DM models

Convention:

$v^i \in \mathbf{2}, \quad v_i \in \bar{\mathbf{2}}, \quad \varepsilon^{12} = +1, \quad \varepsilon_{12} = -1, \quad \varepsilon^{ij} = -\varepsilon^{ji}, \quad \varepsilon_{ij} = -\varepsilon_{ji}$

SM Higgs field $\quad H = H^i = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} G^+(x) \\ \frac{1}{\sqrt{2}}[v + h(x) + iG^0(x)] \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right)$ under $(\mathrm{SU}(2)_\mathrm{L}, \mathrm{U}(1)_\mathrm{Y})$

$$H_i = \varepsilon_{ij} H^j = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} -H^0 \\ H^+ \end{pmatrix} \in \left(\bar{\mathbf{2}}, \frac{1}{2}\right)$$

$$H_i^\dagger = (H^i)^* = \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} \in \left(\bar{\mathbf{2}}, -\frac{1}{2}\right)$$

$$\tilde{H} = H^{\dagger i} = \varepsilon^{ij} H_j^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right)$$

$$\left[\text{Note:} \quad H^\dagger H = H_i^\dagger H^i = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = H^+ H^- + H^0 H^{0*} = -H^{\dagger i} H_i \right]$$

Singlet-Doublet Fermionic Dark Matter (SDFDM)

Ref: D'Eramo, 0705.4493; Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604

Left-handed Weyl fermions:

$$S \in (\mathbf{1}, 0), \quad D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right)$$

$$\mathcal{L}_\mathrm{S} = iS^\dagger \bar{\sigma}^\mu \partial_\mu S - \frac{1}{2}(m_S SS + \text{h.c.}), \quad \mathcal{L}_\mathrm{D} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.})$$

$$\mathcal{L}_\mathrm{HSD} = y_1 H_i S D_1^i - y_2 H_i^\dagger S D_2^i + \text{h.c.}$$

$$-m_D \varepsilon_{ij} D_1^i D_2^j = m_D D_1^0 D_2^0 - m_D D_1^- D_2^+$$

$$y_1 H_i S D_1^i = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} S \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = -y_1 H^0 S D_1^0 + y_1 H^+ S D_1^- \to -\frac{1}{\sqrt{2}} y_1 (v+h) S D_1^0$$

$$-y_2 H_i^\dagger S D_2^i = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} S \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 H^- S D_2^+ - y_2 H^{0*} S D_2^0 \to -\frac{1}{\sqrt{2}} y_2 (v+h) S D_2^0$$

$$\mathcal{L}_\mathrm{mass} = -\frac{1}{2} \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_\mathrm{N} \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} - m_D D_1^- D_2^+ + \text{h.c.} = -\frac{1}{2} \sum_{i=1}^{3} m_{\chi_i^0} \chi_i^0 \chi_i^0 - m_{\chi^\pm} \chi^- \chi^+ + \text{h.c.}$$

$$\mathcal{M}_\mathrm{N} = \begin{pmatrix} m_S & \frac{1}{\sqrt{2}} y_1 v & \frac{1}{\sqrt{2}} y_2 v \\ \frac{1}{\sqrt{2}} y_1 v & 0 & -m_D \\ \frac{1}{\sqrt{2}} y_2 v & -m_D & 0 \end{pmatrix}, \quad m_{\chi^\pm} = m_D, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$$

$$\mathcal{N}^\mathrm{T} \mathcal{M}_\mathrm{N} \mathcal{N} = \mathrm{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}$$

$y = y_1 = y_2 \quad \Rightarrow \quad$ <u>Custodial $\mathrm{SU(2)_R}$ global symmetry</u>

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an } \mathrm{SU(2)_R} \text{ indice}$$

$$H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = -\frac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = -\frac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j$$

$$\mathcal{L}_{\mathrm{HSD}} = y(H_i S D_1^i - H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S (\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_{\mathrm{D}} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A - \frac{1}{2}[m_D \varepsilon_{AB}\varepsilon_{ij}(\mathcal{D}^A)^i(\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_S \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(-D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_{\chi^\pm} = m_D \\ m_{\chi_2^0} = \dfrac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} + m_D - m_S\right] \\ m_{\chi_3^0} = \dfrac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} - m_D + m_S\right] \end{cases}$$

$$m_D > m_S \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = \dfrac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} - m_D + m_S\right] \\ m_{\chi_2^0} = m_{\chi^\pm} = m_D \\ m_{\chi_3^0} = \dfrac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} + m_D - m_S\right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_S)}$$

$$\left[\begin{array}{l} y = y_1 = -y_2 \quad \Rightarrow \quad \text{Another custodial symmetry limit} \\ (\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} -H_i^\dagger \\ H_i \end{pmatrix} \\ H^\dagger H = H_i^\dagger H^i = \dfrac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = \dfrac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = \dfrac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j \\ \mathcal{L}_{\mathrm{HSD}} = y(H_i S D_1^i + H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S (\mathcal{D}^B)^j + \text{h.c.} \end{array}\right]$$

<u>Gauge interactions</u>

$$D_\mu D_i = (\partial_\mu - ig' B_\mu Y_{D_i} - ig W_\mu^a t_{\mathrm{D}}^a) D_i$$

$$Y_{D_1} = -\frac{1}{2}, \quad Y_{D_2} = \frac{1}{2}, \quad t_{\mathrm{D}}^1 = \frac{\sigma^1}{2} = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad t_{\mathrm{D}}^2 = \frac{\sigma^2}{2} = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad t_{\mathrm{D}}^3 = \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$g' B_\mu Y_{D_1} + g W_\mu^a t_{\mathrm{D}}^a = \frac{1}{2}\begin{pmatrix} -g' B_\mu + g W_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -g' B_\mu - g W_\mu^a \end{pmatrix} = \begin{pmatrix} \dfrac{g}{2c_{\mathrm{W}}} Z_\mu & \dfrac{g}{\sqrt{2}} W_\mu^+ \\ \dfrac{g}{\sqrt{2}} W_\mu^- & -eA_\mu + \dfrac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu \end{pmatrix}$$

$$g' B_\mu Y_{D_2} + g W_\mu^a t_{\mathrm{D}}^a = \frac{1}{2}\begin{pmatrix} g' B_\mu + g W_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & g' B_\mu - g W_\mu^a \end{pmatrix} = \begin{pmatrix} eA_\mu - \dfrac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu & \dfrac{g}{\sqrt{2}} W_\mu^+ \\ \dfrac{g}{\sqrt{2}} W_\mu^- & -\dfrac{g}{2c_{\mathrm{W}}} Z_\mu \end{pmatrix}$$

$$\mathcal{L}_{\mathrm{D}} \supset iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2$$

$$= \frac{g}{2c_{\mathrm{W}}} Z_\mu (D_1^0)^\dagger \bar{\sigma}^\mu D_1^0 + \left[-eA_\mu + \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu\right](D_1^-)^\dagger \bar{\sigma}^\mu D_1^- + \frac{g}{\sqrt{2}}[W_\mu^+ (D_1^0)^\dagger \bar{\sigma}^\mu D_1^- + W_\mu^- (D_1^-)^\dagger \bar{\sigma}^\mu D_1^0]$$

$$+ \left[eA_\mu - \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu\right](D_2^+)^\dagger \bar{\sigma}^\mu D_2^+ - \frac{g}{2c_{\mathrm{W}}} Z_\mu (D_2^0)^\dagger \bar{\sigma}^\mu D_2^0 + \frac{g}{\sqrt{2}}[W_\mu^+ (D_2^+)^\dagger \bar{\sigma}^\mu D_2^0 + W_\mu^- (D_2^0)^\dagger \bar{\sigma}^\mu D_2^+]$$

$$X_i^0=\begin{pmatrix}(\chi_{i\mathrm{L}}^0)_\alpha\\(\chi_{i\mathrm{R}}^0)^{\dagger\dot\alpha}\end{pmatrix}=\begin{pmatrix}\chi_{i\mathrm{L}}^0\\(\chi_{i\mathrm{R}}^0)^\dagger\end{pmatrix},\quad \chi_\mathrm{L}^0=\chi_\mathrm{R}^0=\mathcal{N}^\dagger\psi_\mathrm{L}^0=\mathcal{N}^\dagger\psi_\mathrm{R}^0=\begin{pmatrix}\chi_1^0&\chi_2^0&\chi_3^0\end{pmatrix}^\mathrm{T},\quad \bar{X}_i^0=\begin{pmatrix}\chi_{i\mathrm{R}}^0&(\chi_{i\mathrm{L}}^0)^\dagger\end{pmatrix}$$

$$\Psi_i^0=\begin{pmatrix}\psi_{i\mathrm{L}}^0\\(\psi_{i\mathrm{R}}^0)^\dagger\end{pmatrix},\quad \psi_\mathrm{L}^0=\psi_\mathrm{R}^0=\mathcal{N}\chi_\mathrm{L}^0=\mathcal{N}\chi_\mathrm{R}^0=\begin{pmatrix}S&D_1^0&D_2^0\end{pmatrix}^\mathrm{T},\quad \bar{\Psi}_i^0=\begin{pmatrix}\psi_{i\mathrm{R}}^0&(\psi_{i\mathrm{L}}^0)^\dagger\end{pmatrix}$$

$$X^+=\begin{pmatrix}\chi^+\\(\chi^-)^\dagger\end{pmatrix},\quad \chi^+=D_2^+,\quad \chi^-=D_1^-$$

$$\Psi_{i\mathrm{L}}^0=\begin{pmatrix}\psi_{i\mathrm{L}}^0\\0\end{pmatrix}=\begin{pmatrix}(\mathcal{N}\chi_\mathrm{L}^0)_i\\0\end{pmatrix}=\mathcal{N}_{ij}X_{j\mathrm{L}}^0,\quad \Psi_{i\mathrm{R}}^0=\begin{pmatrix}0\\(\psi_{i\mathrm{R}}^0)^\dagger\end{pmatrix}=\begin{pmatrix}0\\(\mathcal{N}\chi_\mathrm{R}^0)_i^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*X_{j\mathrm{R}}^0$$

$$\bar{\Psi}_{i\mathrm{L}}^0=\begin{pmatrix}0&(\psi_{i\mathrm{L}}^0)^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*\bar{X}_{j\mathrm{L}}^0,\quad \bar{\Psi}_{i\mathrm{R}}^0=\begin{pmatrix}\psi_{i\mathrm{R}}^0&0\end{pmatrix}=\mathcal{N}_{ij}\bar{X}_{j\mathrm{R}}^0$$

$$\bar{\Psi}_{i\mathrm{L}}^0\gamma^\mu\Psi_{i\mathrm{L}}^0=\begin{pmatrix}0&(\psi_{i\mathrm{L}}^0)^\dagger\end{pmatrix}\begin{pmatrix}&\sigma^\mu\\\bar\sigma^\mu&\end{pmatrix}\begin{pmatrix}\psi_{i\mathrm{L}}^0\\0\end{pmatrix}=(\psi_{i\mathrm{L}}^0)^\dagger\bar\sigma^\mu\psi_{i\mathrm{L}}^0$$

$$\bar{\Psi}_{i\mathrm{R}}^0\gamma^\mu\Psi_{i\mathrm{R}}^0=\begin{pmatrix}\psi_{i\mathrm{R}}^0&0\end{pmatrix}\begin{pmatrix}&\sigma^\mu\\\bar\sigma^\mu&\end{pmatrix}\begin{pmatrix}0\\(\psi_{i\mathrm{R}}^0)^\dagger\end{pmatrix}=\psi_{i\mathrm{R}}^0\sigma^\mu(\psi_{i\mathrm{R}}^0)^\dagger=-(\psi_{i\mathrm{R}}^0)^\dagger\bar\sigma^\mu\psi_{i\mathrm{R}}^0=-(\psi_{i\mathrm{L}}^0)^\dagger\bar\sigma^\mu\psi_{i\mathrm{L}}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0}=\frac{1}{2}a_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar{\Psi}_{i\mathrm{L}}^0\gamma^\mu\Psi_{i\mathrm{L}}^0+\frac{1}{2}b_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar{\Psi}_{i\mathrm{R}}^0\gamma^\mu\Psi_{i\mathrm{R}}^0$$

$$=\frac{1}{2}a_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}^*\mathcal{N}_{kj}Z_\mu\bar{X}_{i\mathrm{L}}^0\gamma^\mu X_{j\mathrm{L}}^0+\frac{1}{2}b_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}\mathcal{N}_{kj}^*Z_\mu\bar{X}_{i\mathrm{R}}^0\gamma^\mu X_{j\mathrm{R}}^0=\frac{1}{2}(a_{ZX_i^0X_j^0}Z_\mu\bar{X}_{i\mathrm{L}}^0\gamma^\mu X_{j\mathrm{L}}^0+b_{ZX_i^0X_j^0}Z_\mu\bar{X}_{i\mathrm{R}}^0\gamma^\mu X_{j\mathrm{R}}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0}=b_{Z\Psi_1^0\Psi_1^0}=0,\quad a_{Z\Psi_2^0\Psi_2^0}=-b_{Z\Psi_2^0\Psi_2^0}=\frac{g}{2c_W},\quad a_{Z\Psi_3^0\Psi_3^0}=-b_{Z\Psi_3^0\Psi_3^0}=-\frac{g}{2c_W}$$

$$a_{ZX_i^0X_j^0}=a_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}^*\mathcal{N}_{kj},\quad b_{ZX_i^0X_j^0}=b_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}\mathcal{N}_{kj}^*$$

$$\left(\begin{aligned}
&\text{Note: } a_{ZX_j^0X_i^0}=a^*_{ZX_i^0X_j^0},\quad b_{ZX_j^0X_i^0}=b^*_{ZX_i^0X_j^0},\quad a_{ZX_i^0X_j^0}=-b_{ZX_j^0X_i^0}\\
&C^{-1}=-C,\quad X_i^0=C(\bar{X}_i^0)^\mathrm{T},\quad \bar{X}_i^0=(X_i^0)^\mathrm{T}C\\
&C^{-1}(\gamma^\mu P_L)^\mathrm{T}C=\frac{1}{2}C^{-1}(\gamma^\mu)^\mathrm{T}C-\frac{1}{2}C^{-1}(\gamma^\mu\gamma_5)^\mathrm{T}C=-\frac{1}{2}\gamma^\mu-\frac{1}{2}\gamma^\mu\gamma_5=-\gamma^\mu P_R\\
&a_{ZX_1^0X_2^0}\bar{X}_{1\mathrm{L}}^0\gamma^\mu X_{2\mathrm{L}}^0=a_{ZX_1^0X_2^0}(\bar{X}_1^0\gamma^\mu P_\mathrm{L}X_2^0)^\mathrm{T}=-a_{ZX_1^0X_2^0}(X_2^0)^\mathrm{T}CC^{-1}(\gamma^\mu P_\mathrm{L})^\mathrm{T}C^{-1}C(\bar{X}_1^0)^\mathrm{T}=-a_{ZX_1^0X_2^0}\bar{X}_2^0C^{-1}(\gamma^\mu P_\mathrm{L})^\mathrm{T}C^{-1}X_1^0\\
&=a_{ZX_1^0X_2^0}\bar{X}_2^0C^{-1}(\gamma^\mu P_\mathrm{L})^\mathrm{T}CX_1^0=-a_{ZX_1^0X_2^0}\bar{X}_2^0\gamma^\mu P_\mathrm{R}X_1^0=b_{ZX_2^0X_1^0}\bar{X}_{2\mathrm{R}}^0\gamma^\mu X_{1\mathrm{R}}^0\\
&\frac{1}{2}a_{ZX_1^0X_2^0}Z_\mu\bar{X}_{1\mathrm{L}}^0\gamma^\mu X_{2\mathrm{L}}^0\text{ and }\frac{1}{2}b_{ZX_2^0X_1^0}Z_\mu\bar{X}_{2\mathrm{R}}^0\gamma^\mu X_{1\mathrm{R}}^0\text{ give an identical vertex!}
\end{aligned}\right)$$

$$\bar{X}^+_{\mathrm{L}}\gamma^\mu X^+_{\mathrm{L}} = \begin{pmatrix} 0 & (\chi^+)^\dagger \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \chi^+ \\ 0 \end{pmatrix} = (D_2^+)^\dagger \bar{\sigma}^\mu D_2^+$$

$$\bar{X}^+_{\mathrm{R}}\gamma^\mu X^+_{\mathrm{R}} = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 \\ (\chi^-)^\dagger \end{pmatrix} = \chi^-\sigma^\mu(\chi^-)^\dagger = -(\chi^-)^\dagger\bar{\sigma}^\mu\chi^- = -(D_1^-)^\dagger\bar{\sigma}^\mu D_1^-$$

$$\mathcal{L}_{AX^+X^+} = a_{AX^+X^+} A_\mu \bar{X}^+_{\mathrm{L}}\gamma^\mu X^+_{\mathrm{L}} + b_{AX^+X^+} A_\mu \bar{X}^+_{\mathrm{R}}\gamma^\mu X^+_{\mathrm{R}}$$

$$a_{AX^+X^+} = b_{AX^+X^+} = e$$

$$\mathcal{L}_{ZX^+X^+} = a_{ZX^+X^+} Z_\mu \bar{X}^+_{\mathrm{L}}\gamma^\mu X^+_{\mathrm{L}} + b_{ZX^+X^+} Z_\mu \bar{X}^+_{\mathrm{R}}\gamma^\mu X^+_{\mathrm{R}}$$

$$a_{ZX^+X^+} = b_{ZX^+X^+} = -\frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2)$$

$$\bar{X}^+_{\mathrm{L}}\gamma^\mu \Psi^0_{i\mathrm{L}} = \begin{pmatrix} 0 & (\chi^+)^\dagger \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \psi^0_{i\mathrm{L}} \\ 0 \end{pmatrix} = (\chi^+)^\dagger \bar{\sigma}^\mu \psi^0_{i\mathrm{L}}$$

$$\bar{X}^+_{\mathrm{R}}\gamma^\mu \Psi^0_{i\mathrm{R}} = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 \\ (\psi^0_{i\mathrm{R}})^\dagger \end{pmatrix} = \chi^-\sigma^\mu(\psi^0_{i\mathrm{R}})^\dagger = -(\psi^0_{i\mathrm{R}})^\dagger\bar{\sigma}^\mu\chi^-$$

$$\mathcal{L}_{WX^+\Psi^0_i} = a_{WX^+\Psi^0_i}(W^+_\mu \bar{X}^+_{\mathrm{L}}\gamma^\mu \Psi^0_{i\mathrm{L}} + \mathrm{h.c.}) + b_{WX^+\Psi^0_i}(W^+_\mu \bar{X}^+_{\mathrm{R}}\gamma^\mu \Psi^0_{i\mathrm{R}} + \mathrm{h.c.})$$

$$= a_{WX^+\Psi^0_i}[\mathcal{N}_{ij} W^+_\mu \bar{X}^+_{\mathrm{L}}\gamma^\mu X^0_{j\mathrm{L}} + \mathrm{h.c.}] + b_{WX^+\Psi^0_i}[\mathcal{N}^*_{ij} W^+_\mu \bar{X}^+_{\mathrm{R}}\gamma^\mu X^0_{j\mathrm{R}} + \mathrm{h.c.}]$$

$$= a_{WX^+X^0_i} W^+_\mu \bar{X}^+_{\mathrm{L}}\gamma^\mu X^0_{i\mathrm{L}} + a^*_{WX^+X^0_i} W^-_\mu \bar{X}^0_{i\mathrm{L}}\gamma^\mu X^+_{\mathrm{L}} + b_{WX^+X^0_i} W^+_\mu \bar{X}^+_{\mathrm{R}}\gamma^\mu X^0_{i\mathrm{R}} + b^*_{WX^+X^0_i} W^-_\mu \bar{X}^0_{i\mathrm{R}}\gamma^\mu X^+_{\mathrm{R}}$$

$$b_{WX^+\Psi^0_2} = -\frac{g}{\sqrt{2}}, \quad a_{WX^+\Psi^0_3} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0$$

$$a_{WX^+X^0_i} = a_{WX^+\Psi^0_i}\mathcal{N}_{ij}, \quad b_{WX^+X^0_i} = b_{WX^+\Psi^0_i}\mathcal{N}^*_{ij}$$

$$\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} = \begin{pmatrix} \psi^0_{i\mathrm{R}} & 0 \end{pmatrix} \begin{pmatrix} \psi^0_{j\mathrm{L}} \\ 0 \end{pmatrix} = \psi^0_{i\mathrm{R}}\psi^0_{j\mathrm{L}}, \quad \bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}} = \begin{pmatrix} 0 & (\psi^0_{i\mathrm{L}})^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ (\psi^0_{j\mathrm{R}})^\dagger \end{pmatrix} = (\psi^0_{i\mathrm{L}})^\dagger(\psi^0_{j\mathrm{R}})^\dagger$$

$$\mathcal{L}_{h\Psi^0_i\Psi^0_j} = \frac{1}{2} a_{h\Psi^0_i\Psi^0_j} h\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} + \frac{1}{2} b_{h\Psi^0_i\Psi^0_j} h\bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}}$$

$$= \frac{1}{2} a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj} h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + \frac{1}{2} b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj} h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}} = \frac{1}{2}(a_{hX^0_iX^0_j} h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + b_{hX^0_iX^0_j} h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}})$$

$$a_{h\Psi^0_1\Psi^0_2} = b_{h\Psi^0_1\Psi^0_2} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi^0_2\Psi^0_1} = b_{h\Psi^0_2\Psi^0_1}, \quad a_{h\Psi^0_1\Psi^0_3} = b_{h\Psi^0_1\Psi^0_3} = -\frac{y_2}{\sqrt{2}} = a_{h\Psi^0_3\Psi^0_1} = b_{h\Psi^0_3\Psi^0_1}, \quad \text{others} = 0$$

$$a_{hX^0_iX^0_j} = a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj}, \quad b_{hX^0_iX^0_j} = b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj}$$

$$\left(\begin{array}{l}
\text{Note: } a_{hX^0_jX^0_i} = a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{kj}\mathcal{N}_{li} = a_{h\Psi^0_l\Psi^0_k}\mathcal{N}_{kj}\mathcal{N}_{li} = a_{hX^0_iX^0_j}, \quad b_{hX^0_jX^0_i} = b_{hX^0_iX^0_j}, \quad a_{hX^0_iX^0_j} = b^*_{hX^0_iX^0_j} \\
C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C = \frac{1}{2}C^{-1}(1-\gamma_5)^{\mathrm{T}}C = \frac{1}{2}(1-\gamma_5) = P_{\mathrm{L}} \\
a_{hX^0_1X^0_2}\bar{X}^0_{1\mathrm{R}}X^0_{2\mathrm{L}} = a_{hX^0_1X^0_2}(\bar{X}^0_1 P_{\mathrm{L}} X^0_2)^{\mathrm{T}} = -a_{hX^0_1X^0_2}(X^0_2)^{\mathrm{T}}(P_{\mathrm{L}})^{\mathrm{T}}(\bar{X}^0_1)^{\mathrm{T}} = -a_{hX^0_1X^0_2}(X^0_2)^{\mathrm{T}}CC^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C^{-1}C(\bar{X}^0_1)^{\mathrm{T}} = -a_{hX^0_1X^0_2}\bar{X}^0_2 C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C^{-1}X^0_1 \\
= a_{hX^0_1X^0_2}\bar{X}^0_2 C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}CX^0_1 = a_{hX^0_1X^0_2}\bar{X}^0_2 P_{\mathrm{L}} X^0_1 = a_{hX^0_2X^0_1}\bar{X}^0_{2\mathrm{R}}X^0_{1\mathrm{L}} \\
\frac{1}{2}a_{hX^0_1X^0_2}h\bar{X}^0_{1\mathrm{R}}X^0_{2\mathrm{L}}\left(\frac{1}{2}b_{hX^0_1X^0_2}h\bar{X}^0_{1\mathrm{L}}X^0_{2\mathrm{R}}\right) \text{ and } \frac{1}{2}a_{hX^0_2X^0_1}h\bar{X}^0_{2\mathrm{R}}X^0_{1\mathrm{L}}\left(\frac{1}{2}b_{hX^0_2X^0_1}h\bar{X}^0_{2\mathrm{L}}X^0_{1\mathrm{R}}\right) \text{ give an identical vertex!}
\end{array}\right)$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$a_{h\Psi_i^0\Psi_j^0} = b_{h\Psi_i^0\Psi_j^0}$$

$$\mathcal{L}_{hX_i^0X_j^0} = \frac{1}{2}(a_{hX_i^0X_j^0} h\bar{X}_{i\mathrm{R}}^0 X_{j\mathrm{L}}^0 + b_{hX_i^0X_j^0} h\bar{X}_{i\mathrm{L}}^0 X_{j\mathrm{R}}^0) = \frac{1}{2} h\bar{X}_i^0 (a_{hX_i^0X_j^0} P_\mathrm{L} + b_{hX_i^0X_j^0} P_\mathrm{R}) X_j^0$$

$$= \frac{1}{4}(a_{hX_i^0X_j^0} + b_{hX_i^0X_j^0}) h\bar{X}_i^0 X_j^0 + \frac{1}{4}(b_{hX_i^0X_j^0} - a_{hX_i^0X_j^0}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}\mathcal{N}_{lj} + \mathcal{N}_{ki}^*\mathcal{N}_{lj}^*) h\bar{X}_i^0 X_j^0 + \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}^*\mathcal{N}_{lj}^* - \mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 X_j^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Im}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 i\gamma_5 X_j^0$$

$$\operatorname{Im}(\mathcal{N}_{ki}\mathcal{N}_{li}) = 0$$

$$\mathcal{L}_{hX_1^0X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Im}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 i\gamma_5 X_1^0$$

$$= [a_{h\Psi_1^0\Psi_2^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= -\frac{1}{\sqrt{2}}[y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0X_1^0} = \frac{1}{2}(a_{hX_1^0X_1^0} + b_{hX_1^0X_1^0}) = -\sqrt{2}[y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

Effective operators: $\mathcal{L}_{S,q} = \sum_q G_{S,q} \bar{X}_1^0 X_1^0 \bar{q}q, \quad \mathcal{L}_{S,N} = \sum_{N=p,n} G_{S,N} \bar{X}_1^0 X_1^0 \bar{N}N$

$$G_{S,N} = m_N \left(\sum_{q=u,d,s} \frac{G_{S,q}}{m_q} f_q^N + \sum_{q=c,b,t} \frac{G_{S,q}}{m_q} f_Q^N \right), \quad f_Q^N = \frac{2}{27}\left(1 - \sum_{q=u,d,s} f_q^N\right)$$

hep-ph/0001005:

$$f_u^p = 0.020 \pm 0.004, \quad f_d^p = 0.026 \pm 0.005, \quad f_u^n = 0.014 \pm 0.003, \quad f_d^n = 0.036 \pm 0.008, \quad f_s^p = f_s^n = 0.118 \pm 0.062$$

$$\Rightarrow \quad f_Q^p = 0.0619, \quad f_Q^n = 0.0616$$

$$G_{S,q} = -\frac{g_{hX_1^0X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0X_1^0} m_N}{2vm_h^2} \left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N \right) \quad \Rightarrow \quad G_{S,n} \simeq G_{S,p}$$

$$\sigma_{\chi p}^{\mathrm{SI}} = \frac{4}{\pi} \mu_{\chi p}^2 G_{S,p}^2, \quad \mu_{\chi p} \equiv \frac{m_\chi m_p}{m_\chi + m_p}$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$a_{Z\Psi_k^0\Psi_k^0} = -b_{Z\Psi_k^0\Psi_k^0}$$

$$\mathcal{L}_{ZX_i^0X_j^0} = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{L}}^0 \gamma^\mu X_{j\mathrm{L}}^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{R}}^0 \gamma^\mu X_{j\mathrm{R}}^0) = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_\mathrm{L} X_j^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_\mathrm{R} X_j^0)$$

$$= \frac{1}{4}(a_{ZX_i^0X_j^0} + b_{ZX_i^0X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{ZX_i^0X_j^0} - a_{ZX_i^0X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4}(a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} + b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* - a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki}^* \mathcal{N}_{kj} - \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 - \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki} \mathcal{N}_{kj}^* + \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$\mathcal{L}_{ZX_1^0X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 \, Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0X_1^0} = \frac{1}{2}(b_{ZX_1^0X_1^0} - a_{ZX_1^0X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

Effective operators: $\mathcal{L}_{A,q} = \sum_q G_{A,q} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{q} \gamma^\mu \gamma_5 q, \quad \mathcal{L}_{A,N} = \sum_{N=p,n} G_{A,N} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{N} \gamma^\mu \gamma_5 N$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q} \Delta_q^N$$

hep-ex/0609039:

$$\Delta_u^p = \Delta_d^n = 0.842 \pm 0.012, \quad \Delta_d^p = \Delta_u^n = -0.427 \pm 0.013, \quad \Delta_s^p = \Delta_s^n = -0.085 \pm 0.018$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0X_1^0}}{4 c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$\sigma_{\chi N}^{\mathrm{SD}} = \frac{12}{\pi} \mu_{\chi N}^2 G_{A,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Higgs and Z decay widths

$$\mathcal{L} \supset -\frac{1}{2}\sum_{ij} C^{\mathrm{S}}_{h,ij} h \bar{X}^0_i X^0_j + \frac{1}{2}\sum_{ij} C^{\mathrm{P}}_{h,ij} h \bar{X}^0_i i\gamma_5 X^0_j - \frac{1}{2}\sum_{ij} C^{\mathrm{A}}_{Z,ij} Z_\mu \bar{X}^0_i \gamma^\mu \gamma_5 X^0_j + \frac{1}{2}\sum_{ij} C^{\mathrm{V}}_{Z,ij} Z_\mu \bar{X}^0_i \gamma^\mu X^0_j + \frac{g(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} Z_\mu \bar{X}^+ \gamma^\mu X^+$$

$$C^{\mathrm{S}}_{h,ij} = \sqrt{2}\,\mathrm{Re}(y_1 \mathcal{N}_{1i}\mathcal{N}_{2j} + y_2 \mathcal{N}_{1i}\mathcal{N}_{3j}), \quad C^{\mathrm{P}}_{h,ij} = \sqrt{2}\,\mathrm{Im}(y_1 \mathcal{N}_{1i}\mathcal{N}_{2j} + y_2 \mathcal{N}_{1i}\mathcal{N}_{3j})$$

$$C^{\mathrm{A}}_{Z,ij} = \frac{1}{2} a_{Z\Psi^0_k\Psi^0_k} (\mathcal{N}_{ki}\mathcal{N}^*_{kj} + \mathcal{N}^*_{ki}\mathcal{N}_{kj}) = a_{Z\Psi^0_k\Psi^0_k}\,\mathrm{Re}(\mathcal{N}^*_{ki}\mathcal{N}_{kj}) = \frac{g}{2c_{\mathrm{W}}}\mathrm{Re}(\mathcal{N}^*_{2i}\mathcal{N}_{2j} - \mathcal{N}^*_{3i}\mathcal{N}_{3j})$$

$$C^{\mathrm{V}}_{Z,ij} = \frac{1}{2} a_{Z\Psi^0_k\Psi^0_k} (\mathcal{N}^*_{ki}\mathcal{N}_{kj} - \mathcal{N}_{ki}\mathcal{N}^*_{kj}) = i a_{Z\Psi^0_k\Psi^0_k}\,\mathrm{Im}(\mathcal{N}^*_{ki}\mathcal{N}_{kj}) = \frac{ig}{2c_{\mathrm{W}}}\mathrm{Im}(\mathcal{N}^*_{2i}\mathcal{N}_{2j} - \mathcal{N}^*_{3i}\mathcal{N}_{3j})$$

$$C^{-1} = -C, \quad C\gamma_5 C^{-1} = (\gamma_5)^{\mathrm{T}}$$

$$\bar{X}^0_j X^0_i = (X^0_j)^{\mathrm{T}} CC(\bar{X}^0_i)^{\mathrm{T}} = -(X^0_j)^{\mathrm{T}}(\bar{X}^0_i)^{\mathrm{T}} = \bar{X}^0_i X^0_j$$

$$\bar{X}^0_j i\gamma_5 X^0_i = (X^0_j)^{\mathrm{T}} C i\gamma_5 C(\bar{X}^0_i)^{\mathrm{T}} = -(X^0_j)^{\mathrm{T}} i(\gamma_5)^{\mathrm{T}}(\bar{X}^0_i)^{\mathrm{T}} = \bar{X}^0_i i\gamma_5 X^0_j$$

$$i \neq j: \quad C^{\mathrm{S}}_{h,ji} \neq C^{\mathrm{S}}_{h,ij}, \quad C^{\mathrm{P}}_{h,ji} \neq C^{\mathrm{P}}_{h,ij}$$

$$-\frac{1}{2} C^{\mathrm{S}}_{h,ij} h\bar{X}^0_i X^0_j - \frac{1}{2} C^{\mathrm{S}}_{h,ji} h\bar{X}^0_j X^0_i = -\frac{1}{2}(C^{\mathrm{S}}_{h,ij} + C^{\mathrm{S}}_{h,ji}) h\bar{X}^0_i X^0_j$$

$$\frac{1}{2} C^{\mathrm{P}}_{h,ij} h\bar{X}^0_i i\gamma_5 X^0_j + \frac{1}{2} C^{\mathrm{P}}_{h,ji} h\bar{X}^0_j i\gamma_5 X^0_i = \frac{1}{2}(C^{\mathrm{P}}_{h,ij} + C^{\mathrm{P}}_{h,ji}) h\bar{X}^0_i i\gamma_5 X^0_j$$

$$C^{\mathrm{A}}_{Z,ji} = C^{\mathrm{A}}_{Z,ij}, \quad C^{\mathrm{V}}_{Z,ji} = -C^{\mathrm{V}}_{Z,ij}$$

$$C\gamma^\mu\gamma_5 C^{-1} = (\gamma^\mu\gamma_5)^{\mathrm{T}}, \quad C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}$$

$$\bar{X}^0_j \gamma^\mu\gamma_5 X^0_i = (X^0_j)^{\mathrm{T}} C\gamma^\mu\gamma_5 C(\bar{X}^0_i)^{\mathrm{T}} = -(X^0_j)^{\mathrm{T}}(\gamma^\mu\gamma_5)^{\mathrm{T}}(\bar{X}^0_i)^{\mathrm{T}} = \bar{X}^0_i\gamma^\mu\gamma_5 X^0_j \quad \Rightarrow \quad C^{\mathrm{A}}_{Z,ji}\bar{X}^0_j\gamma^\mu\gamma_5 X^0_i = C^{\mathrm{A}}_{Z,ij}\bar{X}^0_i\gamma^\mu\gamma_5 X^0_j$$

$$\bar{X}^0_j \gamma^\mu X^0_i = (X^0_j)^{\mathrm{T}} C\gamma^\mu C(\bar{X}^0_i)^{\mathrm{T}} = (X^0_j)^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}}(\bar{X}^0_i)^{\mathrm{T}} = -\bar{X}^0_i\gamma^\mu X^0_j \quad \Rightarrow \quad C^{\mathrm{V}}_{Z,ji}\bar{X}^0_j\gamma^\mu X^0_i = C^{\mathrm{V}}_{Z,ij}\bar{X}^0_i\gamma^\mu X^0_j$$

$-\frac{1}{2} C^{\mathrm{A}}_{Z,12} Z_\mu \bar{X}^0_1\gamma^\mu\gamma_5 X^0_2 \left(\frac{1}{2} C^{\mathrm{V}}_{Z,12} Z_\mu \bar{X}^0_1 \gamma^\mu X^0_2\right)$ and $-\frac{1}{2} C^{\mathrm{A}}_{Z,21} Z_\mu \bar{X}^0_2\gamma^\mu\gamma_5 X^0_1 \left(\frac{1}{2} C^{\mathrm{V}}_{Z,21} Z_\mu \bar{X}^0_2 \gamma^\mu X^0_1\right)$ give an identical vertex!

Another way using symmetric Higgs couplings:

$$\mathcal{L} \supset -\frac{1}{2}\sum_{ij} c^{\mathrm{S}}_{h,ij} h\bar{X}^0_i X^0_j + \frac{1}{2}\sum_{ij} c^{\mathrm{P}}_{h,ij} h\bar{X}^0_i i\gamma_5 X^0_j$$

$$c^{\mathrm{S}}_{h,ij} = -a_{h\Psi^0_k\Psi^0_l}\,\mathrm{Re}(\mathcal{N}_{ki}\mathcal{N}_{lj}) = \frac{y_1}{\sqrt{2}}\mathrm{Re}(\mathcal{N}_{1i}\mathcal{N}_{2j}) + \frac{y_1}{\sqrt{2}}\mathrm{Re}(\mathcal{N}_{2i}\mathcal{N}_{1j}) + \frac{y_2}{\sqrt{2}}\mathrm{Re}(\mathcal{N}_{1i}\mathcal{N}_{3j}) + \frac{y_2}{\sqrt{2}}\mathrm{Re}(\mathcal{N}_{3i}\mathcal{N}_{1j})$$

$$= \frac{1}{\sqrt{2}}\mathrm{Re}[y_1(\mathcal{N}_{1i}\mathcal{N}_{2j} + \mathcal{N}_{2i}\mathcal{N}_{1j}) + y_2(\mathcal{N}_{1i}\mathcal{N}_{3j} + \mathcal{N}_{3i}\mathcal{N}_{1j})]$$

$$c^{\mathrm{P}}_{h,ij} = -a_{h\Psi^0_k\Psi^0_l}\,\mathrm{Im}(\mathcal{N}_{ki}\mathcal{N}_{lj}) = \frac{1}{\sqrt{2}}\mathrm{Im}[y_1(\mathcal{N}_{1i}\mathcal{N}_{2j} + \mathcal{N}_{2i}\mathcal{N}_{1j}) + y_2(\mathcal{N}_{1i}\mathcal{N}_{3j} + \mathcal{N}_{3i}\mathcal{N}_{1j})]$$

$$c^{\mathrm{S}}_{h,ij} = \frac{1}{2}(C^{\mathrm{S}}_{h,ij} + C^{\mathrm{S}}_{h,ji}), \quad c^{\mathrm{P}}_{h,ij} = \frac{1}{2}(C^{\mathrm{P}}_{h,ij} + C^{\mathrm{P}}_{h,ji})$$

$$c^{\mathrm{S}}_{h,ji} = c^{\mathrm{S}}_{h,ij}, \quad c^{\mathrm{P}}_{h,ji} = c^{\mathrm{P}}_{h,ij} \quad \Rightarrow \quad c^{\mathrm{S}}_{h,ji}\bar{X}^0_j X^0_i = c^{\mathrm{S}}_{h,ij}\bar{X}^0_i X^0_j, \quad c^{\mathrm{P}}_{h,ji}\bar{X}^0_j i\gamma_5 X^0_i = c^{\mathrm{P}}_{h,ij}\bar{X}^0_i i\gamma_5 X^0_j$$

$-\frac{1}{2} c^{\mathrm{S}}_{h,12} h\bar{X}^0_1 X^0_2 \left(\frac{1}{2} c^{\mathrm{P}}_{h,12} h\bar{X}^0_1 i\gamma_5 X^0_2\right)$ and $-\frac{1}{2} c^{\mathrm{S}}_{h,21} h\bar{X}^0_2 X^0_1 \left(\frac{1}{2} c^{\mathrm{P}}_{h,21} h\bar{X}^0_2 i\gamma_5 X^0_1\right)$ give an identical vertex!

(1) $h(p) \to \chi_i^0(k_1) + \chi_j^0(k_2), \quad i \neq j$

$$m_h^2 = p^2 = (k_1+k_2)^2 = m_{\chi_i^0}^2 + m_{\chi_j^0}^2 + 2k_1\cdot k_2, \quad k_1\cdot k_2 = \frac{1}{2}(m_h^2 - m_{\chi_i^0}^2 - m_{\chi_j^0}^2)$$

$$k_1\cdot k_2 - m_{\chi_i^0} m_{\chi_j^0} = \frac{1}{2}[m_h^2 - (m_{\chi_i^0}+m_{\chi_j^0})^2], \quad k_1\cdot k_2 + m_{\chi_i^0} m_{\chi_j^0} = \frac{1}{2}[m_h^2 - (m_{\chi_i^0}-m_{\chi_j^0})^2]$$

$$|\mathbf{k}_1| = \frac{1}{2m_h}\sqrt{[m_h^2-(m_{\chi_i^0}+m_{\chi_j^0})^2][m_h^2-(m_{\chi_i^0}-m_{\chi_j^0})^2]} = \frac{1}{2m_h}\sqrt{m_h^4 + m_{\chi_i^0}^4 + m_{\chi_j^0}^4 - 2m_h^2 m_{\chi_i^0}^2 - 2m_h^2 m_{\chi_j^0}^2 - 2m_{\chi_i^0}^2 m_{\chi_j^0}^2} = \frac{F(m_h^2, m_{\chi_i^0}^2, m_{\chi_j^0}^2)}{2m_h}$$

$$F(x,y,z) \equiv \sqrt{x^2+y^2+z^2-2xy-2xz-2yz}$$

$$i\mathcal{M} = -\frac{i}{2}(C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}})\bar{u}(k_1)v(k_2) + \frac{i}{2}(C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}})\bar{u}(k_1)i\gamma_5 v(k_2) = -\frac{i}{2}\bar{u}(k_1)[(C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}) - i(C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}})\gamma_5]v(k_2)$$

$$(i\mathcal{M})^* = +\frac{i}{2}\bar{v}(k_2)[(C_{h,ij}^{\mathrm{S}*} + C_{h,ji}^{\mathrm{S}*}) - i(C_{h,ij}^{\mathrm{P}*} + C_{h,ji}^{\mathrm{P}*})\gamma_5]u(k_1)$$

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}}\bar{u}(k_1)[(C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}) - i(C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}})\gamma_5]v(k_2)\bar{v}(k_2)[(C_{h,ij}^{\mathrm{S}*} + C_{h,ji}^{\mathrm{S}*}) - i(C_{h,ij}^{\mathrm{P}*} + C_{h,ji}^{\mathrm{P}*})\gamma_5]u(k_1)$$

$$= \frac{1}{4}\mathrm{Tr}\{(\not{k}_1 + m_{\chi_i^0})[(C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}) - i(C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}})\gamma_5](\not{k}_2 - m_{\chi_j^0})[(C_{h,ij}^{\mathrm{S}*} + C_{h,ji}^{\mathrm{S}*}) - i(C_{h,ij}^{\mathrm{P}*} + C_{h,ji}^{\mathrm{P}*})\gamma_5]\}$$

$$= [|C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}|^2 (k_1\cdot k_2 - m_{\chi_i^0}m_{\chi_j^0}) + |C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}}|^2 (k_1\cdot k_2 + m_{\chi_i^0}m_{\chi_j^0})]$$

$$= \frac{1}{2}\{|C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}|^2 [m_h^2 - (m_{\chi_i^0}+m_{\chi_j^0})^2] + |C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}}|^2 [m_h^2 - (m_{\chi_i^0}-m_{\chi_j^0})^2]\}$$

$$\left[\begin{aligned}
& i\mathcal{M} = -ic_{h,ij}^{\mathrm{S}}\bar{u}(k_1)v(k_2) + ic_{h,ij}^{\mathrm{P}}\bar{u}(k_1)i\gamma_5 v(k_2) = -i\bar{u}(k_1)(c_{h,ij}^{\mathrm{S}} - ic_{h,ij}^{\mathrm{P}}\gamma_5)v(k_2) \\
& (i\mathcal{M})^* = +i\bar{v}(k_2)(c_{h,ij}^{\mathrm{S}*} - ic_{h,ij}^{\mathrm{P}*}\gamma_5)u(k_1) \\
& \sum_{\text{spins}}|\mathcal{M}|^2 = \sum_{\text{spins}}\bar{u}(k_1)(c_{h,ij}^{\mathrm{S}} - ic_{h,ij}^{\mathrm{P}}\gamma_5)v(k_2)\bar{v}(k_2)(c_{h,ij}^{\mathrm{S}*} - ic_{h,ij}^{\mathrm{P}*}\gamma_5)u(k_1) \\
& = \mathrm{Tr}[(\not{k}_1 + m_{\chi_i^0})(c_{h,ij}^{\mathrm{S}} - ic_{h,ij}^{\mathrm{P}}\gamma_5)(\not{k}_2 - m_{\chi_j^0})(C_{h,ij}^{\mathrm{S}*} - ic_{h,ij}^{\mathrm{P}*}\gamma_5)] \\
& = 4[|c_{h,ij}^{\mathrm{S}}|^2 (k_1\cdot k_2 - m_{\chi_i^0}m_{\chi_j^0}) + |c_{h,ij}^{\mathrm{P}}|^2 (k_1\cdot k_2 + m_{\chi_i^0}m_{\chi_j^0})] \\
& = 2\{|c_{h,ij}^{\mathrm{S}}|^2 [m_h^2 - (m_{\chi_i^0}+m_{\chi_j^0})^2] + |c_{h,ij}^{\mathrm{P}}|^2 [m_h^2 - (m_{\chi_i^0}-m_{\chi_j^0})^2]\}
\end{aligned}\right]$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_h^2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{F(m_h^2, m_{\chi_i^0}^2, m_{\chi_j^0}^2)}{32\pi m_h^3}\{|C_{h,ij}^{\mathrm{S}} + C_{h,ji}^{\mathrm{S}}|^2 [m_h^2 - (m_{\chi_i^0}+m_{\chi_j^0})^2] + |C_{h,ij}^{\mathrm{P}} + C_{h,ji}^{\mathrm{P}}|^2 [m_h^2 - (m_{\chi_i^0}-m_{\chi_j^0})^2]\}$$

(2) $h(p) \to \chi_i^0(k_1) + \chi_i^0(k_2)$

$$k_1\cdot k_2 = \frac{1}{2}(m_h^2 - 2m_{\chi_i^0}^2), \quad |\mathbf{k}_1| = \frac{1}{2}\sqrt{m_h^2 - 4m_{\chi_i^0}^2}$$

Real y_1 and y_2 $\quad\Rightarrow\quad C_{h,ii}^{\mathrm{P}} = 0$

$$i\mathcal{M} = -iC_{h,ii}^{\mathrm{S}}\bar{u}(k_1)v(k_2), \quad (i\mathcal{M})^* = +iC_{h,ii}^{\mathrm{S}*}\bar{v}(k_2)u(k_1)$$

$$\sum_{\text{spins}}|\mathcal{M}|^2 = \sum_{\text{spins}}|C_{h,ii}^{\mathrm{S}}|^2\bar{u}(k_1)v(k_2)\bar{v}(k_2)u(k_1) = |C_{h,ii}^{\mathrm{S}}|^2\ \mathrm{Tr}[(\not{k}_1 + m_{\chi_i^0})(\not{k}_2 - m_{\chi_i^0})]$$

$$= 4|C_{h,ii}^{\mathrm{S}}|^2 (k_1\cdot k_2 - m_{\chi_i^0}^2) = 2|C_{h,ii}^{\mathrm{S}}|^2 (m_h^2 - 4m_{\chi_i^0}^2)$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_h^2}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{|C_{h,ii}^{\mathrm{S}}|^2}{16\pi m_h^2}(m_h^2 - 4m_{\chi_i^0}^2)^{3/2}$$

(3) $Z(p) \to \chi_i^0(k_1) + \chi_j^0(k_2), \quad i \neq j$

$$|\mathbf{k}_1| = \frac{F(m_Z^2, m_{\chi_i^0}^2, m_{\chi_j^0}^2)}{2m_Z}, \quad k_1^0 = \frac{1}{2m_Z}(m_Z^2 + m_{\chi_i^0}^2 - m_{\chi_j^0}^2), \quad k_2^0 = \frac{1}{2m_Z}(m_Z^2 + m_{\chi_j^0}^2 - m_{\chi_i^0}^2)$$

$$k_1 \cdot k_2 = \frac{1}{2}(m_Z^2 - m_{\chi_i^0}^2 - m_{\chi_j^0}^2), \quad p \cdot k_1 = m_Z k_1^0 = \frac{1}{2}(m_Z^2 + m_{\chi_i^0}^2 - m_{\chi_j^0}^2), \quad p \cdot k_2 = m_Z k_2^0 = \frac{1}{2}(m_Z^2 + m_{\chi_j^0}^2 - m_{\chi_i^0}^2)$$

$$i\mathcal{M} = -i\varepsilon_\mu(p)\overline{u}(k_1)(C_{Z,ij}^{\mathrm{A}}\gamma^\mu\gamma_5 - C_{Z,ij}^{\mathrm{V}}\gamma^\mu)v(k_2), \quad (i\mathcal{M})^* = +i\varepsilon_\nu^*(p)\overline{v}(k_2)(C_{Z,ij}^{\mathrm{A}*}\gamma^\nu\gamma_5 - C_{Z,ij}^{\mathrm{V}*}\gamma^\nu)u(k_1)$$

$$\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{3}\sum_{\text{spins}}\varepsilon_\mu(p)\varepsilon_\nu^*(p)\overline{u}(k_1)(C_{Z,ij}^{\mathrm{A}}\gamma^\mu\gamma_5 - C_{Z,ij}^{\mathrm{V}}\gamma^\mu)v(k_2)\overline{v}(k_2)(C_{Z,ij}^{\mathrm{A}*}\gamma^\nu\gamma_5 - C_{Z,ij}^{\mathrm{V}*}\gamma^\nu)u(k_1)$$

$$= \frac{1}{3}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\right)\mathrm{Tr}[(\not{k}_1 + m_{\chi_i^0})(C_{Z,ij}^{\mathrm{A}}\gamma^\mu\gamma_5 - C_{Z,ij}^{\mathrm{V}}\gamma^\mu)(\not{k}_2 - m_{\chi_j^0})(C_{Z,ij}^{\mathrm{A}*}\gamma^\nu\gamma_5 - C_{Z,ij}^{\mathrm{V}*}\gamma^\nu)]$$

$$= \frac{2}{3m_Z^2}\{(|C_{Z,ij}^{\mathrm{A}}|^2 + |C_{Z,ij}^{\mathrm{V}}|^2)[m_Z^2(2m_Z^2 - m_{\chi_i^0}^2 - m_{\chi_j^0}^2) - (m_{\chi_i^0}^2 - m_{\chi_j^0}^2)^2] + 6(|C_{Z,ij}^{\mathrm{V}}|^2 - |C_{Z,ij}^{\mathrm{A}}|^2)m_Z^2 m_{\chi_i^0} m_{\chi_j^0})\}$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_Z^2}\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{F(m_Z^2, m_{\chi_i^0}^2, m_{\chi_j^0}^2)}{24\pi m_Z^5}\{(|C_{Z,ij}^{\mathrm{A}}|^2 + |C_{Z,ij}^{\mathrm{V}}|^2)[m_Z^2(2m_Z^2 - m_{\chi_i^0}^2 - m_{\chi_j^0}^2) - (m_{\chi_i^0}^2 - m_{\chi_j^0}^2)^2] + 6(|C_{Z,ij}^{\mathrm{V}}|^2 - |C_{Z,ij}^{\mathrm{A}}|^2)m_Z^2 m_{\chi_i^0} m_{\chi_j^0})\}$$

(4) $Z(p) \to \chi_i^0(k_1) + \chi_i^0(k_2)$

$$C_{Z,ii}^{\mathrm{V}} = 0$$

$$|\mathbf{k}_1| = \frac{1}{2}\sqrt{m_Z^2 - 4m_{\chi_i^0}^2}, \quad k_1^0 = k_2^0 = \frac{m_Z}{2}, \quad k_1 \cdot k_2 = \frac{1}{2}(m_Z^2 - 2m_{\chi_i^0}^2), \quad p \cdot k_1 = \frac{m_Z^2}{2}, \quad p \cdot k_2 = \frac{m_Z^2}{2}$$

$$i\mathcal{M} = -iC_{Z,ii}^{\mathrm{A}}\varepsilon_\mu(p)\overline{u}(k_1)\gamma^\mu\gamma_5 v(k_2), \quad (i\mathcal{M})^* = +iC_{Z,ii}^{\mathrm{A}*}\varepsilon_\nu^*(p)\overline{v}(k_2)\gamma^\nu\gamma_5 u(k_1)$$

$$\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{3}\sum_{\text{spins}}|C_{Z,ii}^{\mathrm{A}}|^2\ \varepsilon_\mu(p)\varepsilon_\nu^*(p)\overline{u}(k_1)\gamma^\mu\gamma_5 v(k_2)\overline{v}(k_2)\gamma^\nu\gamma_5 u(k_1)$$

$$= \frac{|C_{Z,ii}^{\mathrm{A}}|^2}{3}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\right)\mathrm{Tr}[(\not{k}_1 + m_{\chi_i^0})\gamma^\mu\gamma_5(\not{k}_2 - m_{\chi_j^0})\gamma^\nu\gamma_5] = \frac{4}{3}|C_{Z,ii}^{\mathrm{A}}|^2\ (m_Z^2 - 4m_{\chi_i^0}^2)$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_Z^2}\frac{1}{2}\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{|C_{Z,ii}^{\mathrm{A}}|^2}{24\pi m_Z^2}(m_Z^2 - 4m_{\chi_i^0}^2)^{3/2}$$

(5) $Z(p) \to \chi^+(k_1) + \chi^-(k_2)$

$$i\mathcal{M} = i\frac{g(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}}\varepsilon_\mu(p)\overline{u}(k_1)\gamma^\mu v(k_2), \quad (i\mathcal{M})^* = -i\frac{g(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}}\varepsilon_\nu^*(p)\overline{v}(k_2)\gamma^\nu u(k_1)$$

$$\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{3}\sum_{\text{spins}}\frac{g^2(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2}{4c_{\mathrm{W}}^2}\varepsilon_\mu(p)\varepsilon_\nu^*(p)\overline{u}(k_1)\gamma^\mu v(k_2)\overline{v}(k_2)\gamma^\nu u(k_1)$$

$$= \frac{g^2(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2}{12c_{\mathrm{W}}^2}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\right)\mathrm{Tr}[(\not{k}_1 + m_{\chi^\pm})\gamma^\mu(\not{k}_2 - m_{\chi^\pm})\gamma^\nu] = \frac{g^2(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2}{3c_{\mathrm{W}}^2}(m_Z^2 + 2m_{\chi^\pm}^2)$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_Z^2}\frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{g^2(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2}{48\pi m_Z^2 c_{\mathrm{W}}^2}\sqrt{m_Z^2 - 4m_{\chi^\pm}^2}(m_Z^2 + 2m_{\chi^\pm}^2)$$

Doublet-Triplet Fermionic Dark Matter (DTFDM)

Ref: Dedes & Karamitros, 1403.7744

Left-handed Weyl fermions:

$$D_1=\begin{pmatrix}D_1^0\\ D_1^-\end{pmatrix}\in\left(\mathbf{2},-\frac{1}{2}\right),\quad D_2=\begin{pmatrix}D_2^+\\ D_2^0\end{pmatrix}\in\left(\mathbf{2},\frac{1}{2}\right),\quad T=\begin{pmatrix}T^+\\ T^0\\ T^-\end{pmatrix}\in(\mathbf{3},0)$$

$$\mathcal{L}_{\mathrm{D}}=iD_1^\dagger\bar\sigma^\mu D_\mu D_1+iD_2^\dagger\bar\sigma^\mu D_\mu D_2+(m_D\varepsilon_{ij}D_1^iD_2^j+\mathrm{h.c.}),\quad \mathcal{L}_{\mathrm{T}}=iT^\dagger\bar\sigma^\mu D_\mu T-\frac{1}{2}(m_TT^aT^a+\mathrm{h.c.})$$

$$\mathcal{L}_{\mathrm{HDT}}=y_1H_iT^a(\sigma^a)^i_{\ j}D_1^j-y_2H_i^\dagger T^a(\sigma^a)^i_{\ j}D_2^j+\mathrm{h.c.},\quad i,j=1,2,\quad a=1,2,3$$

$$m_D\varepsilon_{ij}D_1^iD_2^j=-m_DD_1^0D_2^0+m_DD_1^-D_2^+$$

$$T^\pm=-\frac{1}{\sqrt2}(T^1\mp iT^2),\quad T^0=T^3,\quad T^1=-\frac{1}{\sqrt2}(T^++T^-),\quad T^2=-\frac{i}{\sqrt2}(T^+-T^-)$$

$$-\frac{1}{2}m_TT^aT^a=-\frac{1}{2}m_T\left[\frac{1}{2}(T^++T^-)^2-\frac{1}{2}(T^+-T^-)^2+T^0T^0\right]=-m_TT^-T^+-\frac{1}{2}m_TT^0T^0$$

$$T^a\sigma^a=\begin{pmatrix}T^3 & T^1-iT^2\\ T^1+iT^2 & -T^3\end{pmatrix}=\begin{pmatrix}T^0 & -\sqrt2T^+\\ -\sqrt2T^- & -T^0\end{pmatrix}$$

$$y_1H_iT^a(\sigma^a)^i_{\ j}D_1^j=y_1\begin{pmatrix}-H^0 & H^+\end{pmatrix}\begin{pmatrix}T^0 & -\sqrt2T^+\\ -\sqrt2T^- & -T^0\end{pmatrix}\begin{pmatrix}D_1^0\\ D_1^-\end{pmatrix}=y_1\begin{pmatrix}-H^0 & H^+\end{pmatrix}\begin{pmatrix}T^0D_1^0-\sqrt2T^+D_1^-\\ -\sqrt2T^-D_1^0-T^0D_1^-\end{pmatrix}$$

$$=y_1(-H^0T^0D_1^0+\sqrt2H^0T^+D_1^--\sqrt2H^+T^-D_1^0-H^+T^0D_1^-)\to-\frac{1}{\sqrt2}y_1(v+h)T^0D_1^0+y_1(v+h)T^+D_1^-$$

$$-y_2H_i^\dagger T^a(\sigma^a)^i_{\ j}D_2^j=-y_2\begin{pmatrix}H^- & H^{0*}\end{pmatrix}\begin{pmatrix}T^0 & -\sqrt2T^+\\ -\sqrt2T^- & -T^0\end{pmatrix}\begin{pmatrix}D_2^+\\ D_2^0\end{pmatrix}=-y_2\begin{pmatrix}H^- & H^{0*}\end{pmatrix}\begin{pmatrix}T^0D_2^+-\sqrt2T^+D_2^0\\ -\sqrt2T^-D_2^+-T^0D_2^0\end{pmatrix}$$

$$=-y_2(H^-T^0D_2^+-\sqrt2H^-T^+D_2^0-\sqrt2H^{0*}T^-D_2^+-H^{0*}T^0D_2^0)\to y_2(v+h)T^-D_2^++\frac{1}{\sqrt2}y_2(v+h)T^0D_2^0$$

Note: $T^i_{\ j}=u^iv_j-\frac{1}{2}\delta^i_ju^kv_k=\frac{1}{\sqrt2}T^a(\sigma^a)^i_{\ j}$

$$T^+=-T^1_{\ 2},\quad T^-=-T^2_{\ 1},\quad T^0=\sqrt2T^1_{\ 1}=-\sqrt2T^2_{\ 2}\quad\Rightarrow\quad -\frac{1}{2}m_TT^j_{\ i}T^i_{\ j}=-m_TT^-T^+-\frac{1}{2}m_TT^0T^0$$

$$\sqrt2y_1H_iT^i_{\ j}D_1^j=\sqrt2y_1(H_1T^1_{\ 1}D_1^1+H_1T^1_{\ 2}D_1^2+H_2T^2_{\ 1}D_1^1+H_2T^2_{\ 2}D_1^2)$$

$$=\sqrt2y_1\left(-\frac{1}{\sqrt2}H^0T^0D_1^0+H^0T^+D_1^--H^+T^-D_1^0-\frac{1}{\sqrt2}H^+T^0D_1^-\right)$$

$$=y_1(-H^0T^0D_1^0+\sqrt2H^0T^+D_1^--\sqrt2H^+T^-D_1^0-H^+T^0D_1^-)$$

$$-\sqrt2y_2H_i^\dagger T^i_{\ j}D_2^j=-\sqrt2y_2(H_1^\dagger T^1_{\ 1}D_2^1+H_1^\dagger T^1_{\ 2}D_2^2+H_2^\dagger T^2_{\ 1}D_2^1+H_2^\dagger T^2_{\ 2}D_2^2)$$

$$=-\sqrt2y_2\left(\frac{1}{\sqrt2}H^-T^0D_2^+-H^-T^+D_2^0-H^{0*}T^-D_2^+-\frac{1}{\sqrt2}H^{0*}T^0D_2^0\right)$$

$$=-y_2(H^-T^0D_2^+-\sqrt2H^-T^+D_2^0-\sqrt2H^{0*}T^-D_2^+-H^{0*}T^0D_2^0)$$

$$\mathcal{L}_{\mathrm{mass}}=-\frac{1}{2}\begin{pmatrix}T^0 & D_1^0 & D_2^0\end{pmatrix}\mathcal{M}_{\mathrm{N}}\begin{pmatrix}T^0\\ D_1^0\\ D_2^0\end{pmatrix}-\begin{pmatrix}T^- & D_1^-\end{pmatrix}\mathcal{M}_{\mathrm{C}}\begin{pmatrix}T^+\\ D_2^+\end{pmatrix}+\mathrm{h.c.}=-\frac{1}{2}\sum_{i=1}^3m_{\chi_i^0}\chi_i^0\chi_i^0-\sum_{i=1}^2m_{\chi_i^\pm}\chi_i^-\chi_i^++\mathrm{h.c.}$$

$$\mathcal{M}_{\mathrm{N}}=\begin{pmatrix}m_T & \frac{1}{\sqrt2}y_1v & -\frac{1}{\sqrt2}y_2v\\ \frac{1}{\sqrt2}y_1v & 0 & m_D\\ -\frac{1}{\sqrt2}y_2v & m_D & 0\end{pmatrix},\quad \mathcal{M}_{\mathrm{C}}=\begin{pmatrix}m_T & -y_2v\\ -y_1v & -m_D\end{pmatrix}$$

$$\mathcal{N}^{\mathrm{T}}\mathcal{M}_{\mathrm{N}}\mathcal{N}=\mathrm{diag}(m_{\chi_1^0},m_{\chi_2^0},m_{\chi_3^0}),\quad \mathcal{C}_{\mathrm{R}}^{\mathrm{T}}\mathcal{M}_{\mathrm{C}}\mathcal{C}_{\mathrm{L}}=\mathrm{diag}(m_{\chi_1^\pm},m_{\chi_2^\pm}),\quad \mathcal{N}^{-1}=\mathcal{N}^\dagger,\quad \mathcal{C}_{\mathrm{L}}^{-1}=\mathcal{C}_{\mathrm{L}}^\dagger,\quad \mathcal{C}_{\mathrm{R}}^{-1}=\mathcal{C}_{\mathrm{R}}^\dagger$$

$$\mathcal{C}_{\mathrm{L}}^\dagger\mathcal{M}_{\mathrm{C}}^\dagger\mathcal{M}_{\mathrm{C}}\mathcal{C}_{\mathrm{L}}=(\mathcal{C}_{\mathrm{L}}^\dagger\mathcal{M}_{\mathrm{C}}^\dagger\mathcal{C}_{\mathrm{R}}^*)(\mathcal{C}_{\mathrm{R}}^{\mathrm{T}}\mathcal{M}_{\mathrm{C}}\mathcal{C}_{\mathrm{L}})=\mathrm{diag}(m^2_{\chi_1^\pm},m^2_{\chi_2^\pm}),\quad \mathcal{C}_{\mathrm{R}}^{\mathrm{T}}\mathcal{M}_{\mathrm{C}}\mathcal{M}_{\mathrm{C}}^\dagger\mathcal{C}_{\mathrm{R}}^*=(\mathcal{C}_{\mathrm{R}}^{\mathrm{T}}\mathcal{M}_{\mathrm{C}}\mathcal{C}_{\mathrm{L}})(\mathcal{C}_{\mathrm{L}}^\dagger\mathcal{M}_{\mathrm{C}}^\dagger\mathcal{C}_{\mathrm{R}}^*)=\mathrm{diag}(m^2_{\chi_1^\pm},m^2_{\chi_2^\pm}$$

$$\begin{pmatrix}T^0\\ D_1^0\\ D_2^0\end{pmatrix}=\mathcal{N}\begin{pmatrix}\chi_1^0\\ \chi_2^0\\ \chi_3^0\end{pmatrix},\quad \begin{pmatrix}T^+\\ D_2^+\end{pmatrix}=\mathcal{C}_{\mathrm{L}}\begin{pmatrix}\chi_1^+\\ \chi_2^+\end{pmatrix},\quad \begin{pmatrix}T^-\\ D_1^-\end{pmatrix}=\mathcal{C}_{\mathrm{R}}\begin{pmatrix}\chi_1^-\\ \chi_2^-\end{pmatrix}$$

$y = y_1 = y_2 \quad \Rightarrow \quad \underline{\text{Custodial SU(2)}_\text{R} \text{ global symmetry}}$

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an SU(2)}_\text{R} \text{ indice}$$

$$\mathcal{L}_\text{HDT} = y[H_i T^a (\sigma^a)^i_j D_1^j - H_i^\dagger T^a (\sigma^a)^i_j D_2^j] + \text{h.c.} = y\varepsilon_{AB} (\mathcal{H}^A)_i T^a (\sigma^a)^i_j (\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_\text{D} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A + \frac{1}{2}[m_D \varepsilon_{AB} \varepsilon_{ij} (\mathcal{D}^A)^i (\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_T \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_D \\ m_{\chi_2^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T\right] \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T\right] \end{cases}$$

$$m_D > m_T \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T\right] \\ m_{\chi_2^0} = m_D \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T\right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_T)}$$

<u>Gauge interactions</u>

$$D_\mu T = (\partial_\mu - igW_\mu^a t_\text{T}^a)T$$

$$t_\text{T}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix}, \quad t_\text{T}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & i \\ & -i & \end{pmatrix}, \quad t_\text{T}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$gW_\mu^a t_\text{T}^a = \begin{pmatrix} gW_\mu^3 & g(W_\mu^1 - iW_\mu^2)/\sqrt{2} & 0 \\ g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & 0 & -g(W_\mu^1 - iW_\mu^2)/\sqrt{2} \\ 0 & -g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & -gW_\mu^3 \end{pmatrix} = \begin{pmatrix} eA_\mu + gc_\text{W} Z_\mu & gW_\mu^+ & 0 \\ gW_\mu^- & 0 & -gW_\mu^+ \\ 0 & -gW_\mu^- & -eA_\mu - gc_\text{W} Z_\mu \end{pmatrix}$$

$$\begin{aligned}
\mathcal{L}_\text{T} &\supset T^\dagger \bar{\sigma}^\mu g W_\mu^a t_\text{T}^a T \\
&= (eA_\mu + gc_\text{W} Z_\mu)(T^+)^\dagger \bar{\sigma}^\mu T^+ + gW_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\
&\quad + gW_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - gW_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\
&\quad - gW_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - (eA_\mu + gc_\text{W} Z_\mu)(T^-)^\dagger \bar{\sigma}^\mu T^-
\end{aligned}$$

$$\text{Note: } W_\mu \equiv W_\mu^a \frac{\sigma^a}{2} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}$$

$$(W_\mu)^i_j = \frac{1}{2}W^a (\sigma^a)^i{}_j, \quad W_\mu^+ = \sqrt{2}(W_\mu)^1_2, \quad W_\mu^- = \sqrt{2}(W_\mu)^2_1, \quad W_\mu^0 = W_\mu^3 = 2(W_\mu)^1_1 = -2(W_\mu)^2_2$$

$$\text{tr}(W_\mu W^\mu) = \frac{1}{4}W_\mu^a W^{b\mu} \text{tr}(\sigma^a \sigma^b) = \frac{1}{2}W_\mu^a W^{a\mu} = (W_\mu)^i_j (W^\mu)^j_i = W_\mu^+ W^{-\mu} + \frac{1}{2}W_\mu^0 W^{0\mu}$$

$$\begin{aligned}
&(T^\dagger)^i_j \bar{\sigma}^\mu (W_\mu)^j_k T^k_i \\
&= (T^\dagger)^1_1 \bar{\sigma}^\mu (W_\mu)^1_1 T^1_1 + (T^\dagger)^1_1 \bar{\sigma}^\mu (W_\mu)^1_2 T^2_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu (W_\mu)^2_1 T^1_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu (W_\mu)^2_2 T^2_1 \\
&\quad + (T^\dagger)^2_1 \bar{\sigma}^\mu (W_\mu)^1_1 T^1_2 + (T^\dagger)^2_1 \bar{\sigma}^\mu (W_\mu)^1_2 T^2_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu (W_\mu)^2_1 T^1_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu (W_\mu)^2_2 T^2_2 \\
&= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^0 T^0 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^+ T^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^- T^0 - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^0 T^- \\
&\quad + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^0 T^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^+ T^0 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^- T^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^0 T^0 \\
&= \frac{1}{2}[-W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^0 (T^-)^\dagger \bar{\sigma}^\mu T^- \\
&\qquad + W_\mu^0 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+]
\end{aligned}$$

$$X_i^0=\begin{pmatrix}\chi_{iL}^0\\(\chi_{iR}^0)^\dagger\end{pmatrix},\quad \chi_L^0=\chi_R^0=\mathcal{N}^\dagger\psi_L^0=\mathcal{N}^\dagger\psi_R^0=\begin{pmatrix}\chi_1^0 & \chi_2^0 & \chi_3^0\end{pmatrix}^T,\quad \bar{X}_i^0=\begin{pmatrix}\chi_{iR}^0 & (\chi_{iL}^0)^\dagger\end{pmatrix}$$

$$X_i^+=\begin{pmatrix}\chi_{iL}^+\\(\chi_{iR}^-)^\dagger\end{pmatrix},\quad \chi_L^+=\mathcal{C}_L^\dagger\psi_L^+=\begin{pmatrix}\chi_1^+ & \chi_2^+\end{pmatrix}^T,\quad \chi_R^-=\mathcal{C}_R^\dagger\psi_R^-=\begin{pmatrix}\chi_1^- & \chi_2^-\end{pmatrix}^T,\quad \bar{X}_i^+=\begin{pmatrix}\chi_{iR}^- & (\chi_{iL}^+)^\dagger\end{pmatrix}$$

$$\mathbf{\Psi}_i^0=\begin{pmatrix}\psi_{iL}^0\\(\psi_{iR}^0)^\dagger\end{pmatrix},\quad \psi_L^0=\psi_R^0=\mathcal{N}\chi_L^0=\mathcal{N}\chi_R^0=\begin{pmatrix}T^0 & D_1^0 & D_2^0\end{pmatrix}^T,\quad \bar{\mathbf{\Psi}}_i^0=\begin{pmatrix}\psi_{iR}^0 & (\psi_{iL}^0)^\dagger\end{pmatrix}$$

$$\mathbf{\Psi}_i^+=\begin{pmatrix}\psi_{iL}^+\\(\psi_{iR}^-)^\dagger\end{pmatrix},\quad \psi_L^+=\mathcal{C}_L\chi_L^+=\begin{pmatrix}T^+ & D_2^+\end{pmatrix}^T,\quad \psi_R^-=\mathcal{C}_R\chi_R^-=\begin{pmatrix}T^- & D_1^-\end{pmatrix}^T,\quad \bar{\mathbf{\Psi}}_i^+=\begin{pmatrix}\psi_{iR}^- & (\psi_{iL}^+)^\dagger\end{pmatrix}$$

$$\mathbf{\Psi}_{iL}^0=\begin{pmatrix}\psi_{iL}^0\\0\end{pmatrix}=\begin{pmatrix}(\mathcal{N}\chi_L^0)_i\\0\end{pmatrix}=\mathcal{N}_{ij}X_{jL}^0,\quad \mathbf{\Psi}_{iR}^0=\begin{pmatrix}0\\(\psi_{iR}^0)^\dagger\end{pmatrix}=\begin{pmatrix}0\\(\mathcal{N}\chi_R^0)_i^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*X_{jR}^0$$

$$\bar{\mathbf{\Psi}}_{iL}^0=\begin{pmatrix}0 & (\psi_{iL}^0)^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*\bar{X}_{jL}^0,\quad \bar{\mathbf{\Psi}}_{iR}^0=\begin{pmatrix}\psi_{iR}^0 & 0\end{pmatrix}=\mathcal{N}_{ij}\bar{X}_{jR}^0$$

$$\mathbf{\Psi}_{iL}^+=\begin{pmatrix}\psi_{iL}^+\\0\end{pmatrix}=\begin{pmatrix}(\mathcal{C}_L\chi_L^+)_i\\0\end{pmatrix}=(\mathcal{C}_L)_{ij}X_{jL}^+,\quad \mathbf{\Psi}_{iR}^+=\begin{pmatrix}0\\(\psi_{iR}^-)^\dagger\end{pmatrix}=\begin{pmatrix}0\\(\mathcal{C}_R\chi_R^-)_i^\dagger\end{pmatrix}=(\mathcal{C}_R)_{ij}^*X_{jR}^+$$

$$\bar{\mathbf{\Psi}}_{iL}^+=\begin{pmatrix}0 & (\psi_{iL}^+)^\dagger\end{pmatrix}=(\mathcal{C}_L)_{ij}^*\bar{X}_{jL}^+,\quad \bar{\mathbf{\Psi}}_{iR}^+=\begin{pmatrix}\psi_{iR}^- & 0\end{pmatrix}=(\mathcal{C}_R)_{ij}\bar{X}_{jR}^+$$

$$\bar{\mathbf{\Psi}}_{iL}^0\gamma^\mu\mathbf{\Psi}_{iL}^0=(\psi_{iL}^0)^\dagger\bar{\sigma}^\mu\psi_{iL}^0,\quad \bar{\mathbf{\Psi}}_{iR}^0\gamma^\mu\mathbf{\Psi}_{iR}^0=\psi_{iR}^0\sigma^\mu(\psi_{iR}^0)^\dagger=-(\psi_{iR}^0)^\dagger\bar{\sigma}^\mu\psi_{iR}^0=-(\psi_{iL}^0)^\dagger\bar{\sigma}^\mu\psi_{iL}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0}=\frac{1}{2}a_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar{\mathbf{\Psi}}_{iL}^0\gamma^\mu\mathbf{\Psi}_{iL}^0+\frac{1}{2}b_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar{\mathbf{\Psi}}_{iR}^0\gamma^\mu\mathbf{\Psi}_{iR}^0=\frac{1}{2}(a_{ZX_i^0X_j^0}Z_\mu\bar{X}_{iL}^0\gamma^\mu X_{jL}^0+b_{ZX_i^0X_j^0}Z_\mu\bar{X}_{iR}^0\gamma^\mu X_{jR}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0}=b_{Z\Psi_1^0\Psi_1^0}=0,\quad a_{Z\Psi_2^0\Psi_2^0}=-b_{Z\Psi_2^0\Psi_2^0}=\frac{g}{2c_W},\quad a_{Z\Psi_3^0\Psi_3^0}=-b_{Z\Psi_3^0\Psi_3^0}=-\frac{g}{2c_W}$$

$$a_{ZX_i^0X_j^0}=a_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}^*\mathcal{N}_{kj},\quad b_{ZX_i^0X_j^0}=b_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}\mathcal{N}_{kj}^*$$

$$\bar{\mathbf{\Psi}}_{iL}^+\gamma^\mu\mathbf{\Psi}_{iL}^0=(\psi_{iL}^+)^\dagger\bar{\sigma}^\mu\psi_{iL}^0,\quad \bar{\mathbf{\Psi}}_{iR}^+\gamma^\mu\mathbf{\Psi}_{iR}^0=\psi_{iR}^-\sigma^\mu(\psi_{iR}^0)^\dagger=-(\psi_{iR}^0)^\dagger\bar{\sigma}^\mu\psi_{iR}^-$$

$$\mathcal{L}_{W\Psi_i^+\Psi_j^0}=a_{W\Psi_i^+\Psi_j^0}(W_\mu^+\bar{\mathbf{\Psi}}_{iL}^+\gamma^\mu\mathbf{\Psi}_{jL}^0+\text{h.c.})+b_{W\Psi_i^+\Psi_j^0}(W_\mu^+\bar{\mathbf{\Psi}}_{iR}^+\gamma^\mu\mathbf{\Psi}_{jR}^0+\text{h.c.})$$

$$=a_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_L)_{ki}^*\mathcal{N}_{lj}W_\mu^+\bar{X}_{iL}^+\gamma^\mu X_{jL}^0+\text{h.c.}]+b_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_R)_{ki}\mathcal{N}_{lj}^*W_\mu^+\bar{X}_{iR}^+\gamma^\mu X_{jR}^0+\text{h.c.}]$$

$$=a_{WX_i^+X_j^0}W_\mu^+\bar{X}_{iL}^+\gamma^\mu X_{jL}^0+a_{WX_i^+X_j^0}^*W_\mu^-\bar{X}_{jL}^0\gamma^\mu X_{iL}^++b_{WX_i^+X_j^0}W_\mu^+\bar{X}_{iR}^+\gamma^\mu X_{jR}^0+b_{WX_i^+X_j^0}^*W_\mu^-\bar{X}_{jR}^0\gamma^\mu X_{iR}^+$$

$$a_{W\Psi_1^+\Psi_1^0}=b_{W\Psi_1^+\Psi_1^0}=g,\quad b_{W\Psi_2^+\Psi_2^0}=-\frac{g}{\sqrt{2}},\quad a_{W\Psi_2^+\Psi_3^0}=\frac{g}{\sqrt{2}},\quad \text{others}=0$$

$$a_{WX_i^+X_j^0}=a_{W\Psi_k^+\Psi_l^0}(\mathcal{C}_L)_{ki}^*\mathcal{N}_{lj},\quad b_{WX_i^+X_j^0}=b_{W\Psi_k^+\Psi_l^0}(\mathcal{C}_R)_{ki}\mathcal{N}_{lj}^*$$

$$\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} = (\psi^+_{i\mathrm{L}})^\dagger\bar{\sigma}^\mu\psi^+_{i\mathrm{L}}, \quad \bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}} = \psi^-_{i\mathrm{R}}\sigma^\mu(\psi^-_{i\mathrm{R}})^\dagger = -(\psi^-_{i\mathrm{R}})^\dagger\bar{\sigma}^\mu\psi^-_{i\mathrm{R}}$$

$$\mathcal{L}_{A\Psi^+_i\Psi^+_i} = a_{A\Psi^+_i\Psi^+_i}A_\mu\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} + b_{A\Psi^+_i\Psi^+_i}A_\mu\bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}}$$

$$= a_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}A_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}A_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$= a_{AX^+_iX^+_j}A_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{AX^+_iX^+_j}A_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$a_{A\Psi^+_1\Psi^+_1} = b_{A\Psi^+_1\Psi^+_1} = a_{A\Psi^+_2\Psi^+_2} = b_{A\Psi^+_2\Psi^+_2} = e$$

$$a_{AX^+_iX^+_j} = a_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj} = e\delta_{ij}, \quad b_{AX^+_iX^+_j} = b_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj} = e\delta_{ij}$$

$$\mathcal{L}_{Z\Psi^+_i\Psi^+_i} = a_{Z\Psi^+_i\Psi^+_i}Z_\mu\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} + b_{Z\Psi^+_i\Psi^+_i}Z_\mu\bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}}$$

$$= a_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}Z_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}Z_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$= a_{ZX^+_iX^+_j}Z_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{ZX^+_iX^+_j}Z_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$a_{Z\Psi^+_1\Psi^+_1} = b_{Z\Psi^+_1\Psi^+_1} = gc_\mathrm{W}, \quad a_{Z\Psi^+_2\Psi^+_2} = b_{Z\Psi^+_2\Psi^+_2} = -\frac{g}{2c_\mathrm{W}}(s^2_\mathrm{W} - c^2_\mathrm{W})$$

$$a_{ZX^+_iX^+_j} = a_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}, \quad b_{ZX^+_iX^+_j} = b_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}$$

$$\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} = \psi^0_{i\mathrm{R}}\psi^0_{j\mathrm{L}}, \quad \bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}} = (\psi^0_{i\mathrm{L}})^\dagger(\psi^0_{j\mathrm{R}})^\dagger$$

$$\mathcal{L}_{h\Psi^0_i\Psi^0_j} = \frac{1}{2}a_{h\Psi^0_i\Psi^0_j}h\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} + \frac{1}{2}b_{h\Psi^0_i\Psi^0_j}h\bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}}$$

$$= \frac{1}{2}a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj}h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + \frac{1}{2}b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj}h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}} = \frac{1}{2}(a_{hX^0_iX^0_j}h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + b_{hX^0_iX^0_j}h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}})$$

$$a_{h\Psi^0_1\Psi^0_2} = b_{h\Psi^0_1\Psi^0_2} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi^0_2\Psi^0_1} = b_{h\Psi^0_2\Psi^0_1}, \quad a_{h\Psi^0_1\Psi^0_3} = b_{h\Psi^0_1\Psi^0_3} = \frac{y_2}{\sqrt{2}} = a_{h\Psi^0_3\Psi^0_1} = b_{h\Psi^0_3\Psi^0_1}, \quad \text{others} = 0$$

$$a_{hX^0_iX^0_j} = a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj}, \quad b_{hX^0_iX^0_j} = b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj}$$

$$\bar{\Psi}^+_{i\mathrm{R}}\Psi^+_{j\mathrm{L}} = \begin{pmatrix}\psi^-_{i\mathrm{R}} & 0\end{pmatrix}\begin{pmatrix}\psi^+_{j\mathrm{L}} \\ 0\end{pmatrix} = \psi^-_{i\mathrm{R}}\psi^+_{j\mathrm{L}}, \quad \bar{\Psi}^+_{i\mathrm{L}}\Psi^+_{j\mathrm{R}} = \begin{pmatrix}0 & (\psi^+_{i\mathrm{L}})^\dagger\end{pmatrix}\begin{pmatrix}0 \\ (\psi^-_{j\mathrm{R}})^\dagger\end{pmatrix} = (\psi^+_{i\mathrm{L}})^\dagger(\psi^-_{j\mathrm{R}})^\dagger$$

$$\mathcal{L}_{h\Psi^+_i\Psi^+_j} = a_{h\Psi^+_i\Psi^+_j}h\bar{\Psi}^+_{i\mathrm{R}}\Psi^+_{j\mathrm{L}} + b_{h\Psi^+_i\Psi^+_j}h\bar{\Psi}^+_{i\mathrm{L}}\Psi^+_{j\mathrm{R}}$$

$$= a_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{L})_{lj}h\bar{X}^+_{i\mathrm{R}}X^+_{j\mathrm{L}} + b_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{R})^*_{lj}h\bar{X}^+_{i\mathrm{L}}X^+_{j\mathrm{R}} = a_{hX^+_iX^+_j}h\bar{X}^+_{i\mathrm{R}}X^+_{j\mathrm{L}} + b_{hX^+_iX^+_j}h\bar{X}^+_{i\mathrm{L}}X^+_{j\mathrm{R}}$$

$$a_{h\Psi^+_1\Psi^+_2} = b_{h\Psi^+_2\Psi^+_1} = y_2, \quad a_{h\Psi^+_2\Psi^+_1} = b_{h\Psi^+_1\Psi^+_2} = y_1, \quad \text{others} = 0$$

$$a_{hX^+_iX^+_j} = a_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{L})_{lj}, \quad b_{hX^+_iX^+_j} = b_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{R})^*_{lj}$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$\mathcal{L}_{hX_1^0X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 = [a_{h\Psi_1^0\Psi_2^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= \frac{1}{\sqrt{2}}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0X_1^0} = \frac{1}{2}(a_{hX_1^0X_1^0} + b_{hX_1^0X_1^0}) = \sqrt{2}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

$$G_{S,q} = -\frac{g_{hX_1^0X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0X_1^0} m_N}{2vm_h^2}\left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N\right), \quad \sigma_{\chi p}^{\mathrm{SI}} = \frac{4}{\pi}\mu_{\chi p}^2 G_{S,p}^2$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$\mathcal{L}_{ZX_1^0X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0X_1^0} = \frac{1}{2}(b_{ZX_1^0X_1^0} - a_{ZX_1^0X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q}\Delta_q^N, \quad \sigma_{\chi N}^{\mathrm{SD}} = \frac{12}{\pi}\mu_{\chi N}^2 G_{A,N}^2$$

Higgs and Z decay widths

$$\mathcal{L} \supset \sum_{ij} G^{\mathrm{S}}_{h,ij} h \bar{X}^+_i X^+_j - \sum_{ij} G^{\mathrm{P}}_{h,ij} h \bar{X}^+_i i\gamma_5 X^+_j + \sum_{ij} Z_\mu (G^{\mathrm{L}}_{Z,ij} \bar{X}^+_i \gamma^\mu P_{\mathrm{L}} X^+_j + G^{\mathrm{R}}_{Z,ij} \bar{X}^+_i \gamma^\mu P_{\mathrm{R}} X^+_j)$$

$$- \frac{1}{2}\sum_{ij} C^{\mathrm{S}}_{h,ij} h \bar{X}^0_i X^0_j + \frac{1}{2}\sum_{ij} C^{\mathrm{P}}_{h,ij} h \bar{X}^0_i i\gamma_5 X^0_j - \frac{1}{2}\sum_{ij} C^{\mathrm{A}}_{Z,ij} Z_\mu \bar{X}^0_i \gamma^\mu \gamma_5 X^0_j + \frac{1}{2}\sum_{ij} C^{\mathrm{V}}_{Z,ij} Z_\mu \bar{X}^0_i \gamma^\mu X^0_j$$

$$G^{\mathrm{S}}_{h,ij} = \mathrm{Re}(y_1 \mathcal{C}_{\mathrm{L},1j}\mathcal{C}_{\mathrm{R},2i} - y_2 \mathcal{C}_{\mathrm{L},2j}\mathcal{C}_{\mathrm{R},1i}), \quad G^{\mathrm{P}}_{h,ij} = \mathrm{Im}(y_1 \mathcal{C}_{\mathrm{L},1j}\mathcal{C}_{\mathrm{R},2i} - y_2 \mathcal{C}_{\mathrm{L},2j}\mathcal{C}_{\mathrm{R},1i})$$

$$G^{\mathrm{L}}_{Z,ij} = \frac{g(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} \mathcal{C}^*_{\mathrm{L},2i}\mathcal{C}_{\mathrm{L},2i} + g c_{\mathrm{W}} \mathcal{C}^*_{\mathrm{L},1i}\mathcal{C}_{\mathrm{L},1i}, \quad G^{\mathrm{R}}_{Z,ij} = \frac{g(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} \mathcal{C}^*_{\mathrm{R},2i}\mathcal{C}_{\mathrm{R},2i} + g c_{\mathrm{W}} \mathcal{C}^*_{\mathrm{R},1i}\mathcal{C}_{\mathrm{R},1i}$$

$$C^{\mathrm{S}}_{h,ij} = \sqrt{2}\,\mathrm{Re}(y_1 \mathcal{N}_{1i}\mathcal{N}_{2j} + y_2 \mathcal{N}_{1i}\mathcal{N}_{3j}), \quad C^{\mathrm{P}}_{h,ij} = \sqrt{2}\,\mathrm{Im}(y_1 \mathcal{N}_{1i}\mathcal{N}_{2j} + y_2 \mathcal{N}_{1i}\mathcal{N}_{3j})$$

$$C^{\mathrm{A}}_{Z,ij} = \frac{g}{2c_{\mathrm{W}}} \mathrm{Re}(\mathcal{N}^*_{2i}\mathcal{N}_{2j} - \mathcal{N}^*_{3i}\mathcal{N}_{3j}), \quad C^{\mathrm{V}}_{Z,ij} = \frac{ig}{2c_{\mathrm{W}}} \mathrm{Im}(\mathcal{N}^*_{2i}\mathcal{N}_{2j} - \mathcal{N}^*_{3i}\mathcal{N}_{3j})$$

(1) $h(p) \to \chi^+_i(k_1) + \chi^-_j(k_2)$

$$k_1 \cdot k_2 - m_{\chi^\pm_i} m_{\chi^\pm_j} = \frac{1}{2}[m_h^2 - (m_{\chi^\pm_i} + m_{\chi^\pm_j})^2], \quad k_1 \cdot k_2 + m_{\chi^\pm_i} m_{\chi^\pm_j} = \frac{1}{2}[m_h^2 - (m_{\chi^\pm_i} - m_{\chi^\pm_j})^2], \quad |\mathbf{k}_1| = \frac{F(m_h^2, m^2_{\chi^\pm_i}, m^2_{\chi^\pm_j})}{2m_h}$$

$$i\mathcal{M} = i\bar{u}(k_1)(G^{\mathrm{S}}_{h,ij} - iG^{\mathrm{P}}_{h,ij}\gamma_5) v(k_2), \quad (i\mathcal{M})^* = -i\bar{v}(k_2)(G^{\mathrm{S}*}_{h,ij} - iG^{\mathrm{P}*}_{h,ij}\gamma_5) u(k_1)$$

$$\sum_{\mathrm{spins}} |\mathcal{M}|^2 = \sum_{\mathrm{spins}} \bar{u}(k_1)(G^{\mathrm{S}}_{h,ij} - iG^{\mathrm{P}}_{h,ij}\gamma_5) v(k_2)\bar{v}(k_2)(G^{\mathrm{S}*}_{h,ij} - iG^{\mathrm{P}*}_{h,ij}\gamma_5) u(k_1)$$

$$= \mathrm{Tr}[(\not{k}_1 + m_{\chi^\pm_i})(G^{\mathrm{S}}_{h,ij} - iG^{\mathrm{P}}_{h,ij}\gamma_5)\gamma_5](\not{k}_2 - m_{\chi^\pm_j})(G^{\mathrm{S}*}_{h,ij} - iG^{\mathrm{P}*}_{h,ij}\gamma_5)]$$

$$= 2\{|G^{\mathrm{S}}_{h,ij}|^2 [m_h^2 - (m_{\chi^\pm_i} + m_{\chi^\pm_j})^2] + |G^{\mathrm{P}}_{h,ij}|^2 [m_h^2 - (m_{\chi^\pm_i} - m_{\chi^\pm_j})^2]\}$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_h^2}\sum_{\mathrm{spins}} |\mathcal{M}|^2 = \frac{F(m_h^2, m^2_{\chi^\pm_i}, m^2_{\chi^\pm_j})}{8\pi m_h^3}\{|G^{\mathrm{S}}_{h,ij}|^2 [m_h^2 - (m_{\chi^\pm_i} + m_{\chi^\pm_j})^2] + |G^{\mathrm{P}}_{h,ij}|^2 [m_h^2 - (m_{\chi^\pm_i} - m_{\chi^\pm_j})^2]\}$$

(2) $Z(p) \to \chi^+_i(k_1) + \chi^-_j(k_2)$

$$i\mathcal{M} = i\varepsilon_\mu(p)\bar{u}(k_1)\gamma^\mu (G^{\mathrm{L}}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}}_{Z,ij} P_{\mathrm{R}}) v(k_2), \quad (i\mathcal{M})^* = -i\varepsilon^*_\nu(p)\bar{v}(k_2)\gamma^\nu (G^{\mathrm{L}*}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}*}_{Z,ij} P_{\mathrm{R}}) u(k_1)$$

$$\frac{1}{3}\sum_{\mathrm{spins}} |\mathcal{M}|^2 = \frac{1}{3}\sum_{\mathrm{spins}} \varepsilon_\mu(p)\varepsilon^*_\nu(p)\bar{u}(k_1)\gamma^\mu (G^{\mathrm{L}}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}}_{Z,ij} P_{\mathrm{R}}) v(k_2)\bar{v}(k_2)\gamma^\nu (G^{\mathrm{L}*}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}*}_{Z,ij} P_{\mathrm{R}}) u(k_1)$$

$$= \frac{1}{3}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2}\right)\mathrm{Tr}[(\not{k}_1 + m_{\chi^\pm_i})\gamma^\mu (G^{\mathrm{L}}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}}_{Z,ij} P_{\mathrm{R}})(\not{k}_2 - m_{\chi^\pm_j})\gamma^\nu (G^{\mathrm{L}*}_{Z,ij} P_{\mathrm{L}} + G^{\mathrm{R}*}_{Z,ij} P_{\mathrm{R}})]$$

$$= \frac{1}{3m_Z^2}\{(|G^{\mathrm{L}}_{Z,ij}|^2 + |G^{\mathrm{R}}_{Z,ij}|^2)[m_Z^2(2m_Z^2 - m^2_{\chi^\pm_i} - m^2_{\chi^\pm_j}) - (m^2_{\chi^\pm_i} - m^2_{\chi^\pm_j})^2] + 6(G^{\mathrm{L}}_{Z,ij}G^{\mathrm{R}*}_{Z,ij} + G^{\mathrm{L}*}_{Z,ij}G^{\mathrm{R}}_{Z,ij}) m_Z^2 m_{\chi^\pm_i} m_{\chi^\pm_j}\}$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_Z^2}\frac{1}{3}\sum_{\mathrm{spins}} |\mathcal{M}|^2$$

$$= \frac{F(m_Z^2, m^2_{\chi^\pm_i}, m^2_{\chi^\pm_j})}{48\pi m_Z^5}\{(|G^{\mathrm{L}}_{Z,ij}|^2 + |G^{\mathrm{R}}_{Z,ij}|^2)[m_Z^2(2m_Z^2 - m^2_{\chi^\pm_i} - m^2_{\chi^\pm_j}) - (m^2_{\chi^\pm_i} - m^2_{\chi^\pm_j})^2] + 6(G^{\mathrm{L}}_{Z,ij}G^{\mathrm{R}*}_{Z,ij} + G^{\mathrm{L}*}_{Z,ij}G^{\mathrm{R}}_{Z,ij}) m_Z^2 m_{\chi^\pm_i} m_{\chi^\pm_j}\}$$

Conventions for electroweak gauge couplings

My convention

$$D_\mu = \partial_\mu - ig' B_\mu Y - ig W_\mu^a T^a$$

$$A_\mu = s_\mathrm{W} W_\mu^3 + c_\mathrm{W} B_\mu, \quad Z_\mu = c_\mathrm{W} W_\mu^3 - s_\mathrm{W} B_\mu, \quad W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$$

$$B_\mu = c_\mathrm{W} A_\mu - s_\mathrm{W} Z_\mu, \quad W_\mu^3 = s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu$$

$$Q = T^3 + Y, \quad e = g s_\mathrm{W} = g' c_\mathrm{W}$$

$$iD_\mu \supset g' Y (c_\mathrm{W} A_\mu - s_\mathrm{W} Z_\mu) + g T^3 (s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu)$$

$$= e(Y + T^3) A_\mu + \left(g c_\mathrm{W} T^3 - \frac{g s_\mathrm{W}}{c_\mathrm{W}} s_\mathrm{W} Y \right) Z_\mu = Q e A_\mu + \frac{g}{c_\mathrm{W}} (T^3 c_\mathrm{W}^2 - Y s_\mathrm{W}^2) Z_\mu$$

$$= Q e A_\mu + \frac{g}{c_\mathrm{W}} (T^3 - Q s_\mathrm{W}^2) Z_\mu$$

Denner's convention [0709.1075]

$$D_\mu = \partial_\mu + ig' B_\mu Y - ig W_\mu^a T^a$$

$$A_\mu = -s_\mathrm{W} W_\mu^3 + c_\mathrm{W} B_\mu, \quad Z_\mu = c_\mathrm{W} W_\mu^3 + s_\mathrm{W} B_\mu, \quad W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$$

$$B_\mu = c_\mathrm{W} A_\mu + s_\mathrm{W} Z_\mu, \quad W_\mu^3 = -s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu$$

$$Q = T^3 + Y, \quad e = g s_W = g' c_W$$

$$iD_\mu \supset -g' Y (c_\mathrm{W} A_\mu + s_\mathrm{W} Z_\mu) + g T^3 (-s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu)$$

$$= -e(Y + T^3) A_\mu + \left(g c_\mathrm{W} T^3 - \frac{g s_\mathrm{W}}{c_\mathrm{W}} s_W Y \right) Z_\mu = Q e A_\mu + \frac{g}{c_\mathrm{W}} (T^3 c_\mathrm{W}^2 - Y s_\mathrm{W}^2) Z_\mu$$

$$= -Q e A_\mu + \frac{g}{c_\mathrm{W}} (T^3 - Q s_\mathrm{W}^2) Z_\mu$$

Quantum angular momentum

$[J^a, J^b] = i\varepsilon^{abc} J^c$

$J^a J^a = (J^1)^2 + (J^2)^2 + (J^3)^2, \quad [J^a J^a, J^b] = J^a [J^a, J^b] + [J^a, J^b] J^a = i\varepsilon^{abc}(J^a J^c + J^c J^a) = 0$

$[J^a J^a, J^3] = 0, \quad J^a J^a |\lambda, m\rangle = \lambda |\lambda, m\rangle, \quad J^3 |\lambda, m\rangle = m |\lambda, m\rangle$

$J^\pm = J^1 \pm iJ^2, \quad [J^a J^a, J^\pm] = 0, \quad [J^3, J^\pm] = [J^3, J^1] \pm i[J^3, J^2] = iJ^2 \pm J^1 = \pm J^\pm$

$J^a J^a J^\pm |\lambda, m\rangle = \lambda J^\pm |\lambda, m\rangle, \quad J^3 J^\pm |\lambda, m\rangle = (J^\pm J^3 \pm J^\pm)|\lambda, m\rangle = (m \pm 1) J^\pm |\lambda, m\rangle \quad \Rightarrow \quad J^\pm |\lambda, m\rangle = c_{m,\pm} |\lambda, m \pm 1\rangle$

$\langle \lambda, m | [J^a J^a - (J^3)^2] | \lambda, m\rangle = \langle \lambda, m | (J^1)^2 + (J^2)^2 | \lambda, m\rangle \geq 0 \quad \Rightarrow \quad \lambda - m^2 \geq 0,$ m has a maximal $m_{\max}$ and a minimal $m_{\min}$

$J^+ |\lambda, m_{\max}\rangle = 0, \quad J^- |\lambda, m_{\min}\rangle = 0, \quad J^\mp J^\pm = (J^1 \mp iJ^2)(J^1 \pm iJ^2) = (J^1)^2 + (J^2)^2 \mp i[J^2, J^1] = J^a J^a - (J^3)^2 \mp J^3$

$0 = J^- J^+ |\lambda, m_{\max}\rangle = [J^a J^a - (J^3)^2 - J^3] |\lambda, m_{\max}\rangle = (\lambda - m_{\max}^2 - m_{\max}) |\lambda, m_{\max}\rangle$

$0 = J^+ J^- |\lambda, m_{\min}\rangle = [J^a J^a - (J^3)^2 + J^3] |\lambda, m_{\min}\rangle = (\lambda - m_{\min}^2 + m_{\min}) |\lambda, m_{\min}\rangle$

$\lambda = m_{\max}(m_{\max} + 1) = m_{\min}(m_{\min} - 1) \quad \Rightarrow \quad m_{\max} = -m_{\min}$

$j \equiv m_{\max}, \quad m_{\max} - m_{\min} = 2j$ is an integer, $\quad j = 0, \dfrac{1}{2}, 1, \dfrac{3}{2}, 2, \cdots$

$\lambda = j(j+1), \quad J^a J^a |\lambda, m\rangle = j(j+1) | j, m\rangle, \quad J^3 | j_n, m\rangle = m | j, m\rangle, \quad m = j, j-1, \cdots, -j+1, -j$

$J^3 = \mathrm{diag}(j, j-1, \cdots, -j+1, -j)$

$(J^\pm)^\dagger = J^\mp, \quad |c_{m,\pm}|^2 = \langle \lambda, m | J^\mp J^\pm | \lambda, m\rangle = \langle \lambda, m | [J^a J^a - (J^3)^2 \mp J^3] | \lambda, m\rangle = j(j+1) - m^2 \mp m = (j \mp m)(j \pm m + 1)$

$J^\pm |\lambda, m\rangle = e^{i\phi_{m,\pm}} \sqrt{(j \mp m)(j \pm m + 1)} |\lambda, m \pm 1\rangle, \quad \phi_{m,\pm} = \arg c_{m,\pm} \in \mathbf{R}$

Real multiplet in SU(2)

$\phi = \begin{pmatrix} \phi_j \\ \vdots \\ \phi_{-j} \end{pmatrix} \in \mathbf{n}$ of $\mathrm{SU}(2)_\mathrm{L}, \quad j = \dfrac{n-1}{2}$

$D_\mu \phi = (\partial_\mu - igW_\mu^a t_{(n)}^a)\phi, \quad t_{(n)}^a$ is generators for $\mathbf{n}$, $\quad (t_{(n)}^a)^\dagger = t_{(n)}^a, \quad [t_{(n)}^a, t_{(n)}^b] = i\varepsilon^{abc} t_{(n)}^c$

$[\exp(-i\theta^a t_{(n)}^a)]^* = \exp[+i\theta^a (t_{(n)}^a)^*] \quad \Rightarrow \quad -(t_{(n)}^a)^*$ is generators for $\mathbf{n}^*$

$\phi \to \exp(-i\theta^a t_{(n)}^a)\phi \quad \Rightarrow \quad \phi^* \to \exp[+i\theta^a (t_{(n)}^a)^*]\phi^* \quad \Rightarrow \quad \phi^* \in \mathbf{n}^*$

$D_\mu \phi^* = [\partial_\mu + igW_\mu^a (t_{(n)}^a)^*]\phi^* = [\partial_\mu + igW_\mu^a (t_{(n)}^a)^\mathrm{T}]\phi^*, \quad (D_\mu \phi)^\dagger = \partial_\mu \phi^\dagger + igW_\mu^a \phi^\dagger t_{(n)}^a = (D_\mu \phi^*)^\mathrm{T}$

All SU(2) irreducible representations are real: $S_{(n)} \exp(-i\theta^a t_{(n)}^a) S_{(n)}^{-1} = \exp[+i\theta^a (t_{(n)}^a)^*]$

$\quad \Rightarrow \quad S_{(n)} t_{(n)}^a S_{(n)}^{-1} = -(t_{(n)}^a)^*$

$\tilde{\phi} \equiv S_{(n)}^{-1} \phi^*$

$\tilde{\phi} \to S_{(n)}^{-1} \exp[+i\theta^a (t_{(n)}^a)^*]\phi^* = S_{(n)}^{-1} \exp[+i\theta^a (t_{(n)}^a)^*] S_{(n)} S_{(n)}^{-1} \phi^* = \exp(-i\theta^a t_{(n)}^a)\tilde{\phi} \quad \Rightarrow \quad \tilde{\phi} \in \mathbf{n}$

The condition for a real multiplet ϕ: $\quad \tilde{\phi} = \phi$

Conditions for $t_{(n)}^a$ and S

$t_{(n)}^\pm = t_{(n)}^1 \pm it_{(n)}^2, \quad t_{(n)}^1 = \dfrac{1}{2}(t_{(n)}^+ + t_{(n)}^-), \quad t_{(n)}^2 = -\dfrac{i}{2}(t_{(n)}^+ - t_{(n)}^-)$

Real $t_{(n)}^\pm$ and $S_{(n)}$ $\quad \Rightarrow \quad t_{(n)}^- = (t_{(n)}^+)^\mathrm{T}, \quad S_{(n)}^{-1} = S_{(n)}^\mathrm{T}$

$\quad \Rightarrow \quad (t_{(n)}^1)^* = t_{(n)}^1, \quad (t_{(n)}^2)^* = -t_{(n)}^2, \quad (t_{(n)}^3)^* = t_{(n)}^3$

$$\Rightarrow \quad \begin{cases} S_{(n)} t_{(n)}^1 S_{(n)}^{-1} = -t_{(n)}^1, \quad S_{(n)} t_{(n)}^2 S_{(n)}^{-1} = t_{(n)}^2 \quad \Leftrightarrow \quad S_{(n)} t_{(n)}^+ S_{(n)}^{-1} = -t_{(n)}^-, \quad S_{(n)} t_{(n)}^- S_{(n)}^{-1} = -t_{(n)}^+ \\ S_{(n)} t_{(n)}^3 S_{(n)}^{-1} = -t_{(n)}^3 \end{cases}$$

1) The convention that $t^{+}_{(n)}|j,m\rangle = \begin{cases} -\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, & m \ge 0 \\ \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, & m<0 \end{cases}$ and $S_{(n)}|j,m\rangle = \begin{cases} (-1)^{n+1}|j,-m\rangle, & m\ge 0 \\ |j,-m\rangle, & m<0 \end{cases}$

satisfies the condition $S_{(n)} t^{a}_{(n)} S^{-1}_{(n)} = -(t^{a}_{(n)})^{*}$ [Ref: Hambye et al. 0903.4010]

Proof: $t^{-}_{(n)}|j,m\rangle = \begin{cases} -\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle, & m>0 \\ \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle, & m\le 0 \end{cases}$, $S^{-1}_{(n)}|j,m\rangle = S^{\mathrm{T}}_{(n)}|j,m\rangle = \begin{cases} |j,-m\rangle, & m>0 \\ (-1)^{n+1}|j,-m\rangle, & m\le 0 \end{cases}$

$m>1$: $S_{(n)}t^{+}_{(n)}S^{-1}_{(n)}|j,m\rangle = S_{(n)}t^{+}_{(n)}|j,-m\rangle = \sqrt{(j+m)(j-m+1)}S_{(n)}|j,-m+1\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle = -t^{-}_{(n)}|j,m\rangle$

$S_{(n)}t^{-}_{(n)}S^{-1}_{(n)}|j,m\rangle = S_{(n)}t^{-}_{(n)}|j,-m\rangle = \sqrt{(j-m)(j+m+1)}S_{(n)}|j,-m-1\rangle = \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle = -t^{+}_{(n)}|j,m\rangle$

$m=1$: $(-1)^{n+1}=1$, $S_{(n)}t^{+}_{(n)}S^{-1}_{(n)}|j,1\rangle = S_{(n)}t^{+}_{(n)}|j,-1\rangle = \sqrt{(j+1)j}S_{(n)}|j,0\rangle = \sqrt{(j+1)j}\,|j,0\rangle = -t^{-}_{(n)}|j,1\rangle$

$S_{(n)}t^{-}_{(n)}S^{-1}_{(n)}|j,1\rangle = S_{(n)}t^{-}_{(n)}|j,-1\rangle = \sqrt{(j-1)(j+1+1)}S_{(n)}|j,-2\rangle = \sqrt{(j-1)(j+1+1)}\,|j,2\rangle = -t^{+}_{(n)}|j,1\rangle$

$m=0$: $(-1)^{n+1}=1$, $S_{(n)}t^{+}_{(n)}S^{-1}_{(n)}|j,0\rangle = S_{(n)}t^{+}_{(n)}|j,0\rangle = -\sqrt{j(j+1)}S_{(n)}|j,1\rangle = -\sqrt{j(j+1)}\,|j,-1\rangle = -t^{-}_{(n)}|j,0\rangle$

$S_{(n)}t^{-}_{(n)}S^{-1}_{(n)}|j,0\rangle = S_{(n)}t^{-}_{(n)}|j,0\rangle = \sqrt{j(j+1)}S_{(n)}|j,-1\rangle = \sqrt{j(j+1)}\,|j,1\rangle = -t^{+}_{(n)}|j,0\rangle$

$m=-1$: $(-1)^{n+1}=1$, $S_{(n)}t^{+}_{(n)}S^{-1}_{(n)}|j,-1\rangle = S_{(n)}t^{+}_{(n)}|j,1\rangle = -\sqrt{(j-1)(j+1+1)}S_{(n)}|j,2\rangle = -\sqrt{(j-1)(j+1+1)}\,|j,-2\rangle = -t^{-}_{(n)}|j,-1\rangle$

$S_{(n)}t^{-}_{(n)}S^{-1}_{(n)}|j,-1\rangle = S_{(n)}t^{-}_{(n)}|j,1\rangle = -\sqrt{(j+1)j}S_{(n)}|j,0\rangle = -\sqrt{(j+1)j}\,|j,0\rangle = -t^{+}_{(n)}|j,-1\rangle$

$m<-1$: $S_{(n)}t^{+}_{(n)}S^{-1}_{(n)}|j,m\rangle = (-1)^{n+1}S_{(n)}t^{+}_{(n)}|j,-m\rangle = -(-1)^{n+1}\sqrt{(j+m)(j-m+1)}S_{(n)}|j,-m+1\rangle = -\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle = -t^{-}_{(n)}|j,m\rangle$

$S_{(n)}t^{-}_{(n)}S^{-1}_{(n)}|j,m\rangle = (-1)^{n+1}S_{(n)}t^{-}_{(n)}|j,-m\rangle = -(-1)^{n+1}\sqrt{(j-m)(j+m+1)}S_{(n)}|j,-m-1\rangle = -\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle = -t^{+}_{(n)}|j,m\rangle$

$m>0$: $S_{(n)}t^{3}_{(n)}S^{-1}_{(n)}|j,m\rangle = S_{(n)}t^{3}_{(n)}|j,-m\rangle = -mS_{(n)}|j,-m\rangle = -m|j,m\rangle = -t^{3}_{(n)}|j,m\rangle$

$m=0$: $(-1)^{n+1}=1$, $S_{(n)}t^{3}_{(n)}S^{-1}_{(n)}|j,0\rangle = S_{(n)}t^{3}_{(n)}|j,0\rangle = 0 = -t^{3}_{(n)}|j,0\rangle$

$m<0$: $S_{(n)}t^{3}_{(n)}S^{-1}_{(n)}|j,m\rangle = (-1)^{n+1}S_{(n)}t^{3}_{(n)}|j,-m\rangle = -m(-1)^{n+1}S_{(n)}|j,-m\rangle = -m|j,m\rangle = -t^{3}_{(n)}|j,m\rangle$

Thus, $S_{(n)}t^{\pm}_{(n)}S^{-1}_{(n)} = -t^{\mp}_{(n)}$, $S_{(n)}t^{3}_{(n)}S^{-1}_{(n)} = -t^{3}_{(n)}$

$\underline{n=2}$

$$t^{+}_{(2)}=\begin{pmatrix} & 1\\ 0 & \end{pmatrix},\quad t^{-}_{(2)}=\begin{pmatrix} & 0\\ 1 & \end{pmatrix},\quad S_{(2)}=\begin{pmatrix} & 1\\ -1 & \end{pmatrix},\quad S_{(2)}\begin{pmatrix} a & b\\ c & d\end{pmatrix}S^{-1}_{(2)}=\begin{pmatrix} d & -c\\ -b & a\end{pmatrix}$$

$$t^{1}_{(2)}=\frac{1}{2}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}=\frac{\sigma^1}{2},\quad t^{2}_{(2)}=\frac{1}{2}\begin{pmatrix} & -i\\ i & \end{pmatrix}=\frac{\sigma^2}{2},\quad t^{3}_{(2)}=\frac{1}{2}\begin{pmatrix} 1 & \\ & -1\end{pmatrix}=\frac{\sigma^3}{2},\quad \tilde{\phi}=S^{-1}_{(2)}\begin{pmatrix}\phi_{+1/2}\\ \phi_{-1/2}\end{pmatrix}^{*}=\begin{pmatrix}-\phi^{*}_{-1/2}\\ \phi^{*}_{+1/2}\end{pmatrix}$$

$$\tilde{\phi}=\phi \quad\Rightarrow\quad \phi=\begin{pmatrix}0\\0\end{pmatrix}$$

$\underline{n=3}$

$$t^{+}_{(3)}=\begin{pmatrix} 0 & -\sqrt{2} & \\ & 0 & \sqrt{2}\\ & & 0\end{pmatrix},\quad t^{-}_{(3)}=\begin{pmatrix} 0 & & \\ -\sqrt{2} & 0 & \\ & \sqrt{2} & 0\end{pmatrix},\quad S_{(3)}=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix},\quad S_{(3)}\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}S^{-1}_{(3)}=\begin{pmatrix} i & h & g\\ f & e & d\\ c & b & a\end{pmatrix}$$

$$t^{1}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & \\ -1 & 0 & 1\\ & 1 & 0\end{pmatrix},\quad t^{2}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & \\ -i & 0 & -i\\ & i & 0\end{pmatrix},\quad t^{3}_{(3)}=\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1\end{pmatrix},\quad \tilde{\phi}=S^{-1}_{(3)}\begin{pmatrix}\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}^{*}=\begin{pmatrix}\phi^{*}_{-1}\\ \phi^{*}_{0}\\ \phi^{*}_{+1}\end{pmatrix}$$

$$\left[\begin{array}{l}\text{Note: } t^{+}_{(n)}|j,m\rangle=\begin{cases}\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, & m\geq 0\\ -\sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, & m<0\end{cases} \text{ and } S_{(n)}|j,m\rangle=\begin{cases}(-1)^{n+1}|j,-m\rangle, & m\geq 0\\ |j,-m\rangle, & m<0\end{cases} \text{ will lead to}\\ t^{+}_{(3)}=\begin{pmatrix} 0 & \sqrt{2} & \\ & 0 & -\sqrt{2}\\ & & 0\end{pmatrix},\quad t^{-}_{(3)}=\begin{pmatrix} 0 & & \\ \sqrt{2} & 0 & \\ & -\sqrt{2} & 0\end{pmatrix},\quad t^{1}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & -1\\ & -1 & 0\end{pmatrix},\quad t^{2}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & \\ i & 0 & i\\ & -i & 0\end{pmatrix},\quad \tilde{\phi}=\begin{pmatrix}\phi^{*}_{-1}\\ \phi^{*}_{0}\\ \phi^{*}_{+1}\end{pmatrix}\end{array}\right]$$

ϕ is a real multiplet $\quad\Rightarrow\quad \tilde{\phi}=\phi \quad\Rightarrow\quad \phi^{*}_{i}=\phi_{-i}$

Real multiplets only exist in odd-dimensional respresentations

A real triplet can be expressed as $\phi=\begin{pmatrix}\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}$ with $\phi_{+1}=(\phi_{-1})^{*}$

By contrast, for a complex triplet $\phi=\begin{pmatrix}\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}$, $\phi_{+1}\neq(\phi_{-1})^{*}$ and $\phi_{0}\neq(\phi_{0})^{*}$

$\underline{n=4}$

$$t^{+}_{(4)}=\begin{pmatrix} 0 & -\sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3}\\ & & & 0\end{pmatrix},\quad t^{-}_{(4)}=\begin{pmatrix} 0 & & & \\ -\sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0\end{pmatrix},\quad S_{(4)}=\begin{pmatrix} & & & 1\\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},\quad S_{(4)}\begin{pmatrix} a & b & c & d\\ e & f & g & h\\ i & j & k & l\\ m & n & o & p\end{pmatrix}S^{-1}_{(4)}=\begin{pmatrix} p & o & -n & -m\\ l & k & -j & -i\\ -h & -g & f & e\\ -d & -c & b & a\end{pmatrix}$$

$$t^{1}_{(4)}=\frac{1}{2}\begin{pmatrix} 0 & -\sqrt{3} & & \\ -\sqrt{3} & 0 & 2 & \\ & 2 & 0 & \sqrt{3}\\ & & \sqrt{3} & 0\end{pmatrix},\quad t^{2}_{(4)}=\frac{1}{2}\begin{pmatrix} 0 & \sqrt{3}i & & \\ -\sqrt{3}i & 0 & -2i & \\ & 2i & 0 & -\sqrt{3}i\\ & & \sqrt{3}i & 0\end{pmatrix},\quad t^{3}_{(4)}=\frac{1}{2}\begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3\end{pmatrix},\quad \tilde{\phi}=S^{-1}_{(4)}\begin{pmatrix}\phi_{+3/2}\\ \phi_{+1/2}\\ \phi_{-1/2}\\ \phi_{-3/2}\end{pmatrix}^{*}=\begin{pmatrix}-\phi^{*}_{-3/2}\\ -\phi^{*}_{-1/2}\\ \phi^{*}_{+1/2}\\ \phi^{*}_{+3/2}\end{pmatrix}$$

$$\tilde{\phi}=\phi \quad\Rightarrow\quad \tilde{\phi}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

2) The convention that $t^{+}_{(3)}=\begin{pmatrix} 0 & \sqrt{2} & \\ & 0 & \sqrt{2}\\ & & 0\end{pmatrix}$, $t^{-}_{(3)}=\begin{pmatrix} 0 & & \\ \sqrt{2} & 0 & \\ & \sqrt{2} & 0\end{pmatrix}$, and $S_{(3)}=\begin{pmatrix} & & -1\\ & 1 & \\ -1 & & \end{pmatrix}$

leads to $t^{1}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & 1\\ & 1 & 0\end{pmatrix}$ and $t^{2}_{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & \\ i & 0 & -i\\ & i & 0\end{pmatrix}$, also satisfying $S_{(n)}t^{a}_{(n)}S^{-1}_{(n)}=-(t^{a}_{(n)})^{*}$

$$S_{(3)}\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}S^{-1}_{(3)}=\begin{pmatrix} i & -h & g\\ -f & e & -d\\ c & -b & a\end{pmatrix}$$

$\tilde{\phi}=S^{-1}_{(3)}\begin{pmatrix}\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}^{*}=\begin{pmatrix}-\phi^{*}_{-1}\\ \phi^{*}_{0}\\ -\phi^{*}_{+1}\end{pmatrix}$; therefore, $\tilde{\phi}=\phi \quad\Rightarrow\quad (\phi_{+1})^{*}=-\phi_{-1},\quad \phi^{*}_{0}=\phi_{0}$

In this case, a real triplet can be expressed as $\phi=\begin{pmatrix}\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}$ with $\phi_{+1}=-(\phi_{-1})^{*}$ or $\phi=\begin{pmatrix}-\phi_{+1}\\ \phi_{0}\\ \phi_{-1}\end{pmatrix}$ with $\phi_{+1}=(\phi_{-1})^{*}$

Dirac, Majorana, and Weyl triplets

4-component spinor triplet $\mathcal{T} = \begin{pmatrix} \mathcal{T}^+ \\ \mathcal{T}^0 \\ \mathcal{T}^- \end{pmatrix} \in (\mathbf{3},0)$

$$\mathcal{T}^+ = \mathcal{T}_2^1, \quad \mathcal{T}^- = \mathcal{T}_1^2, \quad \mathcal{T}^0 = \sqrt{2}\mathcal{T}_1^1 = -\sqrt{2}\mathcal{T}_2^2$$

$$(\mathcal{T}^+)^\dagger = (\mathcal{T}^\dagger)_1^2, \quad (\mathcal{T}^-)^\dagger = (\mathcal{T}^\dagger)_2^1, \quad (\mathcal{T}^0)^\dagger = \sqrt{2}(\mathcal{T}^\dagger)_1^1 = -\sqrt{2}(\mathcal{T}^\dagger)_2^2$$

$$\bar{\mathcal{T}}\mathcal{T} = (\mathcal{T}^\dagger)_i^j \gamma^0 \mathcal{T}_j^i = (\mathcal{T}^\dagger)_1^2 \gamma^0 \mathcal{T}_2^1 + (\mathcal{T}^\dagger)_2^1 \gamma^0 \mathcal{T}_1^2 + (\mathcal{T}^\dagger)_1^1 \gamma^0 \mathcal{T}_1^1 + (\mathcal{T}^\dagger)_2^2 \gamma^0 \mathcal{T}_2^2 = \bar{\mathcal{T}}^+\mathcal{T}^+ + \bar{\mathcal{T}}^-\mathcal{T}^- + \bar{\mathcal{T}}^0\mathcal{T}^0$$

$$\mathcal{L}_{\text{DK}} = i\bar{\mathcal{T}}\gamma^\mu\partial_\mu\mathcal{T} = i\bar{\mathcal{T}}_i^{\,j}\gamma^\mu\partial_\mu\mathcal{T}_j^i = i\bar{\mathcal{T}}_1^{\,2}\gamma^\mu\partial_\mu\mathcal{T}_2^1 + i\bar{\mathcal{T}}_2^{\,1}\gamma^\mu\partial_\mu\mathcal{T}_1^2 + i\bar{\mathcal{T}}_1^{\,1}\gamma^\mu\partial_\mu\mathcal{T}_1^1 + i\bar{\mathcal{T}}_2^{\,2}\gamma^\mu\partial_\mu\mathcal{T}_2^2$$
$$= i\bar{\mathcal{T}}^+\gamma^\mu\partial_\mu\mathcal{T}^+ + i\bar{\mathcal{T}}^-\gamma^\mu\partial_\mu\mathcal{T}^- + i\bar{\mathcal{T}}^0\gamma^\mu\partial_\mu\mathcal{T}^0$$

$$D_\mu\mathcal{T} = (\partial_\mu - igW_\mu^a t_{\text{T}}^a)\mathcal{T}$$

$$\mathcal{L}_{\text{DG}} = gW_\mu^a\bar{\mathcal{T}}\gamma^\mu t_{\text{T}}^a\mathcal{T} = [gW_\mu^+(\bar{\mathcal{T}}^+\gamma^\mu\mathcal{T}^0 - \bar{\mathcal{T}}^0\gamma^\mu\mathcal{T}^-) + h.c.] + (eA_\mu + gc_{\text{W}}Z_\mu)(\bar{\mathcal{T}}^+\gamma^\mu\mathcal{T}^+ - \bar{\mathcal{T}}^-\gamma^\mu\mathcal{T}^-)$$

$\mathcal{T}_j^i = \begin{pmatrix} [(T_{\text{L}})_\alpha]_j^i \\ [(T_{\text{R}})^{\dagger\dot\alpha}]_j^i \end{pmatrix}$, $\quad (T_{\text{L,R}})_j^i$ are Weyl spinors

$$\bar{\mathcal{T}}_i^{\,j} = (\mathcal{T}^\dagger)_i^j\gamma^0 = \left([(T_{\text{L}})_{\dot\alpha}^\dagger]_i^j \quad [(T_{\text{R}})^\alpha]_i^j\right)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \left([(T_{\text{R}})^\alpha]_i^j \quad [(T_{\text{L}})_{\dot\alpha}^\dagger]_i^j\right)$$

$$\bar{\mathcal{T}}_i^{\,j}\mathcal{T}_j^i = \left([(T_{\text{R}})^\alpha]_i^j \quad [(T_{\text{L}})_{\dot\alpha}^\dagger]_i^j\right)\begin{pmatrix} [(T_{\text{L}})_\alpha]_j^i \\ [(T_{\text{R}})^{\dagger\dot\alpha}]_j^i \end{pmatrix} = (T_{\text{R}})_i^j(T_{\text{L}})_j^i + (T_{\text{L}}^\dagger)_i^j(T_{\text{R}}^\dagger)_j^i = (T_{\text{R}})_i^j(T_{\text{L}})_j^i + h.c.$$

$$\mathcal{T}^+ = \begin{pmatrix} (T_{\text{L}}^+)_\alpha \\ (T_{\text{R}}^-)^{\dagger\dot\alpha} \end{pmatrix}, \quad \mathcal{T}^- = \begin{pmatrix} (T_{\text{L}}^-)_\alpha \\ (T_{\text{R}}^+)^{\dagger\dot\alpha} \end{pmatrix}, \quad \mathcal{T}^0 = \begin{pmatrix} (T_{\text{L}}^0)_\alpha \\ (T_{\text{R}}^0)^{\dagger\dot\alpha} \end{pmatrix}$$

$$\bar{\mathcal{T}}^+\mathcal{T}^+ = T_{\text{R}}^-T_{\text{L}}^+ + h.c., \quad \bar{\mathcal{T}}^-\mathcal{T}^- = T_{\text{R}}^+T_{\text{L}}^- + h.c., \quad \bar{\mathcal{T}}^0\mathcal{T}^0 = T_{\text{R}}^0T_{\text{L}}^0 + h.c.$$

$$i\bar{\mathcal{T}}^+\gamma^\mu\partial_\mu\mathcal{T}^+ = \left(T_{\text{R}}^- \quad (T_{\text{L}}^+)^\dagger\right)\begin{pmatrix} & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & \end{pmatrix}\begin{pmatrix} T_{\text{L}}^+ \\ (T_{\text{R}}^-)^\dagger \end{pmatrix} = i(T_{\text{L}}^+)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^+ + iT_{\text{R}}^-\sigma^\mu\partial_\mu(T_{\text{R}}^-)^\dagger$$
$$= i(T_{\text{L}}^+)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^+ + i\partial_\mu[T_{\text{R}}^-\sigma^\mu(T_{\text{R}}^-)^\dagger] - i(\partial_\mu T_{\text{R}}^-)\sigma^\mu(T_{\text{R}}^-)^\dagger \to i(T_{\text{L}}^+)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^+ + i(T_{\text{R}}^-)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{R}}^-$$

$$\mathcal{L}_{\text{DK}} = i(T_{\text{L}}^+)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^+ + i(T_{\text{R}}^-)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{R}}^- + i(T_{\text{L}}^-)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^- + i(T_{\text{R}}^+)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{R}}^+ + i(T_{\text{L}}^0)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{L}}^0 + i(T_{\text{R}}^0)^\dagger\bar\sigma^\mu\partial_\mu T_{\text{R}}^0$$

$$\mathcal{L}_{\text{DG}} = gW_\mu^a\bar{\mathcal{T}}\gamma^\mu t_{\text{T}}^a\mathcal{T} = \{gW_\mu^+[(T_{\text{L}}^+)^\dagger\bar\sigma^\mu T_{\text{L}}^0 + (T_{\text{R}}^+)^\dagger\bar\sigma^\mu T_{\text{R}}^0 - (T_{\text{L}}^0)^\dagger\bar\sigma^\mu T_{\text{L}}^- - (T_{\text{R}}^0)^\dagger\bar\sigma^\mu T_{\text{R}}^-] + h.c.\}$$
$$+ (eA_\mu + gc_{\text{W}}Z_\mu)[(T_{\text{L}}^+)^\dagger\bar\sigma^\mu T_{\text{L}}^+ + (T_{\text{R}}^+)^\dagger\bar\sigma^\mu T_{\text{R}}^+ - (T_{\text{L}}^-)^\dagger\bar\sigma^\mu T_{\text{L}}^- - (T_{\text{R}}^-)^\dagger\bar\sigma^\mu T_{\text{R}}^-]$$

Dirac triplet: $\quad \mathcal{L}_{\text{Dirac}} = i\bar{\mathcal{T}}\gamma^\mu D_\mu\mathcal{T} - m_{\mathcal{T}}\bar{\mathcal{T}}\mathcal{T}$

$$= \mathcal{L}_{\text{DG}} + i\bar{\mathcal{T}}^+\gamma^\mu\partial_\mu\mathcal{T}^+ + i\bar{\mathcal{T}}^-\gamma^\mu\partial_\mu\mathcal{T}^- + i\bar{\mathcal{T}}^0\gamma^\mu\partial_\mu\mathcal{T}^0 - m_{\mathcal{T}}(\bar{\mathcal{T}}^+\mathcal{T}^+ + \bar{\mathcal{T}}^-\mathcal{T}^- + \bar{\mathcal{T}}^0\mathcal{T}^0)$$

There are 2 singly charged Dirac femions and 1 neutral Dirac fermion

Majorana conditions: $T_{\mathrm{L}}^{+} = T_{\mathrm{R}}^{+} \equiv T^{+}, \quad T_{\mathrm{L}}^{-} = T_{\mathrm{R}}^{-} \equiv T^{-}, \quad T_{\mathrm{L}}^{0} = T_{\mathrm{R}}^{0} \equiv T^{0}$

$$\mathcal{T}^{+} = \begin{pmatrix} (T^{+})_{\alpha} \\ (T^{-})^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}^{-} = \begin{pmatrix} (T^{-})_{\alpha} \\ (T^{+})^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}^{0} = \begin{pmatrix} (T^{0})_{\alpha} \\ (T^{0})^{\dagger\dot{\alpha}} \end{pmatrix}$$

Charge conjugation matrix $\mathcal{C} = i\gamma^{0}\gamma^{2} = i\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & \sigma^{2} \\ -\sigma^{2} & \end{pmatrix} = \begin{pmatrix} -i\sigma^{2} & \\ & i\sigma^{2} \end{pmatrix} = \begin{pmatrix} \varepsilon_{\alpha\beta} & \\ & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}, \quad \mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{\mathrm{T}}, \quad \mathcal{C}^{-1}\gamma_{5}\mathcal{C} = \gamma_{5}$$

$$\overline{\mathcal{C}\bar{\psi}^{\mathrm{T}}} = (\mathcal{C}\bar{\psi}^{\mathrm{T}})^{\dagger}\gamma^{0} = [(\bar{\psi}\mathcal{C}^{\mathrm{T}})^{\dagger}]^{\mathrm{T}}\gamma^{0} = [(\psi^{\dagger}\gamma^{0}\mathcal{C}^{\mathrm{T}})^{\dagger}]^{\mathrm{T}}\gamma^{0} = (\mathcal{C}\gamma^{0}\psi)^{\mathrm{T}}\gamma^{0} = \psi^{\mathrm{T}}(\gamma^{0})^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\gamma^{0} = -\psi^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\gamma^{0}\gamma^{0} = \psi^{\mathrm{T}}\mathcal{C}$$

$$\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & \\ & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}\left[\begin{pmatrix} (T^{+})^{\dagger}_{\dot{\beta}} & (T^{-})^{\beta} \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right]^{\mathrm{T}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & \\ & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}\begin{pmatrix} (T^{-})^{\beta} \\ (T^{+})^{\dagger}_{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} (T^{-})_{\alpha} \\ (T^{+})^{\dagger\dot{\alpha}} \end{pmatrix} = \mathcal{T}^{-}, \quad \mathcal{C}(\bar{\mathcal{T}}^{0})^{\mathrm{T}} = \mathcal{T}^{0}$$

$$S_{(3)} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \quad \tilde{\mathcal{T}} \equiv S_{(3)}^{-1}\mathcal{C}(\bar{\mathcal{T}})^{\mathrm{T}} = \begin{pmatrix} \mathcal{C}(\bar{\mathcal{T}}^{-})^{\mathrm{T}} \\ \mathcal{C}(\bar{\mathcal{T}}^{0})^{\mathrm{T}} \\ \mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \mathcal{T}^{+} \\ \mathcal{T}^{0} \\ \mathcal{T}^{-} \end{pmatrix} = \mathcal{T}$$

Majorana conditions $\Leftrightarrow$ $\tilde{\mathcal{T}} = \mathcal{T}$

$$\bar{\mathcal{T}}\mathcal{T} = \bar{\mathcal{T}}^{+}\mathcal{T}^{+} + \bar{\mathcal{T}}^{-}\mathcal{T}^{-} + \bar{\mathcal{T}}^{0}\mathcal{T}^{0} = T_{\mathrm{R}}^{-}T_{\mathrm{L}}^{+} + T_{\mathrm{R}}^{+}T_{\mathrm{L}}^{-} + T_{\mathrm{R}}^{0}T_{\mathrm{L}}^{0} + h.c. = 2T^{-}T^{+} + T^{0}T^{0} + h.c.$$

$$i\bar{\mathcal{T}}\gamma^{\mu}\partial_{\mu}\mathcal{T} = i\bar{\mathcal{T}}^{+}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{+} + i\bar{\mathcal{T}}^{-}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{-} + i\bar{\mathcal{T}}^{0}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{0} = 2[i(T^{+})^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}T^{+} + i(T^{-})^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}T^{-} + i(T^{0})^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}T^{0}]$$

$$\mathcal{L}_{\mathrm{MG}} = \frac{1}{2}\mathcal{L}_{\mathrm{DG}} = \{gW_{\mu}^{+}[(T^{+})^{\dagger}\bar{\sigma}^{\mu}T^{0} - (T^{0})^{\dagger}\bar{\sigma}^{\mu}T^{-}] + h.c.\} + (eA_{\mu} + gc_{\mathrm{W}}Z_{\mu})[(T^{+})^{\dagger}\bar{\sigma}^{\mu}T^{+} - (T^{-})^{\dagger}\bar{\sigma}^{\mu}T^{-}]$$

$$\bar{\mathcal{T}}^{-}\mathcal{T}^{-} = \overline{\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}}}\,\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} = (\mathcal{T}^{+})^{\mathrm{T}}\mathcal{C}\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} = -(\mathcal{T}^{+})^{\mathrm{T}}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} = \bar{\mathcal{T}}^{+}\mathcal{T}^{+}$$

$$\begin{aligned} i\bar{\mathcal{T}}^{-}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{-} &= i\overline{\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}}}\gamma^{\mu}\partial_{\mu}[\mathcal{C}(\bar{\mathcal{T}}^{+})^{\mathrm{T}}] = i(\mathcal{T}^{+})^{\mathrm{T}}\mathcal{C}\gamma^{\mu}\mathcal{C}\partial_{\mu}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} = i(\mathcal{T}^{+})^{\mathrm{T}}(\gamma^{\mu})^{\mathrm{T}}\partial_{\mu}(\bar{\mathcal{T}}^{+})^{\mathrm{T}} \\ &= -i(\partial_{\mu}\bar{\mathcal{T}}^{+})\gamma^{\mu}\mathcal{T}^{+} = -i\partial_{\mu}(\bar{\mathcal{T}}^{+}\gamma^{\mu}\mathcal{T}^{+}) + i\bar{\mathcal{T}}^{+}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{+} \to i\bar{\mathcal{T}}^{+}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{+} \end{aligned}$$

Majorana triplet: $\mathcal{L}_{\mathrm{Majorana}} = \frac{i}{2}\bar{\mathcal{T}}\gamma^{\mu}D_{\mu}\mathcal{T} - \frac{1}{2}m_{\mathcal{T}}\bar{\mathcal{T}}\mathcal{T}$

$$= \mathcal{L}_{\mathrm{MG}} + i\bar{\mathcal{T}}^{+}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{+} - m_{\mathcal{T}}\bar{\mathcal{T}}^{+}\mathcal{T}^{+} + \frac{1}{2}(i\bar{\mathcal{T}}^{0}\gamma^{\mu}\partial_{\mu}\mathcal{T}^{0} + m_{\mathcal{T}}\bar{\mathcal{T}}^{0}\mathcal{T}^{0})$$

There are 1 singly charged Dirac femion and 1 Majorana fermion

Two left-handed Weyl triplets $T_1 = \begin{pmatrix} T_1^+ \\ T_1^0 \\ T_1^- \end{pmatrix} \in (\mathbf{3},0), \quad T_2 = \begin{pmatrix} T_2^+ \\ T_2^0 \\ T_2^- \end{pmatrix} \in (\mathbf{3},0)$

$$\mathcal{L}_{\mathrm{T}} = iT_1^\dagger \bar{\sigma}^\mu D_\mu T_1 + iT_2^\dagger \bar{\sigma}^\mu D_\mu T_2 - \left(\frac{1}{2} m_1 T_1 T_1 + \frac{1}{2} m_2 T_2 T_2 + m_{12} T_1 T_2 + h.c. \right)$$

$$T_I T_J = (T_I)_i^j (T_I)_j^i = T_I^- T_J^+ + T_I^+ T_J^- + T_I^0 T_J^0$$

$$\mathcal{L}_{\mathrm{WK}} = \sum_{I=1}^{2} iT_I^\dagger \bar{\sigma}^\mu \partial_\mu T_I = \sum_{I=1}^{2} i(T_I^\dagger)_i^j \bar{\sigma}^\mu \partial_\mu (T_I)_j^i = \sum_{I=1}^{2} [i(T_I^+)^\dagger \bar{\sigma}^\mu \partial_\mu T_I^+ + i(T_I^-)^\dagger \bar{\sigma}^\mu \partial_\mu T_I^- + i(T_I^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_I^0]$$

$$\mathcal{L}_{\mathrm{WG}} = \sum_{I=1}^{2} gW_\mu^a T_I^\dagger \bar{\sigma}^\mu t_{\mathrm{T}}^a T_I = \sum_{I=1}^{2} \{ [gW_\mu^+ (T_I^+)^\dagger \bar{\sigma}^\mu T_I^0 - gW_\mu^+ (T_I^0)^\dagger \bar{\sigma}^\mu T_I^- + h.c.] + (eA_\mu + gc_{\mathrm{W}} Z_\mu)[(T_I^+)^\dagger \bar{\sigma}^\mu T_I^+ - (T_I^-)^\dagger \bar{\sigma}^\mu T_I^-] \}$$

1) $m_1 = m_2 = 0$

$$T_{\mathrm{L}}^+ = T_1^+, \quad T_{\mathrm{R}}^- = T_2^-, \quad T_{\mathrm{R}}^+ = T_2^+, \quad T_{\mathrm{L}}^- = T_1^-, \quad T_{\mathrm{L}}^0 = T_1^0, \quad T_{\mathrm{R}}^0 = T_2^0$$

$$\mathcal{T}^+ = \begin{pmatrix} (T_1^+)_\alpha \\ (T_2^-)^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}^- = \begin{pmatrix} (T_1^-)_\alpha \\ (T_2^+)^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}^0 = \begin{pmatrix} (T_1^0)_\alpha \\ (T_2^0)^{\dagger\dot{\alpha}} \end{pmatrix}$$

$$-m_{12} T_1 T_2 = -m_{12}(T_1^- T_2^+ + T_1^+ T_2^- + T_1^0 T_2^0) = -m_{12}(T_{\mathrm{L}}^- T_{\mathrm{R}}^+ + T_{\mathrm{L}}^+ T_{\mathrm{R}}^- + T_{\mathrm{L}}^0 T_{\mathrm{R}}^0) = -m_{12} \bar{\mathcal{T}} \mathcal{T}$$

$$\mathcal{L}_{\mathrm{WK}} = \mathcal{L}_{\mathrm{DK}}, \quad \mathcal{L}_{\mathrm{WG}} = \mathcal{L}_{\mathrm{DG}}$$

$$\mathcal{L}_{\mathrm{T}} = iT_1^\dagger \bar{\sigma}^\mu D_\mu T_1 + iT_2^\dagger \bar{\sigma}^\mu D_\mu T_2 - (m_{12} T_1 T_2 + h.c.) = i\bar{\mathcal{T}} \gamma^\mu D_\mu \mathcal{T} - m_{12} \bar{\mathcal{T}} \mathcal{T}$$

$\mathcal{T}$ is a Dirac triplet

2) $m_{12} = 0$

$$\mathcal{T}_I^+ = \begin{pmatrix} (T_I^+)_\alpha \\ (T_I^-)^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}_I^- = \begin{pmatrix} (T_I^-)_\alpha \\ (T_I^+)^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \mathcal{T}_I^0 = \begin{pmatrix} (T_I^0)_\alpha \\ (T_I^0)^{\dagger\dot{\alpha}} \end{pmatrix}$$

$$-\frac{1}{2} m_I T_I T_I = -\frac{1}{2} m_I (2T_I^- T_I^+ + T_I^0 T_I^0) = -\frac{1}{2} m_I \bar{\mathcal{T}}_I \mathcal{T}_I, \quad iT_I^\dagger \bar{\sigma}^\mu \partial_\mu T_I = \frac{i}{2} \bar{\mathcal{T}}_I \gamma^\mu \partial_\mu \mathcal{T}_I$$

$$\mathcal{L}_{\mathrm{T}} = iT_1^\dagger \bar{\sigma}^\mu D_\mu T_1 + iT_2^\dagger \bar{\sigma}^\mu D_\mu T_2 - \frac{1}{2}(m_1 T_1 T_1 + m_2 T_2 T_2 + h.c.) = \frac{1}{2} \sum_{I=1}^{2} (i\bar{\mathcal{T}}_I \gamma^\mu D_\mu \mathcal{T}_I - m_I \bar{\mathcal{T}}_I \mathcal{T}_I)$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are two Majorana triplets

3) $m_{12} = 0, \quad m_1 = m_2 \equiv m$

$$\mathcal{T}^0 = \begin{pmatrix} (T_{\mathrm{L}}^0)_\alpha \\ (T_{\mathrm{R}}^0)^{\dagger\dot{\alpha}} \end{pmatrix}, \quad T_{\mathrm{L}}^0 = \frac{1}{\sqrt{2}}(T_1^0 + iT_2^0), \quad T_{\mathrm{R}}^0 = \frac{1}{\sqrt{2}}(T_1^0 - iT_2^0), \quad T_{\mathrm{R}}^0 T_{\mathrm{L}}^0 = \frac{1}{2}(T_1^0 T_1^0 + T_2^0 T_2^0)$$

$$-m T_{\mathrm{R}}^0 T_{\mathrm{L}}^0 = -\frac{m}{2}(T_1^0 T_1^0 + T_2^0 T_2^0)$$

$$\begin{aligned} i(T_{\mathrm{L}}^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_{\mathrm{L}}^0 + i(T_{\mathrm{R}}^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_{\mathrm{R}}^0 &= \frac{i}{2}(T_1^0 + iT_2^0)^\dagger \bar{\sigma}^\mu \partial_\mu (T_1^0 + iT_2^0) + \frac{i}{2}(T_1^0 - iT_2^0)^\dagger \bar{\sigma}^\mu \partial_\mu (T_1^0 - iT_2^0) \\ &= i(T_1^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_1^0 + i(T_2^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_2^0 \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{T}} &= iT_1^\dagger \bar{\sigma}^\mu D_\mu T_1 + iT_2^\dagger \bar{\sigma}^\mu D_\mu T_2 - \frac{m}{2}(T_1 T_1 + T_2 T_2 + h.c.) \\ &\supset i(T_{\mathrm{L}}^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_{\mathrm{L}}^0 + i(T_{\mathrm{R}}^0)^\dagger \bar{\sigma}^\mu \partial_\mu T_{\mathrm{R}}^0 - (m T_{\mathrm{R}}^0 T_{\mathrm{L}}^0 + h.c.) = i\bar{\mathcal{T}}^0 \gamma^\mu \partial_\mu \mathcal{T}^0 - m \bar{\mathcal{T}}^0 \mathcal{T}^0 \end{aligned}$$

$\mathcal{T}^0$ is a neutral Dirac fermion

Passarino-Veltman scalar functions

D-dim one-loop integrals defined by Denner, 0709.1075:

$$T^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1}) = F\int d^Dq\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0D_1\cdots D_{N-1}},\quad F\equiv\frac{(2\pi\mu)^{4-D}}{i\pi^2}$$

N = number of propagator factors in the denominator, P = number of integration momenta in the numerator

$$D_0 = q^2-m_0^2+i\varepsilon,\quad D_i=(q+p_i)^2-m_i^2+i\varepsilon,\quad i=1,\cdots,N-1,\quad \varepsilon=\frac{4-D}{2},\quad D=4-2\varepsilon$$

These integrals give rise to a UV-divergent term $\Delta = \frac{1}{\varepsilon}-\gamma_{\mathrm{E}}+\log 4\pi$

Subtracting Δ corresponds to the $\overline{\mathrm{MS}}$ scheme

$$(2\pi)^{4-D}=(2\pi)^{2\varepsilon}=[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)],\quad F=\frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)]$$

Conventionally, $T^1\equiv A,\quad T^2\equiv B,\quad T^3\equiv C,\quad \cdots$

$$A(m_0^2)=A_0(m_0^2),\quad B(p_1^2,m_0^2,m_1^2)=B_0(p_1^2,m_0^2,m_1^2),\quad B_\mu(p_1^2,m_0^2,m_1^2)=p_{1\mu}B_1(p_1^2,m_0^2,m_1^2)$$

$$B_{\mu\nu}(p_1^2,m_0^2,m_1^2)=g_{\mu\nu}B_{00}(p_1^2,m_0^2,m_1^2)+p_{1\mu}p_{1\nu}B_{11}(p_1^2,m_0^2,m_1^2)$$

$$A_0(m^2)\sim m^2\Delta,\quad B_0(p^2,m_1^2,m_2^2)\sim\Delta,\quad B_1(p^2,m_1^2,m_2^2)\sim-\frac{1}{2}\Delta,\quad B_{00}(p^2,m_1^2,m_2^2)\sim-\frac{1}{12}(p^2-3m_1^2-3m_2^2)\Delta$$

D-dim one-loop integrals defined by LoopTools User's Guide:

$$T'^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1}) = F'\int d^Dq\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0D_1\cdots D_{N-1}},\quad F'\equiv\frac{\mu^{4-D}}{i\pi^{D/2}r_\Gamma},\quad r_\Gamma=\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{1}{r_\Gamma}=1+\gamma_{\mathrm{E}}\varepsilon+\mathcal{O}(\varepsilon^2),\quad \frac{1}{\pi^{D/2}}=\frac{1}{\pi^{2-\varepsilon}}=\frac{1}{\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$F'=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)][1+\gamma_{\mathrm{E}}\varepsilon+\mathcal{O}(\varepsilon^2)]=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\gamma_{\mathrm{E}}\varepsilon+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$\frac{T'^N_{\mu_1\cdots\mu_P}}{T^N_{\mu_1\cdots\mu_P}}=\frac{F'}{F}=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}\frac{i\pi^2}{(2\pi\mu)^{2\varepsilon}}=\frac{1}{(2\pi)^{2\varepsilon}\pi^{-\varepsilon}r_\Gamma}=1+(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon^2)$$

$$\Delta'=\frac{F'}{F}\Delta=\Delta+\Delta(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}-\gamma_{\mathrm{E}}+\log 4\pi+\frac{1}{\varepsilon}(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}+\mathcal{O}(\varepsilon)$$

The UV-divergent term from $T'^N_{\mu_1\cdots\mu_P}$ is Δ', and subtracting Δ' corresponds to the $\overline{\mathrm{MS}}$ scheme

$$\frac{1}{K^{2-D/2}}=1-\frac{4-D}{2}\ln K+\mathcal{O}((4-D)^2),\quad \Gamma(2-D/2)=\frac{2}{4-D}-\gamma_{\mathrm{E}}+\mathcal{O}(4-D)$$

$$\frac{1}{(4\pi)^{D/2}}=\frac{1}{(4\pi)^2(4\pi)^{(D-4)/2}}=\frac{1}{16\pi^2}(4\pi)^{2-D/2}=\frac{1}{16\pi^2}\left[1+\frac{4-D}{2}\ln 4\pi+\mathcal{O}((4-D)^2)\right]$$

$$\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}}=\frac{1}{16\pi^2}\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln K+\mathcal{O}(4-D)\right]$$

$$\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2-K)^n}=\frac{(-1)^n i}{(4\pi)^{D/2}}\frac{\Gamma(n-D/2)}{\Gamma(n)}\frac{1}{K^{n-D/2}}$$

$$\Gamma(2-D/2)=(1-D/2)\Gamma(1-D/2),\quad \Gamma(1)=1,\quad \frac{1}{1-(4-D)/2}=1+\frac{4-D}{2}+\mathcal{O}((4-D)^2)$$

$$\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2-K}=\frac{-i}{(4\pi)^{D/2}}\frac{\Gamma(1-D/2)}{\Gamma(1)}\frac{1}{K^{1-D/2}}=\frac{-iK}{1-D/2}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}}$$

$$=\frac{iK}{1-(4-D)/2}\frac{1}{16\pi^2}\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln K+\mathcal{O}(4-D)\right]$$

$$=\frac{iK}{16\pi^2}\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln K+1+\mathcal{O}(4-D)\right]$$

$$\mu^{4-D}=1+\frac{4-D}{2}\ln\mu^2+\mathcal{O}((4-D)^2)$$

$$A_0(m^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{1}{q^2-m^2+i\varepsilon}=-16i\pi^2\mu^{4-D}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2-(m^2-i\varepsilon)}$$

$$=(m^2-i\varepsilon)\mu^{4-D}\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln(m^2-i\varepsilon)+1+\mathcal{O}(4-D)\right]=m^2\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi+\ln\mu^2-\ln m^2+1+\mathcal{O}(4-D)\right]$$

$$=m^2\left(\Delta+1-\ln\frac{m^2}{\mu^2}\right)+\mathcal{O}(4-D)$$

$$\Delta\equiv\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi$$

$$x(q^2-m_1^2+i\varepsilon)+(1-x)[(q+p)^2-m_2^2+i\varepsilon]=q^2+2(1-x)q\cdot p+(1-x)p^2-xm_1^2-(1-x)m_2^2+i\varepsilon$$

$$=[q+(1-x)p]^2+x(1-x)p^2-xm_1^2-(1-x)m_2^2+i\varepsilon$$

$$=\ell^2-K$$

$$\ell\equiv q+(1-x)p,\quad K=-x(1-x)p^2+xm_1^2+(1-x)m_2^2-i\varepsilon=x^2p^2-x(p^2-m_1^2+m_2^2)+m_2^2-i\varepsilon$$

$$\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}=\int_0^1dx\frac{1}{\{x[q^2-m_1^2+i\varepsilon]+(1-x)[(q+p)^2-m_2^2+i\varepsilon]\}^2}=\int_0^1dx\frac{1}{(\ell^2-K)^2}$$

$$\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2}=i\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}}=\frac{i}{16\pi^2}\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln K+\mathcal{O}(4-D)\right]$$

$$B_0(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}=\frac{16\pi^2\mu^{4-D}}{i}\int_0^1dx\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2}$$

$$=\mu^{4-D}\int_0^1dx\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi-\ln K+\mathcal{O}(4-D)\right]=\int_0^1dx\left[\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln 4\pi+\ln\mu^2-\ln K+\mathcal{O}(4-D)\right]$$

$$=\Delta-\int_0^1dx\ln\frac{x^2p^2-x(p^2-m_1^2+m_2^2)+m_2^2-i\varepsilon}{\mu^2}+\mathcal{O}(4-D)$$

$$x^2p^2-x(p^2-m_1^2+m_2^2)+m_2^2-i\varepsilon=p^2(x-x_+)(x-x_-)$$

$$x_\pm\equiv\frac{1}{2p^2}\left[p^2-m_1^2+m_2^2\pm\sqrt{(p^2-m_1^2+m_2^2)^2-4p^2(m_2^2-i\varepsilon)}\right]$$

$$\int_0^1dx\ln\frac{x^2p^2-x(p^2-m_1^2+m_2^2)+m_2^2-i\varepsilon}{\mu^2}=\int_0^1dx\ln\frac{p^2(x-x_+)(x-x_-)}{\mu^2}=\ln\frac{p^2}{\mu^2}+\int_0^1dx[\ln(x-x_+)+\ln(x-x_-)]$$

$$=\ln\frac{p^2}{\mu^2}+[(x-x_+)\ln(x-x_+)-x+(x-x_-)\ln(x-x_-)-x]\Big|_0^1=\ln\frac{p^2}{\mu^2}+f(x_+)+f(x_-)$$

$$f(z)\equiv(1-z)\ln(1-z)+z\ln(-z)-1=\ln(1-z)-z\ln(1-z^{-1})-1$$

$$B_0(p^2,m_1^2,m_2^2)=\Delta-\ln\frac{p^2}{\mu^2}-f(x_+)-f(x_-)+\mathcal{O}(4-D)$$

$$p_\mu B_1(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{q_\mu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$p^2B_1(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$
$$2q\cdot p=[(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-p^2-m_1^2+m_2^2$$
$$\frac{2q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}=\frac{1}{q^2-m_1^2+i\varepsilon}-\frac{1}{(q+p)^2-m_2^2+i\varepsilon}-\frac{p^2+m_1^2-m_2^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$
$$2p^2B_1(p^2,m_1^2,m_2^2)=A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)$$

$$B_1(p^2,m_1^2,m_2^2)=\frac{1}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$g_{\mu\nu}B_{00}(p^2,m_1^2,m_2^2)+p_\mu p_\nu B_{11}(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{q_\mu q_\nu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$DB_{00}(p^2,m_1^2,m_2^2)+p^2B_{11}(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$
$$q^2=(q^2-m_1^2+i\varepsilon)+m_1^2-i\varepsilon$$
$$\frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}=\frac{1}{(q+p)^2-m_2^2+i\varepsilon}+\frac{m_1^2-i\varepsilon}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$
$$DB_{00}(p^2,m_1^2,m_2^2)+p^2B_{11}(p^2,m_1^2,m_2^2)=A_0(m_2^2)+m_1^2B_0(p^2,m_1^2,m_2^2)$$
$$4Dp^2B_{00}(p^2,m_1^2,m_2^2)+4p^4B_{11}(p^2,m_1^2,m_2^2)=4p^2A_0(m_2^2)+4p^2m_1^2B_0(p^2,m_1^2,m_2^2)$$

$$p^2B_{00}(p^2,m_1^2,m_2^2)+p^4B_{11}(p^2,m_1^2,m_2^2)=\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\frac{(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$
$$\begin{aligned}4(q\cdot p)^2&=\{[(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-[p^2+m_1^2-m_2^2]\}^2\\&=[(q+p)^2-m_2^2+i\varepsilon]\{[(q+p)^2-m_2^2+i\varepsilon]-2[q^2-m_1^2+i\varepsilon]-2[p^2+m_1^2-m_2^2]\}\\&\quad+[q^2-m_1^2+i\varepsilon]\{[q^2-m_1^2+i\varepsilon]+2[p^2+m_1^2-m_2^2]\}+[p^2+m_1^2-m_2^2]^2\\&=[(q+p)^2-m_2^2+i\varepsilon](-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon)+(q^2-m_1^2+i\varepsilon)(q^2+2p^2+m_1^2-2m_2^2+i\varepsilon)+(p^2+m_1^2-m_2^2)^2\end{aligned}$$
$$\begin{aligned}\frac{4(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}&=\frac{-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon}{q^2-m_1^2+i\varepsilon}+\frac{q^2+2p^2+m_1^2-2m_2^2+i\varepsilon}{(q+p)^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}\\&=-1+\frac{2q\cdot p-p^2-m_1^2+m_2^2}{q^2-m_1^2+i\varepsilon}+1+\frac{-2q\cdot p+p^2+m_1^2-m_2^2}{(q+p)^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}\\&=\frac{2q\cdot p-(p^2+m_1^2-m_2^2)}{q^2-m_1^2+i\varepsilon}+\frac{-2q'\cdot p+3p^2+m_1^2-m_2^2}{q'^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}\end{aligned}$$
$$q'=q+p$$
$$4p^2B_{00}(p^2,m_1^2,m_2^2)+4p^4B_{11}(p^2,m_1^2,m_2^2)=-(p^2+m_1^2-m_2^2)A_0(m_1^2)+(3p^2+m_1^2-m_2^2)A_0(m_2^2)+(p^2+m_1^2-m_2^2)^2B_0(p^2,m_1^2,m_2^2)$$

$$4(D-1)p^2 B_{00}(p^2,m_1^2,m_2^2) = 4p^2 A_0(m_2^2) + 4p^2 m_1^2 B_0(p^2,m_1^2,m_2^2)$$
$$+(p^2+m_1^2-m_2^2)A_0(m_1^2) - (3p^2+m_1^2-m_2^2)A_0(m_2^2) - (p^2+m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$
$$= p^2 A_0(m_1^2) + p^2 A_0(m_2^2) + 2p^2(m_1^2+m_2^2)B_0(p^2,m_1^2,m_2^2) - p^4 B_0(p^2,m_1^2,m_2^2)$$
$$+(m_1^2-m_2^2)A_0(m_1^2) - (m_1^2-m_2^2)A_0(m_2^2) - (m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$

$$[1-(4-D)/3]6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$\frac{1}{1-(4-D)/3} = 1+\frac{1}{3}(4-D)+\mathcal{O}((4-D)^2)$$

$$6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{1-(4-D)/3}\left\{\frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left.+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$
$$= \frac{1}{3}(m_1^2+m_2^2) + \frac{2}{3}\left(m_1^2+m_2^2-\frac{1}{2}p^2\right) + \frac{m_1^2-m_2^2}{3p^2}[m_1^2-m_2^2-(m_1^2-m_2^2)]$$
$$+\frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)] + \mathcal{O}(4-D)$$

$$B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{6}\left\{m_1^2+m_2^2-\frac{1}{3}p^2 + \frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left.+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$