aligned}
$$

$$= -\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\Big\{[\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda)-\varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}$$
$$+[\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda)-\varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big\}. \tag{3.204}$$

上式第二步进行了 $\mathbf{p}\to-\mathbf{p}$ 的替换和 $\lambda\leftrightarrow\lambda'$ 的互换，由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和，这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.171)。留意到第一步与第三步的结果互为相反数，可知上式为零。因此，(3.202) 式最后两行方括号中最后两项没有贡献。再利用 (3.161) 式，可得

$$\mathbf{P} = -\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\left[-\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}-\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\right]=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\left[a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}+a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}\right]$$
$$=\sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+\frac{3}{2}\delta^{(3)}(0)\int d^3p\,\mathbf{p}=\sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}. \tag{3.205}$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m=0$ 时，由 (3.100) 和 (3.101) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 的形式不变，但 (3.111) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p},3)$ 的良好定义。实际上，在满足正确归一化的条件下，$m=0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面，由于无质量矢量粒子的动量 p^μ 的内积为 $p^2=0$，也不能像 (3.115) 式那样将类时极化矢量 $e^\mu(\mathbf{p},0)$ 取为正比于 p^μ 的矢量，否则将出现 $e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0)=0$ 而不能得到正确的归一化。因此，我们需要重新定义 $e^\mu(\mathbf{p},3)$ 和 $e^\mu(\mathbf{p},0)$。

在用 (3.100) 和 (3.101) 式定义 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 时，我们已经选取了一个特定的惯性参考系。在这个参考系中，可以定义一个类时单位矢量

$$n^\mu=(1,0,0,0), \tag{3.206}$$

它的 Lorentz 不变内积是

$$n^2=1. \tag{3.207}$$

然后，将类时极化矢量 $e^\mu(\mathbf{p},0)$ 在此参考系中的形式就取为 n^μ，即

$$e^\mu(\mathbf{p},0)=n^\mu. \tag{3.208}$$

$e^\mu(\mathbf{p},0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面，纵向极化矢量 $e^\mu(\mathbf{p},3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式：

$$e^\mu(\mathbf{p},3)=\frac{p^\mu-(p\cdot n)n^\mu}{p\cdot n}. \tag{3.209}$$

$p^2 = (p^0)^2 - |\mathbf{p}|^2 = 0$ 表明

$$p^0 = |\mathbf{p}|, \tag{3.210}$$

从而，$e^\mu(\mathbf{p}, 3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n} = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right). \tag{3.211}$$

这样定义的 $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 满足正交归一关系 (3.98)：

$$\begin{aligned}
e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 0) &= n^2 = 1, \quad e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|^2} = -1; &(3.212)\\
e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 1) &= e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 2) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 3) = 0; &(3.213)\\
e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, i) &= -\frac{1}{|\mathbf{p}|}\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. &(3.214)
\end{aligned}$$

此外，可以验证，由 (3.100)、(3.101)、(3.208) 和 (3.209) 式定义的这组极化矢量确实满足完备性关系 (3.99)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\
&= e_\mu(\mathbf{p}, 0)e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1)e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2)e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3)e_\nu(\mathbf{p}, 3) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_\mathrm{T}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1p^3p^1p^3 & p^1p^3p^2p^3 & -p^1p^3|\mathbf{p}_\mathrm{T}|^2 \\ 0 & p^2p^3p^1p^3 & p^2p^3p^2p^3 & -p^2p^3|\mathbf{p}_\mathrm{T}|^2 \\ 0 & -|\mathbf{p}_\mathrm{T}|^2p^1p^3 & -|\mathbf{p}_\mathrm{T}|^2p^2p^3 & |\mathbf{p}_\mathrm{T}|^4 \end{pmatrix} \\
&\quad - \frac{1}{|\mathbf{p}_\mathrm{T}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2p^2 & -p^2p^1 & 0 \\ 0 & -p^1p^2 & p^1p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1p^1 & p^1p^2 & p^1p^3 \\ 0 & p^2p^1 & p^2p^2 & p^2p^3 \\ 0 & p^3p^1 & p^3p^2 & p^3p^3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2(p^3)^2+(p^2)^2|\mathbf{p}|^2+(p^1)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^3}{|\mathbf{p}|^2} - \frac{p^1p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{(p^2)^2(p^3)^2+(p^1)^2|\mathbf{p}|^2+(p^2)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^2p^3}{|\mathbf{p}|^2} - \frac{p^2p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1p^3}{|\mathbf{p}|^2} - \frac{p^1p^3}{|\mathbf{p}|^2} & \frac{p^2p^3}{|\mathbf{p}|^2} - \frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{|\mathbf{p}_\mathrm{T}|^2+(p^3)^2}{|\mathbf{p}|^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}.
\end{aligned} \tag{3.215}$$

不过，$e^\mu(\mathbf{p},0)$ 和 $e^\mu(\mathbf{p},3)$ 都不满足四维横向条件：

$$p_\mu e^\mu(\mathbf{p},0) = p\cdot n = p^0 = |\mathbf{p}|,\quad p_\mu e^\mu(\mathbf{p},3) = -\frac{\mathbf{p}\cdot\mathbf{p}}{|\mathbf{p}|} = -|\mathbf{p}| = -p\cdot n. \tag{3.216}$$

横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 具有求和关系

$$\begin{aligned}
-\sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) &= \sum_{\sigma=1}^{2} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu} - g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0) - g_{33}e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\
&= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu-(p\cdot n)n_\mu}{p\cdot n}\frac{p_\nu-(p\cdot n)n_\nu}{p\cdot n}\\
&= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu p_\nu-(p\cdot n)p_\mu n_\nu-(p\cdot n)p_\nu n_\mu+(p\cdot n)^2 n_\mu n_\nu}{(p\cdot n)^2}\\
&= g_{\mu\nu} + \frac{p_\mu p_\nu}{(p\cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n},
\end{aligned} \tag{3.217}$$

即

$$\sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}. \tag{3.218}$$

根据 (3.125) 式，作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 满足

$$\begin{aligned}
\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) &= \frac{1}{2}[e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)]\\
&\quad+\frac{1}{2}[-e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][-e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)]\\
&= e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)+e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2) = \sum_{\sigma=1}^{2} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma),
\end{aligned} \tag{3.219}$$

因而具有**求和关系**

$$\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}. \tag{3.220}$$

四维横向条件 $p_\mu\varepsilon^\mu(\mathbf{p},\pm)=0$ 在上式中体现为

$$p^\nu\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -p_\mu - \frac{p_\mu p^2}{(p\cdot n)^2} + \frac{p_\mu(p\cdot n)+p^2 n_\mu}{p\cdot n} = -p_\mu + p_\mu = 0. \tag{3.221}$$

3.4.2　无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中，令参数 $m=0$，就得到自由无质量实矢量场 $A^\mu(x)$ 的**拉氏量**

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.222}$$

其中 $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$。同理，令 Proca 方程中 $m=0$，就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu} = 0. \tag{3.223}$$

根据 1.5 节的讨论，这个方程就是无源的 **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子，**光子**是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作**规范变换** (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu\chi(x), \tag{3.224}$$

其中，作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数，依赖于时空坐标，因而这样的变换是**局域** (local) 变换。在此规范变换下，场强张量不变：

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu[A^\nu(x) + \partial^\nu\chi(x)] - \partial^\nu[A^\mu(x) + \partial^\mu\chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu\partial^\nu\chi(x) - \partial^\nu\partial^\mu\chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \tag{3.225}$$

因而，拉氏量 (3.222) 和无源 Maxwell 方程 (3.223) 都不会改变，这称为**规范对称性** (gauge symmetry)。

在经典电动力学中，这种对称性广为人知，它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定，因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$，也就是说，不依赖于**规范**的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算，常常将规范固定下来，使得计算过程依赖于选取的规范，不过，最后得出的可观测量必须是规范不变 (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**，规范条件为

$$\partial_\mu A^\mu = 0. \tag{3.226}$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同，但是，在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件，而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$，令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2\chi(x) = -\partial_\mu A^\mu(x), \tag{3.227}$$

那么，作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件：

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2\chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \tag{3.228}$$

但是，经过这种变换之后，矢量场仍然没有被唯一地确定：对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$，取满足齐次波动方程

$$\partial^2\tilde{\chi}(x) = 0 \tag{3.229}$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换，都能得到满足 Lorenz 规范条件的另一个矢量场 $A'^\mu(x)$。可见，存在无穷多个规范等价的矢量场，它们描述相同的物理，而且全都满足 Lorenz 规范条件 (3.226)。

矢量场 $A^\mu(x)$ 有 4 个分量，因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是，上述规范等价性表明，$A^\mu(x)$ 并没有 3 个独立的自由度，否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上，无质量矢量场 $A^\mu(x)$ 只具有 *2 个独立的自由度*，也就是说，有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论：电磁波具有 2 种独立的极化态，以螺旋度 λ 来表征的话，就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有质量矢量场 $A^\mu(x)$ 的量子化程序时，由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度，因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难，因为 Proca 方程要求 $A^0(x)$ 不是独立变量，而是由 (3.175) 式决定的：

$$A^0 = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}. \tag{3.230}$$

于是，以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的，自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时，(3.230) 式显然不能成立。因此，对于无质量矢量场，最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度，可以在拉氏量中增加一个不会影响最终物理结果的项：

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.231}$$

其中 ξ 是一个可以自由选取的实参数。可以看出，在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.226) 的情况下，由 (3.231) 式定义的 $\mathcal{L}_1$ 等价于由 (3.222) 式定义的 $\mathcal{L}$。新增的项 $-\frac{1}{2}\xi(\partial_\mu A^\mu)^2$ 破坏了规范对称性，相当于把规范固定下来，因而称为**规范固定项** (gauge-fixing term)。可以将 $\mathcal{L}_1$ 展开为

$$\mathcal{L}_1 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.232}$$

从而，A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}_1}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi(\partial_\nu A^\nu)\frac{\partial(\partial_\sigma A^\sigma)}{\partial(\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0}\partial_\nu A^\nu, \tag{3.233}$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi\partial_\mu A^\mu. \tag{3.234}$$

因此，$\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0。

现在，正则量子化程序要求算符 A^μ 和 π_μ 满足如下等时对易关系：

$$[A^\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)] = i\delta^\mu{}_\nu\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] = [\pi_\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)] = 0. \tag{3.235}$$

但是，这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子，利用 (3.234) 式，可得

$$[A^0(\mathbf{x},t), \partial_\mu A^\mu(\mathbf{y},t)] = -\frac{1}{\xi}[A^0(\mathbf{x},t), \pi_0(\mathbf{y},t)] = -\frac{i}{\xi}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.236}$$

上式在 $\mathbf{x}=\mathbf{y}$ 处非零，因而必有 $\partial_\mu A^\mu \neq 0$。所以，$A^\mu$ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.226)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$，但对于量子场 $A^\mu(x)$ 来说限制太强了，下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - \xi g^{\mu\nu}(\partial_\rho A^\rho), \quad \frac{\partial\mathcal{L}_1}{\partial A_\nu} = 0, \tag{3.237}$$

可得，与 $\mathcal{L}_1$ 对应的 Euler-Lagrange 方程为

$$0 = \partial_\mu \frac{\partial\mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu\partial_\mu A^\mu - \xi g^{\mu\nu}\partial_\mu(\partial_\rho A^\rho) = -\partial^2 A^\nu + (1-\xi)\partial^\nu(\partial_\rho A^\rho), \tag{3.238}$$

即

$$\partial^2 A^\mu - (1-\xi)\partial^\mu(\partial_\nu A^\nu) = 0. \tag{3.239}$$

若取 $\xi = 1$，则上式化为 **d'Alembert 方程**

$$\partial^2 A^\mu(x) = 0, \tag{3.240}$$

可以看作无质量情况下的 Klein-Gordon 方程。可见，将规范固定参数取为

$$\xi = 1 \tag{3.241}$$

将有利于简化计算，这种取法称为 **Feynman 规范**，本节后续计算采用这个规范。在 Feynman 规范下，拉氏量化为

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu) \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu) - \frac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu - \frac{1}{2}\partial^\mu(A_\mu\partial_\nu A^\nu) + \frac{1}{2}A_\mu\partial^\mu\partial_\nu A^\nu \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu - A^\mu\partial_\nu A^\nu).
\end{aligned} \tag{3.242}$$

上式最后一行第二项是一个全散度，它不会影响作用量和运动方程，可以舍弃。因此，可以采用更加简化的拉氏量

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \tag{3.243}$$

此时，共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}_2}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu. \tag{3.244}$$

对于 d'Alembert 方程 (3.240)，平面波解的正能解和负能解分别正比于 $\exp(-ip\cdot x)$ 和 $\exp(ip\cdot x)$，其中

$$p^0 = E_{\mathbf{p}} = |\mathbf{p}|. \tag{3.245}$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p},\sigma)$，可以对无质量矢量场 $A^\mu(\mathbf{x},t)$ 作如下平面波展开：

$$A^\mu(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right). \tag{3.246}$$

容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t). \tag{3.247}$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x},t) = -\partial_0 A_\mu = \int \frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e_\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right), \tag{3.248}$$

它也满足自共轭条件

$$[\pi_\mu(\mathbf{x},t)]^\dagger = \pi_\mu(\mathbf{x},t). \tag{3.249}$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}A^\mu &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\, \frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right]
\end{aligned} \tag{3.250}$$

和

$$\begin{aligned}
&\int d^3x\, e^{iq\cdot x}\partial_0 A^\mu \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{-iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right],
\end{aligned} \tag{3.251}$$

以及正交归一关系 (3.98)，可得

$$\begin{aligned}
e_\mu(\mathbf{q},\sigma')\int d^3x\, e^{iq\cdot x}(\partial_0 A^\mu - iq_0 A^\mu) &= e_\mu(\mathbf{q},\sigma')\,\frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} \\
&= -i\sqrt{2E_\mathbf{q}}\sum_{\sigma=0}^{3} g_{\sigma'\sigma}a_{\mathbf{q};\sigma'} = -i\sqrt{2E_\mathbf{q}}\, g_{\sigma'\sigma'}a_{\mathbf{q};\sigma'}.
\end{aligned} \tag{3.252}$$

注意，虽然上式出现了重复的指标 σ'，但此处不需要对 σ' 求和。于是，有

$$a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_\mathbf{p}}}\, g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)\int d^3x\, e^{ip\cdot x}(\partial_0 A^\mu - ip_0 A^\mu). \tag{3.253}$$

对上式取厄米共轭，得

$$a^\dagger_{\mathbf{p};\sigma} = \frac{-i}{\sqrt{2E_\mathbf{p}}}\, g_{\sigma\sigma} e_\mu(\mathbf{p},\sigma) \int d^3x\, e^{-ip\cdot x}\, (\partial_0 A^\mu + ip_0 A^\mu). \tag{3.254}$$

根据等时对易关系 (3.235)，湮灭算符与产生算符的对易关系为

$$\begin{aligned}
[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\quad \times [\partial_0 A^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\, \partial_0 A^\nu(\mathbf{y},t) + iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\quad \times [-\pi^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\, -\pi^\nu(\mathbf{y},t) + iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\quad \times \{-iq_0\, [\pi^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] + ip_0\, [A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)]\} \\
&= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\, [-(p_0+q_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') e^{i(E_\mathbf{p}-E_\mathbf{q})t} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{p},\sigma') \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{3.255}$$

倒数第二步用到正交归一关系 (3.98)。另一方面，两个湮灭算符之间的对易关系为

$$\begin{aligned}
[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\quad \times [\partial_0 A^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\, \partial_0 A^\nu(\mathbf{y},t) - iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\quad \times [-\pi^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\, -\pi^\nu(\mathbf{y},t) - iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\quad \times \{iq_0\, [\pi^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] + ip_0\, [A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)]\} \\
&= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma) e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)}\, [(q_0-p_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \int d^3x\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') e^{i(E_\mathbf{p}+E_\mathbf{q})t} (2\pi)^3 \delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0.
\end{aligned} \tag{3.256}$$

归纳起来，产生湮灭算符的对易关系为

$$[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p}-\mathbf{q}), \qquad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a^\dagger_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = 0. \tag{3.257}$$

3.4.4 哈密顿量和总动量

根据 (1.119)、(3.244) 和 (3.243) 式，无质量矢量场的哈密顿量密度是

$$
\begin{aligned}
\mathcal{H} &= \pi_\mu \partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu \\
&= -\frac{1}{2}(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_i A_\mu)\partial^i A^\mu = -\frac{1}{2}\left[\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right].
\end{aligned} \tag{3.258}
$$

于是，哈密顿量表达为

$$
\begin{aligned}
H &= \int d^3x\, \mathcal{H} = -\frac{1}{2}\int d^3x \left[\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\quad \times \Big[(ip_0)(iq_0)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\left(a_{\mathbf{q},\sigma'}e^{-iq\cdot x} - a^\dagger_{\mathbf{q},\sigma'}e^{iq\cdot x}\right) \\
&\qquad + \left(i\mathbf{p}\, a_{\mathbf{p};\sigma}e^{-ip\cdot x} - i\mathbf{p}\, a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\, a_{\mathbf{q};\sigma'}e^{-iq\cdot x} - i\mathbf{q}\, a^\dagger_{\mathbf{q};\sigma'}e^{iq\cdot x}\right)\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')\Big[(p_0q_0 + \mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p-q)\cdot x} \\
&\quad + (p_0q_0 + \mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p-q)\cdot x} + (p_0q_0 - \mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p+q)\cdot x} \\
&\quad + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p+q)\cdot x}\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')(p_0q_0 + \mathbf{p}\cdot\mathbf{q}) \\
&\quad \times\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p_0-q_0)t}\right] \\
&\qquad - \delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p_0+q_0)t}\right]\Big\} \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E^2_{\mathbf{p}} + |\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) \\
&\qquad\qquad - e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E^2_{\mathbf{p}} - |\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_{\mathbf{p}}t}\right)\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) \\
&= -\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\, g_{\sigma\sigma'}\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) = \int \frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}\sum_{\sigma=0}^{3}\left(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) + (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2 \\
&= \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) + (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\, 2E_{\mathbf{p}}.
\end{aligned} \tag{3.259}
$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负，与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规，时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论，将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态，满足

$$a_{\mathbf{p};\sigma}|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad H|0\rangle = E_{\mathrm{vac}}|0\rangle, \quad E_{\mathrm{vac}} = 2\delta^{(3)}(0)\int d^3p\, E_{\mathbf{p}}. \tag{3.260}$$

动量为 $\mathbf{p}$、极化态为 σ 的单粒子态定义为

$$|\mathbf{p};\sigma\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, a^\dagger_{\mathbf{p};\sigma}|0\rangle. \tag{3.261}$$

从而，由

$$\begin{aligned}[H, a^\dagger_{\mathbf{p};\sigma}] &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^{3} (-g_{\sigma'\sigma'})[a^\dagger_{\mathbf{q};\sigma'} a_{\mathbf{q};\sigma'}, a^\dagger_{\mathbf{p};\sigma}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^{3} (-g_{\sigma'\sigma'}) a^\dagger_{\mathbf{q};\sigma'}[a_{\mathbf{q};\sigma'}, a^\dagger_{\mathbf{p};\sigma}] \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^{3} (-g_{\sigma'\sigma'}) a^\dagger_{\mathbf{q};\sigma'} (2\pi)^3 (-g_{\sigma'\sigma})\delta^{(3)}(\mathbf{q}-\mathbf{p}) \\ &= E_{\mathbf{p}} \sum_{\sigma'=0}^{3} g_{\sigma'\sigma'} g_{\sigma'\sigma} a^\dagger_{\mathbf{p};\sigma'} = E_{\mathbf{p}} a^\dagger_{\mathbf{p};\sigma}\end{aligned} \tag{3.262}$$

可得

$$\begin{aligned} H|\mathbf{p};\sigma\rangle &= \sqrt{2E_{\mathbf{p}}}\, H a^\dagger_{\mathbf{p};\sigma}|0\rangle = \sqrt{2E_{\mathbf{p}}}\,(E_{\mathbf{p}} a^\dagger_{\mathbf{p};\sigma} + a^\dagger_{\mathbf{p};\sigma} H)|0\rangle \\ &= \sqrt{2E_{\mathbf{p}}}\,(E_{\mathbf{p}} + E_{\mathrm{vac}}) a^\dagger_{\mathbf{p};\sigma}|0\rangle = (E_{\mathbf{p}} + E_{\mathrm{vac}})|\mathbf{p};\sigma\rangle. \end{aligned} \tag{3.263}$$

这似乎是一个正常的结果，说明单粒子态 $|\mathbf{p};\sigma\rangle$ 比真空多了一份能量 $E_{\mathbf{p}}$。

利用产生湮灭算符的对易关系 (3.257)，可以计算单粒子态的内积：

$$\begin{aligned}\langle \mathbf{q};\sigma'|\mathbf{p};\sigma\rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}}\,\langle 0| a_{\mathbf{q};\sigma'} a^\dagger_{\mathbf{p};\sigma}|0\rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}}\,\langle 0|\left[a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} - (2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]|0\rangle \\ &= -2E_{\mathbf{p}}(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{3.264}$$

于是，有

$$\langle \mathbf{p};0|\mathbf{p};0\rangle = -2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(0), \qquad \langle \mathbf{p};i|\mathbf{p};i\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(0), \quad i = 1,2,3. \tag{3.265}$$

上式表明，单粒子态 $|\mathbf{p};0\rangle$ 的自我内积是负的，从而导致它的能量期待值也是负的：

$$\langle \mathbf{p};0|H|\mathbf{p};0\rangle = (E_{\mathbf{p}} + E_{\mathrm{vac}})\langle \mathbf{p};0|\mathbf{p};0\rangle = -2E_{\mathbf{p}}(E_{\mathbf{p}} + E_{\mathrm{vac}})(2\pi)^3\delta^{(3)}(0) < 0. \tag{3.266}$$

这个负能量结果在物理上看起来是不可接受的，它的根源在于不定度规。

不过，如前所述，无质量矢量场只有 2 种独立的极化态，对应于 2 种横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$，纵向极化和类时极化都应该是非物理的。选取一定的规范条件，应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.226) 与正则量子化程序不相容，我们不能直接使用这个条件，而需要将它转换到物理 Hilbert 空间中的态的期待值上，要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle \Psi|\partial_\mu A^\mu(x)|\Psi\rangle = 0. \tag{3.267}$$

上式称为**弱 Lorenz 规范条件**。

$A^\mu(x)$ 的平面波展开式 (3.246) 可以分解成正能解和负能解两个部分：

$$A^\mu(x) = A^{(+)\mu}(x) + A^{(-)\mu}(x). \tag{3.268}$$

其中，正能解部分为

$$A^{(+)\mu}(\mathbf{x},t) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\, a_{\mathbf{p};\sigma}e^{-ip\cdot x}, \tag{3.269}$$

上式的厄米共轭即是负能解部分

$$A^{(-)\mu}(\mathbf{x},t) \equiv [A^{(+)\mu}(\mathbf{x},t)]^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\, a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}. \tag{3.270}$$

如果要求

$$\partial_\mu A^{(+)\mu}(x)\,|\Psi\rangle = 0 \tag{3.271}$$

对任意物理态 $|\Psi\rangle$ 成立，则伴随有

$$\langle\Psi|\,\partial_\mu A^{(-)\mu}(x) = \langle\Psi|\,[\partial_\mu A^{(+)\mu}(x)]^\dagger = 0, \tag{3.272}$$

从而，弱 Lorenz 规范条件 (3.267) 得到满足：

$$\langle\Psi|\,\partial_\mu A^\mu(x)\,|\Psi\rangle = \langle\Psi|\,\partial_\mu A^{(+)\mu}(x)\,|\Psi\rangle + \langle\Psi|\,\partial_\mu A^{(-)\mu}(x)\,|\Psi\rangle = 0. \tag{3.273}$$

利用 (3.109) 和 (3.216) 式，规范条件 (3.271) 可化为

$$\begin{aligned}
0 &= \partial_\mu A^{(+)\mu}(x)\,|\Psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ie^{-ip\cdot x}}{\sqrt{2E_\mathbf{p}}}\left[p_\mu e^\mu(\mathbf{p},0)a_{\mathbf{p};0} + p_\mu e^\mu(\mathbf{p},1)a_{\mathbf{p};1} + p_\mu e^\mu(\mathbf{p},2)a_{\mathbf{p};2} + p_\mu e^\mu(\mathbf{p},3)a_{\mathbf{p};3}\right]|\Psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ie^{-ip\cdot x}}{\sqrt{2E_\mathbf{p}}}\, p\cdot n\,(a_{\mathbf{p};0} - a_{\mathbf{p};3})\,|\Psi\rangle\,.
\end{aligned} \tag{3.274}$$

这意味着

$$(a_{\mathbf{p};0} - a_{\mathbf{p};3})\,|\Psi\rangle = 0 \tag{3.275}$$

对任意物理态 $|\Psi\rangle$ 和任意动量 $\mathbf{p}$ 成立。从而，也有

$$\langle\Psi|\,(a^\dagger_{\mathbf{p};0} - a^\dagger_{\mathbf{p};3}) = 0. \tag{3.276}$$

于是，

$$\langle\Psi|\,a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}\,|\Psi\rangle = \langle\Psi|\,a^\dagger_{\mathbf{p};3}a_{\mathbf{p};3}\,|\Psi\rangle\,. \tag{3.277}$$

这样一来，根据 (3.259) 式计算，$|\Psi\rangle$ 的能量期待值为

$$\langle\Psi|\,H\,|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}E_\mathbf{p}\,\langle\Psi|\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3} a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)|\Psi\rangle + E_{\mathrm{vac}}\,\langle\Psi|\Psi\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \sum_{\sigma=1}^{2} \langle\Psi|\, a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}\, |\Psi\rangle + E_{\text{vac}}\, \langle\Psi|\Psi\rangle\,. \tag{3.278}$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.158) 式可得无质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi_\mu \nabla A^\mu \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times (ip_0) \left(a_{\mathbf{p};\sigma} e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma} e^{ip\cdot x}\right) \left(i\mathbf{q}\, a_{\mathbf{q};\sigma'} e^{-iq\cdot x} - i\mathbf{q}\, a^\dagger_{\mathbf{q};\sigma'} e^{iq\cdot x}\right) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, p_0 \mathbf{q}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \left[-a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{i(p-q)\cdot x} + a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q)\cdot x} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{i(p+q)\cdot x}\right] \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p\, d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, p_0 \mathbf{q}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \Big\{ -\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{i(p_0-q_0)t}\right] \\
&\qquad\quad + \delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{i(p_0+q_0)t}\right] \Big\} \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big[-e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) \\
&\qquad\qquad - e_\mu(\mathbf{p},\sigma) e^\mu(-\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right) \Big].
\end{aligned} \tag{3.279}$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p} \to -\mathbf{p}$ 的替换和 $\sigma \leftrightarrow \sigma'$ 的互换，可得

$$\begin{aligned}
&-\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(\mathbf{p},\sigma) e^\mu(-\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right) \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2}\, e_\mu(-\mathbf{p},\sigma') e^\mu(\mathbf{p},\sigma) \left(a_{-\mathbf{p};\sigma'} a_{\mathbf{p};\sigma} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p};\sigma'} a^\dagger_{\mathbf{p};\sigma} e^{2iE_{\mathbf{p}}t}\right) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(-\mathbf{p},\sigma') e^\mu(\mathbf{p},\sigma) \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right).
\end{aligned} \tag{3.280}$$

可以看出，上式为零。于是，总动量化为

$$\begin{aligned}
\mathbf{P} &= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, g_{\sigma\sigma'} \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \sum_{\sigma=0}^{3} (-g_{\sigma\sigma}) \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}\right)
\end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}) + \delta^{(3)}(0)\int d^3p\,\frac{\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right). \tag{3.281}$$

根据 (3.277) 式，物理态 $|\Psi\rangle$ 的动量期待值为

$$\langle\Psi|\,\mathbf{P}\,|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,\langle\Psi|\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\sum_{\sigma=1}^{2}\langle\Psi|\,a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\,|\Psi\rangle. \tag{3.282}$$

同样，只有两种物理的横向极化才对动量有净贡献。

通过线性组合，可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$a_{\mathbf{p},\pm} \equiv \frac{1}{\sqrt{2}}(\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2}), \tag{3.283}$$

相应的产生算符可以通过取厄米共轭得到。反过来，有

$$a_{\mathbf{p};1} = -\frac{1}{\sqrt{2}}(a_{\mathbf{p},+} - a_{\mathbf{p},-}), \quad a_{\mathbf{p};2} = -\frac{i}{\sqrt{2}}(a_{\mathbf{p},+} + a_{\mathbf{p},-}). \tag{3.284}$$

利用对易关系 (3.257)，可得

$$\begin{aligned}
[a_{\mathbf{p},\pm}, a^\dagger_{\mathbf{q},\pm}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2},\, \mp a^\dagger_{\mathbf{q};1} - ia^\dagger_{\mathbf{q};2}] = \frac{1}{2}[a_{\mathbf{p};1},\, a^\dagger_{\mathbf{q};1}] + \frac{1}{2}[a_{\mathbf{p};2},\, a^\dagger_{\mathbf{q};2}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),\\
[a_{\mathbf{p},\pm}, a^\dagger_{\mathbf{q},\mp}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2},\, \pm a^\dagger_{\mathbf{q};1} - ia^\dagger_{\mathbf{q};2}] = -\frac{1}{2}[a_{\mathbf{p};1},\, a^\dagger_{\mathbf{q};1}] + \frac{1}{2}[a_{\mathbf{p};2},\, a^\dagger_{\mathbf{q};2}] = 0,\\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2},\, \mp a_{\mathbf{q};1} + ia_{\mathbf{q};2}] = 0,\\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + ia_{\mathbf{p};2},\, \pm a_{\mathbf{q};1} + ia_{\mathbf{q};2}] = 0.
\end{aligned} \tag{3.285}$$

于是，这组产生湮灭算符的对易关系可以整理为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0, \quad \lambda, \lambda' = \pm. \tag{3.286}$$

根据 (3.125) 式，可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 表示 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$：

$$e^\mu(\mathbf{p},1) = -\frac{1}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)], \quad e^\mu(\mathbf{p},2) = \frac{i}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)]. \tag{3.287}$$

从而，有

$$\begin{aligned}
\sum_{\sigma=1}^{2} e^\mu(\mathbf{p},\sigma)a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p},1)a_{\mathbf{p};1} + e^\mu(\mathbf{p},2)a_{\mathbf{p};2}\\
&= \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} + a_{\mathbf{p},-})\\
&= \varepsilon^\mu(\mathbf{p},+)a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p},-)a_{\mathbf{p},-} = \sum_{\lambda=\pm}\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},
\end{aligned} \tag{3.288}$$

取厄米共轭，得

$$\sum_{\sigma=1}^{2} e^{\mu}(\mathbf{p},\sigma)a^{\dagger}_{\mathbf{p};\sigma} = \sum_{\lambda=\pm} \varepsilon^{\mu*}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}. \tag{3.289}$$

于是，可以把 $A^{\mu}(x)$ 的平面波展开式 (3.246) 改写成

$$\begin{aligned} A^{\mu}(\mathbf{x},t) =& \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0,3} e^{\mu}(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} + a^{\dagger}_{\mathbf{p};\sigma}e^{ip\cdot x}\right) \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[\varepsilon^{\mu}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}e^{ip\cdot x}\right], \end{aligned} \tag{3.290}$$

第一行对应于非物理极化态，第二行对应于两种物理的螺旋度本征极化态。可见，(3.283) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda = \pm$ 对应的湮灭算符。

此外，由 (3.284) 式可得

$$\begin{aligned} \sum_{\sigma=1}^{2} a^{\dagger}_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma} =& a^{\dagger}_{\mathbf{p};1}a_{\mathbf{p};1} + a^{\dagger}_{\mathbf{p};2}a_{\mathbf{p};2} = \frac{1}{2}(a^{\dagger}_{\mathbf{p},+} - a^{\dagger}_{\mathbf{p},-})(a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}(a^{\dagger}_{\mathbf{p},+} + a^{\dagger}_{\mathbf{p},-})(a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ =& a^{\dagger}_{\mathbf{p},+}a_{\mathbf{p},+} + a^{\dagger}_{\mathbf{p},-}a_{\mathbf{p},-} = \sum_{\lambda=\pm} a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}, \end{aligned} \tag{3.291}$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle\Psi| H |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\lambda=\pm} \langle\Psi| a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} |\Psi\rangle + E_{\mathrm{vac}}\langle\Psi|\Psi\rangle, \tag{3.292}$$

$$\langle\Psi| \mathbf{P} |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\lambda=\pm} \langle\Psi| a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} |\Psi\rangle. \tag{3.293}$$

第 4 章　旋量场

4.1　Lorentz 群的旋量表示

旋量表示 (spinor representation) 是 Lorentz 群的一个线性表示，它在物理上扮演着非常重要的角色，P. A. M. Dirac 在 1928 年首次将它引入到描述电子的理论中。3.1 节提到，Lorentz 群的线性表示可以通过构造满足 Lorentz 代数关系 (3.20) 的生成元矩阵来得到，下面我们就用这样的方式来建立旋量表示。

首先，我们假设能够找到一组满足如下**反对易关系**的 $N\times N$ 矩阵 γ^μ $(\mu=0,1,2,3)$：

$$\{\gamma^\mu,\gamma^\nu\}\equiv\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}\mathbf{1}=2g^{\mu\nu}. \tag{4.1}$$

最后一步是一种简写，省略了 $N\times N$ 单位矩阵 $\mathbf{1}$。这样的 γ^μ 称为 **Dirac 矩阵**。然后，以 Dirac 矩阵的对易子定义另一组 $N\times N$ 矩阵

$$S^{\mu\nu}\equiv\frac{i}{4}[\gamma^\mu,\gamma^\nu]. \tag{4.2}$$

显然，$S^{\mu\nu}$ 关于 μ 和 ν 反对称：

$$S^{\mu\nu}=-S^{\nu\mu}. \tag{4.3}$$

因而 $S^{\mu\nu}$ 的独立分量有 6 个。

利用对易子公式

$$[AB,C]=ABC+ACB-ACB-CAB=A\{B,C\}-\{A,C\}B, \tag{4.4}$$

可得

$$\begin{aligned}[S^{\mu\nu},\gamma^\rho]&=\frac{i}{4}[\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu,\gamma^\rho]=\frac{i}{4}[\gamma^\mu\gamma^\nu-(2g^{\nu\mu}-\gamma^\mu\gamma^\nu),\gamma^\rho]=\frac{i}{2}[\gamma^\mu\gamma^\nu,\gamma^\rho]-\frac{i}{2}[g^{\nu\mu},\gamma^\rho]\\&=\frac{i}{2}[\gamma^\mu\gamma^\nu,\gamma^\rho]=\frac{i}{2}(\gamma^\mu\{\gamma^\nu,\gamma^\rho\}-\{\gamma^\mu,\gamma^\rho\}\gamma^\nu)=i(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho}).\end{aligned} \tag{4.5}$$

从而，根据对易子公式 (2.11)，有

$$\begin{aligned}[S^{\mu\nu},S^{\rho\sigma}]&=\frac{i}{4}[S^{\mu\nu},\gamma^\rho\gamma^\sigma-\gamma^\sigma\gamma^\rho]=\frac{i}{4}([S^{\mu\nu},\gamma^\rho\gamma^\sigma]-[S^{\mu\nu},\gamma^\sigma\gamma^\rho])\\&=\frac{i}{4}([S^{\mu\nu},\gamma^\rho]\gamma^\sigma+\gamma^\rho[S^{\mu\nu},\gamma^\sigma]-[S^{\mu\nu},\gamma^\sigma]\gamma^\rho-\gamma^\sigma[S^{\mu\nu},\gamma^\rho])\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{4}[i(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})\gamma^\sigma + i\gamma^\rho(\gamma^\mu g^{\nu\sigma}-\gamma^\nu g^{\mu\sigma})\\
&\qquad - i(\gamma^\mu g^{\nu\sigma}-\gamma^\nu g^{\mu\sigma})\gamma^\rho - i\gamma^\sigma(\gamma^\mu g^{\nu\rho}-\gamma^\nu g^{\mu\rho})]\\
&= \frac{i^2}{4}(\gamma^\mu\gamma^\sigma g^{\nu\rho}-\gamma^\nu\gamma^\sigma g^{\mu\rho}+\gamma^\rho\gamma^\mu g^{\nu\sigma}-\gamma^\rho\gamma^\nu g^{\mu\sigma}\\
&\qquad -\gamma^\mu\gamma^\rho g^{\nu\sigma}+\gamma^\nu\gamma^\rho g^{\mu\sigma}-\gamma^\sigma\gamma^\mu g^{\nu\rho}+\gamma^\sigma\gamma^\nu g^{\mu\rho})\\
&= \frac{i^2}{4}[g^{\nu\rho}(\gamma^\mu\gamma^\sigma-\gamma^\sigma\gamma^\mu)-g^{\mu\rho}(\gamma^\nu\gamma^\sigma-\gamma^\sigma\gamma^\nu)-g^{\nu\sigma}(\gamma^\mu\gamma^\rho-\gamma^\rho\gamma^\mu)+g^{\mu\sigma}(\gamma^\nu\gamma^\rho-\gamma^\rho\gamma^\nu)]\\
&= i(g^{\nu\rho}S^{\mu\sigma}-g^{\mu\rho}S^{\nu\sigma}-g^{\nu\sigma}S^{\mu\rho}+g^{\mu\sigma}S^{\nu\rho}).
\end{aligned}
\tag{4.6}
$$

可见，$S^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)，因而是 Lorentz 群某个线性表示的生成元。以 $S^{\mu\nu}$ 生成的线性表示就是**旋量表示**。

根据 (3.2.1) 小节的讨论，一组变换参数 $\omega_{\mu\nu}$ 在 Lorentz 群的矢量表示中可以生成固有保时向的有限变换 (3.51)：

$$
\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^X,\quad X\equiv -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \tag{4.7}
$$

类似地，这组参数在旋量表示中生成了固有保时向的有限变换

$$
D(\Lambda) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = e^Y,\quad Y\equiv -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}. \tag{4.8}
$$

这样定义的 $D(\Lambda)$ 是旋量表示中的 Lorentz 变换矩阵，对于任意的 Lorentz 变换 Λ_1 和 Λ_2，满足同态关系

$$
D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1). \tag{4.9}
$$

由 (3.48) 式可得

$$
e^{-Y}e^{Y} = e^{-Y+Y} = e^{\mathbf{0}} = \mathbf{1}, \tag{4.10}
$$

故 $D(\Lambda)$ 的逆矩阵为

$$
D(\Lambda^{-1}) = D^{-1}(\Lambda) = e^{-Y} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{4.11}
$$

这里先来介绍一些将会用到的对易子公式。以如下方式定义 B 与 A 的多重对易子 $[B, A^{(n)}]$：

$$
\begin{aligned}
&[B,A^{(0)}] = B,\quad [B,A^{(1)}] = [B,A] = [[B,A^{(0)}],A]\\
&[B,A^{(2)}] = [[B,A],A] = [[B,A^{(1)}],A],\quad \cdots,\quad [B,A^{(n)}] = [[B,A^{(n-1)}],A].
\end{aligned}
\tag{4.12}
$$

于是，下式成立：

$$
BA^k = \sum_{n=0}^{k}\frac{k!}{(k-n)!n!}A^{k-n}[B,A^{(n)}]. \tag{4.13}
$$

下面用数学归纳法证明这个等式。

证明　当 $k=0$ 和 $k=1$ 时，(4.13) 式明显成立：

$$
BA^0 = B = [B,A^{(0)}] = \frac{0!}{(0-0)!0!}A^{0-0}[B,A^{(0)}], \tag{4.14}
$$

$$BA^1 = BA = AB + [B, A] = \frac{1!}{(1-0)!0!}A^{1-0}[B, A^{(0)}] + \frac{1!}{(1-1)!1!}A^{1-1}[B, A^{(1)}]. \tag{4.15}$$

假设 $k = m$ 时 (4.13) 式成立，则有

$$\begin{aligned}
BA^{m+1} &= \sum_{n=0}^{m}\frac{m!}{(m-n)!n!}A^{m-n}[B, A^{(n)}]A = \sum_{n=0}^{m}\frac{m!}{(m-n)!n!}A^{m-n}(A[B, A^{(n)}] + [[B, A^{(n)}], A]) \\
&= \sum_{n=0}^{m}\frac{m!}{(m-n)!n!}A^{m+1-n}[B, A^{(n)}] + \sum_{n=0}^{m}\frac{m!}{(m-n)!n!}A^{m-n}[B, A^{(n+1)}] \\
&= \sum_{n=0}^{m}\frac{m!}{(m-n)!n!}A^{m+1-n}[B, A^{(n)}] + \sum_{j=1}^{m+1}\frac{m!}{(m-j+1)!(j-1)!}A^{m-j+1}[B, A^{(j)}] \\
&= \frac{m!}{(m-0)!0!}A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m}\left[\frac{m!}{(m-n)!n!} + \frac{m!}{(m-n+1)!(n-1)!}\right]A^{m+1-n}[B, A^{(n)}] \\
&\quad + \frac{m!}{[m-(m+1)+1]![(m+1)-1]!}A^{m-(m+1)+1}[B, A^{(m+1)}] \\
&= A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m}\left[\frac{m!}{(m-n)!n!} + \frac{n}{m-n+1}\frac{m!}{(m-n)!n!}\right]A^{m+1-n}[B, A^{(n)}] \\
&\quad + A^{m-(m+1)+1}[B, A^{(m+1)}] \\
&= \frac{(m+1)!}{[(m+1)-0]!0!}A^{m+1}[B, A^{(0)}] + \sum_{n=1}^{m}\frac{(m+1)!}{(m-n+1)!n!}A^{m+1-n}[B, A^{(n)}] \\
&\quad + \frac{(m+1)!}{[(m+1)-(m+1)]!(m+1)!}A^{m-(m+1)+1}[B, A^{(m+1)}] \\
&= \sum_{n=0}^{m+1}\frac{(m+1)!}{[(m+1)-n]!n!}A^{(m+1)-n}[B, A^{(n)}],
\end{aligned} \tag{4.16}$$

即 $k = m + 1$ 时 (4.13) 式也成立。于是，(4.13) 式对任意非负整数 k 成立。**证毕**。

根据推广的阶乘定义 (3.46) 可以将 (4.13) 式右边的级数化成无穷级数：

$$BA^k = \sum_{n=0}^{\infty}\frac{k!}{(k-n)!n!}A^{k-n}[B, A^{(n)}]. \tag{4.17}$$

利用上式，可得

$$\begin{aligned}
e^{-A}Be^{A} &= e^{-A}\sum_{k=0}^{\infty}\frac{1}{k!}BA^k = e^{-A}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{\infty}\frac{k!}{(k-n)!n!}A^{k-n}[B, A^{(n)}] \\
&= e^{-A}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{(k-n)!}A^{k-n}[B, A^{(n)}] = e^{-A}\sum_{n=0}^{\infty}\frac{1}{n!}e^{A}[B, A^{(n)}] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}[B, A^{(n)}].
\end{aligned} \tag{4.18}$$

现在，我们继续讨论 Lorentz 群的旋量表示。由 (4.5) 和 (3.34) 式可得

$$[\gamma^\mu, S^{\rho\sigma}] = -[S^{\rho\sigma}, \gamma^\mu] = [S^{\sigma\rho}, \gamma^\mu] = i(\gamma^\sigma g^{\rho\mu} - \gamma^\rho g^{\sigma\mu}) = i(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu)\gamma^\nu = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu. \tag{4.19}$$

从而，有

$$
\begin{aligned}
[\gamma^\mu, Y^{(1)}] &= [\gamma^\mu, Y] = -\frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] = -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu = X^\mu{}_\nu\gamma^\nu,\\
[\gamma^\mu, Y^{(2)}] &= [[\gamma^\mu, Y^{(1)}], Y] = X^\mu{}_\nu[\gamma^\nu, Y] = X^\mu{}_\nu X^\nu{}_\rho\gamma^\rho = (X^2)^\mu{}_\nu\gamma^\nu,\\
&\cdots\\
[\gamma^\mu, Y^{(n)}] &= (X^n)^\mu{}_\nu\gamma^\nu.
\end{aligned}
\tag{4.20}
$$

于是，利用 (4.18) 式可以推出

$$
D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = e^{-Y}\gamma^\mu e^Y = \sum_{n=0}^{\infty}\frac{1}{n!}[\gamma^\mu, Y^{(n)}] = \sum_{n=0}^{\infty}\frac{1}{n!}(X^n)^\mu{}_\nu\gamma^\nu = (e^X)^\mu{}_\nu\gamma^\nu, \tag{4.21}
$$

即

$$
D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{4.22}
$$

上式是 γ^μ 在旋量表示中的 Lorentz 变换关系，它说明 γ^μ 是一个 Lorentz 矢量。$N\times N$ 单位矩阵 $\mathbf{1}$ 满足

$$
D^{-1}(\Lambda)\mathbf{1}D(\Lambda) = \mathbf{1}, \tag{4.23}
$$

因而 $\mathbf{1}$ 是一个 Lorentz 标量。生成元 $S^{\mu\nu}$ 的 Lorentz 变换形式为

$$
D^{-1}(\Lambda)S^{\mu\nu}D(\Lambda) = \frac{i}{4}[D^{-1}(\Lambda)\gamma^\mu D(\Lambda), D^{-1}(\Lambda)\gamma^\nu D(\Lambda)] = \frac{i}{4}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma[\gamma^\rho, \gamma^\sigma] = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma S^{\rho\sigma}, \tag{4.24}
$$

可见，$S^{\mu\nu}$ 是一个 2 阶反对称 Lorentz 张量。

$S^{\mu\nu}$ 是用 2 个 Dirac 矩阵的乘积构造出来的反对称张量，类似地，我们也可以用 3 个 Dirac 矩阵的乘积来构造一个 3 阶全反对称张量

$$
\Gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \equiv \frac{1}{3!}(\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho). \tag{4.25}
$$

上式第二步中的中括号表示对 μ、ν、ρ 三个指标作全反对称操作：在偶次置换前面加上正号，奇次置换前面加上负号，然后对所有置换求和并除以置换方式的数目。$\Gamma^{\mu\nu\rho}$ 的 Lorentz 变换形式是

$$
D^{-1}(\Lambda)\Gamma^{\mu\nu\rho}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Gamma^{\alpha\beta\gamma}. \tag{4.26}
$$

由全反对称性可知，$\Gamma^{\mu\nu\rho}$ 的独立分量只有 4 个，可取为 Γ^{012}、Γ^{023}、Γ^{013} 和 Γ^{123}。从 Dirac 矩阵的反对易关系 (4.1) 可知，当 $\mu \neq \nu$ 时，γ^μ 与 γ^ν 是反对易的，即

$$
\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad \mu \neq \nu. \tag{4.27}
$$

于是，从定义式 (4.25) 可得

$$
\Gamma^{012} = \gamma^0\gamma^1\gamma^2, \quad \Gamma^{023} = \gamma^0\gamma^2\gamma^3, \quad \Gamma^{013} = \gamma^0\gamma^1\gamma^3, \quad \Gamma^{123} = \gamma^1\gamma^2\gamma^3. \tag{4.28}
$$

更进一步，可以用 4 个 Dirac 矩阵的乘积来构造一个 4 阶全反对称张量

$$\begin{aligned}\Gamma^{\mu\nu\rho\sigma} &\equiv \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}\\ &\equiv \frac{1}{4!}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma+\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho+\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu-\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho-\gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu-\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\\ &\quad-\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho-\gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu-\gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu+\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu+\gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho\\ &\quad+\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu+\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu+\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma-\gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu-\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma-\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\\ &\quad-\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu-\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma-\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho+\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma+\gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho+\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu).\end{aligned} \tag{4.29}$$

从而，$\Gamma^{\mu\nu\rho\sigma}$ 具有如下性质：

$$\Gamma^{\mu\nu\rho\sigma}=\begin{cases}+\Gamma^{0123}, & (\mu,\nu,\rho,\sigma)\text{ 是 }(0,1,2,3)\text{ 的偶次置换，}\\ -\Gamma^{0123}, & (\mu,\nu,\rho,\sigma)\text{ 是 }(0,1,2,3)\text{ 的奇次置换，}\\ 0, & \text{其它情况。}\end{cases} \tag{4.30}$$

可见，它只有 1 个独立分量，可取为

$$\Gamma^{0123}=\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.31}$$

结合四维 Levi-Civita 符号的定义 (1.65)，可得

$$\Gamma^{\mu\nu\rho\sigma}=\varepsilon^{\mu\nu\rho\sigma}\Gamma^{0123}=\varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.32}$$

受到四维时空的维度限制，我们不能以同样的方式定义高于 4 阶的全反对称张量。现在，我们拥有一组矩阵

$$\{\mathbf{1},\gamma^\mu,S^{\mu\nu},\Gamma^{\mu\nu\rho},\Gamma^{\mu\nu\rho\sigma}\}, \tag{4.33}$$

它们各自的独立分量个数之和为 $1+4+6+4+1=16$。利用反对易关系 (4.1)，可以将任意多个 Dirac 矩阵的乘积转化为集合 (4.33) 中的矩阵与度规张量乘积的线性组合。例如，

$$\gamma^\mu\gamma^\nu=\frac{1}{2}\gamma^\mu\gamma^\nu-\frac{1}{2}\gamma^\nu\gamma^\mu+g^{\mu\nu}=\frac{1}{2}[\gamma^\mu,\gamma^\nu]+g^{\mu\nu}=-2iS^{\mu\nu}+g^{\mu\nu}. \tag{4.34}$$

又如，

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\rho &= \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho+\frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho=\frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho-\frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu+g^{\rho\nu}\gamma^\mu\\ &=\frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho-\frac{1}{4}\gamma^\nu\gamma^\mu\gamma^\rho+\frac{1}{2}g^{\mu\nu}\gamma^\rho-\frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu+\frac{1}{4}\gamma^\rho\gamma^\mu\gamma^\nu-\frac{1}{2}g^{\mu\rho}\gamma^\nu+g^{\rho\nu}\gamma^\mu\\ &=\frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho-\frac{1}{8}\gamma^\nu\gamma^\mu\gamma^\rho+\frac{1}{8}\gamma^\nu\gamma^\rho\gamma^\mu-\frac{1}{4}g^{\rho\mu}\gamma^\nu+\frac{1}{2}g^{\mu\nu}\gamma^\rho-\frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu\\ &\quad+\frac{1}{8}\gamma^\rho\gamma^\mu\gamma^\nu-\frac{1}{8}\gamma^\rho\gamma^\nu\gamma^\mu+\frac{1}{4}g^{\mu\nu}\gamma^\rho-\frac{1}{2}g^{\mu\rho}\gamma^\nu+g^{\rho\nu}\gamma^\mu\\ &=\frac{3!}{8}\Gamma^{\mu\nu\rho}+\frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho-\frac{1}{8}\gamma^\mu\gamma^\rho\gamma^\nu-\frac{3}{4}g^{\rho\mu}\gamma^\nu+\frac{3}{4}g^{\mu\nu}\gamma^\rho+g^{\rho\nu}\gamma^\mu\\ &=\frac{3}{4}\Gamma^{\mu\nu\rho}+\frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho+\frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho-\frac{1}{4}g^{\rho\nu}\gamma^\mu-\frac{3}{4}g^{\rho\mu}\gamma^\nu+\frac{3}{4}g^{\mu\nu}\gamma^\rho+g^{\rho\nu}\gamma^\mu\end{aligned}$$

$$= \frac{3}{4}\Gamma^{\mu\nu\rho} + \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + \frac{3}{4}g^{\rho\nu}\gamma^\mu, \tag{4.35}$$

故

$$\gamma^\mu\gamma^\nu\gamma^\rho = \Gamma^{\mu\nu\rho} - g^{\rho\mu}\gamma^\nu + g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu. \tag{4.36}$$

因此，对于由 Dirac 矩阵乘积的线性组合构造的矩阵，集合 (4.33) 构成一组完备的基底。

定义 γ^μ 对应的协变矢量

$$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu, \tag{4.37}$$

则

$$\gamma_0 = \gamma^0, \quad \gamma_i = -\gamma^i, \quad i = 1, 2, 3. \tag{4.38}$$

由反对易关系 (4.1) 有

$$(\gamma^0)^2 = \frac{1}{2}\{\gamma^0, \gamma^0\} = g^{00} = \mathbf{1}, \quad (\gamma^i)^2 = \frac{1}{2}\{\gamma^i, \gamma^i\} = g^{ii} = -\mathbf{1}. \tag{4.39}$$

我们约定 γ^0 是厄米矩阵，γ^i 是反厄米矩阵，即

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \tag{4.40}$$

则可得

$$(\gamma^0)^\dagger\gamma^0 = (\gamma^0)^2 = \mathbf{1}, \quad (\gamma^i)^\dagger\gamma^i = -(\gamma^i)^2 = \mathbf{1}. \tag{4.41}$$

可见，在此约定下，γ^0 和 γ^i 都是幺正矩阵。

这里引入一个新的矩阵

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.42}$$

从 (4.27) 式可得

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{cases} +\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的偶次置换，} \\ -\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的奇次置换。} \end{cases} \tag{4.43}$$

这种置换性质与四维 Levi-Civita 符号 (1.65) 相同，因而置换操作带来的符号在 $\varepsilon_{\mu\nu\rho\sigma}$ 与 $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ 的乘积中相互抵消，如

$$\varepsilon_{1023}\gamma^1\gamma^0\gamma^2\gamma^3 = -\varepsilon_{0123}(-\gamma^0\gamma^1\gamma^2\gamma^3) = \varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.44}$$

由此可得

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{4.45}$$

对于固有保时向 Lorentz 变换 (4.7)，用度规对 (1.73) 式升降指标，有

$$\varepsilon_{\mu\nu\rho\sigma} = \Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta\Lambda_\rho{}^\gamma\Lambda_\sigma{}^\delta\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}(\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu(\Lambda^{-1})^\gamma{}_\rho(\Lambda^{-1})^\delta{}_\sigma. \tag{4.46}$$

于是，γ^5 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)\gamma^5 D(\Lambda) = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$$

$$
\begin{aligned}
&= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}(\Lambda^{-1})^{\kappa}{}_{\mu}(\Lambda^{-1})^{\lambda}{}_{\nu}(\Lambda^{-1})^{\tau}{}_{\rho}(\Lambda^{-1})^{\varepsilon}{}_{\sigma}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} \\
&= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}\delta^{\kappa}{}_{\alpha}\delta^{\lambda}{}_{\beta}\delta^{\tau}{}_{\gamma}\delta^{\varepsilon}{}_{\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} = -\frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} = \gamma^5.
\end{aligned} \tag{4.47}
$$

可见，γ^5 是一个 Lorentz 标量。γ^5 的平方为

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0 = -(-\mathbf{1})^3 = \mathbf{1}. \tag{4.48}$$

根据约定 (4.40)，γ^5 是厄米矩阵：

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{4.49}$$

γ^5 与 γ^μ 反对易：

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu - \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu) = 0. \tag{4.50}$$

由 (4.32) 式可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon^{\mu\nu\rho\sigma}\gamma^5. \tag{4.51}$$

可见，$\Gamma^{\mu\nu\rho\sigma}$ 正比于 γ^5。此外，由 (4.28) 式有

$$\Gamma^{012} = \gamma^0\gamma^1\gamma^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^3\gamma^5 = i\gamma_3\gamma^5 = i\varepsilon^{0123}\gamma_3\gamma^5, \tag{4.52}$$

$$\Gamma^{023} = \gamma^0\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^5 = i\gamma_1\gamma^5 = i\varepsilon^{0231}\gamma_1\gamma^5, \tag{4.53}$$

$$\Gamma^{013} = \gamma^0\gamma^1\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = -\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^2\gamma^5 = -i\gamma_2\gamma^5 = i\varepsilon^{0132}\gamma_2\gamma^5, \tag{4.54}$$

$$\Gamma^{123} = \gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5 = -i\gamma_0\gamma^5 = i\varepsilon^{1230}\gamma_0\gamma^5. \tag{4.55}$$

综合起来，得

$$\Gamma^{\mu\nu\rho} = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5. \tag{4.56}$$

根据上式，$\Gamma^{\mu\nu\rho}$ 可以写成 $\gamma^\mu\gamma^5$ 的 4 个独立分量的线性组合。$\gamma^\mu\gamma^5$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)\gamma^\mu\gamma^5 D(\Lambda) = D^{-1}(\Lambda)\gamma^\mu D(\Lambda)D^{-1}(\Lambda)\gamma^5 D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu\gamma^5, \tag{4.57}$$

因而它是一个 Lorentz 矢量。再引入

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2S^{\mu\nu}, \tag{4.58}$$

它正比于 $S^{\mu\nu}$，所以也是一个 2 阶反对称 Lorentz 张量：

$$D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\sigma^{\mu\nu}. \tag{4.59}$$

于是，我们可以用 γ^5、$\gamma^\mu\gamma^5$ 和 $\sigma^{\mu\nu}$ 分别代替集合 (4.33) 中的 $\Gamma^{\mu\nu\rho\sigma}$、$\Gamma^{\mu\nu\rho}$ 和 $S^{\mu\nu}$ 作为基底，从而得到另一组完备的矩阵基底

$$\{\mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}, \tag{4.60}$$

它们各自的独立分量个数之和仍是 16。

依照约定 (4.40)，γ^0 即是厄米的又是幺正的，我们可以用它定义一个幺正变换矩阵 β：

$$\beta^{-1} = \beta^\dagger = \beta \equiv \gamma^0. \tag{4.61}$$

从而，有

$$\beta^{-1}\gamma^0\beta = \gamma^0\gamma^0\gamma^0 = +\gamma^0, \quad \beta^{-1}\gamma^i\beta = \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i. \tag{4.62}$$

根据宇称变换 $\mathcal{P}$ 的定义 (1.46)，可以将这两个式子合写为

$$\beta^{-1}\gamma^\mu\beta = \mathcal{P}^\mu{}_\nu\gamma^\nu. \tag{4.63}$$

这表明 β 相当于旋量表示中的宇称变换矩阵 $D(\mathcal{P})$，它是非固有保时向的，上式就是 γ^μ 的宇称变换形式。(4.62) 式说明 γ^0 是宇称本征态，本征值为 $+$，即具有**偶宇称**；γ^i 也是宇称本征态，本征值为 $-$，即具有**奇宇称**。虽然单位矩阵 $\mathbf{1}$ 与 γ_5 都是 Lorentz 标量，但它们的宇称是不同的：

$$\beta^{-1}\mathbf{1}\beta = +\mathbf{1}, \quad \beta^{-1}\gamma^5\beta = \gamma^0\gamma^5\gamma^0 = -\gamma^5\gamma^0\gamma^0 = -\gamma^5. \tag{4.64}$$

像 γ_5 这样具有奇宇称的 Lorentz 标量，称为**赝标量** (pseudoscalar)。此外，$\gamma^\mu\gamma^5$ 的宇称变换形式是

$$\beta^{-1}\gamma^\mu\gamma^5\beta = \beta^{-1}\gamma^\mu\beta\beta^{-1}\gamma^5\beta = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \tag{4.65}$$

即

$$\beta^{-1}\gamma^0\gamma^5\beta = -\gamma^0\gamma^5, \quad \beta^{-1}\gamma^i\gamma^5\beta = +\gamma^i\gamma^5. \tag{4.66}$$

可以看出，虽然 $\gamma^\mu\gamma^5$ 也是 Lorentz 矢量，但它的分量的宇称性质与 γ^μ 相反。宇称变换性质像 $\gamma^\mu\gamma^5$ 这样的 Lorentz 矢量称为**轴矢量** (axial vector)。最后，$\sigma^{\mu\nu}$ 的宇称变换形式为

$$\beta^{-1}\sigma^{\mu\nu}\beta = \frac{i}{2}[\beta^{-1}\gamma^\mu\beta, \beta^{-1}\gamma^\nu\beta] = \frac{i}{2}\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta[\gamma^\alpha, \gamma^\beta] = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\sigma^{\alpha\beta}, \tag{4.67}$$

即

$$\beta^{-1}\sigma^{0i}\beta = \mathcal{P}^0{}_\alpha\mathcal{P}^i{}_\beta\sigma^{\alpha\beta} = -\sigma^{0i}, \quad \beta^{-1}\sigma^{ij}\beta = \mathcal{P}^i{}_\alpha\mathcal{P}^j{}_\beta\sigma^{\alpha\beta} = +\sigma^{ij}. \tag{4.68}$$

可见，基底集合 (4.60) 是由标量 $\mathbf{1}$、赝标量 γ^5、矢量 γ^μ、轴矢量 $\gamma^\mu\gamma^5$ 和 2 阶反对称张量 $\sigma^{\mu\nu}$ 组成的，综合考虑固有保时向 Lorentz 变换和宇称变换，则这些基底的变换性质各不相同，因而它们彼此之间是相互独立的，总共有 16 个独立而完备的基底。由于独立的 $N \times N$ 矩阵最多有 N^2 个，为了得到 16 个这样的基底，需要 $N \geq 4$。我们考虑最简单的情况，将 Dirac 矩阵取为 4×4 矩阵。

4.2　Dirac 旋量场

在 Lorentz 群的旋量表示中，被变换矩阵 $D(\Lambda)$ 作用的态称为 **Dirac 旋量** (spinor)。由于 $D(\Lambda)$ 是 4×4 矩阵，一个 Dirac 旋量 ψ_a 应当具有 4 个分量 $(a = 1, 2, 3, 4)$，相应的 Lorentz 变

换形式为

$$\psi'_a = D_{ab}(\Lambda)\psi_b. \tag{4.69}$$

隐去旋量指标 a 和 b，上式化为

$$\psi' = D(\Lambda)\psi. \tag{4.70}$$

我们可以将 ψ 和 ψ' 看作列矢量，而上式右边的乘积就是线性代数中矩阵与列矢量的乘积。

进一步，如果 ψ_a 依赖于时空坐标 x^μ，它就成为 **Dirac 旋量场** $\psi_a(x)$。类似于 (3.67) 式，量子 Dirac 旋量场的 Lorentz 变换形式是

$$\psi'_a(x') = U^{-1}(\Lambda)\psi_a(x')U(\Lambda) = D_{ab}(\Lambda)\psi_b(x). \tag{4.71}$$

对于固有保时向 Lorentz 变换，由 (4.8) 式可得 $D_{ab}(\Lambda)$ 的无穷小形式为

$$D_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}, \tag{4.72}$$

于是，(4.71) 式的无穷小形式是

$$\psi'_a(x') = \psi_a(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}\psi_b(x). \tag{4.73}$$

将上式与 (1.168) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在旋量表示中对应于 $S^{\mu\nu}$。隐去旋量指标，则 (4.71) 式化为

$$\psi'(x') = U^{-1}(\Lambda)\psi(x')U(\Lambda) = D(\Lambda)\psi(x), \tag{4.74}$$

也可以写成

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x). \tag{4.75}$$

对于无穷小变换，根据 (3.59) 式，将 $\psi(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned}\psi(\Lambda^{-1}x) &= \psi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi(x) = \psi(x) - \omega_{\mu\nu}x^\nu\partial^\mu\psi(x) = \psi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi(x)\\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu}i(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x),\end{aligned} \tag{4.76}$$

其中 $L^{\mu\nu}$ 是 (3.63) 式定义的微分算符。从而，(4.75) 式右边展开到 ω 一阶项的形式为

$$D(\Lambda)\psi(\Lambda^{-1}x) = \left(\mathbf{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left[\psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x)\right] = \psi(x) - \frac{i}{2}\omega_{\mu\nu}(L^{\mu\nu} + S^{\mu\nu})\psi(x). \tag{4.77}$$

另一方面，根据 (3.6) 式可以将 (4.75) 式左边展开为

$$\begin{aligned}U^{-1}(\Lambda)\psi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\psi(x)\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu}\psi(x)J^{\mu\nu} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu}[\psi(x), J^{\mu\nu}].\end{aligned} \tag{4.78}$$

两相比较，得到

$$[\psi(x), J^{\mu\nu}] = (L^{\mu\nu} + S^{\mu\nu})\psi(x). \tag{4.79}$$

$S^{\mu\nu}$ 的空间分量等价于三维矢量

$$S^i \equiv \frac{1}{2}\varepsilon^{ijk}S^{jk}, \quad \mathbf{S} = (S^{23}, S^{31}, S^{12}). \tag{4.80}$$

再根据 (3.21) 和 (3.64) 式，(4.79) 式的纯空间分量部分可以改写为

$$[\psi(x), \mathbf{J}] = (\mathbf{L} + \mathbf{S})\psi(x). \tag{4.81}$$

上式表明，除了轨道角动量 **L**，总角动量算符 **J** 还生成了由 **S** 描述的自旋角动量。

描述半整数自旋经常用到 3 个 2×2 的 **Pauli 矩阵**

$$\sigma^1 \equiv \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{4.82}$$

它们都是既厄米又幺正的：

$$(\sigma^i)^{-1} = (\sigma^i)^\dagger = \sigma^i. \tag{4.83}$$

Pauli 矩阵的两两乘积为

$$\begin{aligned}
(\sigma^1)^2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad (\sigma^2)^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\
(\sigma^3)^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \sigma^1\sigma^2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\sigma^3, \\
\sigma^2\sigma^1 &= \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} -i & \\ & i \end{pmatrix} = -i\sigma^3, \quad \sigma^2\sigma^3 = \begin{pmatrix} & -i \\ i & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma^1, \\
\sigma^3\sigma^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} & -i \\ -i & \end{pmatrix} = -i\sigma^1, \quad \sigma^3\sigma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma^2, \\
\sigma^1\sigma^3 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i\sigma^2.
\end{aligned} \tag{4.84}$$

归纳起来，有

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k. \tag{4.85}$$

从而可得

$$[\sigma^i, \sigma^j] = i\varepsilon^{ijk}\sigma^k - i\varepsilon^{jik}\sigma^k = 2i\varepsilon^{ijk}\sigma^k, \tag{4.86}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} + i\varepsilon^{ijk}\sigma^k + i\varepsilon^{jik}\sigma^k = 2\delta^{ij}. \tag{4.87}$$

利用 Pauli 矩阵可以将 Dirac 矩阵表示成 2×2 分块形式：

$$\gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}, \tag{4.88}$$

其中 $\mathbf{1}$ 表示 2×2 单位矩阵。容易验证，这样表示的 Dirac 矩阵符合约定 (4.40)，而且满足反对易关系 (4.1)：

$$\begin{aligned}
\{\gamma^0,\gamma^0\} &= 2\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} = 2\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{00}, &(4.89)\\
\{\gamma^0,\gamma^i\} &= \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} = 0 = 2g^{0i}, &(4.90)\\
\{\gamma^i,\gamma^j\} &= \begin{pmatrix} -\sigma^i\sigma^j-\sigma^j\sigma^i & \\ & -\sigma^i\sigma^j-\sigma^j\sigma^i \end{pmatrix} = -2\delta^{ij}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{ij}. &(4.91)
\end{aligned}$$

实际上，Dirac 矩阵有多种表示方式，(4.88) 式这种表示方式称为 **Weyl 表象**，也称为**手征表象** (chiral representation)。Dirac 矩阵的所有表示方式都是等价的，彼此可以通过相似变换联系起来。

在 Weyl 表象中，由 (4.86) 式可得 $S^{\mu\nu}$ 的空间分量为

$$\begin{aligned}
S^{ij} &= \frac{i}{4}[\gamma^i,\gamma^j] = \frac{i}{4}\begin{pmatrix} -\sigma^i\sigma^j+\sigma^j\sigma^i & \\ & -\sigma^i\sigma^j+\sigma^j\sigma^i \end{pmatrix} \\
&= \frac{i}{4}\begin{pmatrix} -2i\varepsilon^{ijk}\sigma^k & \\ & -2i\varepsilon^{ijk}\sigma^k \end{pmatrix} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}, \qquad (4.92)
\end{aligned}$$

从 Pauli 矩阵的厄米性可知，S^{ij} 是厄米矩阵：

$$(S^{ij})^\dagger = S^{ij}. \tag{4.93}$$

由 (1.98) 式可得

$$S^i = \frac{1}{2}\varepsilon^{ijk}S^{jk} = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{jkl}\begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{4}2\delta^{il}\begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}. \tag{4.94}$$

于是，自旋角动量矩阵的平方为

$$\mathbf{S}^2 = S^iS^i = \frac{1}{4}\begin{pmatrix} \sigma^i\sigma^i & \\ & \sigma^i\sigma^i \end{pmatrix} = \frac{3}{4}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right) = s(s+1). \tag{4.95}$$

可见，Dirac 旋量场 $\psi(x)$ 的自旋量子数是

$$s = \frac{1}{2}. \tag{4.96}$$

经过量子化程序之后，$\psi(x)$ 应当描述**自旋为 1/2** 的粒子。

4.3 Dirac 方程

为了写下 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量，我们需要结合两个旋量场来得到 Lorentz 标量。在 Weyl 表象中，$S^{\mu\nu}$ 的 $0i$ 分量为

$$S^{0i}=\frac{i}{4}[\gamma^0,\gamma^i]=\frac{i}{2}\gamma^0\gamma^i=\frac{i}{2}\begin{pmatrix}-\sigma^i & \\ & \sigma^i\end{pmatrix}. \tag{4.97}$$

由 Pauli 矩阵的厄米性可得

$$(S^{0i})^\dagger=-\frac{i}{2}\begin{pmatrix}-(\sigma^i)^\dagger & \\ & (\sigma^i)^\dagger\end{pmatrix}=-\frac{i}{2}\begin{pmatrix}-\sigma^i & \\ & \sigma^i\end{pmatrix}=-S^{0i}. \tag{4.98}$$

可见，S^{0i} 不是厄米矩阵。于是，当 $\omega_{0i}\neq 0$ 时，

$$D^\dagger(\Lambda)=\left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\right]^\dagger=\exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\neq\exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)=D^{-1}(\Lambda), \tag{4.99}$$

即 $D(\Lambda)$ 不是幺正矩阵。因此，一般地，$\psi^\dagger(x)\psi(x)$ 不是 Lorentz 标量：

$$\psi'^\dagger(x')\psi'(x')=\psi^\dagger(x)D^\dagger(\Lambda)D(\Lambda)\psi(x)\neq\psi^\dagger(x)\psi(x). \tag{4.100}$$

根据约定 (4.40)，可得

$$(\gamma^0)^\dagger\gamma^0=\gamma^0\gamma^0,\quad (\gamma^i)^\dagger\gamma^0=-\gamma^i\gamma^0=\gamma^0\gamma^i. \tag{4.101}$$

这两条式子可以合起来写成

$$(\gamma^\mu)^\dagger\gamma^0=\gamma^0\gamma^\mu. \tag{4.102}$$

从而，有

$$(S^{\mu\nu})^\dagger\gamma^0=-\frac{i}{4}[\gamma^\mu,\gamma^\nu]^\dagger\gamma^0=-\frac{i}{4}[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger-(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0=-\frac{i}{4}\gamma^0(\gamma^\nu\gamma^\mu-\gamma^\mu\gamma^\nu)=\gamma^0S^{\mu\nu}. \tag{4.103}$$

于是，可得

$$\begin{aligned}D^\dagger(\Lambda)\gamma^0 &= \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\gamma^0=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]^n\gamma^0=\gamma^0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n\\ &= \gamma^0\exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)=\gamma^0D^{-1}(\Lambda).\end{aligned} \tag{4.104}$$

根据上式，定义

$$\bar{\psi}(x)\equiv\psi^\dagger(x)\gamma^0, \tag{4.105}$$

则它的 Lorentz 变换形式为

$$\bar{\psi}'(x')=\psi'^\dagger(x')\gamma^0=\psi^\dagger(x)D^\dagger(\Lambda)\gamma^0=\psi^\dagger(x)\gamma^0D^{-1}(\Lambda)=\bar{\psi}(x)D^{-1}(\Lambda). \tag{4.106}$$

这样一来，$\bar{\psi}(x)\psi(x)$ 就是一个 Lorentz 标量：

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)D(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x). \tag{4.107}$$

$\bar{\psi}(x)\psi(x)$ 这种形式的量属于旋量双线性型 (spinor bilinear)，我们可以使用 $\bar{\psi}(x)$ 构造一些 Lorentz 协变的其它旋量双线性型。$\bar{\psi}(x)i\gamma^5\psi(x)$ 是一个 Lorentz 标量：

$$\bar{\psi}'(x')i\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)i\gamma^5 D(\Lambda)\psi(x) = \bar{\psi}(x)i\gamma^5\psi(x). \tag{4.108}$$

$\bar{\psi}(x)\gamma^\mu\psi(x)$ 和 $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ 都是 Lorentz 矢量：

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x), \tag{4.109}$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu\gamma^5 D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x). \tag{4.110}$$

$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ 是一个 2 阶反对称 Lorentz 张量：

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda)\psi(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x) \tag{4.111}$$

如果将 $\psi(x)$ 看作旋量空间中的列矢量，则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 都是行矢量，因而这些旋量双线性型都只是旋量空间中的 1×1 矩阵，也就是数。由 γ^0 和 γ^5 的厄米性及 (4.102) 式可知，这些旋量双线性型都是厄米的，或者说，都是实数：

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \tag{4.112}$$

$$(\bar{\psi}i\gamma^5\psi)^\dagger = -i\psi^\dagger\gamma^5\gamma^0\psi = i\psi^\dagger\gamma^0\gamma^5\psi = \bar{\psi}i\gamma^5\psi, \tag{4.113}$$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = \psi^\dagger(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \tag{4.114}$$

$$(\bar{\psi}\gamma^\mu\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \tag{4.115}$$

$$(\bar{\psi}\sigma^{\mu\nu}\psi)^\dagger = -\frac{i}{2}\psi^\dagger[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0\psi = -\frac{i}{2}\psi^\dagger\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\psi = \bar{\psi}\sigma^{\mu\nu}\psi. \tag{4.116}$$

此外，包含时空导数的旋量双线性型 $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ 是 Lorentz 标量：

$$\begin{aligned}\bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x') &= \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) = \bar{\psi}(x)\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x)\\ &= \bar{\psi}(x)\delta^\nu{}_\rho\gamma^\rho\partial_\nu\psi(x) = \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x).\end{aligned} \tag{4.117}$$

利用 $\bar{\psi}\gamma^\mu\partial_\mu\psi$ 和 $\bar{\psi}\psi$ 可以写下自由 Dirac 旋量场 $\psi(x)$ 的 **Lorentz 不变拉氏量**

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{4.118}$$

于是，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}. \tag{4.119}$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi}. \tag{4.120}$$

对上式取厄米共轭，得到

$$0=-i(\gamma^\mu)^\dagger\partial_\mu(\psi^\dagger\gamma^0)^\dagger+m(\psi^\dagger\gamma^0)^\dagger=-i(\gamma^\mu)^\dagger\gamma^0\partial_\mu\psi+m\gamma^0\psi=-\gamma^0(i\gamma^\mu\partial_\mu-m)\psi, \tag{4.121}$$

即

$$(i\gamma^\mu\partial_\mu-m)\psi(x)=0. \tag{4.122}$$

上式就是 **Dirac 方程**，标明旋量指标的形式为

$$[i(\gamma^\mu)_{ab}\partial_\mu-m\delta_{ab}]\psi_b(x)=0. \tag{4.123}$$

可以验证，Dirac 方程具有 Lorentz 协变性：

$$\begin{aligned}(i\gamma^\mu\partial'_\mu-m)\psi'(x')&=[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu-m]D(\Lambda)\psi(x)=D(\Lambda)[iD^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu-m]\psi(x)\\&=D(\Lambda)[i\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu-m]\psi(x)=D(\Lambda)(i\delta^\nu{}_\rho\gamma^\rho\partial_\nu-m)\psi(x)\\&=D(\Lambda)(i\gamma^\nu\partial_\nu-m)\psi(x)=0.\end{aligned} \tag{4.124}$$

对 Dirac 方程 (4.122) 左边乘以 $(-i\gamma^\mu\partial_\mu-m)$，利用反对易关系 (4.1)，可得

$$\begin{aligned}0&=(-i\gamma^\mu\partial_\mu-m)(i\gamma^\nu\partial_\nu-m)\psi=(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu+m^2)\psi=\left[\frac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu+\partial_\nu\partial_\mu)+m^2\right]\psi\\&=\left[\frac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu+m^2\right]\psi=(g^{\mu\nu}\partial_\mu\partial_\nu+m^2)\psi=(\partial^2+m^2)\psi.\end{aligned} \tag{4.125}$$

也就是说，自由的 Dirac 旋量场 $\psi(x)$ 满足 *Klein-Gordon 方程*

$$(\partial^2+m^2)\psi(x)=0. \tag{4.126}$$

由 (4.92) 和 (4.97) 式可以看出，旋量表示的生成元在 Weyl 表象中都是分块对角的，因而它可以分解为两个 2 维表示的直和。相应地，可以把具有 4 个分量的 Dirac 旋量场 ψ 分解为两个二分量旋量 φ_L 和 φ_R：

$$\psi=\begin{pmatrix}\varphi_\mathrm{L}\\ \varphi_\mathrm{R}\end{pmatrix}. \tag{4.127}$$

这样的二分量旋量称为 **Weyl 旋量**，其中，φ_L 称为**左手** (left-handed) Weyl 旋量，φ_R 称为**右手** (right-handed) Weyl 旋量。

用 2×2 单位矩阵和 Pauli 矩阵定义

$$\sigma^\mu\equiv(\mathbf{1},\boldsymbol{\sigma}),\quad \bar\sigma^\mu\equiv(\mathbf{1},-\boldsymbol{\sigma}), \tag{4.128}$$

那么，Weyl 表象中的 Dirac 矩阵 (4.88) 可以简洁地表示成

$$\gamma^\mu=\begin{pmatrix} & \sigma^\mu\\ \bar\sigma^\mu & \end{pmatrix}. \tag{4.129}$$

从而，Dirac 方程 (4.122) 化为

$$0 = (i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \varphi_\mathrm{L} \\ \varphi_\mathrm{R} \end{pmatrix} = \begin{pmatrix} i\sigma^\mu\partial_\mu\varphi_\mathrm{R} - m\varphi_\mathrm{L} \\ i\bar{\sigma}^\mu\partial_\mu\varphi_\mathrm{L} - m\varphi_\mathrm{R} \end{pmatrix}, \tag{4.130}$$

即

$$\begin{cases} i\bar{\sigma}^\mu\partial_\mu\varphi_\mathrm{L} - m\varphi_\mathrm{R} = 0, \\ i\sigma^\mu\partial_\mu\varphi_\mathrm{R} - m\varphi_\mathrm{L} = 0. \end{cases} \tag{4.131}$$

这是一组相互耦合的方程。如果 $m = 0$，方程组中的两个方程就变得相互独立了：

$$i\bar{\sigma}^\mu\partial_\mu\varphi_\mathrm{L} = 0, \quad i\sigma^\mu\partial_\mu\varphi_\mathrm{R} = 0. \tag{4.132}$$

这两个独立的方程称为 **Weyl 方程**。可见，非零质量 m 的存在将左手和右手 Weyl 旋量耦合起来。

4.4 Dirac 旋量场的平面波展开

4.4.1 平面波解的一般形式

本小节讨论与表象选取无关。

对于确定的动量 $\mathbf{p}$，我们假设 Dirac 方程具有如下形式的平面波解：

$$\psi_a(x;\mathbf{k}) = w_a(k^0,\mathbf{k})e^{-ik\cdot x}. \tag{4.133}$$

其中，系数 $w_a(k^0,\mathbf{k})$ 是四分量旋量，带着一个旋量指标 a。隐去旋量指标，将这个平面波解代入到 Dirac 方程 (4.122) 中，可得

$$0 = (i\gamma^\mu\partial_\mu - m)\psi(x;\mathbf{k}) = (\gamma^\mu k_\mu - m)w(k^0,\mathbf{k})e^{-ik\cdot x} = (k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0,\mathbf{k})e^{-ik\cdot x}. \tag{4.134}$$

因此，有

$$(k^0\gamma^0 - \mathbf{k}\cdot\boldsymbol{\gamma} - m)w(k^0,\mathbf{k}) = 0. \tag{4.135}$$

对上式左乘 γ^0，可得

$$[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0]w(k^0,\mathbf{k}) = 0. \tag{4.136}$$

通过移项，上式化为

$$[\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) + m\gamma^0]w(k^0,\mathbf{k}) = k^0 w_a(k^0,\mathbf{k}). \tag{4.137}$$

这是一个本征值方程，它具有非平庸解的条件是特征多项式 $\det[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0]$ 为零，即

$$\det[k^0 - \gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma}) - m\gamma^0] = 0. \tag{4.138}$$

这个方程的根给出 k^0 的本征值，相应的非平庸解是本征矢量。

方程 (4.138) 可化为

$$0=\det[k^0-\gamma^0(\mathbf{k}\cdot\boldsymbol{\gamma})-m\gamma^0]=\det[\gamma^0(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det(\gamma^0)\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m).\quad(4.139)$$

由 (4.39) 式可得 $[\det(\gamma^0)]^2=\det(\gamma^0\gamma^0)=\det(\mathbf{1})=1$，故 $\det(\gamma^0)\neq 0$。因而方程 (4.138) 等价于

$$\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=0.\quad(4.140)$$

利用 (4.48) 式，上式左边可化为

$$\begin{aligned}\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)&=\det[(\gamma^5)^2(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det[\gamma^5(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)\gamma^5]\\&=\det[(\gamma^5)^2(-k^0\gamma^0+\mathbf{k}\cdot\boldsymbol{\gamma}-m)]=\det[-(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m].\end{aligned}\quad(4.141)$$

这里第二步用到行列式性质

$$\det(AB)=\det(BA),\quad(4.142)$$

第三步用到 γ^5 与 γ^μ 反对易的性质 (4.50)。由反对易关系 (4.1) 有

$$(k_\mu\gamma^\mu)^2=k_\mu k_\nu\gamma^\mu\gamma^\nu=\frac{1}{2}k_\mu k_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)=k_\mu k_\nu g^{\mu\nu}=k^2=(k^0)^2-|\mathbf{k}|^2.\quad(4.143)$$

从而，可得

$$\begin{aligned}[\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)]^2&=\det[(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m]\det[-(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma})-m]\\&=\det(k_\mu\gamma^\mu-m)\det(-k_\mu\gamma^\mu-m)=\det[(k_\mu\gamma^\mu-m)(-k_\mu\gamma^\mu-m)]\\&=\det[-(k_\mu\gamma^\mu)^2+m^2]=\det\{[-(k^0)^2+|\mathbf{k}|^2+m^2]\mathbf{1}\}\\&=[-(k^0)^2+|\mathbf{k}|^2+m^2]^4=[E_\mathbf{k}^2-(k^0)^2]^4,\end{aligned}\quad(4.144)$$

其中 $E_\mathbf{k}\equiv\sqrt{|\mathbf{k}|^2+m^2}$。于是，方程 (4.140) 化为

$$0=\det(k^0\gamma^0-\mathbf{k}\cdot\boldsymbol{\gamma}-m)=[E_\mathbf{k}^2-(k^0)^2]^2=(E_\mathbf{k}+k^0)^2(E_\mathbf{k}-k^0)^2.\quad(4.145)$$

这个方程有 2 个根 $k^0=\pm E_\mathbf{k}$；这 2 个根都是 2 重根，各自对应于 2 个独立的本征矢量，共有 4 个线性无关的本征矢量。

(1) $k^0=E_\mathbf{k}$ 对应于 2 个本征矢量

$$w^{(+)}(E_\mathbf{k},\mathbf{k};\sigma),\quad\sigma=1,2.\quad(4.146)$$

因而平面波解中有 2 个正能解，形式为

$$w^{(+)}(E_\mathbf{k},\mathbf{k};\sigma)\exp[-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})],\quad\sigma=1,2.\quad(4.147)$$

(2) $k^0=-E_\mathbf{k}$ 对应于 2 个本征矢量

$$w^{(-)}(-E_\mathbf{k},\mathbf{k};\sigma),\quad\sigma=1,2.\quad(4.148)$$

因而平面波解中有 2 个负能解，形式为

$$w^{(-)}(-E_\mathbf{k},\mathbf{k};\sigma)\exp[i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})],\quad\sigma=1,2.\quad(4.149)$$

可以将这 4 个本征矢量的正交归一关系取为

$$\begin{aligned} w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma') &= 2E_{\mathbf{k}}\delta_{\sigma\sigma'}, \quad w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma') = 2E_{\mathbf{k}}\delta_{\sigma\sigma'}, \\ w^{(+)\dagger}(E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}},\mathbf{k};\sigma)\,w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma') = 0. \end{aligned} \tag{4.150}$$

按如下定义引入四分量旋量 $u(\mathbf{k};\sigma)$ 和 $v(\mathbf{k};\sigma)$：

$$u(\mathbf{k};\sigma) \equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma), \quad v(-\mathbf{k};\sigma) \equiv w^{(-)}(-E_{\mathbf{k}},\mathbf{k};\sigma), \quad \sigma = 1,2. \tag{4.151}$$

第二个定义式等价于

$$v(\mathbf{k};\sigma) = w^{(-)}(-E_{\mathbf{k}},-\mathbf{k};\sigma). \tag{4.152}$$

于是，Dirac 方程的正能解和负能解可以分别写作

$$\psi^{(+)}(x;\mathbf{k};\sigma) \equiv w^{(+)}(E_{\mathbf{k}},\mathbf{k};\sigma)\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = u(\mathbf{k};\sigma)\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})], \tag{4.153}$$

$$\psi^{(-)}(x;\mathbf{k};\sigma) \equiv w^{(-)}(-E_{\mathbf{k}},-\mathbf{k};\sigma)\exp[i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})] = v(\mathbf{k};\sigma)\exp[i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]. \tag{4.154}$$

替换一下动量记号，可得

$$\psi^{(+)}(x;\mathbf{p};\sigma) = u(\mathbf{p};\sigma)e^{-ip\cdot x}, \quad \psi^{(-)}(x;\mathbf{p};\sigma) = v(\mathbf{p};\sigma)e^{ip\cdot x}, \quad p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2+m^2}. \tag{4.155}$$

从而，Dirac 旋量场算符 $\psi(\mathbf{x},t)$ 的平面波展开式可写作

$$\begin{aligned} \psi(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=1}^{2}\left[\psi^{(+)}(x;\mathbf{p};\sigma)a_{\mathbf{p};\sigma} + \psi^{(-)}(x;\mathbf{p};\sigma)b^\dagger_{\mathbf{p};\sigma}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=1}^{2}\left[u(\mathbf{p};\sigma)a_{\mathbf{p};\sigma}e^{-ip\cdot x} + v(\mathbf{p};\sigma)b^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right]. \end{aligned} \tag{4.156}$$

其中，$a_{\mathbf{p};\sigma}$ 是湮灭算符，$b^\dagger_{\mathbf{p};\sigma}$ 是产生算符。一般地，$a_{\mathbf{p};\sigma} \neq b_{\mathbf{p};\sigma}$。

旋量系数 $u(\mathbf{p};\sigma)$ 和 $v(\mathbf{p};\sigma)$ 的正交归一关系为

$$u^\dagger(\mathbf{p};\sigma)u(\mathbf{p};\sigma') = w^{(+)\dagger}(E_{\mathbf{p}},\mathbf{p};\sigma)\,w^{(+)}(E_{\mathbf{p}},\mathbf{p};\sigma') = 2E_{\mathbf{p}}\delta_{\sigma\sigma'}, \tag{4.157}$$

$$v^\dagger(\mathbf{p};\sigma)v(\mathbf{p};\sigma') = w^{(-)\dagger}(-E_{\mathbf{p}},-\mathbf{p};\sigma)\,w^{(-)}(-E_{\mathbf{p}},-\mathbf{p};\sigma') = 2E_{\mathbf{p}}\delta_{\sigma\sigma'}, \tag{4.158}$$

$$u^\dagger(\mathbf{p};\sigma)v(-\mathbf{p};\sigma') = w^{(+)\dagger}(E_{\mathbf{p}},\mathbf{p};\sigma)\,w^{(-)}(-E_{\mathbf{p}},\mathbf{p};\sigma') = 0. \tag{4.159}$$

4.4.2 Weyl 表象中的平面波解

本小节在 *Weyl* 表象中讨论 Dirac 方程的平面波解。

Dirac 旋量场描述自旋为 1/2 的粒子，因而粒子的自旋在动量方向上的投影有 2 种取值，+1/2 和 −1/2，归一化后对应于 2 种螺旋度 $\lambda = \pm$。类似于矢量场的情况，Dirac 旋量场所描述的粒子的状态可以用螺旋度本征值 λ 来表征。因此，无论是平面波解的正能解还是负能解，都能够以 2 种螺旋度本征态作为 2 个独立的本征矢量。

按照这个思路，可以把正能解的 2 个本征矢量记作

$$\psi^{(+)}(x;\mathbf{p},\lambda)=u(\mathbf{p},\lambda)e^{-ip\cdot x},\quad \lambda=\pm,\quad p^0=E_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}. \tag{4.160}$$

根据 Dirac 方程 (4.122)，有

$$0=(i\gamma^\mu\partial_\mu-m)\psi^{(+)}(x;\mathbf{p},\lambda)=(p_\mu\gamma^\mu-m)u(\mathbf{p},\lambda)e^{-ip\cdot x}, \tag{4.161}$$

即

$$(\not{p}-m)u(\mathbf{p},\lambda)=0, \tag{4.162}$$

其中，$\not{p}$ 的定义为

$$\not{p}\equiv p_\mu\gamma^\mu. \tag{4.163}$$

这种斜线记号称为 **Dirac 斜线** (slash)，是 R. Feynman 引进的。

将四分量旋量 $u(\mathbf{p},\lambda)$ 分解为两个二分量旋量 $f_\lambda(\mathbf{p})$ 和 $g_\lambda(\mathbf{p})$，

$$u(\mathbf{p},\lambda)=\begin{pmatrix} f_\lambda(\mathbf{p})\\ g_\lambda(\mathbf{p})\end{pmatrix}, \tag{4.164}$$

那么，根据 Weyl 表象中的 Dirac 矩阵表达式 (4.129)，方程 (4.162) 化为

$$0=(\not{p}-m)u(\mathbf{p},\lambda)=\begin{pmatrix} -m & \sigma^\mu p_\mu\\ \bar{\sigma}^\mu p_\mu & -m\end{pmatrix}\begin{pmatrix} f_\lambda(\mathbf{p})\\ g_\lambda(\mathbf{p})\end{pmatrix}=\begin{pmatrix} p_\mu\sigma^\mu g_\lambda(\mathbf{p})-mf_\lambda(\mathbf{p})\\ p_\mu\bar{\sigma}^\mu f_\lambda(\mathbf{p})-mg_\lambda(\mathbf{p})\end{pmatrix}, \tag{4.165}$$

即

$$(p\cdot\sigma)g_\lambda(\mathbf{p})-mf_\lambda(\mathbf{p})=0, \tag{4.166}$$

$$(p\cdot\bar{\sigma})f_\lambda(\mathbf{p})-mg_\lambda(\mathbf{p})=0. \tag{4.167}$$

将 (4.129) 式代入反对易关系 (4.1)，可得

$$2g^{\mu\nu}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{1}\end{pmatrix}=\{\gamma^\mu,\gamma^\nu\}=\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu+\sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu+\bar{\sigma}^\nu\sigma^\mu\end{pmatrix}, \tag{4.168}$$

故

$$\sigma^\mu\bar{\sigma}^\nu+\sigma^\nu\bar{\sigma}^\mu=2g^{\mu\nu}, \tag{4.169}$$

$$\bar{\sigma}^\mu\sigma^\nu+\bar{\sigma}^\nu\sigma^\mu=2g^{\mu\nu}. \tag{4.170}$$

因而，有

$$(p\cdot\sigma)(p\cdot\bar{\sigma})=p_\mu p_\nu\sigma^\mu\bar{\sigma}^\nu=\frac{1}{2}p_\mu p_\nu(\sigma^\mu\bar{\sigma}^\nu+\sigma^\nu\bar{\sigma}^\mu)=\frac{1}{2}p_\mu p_\nu 2g^{\mu\nu}=p^2. \tag{4.171}$$

由方程 (4.167) 可得

$$g_\lambda(\mathbf{p})=\frac{p\cdot\bar{\sigma}}{m}f_\lambda(\mathbf{p}). \tag{4.172}$$

将上式代入到由方程 (4.166) 得出的关系中，有

$$f_\lambda(\mathbf{p}) = \frac{p\cdot\sigma}{m} g_\lambda(\mathbf{p}) = \frac{1}{m^2}(p\cdot\sigma)(p\cdot\bar{\sigma}) f_\lambda(\mathbf{p}) = \frac{p^2}{m^2} f_\lambda(\mathbf{p}) = f_\lambda(\mathbf{p}). \tag{4.173}$$

可见，关系式 (4.172) 是自洽的。这样的话，只要选取合适的 $f_\lambda(\mathbf{p})$，然后由 (4.164) 和 (4.172) 式得到

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.174}$$

就可以满足方程 (4.162)。

在 Weyl 表象中，根据 (4.94) 式，自旋角动量矩阵 $\mathbf{S}$ 在动量 $\mathbf{p}$ 方向上的投影为

$$\hat{\mathbf{p}}\cdot\mathbf{S} = \frac{1}{2}\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{4.175}$$

归一化后，得到**螺旋度**矩阵

$$2\,\hat{\mathbf{p}}\cdot\mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & \\ & \hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{4.176}$$

上式的两个分块相同，因此，左手和右手 Weyl 旋量对应的螺旋度矩阵是相同的，都是

$$\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}. \tag{4.177}$$

引入作为螺旋度本征态的二分量旋量 $\xi_\lambda(\mathbf{p})$，满足

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda\,\xi_\lambda(\mathbf{p}), \quad \lambda = \pm. \tag{4.178}$$

我们要求 $\xi_\lambda(\mathbf{p})$ 具有正交归一关系

$$\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'} \tag{4.179}$$

和完备性关系

$$\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}. \tag{4.180}$$

此外，由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 可得

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = \lambda|\mathbf{p}|\xi_\lambda(\mathbf{p}) \tag{4.181}$$

我们将 $\xi_\lambda(\mathbf{p})$ 称为**螺旋态**。在实际应用中，可以把螺旋态 $\xi_\lambda(\mathbf{p})$ 取为如下形式：

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+ip^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix}. \tag{4.182}$$

可以验证，它们确实是 $\lambda = \pm$ 的本征态：

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_+(\mathbf{p}) = \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+ip^2 \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3(|\mathbf{p}|+p^3)+(p^1-ip^2)(p^1+ip^2) \\ (p^1+ip^2)(|\mathbf{p}|+p^3)-p^3(p^1+ip^2) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3|\mathbf{p}|+|\mathbf{p}|^2 \\ (p^1+ip^2)|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3+|\mathbf{p}| \\ p^1+ip^2 \end{pmatrix} \\
&= +\xi_+(\mathbf{p}),
\end{aligned} \tag{4.183}
$$

$$
\begin{aligned}
(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_-(\mathbf{p}) &= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} -p^3(p^1-ip^2)+(p^1-ip^2)(|\mathbf{p}|+p^3) \\ (p^1+ip^2)(-p^1+ip^2)-p^3(|\mathbf{p}|+p^3) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} (p^1-ip^2)|\mathbf{p}| \\ -|\mathbf{p}|^2-p^3|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix} p^1-ip^2 \\ -|\mathbf{p}|-p^3 \end{pmatrix} \\
&= -\xi_-(\mathbf{p}).
\end{aligned} \tag{4.184}
$$

而且，满足正交归一关系：

$$
\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_+(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2 \end{pmatrix}\begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+ip^2 \end{pmatrix} \\
&= \frac{(|\mathbf{p}|+p^3)^2+|p^1+ip^2|^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2+2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1,
\end{aligned} \tag{4.185}
$$

$$
\begin{aligned}
\xi_-^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} -p^1-ip^2 & |\mathbf{p}|+p^3 \end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{|-p^1+ip^2|^2+(|\mathbf{p}|+p^3)^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2+2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1,
\end{aligned} \tag{4.186}
$$

$$
\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2 \end{pmatrix}\begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{-(|\mathbf{p}|+p^3)(p^1-ip^2)+(|\mathbf{p}|+p^3)(p^1-ip^2)}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 0.
\end{aligned} \tag{4.187}
$$

也满足完备性关系：

$$
\begin{aligned}
&\sum_{\lambda=\pm}\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) = \xi_+(\mathbf{p})\xi_+^\dagger(\mathbf{p})+\xi_-(\mathbf{p})\xi_-^\dagger(\mathbf{p}) \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} (|\mathbf{p}|+p^3)^2+|-p^1+ip^2|^2 & (|\mathbf{p}|+p^3)(p^1-ip^2)+(|\mathbf{p}|+p^3)(-p^1+ip^2) \\ (|\mathbf{p}|+p^3)(p^1+ip^2)+(|\mathbf{p}|+p^3)(-p^1-ip^2) & |p^1+ip^2|^2+(|\mathbf{p}|+p^3)^2 \end{pmatrix} \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)}\begin{pmatrix} 2|\mathbf{p}|^2+2p^3|\mathbf{p}| & \\ & 2|\mathbf{p}|^2+2p^3|\mathbf{p}| \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \mathbf{1}.
\end{aligned} \tag{4.188}
$$

当 $p^3 = -|\mathbf{p}|$ 时，(4.182) 式失去良好的定义，此时我们可以将螺旋态取成

$$
\xi_+(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{4.189}
$$

现在，将 $f_\lambda(\mathbf{p})$ 取为

$$f_\lambda(\mathbf{p}) = C_\lambda\,\xi_\lambda(\mathbf{p}), \tag{4.190}$$

其中 C_λ 是常数。从而，利用 (4.181) 式，(4.174) 式可化为

$$u(\mathbf{p},\lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \dfrac{p\cdot\bar{\sigma}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{E_{\mathbf{p}} + \mathbf{p}\cdot\boldsymbol{\sigma}}{m}\xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \dfrac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m}\xi_\lambda(\mathbf{p}) \end{pmatrix}. \tag{4.191}$$

再取

$$C_\lambda = \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \tag{4.192}$$

则由

$$\sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} = \sqrt{E_{\mathbf{p}}^2 - \lambda^2|\mathbf{p}|^2} = \sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2} = m, \tag{4.193}$$

有

$$\begin{aligned} C_\lambda \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} &= \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}\,\frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} = \frac{\sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}}{m}\sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} \\ &= \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}. \end{aligned} \tag{4.194}$$

于是，得到 $u(\mathbf{p},\lambda)$ **的螺旋态表达式**

$$u(\mathbf{p},\lambda) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}\,\xi_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix}, \tag{4.195}$$

其中，$\omega_\lambda(\mathbf{p})$ 定义为

$$\omega_\lambda(\mathbf{p}) \equiv \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}, \tag{4.196}$$

它是关于 $\mathbf{p}$ 的偶函数：

$$\omega_\lambda(-\mathbf{p}) = \omega_\lambda(\mathbf{p}). \tag{4.197}$$

这样的话，根据 (4.176) 式，$u(\mathbf{p},\lambda)$ 是螺旋度本征态，本征值为 λ：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{S})u(\mathbf{p},\lambda) = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda\,u(\mathbf{p},\lambda). \tag{4.198}$$

另一方面，可以把负能解的 2 个本征矢量记作

$$\psi^{(-)}(x;\mathbf{p},\lambda) = v(\mathbf{p},\lambda)e^{ip\cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{4.199}$$

根据 Dirac 方程 (4.122)，有

$$0 = (i\gamma^\mu\partial_\mu - m)\psi^{(-)}(x;\mathbf{p},\lambda) = (-p_\mu\gamma^\mu - m)v(\mathbf{p},\lambda)e^{ip\cdot x}, \tag{4.200}$$

即

$$(\not{p} + m)v(\mathbf{p},\lambda) = 0. \tag{4.201}$$

同样，将四分量旋量 $v(\mathbf{p},\lambda)$ 分解为两个二分量旋量 $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$，

$$v(\mathbf{p},\lambda)=\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix},\tag{4.202}$$

则有

$$0=(\not{p}+m)v(\mathbf{p},\lambda)=\begin{pmatrix}m & \sigma^\mu p_\mu\\ \bar{\sigma}^\mu p_\mu & m\end{pmatrix}\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ \tilde{g}_\lambda(\mathbf{p})\end{pmatrix}=\begin{pmatrix}p_\mu\sigma^\mu\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})\\ p_\mu\bar{\sigma}^\mu\tilde{f}_\lambda(\mathbf{p})+m\tilde{g}_\lambda(\mathbf{p})\end{pmatrix},\tag{4.203}$$

即

$$(p\cdot\sigma)\tilde{g}_\lambda(\mathbf{p})+m\tilde{f}_\lambda(\mathbf{p})=0,\tag{4.204}$$

$$(p\cdot\bar{\sigma})\tilde{f}_\lambda(\mathbf{p})+m\tilde{g}_\lambda(\mathbf{p})=0.\tag{4.205}$$

由方程 (4.205) 可得

$$\tilde{g}_\lambda(\mathbf{p})=-\frac{p\cdot\bar{\sigma}}{m}\tilde{f}_\lambda(\mathbf{p}).\tag{4.206}$$

将上式代入到由方程 (4.204) 得出的关系中，根据 (4.171) 式，有

$$\tilde{f}_\lambda(\mathbf{p})=-\frac{p\cdot\sigma}{m}\tilde{g}_\lambda(\mathbf{p})=\frac{1}{m^2}(p\cdot\sigma)(p\cdot\bar{\sigma})\tilde{f}_\lambda(\mathbf{p})=\frac{p^2}{m^2}\tilde{f}_\lambda(\mathbf{p})=\tilde{f}_\lambda(\mathbf{p}).\tag{4.207}$$

可见，关系式 (4.206) 是自洽的。

　　现在，将 $\tilde{f}_\lambda(\mathbf{p})$ 取为

$$\tilde{f}_\lambda(\mathbf{p})=\tilde{C}_\lambda\,\xi_{-\lambda}(\mathbf{p}),\tag{4.208}$$

其中 $\tilde{C}_\lambda$ 是常数。在这里，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，而非 $\xi_\lambda(\mathbf{p})$。这种取法的原因将在 4.5.4 小节中说明，现在姑且接受这种选择。从而，有

$$v(\mathbf{p},\lambda)=\begin{pmatrix}\tilde{f}_\lambda(\mathbf{p})\\ -\dfrac{p\cdot\bar{\sigma}}{m}\tilde{f}_\lambda(\mathbf{p})\end{pmatrix}=\tilde{C}_\lambda\begin{pmatrix}\xi_{-\lambda}(\mathbf{p})\\ -\dfrac{E_\mathbf{p}+\mathbf{p}\cdot\boldsymbol{\sigma}}{m}\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=\tilde{C}_\lambda\begin{pmatrix}\xi_{-\lambda}(\mathbf{p})\\ -\dfrac{E_\mathbf{p}-\lambda|\mathbf{p}|}{m}\xi_{-\lambda}(\mathbf{p})\end{pmatrix}.\tag{4.209}$$

再取

$$\tilde{C}_\lambda=-\lambda\sqrt{E_\mathbf{p}+\lambda|\mathbf{p}|},\tag{4.210}$$

则由

$$\begin{aligned}-\tilde{C}_\lambda\frac{E_\mathbf{p}-\lambda|\mathbf{p}|}{m}&=\lambda\sqrt{E_\mathbf{p}+\lambda|\mathbf{p}|}\,\frac{E_\mathbf{p}-\lambda|\mathbf{p}|}{m}=\lambda\frac{\sqrt{E_\mathbf{p}-\lambda|\mathbf{p}|}}{m}\sqrt{(E_\mathbf{p}+\lambda|\mathbf{p}|)(E_\mathbf{p}-\lambda|\mathbf{p}|)}\\&=\lambda\sqrt{E_\mathbf{p}-\lambda|\mathbf{p}|},\end{aligned}\tag{4.211}$$

可得 $v(\mathbf{p},\lambda)$ 的螺旋态表达式

$$v(\mathbf{p},\lambda)=\begin{pmatrix}-\lambda\sqrt{E_\mathbf{p}+\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\\ \lambda\sqrt{E_\mathbf{p}-\lambda|\mathbf{p}|}\,\xi_{-\lambda}(\mathbf{p})\end{pmatrix}=\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}.\tag{4.212}$$

这样一来，$v(\mathbf{p},\lambda)$ 是螺旋度本征态，本征值为 $-\lambda$：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{S})v(\mathbf{p},\lambda) = \begin{pmatrix} -\lambda\,\omega_\lambda(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) \\ \lambda\,\omega_{-\lambda}(\mathbf{p})\,(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda \begin{pmatrix} -\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda\, v(\mathbf{p},\lambda). \tag{4.213}$$

根据 $\xi_\lambda(\mathbf{p})$ 的正交归一关系 (4.179)，可以验证，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足 (4.157) 和 (4.158) 式表示的正交归一关系：

$$\begin{aligned}
&u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = \begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix} \\
&= [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_\lambda(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})\delta_{\lambda\lambda'} + \omega_\lambda(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\
&= [\omega_{-\lambda}^2(\mathbf{p}) + \omega_\lambda^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_\mathbf{p} - \lambda|\mathbf{p}|) + (E_\mathbf{p} + \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_\mathbf{p}\delta_{\lambda\lambda'},
\end{aligned} \tag{4.214}$$

$$\begin{aligned}
&v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = \begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix} \\
&= \lambda\lambda'[\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) = \lambda\lambda'[\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\
&= \lambda^2[\omega_\lambda^2(\mathbf{p}) + \omega_{-\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_\mathbf{p} + \lambda|\mathbf{p}|) + (E_\mathbf{p} - \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_\mathbf{p}\delta_{\lambda\lambda'},
\end{aligned} \tag{4.215}$$

依照螺旋态的本征值方程 (4.178)，可得

$$(-\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = -\lambda\,\xi_{-\lambda}(-\mathbf{p}), \tag{4.216}$$

从而，有

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = \lambda\,\xi_{-\lambda}(-\mathbf{p}). \tag{4.217}$$

可见，$\xi_{-\lambda}(-\mathbf{p})$ 与 $\xi_\lambda(\mathbf{p})$ 服从相同的本征值方程，这意味着 $\xi_{-\lambda}(-\mathbf{p}) \propto \xi_\lambda(\mathbf{p})$，故

$$\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \propto \xi_\lambda^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}. \tag{4.218}$$

于是，(4.159) 式表示的正交关系也成立：

$$\begin{aligned}
u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda') &= \begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\ \lambda'\,\omega_{-\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p})\end{pmatrix} \\
&= \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(-\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda'}(-\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\
&\propto \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\
&\propto \lambda[-\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{\lambda\lambda'} = 0.
\end{aligned} \tag{4.219}$$

整理一下，旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足如下**正交归一关系**：

$$u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = 2E_\mathbf{p}\delta_{\lambda\lambda'}, \quad u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda') = v^\dagger(-\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 0. \tag{4.220}$$

此外，由 (4.193) 式有

$$\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = \sqrt{(E_\mathbf{p} + \lambda|\mathbf{p}|)(E_\mathbf{p} - \lambda|\mathbf{p}|)} = m. \tag{4.221}$$

从而，利用

$$
\begin{aligned}
\bar{u}(\mathbf{p},\lambda) &= u^\dagger(\mathbf{p},\lambda)\gamma^0 = \begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\\
&= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}, \qquad (4.222)\\
\bar{v}(\mathbf{p},\lambda) &= v^\dagger(\mathbf{p},\lambda)\gamma^0 = \begin{pmatrix}-\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\\
&= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}, \qquad (4.223)
\end{aligned}
$$

可得

$$
\begin{aligned}
\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix}\\
&= [\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\,\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = 2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} = 2m\delta_{\lambda\lambda'}, \quad (4.224)\\
\bar{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix}\\
&= -\lambda\lambda'[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = -2\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'}\\
&= -2m\delta_{\lambda\lambda'}, \qquad (4.225)\\
\bar{u}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= \begin{pmatrix}\omega_{\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}-\lambda'\,\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\ \lambda'\,\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\end{pmatrix}\\
&= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\,\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p})\\
&= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\,\delta_{\lambda,-\lambda'}\\
&= -\lambda[-\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p})\,]\,\delta_{\lambda,-\lambda'} = 0, \qquad (4.226)\\
\bar{v}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= \begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\end{pmatrix}\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\,\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p})\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{-\lambda,\lambda'}\\
&= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\,\delta_{-\lambda,\lambda'} = 0. \qquad (4.227)
\end{aligned}
$$

整理一下，有

$$
\bar{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 2m\delta_{\lambda\lambda'},\quad \bar{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = -2m\delta_{\lambda\lambda'},\quad \bar{u}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') = \bar{v}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') = 0. \qquad (4.228)
$$

另一方面，利用等式

$$
\begin{aligned}
(p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p} + \mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p} + \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \qquad (4.229)\\
(p\cdot\sigma)\xi_\lambda(\mathbf{p}) &= (E_\mathbf{p} - \mathbf{p}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_\mathbf{p} - \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \qquad (4.230)
\end{aligned}
$$

以及 (4.221) 式和 $\xi_\lambda(\mathbf{p})$ 的完备性关系 (4.180)，可得

$$
\begin{aligned}
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm}\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\end{pmatrix}\begin{pmatrix}\omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\\ \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}m\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\\ (p\cdot\bar{\sigma})\xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & m\xi_\lambda^\dagger(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p})\end{pmatrix}\\
&= \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar{\sigma} & m\end{pmatrix} = p_\mu\gamma^\mu + m.
\end{aligned}\tag{4.231}
$$

通过等式

$$
\begin{aligned}
(p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p}) &= (E_\mathbf{p}+\mathbf{p}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}-\lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) && (4.232)\\
(p\cdot\sigma)\xi_{-\lambda}(\mathbf{p}) &= (E_\mathbf{p}-\mathbf{p}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(\mathbf{p}) = (E_\mathbf{p}+\lambda|\mathbf{p}|)\xi_{-\lambda}(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p}), && (4.233)
\end{aligned}
$$

则可以得到

$$
\begin{aligned}
\sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) &= \sum_{\lambda=\pm}\begin{pmatrix}-\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\\ \lambda\,\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\end{pmatrix}\begin{pmatrix}\lambda\,\omega_{-\lambda}(\mathbf{p})\,\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\,\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}-\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda^2\omega_\lambda^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ \lambda^2\omega_{-\lambda}^2(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda^2\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \sum_{\lambda=\pm}\begin{pmatrix}-m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & (p\cdot\sigma)\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\\ (p\cdot\bar{\sigma})\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -m\xi_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p})\end{pmatrix}\\
&= \begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar{\sigma} & -m\end{pmatrix} = p_\mu\gamma^\mu - m.
\end{aligned}\tag{4.234}
$$

整理一下，有如下螺旋度求和关系，或者说，**自旋求和关系**：

$$
\sum_{\lambda=\pm} u(\mathbf{p},\lambda)\bar{u}(\mathbf{p},\lambda) = \not{p}+m,\qquad \sum_{\lambda=\pm} v(\mathbf{p},\lambda)\bar{v}(\mathbf{p},\lambda) = \not{p}-m.\tag{4.235}
$$

用 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 可以把 Dirac 旋量场算符 $\psi(\mathbf{x},t)$ 的平面波展开式写作

$$
\begin{aligned}
\psi(\mathbf{x},t) &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[\psi^{(+)}(x;\mathbf{p},\lambda)a_{\mathbf{p},\lambda}+\psi^{(-)}(x;\mathbf{p},\lambda)b_{\mathbf{p},\lambda}^\dagger\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+v(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x}\right].
\end{aligned}\tag{4.236}
$$

从而，有

$$
\psi^\dagger(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[u^\dagger(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x}+v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right],\tag{4.237}
$$

$$
\bar{\psi}(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\left[\bar{u}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x}+\bar{v}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right].\tag{4.238}
$$

4.4.3　哈密顿量和产生湮灭算符

根据 (4.119) 式，$\psi(x)$ 对应的共轭动量密度是

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \tag{4.239}$$

它的平面波展开式为

$$\pi(\mathbf{x},t) = i\psi^\dagger(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{i}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm}\Big[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\Big]. \tag{4.240}$$

自由运动的旋量场 $\psi(x)$ 满足 Dirac 方程 (4.122)，相应地，拉氏量 (4.118) 化为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{4.241}$$

于是，根据 (1.119) 式，自由 Dirac 旋量场的哈密顿量密度为

$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L} = \pi\partial_0\psi = i\psi^\dagger\partial_0\psi. \tag{4.242}$$

从而，哈密顿量为

$$\begin{aligned}
H &= \int d^3x\,\mathcal{H} = \int d^3x\,\psi^\dagger i\partial_0\psi \\
&= \sum_{\lambda\lambda'}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\Big[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\Big] \\
&\qquad\times\Big[q_0u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - q_0v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\Big] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,q_0\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} \\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} - u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x} \\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\Big] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\,E_\mathbf{q}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(E_\mathbf{p}-E_\mathbf{q})t} \\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_\mathbf{p}-E_\mathbf{q})t}\Big] \\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(E_\mathbf{p}+E_\mathbf{q})t} \\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(E_\mathbf{p}+E_\mathbf{q})t}\Big]\Big\} \\
&= \sum_{\lambda\lambda'}\int \frac{d^3p}{(2\pi)^3 2E_\mathbf{p}}\,E_\mathbf{p}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'} \\
&\qquad - u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_\mathbf{p}t} + v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_\mathbf{p}t}\Big] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3p}{(2\pi)^3 2E_\mathbf{p}}\,E_\mathbf{p}\Big(2E_\mathbf{p}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - 2E_\mathbf{p}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\Big)
\end{aligned}$$

$$= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b^\dagger_{\mathbf{p},\lambda} \right). \tag{4.243}$$

倒数第二步用到正交归一关系 (4.220)。

另一方面，利用正交归一关系 (4.220)，可得

$$\begin{aligned}
&\int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{i(p-q)\cdot x} + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{i(p+q)\cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})\cdot t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\
&\qquad\qquad \left. + u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a_{\mathbf{p},\lambda'} + u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')b^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t} \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}}\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda'}) = \sqrt{2E_{\mathbf{p}}}\, a_{\mathbf{p},\lambda}.
\end{aligned} \tag{4.244}$$

从而，湮灭算符 $a_{\mathbf{p},\lambda}$ 和产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 可以表示为

$$a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} u^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad a^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \psi^\dagger(\mathbf{x},t)u(\mathbf{p},\lambda). \tag{4.245}$$

同理，可以推出

$$\begin{aligned}
&\int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t) \\
&= \int \frac{d^3x\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{-i(p-q)\cdot x} \right] \\
&= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}+\mathbf{q}) \right. \\
&\qquad\qquad \left. + v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b^\dagger_{\mathbf{p},\lambda'} \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}b^\dagger_{\mathbf{p},\lambda'} \right) = \sqrt{2E_{\mathbf{p}}}\, b^\dagger_{\mathbf{p},\lambda}.
\end{aligned} \tag{4.246}$$

于是，产生算符 $b^\dagger_{\mathbf{p},\lambda}$ 和湮灭算符 $b_{\mathbf{p},\lambda}$ 可以表示为

$$b^\dagger_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} v^\dagger(\mathbf{p},\lambda)\psi(\mathbf{x},t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \psi^\dagger(\mathbf{x},t)v(\mathbf{p},\lambda). \tag{4.247}$$

4.5　Dirac 旋量场的正则量子化

4.5.1　用等时对易关系量子化 Dirac 旋量场的困难

在标量场和矢量场的正则量子化程序中，我们先假设场算符与其共轭动量密度算符满足等时对易关系 (2.51)，然后推导出产生湮灭算符的对易关系，再通过计算给出正定的哈密顿量（对于无质量矢量场，需要用弱 Lorenz 规范条件来得到正定的哈密顿量期待值），从而说明在量子场论中使用正则量子化方法是合理的。在本小节中，我们将尝试用类似的等时对易关系对 Dirac 旋量场进行量子化，不过，我们会发现这种方法并不能给出正定的哈密顿量，因而是不可行的。

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=0. \tag{4.248}$$

这里已经将旋量指标明显地写出来。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=[\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]=0. \tag{4.249}$$

接下来，我们计算产生湮灭算符的对易关系。由 (4.245) 式和正交归一关系 (4.220)，可得

$$\begin{aligned}
[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)[\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)]u_b(\mathbf{q},\lambda')\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')\\
&= \frac{1}{2E_\mathbf{p}}u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{4.250}$$

另外，有

$$[a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}]=\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b^\dagger(\mathbf{q},\lambda')[\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)]=0. \tag{4.251}$$

由 (4.247) 式和正交归一关系 (4.220)，可得

$$\begin{aligned}
[b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')[\psi_a^\dagger(\mathbf{x},t),\psi_b(\mathbf{y},t)]v_a(\mathbf{p},\lambda)\\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')v_a(\mathbf{p},\lambda)(-\delta_{ba})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\
&= -\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}v^\dagger(\mathbf{q},\lambda')v(\mathbf{p},\lambda)\\
&= -\frac{1}{2E_\mathbf{p}}v^\dagger(\mathbf{p},\lambda')v(\mathbf{p},\lambda)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=-(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{4.252}$$

注意，这个结果非同寻常地多了一个负号。此外，还有

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}} \int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}[\psi_a^\dagger(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)]v_a(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda') = 0, \tag{4.253}$$

$$[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}} \int d^3x\, d^3y\, e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b^\dagger(\mathbf{q},\lambda')[\psi_a(\mathbf{x},t), \psi_b(\mathbf{y},t)] = 0, \tag{4.254}$$

以及

$$\begin{aligned}
[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}} \int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)[\psi_a(\mathbf{x},t), \psi_b^\dagger(\mathbf{y},t)]v_b(\mathbf{q},\lambda') \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}} \int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}} \int d^3x\, e^{i(E_\mathbf{p}+E_\mathbf{q})t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda') \\
&= \frac{1}{2E_\mathbf{p}}e^{2iE_\mathbf{p}t}u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0.
\end{aligned} \tag{4.255}$$

上式最后一步用到正交归一关系 (4.220)。

整理起来，通过等时对易关系 (4.248) 得到的产生湮灭算符对易关系为

$$\begin{aligned}
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0, \\
[b_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] &= -(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0, \\
[a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] &= [b_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}] = 0.
\end{aligned} \tag{4.256}$$

利用这样的对易关系，可以把哈密顿量 (4.243) 化为

$$\begin{aligned}
H &= \sum_{\lambda=\pm}\int \frac{d^3p}{(2\pi)^3}\, E_\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right) \\
&= \sum_{\lambda=\pm}\int \frac{d^3p}{(2\pi)^3}\, E_\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right) + (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\, 2E_\mathbf{p}.
\end{aligned} \tag{4.257}$$

上式最后一行第二项是零点能。在第一项中由 $a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为正，但由 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为负。从而，粒子数密度 $b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}$ 越大，场的总能量越少，这显然是非物理的。因此，用正则对易关系 (4.248) 对 Dirac 旋量场进行量子化是不可行的。

4.5.2 用等时反对易关系量子化 Dirac 旋量场

从 (4.257) 式的计算过程可以看出，如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 位置的同时可以改变圆括号中第二项的符号，就可以得到正定的哈密顿量。这意味着我们需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的对易关系，而是反对易关系。为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b^\dagger_{\mathbf{p},\lambda}$ 的反对易关系，则需要舍弃等时对易关系 (4.248)，代之以**等时反对易关系**

$$\{\psi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)\} = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad \{\psi_a(\mathbf{x},t), \psi_b(\mathbf{y},t)\} = \{\pi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)\} = 0. \tag{4.258}$$

采用反对易关系进行量子化的方法称为 **Jordan-Wigner 量子化**。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时反对易关系

$$\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}=\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad \{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=\{\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}=0. \tag{4.259}$$

接下来，我们计算产生湮灭算符的反对易关系。计算过程与上一小节类似，只是我们要将 (4.250) 至 (4.255) 式中表示对易的方括号改成表示反对易的花括号。因此，可得

$$\begin{aligned}\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}u_b(\mathbf{q},\lambda')\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\end{aligned} \tag{4.260}$$

和

$$\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\} = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)u_b^\dagger(\mathbf{q},\lambda')\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=0. \tag{4.261}$$

另外，有

$$\begin{aligned}\{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')\{\psi_a^\dagger(\mathbf{x},t),\psi_b(\mathbf{y},t)\}v_a(\mathbf{p},\lambda)\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}v_b^\dagger(\mathbf{q},\lambda')v_a(\mathbf{p},\lambda)\delta_{ba}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(E_\mathbf{p}-E_\mathbf{q})t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}v^\dagger(\mathbf{q},\lambda')v(\mathbf{p},\lambda)\\ &= \frac{1}{2E_\mathbf{p}}v^\dagger(\mathbf{p},\lambda')v(\mathbf{p},\lambda)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{4.262}$$

与 (4.252) 式不同，上式的结果具有正常的符号。此外，还有

$$\{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\} = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\{\psi_a^\dagger(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}v_a(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')=0, \tag{4.263}$$

$$\{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger\} = \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b^\dagger(\mathbf{q},\lambda')\{\psi_a(\mathbf{x},t),\psi_b(\mathbf{y},t)\}=0, \tag{4.264}$$

以及

$$\begin{aligned}\{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\} &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)\{\psi_a(\mathbf{x},t),\psi_b^\dagger(\mathbf{y},t)\}v_b(\mathbf{q},\lambda')\\ &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}u_a^\dagger(\mathbf{p},\lambda)v_b(\mathbf{q},\lambda')\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})=0.\end{aligned} \tag{4.265}$$

整理起来，通过等时反对易关系 (4.258) 得到的产生湮灭算符反对易关系为

$$\begin{aligned}\{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger\} &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a_{\mathbf{p},\lambda}^\dagger,a_{\mathbf{q},\lambda'}^\dagger\}=0,\\ \{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger\} &= (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{b_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\}=\{b_{\mathbf{p},\lambda}^\dagger,b_{\mathbf{q},\lambda'}^\dagger\}=0,\\ \{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}^\dagger\} &= \{b_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}^\dagger\}=\{a_{\mathbf{p},\lambda},b_{\mathbf{q},\lambda'}\}=\{a_{\mathbf{p},\lambda}^\dagger,b_{\mathbf{q},\lambda'}^\dagger\}=0.\end{aligned} \tag{4.266}$$

$a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 各自描述一种粒子。利用这样的反对易关系，可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)\\ &= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right)-(2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}\,2E_{\mathbf{p}}. \end{aligned} \tag{4.267}$$

上式最后一行第二项是零点能。第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和，它是**正定**的。可见，用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的。

利用 (4.4) 式和反对易关系 (4.266)，可得哈密顿量 H 与产生湮灭算符的对易子为

$$\begin{aligned} [H,a^\dagger_{\mathbf{p},\lambda}] &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'}+b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{q},\lambda'},\,a^\dagger_{\mathbf{p},\lambda}\right]\\ &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\Big(a^\dagger_{\mathbf{q},\lambda'}\{a_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}\}-\{a^\dagger_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}\}a_{\mathbf{q},\lambda'}\\ &\qquad\qquad +b^\dagger_{\mathbf{q},\lambda'}\{b_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}\}-\{b^\dagger_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}\}b_{\mathbf{q},\lambda'}\Big)\\ &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}a^\dagger_{\mathbf{q},\lambda'}\{a_{\mathbf{q},\lambda'},a^\dagger_{\mathbf{p},\lambda}\}\\ &= \sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}a^\dagger_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda}, \end{aligned} \tag{4.268}$$

$$\begin{aligned} [H,a_{\mathbf{p},\lambda}] &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'}+b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{q},\lambda'},\,a_{\mathbf{p},\lambda}\right]=\sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left(-\{a^\dagger_{\mathbf{q},\lambda'},a_{\mathbf{p},\lambda}\}a_{\mathbf{q},\lambda'}\right)\\ &= -\sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}a_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=-E_{\mathbf{p}}a_{\mathbf{p},\lambda}, \end{aligned} \tag{4.269}$$

$$\begin{aligned} [H,b^\dagger_{\mathbf{p},\lambda}] &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'}+b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{q},\lambda'},\,b^\dagger_{\mathbf{p},\lambda}\right]=\sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}b^\dagger_{\mathbf{q},\lambda'}\{b_{\mathbf{q},\lambda'},b^\dagger_{\mathbf{p},\lambda}\}\\ &= \sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}b^\dagger_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=E_{\mathbf{p}}b^\dagger_{\mathbf{p},\lambda}, \end{aligned} \tag{4.270}$$

$$\begin{aligned} [H,b_{\mathbf{p},\lambda}] &= \sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left[a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{q},\lambda'}+b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{q},\lambda'},b_{\mathbf{p},\lambda}\right]=\sum_{\lambda'}\int\frac{d^3q}{(2\pi)^3}\,E_{\mathbf{q}}\left(-\{b^\dagger_{\mathbf{q},\lambda'},b_{\mathbf{p},\lambda}\}b_{\mathbf{q},\lambda'}\right)\\ &= -\sum_{\lambda'}\int d^3q\,E_{\mathbf{q}}b_{\mathbf{q},\lambda'}\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{q}-\mathbf{p})=-E_{\mathbf{p}}b_{\mathbf{p},\lambda}. \end{aligned} \tag{4.271}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则

$$H\,|E\rangle = E\,|E\rangle\,. \tag{4.272}$$

从而，可得

$$\begin{aligned} Ha^\dagger_{\mathbf{p},\lambda}\,|E\rangle &= (a^\dagger_{\mathbf{p},\lambda}H+E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda})\,|E\rangle=(E+E_{\mathbf{p}})a^\dagger_{\mathbf{p},\lambda}\,|E\rangle\,,\\ Ha_{\mathbf{p},\lambda}\,|E\rangle &= (a_{\mathbf{p},\lambda}H-E_{\mathbf{p}}a_{\mathbf{p},\lambda})\,|E\rangle=(E-E_{\mathbf{p}})a_{\mathbf{p},\lambda}\,|E\rangle\,, \end{aligned}$$

$$
\begin{aligned}
Hb^\dagger_{\mathbf{p},\lambda}\left|E\right\rangle &= (b^\dagger_{\mathbf{p},\lambda}H + E_{\mathbf{p}}b^\dagger_{\mathbf{p},\lambda})\left|E\right\rangle = (E+E_{\mathbf{p}})b^\dagger_{\mathbf{p},\lambda}\left|E\right\rangle,\\
Hb_{\mathbf{p},\lambda}\left|E\right\rangle &= (b_{\mathbf{p},\lambda}H - E_{\mathbf{p}}b_{\mathbf{p},\lambda})\left|E\right\rangle = (E-E_{\mathbf{p}})b_{\mathbf{p},\lambda}\left|E\right\rangle.
\end{aligned}
\tag{4.273}
$$

可见，当 $a^\dagger_{\mathbf{p},\lambda}\left|E\right\rangle$ 和 $b^\dagger_{\mathbf{p},\lambda}\left|E\right\rangle$ 不为零时，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 和 $b^\dagger_{\mathbf{p},\lambda}$ 的作用都是使能量本征值增加 $E_{\mathbf{p}}$；当 $a_{\mathbf{p},\lambda}\left|E\right\rangle$ 和 $b_{\mathbf{p},\lambda}\left|E\right\rangle$ 不为零时，湮灭算符 $a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的作用都是使能量本征值减少 $E_{\mathbf{p}}$。

根据 (1.158) 式，Dirac 旋量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\,\pi\nabla\psi = \int d^3x\,\psi^\dagger(-i\nabla)\psi\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]\\
&\qquad\times\left[\mathbf{q}\,u(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x} - \mathbf{q}\,v(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\mathbf{q}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}\\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} - u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\Big]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\,\mathbf{q}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\\
&\qquad - v^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big]\\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[-u^\dagger(\mathbf{p},\lambda)v(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\\
&\qquad + v^\dagger(\mathbf{p},\lambda)u(\mathbf{q},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big]\Big\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[\mathbf{p}\,u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - \mathbf{p}\,v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\\
&\qquad + \mathbf{p}\,u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t} - \mathbf{p}\,v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big]\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\,\mathbf{p}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} - 2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right) - 2\delta^{(3)}(0)\int d^3p\,\mathbf{p}\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} + b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right).
\end{aligned}
\tag{4.274}
$$

倒数第四步用到正交归一关系 (4.220)，倒数第二步用到反对易关系 (4.266)。总动量是所有动量模式所有螺旋度所有粒子贡献的动量之和。

4.5.3 U(1) 整体对称性

类似于复标量场，Dirac 旋量场也具有 U(1) 整体对称性。对 Dirac 旋量场 $\psi(x)$ 作 U(1) 整体变换

$$\psi'(x) = e^{iq\theta}\psi(x), \tag{4.275}$$

则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x)e^{-iq\theta}, \quad [\bar{\psi}(x)]' = \bar{\psi}'(x) = [\psi'(x)]^\dagger\gamma^0 = \bar{\psi}(x)e^{-iq\theta}. \tag{4.276}$$

在此变换下，拉氏量 (4.118) 不变：

$$\begin{aligned}\mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi'(x)\\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x).\end{aligned} \tag{4.277}$$

容易验证，4.3 节中列举的旋量双线性型都在这种 U(1) 整体变换下不变。因此，用这些旋量双线性型构造的拉氏量都具有 U(1) 整体对称性。

U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + iq\theta\psi(x). \tag{4.278}$$

由于 $\delta x^\mu = 0$，根据 (1.136) 式可得

$$\bar{\delta}\psi = \delta\psi = iq\theta\psi. \tag{4.279}$$

按照 (1.141) 式，相应的 Noether 守恒流为

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar{\delta}\psi = i\bar{\psi}\gamma^\mu(iq\theta\psi) = -q\theta\bar{\psi}\gamma^\mu\psi. \tag{4.280}$$

扔掉无穷小参数 $-\theta$，定义

$$J^\mu \equiv q\bar{\psi}\gamma^\mu\psi, \tag{4.281}$$

则 Noether 定理给出

$$\partial_\mu J^\mu = 0. \tag{4.282}$$

相应的守恒荷为

$$\begin{aligned}Q &= \int d^3x\, J^0 = q\int d^3x\,\bar{\psi}\gamma^0\psi = q\int d^3x\,\psi^\dagger\psi\\ &= q\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\left[u^\dagger(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x} + v^\dagger(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}e^{-ip\cdot x}\right]\\ &\qquad\times\left[u(\mathbf{k},\lambda')a_{\mathbf{k},\lambda'}e^{-ik\cdot x} + v(\mathbf{k},\lambda')b^\dagger_{\mathbf{k},\lambda'}e^{ik\cdot x}\right]\\ &= q\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\left[u^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{i(p-k)\cdot x}\right.\end{aligned}$$

$$
\begin{aligned}
&\quad +v^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{-i(p-k)\cdot x}+u^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{i(p+k)\cdot x}\\
&\quad +v^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(p+k)\cdot x}\Big]\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{k})\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\\
&\qquad\qquad +v^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t}\Big]\\
&\qquad +\delta^{(3)}(\mathbf{p}+\mathbf{k})\Big[u^\dagger(\mathbf{p},\lambda)v(\mathbf{k},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\\
&\qquad\qquad +v^\dagger(\mathbf{p},\lambda)u(\mathbf{k},\lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\Big]\Big\}\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[u^\dagger(\mathbf{p},\lambda)u(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+v^\dagger(\mathbf{p},\lambda)v(\mathbf{p},\lambda')b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\\
&\quad +u^\dagger(\mathbf{p},\lambda)v(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}+v^\dagger(\mathbf{p},\lambda)u(-\mathbf{p},\lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}\Big]\\
&= q\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}+2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda'}\right)\\
&= q\sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\right)\\
&= \sum_{\lambda=\pm}\int\frac{d^3p}{(2\pi)^3}\left(q\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}-q\,b^\dagger_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}\right)+2\delta^{(3)}(0)\int d^3p\,q. \qquad (4.283)
\end{aligned}
$$

上式第二项是零点荷。从第一项的形式可以看出，由 $a^\dagger_{\mathbf{p},\lambda}, a_{\mathbf{p},\lambda}$ 描述的粒子是**正粒子**，具有的荷为 q；由 $b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 描述的粒子是**反粒子**，具有的荷为 $-q$。除去零点荷，总荷是所有动量模式所有螺旋度所有正反粒子贡献的荷之和。

4.5.4 粒子态

对于自由的 Dirac 旋量场，真空态定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$a_{\mathbf{p},\lambda}|0\rangle = b_{\mathbf{p},\lambda}|0\rangle = 0, \qquad (4.284)$$

满足

$$\langle 0|0\rangle = 1,\quad H|0\rangle = E_{\text{vac}}|0\rangle,\quad E_{\text{vac}} = -2\delta^{(3)}(0)\int d^3p\,E_{\mathbf{p}}. \qquad (4.285)$$

动量为 $\mathbf{p}$、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$|\mathbf{p},\lambda,+\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,a^\dagger_{\mathbf{p},\lambda}|0\rangle,\quad |\mathbf{p},\lambda,-\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,b^\dagger_{\mathbf{p},\lambda}|0\rangle. \qquad (4.286)$$

根据 (4.268) 和 (4.270) 式，有

$$
\begin{aligned}
H|\mathbf{p},\lambda,+\rangle &= \sqrt{2E_{\mathbf{p}}}\,Ha^\dagger_{\mathbf{p},\lambda}|0\rangle = \sqrt{2E_{\mathbf{p}}}\,(a^\dagger_{\mathbf{p},\lambda}H+E_{\mathbf{p}}a^\dagger_{\mathbf{p},\lambda})|0\rangle\\
&= \sqrt{2E_{\mathbf{p}}}\,(E_{\text{vac}}+E_{\mathbf{p}})a^\dagger_{\mathbf{p},\lambda}|0\rangle = (E_{\text{vac}}+E_{\mathbf{p}})|\mathbf{p},\lambda,+\rangle, \qquad (4.287)\\
H|\mathbf{p},\lambda,-\rangle &= \sqrt{2E_{\mathbf{p}}}\,Hb^\dagger_{\mathbf{p},\lambda}|0\rangle = \sqrt{2E_{\mathbf{p}}}\,(b^\dagger_{\mathbf{p},\lambda}H+E_{\mathbf{p}}b^\dagger_{\mathbf{p},\lambda})|0\rangle
\end{aligned}
$$

$$= \sqrt{2E_{\mathbf{p}}}\,(E_{\text{vac}} + E_{\mathbf{p}})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle = (E_{\text{vac}} + E_{\mathbf{p}})\,|\mathbf{p},\lambda,-\rangle\,. \tag{4.288}$$

可见，$|\mathbf{p},\lambda,+\rangle$ 和 $|\mathbf{p},\lambda,-\rangle$ 都比真空态多了一份能量 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}$。

将 $\psi(x)$ 的平面波解 (4.236) 代入 (4.81) 式左边，得

$$[\psi(x),\mathbf{J}] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\Big\{u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}]e^{-ip\cdot x} + v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}]e^{ip\cdot x}\Big\}, \tag{4.289}$$

代入右边，得

$$(\mathbf{L}+\mathbf{S})\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}(-i\mathbf{x}\times\nabla+\mathbf{S})\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[(\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + (-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]. \tag{4.290}$$

可见，对于动量模式 $\mathbf{p}$ 和螺旋度 λ，有

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\mathbf{J}] = (\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda},\quad v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\mathbf{J}] = (-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}. \tag{4.291}$$

根据 (4.198) 和 (4.213) 式，$u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态，因而

$$u(\mathbf{p},\lambda)[a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = 2\,\hat{\mathbf{p}}\cdot(\mathbf{x}\times\mathbf{p}+\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = (2\,\hat{\mathbf{p}}\cdot\mathbf{S})u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda} = \lambda\,u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}, \tag{4.292}$$

$$v(\mathbf{p},\lambda)[b^\dagger_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = 2\,\hat{\mathbf{p}}\cdot(-\mathbf{x}\times\mathbf{p}+\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda} = (2\,\hat{\mathbf{p}}\cdot\mathbf{S})v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda} = -\lambda\,v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}. \tag{4.293}$$

故

$$[a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = \lambda\,a_{\mathbf{p},\lambda},\quad [b^\dagger_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}] = -\lambda\,b^\dagger_{\mathbf{p},\lambda}. \tag{4.294}$$

由于 $\mathbf{J}$ 是厄米算符，对第一式取厄米共轭可得

$$\lambda\,a^\dagger_{\mathbf{p},\lambda} = [a_{\mathbf{p},\lambda},\,2\,\hat{\mathbf{p}}\cdot\mathbf{J}]^\dagger = (2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda} - a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) = [2\,\hat{\mathbf{p}}\cdot\mathbf{J},a^\dagger_{\mathbf{p},\lambda}]. \tag{4.295}$$

于是，有

$$[2\,\hat{\mathbf{p}}\cdot\mathbf{J},a^\dagger_{\mathbf{p},\lambda}] = \lambda\,a^\dagger_{\mathbf{p},\lambda},\quad [2\,\hat{\mathbf{p}}\cdot\mathbf{J},b^\dagger_{\mathbf{p},\lambda}] = \lambda\,b^\dagger_{\mathbf{p},\lambda}. \tag{4.296}$$

$\mathbf{J}$ 是总角动量算符，真空态 $|0\rangle$ 不具有角动量，所以满足

$$\mathbf{J}\,|0\rangle = 0. \tag{4.297}$$

由此，可得

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})a^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [a^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) + \lambda\,a^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\,a^\dagger_{\mathbf{p},\lambda}\,|0\rangle\,, \tag{4.298}$$

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})b^\dagger_{\mathbf{p},\lambda}\,|0\rangle = [b^\dagger_{\mathbf{p},\lambda}(2\,\hat{\mathbf{p}}\cdot\mathbf{J}) + \lambda\,b^\dagger_{\mathbf{p},\lambda}]\,|0\rangle = \lambda\,b^\dagger_{\mathbf{p},\lambda}\,|0\rangle\,. \tag{4.299}$$

在没有轨道角动量的情况下，$2\,\hat{\mathbf{p}}\cdot\mathbf{J}$ 是螺旋度算符。因此，上面两式说明 $|\mathbf{p},\lambda,+\rangle$ 和 $|\mathbf{p},\lambda,-\rangle$ 都是螺旋度本征态，本征值为 λ：

$$(2\,\hat{\mathbf{p}}\cdot\mathbf{J})\,|\mathbf{p},\lambda,\pm\rangle = \lambda\,|\mathbf{p},\lambda,\pm\rangle\,. \tag{4.300}$$

这正是我们所期望的。

以上讨论表明，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的正粒子，另一个产生算符 $b^\dagger_{\mathbf{p},\lambda}$ 的作用是产生一个动量为 $\mathbf{p}$、螺旋度为 λ 的反粒子。正粒子和反粒子具有相同的质量 m。

在 (4.208) 式中，我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$，使得 $v(\mathbf{p},\lambda)$ 的螺旋度本征值为 $-\lambda$，从而得到 $b^\dagger_{\mathbf{p},\lambda}|0\rangle$ 的螺旋度本征值为 λ 的结果。如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$，依照上述推导，$b^\dagger_{\mathbf{p},\lambda}|0\rangle$ 的螺旋度本征值就会变成 $-\lambda$；也就是说，$b^\dagger_{\mathbf{p},\lambda}, b_{\mathbf{p},\lambda}$ 将描述螺旋度为 $-\lambda$ 的反粒子。这不符合我们的记号，因此，我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 (4.208) 式的形式。

由反对易关系 (4.266)，可得

$$\begin{aligned}
a_{\mathbf{p},\lambda}|\mathbf{q},\lambda',+\rangle &= \sqrt{2E_\mathbf{q}}\,a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{q},\lambda'}a_{\mathbf{p},\lambda}]|0\rangle \\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})|0\rangle, \qquad (4.301)\\
b_{\mathbf{p},\lambda}|\mathbf{q},\lambda',-\rangle &= \sqrt{2E_\mathbf{q}}\,b_{\mathbf{p},\lambda}b^\dagger_{\mathbf{q},\lambda'}|0\rangle = \sqrt{2E_\mathbf{q}}\,[(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - b^\dagger_{\mathbf{q},\lambda'}b_{\mathbf{p},\lambda}]|0\rangle \\
&= \sqrt{2E_\mathbf{q}}\,(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q})|0\rangle. \qquad (4.302)
\end{aligned}$$

可以看出，湮灭算符 $a_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 $\mathbf{p}$、螺旋度为 λ 的正粒子，湮灭算符 $b_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 $\mathbf{p}$、螺旋度为 λ 的反粒子。

将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle \equiv \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{p}_3}E_{\mathbf{p}_4}}\,a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle. \qquad (4.303)$$

根据反对易关系 (4.266)，有

$$\begin{aligned}
a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle &= -a^\dagger_{\mathbf{p}_2,\lambda_2}a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_3,\lambda_3}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle = -a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle \\
&= -a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_4,\lambda_4}b^\dagger_{\mathbf{p}_3,\lambda_3}|0\rangle = -b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_4,\lambda_4}|0\rangle \\
&= -a^\dagger_{\mathbf{p}_1,\lambda_1}b^\dagger_{\mathbf{p}_4,\lambda_4}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_2,\lambda_2}|0\rangle = -b^\dagger_{\mathbf{p}_4,\lambda_4}a^\dagger_{\mathbf{p}_2,\lambda_2}b^\dagger_{\mathbf{p}_3,\lambda_3}a^\dagger_{\mathbf{p}_1,\lambda_1}|0\rangle. \qquad (4.304)
\end{aligned}$$

从而，可得

$$\begin{aligned}
|\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle, \\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle, \\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_3,\lambda_3,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle, \\
|\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_4,\lambda_4,-\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle, \\
|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_2,\lambda_2,+\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle, \\
|\mathbf{p}_4,\lambda_4,-;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_1,\lambda_1,+\rangle &= -|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+;\,\mathbf{p}_3,\lambda_3,-;\,\mathbf{p}_4,\lambda_4,-\rangle. \qquad (4.305)
\end{aligned}$$

也就是说，交换任意两个粒子，得到的态相差一个负号，故多粒子态对于全同粒子交换是**反对称**的。这说明旋量场描述的粒子是**费米子** (fermion)，服从 Fermi-Dirac 统计。得到这个结论的

关键在于两个产生算符相互**反对易**。对于两个相同的产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 或 $b^\dagger_{\mathbf{p},\lambda}$，反对易关系导致

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = -a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = -b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle, \tag{4.306}$$

故

$$a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = 0, \quad b^\dagger_{\mathbf{p},\lambda}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle = 0. \tag{4.307}$$

这说明在没有其它自由度的情况下，不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态，这就是 **Pauli 不相容原理**。

在第 2 章和第 3 章中，我们分别讨论了自旋为 0 的标量场和自旋为 1 的矢量场，合适的处理方式是通过对易关系对它们进行量子化，因而它们都描述**玻色子**。另一方面，在本章中，我们需要采用**反对易**关系才能对自旋为 1/2 的旋量场进行合适的量子化，因而旋量场描述的粒子是**费米子**。实际上，这样的状况是普遍的，存在**自旋–统计定理**：整数自旋的物理场必须用对易关系进行量子化，对应的粒子是玻色子；半整数自旋的物理场必须用反对易关系进行量子化，对应的粒子是费米子。

将两个正费米子组成的双粒子态记为

$$\left|\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+\right\rangle \equiv \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\,a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}\left|0\right\rangle, \tag{4.308}$$

则双粒子态的内积关系是

$$\begin{aligned}
&\left\langle\mathbf{q}_1,\lambda_1',+;\,\mathbf{q}_2,\lambda_2',+\,\middle|\,\mathbf{p}_1,\lambda_1,+;\,\mathbf{p}_2,\lambda_2,+\right\rangle\\
={}&\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\left\langle 0\right| a_{\mathbf{q}_2,\lambda_2'}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_1,\lambda_1}a^\dagger_{\mathbf{p}_2,\lambda_2}\left|0\right\rangle\\
={}&\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\left\langle 0\right| a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}\left|0\right\rangle\\
&\qquad -\left\langle 0\right| a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}a_{\mathbf{q}_1,\lambda_1'}a^\dagger_{\mathbf{p}_2,\lambda_2}\left|0\right\rangle\Big]\\
={}&\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\Big[(2\pi)^3\delta_{\lambda_1\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\left\langle 0\right| a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_2,\lambda_2}\left|0\right\rangle\\
&\qquad -(2\pi)^3\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\left\langle 0\right| a_{\mathbf{q}_2,\lambda_2'}a^\dagger_{\mathbf{p}_1,\lambda_1}\left|0\right\rangle\Big]\\
={}&\sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\big[(2\pi)^6\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\\
&\qquad -(2\pi)^6\delta_{\lambda_2\lambda_1'}\delta_{\lambda_1\lambda_2'}\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\big]\\
={}&4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\,\big[\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\\
&\qquad -\delta_{\lambda_1\lambda_2'}\delta_{\lambda_2\lambda_1'}\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\big].
\end{aligned}\tag{4.309}$$

上式最后两行方括号中第二项前面有一个**负号**，由产生湮灭算符的反对易关系引起。这是双费米子态内积关系与双玻色子态内积关系 (2.119) 在形式上的不同之处。

附录 A　英汉对照

Annihilation operator: 湮灭算符
Antichronous: 反时向
Anti-particle: 反粒子
Axial vector: 轴矢量
Boost: 增速
Boson: 玻色子
Canonical quantization: 正则量子化
Chiral representation: 手征表象
Conjugate momentum density: 共轭动量密度
Conserved charge: 守恒荷
Conserved current: 守恒流
Contraction: 缩并
Contravariant vector: 逆变矢量
Covariant vector: 协变矢量
Creation operator: 产生算符
Dirac slash: Dirac 斜线
Electron: 电子
Energy-momentum tensor: 能动张量
Expectation value: 期待值
Fermion: 费米子
Field strength tensor: 场强张量
Gauge-fixing term: 规范固定项
Gauge invariant: 规范不变量
Gauge symmetry: 规范对称性
Gauge transformation: 规范变换
Generalized coordinate: 广义坐标
Generator: 生成元
Global: 整体
Hamiltonian: 哈密顿量
Helicity: 螺旋度
Hermitian conjugate: 厄米共轭
Hermitian operator: 厄米算符
Homomorphic: 同态
Improper: 非固有
Lagrangian: 拉格朗日量
Left-handed: 左手
Local: 局域
Lowering operator: 降算符
Metric: 度规
Mode: 模式
Orthochronous: 保时向
Parity: 宇称
Phonon: 声子
Picture: 绘景
Plane-wave solution: 平面波解
Polarization vector: 极化矢量
Positron: 正电子
Proper: 固有
Pseudoscalar: 赝标量
Raising operator: 升算符
Right-handed: 右手
Real orthogonal matrix: 实正交矩阵
Scalar: 标量
Self-conjugate: 自共轭
Simple harmonic oscillator: 简谐振子
Space inversion: 空间反射
Spinor: 旋量
Spinor bilinear: 旋量双线性型
Spinor representation: 旋量表示
Step function: 阶跃函数
Tensor: 张量
Time reversal: 时间反演

Unitary: 幺正
Vacuum: 真空
Vector: 矢量
Zero-point energy: 零点能