Electroweak oblique parameters

Ref: Peskin & Takeuchi, PRD **46**, 381 (1992)

Vacuum polarization amplitude of gauge bosons I and J:

$$i\Pi_{IJ}^{\mu\nu}(p) = ig^{\mu\nu}\Pi_{IJ}(p^2) + (p^\mu p^\nu \text{ terms})$$

$$\Pi_{IJ}(p^2) \equiv \Pi_{IJ}(0) + p^2\Pi'_{IJ}(p^2) = \Pi_{IJ}(0) + p^2\Pi'_{IJ}(0) + \mathcal{O}(p^4), \quad \Pi'_{IJ}(0) = \frac{\partial\Pi_{IJ}(p^2)}{\partial p^2}\bigg|_{p^2=0}$$

$$\Pi_{AA}(p^2) = e^2\Pi_{QQ}(p^2), \quad \Pi_{ZA}(p^2) = \frac{e^2}{s_W c_W}[\Pi_{3Q}(p^2) - s_W^2\Pi_{QQ}(p^2)]$$

$$\Pi_{ZZ}(p^2) = \frac{e^2}{s_W^2 c_W^2}[\Pi_{33}(p^2) - 2s_W^2\Pi_{3Q}(p^2) + s_W^4\Pi_{QQ}(p^2)], \quad \Pi_{WW}(p^2) = \frac{e^2}{s_W^2}\Pi_{11}(p^2)$$

QED Ward identity $\Rightarrow$ $\Pi_{3Q}(0) = \Pi_{QQ}(0) = 0, \quad \Pi_{AA}(0) = \Pi_{ZA}(0) = 0$

$$S \equiv \frac{4e^2}{\alpha}[\Pi'_{33}(0) - \Pi'_{3Q}(0)] = \frac{4s_W^2 c_W^2}{\alpha}\left[\Pi'_{ZZ}(0) - \frac{c_W^2 - s_W^2}{s_W c_W}\Pi'_{ZA}(0) - \Pi'_{AA}(0)\right]$$

$$T \equiv \frac{e^2}{\alpha s_W^2 c_W^2 m_Z^2}[\Pi_{11}(0) - \Pi_{33}(0)] = \frac{1}{\alpha}\left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2}\right]$$

$$U \equiv \frac{4e^2}{\alpha}[\Pi'_{11}(0) - \Pi'_{33}(0)] = \frac{4s_W^2}{\alpha}\left[\Pi'_{WW}(0) - c_W^2\Pi'_{ZZ}(0) - 2s_W c_W\Pi'_{ZA}(0) - s_W^2\Pi'_{AA}(0)\right]$$

$$c_W \equiv \cos\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad s_W \equiv \sin\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}} = gs_W = g'c_W, \quad m_W = m_Z c_W$$

Fan, Reece & Wang, 1411.1054:

S is contributed by $SU(2)_L$ multiplets that are split by the electroweak symmetry breaking

T and U are contributed by $SU(2)_L$ multiplets that are split by the electroweak symmetry
breaking with custodial symmetry [firstly called in Sikivie et al., NPB **173**, 189 (1980)] violating effects

EFT viewpoint [Zhenyu Han, 0807.0490]

$$\frac{1}{\Lambda^2}H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu} \to S, \quad \frac{1}{\Lambda^2}H^\dagger D_\mu H(D^\mu H)^\dagger H \to T, \quad \frac{1}{\Lambda^4}H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu}\sigma^b H \to U$$

U corresponds to higher dimensional operators and is typically much smaller than S and T

<u>$H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu}$ operator</u>

$$H \to \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \quad W_{\mu\nu}^3 = s_W A_{\mu\nu} + c_W Z_{\mu\nu} - g W_\mu^1 W_\nu^2 + g W_\mu^2 W_\nu^1, \quad B_{\mu\nu} = c_W A_{\mu\nu} - s_W Z_{\mu\nu}$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$H^\dagger W_{\mu\nu}^a \sigma^a H \supset H^\dagger W_{\mu\nu}^3 \sigma^3 H \to \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}W_{\mu\nu}^3 = -\frac{v^2}{2}W_{\mu\nu}^3$$

$$\mathcal{O}_{WB} = a_{WB} H^\dagger W_{\mu\nu}^a \sigma^a H B^{\mu\nu} \to -\frac{a_{WB}v^2}{2}W_{\mu\nu}^3 B^{\mu\nu} \supset -\frac{a_{WB}v^2}{2}(s_W A_{\mu\nu} + c_W Z_{\mu\nu})(c_W A^{\mu\nu} - s_W Z^{\mu\nu})$$

$$= -\frac{a_{WB}v^2}{2}[s_W c_W(A_{\mu\nu}A^{\mu\nu} - Z_{\mu\nu}Z^{\mu\nu}) + (c_W^2 - s_W^2)Z_{\mu\nu}A^{\mu\nu}]$$

$\left[\text{Note: Feynman rule for momenta pointing into the vertex :} \quad \partial_\mu \to -ip_\mu \right]$

Feynman rule for $Z_\mu(p) -\times- Z_\nu(p)$ from $Z_{\mu\nu}Z^{\mu\nu}$

$$Z_{\mu\nu}Z^{\mu\nu} = 2(\partial_\mu Z_\nu \partial^\mu Z^\nu - \partial_\mu Z_\nu \partial^\nu Z^\mu)$$

$$\to 2[g^{\rho\sigma}g^{\mu\nu}\partial_\rho Z_\mu(p)\partial_\sigma Z_\nu(p) + g^{\rho\sigma}g^{\nu\mu}\partial_\rho Z_\nu(p)\partial_\sigma Z_\mu(p)$$

$$- g^{\rho\nu}g^{\mu\sigma}\partial_\rho Z_\mu(p)\partial_\sigma Z_\nu(p) - g^{\rho\mu}g^{\nu\sigma}\partial_\rho Z_\nu(p)\partial_\sigma Z_\mu(p)]$$

$$\to 2i[(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})(-ip_\rho)(ip_\sigma) + (g^{\rho\sigma}g^{\nu\mu} - g^{\rho\mu}g^{\nu\sigma})(ip_\rho)(-ip_\sigma)] = 4i(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

Contribution to $\Pi_{ZZ}^{\mu\nu}$ from $\mathcal{O}_{WB}$:

$$i\Pi_{ZZ}^{\mu\nu}(p^2) = \frac{s_W c_W a_{WB}v^2}{2}4i(g^{\mu\nu}p^2 - p^\mu p^\nu) = 2is_W c_W a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{ZZ}(p^2) = 2s_W c_W a_{WB}v^2 p^2 \implies \Pi_{ZZ}'(0) = 2s_W c_W a_{WB}v^2$$

Feynman rule for $A_\mu(p) -\times- A_\nu(p)$ from $A_{\mu\nu}A^{\mu\nu}$: $\quad A_{\mu\nu}A^{\mu\nu} \to 4i(g^{\mu\nu}p^2 - p^\mu p^\nu)$

Contribution to $\Pi_{AA}^{\mu\nu}$ from $\mathcal{O}_{WB}$:

$$i\Pi_{AA}^{\mu\nu}(p^2) = -\frac{s_W c_W a_{WB}v^2}{2}4i(g^{\mu\nu}p^2 - p^\mu p^\nu) = -2is_W c_W a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{AA}(p^2) = -2s_W c_W a_{WB}v^2 p^2 \implies \Pi_{AA}'(0) = -2s_W c_W a_{WB}v^2$$

Feynman rule for $Z_\mu(p) -\times- A_\nu(p)$ from $Z_{\mu\nu}A^{\mu\nu}$:

$$Z_{\mu\nu}A^{\mu\nu} = 2(\partial_\mu Z_\nu \partial^\mu A^\nu - \partial_\mu Z_\nu \partial^\nu A^\mu) \to 2[g^{\rho\sigma}g^{\mu\nu}\partial_\rho Z_\mu(p)\partial_\sigma A_\nu(p) - g^{\rho\nu}g^{\mu\sigma}\partial_\rho Z_\mu(p)\partial_\sigma A_\nu(p)]$$

$$\to 2i(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})(-ip_\rho)(ip_\sigma) = 2i(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

Contribution to $\Pi_{ZA}^{\mu\nu}$ from $\mathcal{O}_{WB}$:

$$i\Pi_{ZA}^{\mu\nu}(p^2) = -\frac{a_{WB}v^2}{2}(c_W^2 - s_W^2)2i(g^{\mu\nu}p^2 - p^\mu p^\nu) = -i(c_W^2 - s_W^2)a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{ZA}(p^2) = -(c_W^2 - s_W^2)a_{WB}v^2 p^2 \implies \Pi_{ZA}'(0) = -(c_W^2 - s_W^2)a_{WB}v^2$$

$$S = \frac{4s_W^2 c_W^2}{\alpha}\left[\Pi_{ZZ}'(0) - \frac{c_W^2 - s_W^2}{s_W c_W}\Pi_{ZA}'(0) - \Pi_{AA}'(0)\right] = \frac{4s_W^2 c_W^2}{\alpha}a_{WB}v^2\left[2s_W c_W + \frac{(c_W^2 - s_W^2)^2}{s_W c_W} + 2s_W c_W\right] = \frac{4s_W c_W}{\alpha}a_{WB}v^2$$

$$T = \frac{1}{\alpha}\left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2}\right] = -\frac{\Pi_{ZZ}(0)}{\alpha m_Z^2} = 0$$

$$U = \frac{4s_W^2}{\alpha}\left[\Pi_{WW}'(0) - c_W^2\Pi_{ZZ}'(0) - 2s_W c_W\Pi_{ZA}'(0) - s_W^2\Pi_{AA}'(0)\right] = \frac{4s_W^2}{\alpha}a_{WB}v^2\left[-2s_W c_W^3 + 2s_W c_W(c_W^2 - s_W^2) + 2s_W^3 c_W\right] = 0$$

$H^\dagger D_\mu H (D^\mu H)^\dagger H$ operator

$$D_\mu H = (\partial_\mu - ig'B_\mu Y_H - igW_\mu^a t_{\rm D}^a)H, \quad Y_H = \frac{1}{2}, \quad t_{\rm D}^a = \frac{\sigma^a}{2}$$

$$D_\mu H \supset -\frac{i}{2}\begin{pmatrix} g'B_\mu + gW_\mu^3 & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & g'B_\mu - gW_\mu^3 \end{pmatrix}H \to -\frac{iv}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}gW_\mu^+ \\ g'B_\mu - gW_\mu^3 \end{pmatrix}, \quad H^\dagger D_\mu H \to -\frac{iv}{4}(g'B_\mu - gW_\mu^3)$$

$$g'B_\mu - gW_\mu^3 = -\sqrt{g^2 + g'^2}\,Z_\mu, \quad m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$$

$$\mathcal{O}_h = a_h H^\dagger D_\mu H (D^\mu H)^\dagger H \to \frac{a_h v^2}{16}(g'B_\mu - gW_\mu^3)^2 = \frac{a_h v^2}{16}(g^2 + g'^2)Z_\mu Z^\mu = \frac{a_h v^2}{4}m_Z^2 Z_\mu Z^\mu$$

Contribution to $\Pi_{ZZ}^{\mu\nu}$ from $\mathcal{O}_{WB}$: $\quad i\Pi_{ZZ}^{\mu\nu}(p^2) = i\frac{a_h v^2}{2}m_Z^2 g^{\mu\nu}$

$$\Pi_{ZZ}(p^2) = \frac{a_h v^2}{2}m_Z^2$$

$$S = \frac{4s_{\rm W}^2 c_{\rm W}^2}{\alpha}\Pi_{ZZ}'(0) = 0, \quad T = -\frac{1}{\alpha}\frac{1}{m_Z^2}\frac{a_h v^2}{2}m_Z^2 = -\frac{1}{2\alpha}a_h v^2, \quad U = -\frac{4s_{\rm W}^2}{\alpha}c_{\rm W}^2\Pi_{ZZ}'(0) = 0$$

$H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu}\sigma^b H$ operator

$$H^\dagger W_{\mu\nu}^a \sigma^a H \to -\frac{v^2}{2}W_{\mu\nu}^3$$

$$\mathcal{O}_{WW} = a_{WW} H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu}\sigma^b H \to \frac{a_{WW}v^4}{4}W_{\mu\nu}^3 W^{3,\mu\nu}$$

$$= \frac{a_{WW}v^4}{4}(s_{\rm W}A_{\mu\nu} + c_{\rm W}Z_{\mu\nu})(s_{\rm W}A^{\mu\nu} + c_{\rm W}Z^{\mu\nu}) = \frac{a_{WW}v^4}{4}(s_{\rm W}^2 A_{\mu\nu}A^{\mu\nu} + 2s_{\rm W}c_{\rm W}Z_{\mu\nu}A^{\mu\nu} + c_{\rm W}^2 Z_{\mu\nu}Z^{\mu\nu})$$

$$\Pi_{ZZ}(p^2) = c_{\rm W}^2 a_{WW}v^4 p^2, \quad \Pi_{ZA}(p^2) = s_{\rm W}c_{\rm W}a_{WW}v^4 p^2, \quad \Pi_{AA}(p^2) = s_{\rm W}^2 a_{WW}v^4 p^2$$

$$\Pi_{ZZ}'(0) = c_{\rm W}^2 a_{WW}v^4, \quad \Pi_{ZA}'(0) = s_{\rm W}c_{\rm W}a_{WW}v^4, \quad \Pi_{AA}'(0) = s_{\rm W}^2 a_{WW}v^4$$

$$S = \frac{4s_{\rm W}^2 c_{\rm W}^2}{\alpha}a_{WW}v^4\left[c_{\rm W}^2 - \frac{c_{\rm W}^2 - s_{\rm W}^2}{s_{\rm W}c_{\rm W}}s_{\rm W}c_{\rm W} - s_{\rm W}^2\right] = 0, \quad T = -\frac{\Pi_{ZZ}(0)}{\alpha m_Z^2} = 0$$

$$U = \frac{4s_{\rm W}^2}{\alpha}a_{WW}v^4(0 - c_{\rm W}^4 - 2s_{\rm W}^2 c_{\rm W}^2 - s_{\rm W}^4) = -\frac{4s_{\rm W}^2}{\alpha}a_{WW}v^4$$

<u>Normal and χ^2 distributions</u>

n-dim normal distribution $f(\mathbf{x}) = \dfrac{1}{(2\pi)^{n/2} |\mathbf{V}|^{1/2}} e^{-Q/2}$

Quadratic form $Q = (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{x} - \boldsymbol{\mu}) = (x_i - \mu_i) V_{ij}^{-1} (x_j - \mu_j)$

$\mathbf{V}$ is the positive definite symmetric covariance matrix

$$V_{ij} = \text{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \rho_{1n}\sigma_1\sigma_n & \cdots & & \sigma_n^2 \end{pmatrix}$$

Correlation coefficients $\rho_{ij} = \dfrac{\text{cov}(X_i, X_j)}{\sqrt{V(X_i)V(X_j)}}$

Probability inside the n-dim ellipsoid $Q = Q_0$: $\quad P(Q < Q_0) = F_{\chi^2(n)}(Q_0)$

$F_{\chi^2(n)}(x)$ is the cumulative χ^2 distribution function with n d.o.f.

$F_{\chi^2(1)}(1) = 68.3\%, \quad F_{\chi^2(1)}(4) = 95.4\%, \quad F_{\chi^2(1)}(9) = 99.7\%$

$F_{\chi^2(2)}(2.295749) = 68.3\%, \quad F_{\chi^2(2)}(6.180074) = 95.4\%, \quad F_{\chi^2(2)}(11.82916) = 99.7\%$

$F_{\chi^2(3)}(3.526741) = 68.3\%, \quad F_{\chi^2(3)}(8.024882) = 95.4\%, \quad F_{\chi^2(3)}(14.15641) = 99.7\%$

2-dim normal distribution

$$V_{ij} = \text{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad V_{ij}^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-Q/2}$$

$$Q = \frac{1}{1-\rho^2}\left[\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 - 2\rho \frac{x_1 - \mu_1}{\sigma_1} \frac{x_2 - \mu_2}{\sigma_2} \right]$$

Ellipse parametric equation with parameter t:

$$\begin{cases} x_1 = \mu_1 + a\cos\phi\cos t - b\sin\phi\sin t \\ x_2 = \mu_2 + a\sin\phi\cos t + b\cos\phi\sin t \end{cases}$$

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right), \quad a = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2\cos^2\phi - 2\rho\sigma_1\sigma_2\sin\phi\cos\phi + \sigma_1^2\sin^2\phi}}$$

$$b = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2\sin^2\phi + 2\rho\sigma_1\sigma_2\sin\phi\cos\phi + \sigma_1^2\cos^2\phi}}$$

<u>Precision directly extracted from previous works</u>

<u>Gfitter, 1407.3792</u>

Current constraints on S and T with $U = 0$ fixed [below Eq. (4)]:

$S = 0.06 \pm 0.09, \quad T = 0.10 \pm 0.07, \quad \rho_{ST} = +0.91$

Current precision on S and T [Fig. 7]:

$\sigma_S = 0.093, \quad \sigma_T = 0.082, \quad \rho_{ST} = +0.93$

<u>Fan, Reece & Wang, 1411.1054</u>

Current precision [Fig. 1]:

$\sigma_S = 0.086, \quad \sigma_T = 0.074, \quad \rho_{ST} = +0.91$

CEPC optimistic baseline precision [Fig. 3]:

$\sigma_S = 0.024, \quad \sigma_T = 0.019, \quad \rho_{ST} = +0.84$

CEPC precision with improvements of m_t, Γ_Z and $\sin^2 \theta_{\text{eff}}^{\ell}$ [Fig. 3]:

$\sigma_S = 0.011, \quad \sigma_T = 0.0072, \quad \rho_{ST} = +0.78$

<u>CEPC-SPPC Preliminary Conceptual Design Report</u>

CEPC baseline precision [Fig. 4.2]:

$\sigma_S = 0.016, \quad \sigma_T = 0.015, \quad \rho_{ST} = +0.82$

CEPC precision with improvements of m_t, m_Z and Γ_Z [Fig. 4.2]:

$\sigma_S = 0.011, \quad \sigma_T = 0.0073, \quad \rho_{ST} = +0.78$

Ref: Ciuchini et al., 1306.4644; Fan, Reece & Wang, 1411.1054

Tree level:

$$m_W = 80.385 \text{ GeV}, \quad m_Z = 91.1876 \text{ GeV}, \quad c_W^2 = \frac{m_W^2}{m_Z^2} = 0.77737, \quad s_W^2 = 1 - c_W^2 = 0.22263$$

Loop level:

$$m_W = m_W^{\text{SM}} + \Delta m_W, \quad \Gamma_W = \Gamma_W^{\text{SM}} + \Delta \Gamma_W, \quad \sin^2 \theta_{\text{eff}}^{\ell} = (\sin^2 \theta_{\text{eff}}^{\ell})^{\text{SM}} + \Delta \sin^2 \theta_{\text{eff}}^{\ell}, \quad \Gamma_Z = \Gamma_Z^{\text{SM}} + \Delta \Gamma_Z$$

$$F_1(S,T,U) \equiv S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U = S - 1.55T - 1.24U$$

$$F_2(S,T) \equiv S - 4s_W^2 c_W^2 T = S - 0.693T$$

$$F_3(S,T) \equiv -10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T = -12.2S + 32.9T = -12.2(S - 2.71T)$$

$$\Delta m_W = -\frac{\alpha m_W^{\text{SM}}}{4(c_W^2 - s_W^2)}\left(S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U \right) = -\frac{\alpha m_W^{\text{SM}}}{4(c_W^2 - s_W^2)} F_1(S,T,U)$$

$$\Delta \Gamma_W = -\frac{3\alpha \Gamma_W^{\text{SM}}}{4(c_W^2 - s_W^2)}\left(S - 2c_W^2 T - \frac{c_W^2 - s_W^2}{2s_W^2} U \right) = -\frac{3\alpha \Gamma_W^{\text{SM}}}{4(c_W^2 - s_W^2)} F_1(S,T,U)$$

$$\mathcal{L} \supset \sum_f \frac{g}{2c_W} \bar{f} \gamma^\mu (g_V^f - g_A^f \gamma_5) f Z_\mu, \quad g_V^f \equiv T_f^3 - 2Q_f s_W^2, \quad g_A^f \equiv T_f^3$$

$$\left[\begin{array}{l} Q_{\ell_i} = -1, \quad Q_{\nu_i} = 0, \quad Q_{d_i} = -\dfrac{1}{3}, \quad Q_{u_i} = \dfrac{2}{3} \\[2mm] g_V^{\ell_i} = -\dfrac{1}{2} + 2s_W^2, \quad g_A^{\ell_i} = -\dfrac{1}{2}; \quad g_V^{\nu_i} = \dfrac{1}{2}, \quad g_A^{\nu_i} = \dfrac{1}{2} \\[2mm] g_V^{d_i} = -\dfrac{1}{2} + \dfrac{2}{3} s_W^2, \quad g_A^{d_i} = -\dfrac{1}{2}; \quad g_V^{u_i} = \dfrac{1}{2} - \dfrac{4}{3} s_W^2, \quad g_A^{u_i} = \dfrac{1}{2} \end{array} \right]$$

$$\delta g_V^f = \frac{g_V^f \alpha}{2} T + \frac{(g_V^f - g_A^f)\alpha}{4s_W^2 (c_W^2 - s_W^2)}(S - 4s_W^2 c_W^2 T), \quad \delta g_A^f = \frac{g_A^f \alpha}{2} T$$

$$\Delta \sin^2 \theta_{\text{eff}}^{\ell} = -\frac{g_A^e \delta g_V^e - g_V^e \delta g_A^e}{4(g_A^e)^2} = -\frac{1}{4(g_A^e)^2}\left[g_A^e \left(\frac{g_V^e \alpha}{2} T + \frac{(g_V^e - g_A^e)\alpha}{4s_W^2 (c_W^2 - s_W^2)}(S - 4s_W^2 c_W^2 T) \right) - g_V^e \frac{g_A^e \alpha}{2} T \right]$$

$$= -\frac{(g_V^e - g_A^e)\alpha}{16 g_A^e s_W^2 (c_W^2 - s_W^2)}(S - 4s_W^2 c_W^2 T) = \frac{\alpha}{4(c_W^2 - s_W^2)} F_2(S,T)$$

$$G_f \equiv (g_V^f)^2 + (g_A^f)^2$$

$$\delta G_f \equiv 2(g_V^f \delta g_V^f + g_A^f \delta g_A^f) = \frac{g_V^f (g_V^f - g_A^f)\alpha}{2s_W^2 (c_W^2 - s_W^2)}(S - 4s_W^2 c_W^2 T) + [(g_V^f)^2 + (g_A^f)^2]\alpha T$$

$$\left[\begin{array}{ll} g_V^{\ell_i}(g_V^{\ell_i} - g_A^{\ell_i}) = -s_W^2 + 4s_W^4, & (g_V^{\ell_i})^2 + (g_A^{\ell_i})^2 = \dfrac{1}{2} - 2s_W^2 + 4s_W^4 \\[3mm] g_V^{\nu_i}(g_V^{\nu_i} - g_A^{\nu_i}) = 0, & (g_V^{\nu_i})^2 + (g_A^{\nu_i})^2 = \dfrac{1}{2} \\[3mm] g_V^{d_i}(g_V^{d_i} - g_A^{d_i}) = -\dfrac{1}{3} s_W^2 + \dfrac{4}{9} s_W^4, & (g_V^{d_i})^2 + (g_A^{d_i})^2 = \dfrac{1}{2} - \dfrac{2}{3} s_W^2 + \dfrac{4}{9} s_W^4 \\[3mm] g_V^{u_i}(g_V^{u_i} - g_A^{u_i}) = -\dfrac{2}{3} s_W^2 + \dfrac{16}{9} s_W^4, & (g_V^{u_i})^2 + (g_A^{u_i})^2 = \dfrac{1}{2} - \dfrac{4}{3} s_W^2 + \dfrac{16}{9} s_W^4 \end{array} \right]$$

$$f = e, \nu_e, \mu, \nu_\mu, \tau, \mu_\tau, d, u, s, c, b$$

$$\sum_f N_c^f g_V^f (g_V^f - g_A^f) = 3(-s_W^2 + 4s_W^4) + 3 \cdot 3\left(-\frac{1}{3} s_W^2 + \frac{4}{9} s_W^4 \right) + 2 \cdot 3\left(-\frac{2}{3} s_W^2 + \frac{16}{9} s_W^4 \right) = -\frac{10}{3} s_W^2 (3 - 8s_W^2)$$

$$\sum_f N_c^f [(g_V^f)^2 + (g_A^f)^2] = \left[3\left(\frac{1}{2} + \frac{1}{2} - 2s_W^2 + 4s_W^4 \right) + 3 \cdot 3\left(\frac{1}{2} - \frac{2}{3} s_W^2 + \frac{4}{9} s_W^4 \right) + 2 \cdot 3\left(\frac{1}{2} - \frac{4}{3} s_W^2 + \frac{16}{9} s_W^4 \right) \right] = \frac{21}{2} - 20s_W^2 + \frac{80}{3} s_W^4$$

$$\sum_f N_c^f \delta G_f = \frac{\alpha}{2s_W^2 (c_W^2 - s_W^2)}(S - 4c_W^2 s_W^2 T) \sum_f N_c^f g_V^f (g_V^f - g_A^f) + \alpha T \sum_f N_c^f [(g_V^f)^2 + (g_A^f)^2]$$

$$= -\frac{5\alpha}{3(c_W^2 - s_W^2)}(S - 4c_W^2 s_W^2 T)(3 - 8s_W^2) + \alpha T\left(\frac{21}{2} - 20s_W^2 + \frac{80}{3} s_W^4 \right)$$

$$= -\frac{5(3 - 8s_W^2)}{3(c_W^2 - s_W^2)}\alpha S + \frac{20\alpha T}{3(c_W^2 - s_W^2)}\left[c_W^2 s_W^2 (3 - 8s_W^2) + \frac{3(c_W^2 - s_W^2)}{20}\left(\frac{21}{2} - 20s_W^2 + \frac{80}{3} s_W^4 \right) \right]$$

$$= \frac{\alpha}{6(c_W^2 - s_W^2)}[-10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T]$$

$$\Delta \Gamma_Z = \frac{\alpha m_Z}{12 s_W^2 c_W^2} \sum_f N_c^f \delta G_f = \frac{\alpha^2 m_Z}{72 s_W^2 c_W^2 (c_W^2 - s_W^2)}[-10(3 - 8s_W^2)S + (63 - 126s_W^2 - 40s_W^4)T] = \frac{\alpha^2 m_Z}{72 s_W^2 c_W^2 (c_W^2 - s_W^2)} F_3(S,T)$$

Parametrization of the SM correction to m_W

Ref: Awramik et al., hep-ph/0311148

Loop level: $m_W^2\left(1-\dfrac{m_W^2}{m_Z^2}\right)=\dfrac{\pi\alpha}{\sqrt{2}G_F}(1+\Delta r) \quad\Rightarrow\quad m_W^2=m_Z^2\left[\dfrac{1}{2}+\sqrt{\dfrac{1}{4}-\dfrac{\pi\alpha}{\sqrt{2}G_F m_Z^2}(1+\Delta r)}\right]$

Parametrization formula:

$$m_W^{\mathrm{SM}}=m_W^0-c_1 dH-c_2 dH^2+c_3 dH^4+c_4(dh-1)-c_5 d\alpha+c_6 dt-c_7 dt^2-c_8 dHdt+c_9 dhdt-c_{10}d\alpha_s-c_{11}dZ$$

$$dH=\ln\frac{m_h}{100\ \mathrm{GeV}},\quad dh=\left(\frac{m_h}{100\ \mathrm{GeV}}\right)^2,\quad dt=\left(\frac{m_t}{174.3\ \mathrm{GeV}}\right)^2-1$$

$$dZ=\frac{m_Z}{91.1875\ \mathrm{GeV}}-1,\quad d\alpha=\frac{\Delta\alpha}{0.05907}-1,\quad d\alpha_s=\frac{\alpha_s(m_Z)}{0.119}-1$$

$$\Delta\alpha\equiv\Delta\alpha_{\mathrm{lep}}+\Delta\alpha_{\mathrm{had}}^{(5)},\quad \Delta\alpha_{\mathrm{lep}}=0.0314977$$

Coefficient values can be found in Eq.(9) for $m_h>100$ GeV

Parametrization of the SM correction to $\sin^2\theta_{\mathrm{eff}}^{\ell}$

Ref: Awramik et al., hep-ph/0608099

Loop level: $\sin^2\theta_{\mathrm{eff}}^{\ell}=\left(1-\dfrac{m_W^2}{m_Z^2}\right)(1+\Delta\kappa)$

Parametrization formula:

$$(\sin^2\theta_{\mathrm{eff}}^{\ell})^{\mathrm{SM}}=s_0+d_1 L_H+d_2 L_H^2+d_3 L_H^4+d_4(\Delta_H^2-1)+d_5\Delta_\alpha+d_6\Delta_t+d_7\Delta_t^2+d_8\Delta_t(\Delta_H-1)+d_9\Delta_{\alpha_s}+d_{10}\Delta_Z$$

$$L_H=\ln\frac{m_h}{100\ \mathrm{GeV}},\quad \Delta_H=\frac{m_h}{100\ \mathrm{GeV}},\quad \Delta_\alpha=\frac{\Delta\alpha}{0.05907}-1$$

$$\Delta_t=\left(\frac{m_t}{178\ \mathrm{GeV}}\right)^2-1,\quad \Delta_{\alpha_s}=\frac{\alpha_s(m_Z)}{0.117}-1,\quad \Delta_Z=\frac{m_Z}{91.1876\ \mathrm{GeV}}-1$$

Coefficient values can be found in the 2nd column of Table 5

Parametrization of the SM correction to Γ_Z

Ref: Freitas, 1401.2447

Parametrization formula:

$$\Gamma_Z^{\mathrm{SM}}=\Gamma_Z^0+c_1 L_H+c_2\Delta_t+c_3\Delta_{\alpha_s}+c_4\Delta_{\alpha_s}^2+c_5\Delta_{\alpha_s}\Delta_t+c_6\Delta_\alpha+c_7\Delta_Z$$

$$L_H=\ln\frac{m_h}{125.7\ \mathrm{GeV}},\quad \Delta_t=\left(\frac{m_t}{173.2\ \mathrm{GeV}}\right)^2-1$$

$$\Delta_{\alpha_s}=\frac{\alpha_s(m_Z)}{0.1184}-1,\quad \Delta_\alpha=\frac{\Delta\alpha}{0.059}-1,\quad \Delta_Z=\frac{m_Z}{91.1876\ \mathrm{GeV}}-1$$

$$\Delta\alpha\equiv\Delta\alpha_{\mathrm{lep}}+\Delta\alpha_{\mathrm{had}}^{(5)},\quad \Delta\alpha_{\mathrm{lep}}=0.0314976$$

Coefficient values can be found in the 9th row of Table 5

<u>Least squares fitting</u>

Ref: Fan, Reece & Wang, 1411.1054

Likelihood for n normal variables $x_i \sim N(\mu_i, \sigma_i)$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \right]$$

$$\ln L = \sum_{i=1}^{n} \left[-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 - \ln(\sqrt{2\pi}\sigma_i) \right]$$

$$\chi^2 \equiv \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \quad \Rightarrow \quad \chi^2 = -2\ln L + \text{const.}$$

Assume a flat theoretical uncertainty δ_j for a observable O_j with theoretical parameters $\{\alpha_k\}$:

$$P(O_j \mid \{\alpha_k\}) = \begin{cases} \dfrac{1}{2\delta_j}, & \left| O_j - O_j^{\text{pred}}(\{\alpha_k\}) \right| \le \delta_j \\[2ex] 0, & \left| O_j - O_j^{\text{pred}}(\{\alpha_k\}) \right| > \delta_j \end{cases}$$

O_j = true value, O_j^{pred} = theoretically predicted value

Assume a normal experimental uncertainty σ_j for the measured value O_j^{meas}:

$$P(O_j^{\text{meas}} \mid O_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}} - O_j}{\sigma_j} \right)^2 \right]$$

Extract the probability for O_j^{meas} in terms of $\{\alpha_k\}$ as a convolution, integrating out the unknown true value O_j:

$$P(O_j^{\text{meas}} \mid \{\alpha_k\}) = \int dO_j \, P(O_j^{\text{meas}} \mid O_j) P(O_j \mid \{\alpha_k\}) = q(O_j^{\text{meas}}; O_j^{\text{pred}}(\{\alpha_k\}), \sigma_j, \delta_j) \quad \Rightarrow \quad \chi^2_{\text{mod}} \supset -2\ln q(O_j^{\text{meas}}; O_j^{\text{pred}}(\{\alpha_k\}), \sigma_j, \delta_j)$$

$$q(x; \mu, \sigma, \delta) \equiv \frac{1}{4\delta}\left[\text{erf}\left(\frac{x - \mu + \delta}{\sqrt{2}\sigma} \right) - \text{erf}\left(\frac{x - \mu - \delta}{\sqrt{2}\sigma} \right) \right]$$

$$\left[\begin{array}{l} \text{Cumulative distribution function for } x \sim N(\mu, \sigma): \quad \text{CDF}(x; \mu, \sigma) = \Phi\left(\frac{x - \mu}{\sigma} \right) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x - \mu}{\sqrt{2}\sigma} \right), \quad \text{erf}(-x) = -\text{erf}(x) \\[3ex] \displaystyle \int dO_j \, P(O_j^{\text{meas}} \mid O_j) P(O_j \mid \{\alpha_k\}) = \int dO_j \, \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}} - O_j}{\sigma_j} \right)^2 \right] \frac{\theta\left(\delta_j - \left| O_j - O_j^{\text{pred}} \right| \right)}{2\delta_j} \\[3ex] \displaystyle = \frac{1}{2\delta_j} \int_{O_j^{\text{pred}} - \delta_j}^{O_j^{\text{pred}} + \delta_j} dO_j \, \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{1}{2}\left(\frac{O_j - O_j^{\text{meas}}}{\sigma_j} \right)^2 \right] = \frac{1}{2\delta_j}\left[\Phi\left(\frac{O_j^{\text{pred}} + \delta_j - O_j^{\text{meas}}}{\sigma_j} \right) - \Phi\left(\frac{O_j^{\text{pred}} - \delta_j - O_j^{\text{meas}}}{\sigma_j} \right) \right] \\[3ex] \displaystyle = \frac{1}{4\delta_j}\left[\text{erf}\left(\frac{O_j^{\text{pred}} + \delta_j - O_j^{\text{meas}}}{\sqrt{2}\sigma_j} \right) - \text{erf}\left(\frac{O_j^{\text{pred}} - \delta_j - O_j^{\text{meas}}}{\sqrt{2}\sigma_j} \right) \right] = \frac{1}{4\delta_j}\left[\text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} + \delta_j}{\sqrt{2}\sigma_j} \right) - \text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} - \delta_j}{\sqrt{2}\sigma_j} \right) \right] \end{array} \right]$$

Define modified χ^2 for n observables O_i without theoretical uncertainties
and m observables O_j with flat theoretical uncertainties:

$$\chi^2_{\text{mod}} = \sum_{i=1}^{n} \left(\frac{O_i^{\text{meas}} - O_i^{\text{pred}}}{\sigma_i} \right)^2 + \sum_{j=1}^{m} \left\{ -2\ln\left[\text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} + \delta_j}{\sqrt{2}\sigma_j} \right) - \text{erf}\left(\frac{O_j^{\text{meas}} - O_j^{\text{pred}} - \delta_j}{\sqrt{2}\sigma_j} \right) \right] \right\}$$

<u>Current data</u>

$\alpha_s(m_Z^2) = 0.1185 \pm 0.0006$ [PDG 2014]

$\Delta\alpha_{\text{had}}^{(5)}(m_Z^2) = 0.02765 \pm 0.00008$ [1209.4802]

$m_Z = 91.1875 \pm 0.0021$ GeV [hep-ex/0509008]

$m_t = 173.34 \pm 0.76_{\text{exp}}$ [1403.4427] $\pm 0.5_{\text{the}}$ [1412.4435] GeV

$m_h = 125.09 \pm 0.24$ GeV [1503.07589]

$m_W = 80.385 \pm 0.015_{\text{exp}}$ [PDG 2014] $\pm 0.004_{\text{the}}$ GeV [hep-ph/0311148]

$\sin^2\theta_{\text{eff}}^{\ell} = 0.23153 \pm 0.00016$ [hep-ex/0509008]

$\Gamma_Z = 2.4952 \pm 0.0023$ GeV [hep-ex/0509008]

$\underline{\sigma_S = 0.10, \quad \sigma_T = 0.12, \quad \sigma_U = 0.094, \quad \rho_{ST} = +0.89, \quad \rho_{SU} = -0.55, \quad \rho_{TU} = -0.80}$

$U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.085, \quad \sigma_T = 0.072, \quad \rho_{ST} = +0.90}$

$S = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.054, \quad \sigma_U = 0.078, \quad \rho_{TU} = -0.81}$

$T = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.037}$

$S = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.032}$

<u>CEPC baseline precision</u>

$\alpha_s(m_Z^2)$: $\pm 1.0 \times 10^{-4}$ [1404.0319]

$\Delta\alpha_{\text{had}}^{(5)}(m_Z^2)$: $\pm 4.7 \times 10^{-5}$ [1407.3792]

m_Z: $\pm 5 \times 10^{-4}$ GeV [CEPC-SPPC pre-CDR]

m_t: $\pm 0.2_{\text{exp}}$ [CMS-PAS-FTR-13-017] $\pm 0.5_{\text{the}}$ [1412.4435] GeV

m_h: $\pm 5.9 \times 10^{-3}$ GeV [CEPC-SPPC pre-CDR]

m_W: $(\pm 3_{\text{exp}}$ [CEPC-SPPC pre-CDR] $\pm 1_{\text{the}}$ [1307.3962]$) \times 10^{-3}$ GeV

$\sin^2\theta_{\text{eff}}^{\ell}$: $(\pm 2.3_{\text{exp}}$ [CEPC-SPPC pre-CDR] $\pm 1.5_{\text{the}}$ [1307.3962]$) \times 10^{-5}$

Γ_Z: $(\pm 5_{\text{exp}}$ [CEPC-SPPC pre-CDR]$\pm 0.8_{\text{the}}$ [1411.1054, Mishima's talk]$) \times 10^{-4}$ GeV

$\underline{\sigma_S = 0.021, \quad \sigma_T = 0.026, \quad \sigma_U = 0.020, \quad \rho_{ST} = +0.90, \quad \rho_{SU} = -0.68, \quad \rho_{TU} = -0.84}$

$U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.015, \quad \sigma_T = 0.014, \quad \rho_{ST} = +0.83}$

$S = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.011, \quad \sigma_U = 0.015, \quad \rho_{TU} = -0.72}$

$T = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.0085}$

$S = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.0079}$

<u>Potential improvements for CEPC</u>

Reduced systematic uncertainty in the CEPC measurement

$\rightarrow \quad m_Z$: $\pm 1 \times 10^{-4}$ GeV [CEPC-SPPC pre-CDR]

Γ_Z: $(\pm 1_{\text{exp}}$ [CEPC-SPPC pre-CDR] $\pm 0.8_{\text{the}}$ [1411.1054, Mishima's talk]$) \times 10^{-4}$ GeV

ILC top threshold scan $\rightarrow \quad m_t$: $\pm 0.03_{\text{exp}} \pm 0.1_{\text{the}}$ GeV [1306.6352]

m_Z, Γ_Z & m_t improvements:

$\underline{\sigma_S = 0.011, \quad \sigma_T = 0.0071, \quad \sigma_U = 0.010, \quad \rho_{ST} = +0.74, \quad \rho_{SU} = +0.15, \quad \rho_{TU} = -0.21}$

$U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.011, \quad \sigma_T = 0.0069, \quad \rho_{ST} = +0.80}$

$S = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.0048, \quad \sigma_U = 0.010, \quad \rho_{TU} = -0.48}$

$T = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_S = 0.0068}$

$S = U = 0$ fixed $\Rightarrow \quad \underline{\sigma_T = 0.0042}$

DM models

Convention:

$v^i \in \mathbf{2}, \quad v_i \in \bar{\mathbf{2}}, \quad \varepsilon^{12} = +1, \quad \varepsilon_{12} = -1, \quad \varepsilon^{ij} = -\varepsilon^{ji}, \quad \varepsilon_{ij} = -\varepsilon_{ji}$

SM Higgs field $\quad H = H^i = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} G^+(x) \\ \dfrac{1}{\sqrt{2}}[v + h(x) + iG^0(x)] \end{pmatrix} \in \left(\mathbf{2}, \dfrac{1}{2}\right)$ under $(SU(2)_L, U(1)_Y)$

$H_i = \varepsilon_{ij} H^j = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} -H^0 \\ H^+ \end{pmatrix} \in \left(\bar{\mathbf{2}}, \dfrac{1}{2}\right)$

$H_i^\dagger = (H^i)^* = \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} \in \left(\bar{\mathbf{2}}, -\dfrac{1}{2}\right)$

$\tilde{H} = H^{\dagger i} = \varepsilon^{ij} H_j^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} \in \left(\mathbf{2}, -\dfrac{1}{2}\right)$

$\left[\text{Note:} \quad H^\dagger H = H_i^\dagger H^i = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = H^+ H^- + H^0 H^{0*} = -H^{\dagger i} H_i \right]$

Singlet-Doublet Fermionic Dark Matter (SDFDM)

Ref: D'Eramo, 0705.4493; Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604

Left-handed Weyl fermions:

$S \in (\mathbf{1}, 0), \quad D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\dfrac{1}{2}\right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \dfrac{1}{2}\right)$

$\mathcal{L}_S = iS^\dagger \bar{\sigma}^\mu \partial_\mu S - \dfrac{1}{2}(m_S SS + \text{h.c.}), \quad \mathcal{L}_D = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.})$

$\mathcal{L}_{HSD} = y_1 H_i SD_1^i - y_2 H_i^\dagger SD_2^i + \text{h.c.}$

$-m_D \varepsilon_{ij} D_1^i D_2^j = m_D D_1^0 D_2^0 - m_D D_1^- D_2^+$

$y_1 H_i SD_1^i = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} S \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = -y_1 H^0 SD_1^0 + y_1 H^+ SD_1^- \rightarrow -\dfrac{1}{\sqrt{2}} y_1(v+h)SD_1^0$

$-y_2 H_i^\dagger SD_2^i = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} S \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 H^- SD_2^+ - y_2 H^{0*} SD_2^0 \rightarrow -\dfrac{1}{\sqrt{2}} y_2(v+h)SD_2^0$

$\mathcal{L}_{\text{mass}} = -\dfrac{1}{2} \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_N \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} - m_D D_1^- D_2^+ + \text{h.c.} = -\dfrac{1}{2} \sum_{i=1}^{3} m_{\chi_i^0} \chi_i^0 \chi_i^0 - m_{\chi^\pm} \chi^- \chi^+ + \text{h.c.}$

$\mathcal{M}_N = \begin{pmatrix} m_S & \dfrac{1}{\sqrt{2}} y_1 v & \dfrac{1}{\sqrt{2}} y_2 v \\ \dfrac{1}{\sqrt{2}} y_1 v & 0 & -m_D \\ \dfrac{1}{\sqrt{2}} y_2 v & -m_D & 0 \end{pmatrix}, \quad m_{\chi^\pm} = m_D, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$

$\mathcal{N}^T \mathcal{M}_N \mathcal{N} = \text{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}$

$$y = y_1 = y_2 \quad \Rightarrow \quad \underline{\text{Custodial SU(2)}_\text{R} \text{ global symmetry}}$$

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an SU(2)}_\text{R} \text{ indice}$$

$$H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = -\frac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = -\frac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j$$

$$\mathcal{L}_\text{HSD} = y(H_i S D_1^i - H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S(\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_\text{D} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A - \frac{1}{2}[m_D \varepsilon_{AB}\varepsilon_{ij}(\mathcal{D}^A)^i(\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_S \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(-D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_{\chi^\pm} = m_D \\ m_{\chi_2^0} = \frac{1}{2}\left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} + m_D - m_S\right] \\ m_{\chi_3^0} = \frac{1}{2}\left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} - m_D + m_S\right] \end{cases}$$

$$m_D > m_S \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = \frac{1}{2}\left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} - m_D + m_S\right] \\ m_{\chi_2^0} = m_{\chi^\pm} = m_D \\ m_{\chi_3^0} = \frac{1}{2}\left[\sqrt{(m_D + m_S)^2 + 4y^2 v^2} + m_D - m_S\right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_S)}$$

$$\left[\begin{array}{l} y = y_1 = -y_2 \quad \Rightarrow \quad \text{Another custodial symmetry limit} \\[2mm] (\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} -H_i^\dagger \\ H_i \end{pmatrix} \\[4mm] H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = \frac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = \frac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j \\[4mm] \mathcal{L}_\text{HSD} = y(H_i S D_1^i + H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S(\mathcal{D}^B)^j + \text{h.c.} \end{array}\right]$$

Gauge interactions

$$D_\mu D_i = (\partial_\mu - ig' B_\mu Y_{D_i} - ig W_\mu^a t_\text{D}^a) D_i$$

$$Y_{D_1} = -\frac{1}{2}, \quad Y_{D_2} = \frac{1}{2}, \quad t_\text{D}^1 = \frac{\sigma^1}{2} = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad t_\text{D}^2 = \frac{\sigma^2}{2} = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad t_\text{D}^3 = \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$g' B_\mu Y_{D_1} + g W_\mu^a t_\text{D}^a = \frac{1}{2}\begin{pmatrix} -g'B_\mu + gW_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -g'B_\mu - gW_\mu^a \end{pmatrix} = \begin{pmatrix} \frac{g}{2c_\text{W}}Z_\mu & \frac{g}{\sqrt{2}}W_\mu^+ \\ \frac{g}{\sqrt{2}}W_\mu^- & -eA_\mu + \frac{g}{2c_\text{W}}(s_\text{W}^2 - c_\text{W}^2)Z_\mu \end{pmatrix}$$

$$g' B_\mu Y_{D_2} + g W_\mu^a t_\text{D}^a = \frac{1}{2}\begin{pmatrix} g'B_\mu + gW_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & g'B_\mu - gW_\mu^a \end{pmatrix} = \begin{pmatrix} eA_\mu - \frac{g}{2c_\text{W}}(s_\text{W}^2 - c_\text{W}^2)Z_\mu & \frac{g}{\sqrt{2}}W_\mu^+ \\ \frac{g}{\sqrt{2}}W_\mu^- & -\frac{g}{2c_\text{W}}Z_\mu \end{pmatrix}$$

$$\mathcal{L}_\text{D} \supset iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2$$

$$= \frac{g}{2c_\text{W}}Z_\mu(D_1^0)^\dagger \bar{\sigma}^\mu D_1^0 + \left[-eA_\mu + \frac{g}{2c_\text{W}}(s_\text{W}^2 - c_\text{W}^2)Z_\mu\right](D_1^-)^\dagger \bar{\sigma}^\mu D_1^- + \frac{g}{\sqrt{2}}[W_\mu^+(D_1^0)^\dagger \bar{\sigma}^\mu D_1^- + W_\mu^-(D_1^-)^\dagger \bar{\sigma}^\mu D_1^0]$$

$$+ \left[eA_\mu - \frac{g}{2c_\text{W}}(s_\text{W}^2 - c_\text{W}^2)Z_\mu\right](D_2^+)^\dagger \bar{\sigma}^\mu D_2^+ - \frac{g}{2c_\text{W}}Z_\mu(D_2^0)^\dagger \bar{\sigma}^\mu D_2^0 + \frac{g}{\sqrt{2}}[W_\mu^+(D_2^+)^\dagger \bar{\sigma}^\mu D_2^0 + W_\mu^-(D_2^0)^\dagger \bar{\sigma}^\mu D_2^+]$$

$$X_i^0 = \begin{pmatrix} (\chi_{iL}^0)_\alpha \\ (\chi_{iR}^0)^{\dagger\dot\alpha} \end{pmatrix} = \begin{pmatrix} \chi_{iL}^0 \\ (\chi_{iR}^0)^\dagger \end{pmatrix}, \quad \chi_L^0 = \chi_R^0 = \mathcal{N}^\dagger \psi_L^0 = \mathcal{N}^\dagger \psi_R^0 = \begin{pmatrix} \chi_1^0 & \chi_2^0 & \chi_3^0 \end{pmatrix}^T, \quad \bar{X}_i^0 = \begin{pmatrix} \chi_{iR}^0 & (\chi_{iL}^0)^\dagger \end{pmatrix}$$

$$\Psi_i^0 = \begin{pmatrix} \psi_{iL}^0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix}, \quad \psi_L^0 = \psi_R^0 = \mathcal{N}\chi_L^0 = \mathcal{N}\chi_R^0 = \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix}^T, \quad \bar{\Psi}_i^0 = \begin{pmatrix} \psi_{iR}^0 & (\psi_{iL}^0)^\dagger \end{pmatrix}$$

$$X^+ = \begin{pmatrix} \chi^+ \\ (\chi^-)^\dagger \end{pmatrix}, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$$

$$\Psi_{iL}^0 = \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{N}\chi_L^0)_i \\ 0 \end{pmatrix} = \mathcal{N}_{ij} X_{jL}^0, \quad \Psi_{iR}^0 = \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{N}\chi_R^0)_i^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* X_{jR}^0$$

$$\bar{\Psi}_{iL}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* \bar{X}_{jL}^0, \quad \bar{\Psi}_{iR}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} = \mathcal{N}_{ij} \bar{X}_{jR}^0$$

$$\bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = (\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \psi_{iR}^0 \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^0 = -(\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0} = \frac{1}{2} a_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 + \frac{1}{2} b_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0$$

$$= \frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + \frac{1}{2} b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0 = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0} = b_{Z\Psi_1^0\Psi_1^0} = 0, \quad a_{Z\Psi_2^0\Psi_2^0} = -b_{Z\Psi_2^0\Psi_2^0} = \frac{g}{2c_W}, \quad a_{Z\Psi_3^0\Psi_3^0} = -b_{Z\Psi_3^0\Psi_3^0} = -\frac{g}{2c_W}$$

$$a_{ZX_i^0X_j^0} = a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}, \quad b_{ZX_i^0X_j^0} = b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*$$

$$\left(\begin{array}{l} \text{Note: } a_{ZX_j^0X_i^0} = a_{ZX_i^0X_j^0}^*, \quad b_{ZX_j^0X_i^0} = b_{ZX_i^0X_j^0}^*, \quad a_{ZX_i^0X_j^0} = -b_{ZX_j^0X_i^0} \\[4pt] C^{-1} = -C, \quad X_i^0 = C(\bar{X}_i^0)^T, \quad \bar{X}_i^0 = (X_i^0)^T C \\[4pt] C^{-1}(\gamma^\mu P_L)^T C = \frac{1}{2} C^{-1}(\gamma^\mu)^T C - \frac{1}{2} C^{-1}(\gamma^\mu \gamma_5)^T C = -\frac{1}{2}\gamma^\mu - \frac{1}{2}\gamma^\mu \gamma_5 = -\gamma^\mu P_R \\[4pt] a_{ZX_1^0X_2^0} \bar{X}_{1L}^0 \gamma^\mu X_{2L}^0 = a_{ZX_1^0X_2^0}(\bar{X}_1^0 \gamma^\mu P_L X_2^0)^T = -a_{ZX_1^0X_2^0}(X_2^0)^T C C^{-1}(\gamma^\mu P_L)^T C^{-1} C (\bar{X}_1^0)^T = -a_{ZX_1^0X_2^0} \bar{X}_2^0 C^{-1}(\gamma^\mu P_L)^T C^{-1} X_1^0 \\[4pt] = a_{ZX_1^0X_2^0} \bar{X}_2^0 C^{-1}(\gamma^\mu P_L)^T C X_1^0 = -a_{ZX_1^0X_2^0} \bar{X}_2^0 \gamma^\mu P_R X_1^0 = b_{ZX_2^0X_1^0} \bar{X}_{2R}^0 \gamma^\mu X_{1R}^0 \\[4pt] \frac{1}{2} a_{ZX_1^0X_2^0} Z_\mu \bar{X}_{1L}^0 \gamma^\mu X_{2L}^0 \text{ and } \frac{1}{2} b_{ZX_2^0X_1^0} Z_\mu \bar{X}_{2R}^0 \gamma^\mu X_{1R}^0 \text{ give an identical vertex!} \end{array}\right.$$

$$\bar{X}_L^+ \gamma^\mu X_L^+ = \begin{pmatrix} 0 & (\chi^+)^\dagger \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \chi^+ \\ 0 \end{pmatrix} = (D_2^+)^\dagger \bar{\sigma}^\mu D_2^+$$

$$\bar{X}_R^+ \gamma^\mu X_R^+ = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 \\ (\chi^-)^\dagger \end{pmatrix} = \chi^- \sigma^\mu (\chi^-)^\dagger = -(\chi^-)^\dagger \bar{\sigma}^\mu \chi^- = -(D_1^-)^\dagger \bar{\sigma}^\mu D_1^-$$

$$\mathcal{L}_{AX^+X^+} = a_{AX^+X^+} A_\mu \bar{X}_L^+ \gamma^\mu X_L^+ + b_{AX^+X^+} A_\mu \bar{X}_R^+ \gamma^\mu X_R^+$$

$$a_{AX^+X^+} = b_{AX^+X^+} = e$$

$$\mathcal{L}_{ZX^+X^+} = a_{ZX^+X^+} Z_\mu \bar{X}_L^+ \gamma^\mu X_L^+ + b_{ZX^+X^+} Z_\mu \bar{X}_R^+ \gamma^\mu X_R^+$$

$$a_{ZX^+X^+} = b_{ZX^+X^+} = -\frac{g}{2c_W}(s_W^2 - c_W^2)$$

$$\bar{X}_L^+ \gamma^\mu \Psi_{iL}^0 = \begin{pmatrix} 0 & (\chi^+)^\dagger \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = (\chi^+)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\bar{X}_R^+ \gamma^\mu \Psi_{iR}^0 = \begin{pmatrix} \chi^- & 0 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \chi^- \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \chi^-$$

$$\mathcal{L}_{WX^+\Psi_i^0} = a_{WX^+\Psi_i^0}(W_\mu^+ \bar{X}_L^+ \gamma^\mu \Psi_{iL}^0 + \text{h.c.}) + b_{WX^+\Psi_i^0}(W_\mu^+ \bar{X}_R^+ \gamma^\mu \Psi_{iR}^0 + \text{h.c.})$$

$$= a_{WX^+\Psi_i^0}[\mathcal{N}_{ij} W_\mu^+ \bar{X}_L^+ \gamma^\mu X_{jL}^0 + \text{h.c.}] + b_{WX^+\Psi_i^0}[\mathcal{N}_{ij}^* W_\mu^+ \bar{X}_R^+ \gamma^\mu X_{jR}^0 + \text{h.c.}]$$

$$= a_{WX^+X_i^0} W_\mu^+ \bar{X}_L^+ \gamma^\mu X_{iL}^0 + a_{WX^+X_i^0}^* W_\mu^- \bar{X}_{iL}^0 \gamma^\mu X_L^+ + b_{WX^+X_i^0} W_\mu^+ \bar{X}_R^+ \gamma^\mu X_{iR}^0 + b_{WX^+X_i^0}^* W_\mu^- \bar{X}_{iR}^0 \gamma^\mu X_R^+$$

$$b_{WX^+\Psi_2^0} = -\frac{g}{\sqrt{2}}, \quad a_{WX^+\Psi_3^0} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0$$

$$a_{WX^+X_i^0} = a_{WX^+\Psi_i^0} \mathcal{N}_{ij}, \quad b_{WX^+X_i^0} = b_{WX^+\Psi_i^0} \mathcal{N}_{ij}^*$$

$$\bar{\Psi}_{iR}^0 \Psi_{jL}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{jL}^0 \\ 0 \end{pmatrix} = \psi_{iR}^0 \psi_{jL}^0, \quad \bar{\Psi}_{iL}^0 \Psi_{jR}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ (\psi_{jR}^0)^\dagger \end{pmatrix} = (\psi_{iL}^0)^\dagger (\psi_{jR}^0)^\dagger$$

$$\mathcal{L}_{h\Psi_i^0\Psi_j^0} = \frac{1}{2} a_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iR}^0 \Psi_{jL}^0 + \frac{1}{2} b_{h\Psi_i^0\Psi_j^0} h \bar{\Psi}_{iL}^0 \Psi_{jR}^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj} h \bar{X}_{iR}^0 X_{jL}^0 + \frac{1}{2} b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^* h \bar{X}_{iL}^0 X_{jR}^0 = \frac{1}{2}(a_{hX_i^0X_j^0} h \bar{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0X_j^0} h \bar{X}_{iL}^0 X_{jR}^0)$$

$$a_{h\Psi_1^0\Psi_2^0} = b_{h\Psi_1^0\Psi_2^0} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi_2^0\Psi_1^0} = b_{h\Psi_2^0\Psi_1^0}, \quad a_{h\Psi_1^0\Psi_3^0} = b_{h\Psi_1^0\Psi_3^0} = -\frac{y_2}{\sqrt{2}} = a_{h\Psi_3^0\Psi_1^0} = b_{h\Psi_3^0\Psi_1^0}, \quad \text{others} = 0$$

$$a_{hX_i^0X_j^0} = a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki} \mathcal{N}_{lj}, \quad b_{hX_i^0X_j^0} = b_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{ki}^* \mathcal{N}_{lj}^*$$

$$\left\{ \begin{aligned}
& \text{Note: } a_{hX_j^0X_i^0} = a_{h\Psi_k^0\Psi_l^0} \mathcal{N}_{kj} \mathcal{N}_{li} = a_{h\Psi_l^0\Psi_k^0} \mathcal{N}_{kj} \mathcal{N}_{li} = a_{hX_i^0X_j^0}, \quad b_{hX_j^0X_i^0} = b_{hX_i^0X_j^0}, \quad a_{hX_i^0X_j^0} = b_{hX_i^0X_j^0}^* \\
& C^{-1}(P_L)^T C = \frac{1}{2} C^{-1}(1-\gamma_5)^T C = \frac{1}{2}(1-\gamma_5) = P_L \\
& a_{hX_1^0X_2^0} \bar{X}_{1R}^0 X_{2L}^0 = a_{hX_1^0X_2^0} (\bar{X}_1^0 P_L X_2^0)^T = -a_{hX_1^0X_2^0} (X_2^0)^T (P_L)^T (\bar{X}_1^0)^T = -a_{hX_1^0X_2^0} (X_2^0)^T C C^{-1} (P_L)^T C^{-1} C (\bar{X}_1^0)^T = -a_{hX_1^0X_2^0} \bar{X}_2^0 C^{-1} (P_L)^T C^{-1} X_1^0 \\
& = a_{hX_1^0X_2^0} \bar{X}_2^0 C^{-1} (P_L)^T C X_1^0 = a_{hX_1^0X_2^0} \bar{X}_2^0 P_L X_1^0 = a_{hX_2^0X_1^0} \bar{X}_{2R}^0 X_{1L}^0 \\
& \frac{1}{2} a_{hX_1^0X_2^0} h \bar{X}_{1R}^0 X_{2L}^0 \left(\frac{1}{2} b_{hX_1^0X_2^0} h \bar{X}_{1L}^0 X_{2R}^0 \right) \text{ and } \frac{1}{2} a_{hX_2^0X_1^0} h \bar{X}_{2R}^0 X_{1L}^0 \left(\frac{1}{2} b_{hX_2^0X_1^0} h \bar{X}_{2L}^0 X_{1R}^0 \right) \text{ give an identical vertex!}
\end{aligned} \right.$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$a_{h\Psi_i^0\Psi_j^0} = b_{h\Psi_i^0\Psi_j^0}$$

$$\mathcal{L}_{hX_i^0 X_j^0} = \frac{1}{2}(a_{hX_i^0 X_j^0} h\bar{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0 X_j^0} h\bar{X}_{iL}^0 X_{jR}^0) = \frac{1}{2} h\bar{X}_i^0 (a_{hX_i^0 X_j^0} P_L + b_{hX_i^0 X_j^0} P_R) X_j^0$$

$$= \frac{1}{4}(a_{hX_i^0 X_j^0} + b_{hX_i^0 X_j^0}) h\bar{X}_i^0 X_j^0 + \frac{1}{4}(b_{hX_i^0 X_j^0} - a_{hX_i^0 X_j^0}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}\mathcal{N}_{lj} + \mathcal{N}_{ki}^*\mathcal{N}_{lj}^*) h\bar{X}_i^0 X_j^0 + \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}^*\mathcal{N}_{lj}^* - \mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathrm{Re}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 X_j^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathrm{Im}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 i\gamma_5 X_j^0$$

$$\mathrm{Im}(\mathcal{N}_{ki}\mathcal{N}_{li}) = 0$$

$$\mathcal{L}_{hX_1^0 X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathrm{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \mathrm{Im}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 i\gamma_5 X_1^0$$

$$= [a_{h\Psi_1^0\Psi_2^0} \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= -\frac{1}{\sqrt{2}}[y_1 \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0 X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0 X_1^0} = \frac{1}{2}(a_{hX_1^0 X_1^0} + b_{hX_1^0 X_1^0}) = -\sqrt{2}[y_1 \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \mathrm{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

Effective operators: $\mathcal{L}_{S,q} = \sum_q G_{S,q} \bar{X}_1^0 X_1^0 \bar{q}q, \quad \mathcal{L}_{S,N} = \sum_{N=p,n} G_{S,N} \bar{X}_1^0 X_1^0 \bar{N}N$

$$G_{S,N} = m_N \left(\sum_{q=u,d,s} \frac{G_{S,q}}{m_q} f_q^N + \sum_{q=c,b,t} \frac{G_{S,q}}{m_q} f_Q^N \right), \quad f_Q^N = \frac{2}{27}\left(1 - \sum_{q=u,d,s} f_q^N\right)$$

hep-ph/0001005 :

$$f_u^p = 0.020 \pm 0.004, \quad f_d^p = 0.026 \pm 0.005, \quad f_u^n = 0.014 \pm 0.003, \quad f_d^n = 0.036 \pm 0.008, \quad f_s^p = f_s^n = 0.118 \pm 0.062$$

$$\Rightarrow \quad f_Q^p = 0.0619, \quad f_Q^n = 0.0616$$

$$G_{S,q} = -\frac{g_{hX_1^0 X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0 X_1^0} m_N}{2vm_h^2}\left(\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \quad \Rightarrow \quad G_{S,n} \simeq G_{S,p}$$

$$\sigma_{\chi N}^{SI} = \frac{4}{\pi} \mu_{\chi N}^2 G_{S,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$a_{Z\Psi_k^0\Psi_k^0} = -b_{Z\Psi_k^0\Psi_k^0}$$

$$\mathcal{L}_{ZX_i^0 X_j^0} = \frac{1}{2}(a_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0) = \frac{1}{2}(a_{ZX_i^0 X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_L X_j^0 + b_{ZX_i^0 X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_R X_j^0)$$

$$= \frac{1}{4}(a_{ZX_i^0 X_j^0} + b_{ZX_i^0 X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{ZX_i^0 X_j^0} - a_{ZX_i^0 X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4}(a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} + b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* - a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki}^* \mathcal{N}_{kj} - \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 - \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki} \mathcal{N}_{kj}^* + \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$\mathcal{L}_{ZX_1^0 X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0 X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0 X_1^0} = \frac{1}{2}(b_{ZX_1^0 X_1^0} - a_{ZX_1^0 X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

Effective operators: $\mathcal{L}_{A,q} = \sum_q G_{A,q} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{q} \gamma^\mu \gamma_5 q, \quad \mathcal{L}_{A,N} = \sum_{N=p,n} G_{A,N} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{N} \gamma^\mu \gamma_5 N$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q} \Delta_q^N$$

hep-ex/0609039 :

$$\Delta_u^p = \Delta_d^n = 0.842 \pm 0.012, \quad \Delta_d^p = \Delta_u^n = -0.427 \pm 0.013, \quad \Delta_s^p = \Delta_s^n = -0.085 \pm 0.018$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0 X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$\sigma_{\chi N}^{SD} = \frac{12}{\pi} \mu_{\chi N}^2 G_{A,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

<u>Doublet-Triplet Fermionic Dark Matter (DTFDM)</u>

Ref: Dedes & Karamitros, 1403.7744

Left-handed Weyl fermions:

$$D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right), \quad T = \begin{pmatrix} T^+ \\ T^0 \\ T^- \end{pmatrix} \in (\mathbf{3}, 0)$$

$$\mathcal{L}_D = iD_1^\dagger \bar\sigma^\mu D_\mu D_1 + iD_2^\dagger \bar\sigma^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}), \quad \mathcal{L}_T = iT^\dagger \bar\sigma^\mu D_\mu T - \frac{1}{2}(m_T T^a T^a + \text{h.c.})$$

$$\mathcal{L}_{\text{HDT}} = y_1 H_i T^a (\sigma^a)^i_j D_1^j - y_2 H_i^\dagger T^a (\sigma^a)^i_j D_2^j + \text{h.c.}, \quad i,j = 1,2, \quad a = 1,2,3$$

$$m_D \varepsilon_{ij} D_1^i D_2^j = -m_D D_1^0 D_2^0 + m_D D_1^- D_2^+$$

$$T^\pm = -\frac{1}{\sqrt{2}}(T^1 \mp iT^2), \quad T^0 = T^3, \quad T^1 = -\frac{1}{\sqrt{2}}(T^+ + T^-), \quad T^2 = -\frac{i}{\sqrt{2}}(T^+ - T^-)$$

$$-\frac{1}{2}m_T T^a T^a = -\frac{1}{2}m_T \left[\frac{1}{2}(T^+ + T^-)^2 - \frac{1}{2}(T^+ - T^-)^2 + T^0 T^0\right] = -m_T T^- T^+ - \frac{1}{2}m_T T^0 T^0$$

$$T^a \sigma^a = \begin{pmatrix} T^3 & T^1 - iT^2 \\ T^1 + iT^2 & -T^3 \end{pmatrix} = \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix}$$

$$y_1 H_i T^a (\sigma^a)^i_j D_1^j = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 D_1^0 - \sqrt{2}T^+ D_1^- \\ -\sqrt{2}T^- D_1^0 - T^0 D_1^- \end{pmatrix}$$

$$= y_1(-H^0 T^0 D_1^0 + \sqrt{2}H^0 T^+ D_1^- - \sqrt{2}H^+ T^- D_1^0 - H^+ T^0 D_1^-) \to -\frac{1}{\sqrt{2}}y_1(v+h)T^0 D_1^0 + y_1(v+h)T^+ D_1^-$$

$$-y_2 H_i^\dagger T^a (\sigma^a)^i_j D_2^j = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 D_2^+ - \sqrt{2}T^+ D_2^0 \\ -\sqrt{2}T^- D_2^+ - T^0 D_2^0 \end{pmatrix}$$

$$= -y_2(H^- T^0 D_2^+ - \sqrt{2}H^- T^+ D_2^0 - \sqrt{2}H^{0*}T^- D_2^+ - H^{0*}T^0 D_2^0) \to y_2(v+h)T^- D_2^+ + \frac{1}{\sqrt{2}}y_2(v+h)T^0 D_2^0$$

$$\boxed{\begin{aligned}
&\text{Note: } T_j^i = u^i v_j - \frac{1}{2}\delta_j^i u^k v_k = \frac{1}{\sqrt{2}}T^a(\sigma^a)^i_j \\[4pt]
&T^+ = -T_2^1, \quad T^- = -T_1^2, \quad T^0 = \sqrt{2}T_1^1 = -\sqrt{2}T_2^2 \quad \Rightarrow \quad -\frac{1}{2}m_T T_i^j T_j^i = -m_T T^- T^+ - \frac{1}{2}m_T T^0 T^0 \\[4pt]
&\sqrt{2}y_1 H_i T_j^i D_1^j = \sqrt{2}y_1(H_1 T_1^1 D_1^1 + H_1 T_2^1 D_1^2 + H_2 T_1^2 D_1^1 + H_2 T_2^2 D_1^2) \\[4pt]
&\quad = \sqrt{2}y_1\left(-\frac{1}{\sqrt{2}}H^0 T^0 D_1^0 + H^0 T^+ D_1^- - H^+ T^- D_1^0 - \frac{1}{\sqrt{2}}H^+ T^0 D_1^-\right) \\[4pt]
&\quad = y_1(-H^0 T^0 D_1^0 + \sqrt{2}H^0 T^+ D_1^- - \sqrt{2}H^+ T^- D_1^0 - H^+ T^0 D_1^-) \\[4pt]
&-\sqrt{2}y_2 H_i^\dagger T_j^i D_2^j = -\sqrt{2}y_2(H_1^\dagger T_1^1 D_2^1 + H_1^\dagger T_2^1 D_2^2 + H_2^\dagger T_1^2 D_2^1 + H_2^\dagger T_2^2 D_2^2) \\[4pt]
&\quad = -\sqrt{2}y_2\left(\frac{1}{\sqrt{2}}H^- T^0 D_2^+ - H^- T^+ D_2^0 - H^{0*}T^- D_2^+ - \frac{1}{\sqrt{2}}H^{0*}T^0 D_2^0\right) \\[4pt]
&\quad = -y_2(H^- T^0 D_2^+ - \sqrt{2}H^- T^+ D_2^0 - \sqrt{2}H^{0*}T^- D_2^+ - H^{0*}T^0 D_2^0)
\end{aligned}}$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix} T^0 & D_1^0 & D_2^0 \end{pmatrix}\mathcal{M}_N \begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} - \begin{pmatrix} T^- & D_1^- \end{pmatrix}\mathcal{M}_C \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} + \text{h.c.} = -\frac{1}{2}\sum_{i=1}^3 m_{\chi_i^0} \chi_i^0 \chi_i^0 - \sum_{i=1}^2 m_{\chi_i^\pm} \chi_i^- \chi_i^+ + \text{h.c.}$$

$$\mathcal{M}_N = \begin{pmatrix} m_T & \frac{1}{\sqrt{2}}y_1 v & -\frac{1}{\sqrt{2}}y_2 v \\ \frac{1}{\sqrt{2}}y_1 v & 0 & m_D \\ -\frac{1}{\sqrt{2}}y_2 v & m_D & 0 \end{pmatrix}, \quad \mathcal{M}_C = \begin{pmatrix} m_T & -y_2 v \\ -y_1 v & -m_D \end{pmatrix}$$

$$\mathcal{N}^T \mathcal{M}_N \mathcal{N} = \text{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L = \text{diag}(m_{\chi_1^\pm}, m_{\chi_2^\pm}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \mathcal{C}_L^{-1} = \mathcal{C}_L^\dagger, \quad \mathcal{C}_R^{-1} = \mathcal{C}_R^\dagger$$

$$\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{M}_C \mathcal{C}_L = (\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{C}_R^*)(\mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L) = \text{diag}(m_{\chi_1^\pm}^2, m_{\chi_2^\pm}^2), \quad \mathcal{C}_R^T \mathcal{M}_C \mathcal{M}_C^\dagger \mathcal{C}_R^* = (\mathcal{C}_R^T \mathcal{M}_C \mathcal{C}_L)(\mathcal{C}_L^\dagger \mathcal{M}_C^\dagger \mathcal{C}_R^*) = \text{diag}(m_{\chi_1^\pm}^2, m_{\chi_2^\pm}^2)$$

$$\begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N}\begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}, \quad \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} = \mathcal{C}_L \begin{pmatrix} \chi_1^+ \\ \chi_2^+ \end{pmatrix}, \quad \begin{pmatrix} T^- \\ D_1^- \end{pmatrix} = \mathcal{C}_R \begin{pmatrix} \chi_1^- \\ \chi_2^- \end{pmatrix}$$

$y = y_1 = y_2 \quad \Rightarrow \quad \underline{\text{Custodial SU(2)}_\text{R} \text{ global symmetry}}$

$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an SU(2)}_\text{R} \text{ indice}$

$\mathcal{L}_\text{HDT} = y[H_i T^a (\sigma^a)^i_j D_1^j - H_i^\dagger T^a (\sigma^a)^i_j D_2^j] + \text{h.c.} = y\varepsilon_{AB} (\mathcal{H}^A)_i T^a (\sigma^a)^i_j (\mathcal{D}^B)^j + \text{h.c.}$

$\mathcal{L}_\text{D} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A + \frac{1}{2}[m_D \varepsilon_{AB} \varepsilon_{ij} (\mathcal{D}^A)^i (\mathcal{D}^B)^j + \text{h.c.}]$

$m_D < m_T \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_D \\[2mm] m_{\chi_2^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T \right] \\[2mm] m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T \right] \end{cases}$

$m_D > m_T \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} - m_D + m_T \right] \\[2mm] m_{\chi_2^0} = m_D \\[2mm] m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2 v^2} + m_D - m_T \right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_T)}$

Gauge interactions

$D_\mu T = (\partial_\mu - igW_\mu^a t_\text{T}^a)T$

$t_\text{T}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix}, \quad t_\text{T}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} & -i & \\ i & & i \\ & -i & \end{pmatrix}, \quad t_\text{T}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$gW_\mu^a t_\text{T}^a = \begin{pmatrix} gW_\mu^3 & g(W_\mu^1 - iW_\mu^2)/\sqrt{2} & 0 \\ g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & 0 & -g(W_\mu^1 - iW_\mu^2)/\sqrt{2} \\ 0 & -g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & -gW_\mu^3 \end{pmatrix}$

$= \begin{pmatrix} eA_\mu + gc_\text{W} Z_\mu & gW_\mu^+ & 0 \\ gW_\mu^- & 0 & -gW_\mu^+ \\ 0 & -gW_\mu^- & -eA_\mu - gc_\text{W} Z_\mu \end{pmatrix}$

$\mathcal{L}_\text{T} \supset T^\dagger \bar{\sigma}^\mu gW_\mu^a t_\text{T}^a T$

$= g[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0$

$\quad + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^-$

$\quad - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]$

$= (eA_\mu + gc_\text{W} Z_\mu)(T^+)^\dagger \bar{\sigma}^\mu T^+ + gW_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0$

$\quad + gW_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - gW_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^-$

$\quad - gW_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - (eA_\mu + gc_\text{W} Z_\mu)(T^-)^\dagger \bar{\sigma}^\mu T^-$

$$X_i^0 = \begin{pmatrix} \chi_{iL}^0 \\ (\chi_{iR}^0)^\dagger \end{pmatrix}, \quad \chi_L^0 = \chi_R^0 = \mathcal{N}^\dagger \psi_L^0 = \mathcal{N}^\dagger \psi_R^0 = \begin{pmatrix} \chi_1^0 & \chi_2^0 & \chi_3^0 \end{pmatrix}^T, \quad \bar{X}_i^0 = \begin{pmatrix} \chi_{iR}^0 & (\chi_{iL}^0)^\dagger \end{pmatrix}$$

$$X_i^+ = \begin{pmatrix} \chi_{iL}^+ \\ (\chi_{iR}^-)^\dagger \end{pmatrix}, \quad \chi_L^+ = \mathcal{C}_L^\dagger \psi_L^+ = \begin{pmatrix} \chi_1^+ & \chi_2^+ \end{pmatrix}^T, \quad \chi_R^- = \mathcal{C}_R^\dagger \psi_R^- = \begin{pmatrix} \chi_1^- & \chi_2^- \end{pmatrix}^T, \quad \bar{X}_i^+ = \begin{pmatrix} \chi_{iR}^- & (\chi_{iL}^+)^\dagger \end{pmatrix}$$

$$\Psi_i^0 = \begin{pmatrix} \psi_{iL}^0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix}, \quad \psi_L^0 = \psi_R^0 = \mathcal{N}\chi_L^0 = \mathcal{N}\chi_R^0 = \begin{pmatrix} T^0 & D_1^0 & D_2^0 \end{pmatrix}^T, \quad \bar{\Psi}_i^0 = \begin{pmatrix} \psi_{iR}^0 & (\psi_{iL}^0)^\dagger \end{pmatrix}$$

$$\Psi_i^+ = \begin{pmatrix} \psi_{iL}^+ \\ (\psi_{iR}^-)^\dagger \end{pmatrix}, \quad \psi_L^+ = \mathcal{C}_L\chi_L^+ = \begin{pmatrix} T^+ & D_2^+ \end{pmatrix}^T, \quad \psi_R^- = \mathcal{C}_R\chi_R^- = \begin{pmatrix} T^- & D_1^- \end{pmatrix}^T, \quad \bar{\Psi}_i^+ = \begin{pmatrix} \psi_{iR}^- & (\psi_{iL}^+)^\dagger \end{pmatrix}$$

$$\Psi_{iL}^0 = \begin{pmatrix} \psi_{iL}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{N}\chi_L^0)_i \\ 0 \end{pmatrix} = \mathcal{N}_{ij} X_{jL}^0, \quad \Psi_{iR}^0 = \begin{pmatrix} 0 \\ (\psi_{iR}^0)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{N}\chi_R^0)_i^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* X_{jR}^0$$

$$\bar{\Psi}_{iL}^0 = \begin{pmatrix} 0 & (\psi_{iL}^0)^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* \bar{X}_{jL}^0, \quad \bar{\Psi}_{iR}^0 = \begin{pmatrix} \psi_{iR}^0 & 0 \end{pmatrix} = \mathcal{N}_{ij} \bar{X}_{jR}^0$$

$$\Psi_{iL}^+ = \begin{pmatrix} \psi_{iL}^+ \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{C}_L\chi_L^+)_i \\ 0 \end{pmatrix} = (\mathcal{C}_L)_{ij} X_{jL}^+, \quad \Psi_{iR}^+ = \begin{pmatrix} 0 \\ (\psi_{iR}^-)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{C}_R\chi_R^-)_i^\dagger \end{pmatrix} = (\mathcal{C}_R)_{ij}^* X_{jR}^+$$

$$\bar{\Psi}_{iL}^+ = \begin{pmatrix} 0 & (\psi_{iL}^+)^\dagger \end{pmatrix} = (\mathcal{C}_L)_{ij}^* \bar{X}_{jL}^+, \quad \bar{\Psi}_{iR}^+ = \begin{pmatrix} \psi_{iR}^- & 0 \end{pmatrix} = (\mathcal{C}_R)_{ij} \bar{X}_{jR}^+$$

$$\bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 = (\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0, \quad \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \psi_{iR}^0 \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^0 = -(\psi_{iL}^0)^\dagger \bar{\sigma}^\mu \psi_{iL}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0} = \frac{1}{2} a_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iL}^0 \gamma^\mu \Psi_{iL}^0 + \frac{1}{2} b_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{iR}^0 \gamma^\mu \Psi_{iR}^0 = \frac{1}{2}(a_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iL}^0 \gamma^\mu X_{jL}^0 + b_{ZX_i^0 X_j^0} Z_\mu \bar{X}_{iR}^0 \gamma^\mu X_{jR}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0} = b_{Z\Psi_1^0\Psi_1^0} = 0, \quad a_{Z\Psi_2^0\Psi_2^0} = -b_{Z\Psi_2^0\Psi_2^0} = \frac{g}{2c_W}, \quad a_{Z\Psi_3^0\Psi_3^0} = -b_{Z\Psi_3^0\Psi_3^0} = -\frac{g}{2c_W}$$

$$a_{ZX_i^0 X_j^0} = a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}, \quad b_{ZX_i^0 X_j^0} = b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*$$

$$\bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{iL}^0 = (\psi_{iL}^+)^\dagger \bar{\sigma}^\mu \psi_{iL}^0, \quad \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{iR}^0 = \psi_{iR}^- \sigma^\mu (\psi_{iR}^0)^\dagger = -(\psi_{iR}^0)^\dagger \bar{\sigma}^\mu \psi_{iR}^-$$

$$\mathcal{L}_{W\Psi_i^+\Psi_j^0} = a_{W\Psi_i^+\Psi_j^0}(W_\mu^+ \bar{\Psi}_{iL}^+ \gamma^\mu \Psi_{jL}^0 + \text{h.c.}) + b_{W\Psi_i^+\Psi_j^0}(W_\mu^+ \bar{\Psi}_{iR}^+ \gamma^\mu \Psi_{jR}^0 + \text{h.c.})$$

$$= a_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_L)_{ki}^* \mathcal{N}_{lj} W_\mu^+ \bar{X}_{iL}^+ \gamma^\mu X_{jL}^0 + \text{h.c.}] + b_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_R)_{ki} \mathcal{N}_{lj}^* W_\mu^+ \bar{X}_{iR}^+ \gamma^\mu X_{jR}^0 + \text{h.c.}]$$

$$= a_{WX_i^+ X_j^0} W_\mu^+ \bar{X}_{iL}^+ \gamma^\mu X_{jL}^0 + a_{WX_i^+ X_j^0}^* W_\mu^- \bar{X}_{jL}^0 \gamma^\mu X_{iL}^+ + b_{WX_i^+ X_j^0} W_\mu^+ \bar{X}_{iR}^+ \gamma^\mu X_{jR}^0 + b_{WX_i^+ X_j^0}^* W_\mu^- \bar{X}_{jR}^0 \gamma^\mu X_{iR}^+$$

$$a_{W\Psi_1^+\Psi_1^0} = b_{W\Psi_1^+\Psi_1^0} = g, \quad b_{W\Psi_2^+\Psi_2^0} = -\frac{g}{\sqrt{2}}, \quad a_{W\Psi_2^+\Psi_3^0} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0$$

$$a_{WX_i^+ X_j^0} = a_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_L)_{ki}^* \mathcal{N}_{lj}, \quad b_{WX_i^+ X_j^0} = b_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_R)_{ki} \mathcal{N}_{lj}^*$$

$$\overline{\Psi}_{iL}^+\gamma^\mu\Psi_{iL}^+ = (\psi_{iL}^+)^\dagger\overline{\sigma}^\mu\psi_{iL}^+, \quad \overline{\Psi}_{iR}^+\gamma^\mu\Psi_{iR}^+ = \psi_{iR}^-\sigma^\mu(\psi_{iR}^-)^\dagger = -(\psi_{iR}^-)^\dagger\overline{\sigma}^\mu\psi_{iR}^-$$

$$\mathcal{L}_{A\Psi_i^+\Psi_i^+} = a_{A\Psi_i^+\Psi_i^+}A_\mu\overline{\Psi}_{iL}^+\gamma^\mu\Psi_{iL}^+ + b_{A\Psi_i^+\Psi_i^+}A_\mu\overline{\Psi}_{iR}^+\gamma^\mu\Psi_{iR}^+$$

$$= a_{A\Psi_k^+\Psi_k^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_L)_{kj}A_\mu\overline{X}_{iL}^+\gamma^\mu X_{jL}^+ + b_{A\Psi_k^+\Psi_k^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_R)_{kj}^*A_\mu\overline{X}_{iR}^+\gamma^\mu X_{jR}^+$$

$$= a_{AX_i^+X_j^+}A_\mu\overline{X}_{iL}^+\gamma^\mu X_{jL}^+ + b_{AX_i^+X_j^+}A_\mu\overline{X}_{iR}^+\gamma^\mu X_{jR}^+$$

$$a_{A\Psi_1^+\Psi_1^+} = b_{A\Psi_1^+\Psi_1^+} = a_{A\Psi_2^+\Psi_2^+} = b_{A\Psi_2^+\Psi_2^+} = e$$

$$a_{AX_i^+X_j^+} = a_{A\Psi_k^+\Psi_k^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_L)_{kj} = e\delta_{ij}, \quad b_{AX_i^+X_j^+} = b_{A\Psi_k^+\Psi_k^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_R)_{kj}^* = e\delta_{ij}$$

$$\mathcal{L}_{Z\Psi_i^+\Psi_i^+} = a_{Z\Psi_i^+\Psi_i^+}Z_\mu\overline{\Psi}_{iL}^+\gamma^\mu\Psi_{iL}^+ + b_{Z\Psi_i^+\Psi_i^+}Z_\mu\overline{\Psi}_{iR}^+\gamma^\mu\Psi_{iR}^+$$

$$= a_{Z\Psi_k^+\Psi_k^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_L)_{kj}Z_\mu\overline{X}_{iL}^+\gamma^\mu X_{jL}^+ + b_{Z\Psi_k^+\Psi_k^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_R)_{kj}^*Z_\mu\overline{X}_{iR}^+\gamma^\mu X_{jR}^+$$

$$= a_{ZX_i^+X_j^+}Z_\mu\overline{X}_{iL}^+\gamma^\mu X_{jL}^+ + b_{ZX_i^+X_j^+}Z_\mu\overline{X}_{iR}^+\gamma^\mu X_{jR}^+$$

$$a_{Z\Psi_1^+\Psi_1^+} = b_{Z\Psi_1^+\Psi_1^+} = gc_W, \quad a_{Z\Psi_2^+\Psi_2^+} = b_{Z\Psi_2^+\Psi_2^+} = -\frac{g}{2c_W}(s_W^2 - c_W^2)$$

$$a_{ZX_i^+X_j^+} = a_{Z\Psi_k^+\Psi_k^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_L)_{kj}, \quad b_{ZX_i^+X_j^+} = b_{Z\Psi_k^+\Psi_k^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_R)_{kj}^*$$

$$\overline{\Psi}_{iR}^0\Psi_{jL}^0 = \psi_{iR}^0\psi_{jL}^0, \quad \overline{\Psi}_{iL}^0\Psi_{jR}^0 = (\psi_{iL}^0)^\dagger(\psi_{jR}^0)^\dagger$$

$$\mathcal{L}_{h\Psi_i^0\Psi_j^0} = \frac{1}{2}a_{h\Psi_i^0\Psi_j^0}h\overline{\Psi}_{iR}^0\Psi_{jL}^0 + \frac{1}{2}b_{h\Psi_i^0\Psi_j^0}h\overline{\Psi}_{iL}^0\Psi_{jR}^0$$

$$= \frac{1}{2}a_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}\mathcal{N}_{lj}h\overline{X}_{iR}^0 X_{jL}^0 + \frac{1}{2}b_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}^*\mathcal{N}_{lj}^*h\overline{X}_{iL}^0 X_{jR}^0 = \frac{1}{2}(a_{hX_i^0X_j^0}h\overline{X}_{iR}^0 X_{jL}^0 + b_{hX_i^0X_j^0}h\overline{X}_{iL}^0 X_{jR}^0)$$

$$a_{h\Psi_1^0\Psi_2^0} = b_{h\Psi_1^0\Psi_2^0} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi_2^0\Psi_1^0} = b_{h\Psi_2^0\Psi_1^0}, \quad a_{h\Psi_1^0\Psi_3^0} = b_{h\Psi_1^0\Psi_3^0} = \frac{y_2}{\sqrt{2}} = a_{h\Psi_3^0\Psi_1^0} = b_{h\Psi_3^0\Psi_1^0}, \quad \text{others} = 0$$

$$a_{hX_i^0X_j^0} = a_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}\mathcal{N}_{lj}, \quad b_{hX_i^0X_j^0} = b_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}^*\mathcal{N}_{lj}^*$$

$$\overline{\Psi}_{iR}^+\Psi_{jL}^+ = \begin{pmatrix} \psi_{iR}^- & 0 \end{pmatrix}\begin{pmatrix} \psi_{jL}^+ \\ 0 \end{pmatrix} = \psi_{iR}^-\psi_{jL}^+, \quad \overline{\Psi}_{iL}^+\Psi_{jR}^+ = \begin{pmatrix} 0 & (\psi_{iL}^+)^\dagger \end{pmatrix}\begin{pmatrix} 0 \\ (\psi_{jR}^-)^\dagger \end{pmatrix} = (\psi_{iL}^+)^\dagger(\psi_{jR}^-)^\dagger$$

$$\mathcal{L}_{h\Psi_i^+\Psi_j^+} = a_{h\Psi_i^+\Psi_j^+}h\overline{\Psi}_{iR}^+\Psi_{jL}^+ + b_{h\Psi_i^+\Psi_j^+}h\overline{\Psi}_{iL}^+\Psi_{jR}^+$$

$$= a_{h\Psi_k^+\Psi_l^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_L)_{lj}h\overline{X}_{iR}^+ X_{jL}^+ + b_{h\Psi_k^+\Psi_l^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_R)_{lj}^*h\overline{X}_{iL}^+ X_{jR}^+ = a_{hX_i^+X_j^+}h\overline{X}_{iR}^+ X_{jL}^+ + b_{hX_i^+X_j^+}h\overline{X}_{iL}^+ X_{jR}^+$$

$$a_{h\Psi_1^+\Psi_2^+} = b_{h\Psi_2^+\Psi_1^+} = y_2, \quad a_{h\Psi_2^+\Psi_1^+} = b_{h\Psi_1^+\Psi_2^+} = y_1, \quad \text{others} = 0$$

$$a_{hX_i^+X_j^+} = a_{h\Psi_k^+\Psi_l^+}(\mathcal{C}_R)_{ki}(\mathcal{C}_L)_{lj}, \quad b_{hX_i^+X_j^+} = b_{h\Psi_k^+\Psi_l^+}(\mathcal{C}_L)_{ki}^*(\mathcal{C}_R)_{lj}^*$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$\mathcal{L}_{hX_1^0 X_1^0} = \frac{1}{2} a_{h\Psi_k^0 \Psi_l^0} \operatorname{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 = [a_{h\Psi_1^0 \Psi_2^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0 \Psi_3^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= \frac{1}{\sqrt{2}}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0 X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0 X_1^0} = \frac{1}{2}(a_{hX_1^0 X_1^0} + b_{hX_1^0 X_1^0}) = \sqrt{2}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

$$G_{S,q} = -\frac{g_{hX_1^0 X_1^0} m_q}{2v m_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0 X_1^0} m_N}{2v m_h^2}\left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N\right), \quad \sigma_{\chi N}^{\mathrm{SI}} = \frac{4}{\pi}\mu_{\chi N}^2 G_{S,N}^2$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$\mathcal{L}_{ZX_1^0 X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0 \Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0 X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0 X_1^0} = \frac{1}{2}(b_{ZX_1^0 X_1^0} - a_{ZX_1^0 X_1^0}) = -a_{Z\Psi_k^0 \Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0 X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q} \Delta_q^N, \quad \sigma_{\chi N}^{\mathrm{SD}} = \frac{12}{\pi}\mu_{\chi N}^2 G_{A,N}^2$$

Ref: Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604; Cheung & Sanford, 1311.5896

CP-even real scalar singlet S and complex scalar doublet Φ :

$$S \in (\mathbf{1}, 0), \quad \Phi = \begin{pmatrix} \phi^+ \\ \dfrac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} \in \left(\mathbf{2}, \dfrac{1}{2}\right)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)\partial^\mu S + (D_\mu \Phi)^\dagger D^\mu \Phi - V(S,\Phi)$$

$$V(S,\Phi) = \frac{1}{2}m_S^2 S^2 + m_D^2 |\Phi|^2 + (\kappa S\Phi^\dagger H + h.c.) + \frac{1}{2}\lambda_{Sh}S^2 |H|^2 + \lambda_1 |H|^2|\Phi|^2$$

$$+ (\lambda_2 \Phi^\dagger H \Phi^\dagger H + h.c.) + \lambda_3 |\Phi^\dagger H|^2 + (\text{irrelevant terms})$$

$$|H|^2 \to \frac{1}{2}(v+h)^2, \quad |\Phi|^2 = \phi^+\phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2, \quad \kappa S\Phi^\dagger H + h.c. \to \kappa(v+h)S\phi^0$$

$$\frac{1}{2}\lambda_{Sh}S^2 |H|^2 \to \frac{\lambda_{Sh}}{4}S^2(v+h)^2, \quad \lambda_1 |H|^2|\Phi|^2 = \frac{\lambda_1}{2}(v+h)^2\left[\phi^+\phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2\right]$$

$$\lambda_2\Phi^\dagger H\Phi^\dagger H + h.c. \to \frac{\lambda_2}{2}(v+h)^2[(\phi^0)^2 - (a^0)^2], \quad \lambda_3 |\Phi^\dagger H|^2 = \frac{\lambda_3}{4}(v+h)^2[(\phi^0)^2 + (a^0)^2]$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix} S & \phi^0 \end{pmatrix} M_0^2 \begin{pmatrix} S \\ \phi^0 \end{pmatrix} - \frac{1}{2}m_a^2 (a^0)^2 - m_C^2 |\phi^+|^2$$

$$M_0^2 = \begin{pmatrix} m_S^2 + \dfrac{1}{2}\lambda_{Sh}v^2 & \kappa v \\ \kappa v & m_D^2 + \dfrac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)v^2 \end{pmatrix}, \quad m_a^2 = m_D^2 + \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)v^2, \quad m_C^2 = m_D^2 + \frac{1}{2}\lambda_1 v^2$$

$$U^{\mathrm{T}} M_0^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix} = U\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} c_\theta X_1 - s_\theta X_2 \\ s_\theta X_1 + c_\theta X_2 \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{ (M_0^2)_{11} + (M_0^2)_{22} \mp \sqrt{[(M_0^2)_{11} - (M_0^2)_{22}]^2 + 4(M_0^2)_{12}^2} \right\}$$

$$\text{DM candidate} = \begin{cases} X_1, & m_1^2 < m_a^2, m_C^2 \\ a^0, & m_a^2 < m_1^2, m_C^2 \end{cases}$$

$$\mathcal{L}_{hXX,haa} = -\kappa hS\phi^0 - \frac{\lambda_{Sh}v}{2}hS^2 - \frac{\lambda_1}{2}vh[(\phi^0)^2 + (a^0)^2] - \lambda_2 vh[(\phi^0)^2 - (a^0)^2] - \frac{\lambda_3}{2}vh[(\phi^0)^2 + (a^0)^2]$$

$$= -\kappa h(c_\theta X_1 - s_\theta X_2)(s_\theta X_1 + c_\theta X_2) - \frac{\lambda_{Sh}v}{2}h(c_\theta X_1 - s_\theta X_2)^2$$

$$- \frac{1}{2}vh(\lambda_1 + 2\lambda_2 + \lambda_3)(s_\theta X_1 + c_\theta X_2)^2 - \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)vh(a^0)^2$$

$$= \frac{1}{2}\lambda_{hX_1X_1}vhX_1^2 + \frac{1}{2}\lambda_{hX_2X_2}vhX_2^2 + \lambda_{hX_1X_2}vhX_1X_2 + \frac{1}{2}\lambda_{haa}vh(a^0)^2$$

$$\lambda_{hX_1X_1} = -2\frac{\kappa}{v}s_\theta c_\theta - \lambda_{Sh}c_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3)s_\theta^2, \quad \lambda_{hX_2X_2} = 2\frac{\kappa}{v}s_\theta c_\theta - \lambda_{Sh}s_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3)c_\theta^2$$

$$\lambda_{hX_1X_2} = -\frac{\kappa}{v}(c_\theta^2 - s_\theta^2) - (-\lambda_{Sh} + \lambda_1 + 2\lambda_2 + \lambda_3)s_\theta c_\theta, \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

Direct detection

Effective operators: $\quad \mathcal{L}_{S,q} = \dfrac{1}{2}\sum_q F_{S,q}\chi^2 \bar{q}q, \quad \mathcal{L}_{S,N} = \dfrac{1}{2}\sum_{N=p,n} F_{S,N}\chi^2 \bar{N}N, \quad \chi = X_1, a^0$

$$F_{S,N} = m_N\left(\sum_{q=u,d,s}\frac{F_{S,q}}{m_q}f_q^N + \sum_{q=c,b,t}\frac{F_{S,q}}{m_q}f_Q^N\right), \quad f_Q^N = \frac{2}{27}\left(1 - \sum_{q=u,d,s}f_q^N\right)$$

$$\chi = X_1 \quad\Rightarrow\quad F_{S,q} = \frac{1}{i}i\lambda_{hX_1X_1}v\frac{i}{-m_h^2}\left(-i\frac{m_q}{v}\right) = -\frac{\lambda_{hX_1X_1}m_q}{m_h^2}, \quad F_{S,N} = -\frac{\lambda_{hX_1X_1}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N + 3f_Q^N\right)$$

$$\chi = a^0 \quad\Rightarrow\quad F_{S,q} = -\frac{\lambda_{haa}m_q}{m_h^2}, \quad F_{S,N} = -\frac{\lambda_{haa}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N + 3f_Q^N\right)$$

$$\sigma_{\chi N}^{\mathrm{SI}} = \frac{m_N^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}$$

Gauge interactions

$$D^\mu\Phi = \left(\partial_\mu - \frac{1}{2}ig'B_\mu - \frac{1}{2}igW_\mu^a\sigma^a\right)\Phi$$

$$\frac{1}{2}(g'B_\mu + gW_\mu^a\sigma^a)\Phi = \begin{pmatrix} eA_\mu - \dfrac{g}{2c_W}(s_W^2 - c_W^2)Z_\mu & \dfrac{g}{\sqrt{2}}W_\mu^+ \\[2mm] \dfrac{g}{\sqrt{2}}W_\mu^- & -\dfrac{g}{2c_W}Z_\mu \end{pmatrix}\Phi = \begin{pmatrix} \left[eA_\mu + \dfrac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]\phi^+ + \dfrac{g}{2}W_\mu^+(\phi^0 + ia^0) \\[2mm] \dfrac{g}{\sqrt{2}}W_\mu^-\phi^+ - \dfrac{g}{2\sqrt{2}c_W}Z_\mu(\phi^0 + ia^0) \end{pmatrix}$$

$$\varphi_1\overleftrightarrow{\partial^\mu}\varphi_2 \equiv \varphi_1\partial^\mu\varphi_2 - (\partial^\mu\varphi_1)\varphi_2 = -\varphi_2\overleftrightarrow{\partial^\mu}\varphi_1, \quad (\varphi_1 i\overleftrightarrow{\partial^\mu}\varphi_2)^\dagger = \varphi_1 i\overleftrightarrow{\partial^\mu}\varphi_2$$

$$\mathcal{L}_{\mathrm{gauge}} = -\frac{i}{2}(\partial^\mu\Phi^\dagger)(g'B_\mu + gW_\mu^a\sigma^a)\Phi + \frac{i}{2}[(g'B_\mu + gW_\mu^a\sigma^a)\Phi]^\dagger\partial^\mu\Phi + \left|\frac{1}{2}(g'B_\mu + gW_\mu^a\sigma^a)\Phi\right|^2$$

$$= -i\partial^\mu\phi^-\left\{\left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]\phi^+ + \frac{g}{2}W_\mu^+(\phi^0 + ia^0)\right\} - \frac{i}{\sqrt{2}}(\partial^\mu\phi^0 - i\partial^\mu a^0)\left[\frac{g}{\sqrt{2}}W_\mu^-\phi^+ - \frac{g}{2\sqrt{2}c_W}Z_\mu(\phi^0 + ia^0)\right]$$

$$+ i\partial^\mu\phi^+\left\{\left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]\phi^- + \frac{g}{2}W_\mu^-(\phi^0 - ia^0)\right\} + \frac{i}{\sqrt{2}}(\partial^\mu\phi^0 + i\partial^\mu a^0)\left[\frac{g}{\sqrt{2}}W_\mu^+\phi^- - \frac{g}{2\sqrt{2}c_W}Z_\mu(\phi^0 - ia^0)\right]$$

$$+ \left\|\left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]\phi^+ + \frac{g}{2}W_\mu^+(\phi^0 + iA^0)\right\|^2 + \left|\frac{g}{\sqrt{2}}W_\mu^-\phi^+ - \frac{g}{2\sqrt{2}c_W}Z_\mu(\phi^0 + ia^0)\right|^2$$

$$= \frac{g}{2}[W_\mu^+\phi^- i\overleftrightarrow{\partial^\mu}(\phi^0 + ia^0) + h.c.] + eA_\mu\phi^- i\overleftrightarrow{\partial^\mu}\phi^+ + \frac{g}{2c_W}Z_\mu[ia^0 i\overleftrightarrow{\partial^\mu}\phi^0 + (c_W^2 - s_W^2)\phi^- i\overleftrightarrow{\partial^\mu}\phi^+]$$

$$+ \frac{g^2}{4}W_\mu^+W^{-\mu}[2\phi^+\phi^- + (\phi^0)^2 + (a^0)^2] + e^2 A_\mu A^\mu\phi^+\phi^- + \frac{g^2}{4c_W^2}Z_\mu Z^\mu\left[(c_W^2 - s_W^2)^2\phi^+\phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2\right]$$

$$+ \left[\frac{eg}{2}W_\mu^+ A^\mu\phi^-(\phi^0 + ia^0) - \frac{g^2 s_W^2}{2c_W}W_\mu^+ Z^\mu\phi^-(\phi^0 + ia^0) + h.c.\right] + \frac{eg}{c_W}(c_W^2 - s_W^2)A_\mu Z^\mu\phi^+\phi^-$$

Custodial symmetry: $T = U = 0$

1) $\lambda_3 = 2\lambda_2$

Bidoublet formalism: $\quad \mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix}$

$$\tilde{H}^\dagger \tilde{H} = \begin{pmatrix} H^0 & -H^+ \end{pmatrix} \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = H^0 H^{0*} + H^+ H^- = H^\dagger H$$

$$\mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = \mathrm{tr}\left(\begin{pmatrix} \tilde{H}^\dagger \\ H^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2\,|H|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2\,|\Phi|^2$$

$$\tilde{\Phi}^\dagger \tilde{H} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi^0 + ia^0) & -\phi^+ \end{pmatrix} \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)H^{0*} + \phi^+ H^- = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} = H^\dagger \Phi$$

$$\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \left(\begin{pmatrix} \tilde{\Phi}^\dagger \\ \Phi^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{\Phi}^\dagger \tilde{H} + \Phi^\dagger H = \Phi^\dagger H + H^\dagger \Phi$$

$$[\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2 = (\Phi^\dagger H + H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi + 2\,|\Phi^\dagger H|^2$$

$$V(S,\Phi) \supset \frac{1}{2}m_D^2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \kappa S\,\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_{Sh} S^2 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_1 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1)\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 [\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2$$

$$\boxed{\begin{array}{l} \text{Tensor formalism:} \quad \mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \mathcal{F}_i^A = \begin{pmatrix} \Phi_i^\dagger \\ \Phi_i \end{pmatrix}, \quad A \text{ is an } SU(2)_R \text{ indice} \\[2mm] \varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A \mathcal{H}_j^B = \varepsilon^{ij}(\varepsilon_{12}\mathcal{H}_i^1 \mathcal{H}_j^2 + \varepsilon_{21}\mathcal{H}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij}(-H_i^\dagger H_j + H_i H_j^\dagger) \\[1mm] \quad = -H_i^\dagger H^i - H^j H_j^\dagger = -2\,|H|^2 \\[1mm] \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = -\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A \mathcal{H}_j^B, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = -\varepsilon_{AB}\varepsilon^{ij}\mathcal{F}_i^A \mathcal{F}_j^B \\[1mm] \varepsilon_{AB}\varepsilon^{ij}\mathcal{F}_i^A \mathcal{H}_j^B = \varepsilon^{ij}(\varepsilon_{12}\mathcal{F}_i^1 \mathcal{H}_j^2 + \varepsilon_{21}\mathcal{F}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij}(-\Phi_i^\dagger H_j + \Phi_i H_j^\dagger) \\[1mm] \quad = -\Phi_i^\dagger H^i - \Phi^j H_j^\dagger = -\Phi^\dagger H - H^\dagger \Phi = -\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) \end{array}}$$

2) $\lambda_3 = -2\lambda_2$ and $\kappa = 0$

$$\mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} -\tilde{\Phi} & \Phi \end{pmatrix}$$

$$\mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = 2\,|H|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2\,|\Phi|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \Phi^\dagger H - H^\dagger \Phi$$

$$\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2 = (\Phi^\dagger H - H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi - 2\,|\Phi^\dagger H|^2$$

$$V(S,\Phi) \supset \frac{1}{2}m_D^2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \frac{1}{4}\lambda_{Sh} S^2 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_1 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1)\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2$$

$\lambda_2 = \lambda_3 = 0$ and $\kappa = 0$: $\quad S = T = U = 0$

$$\mathcal{L}_{\mathrm{mass}} = -\frac{1}{2}\left(m_S^2 + \frac{1}{2}\lambda_{Sh}v^2 \right)S^2 - \left(m_D^2 + \frac{1}{2}\lambda_1 v^2 \right)\left[\frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + |\phi^+|^2 \right]$$

$\Rightarrow$ No mixing, components of Φ have exactly degenerate masses

$\Rightarrow$ $S = T = U = 0$

$\kappa = 0$ and $m_S \to \infty$: inert Higgs doublet model [Deshpande & Ma, PRD **18**, 2574 (1978)]

$$s_\theta = 0, \quad c_\theta = 1$$

$$m_{\phi^0}^2 = m_D^2 + \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)v^2, \quad m_a^2 = m_D^2 + \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)v^2, \quad m_C^2 = m_D^2 + \frac{1}{2}\lambda_1 v^2$$

$$\lambda_3 < 2|\lambda_2|, \quad \text{DM candidate} = \begin{cases} \phi^0, & \lambda_2 < 0 \\ a^0, & \lambda_2 > 0 \end{cases}$$

$$\lambda_{h\phi^0\phi^0} = -(\lambda_1 + 2\lambda_2 + \lambda_3), \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

Singlet-Triplet Scalar Dark Matter (STSDM)

CP-even real scalar singlet S and complex scalar doublet Φ:

$$S \in (\mathbf{1}, 0), \quad \Delta = \begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} \in (\mathbf{3}, 0), \quad \Delta^0 = \frac{1}{\sqrt{2}}(\phi^0 + i a^0)$$

$$\Delta^+ = \Delta_2^1, \quad \Delta^- = \Delta_1^2, \quad \Delta^0 = \sqrt{2}\Delta_2^2 = -\sqrt{2}\Delta_1^1$$

$$\sum_i \Delta_i^i = 0, \quad \Delta_j^i = \varepsilon^{ik}\varepsilon_{jl}\Delta_k^l, \quad (\Delta^\dagger)_i^j = (\Delta_j^i)^\dagger$$

$$(\Delta^+)^* = (\Delta^\dagger)_1^2, \quad (\Delta^-)^* = (\Delta^\dagger)_2^1, \quad (\Delta^0)^* = \sqrt{2}(\Delta^\dagger)_2^2 = -\sqrt{2}(\Delta^\dagger)_1^1$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)\partial^\mu S + (D_\mu \Delta)^\dagger D^\mu \Delta - V(S, \Delta)$$

$$V(S, \Delta) = \frac{1}{2}m_S^2 S^2 + m_\Delta^2 |\Delta|^2 + \frac{1}{2}\lambda_{Sh} S^2 |H|^2 + \lambda_0 |H|^2 |\Delta|^2 + \lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k + \lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k$$

$$+ (-\lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k - \lambda_3' |H|^2 \Delta_j^i \Delta_i^j - \lambda_4 S H_i^\dagger \Delta_j^i H^j + h.c.) + (\text{irrelevant terms})$$

$$|\Delta|^2 = (\Delta^\dagger)_i^j \Delta_j^i = (\Delta^+)^* \Delta^+ + \Delta^-(\Delta^-)^* + (\Delta^0)^* \Delta^0, \quad (\Delta^0)^* \Delta^0 = \frac{1}{2}[(\phi^0)^2 + (a^0)^2], \quad (\Delta^0)^2 + h.c. = (\phi^0)^2 - (a^0)^2$$

$$H^1 \to 0, \quad H^2 \to \frac{v+h}{\sqrt{2}}, \quad H_1^\dagger \to 0, \quad H_2^\dagger \to \frac{v+h}{\sqrt{2}}$$

$$\lambda_0 |H|^2 |\Delta|^2 \to \frac{\lambda_0}{2}(v+h)^2 [(\Delta^+)^* \Delta^+ + \Delta^-(\Delta^-)^* + (\Delta^0)^* \Delta^0]$$

$$\lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k \to \lambda_1 H_2^\dagger \Delta_j^2 (\Delta^\dagger)_2^j H^2 = \frac{\lambda_1}{2}(v+h)^2 \left[\Delta^-(\Delta^-)^* + \frac{1}{2}(\Delta^0)^* \Delta^0\right]$$

$$\lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k \to \lambda_2 H_2^\dagger (\Delta^\dagger)_j^2 \Delta_2^j H^2 = \frac{\lambda_2}{2}(v+h)^2 \left[(\Delta^+)^* \Delta^+ + \frac{1}{2}(\Delta^0)^* \Delta^0\right]$$

$$-\lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k \to -\lambda_3 H_2^\dagger \Delta_j^2 \Delta_2^j H^2 = -\frac{\lambda_3}{2}(v+h)^2 \left[\Delta^- \Delta^+ + \frac{1}{2}(\Delta^0)^2\right]$$

$$-\lambda_3' |H|^2 \Delta_j^i \Delta_i^j \to -\frac{\lambda_3'}{2}(v+h)^2 [2\Delta^- \Delta^+ + (\Delta^0)^2] \propto \lambda_3 H_i^\dagger \Delta_j^i \Delta_k^j H^k \quad [\lambda_3' \text{ can be absorbed into } \lambda_3 \text{ in the unitary gauge}]$$

$$-\lambda_4 S H_i^\dagger \Delta_j^i H^j \to -\lambda_4 S H_2^\dagger \Delta_2^2 H^2 = -\frac{\lambda_4}{2\sqrt{2}}(v+h)^2 S\Delta^0 = -\frac{\lambda_4}{4}(v+h)^2 S(\phi^0 + i a^0)$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}m_a^2(a^0)^2 - \frac{1}{2}\begin{pmatrix} S & \phi^0 \end{pmatrix} M_0^2 \begin{pmatrix} S \\ \phi^0 \end{pmatrix} - \begin{pmatrix} (\Delta^+)^* & \Delta^- \end{pmatrix} M_C^2 \begin{pmatrix} \Delta^+ \\ (\Delta^-)^* \end{pmatrix}$$

$$m_a^2 = m_\Delta^2 + \frac{1}{4}(2\lambda_0 + \lambda_1 + \lambda_2 + 2\lambda_3 + 4\lambda_3')v^2$$

$$M_0^2 = \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{Sh} v^2 & -\frac{1}{2}\lambda_4 v^2 \\ -\frac{1}{2}\lambda_4 v^2 & m_\Delta^2 + \frac{1}{4}(2\lambda_0 + \lambda_1 + \lambda_2 - 2\lambda_3 - 4\lambda_3')v^2 \end{pmatrix}, \quad M_C^2 = \begin{pmatrix} m_\Delta^2 + \frac{1}{2}(\lambda_0 + \lambda_2)v^2 & -\frac{1}{2}(\lambda_3 + 2\lambda_3')v^2 \\ -\frac{1}{2}(\lambda_3 + 2\lambda_3')v^2 & m_\Delta^2 + \frac{1}{2}(\lambda_0 + \lambda_1)v^2 \end{pmatrix}$$

$$\lambda_\pm \equiv \lambda_1 \pm \lambda_2$$

$$\lambda_1 H_i^\dagger \Delta_j^i (\Delta^\dagger)_k^j H^k + \lambda_2 H_i^\dagger (\Delta^\dagger)_j^i \Delta_k^j H^k \to \frac{\lambda_+}{4}(v+h)^2 [(\Delta^+)^* \Delta^+ + \Delta^-(\Delta^-)^* + (\Delta^0)^* \Delta^0] + \frac{\lambda_-}{4}(v+h)^2 [\Delta^-(\Delta^-)^* - (\Delta^+)^* \Delta^+]$$

$$\Rightarrow \quad \lambda_0 \text{ can be absorbed into } \lambda_+ \text{ in the unitary gauge}$$

$$m_a^2 \to m_\Delta^2 + \frac{1}{4}(\lambda_+ + 2\lambda_3)v^2, \quad M_0^2 \to \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{Sh} v^2 & -\frac{1}{2}\lambda_4 v^2 \\ -\frac{1}{2}\lambda_4 v^2 & m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\lambda_3)v^2 \end{pmatrix}, \quad M_C^2 \to \begin{pmatrix} m_\Delta^2 + \frac{1}{4}(\lambda_+ - \lambda_-)v^2 & -\frac{1}{2}\lambda_3 v^2 \\ -\frac{1}{2}\lambda_3 v^2 & m_\Delta^2 + \frac{1}{4}(\lambda_+ + \lambda_-)v^2 \end{pmatrix}$$

$$U^{\mathrm{T}}M_{\mathrm{C}}^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} \Delta^+ \\ (\Delta^-)^* \end{pmatrix} = U \begin{pmatrix} \Delta_1^+ \\ \Delta_2^+ \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{ (M_{\mathrm{C}}^2)_{11} + (M_{\mathrm{C}}^2)_{22} \mp \sqrt{[(M_{\mathrm{C}}^2)_{11} - (M_{\mathrm{C}}^2)_{22}]^2 + 4(M_{\mathrm{C}}^2)_{12}^2} \right\} = m_\Delta^2 + \frac{v^2}{4}\left(\lambda_+ \mp \sqrt{\lambda_-^2 + 4\lambda_3^2} \right)$$

$$V^{\mathrm{T}}M_0^2 V = \begin{pmatrix} \mu_1^2 & \\ & \mu_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix} = V \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} c_\alpha X_1 - s_\alpha X_2 \\ s_\alpha X_1 + c_\alpha X_2 \end{pmatrix}$$

$$\mu_{1,2}^2 = \frac{1}{2}\left\{ (M_0^2)_{11} + (M_0^2)_{22} \mp \sqrt{[(M_0^2)_{11} - (M_0^2)_{22}]^2 + 4(M_0^2)_{12}^2} \right\}$$

$$\text{DM candidate} = \begin{cases} X_1, & \mu_1^2 < m_a^2, m_1^2 \\ a^0, & m_a^2 < \mu_1^2, m_{\mathrm{C}}^2 \end{cases}$$

$$\mathcal{L}_{hXX,haa} = -\frac{1}{2}\lambda_{Sh}vhS^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(\phi^0)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4 vhS\phi^0$$

$$= -\frac{1}{2}\lambda_{Sh}vh(c_\alpha X_1 - s_\alpha X_2)^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(s_\alpha X_1 + c_\alpha X_2)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4 vh(c_\alpha X_1 - s_\alpha X_2)(s_\alpha X_1 + c_\alpha X_2)$$

$$= \frac{1}{2}\lambda_{hX_1X_1}vhX_1^2 + \frac{1}{2}\lambda_{hX_2X_2}vhX_2^2 + \lambda_{hX_1X_2}vhX_1X_2 + \frac{1}{2}\lambda_{haa}vh(a^0)^2$$

$$\lambda_{hX_1X_1} = -\left[\lambda_{Sh}c_\alpha^2 + \frac{1}{2}(\lambda_+ - 2\lambda_3)s_\alpha^2 - 2\lambda_4 s_\alpha c_\alpha \right], \quad \lambda_{hX_2X_2} = -\left[\lambda_{Sh}s_\alpha^2 + \frac{1}{2}(\lambda_+ - 2\lambda_3)c_\alpha^2 + 2\lambda_4 s_\alpha c_\alpha \right]$$

$$\lambda_{hX_1X_2} = \left[\lambda_{Sh} - \frac{1}{2}(\lambda_+ - 2\lambda_3) \right]s_\alpha c_\alpha + \lambda_4(c_\alpha^2 - s_\alpha^2), \quad \lambda_{haa} = -\frac{1}{2}(\lambda_+ + 2\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{\mathrm{SI}} = \frac{m_p^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}, \quad \begin{cases} \text{DM candidate } \chi = X_1 \;\Rightarrow\; F_{S,N} = -\dfrac{\lambda_{hX_1X_1}m_N}{m_h^2}\left(\displaystyle\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \\[4mm] \text{DM candidate } \chi = a^0 \;\Rightarrow\; F_{S,N} = -\dfrac{\lambda_{haa}m_N}{m_h^2}\left(\displaystyle\sum_{q=u,d,s} f_q^N + 3f_Q^N \right) \end{cases}$$

Gauge interactions

$$D^\mu \Delta = (\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a)\Delta$$

$$t_{\mathrm{T}}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} & -1 & \\ -1 & & 1 \\ & 1 & \end{pmatrix}, \quad t_{\mathrm{T}}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} & i & \\ -i & & -i \\ & i & \end{pmatrix}, \quad t_{\mathrm{T}}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$gW_\mu^a t_{\mathrm{T}}^a \Delta = \begin{pmatrix} gW_\mu^3 & -g(W_\mu^1 - iW_\mu^2)/\sqrt{2} & 0 \\ -g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & 0 & g(W_\mu^1 - iW_\mu^2)/\sqrt{2} \\ 0 & g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & -gW_\mu^3 \end{pmatrix}\begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} = \begin{pmatrix} (eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^+ - gW_\mu^+\Delta^0 \\ -gW_\mu^-\Delta^+ + gW_\mu^+\Delta^- \\ gW_\mu^-\Delta^0 - (eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^- \end{pmatrix}$$

$$\mathcal{L}_{\mathrm{gauge}} = -i(\partial^\mu \Delta^\dagger)gW_\mu^a t_{\mathrm{T}}^a \Delta + i(gW_\mu^a t_{\mathrm{T}}^a \Phi)^\dagger \partial^\mu \Delta + |gW_\mu^a t_{\mathrm{T}}^a \Delta|^2$$

$$= -i\partial^\mu(\Delta^+)^*[(eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^+ - gW_\mu^+\Delta^0] - i\partial^\mu(\Delta^0)^*(-gW_\mu^-\Delta^+ + gW_\mu^+\Delta^-) - i\partial^\mu(\Delta^-)^*[gW_\mu^-\Delta^0 - (eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^-]$$

$$+ i\partial^\mu\Delta^+[(eA_\mu + gc_{\mathrm{W}}Z_\mu)(\Delta^+)^* - gW_\mu^-(\Delta^0)^*] + i\partial^\mu\Delta^0[-gW_\mu^+(\Delta^+)^* + gW_\mu^-(\Delta^-)^*] + i\partial^\mu\Delta^-[gW_\mu^+(\Delta^0)^* - (eA_\mu + gc_{\mathrm{W}}Z_\mu)(\Delta^-)^*]$$

$$+ |(eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^+ - gW_\mu^+\Delta^0|^2 + |-gW_\mu^-\Delta^+ + gW_\mu^+\Delta^-|^2 + |gW_\mu^-\Delta^0 - (eA_\mu + gc_{\mathrm{W}}Z_\mu)\Delta^-|^2$$

$$= \frac{g}{\sqrt{2}}[W_\mu^+(\phi^0 + ia^0)i\overrightarrow{\partial^\mu}(\Delta^+)^* + W_\mu^+(\phi^0 - ia^0)i\overrightarrow{\partial^\mu}\Delta^- + h.c.] + (eA_\mu + gc_{\mathrm{W}}Z_\mu)[(\Delta^+)^*i\overrightarrow{\partial^\mu}\Delta^+ + \Delta^- i\overrightarrow{\partial^\mu}(\Delta^-)^*]$$

$$+ g^2 W_\mu^+ W^{-\mu}[|\Delta^+|^2 + |\Delta^-|^2 + (\phi^0)^2 + (a^0)^2] - g^2[W_\mu^+ W^{+\mu}(\Delta^+)^*\Delta^- + h.c.]$$

$$+ (e^2 A_\mu A^\mu + g^2 c_{\mathrm{W}}^2 Z_\mu Z^\mu + 2egc_{\mathrm{W}}A_\mu Z^\mu)(|\Delta^+|^2 + |\Delta^-|^2)$$

$$- \frac{g}{\sqrt{2}}[W_\mu^+(\Delta^+)^*(\phi^0 + ia^0) + W_\mu^+\Delta^-(\phi^0 - ia^0) + h.c.](eA^\mu + gc_{\mathrm{W}}Z^\mu)$$

Custodial symmetry limit $\lambda_- = \lambda_4 = 0$: $T = U = 0$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}$$

$$\varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B = \varepsilon_{12}\mathcal{H}_i^1\mathcal{H}_j^2 + \varepsilon_{21}\mathcal{H}_i^2\mathcal{H}_j^1 = -H_i^\dagger H_j + H_i H_j^\dagger$$

$$\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = \varepsilon^{ij}(-H_i^\dagger H_j + H_i H_j^\dagger) = -H_i^\dagger H^i - H^j H_j^\dagger = -2\,|H|^2$$

Custodial symmetric potential:

$$V_{\text{cust}} \supset \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B\left[-\frac{1}{4}\lambda_{Sh}S^2\varepsilon^{ij} - \frac{1}{2}\lambda_0\,|\Delta|^2\,\varepsilon^{ij} + \frac{1}{2}\lambda_3'\Delta_n^m\Delta_m^n\varepsilon^{ij} + (\lambda_a S\Delta_k^i\varepsilon^{kj} + h.c.) + \lambda_b\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl} + (\lambda_c\Delta_k^i\Delta_l^j\varepsilon^{kl} + h.c.)\right]$$

$$\left[\begin{array}{l} \text{For scalars } A^i \text{ and } B^i, \quad A^iB_i = A^i\varepsilon_{ij}B^j = -A^i\varepsilon_{ji}B^j = -B^j\varepsilon_{ji}A^i = -B^iA_i = -A_iB^i \\ \Rightarrow \quad \text{Raising an index and lowering the index contracted to it give a minus sign} \end{array}\right]$$

$$V_a = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_a S\Delta_k^i\varepsilon^{kj}] = \lambda_a S(-H_i^\dagger H_j + H_i H_j^\dagger)\Delta_k^i\varepsilon^{kj} = \lambda_a S(-H_i^\dagger\Delta_j^i H^j + H^{\dagger i}\Delta_i^j H_j) = \lambda_a S(-H_i^\dagger\Delta_j^i H^j + H_i^\dagger\Delta_j^i H^j) = 0$$

$$V_b = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_b\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl}] = \lambda_b(-H_i^\dagger H_j + H_i H_j^\dagger)\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl} = \lambda_b[-H_i^\dagger\Delta_k^i\varepsilon^{kl}(\Delta^\dagger)_l^j H_j - H_j^\dagger(\Delta^\dagger)_l^j\varepsilon^{lk}\Delta_k^i H_i]$$

$$= \lambda_b[H_i^\dagger\Delta_k^i(\Delta^\dagger)_j^k H^j + H_j^\dagger(\Delta^\dagger)_l^j\Delta_i^l H^i] = \lambda_b[H_i^\dagger\Delta_j^i(\Delta^\dagger)_k^j H^k + H_i^\dagger(\Delta^\dagger)_j^i\Delta_k^j H^k]$$

$$V_c = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_c\Delta_k^i\Delta_l^j\varepsilon^{kl}] = \lambda_c(-H_i^\dagger H_j + H_i H_j^\dagger)\Delta_k^i\Delta_l^j\varepsilon^{kl} = \lambda_c(-H_i^\dagger\Delta_k^i\varepsilon^{kl}\Delta_l^j H_j - H_j^\dagger\Delta_l^j\varepsilon^{lk}\Delta_k^i H_i)$$

$$= \lambda_c(H_i^\dagger\Delta_k^i\Delta_j^k H^j + H_j^\dagger\Delta_l^j\Delta_i^l H^i) = 2\lambda_c H_i^\dagger\Delta_j^i\Delta_k^j H^k$$

$$\lambda_4 = 0, \quad \lambda_1 = \lambda_2 = \lambda_b(\Rightarrow \lambda_- = 0), \quad -\lambda_3 = 2\lambda_c \quad \Rightarrow \quad \text{Custodial symmetry}$$

$\lambda_- = \lambda_4 = \lambda_3 = 0$: components of Δ have exactly degenerate masses

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\left(m_S^2 + \frac{1}{2}\lambda_{Sh}v^2\right)S^2 - \left(m_\Delta^2 + \frac{1}{4}\lambda_+ v^2\right)^2\left[\frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + |\Delta^+|^2 + |\Delta^-|^2\right]$$

$\lambda_4 = 0$ and $m_S \to \infty$: complex triplet model

$$s_\alpha = 0, \quad c_\alpha = 1$$

$$m_{\phi^0}^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\lambda_3)v^2, \quad m_a^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ + 2\lambda_3)v^2$$

$$m_1^2 = m_\Delta^2 + \frac{1}{4}\lambda_+ v^2 - \frac{1}{4}\sqrt{\lambda_-^2 + 4\lambda_3^2}\,v^2 \le m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\,|\lambda_3|)v^2 \le \min(m_{\phi^0}^2, m_a^2)$$

$$\lambda_- = 0 \quad \Rightarrow \quad m_1^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\,|\lambda_3|)v^2 = \min(m_{\phi^0}^2, m_a^2)$$

$$\lambda_- \ne 0 \quad \Rightarrow \quad \text{unstable DM candidate}$$

Quadruplet Scalar Dark Matter (QSDM)

Complex scalar quadruplet X :

$$X = \begin{pmatrix} X^{++} \\ X^{+} \\ X^{0} \\ X^{-} \end{pmatrix} \in \left(\mathbf{4}, \frac{1}{2}\right), \quad X^0 = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)$$

$$X^{++} = X_2^{11}, \quad X^+ = \sqrt{3}X_1^{11} = -\sqrt{3}X_2^{12} = -\sqrt{3}X_2^{21}, \quad X^0 = \sqrt{3}X_2^{22} = -\sqrt{3}X_1^{12} = -\sqrt{3}X_1^{21}, \quad X^- = X_1^{22}$$

$$X_k^{ij} = X_k^{ji}, \quad \sum_k X_k^{ik} = \sum_k X_k^{kj} = 0, \quad (X^\dagger)_{ij}^k = (X_k^{ij})^\dagger$$

$$(X^{++})^* = (X^\dagger)_{11}^2, \quad (X^+)^* = \sqrt{3}(X^\dagger)_{11}^1 = -\sqrt{3}(X^\dagger)_{12}^2 = -\sqrt{3}(X^\dagger)_{21}^2, \quad (X^0)^* = \sqrt{3}(X^\dagger)_{22}^2 = -\sqrt{3}(X^\dagger)_{12}^1 = -\sqrt{3}(X^\dagger)_{21}^1, \quad (X^-)^* = (X^\dagger)_{22}^1$$

$$\mathcal{L} = (D_\mu X)^\dagger D^\mu X - V(X)$$

$$V(X) = m_X^2 \,|X|^2 + \lambda_0 \,|H|^2|X|^2 + \lambda_1 H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l - (\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) + \text{(irrelevant terms)}$$

$$|X|^2 = (X^\dagger)_{ij}^k X_k^{ij} = (X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-, \quad (X^0)^* X^0 = \frac{1}{2}[(\phi^0)^2 + (a^0)^2], \quad (X^0)^2 + h.c. = (\phi^0)^2 - (a^0)^2$$

$$H^1 \to 0, \quad H^2 \to \frac{v+h}{\sqrt{2}}, \quad H_1^\dagger \to 0, \quad H_2^\dagger \to \frac{v+h}{\sqrt{2}}$$

$$\lambda_0 \,|H|^2|X|^2 \to \frac{\lambda_0}{2}(v+h)^2[(X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-]$$

$$\lambda_1 H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l \to \lambda_1 H_2^\dagger X_k^{2j}(X^\dagger)_{j2}^k H^2 = \frac{\lambda_1}{2}(v+h)^2[X_1^{21}(X^\dagger)_{12}^1 + X_1^{22}(X^\dagger)_{22}^1 + X_2^{21}(X^\dagger)_{12}^2 + X_2^{22}(X^\dagger)_{22}^2]$$

$$= \frac{\lambda_1}{2}(v+h)^2\left[\frac{1}{3}(X^+)^* X^+ + \frac{2}{3}(X^0)^* X^0 + (X^-)^* X^-\right]$$

$$\lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l \to \lambda_2 H_2^\dagger (X^\dagger)_{jk}^2 X_2^{jk} H^2 = \frac{\lambda_2}{2}(v+h)^2[(X^\dagger)_{11}^2 X_2^{11} + (X^\dagger)_{12}^2 X_2^{12} + (X^\dagger)_{21}^2 X_2^{21} + (X^\dagger)_{22}^2 X_2^{22}]$$

$$= \frac{\lambda_2}{2}(v+h)^2[(X^{++})^* X^{++} + \frac{2}{3}(X^+)^* X^+ + \frac{1}{3}(X^0)^* X^0]$$

$$-\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} \to -\lambda_3 H_2^\dagger H_2^\dagger X_l^{2k} X_k^{2l} = -\frac{\lambda_3}{2}(v+h)^2(X_1^{21}X_1^{21} + X_2^{21}X_1^{22} + X_1^{22}X_2^{21} + X_2^{22}X_2^{22}) = -\frac{\lambda_3}{2}(v+h)^2\left[\frac{2}{3}(X^0)^2 - \frac{2}{\sqrt{3}}X^+X^-\right]$$

$$\mathcal{L}_{\text{mass}} = -m_{++}^2 \,|X^{++}|^2 - \left((X^+)^* \quad X^-\right) M_C^2 \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} - \frac{1}{2}m_\phi^2(\phi^0)^2 - \frac{1}{2}m_a^2(a^0)^2$$

$$m_{++}^2 = m_X^2 + \frac{1}{2}(\lambda_0 + \lambda_2)v^2, \quad m_\phi^2 = m_X^2 + \left(\frac{1}{2}\lambda_0 + \frac{1}{3}\lambda_1 + \frac{1}{6}\lambda_2 - \frac{2}{3}\lambda_3\right)v^2, \quad m_a^2 = m_X^2 + \left(\frac{1}{2}\lambda_0 + \frac{1}{3}\lambda_1 + \frac{1}{6}\lambda_2 + \frac{2}{3}\lambda_3\right)v^2$$

$$M_C^2 = \begin{pmatrix} m_X^2 + \left(\dfrac{\lambda_0}{2} + \dfrac{\lambda_1}{6} + \dfrac{\lambda_2}{3}\right)v^2 & \dfrac{\lambda_3}{\sqrt{3}}v^2 \\ \dfrac{\lambda_3}{\sqrt{3}}v^2 & m_X^2 + \dfrac{1}{2}(\lambda_0 + \lambda_1)v^2 \end{pmatrix}$$

$$\lambda_\pm \equiv \lambda_1 \pm \lambda_2$$

$$\lambda_1 H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l \to \frac{\lambda_+}{4}(v+h)^2\left[(X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^- + (X^{++})^* X^{++}\right]$$

$$+ \frac{\lambda_-}{4}(v+h)^2\left[-(X^{++})^* X^{++} - \frac{1}{3}(X^+)^* X^+ + \frac{1}{3}(X^0)^* X^0 + (X^-)^* X^-\right]$$

$\Rightarrow$ λ_0 can be absorbed into λ_+ in the unitary gauge

$$m_{++}^2 \to m_X^2 + \frac{1}{4}(\lambda_+ - \lambda_-)v^2, \quad m_\phi^2 \to m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- - 8\lambda_3)v^2, \quad m_a^2 \to m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- + 8\lambda_3)v^2$$

$$M_C^2 \to \begin{pmatrix} m_X^2 + \frac{1}{12}(3\lambda_+ - \lambda_-)v^2 & \frac{\lambda_3}{\sqrt{3}}v^2 \\ \frac{\lambda_3}{\sqrt{3}}v^2 & m_X^2 + \frac{1}{4}(\lambda_+ + \lambda_-)v^2 \end{pmatrix}$$

$$U^T M_C^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} = U \begin{pmatrix} X_1^+ \\ X_2^+ \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{(M_C^2)_{11} + (M_C^2)_{22} \mp \sqrt{[(M_C^2)_{11} - (M_C^2)_{22}]^2 + 4(M_C^2)_{12}^2}\right\} = m_X^2 + \frac{v^2}{12}\left(3\lambda_+ + \lambda_- \mp 2\sqrt{\lambda_-^2 + 12\lambda_3^2}\right)$$

$$\text{DM candidate} = \begin{cases} \phi^0, & \lambda_3 > 0 \text{ and } |\lambda_-| < 2\lambda_3 \\ a^0, & \lambda_3 < 0 \text{ and } |\lambda_-| < -2\lambda_3 \end{cases}$$

$$\mathcal{L}_{h\phi\phi, haa} = \frac{1}{2}\lambda_{h\phi\phi}vh(\phi^0)^2 + \frac{1}{2}\lambda_{haa}vh(a^0)^2, \quad \lambda_{h\phi\phi} = -\frac{1}{6}(3\lambda_+ + \lambda_- - 8\lambda_3), \quad \lambda_{haa} = -\frac{1}{6}(3\lambda_+ + \lambda_- + 8\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{SI} = \frac{m_p^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}, \quad \begin{cases} \text{DM candidate } \chi = \phi^0 \Rightarrow F_{S,N} = -\frac{\lambda_{h\phi\phi}m_N}{m_h^2}\left(\sum_{q=u,d,s} f_q^N + 3f_Q^N\right) \\ \text{DM candidate } \chi = a^0 \Rightarrow F_{S,N} = -\frac{\lambda_{haa}m_N}{m_h^2}\left(\sum_{q=u,d,s} f_q^N + 3f_Q^N\right) \end{cases}$$

$\underline{\lambda_- = \lambda_3 = 0:\ S = T = U = 0}$

$$\mathcal{L}_{mass} = -\left(m_X^2 + \frac{1}{4}\lambda_+ v^2\right)\left[|X^{++}|^2 + |X^+|^2 + |X^-|^2 + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2\right]$$

$\Rightarrow$ No mixing between X^+ and $(X^-)^*$, components of X have exactly degenerate masses

$\Rightarrow$ $S = T = U = 0$

$$D^\mu X = (\partial_\mu - \tfrac{1}{2}ig'B_\mu - igW_\mu^a t_Q^a)X$$

$$t_Q^1 = \begin{pmatrix} & -\sqrt{3}/2 & & \\ -\sqrt{3}/2 & & -1 & \\ & -1 & & -\sqrt{3}/2 \\ & & -\sqrt{3}/2 & \end{pmatrix}, \quad t_Q^2 = \begin{pmatrix} & \sqrt{3}i/2 & & \\ -\sqrt{3}i/2 & & i & \\ & -i & & \sqrt{3}i/2 \\ & & -\sqrt{3}i/2 & \end{pmatrix}, \quad t_Q^3 = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix}$$

$$\left(\tfrac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X = \begin{pmatrix} 2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu & -\frac{\sqrt{6}}{2}gW_\mu^+ & 0 & 0 \\ -\frac{\sqrt{6}}{2}gW_\mu^- & eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu & -\sqrt{2}gW_\mu^+ & 0 \\ 0 & -\sqrt{2}gW_\mu^- & -\frac{g}{2c_W}Z_\mu & -\frac{\sqrt{6}}{2}gW_\mu^+ \\ 0 & 0 & -\frac{\sqrt{6}}{2}gW_\mu^- & -eA_\mu - \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu \end{pmatrix}\begin{pmatrix} X^{++} \\ X^+ \\ X^0 \\ X^- \end{pmatrix}$$

$$= \begin{pmatrix} \left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+ X^+ \\ -\frac{\sqrt{6}}{2}gW_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+ X^0 \\ -\sqrt{2}gW_\mu^- X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+ X^- \\ -\frac{\sqrt{6}}{2}gW_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^- \end{pmatrix}$$

$$\mathcal{L}_{\text{gauge}} = \left[-i(\partial^\mu\Delta^\dagger)\left(\tfrac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X + h.c.\right] + \left|\left(\tfrac{1}{2}g'B_\mu + gW_\mu^a t_Q^a\right)X\right|^2$$

$$= \left\{-i\partial^\mu(X^{++})^*\left(\left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+ X^+\right) - i\partial^\mu(X^+)^*\left(-\frac{\sqrt{6}}{2}gW_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+ X^0\right)\right.$$

$$\left. -i\partial^\mu(X^0)^*\left(-\sqrt{2}gW_\mu^- X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+ X^-\right) - i\partial^\mu(X^-)^*\left(-\frac{\sqrt{6}}{2}gW_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^-\right) + h.c.\right\}$$

$$+ \left|\left[2eA_\mu + \frac{g}{2c_W}(3c_W^2 - s_W^2)Z_\mu\right]X^{++} - \frac{\sqrt{6}}{2}gW_\mu^+ X^+\right|^2 + \left|-\frac{\sqrt{6}}{2}gW_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_W}(c_W^2 - s_W^2)Z_\mu\right]X^+ - \sqrt{2}gW_\mu^+ X^0\right|^2$$

$$+ \left|-\sqrt{2}gW_\mu^- X^+ - \frac{g}{2c_W}Z_\mu X^0 - \frac{\sqrt{6}}{2}gW_\mu^+ X^-\right|^2 + \left|-\frac{\sqrt{6}}{2}gW_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_W}(3c_W^2 + s_W^2)Z_\mu\right]X^-\right|^2$$

$$= g\left[\frac{\sqrt{6}}{2}W_\mu^+ X^+ i\overleftrightarrow{\partial^\mu}(X^{++})^* + W_\mu^+(\phi^0 + ia^0)i\overleftrightarrow{\partial^\mu}(X^+)^* + \frac{\sqrt{3}}{2}W_\mu^+ X^- i\overleftrightarrow{\partial^\mu}(\phi^0 - ia^0) + h.c.\right]$$

$$+ eA_\mu[2(X^{++})^* i\overleftrightarrow{\partial^\mu}X^{++} + (X^+)^* i\overleftrightarrow{\partial^\mu}X^+ - (X^-)^* i\overleftrightarrow{\partial^\mu}X^-]$$

$$+ \frac{g}{2c_W}Z_\mu[(3c_W^2 - s_W^2)(X^{++})^* i\overleftrightarrow{\partial^\mu}X^{++} + (c_W^2 - s_W^2)(X^+)^* i\overleftrightarrow{\partial^\mu}X^+ + ia^0 i\overleftrightarrow{\partial^\mu}\phi^0 - (3c_W^2 + s_W^2)(X^-)^* i\overleftrightarrow{\partial^\mu}X^-]$$

$$+ g^2 W_\mu^+ W^{-\mu}\left[\frac{3}{2}|X^{++}|^2 + \frac{7}{2}|X^+|^2 + \frac{7}{4}(\phi^0)^2 + \frac{7}{4}(a^0)^2 + \frac{3}{2}|X^-|^2\right] + g^2\left[\sqrt{3}W_\mu^+ W^{+\mu}(X^{++})^* X^0 + \sqrt{3}W_\mu^+ W^{+\mu}(X^+)^* X^- + h.c.\right]$$

$$+ e^2 A_\mu A^\mu(4|X^{++}|^2 + |X^+|^2 + |X^-|^2) + \frac{eg}{c_W}A_\mu Z^\mu[2(3c_W^2 - s_W^2)|X^{++}|^2 + (c_W^2 - s_W^2)|X^+|^2 + (3c_W^2 + s_W^2)|X^-|^2]$$

$$+ \frac{g^2}{4c_W^2}Z_\mu Z^\mu\left[(3c_W^2 - s_W^2)^2|X^{++}|^2 + (c_W^2 - s_W^2)^2|X^+|^2 + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + (3c_W^2 + s_W^2)^2|X^-|^2\right]$$

$$+ \left\{-\frac{\sqrt{6}}{2}\left[2egA^\mu + \frac{g^2(3c_W^2 - s_W^2)}{2c_W}Z^\mu\right]W_\mu^+ X^+(X^{++})^* - \left[egA^\mu + \frac{g^2(c_W^2 - s_W^2)}{2c_W}Z^\mu\right]W_\mu^+\left(\frac{\sqrt{6}}{2}X^+(X^{++})^* + (X^+)^*(\phi^0 + ia^0)\right)\right.$$

$$\left. + \frac{\sqrt{3}g^2}{4c_W}Z^\mu W_\mu^+ X^-(\phi^0 - ia^0) + \frac{g^2}{2c_W}Z^\mu W_\mu^+(X^+)^*(\phi^0 + ia^0) + \frac{\sqrt{3}}{2}\left[egA^\mu + \frac{g^2(3c_W^2 + s_W^2)}{2c_W}Z^\mu\right]W_\mu^+ X^-(\phi^0 - ia^0) + h.c.\right\}$$

Custodial symmetry: $T = U = 0$

1) $\lambda_- = -2\lambda_3$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B = -H_i^\dagger H_j + H_i H_j^\dagger, \quad \varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = -2\,|H|^2$$

$$(\mathcal{X}^A)_k^{ij} = \begin{pmatrix} (X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}, \quad (X^\dagger)_k^{ij} = \varepsilon^{il}(X^\dagger)_{lk}^j$$

$$\varepsilon_{AB}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = \varepsilon_{12}(\mathcal{X}^1)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{21}(\mathcal{X}^2)_l^{ik}(\mathcal{X}^1)_k^{jl} = -(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}$$

$$\varepsilon_{AB}\varepsilon_{ij}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = \varepsilon_{ij}[-(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}] = (X^\dagger)_{jl}^k X_k^{jl} + X_l^{ik}(X^\dagger)_{ik}^l = 2\,|X|^2$$

Custodial symmetric potential:

$$V_{\text{cust}} \supset \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} + \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$

$$+ \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} + \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$

[Note: raising an index and lowering the index contracted to it give a minus sign]

$$V_{0a} = \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} = -4\lambda_{0a}\,|H|^2|X|^2$$

$$V_{0b} = \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$

$$= \lambda_{0b}[\varepsilon_{12}\varepsilon_{12}\varepsilon^{ij}\mathcal{H}_i^1\mathcal{H}_j^1\varepsilon_{mn}(\mathcal{X}^2)_l^{mk}(\mathcal{X}^2)_k^{nl} + \varepsilon_{12}\varepsilon_{21}\varepsilon^{ij}\mathcal{H}_i^1\mathcal{H}_j^2\varepsilon_{mn}(\mathcal{X}^2)_l^{mk}(\mathcal{X}^1)_k^{nl}$$

$$+ \varepsilon_{21}\varepsilon_{12}\varepsilon^{ij}\mathcal{H}_i^2\mathcal{H}_j^1\varepsilon_{mn}(\mathcal{X}^1)_l^{mk}(\mathcal{X}^2)_k^{nl} + \varepsilon_{21}\varepsilon_{21}\varepsilon^{ij}\mathcal{H}_i^2\mathcal{H}_j^2\varepsilon_{mn}(\mathcal{X}^1)_l^{mk}(\mathcal{X}^1)_k^{nl}]$$

$$= \lambda_{0b}[\varepsilon^{ij}H_i^\dagger H_j^\dagger\varepsilon_{mn}X_l^{mk}X_k^{nl} - \varepsilon^{ij}H_i^\dagger H_j\varepsilon_{mn}X_l^{mk}(X^\dagger)_k^{nl} - \varepsilon^{ij}H_i H_j^\dagger\varepsilon_{mn}(X^\dagger)_l^{mk}X_k^{nl} + \varepsilon^{ij}H_i H_j\varepsilon_{mn}(X^\dagger)_l^{mk}(X^\dagger)_k^{nl}]$$

$$= \lambda_{0b}[-H_i^\dagger H^i(X^\dagger)_{mk}^l X_l^{mk} - H_j^\dagger H^j\varepsilon_{nm}(X^\dagger)_{nl}^k X_k^{nl}] = -2\lambda_{0b}\,|H|^2|X|^2$$

$$V_a = \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} = \lambda_a(-H_i^\dagger H_j + H_i H_j^\dagger)[-(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}]$$

$$= \lambda_a[H_i^\dagger(X^\dagger)_l^{ik}X_k^{jl}H_j - H_i^\dagger X_l^{ik}(X^\dagger)_k^{jl}H_j - H_j^\dagger X_k^{jl}(X^\dagger)_l^{ik}H_i + H_j^\dagger(X^\dagger)_k^{jl}X_l^{ik}H_i]$$

$$= 2\lambda_a[H_i^\dagger(X^\dagger)_l^{ik}X_k^{jl}H_j - H_i^\dagger X_l^{ik}(X^\dagger)_k^{jl}H_j] = 2\lambda_a[H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + H_i X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_b = \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$

$$= \lambda_b[\varepsilon_{12}\varepsilon_{12}\mathcal{H}_i^1\mathcal{H}_j^1(\mathcal{X}^2)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{12}\varepsilon_{21}\mathcal{H}_i^1\mathcal{H}_j^2(\mathcal{X}^2)_l^{ik}(\mathcal{X}^1)_k^{jl} + \varepsilon_{21}\varepsilon_{12}\mathcal{H}_i^2\mathcal{H}_j^1(\mathcal{X}^1)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{21}\varepsilon_{21}\mathcal{H}_i^2\mathcal{H}_j^2(\mathcal{X}^1)_l^{ik}(\mathcal{X}^1)_k^{jl}]$$

$$= \lambda_b[H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} - H_i^\dagger H_j X_l^{ik}(X^\dagger)_k^{jl} - H_i H_j^\dagger(X^\dagger)_l^{ik}X_k^{jl} + H_i H_j(X^\dagger)_l^{ik}(X^\dagger)_k^{jl}]$$

$$= \lambda_b[(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.) + 2H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset -(4\lambda_{0a} + 2\lambda_{0b})\,|H|^2|X|^2 + 2(\lambda_a + \lambda_b)H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + 2\lambda_a H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + \lambda_b(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.)$$

Custodial symmetry $\Rightarrow$
$$\begin{cases} \lambda_0 = -(4\lambda_{0a} + 2\lambda_{0b}) \\ \lambda_1 = 2(\lambda_a + \lambda_b) \\ \lambda_2 = 2\lambda_a \\ -\lambda_3 = \lambda_b \end{cases} \Rightarrow \lambda_- = \lambda_1 - \lambda_2 = 2\lambda_b = -2\lambda_3$$

2) $\lambda_- = 2\lambda_3$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad (\mathcal{X}^A)_k^{ij} = \begin{pmatrix} -(X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}$$

$$\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = -2\,|H|^2, \quad \varepsilon_{AB}\varepsilon_{ij}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = -2\,|X|^2$$

$$V_{\text{cust}} \supset \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} + \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$

$$+ \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} + \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$

$$V_{0a} = \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} = 4\lambda_{0a}\,|H|^2|X|^2$$

$$V_{0b} = \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl} = 2\lambda_{0b}\,|H|^2|X|^2$$

$$V_a = \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} = -2\lambda_a[H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_b = \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl} = \lambda_b[(H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) - 2H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset (4\lambda_{0a} + 2\lambda_{0b})\,|H|^2|X|^2 - 2(\lambda_a + \lambda_b)H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l - 2\lambda_a H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + \lambda_b(H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.)$$

Custodial symmetry $\Rightarrow$
$$\begin{cases} \lambda_0 = 4\lambda_{0a} + 2\lambda_{0b} \\ \lambda_1 = -2(\lambda_a + \lambda_b) \\ \lambda_2 = -2\lambda_a \\ -\lambda_3 = \lambda_b \end{cases} \Rightarrow \quad \lambda_- = \lambda_1 - \lambda_2 = -2\lambda_b = 2\lambda_3$$

Passarino-Veltman scalar functions

D-dim one-loop integrals defined by Denner, 0709.1075:

$$T^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1})=F\int d^D q\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0 D_1\cdots D_{N-1}},\quad F\equiv\frac{(2\pi\mu)^{4-D}}{i\pi^2}$$

N = number of propagator factors in the denominator, $\quad P$ = number of integration momenta in the numerator

$$D_0=q^2-m_0^2+i\varepsilon,\quad D_i=(q+p_i)^2-m_i^2+i\varepsilon,\quad i=1,\cdots,N-1,\quad \varepsilon=\frac{4-D}{2},\quad D=4-2\varepsilon$$

These integrals give rise to a UV-divergent term $\Delta=\dfrac{1}{\varepsilon}-\gamma_E+\log 4\pi$

Subtracting Δ corresponds to the $\overline{\text{MS}}$ scheme

$$(2\pi)^{4-D}=(2\pi)^{2\varepsilon}=[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)],\quad F=\frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)]$$

Conventionally, $\quad T^1\equiv A,\quad T^2\equiv B,\quad T^3\equiv C,\quad \cdots$

$$A(m_0^2)=A_0(m_0^2),\quad B(p_1^2,m_0^2,m_1^2)=B_0(p_1^2,m_0^2,m_1^2),\quad B_\mu(p_1^2,m_0^2,m_1^2)=p_{1\mu}B_1(p_1^2,m_0^2,m_1^2)$$

$$B_{\mu\nu}(p_1^2,m_0^2,m_1^2)=g_{\mu\nu}B_{00}(p_1^2,m_0^2,m_1^2)+p_{1\mu}p_{1\nu}B_{11}(p_1^2,m_0^2,m_1^2)$$

$$A_0(m^2)\sim m^2\Delta,\quad B_0(p^2,m_1^2,m_2^2)\sim\Delta,\quad B_1(p^2,m_1^2,m_2^2)\sim-\frac{1}{2}\Delta,\quad B_{00}(p^2,m_1^2,m_2^2)\sim-\frac{1}{12}(p^2-3m_1^2-3m_2^2)\Delta$$

D-dim one-loop integrals defined by LoopTools User's Guide:

$$T'^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1})=F'\int d^D q\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0 D_1\cdots D_{N-1}},\quad F'\equiv\frac{\mu^{4-D}}{i\pi^{D/2}r_\Gamma},\quad r_\Gamma=\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{1}{r_\Gamma}=1+\gamma_E\varepsilon+\mathcal{O}(\varepsilon^2),\quad \frac{1}{\pi^{D/2}}=\frac{1}{\pi^{2-\varepsilon}}=\frac{1}{\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$F'=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)][1+\gamma_E\varepsilon+\mathcal{O}(\varepsilon^2)]=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\gamma_E\varepsilon+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$\frac{T'^N_{\mu_1\cdots\mu_P}}{T^N_{\mu_1\cdots\mu_P}}=\frac{F'}{F}=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}\frac{i\pi^2}{(2\pi\mu)^{2\varepsilon}}=\frac{1}{(2\pi)^{2\varepsilon}\pi^{-\varepsilon}r_\Gamma}=1+(\gamma_E-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon^2)$$

$$\Delta'=\frac{F'}{F}\Delta=\Delta+\Delta(\gamma_E-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\frac{1}{\varepsilon}(\gamma_E-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}+\mathcal{O}(\varepsilon)$$

The UV-divergent term from $T'^N_{\mu_1\cdots\mu_P}$ is Δ', and subtracting Δ' corresponds to the $\overline{\text{MS}}$ scheme

$$\frac{1}{K^{2-D/2}} = 1 - \frac{4-D}{2}\ln K + \mathcal{O}((4-D)^2), \quad \Gamma(2-D/2) = \frac{2}{4-D} - \gamma_E + \mathcal{O}(4-D)$$

$$\frac{1}{(4\pi)^{D/2}} = \frac{1}{(4\pi)^2(4\pi)^{(D-4)/2}} = \frac{1}{16\pi^2}(4\pi)^{2-D/2} = \frac{1}{16\pi^2}\left[1 + \frac{4-D}{2}\ln 4\pi + \mathcal{O}((4-D)^2)\right]$$

$$\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}} = \frac{1}{16\pi^2}\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2-K)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}}\frac{\Gamma(n-D/2)}{\Gamma(n)}\frac{1}{K^{n-D/2}}$$

$$\Gamma(2-D/2) = (1-D/2)\Gamma(1-D/2), \quad \Gamma(1) = 1, \quad \frac{1}{1-(4-D)/2} = 1 + \frac{4-D}{2} + \mathcal{O}((4-D)^2)$$

$$\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2-K} = \frac{-i}{(4\pi)^{D/2}}\frac{\Gamma(1-D/2)}{\Gamma(1)}\frac{1}{K^{1-D/2}} = \frac{-iK}{1-D/2}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}}$$

$$= \frac{iK}{1-(4-D)/2}\frac{1}{16\pi^2}\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$= \frac{iK}{16\pi^2}\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + 1 + \mathcal{O}(4-D)\right]$$

$$\mu^{4-D} = 1 + \frac{4-D}{2}\ln\mu^2 + \mathcal{O}((4-D)^2)$$

$$A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q\frac{1}{q^2-m^2+i\varepsilon} = -16i\pi^2\mu^{4-D}\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2-(m^2-i\varepsilon)}$$

$$= (m^2-i\varepsilon)\mu^{4-D}\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln(m^2-i\varepsilon) + 1 + \mathcal{O}(4-D)\right] = m^2\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi + \ln\mu^2 - \ln m^2 + 1 + \mathcal{O}(4-D)\right]$$

$$= m^2\left(\Delta + 1 - \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}(4-D)$$

$$\Delta \equiv \frac{2}{4-D} - \gamma_E + \ln 4\pi$$

$$x(q^2-m_1^2+i\varepsilon) + (1-x)[(q+p)^2 - m_2^2 + i\varepsilon] = q^2 + 2(1-x)q\cdot p + (1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon$$

$$= [q+(1-x)p]^2 + x(1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon$$

$$= \ell^2 - K$$

$$\ell \equiv q + (1-x)p, \quad K = -x(1-x)p^2 + xm_1^2 + (1-x)m_2^2 - i\varepsilon = x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon$$

$$\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \int_0^1 dx\frac{1}{\{x[q^2-m_1^2+i\varepsilon]+(1-x)[(q+p)^2-m_2^2+i\varepsilon]\}^2} = \int_0^1 dx\frac{1}{(\ell^2-K)^2}$$

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2} = i\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}} = \frac{i}{16\pi^2}\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$B_0(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{16\pi^2\mu^{4-D}}{i}\int_0^1 dx\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2}$$

$$= \mu^{4-D}\int_0^1 dx\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right] = \int_0^1 dx\left[\frac{2}{4-D} - \gamma_E + \ln 4\pi + \ln\mu^2 - \ln K + \mathcal{O}(4-D)\right]$$

$$= \Delta - \int_0^1 dx\ln\frac{x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon}{\mu^2} + \mathcal{O}(4-D)$$

$$x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2 - i\varepsilon = p^2(x-x_+)(x-x_-)$$

$$x_\pm \equiv \frac{1}{2p^2}\left[p^2 - m_1^2 + m_2^2 \pm \sqrt{(p^2-m_1^2+m_2^2)^2 - 4p^2(m_2^2-i\varepsilon)}\right]$$

$$\int_0^1 dx\ln\frac{x^2 p^2 - x(p^2-m_1^2+m_2^2)+m_2^2-i\varepsilon}{\mu^2} = \int_0^1 dx\ln\frac{p^2(x-x_+)(x-x_-)}{\mu^2} = \ln\frac{p^2}{\mu^2} + \int_0^1 dx[\ln(x-x_+) + \ln(x-x_-)]$$

$$= \ln\frac{p^2}{\mu^2} + [(x-x_+)\ln(x-x_+) - x + (x-x_-)\ln(x-x_-) - x]\big|_0^1 = \ln\frac{p^2}{\mu^2} + f(x_+) + f(x_-)$$

$$f(z) \equiv (1-z)\ln(1-z) + z\ln(-z) - 1 = \ln(1-z) - z\ln(1-z^{-1}) - 1$$

$$B_0(p^2,m_1^2,m_2^2) = \Delta - \ln\frac{p^2}{\mu^2} - f(x_+) - f(x_-) + \mathcal{O}(4-D)$$

$$p_\mu B_1(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q_\mu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$p^2 B_1(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$2q\cdot p = [(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-p^2-m_1^2+m_2^2$$

$$\frac{2q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{1}{q^2-m_1^2+i\varepsilon} - \frac{1}{(q+p)^2-m_2^2+i\varepsilon} - \frac{p^2+m_1^2-m_2^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$2p^2 B_1(p^2,m_1^2,m_2^2) = A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)$$

$$B_1(p^2,m_1^2,m_2^2) = \frac{1}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$g_{\mu\nu}B_{00}(p^2,m_1^2,m_2^2)+p_\mu p_\nu B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q_\mu q_\nu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$DB_{00}(p^2,m_1^2,m_2^2)+p^2 B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$q^2 = (q^2-m_1^2+i\varepsilon)+m_1^2-i\varepsilon$$

$$\frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{1}{(q+p)^2-m_2^2+i\varepsilon} + \frac{m_1^2-i\varepsilon}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$DB_{00}(p^2,m_1^2,m_2^2)+p^2 B_{11}(p^2,m_1^2,m_2^2) = A_0(m_2^2)+m_1^2 B_0(p^2,m_1^2,m_2^2)$$

$$4Dp^2 B_{00}(p^2,m_1^2,m_2^2)+4p^4 B_{11}(p^2,m_1^2,m_2^2) = 4p^2 A_0(m_2^2)+4p^2 m_1^2 B_0(p^2,m_1^2,m_2^2)$$

$$p^2 B_{00}(p^2,m_1^2,m_2^2)+p^4 B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$4(q\cdot p)^2 = \{[(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-[p^2+m_1^2-m_2^2]\}^2$$

$$= [(q+p)^2-m_2^2+i\varepsilon]\{[(q+p)^2-m_2^2+i\varepsilon]-2[q^2-m_1^2+i\varepsilon]-2[p^2+m_1^2-m_2^2]\}$$

$$+[q^2-m_1^2+i\varepsilon]\{[q^2-m_1^2+i\varepsilon]+2[p^2+m_1^2-m_2^2]\}+[p^2+m_1^2-m_2^2]^2$$

$$= [(q+p)^2-m_2^2+i\varepsilon](-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon)+(q^2-m_1^2+i\varepsilon)(q^2+2p^2+m_1^2-2m_2^2+i\varepsilon)+(p^2+m_1^2-m_2^2)^2$$

$$\frac{4(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon}{q^2-m_1^2+i\varepsilon} + \frac{q^2+2p^2+m_1^2-2m_2^2+i\varepsilon}{(q+p)^2-m_2^2+i\varepsilon} + \frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$= -1+\frac{2q\cdot p-p^2-m_1^2+m_2^2}{q^2-m_1^2+i\varepsilon}+1+\frac{-2q\cdot p+p^2+m_1^2-m_2^2}{(q+p)^2-m_2^2+i\varepsilon} + \frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$= \frac{2q\cdot p-(p^2+m_1^2-m_2^2)}{q^2-m_1^2+i\varepsilon}+\frac{-2q'\cdot p+3p^2+m_1^2-m_2^2}{q'^2-m_2^2+i\varepsilon} + \frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$q' = q+p$$

$$4p^2 B_{00}(p^2,m_1^2,m_2^2)+4p^4 B_{11}(p^2,m_1^2,m_2^2) = -(p^2+m_1^2-m_2^2)A_0(m_1^2)+(3p^2+m_1^2-m_2^2)A_0(m_2^2)+(p^2+m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$

$$4(D-1)p^2 B_{00}(p^2,m_1^2,m_2^2) = 4p^2 A_0(m_2^2) + 4p^2 m_1^2 B_0(p^2,m_1^2,m_2^2)$$
$$+ (p^2 + m_1^2 - m_2^2)A_0(m_1^2) - (3p^2 + m_1^2 - m_2^2)A_0(m_2^2) - (p^2 + m_1^2 - m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$
$$= p^2 A_0(m_1^2) + p^2 A_0(m_2^2) + 2p^2(m_1^2 + m_2^2)B_0(p^2,m_1^2,m_2^2) - p^4 B_0(p^2,m_1^2,m_2^2)$$
$$+ (m_1^2 - m_2^2)A_0(m_1^2) - (m_1^2 - m_2^2)A_0(m_2^2) - (m_1^2 - m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$

$$[1-(4-D)/3]6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+ \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$\frac{1}{1-(4-D)/3} = 1 + \frac{1}{3}(4-D) + \mathcal{O}((4-D)^2)$$

$$6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{1-(4-D)/3}\left\{\frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left. + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(\boldsymbol{m_2^2}) - (m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$

$$= \frac{1}{3}(m_1^2 + m_2^2) + \frac{2}{3}\left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right) + \frac{m_1^2 - m_2^2}{3p^2}[m_1^2 - m_2^2 - (m_1^2 - m_2^2)]$$
$$+ \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+ \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(\boldsymbol{m_2^2}) - (m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2)] + \mathcal{O}(4-D)$$

$$B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{6}\left\{m_1^2 + m_2^2 - \frac{1}{3}p^2 + \frac{1}{2}[A_0(m_1^2) + A_0(m_2^2)] + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left. + \frac{m_1^2 - m_2^2}{2p^2}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$

$$J_1(p^2,m_1^2,m_2^2) \equiv A_0(m_1^2) + A_0(m_2^2) - (p^2 - m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2) - 4B_{00}(p^2,m_1^2,m_2^2)$$

$$J_1'(p^2,m_1^2,m_2^2) = -B_0(p^2,m_1^2,m_2^2) - (p^2 - m_1^2 - m_2^2)B_0'(p^2,m_1^2,m_2^2) - 4B_{00}'(p^2,m_1^2,m_2^2)$$

$$B_0'(p^2,m_1^2,m_2^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - x(p^2 - m_1^2 + m_2^2) + m_2^2}$$

$$B_{00}'(p^2,m_1^2,m_2^2) = \frac{1}{6}\left\{ -\frac{1}{3} - \frac{1}{2}B_0(p^2,m_1^2,m_2^2) + \left(m_1^2 + m_2^2 - \frac{1}{2}p^2\right)B_0'(p^2,m_1^2,m_2^2) \right.$$
$$\left. -\frac{m_1^2 - m_2^2}{2p^4}[A_0(m_1^2) - A_0(m_2^2) - (m_1^2 - m_2^2)B_0(p^2,m_1^2,m_2^2)] - \frac{(m_1^2 - m_2^2)^2}{2p^2}B_0'(p^2,m_1^2,m_2^2) \right\}$$

$\underline{m = m_1 = m_2}$

$$-\ln\frac{x^2 p^2 - xp^2 + m^2}{\mu^2} = -\ln\frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$$B_0(p^2,m^2,m^2) = \Delta - \int_0^1 dx \ln\frac{x^2 p^2 - xp^2 + m^2 - i\varepsilon}{\mu^2} = \Delta - \ln\frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$$-\frac{x^2 - x}{x^2 p^2 - xp^2 + m^2} = -\frac{x^2 - x}{m^2} + \mathcal{O}(p^2)$$

$$B_0'(p^2,m^2,m^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - xp^2 + m^2} = \frac{1}{6m^2} + \mathcal{O}(p^2)$$

$$B_{00}(p^2,m^2,m^2) = \frac{1}{3}m^2 - \frac{1}{18}p^2 + \frac{1}{6}A_0(m^2) + \left(\frac{1}{3}m^2 - \frac{1}{12}p^2\right)B_0(p^2,m^2,m^2)$$

$$B_{00}'(p^2,m^2,m^2) = -\frac{1}{18} - \frac{1}{12}B_0(p^2,m^2,m^2) + \left(\frac{1}{3}m^2 - \frac{1}{12}p^2\right)B_0'(p^2,m^2,m^2)$$

$$J_1(p^2,m^2,m^2) = 2A_0(m^2) - (p^2 - 2m^2)B_0(p^2,m^2,m^2) - 4B_{00}(p^2,m^2,m^2)$$

$$J_1'(p^2,m^2,m^2) = -B_0(p^2,m^2,m^2) - (p^2 - 2m^2)B_0'(p^2,m^2,m^2) + \frac{2}{9} + \frac{1}{3}B_0(p^2,m^2,m^2) - \left(\frac{4}{3}m^2 - \frac{1}{3}p^2\right)B_0'(p^2,m^2,m^2)$$

$$= \frac{2}{9} - \frac{2}{3}B_0(p^2,m^2,m^2) + \frac{2}{3}(m^2 - p^2)B_0'(p^2,m^2,m^2)$$

$$= \frac{2}{9} - \frac{2}{3}\left(\Delta - \ln\frac{m^2}{\mu^2}\right) + \frac{2}{3}m^2 \frac{1}{6m^2} + \mathcal{O}(p^2) = -\frac{2}{3}\Delta + \frac{1}{3} + \frac{2}{3}\ln\frac{m^2}{\mu^2} + \mathcal{O}(p^2)$$

$\underline{m_1 = 0}$

$$-\ln\frac{x^2 p^2 - x(p^2 + m_2^2) + m_2^2}{\mu^2} = -\ln\frac{(1-x)m_2^2}{\mu^2} + \frac{xp^2}{m_2^2} + \frac{x^2 p^4}{2m_2^4} + \mathcal{O}(p^6)$$

$$B_0(p^2,0,m_2^2) = \Delta - \int_0^1 dx \ln\frac{x^2 p^2 - x(p^2 + m_2^2) + m_2^2 - i\varepsilon}{\mu^2} = \Delta + 1 - \ln\frac{m_2^2}{\mu^2} + \frac{p^2}{2m_2^2} + \frac{p^4}{6m_2^4} + \mathcal{O}(p^6)$$

$$-\frac{x^2 - x}{x^2 p^2 - x(p^2 + m_2^2) + m_2^2} = \frac{x}{m_2^2} + \frac{x^2 p^2}{m_2^4} + \frac{x^3 p^4}{m_2^6} + \mathcal{O}(p^6)$$

$$B_0'(p^2,0,m_2^2) = -\int_0^1 dx \frac{x^2 - x}{x^2 p^2 - x(p^2 + m_2^2) + m_2^2 - i\varepsilon} = \frac{1}{2m_2^2} + \frac{p^2}{3m_2^4} + \frac{p^4}{4m_2^6} + \mathcal{O}(p^6)$$

$$-A_0(m_2^2) + m_2^2 B_0(p^2,0,m_2^2) = \frac{p^2}{2} + \frac{p^4}{6m_2^2} + \mathcal{O}(p^6)$$

$$B_{00}'(p^2,0,m_2^2) = \frac{1}{6}\left[-\frac{1}{3} - \frac{1}{2}B_0(p^2,0,m_2^2) + \left(m_2^2 - \frac{1}{2}p^2\right)B_0'(p^2,0,m_2^2) + \frac{m_2^2}{2p^4}[-A_0(m_2^2) + m_2^2 B_0(p^2,0,m_2^2)] - \frac{m_2^4}{2p^2}B_0'(p^2,0,m_2^2) \right]$$

$$= \frac{1}{6}\left[-\frac{1}{3} - \frac{1}{2}\left(\Delta + 1 - \ln\frac{m_2^2}{\mu^2}\right) + m_2^2 \frac{1}{2m_2^2} + \frac{m_2^2}{2p^4}\left(\frac{p^2}{2} + \frac{p^4}{6m_2^2}\right) - \frac{m_2^4}{2p^2}\left(\frac{1}{2m_2^2} + \frac{p^2}{3m_2^4}\right) \right] + \mathcal{O}(p^2)$$

$$= -\frac{1}{12}\Delta - \frac{5}{72} + \frac{1}{12}\ln\frac{m_2^2}{\mu^2} + \mathcal{O}(p^2)$$

$$J_1'(p^2,0,m_2^2) = -B_0(p^2,0,m_2^2) - (p^2 - m_2^2)B_0'(p^2,0,m_2^2) - 4B_{00}'(p^2,0,m_2^2)$$

$$= -\left(\Delta + 1 - \ln\frac{m_2^2}{\mu^2}\right) + m_2^2 \frac{1}{2m_2^2} - 4\left(-\frac{1}{12}\Delta - \frac{5}{72} + \frac{1}{12}\ln\frac{m_2^2}{\mu^2}\right) + \mathcal{O}(p^2)$$

$$= -\frac{2}{3}\Delta - \frac{2}{9} + \frac{2}{3}\ln\frac{m_2^2}{\mu^2} + \mathcal{O}(p^2)$$

<u>Custodial symmetry</u>

Ref: Jose Santiago's lecture note "the Physics of Electroweak Symmetry Breaking", 2009;
Montero & Pleitez, hep-ph/0607144; Branco, Ferreira, Lavoura, Rebelo, Sher & Silva, 1106.0034

$\mathrm{SU(2)_L \times SU(2)_R}$ bidoublet $\mathbf{H} = \begin{pmatrix} \tilde{H} & H \end{pmatrix} = \begin{pmatrix} H^{0*} & H^+ \\ -H^- & H^0 \end{pmatrix}$

$\tilde{H}^\dagger \tilde{H} = H^0 H^{0*} + H^+ H^- = H^\dagger H, \quad \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2\,|\,H\,|^2$

$U_\mathrm{L} \in$ global $\mathrm{SU(2)_L}, \quad U_\mathrm{R} \in$ global $\mathrm{SU(2)_R}, \quad \mathbf{H} \to U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger \quad \Rightarrow \quad \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) \to \mathrm{tr}(U_\mathrm{R} \mathbf{H}^\dagger U_\mathrm{L}^\dagger U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger) = \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H})$

SM scalar potential $V = -\mu^2\,|\,H\,|^2 + \lambda\,|\,H\,|^4 = -\dfrac{\mu^2}{2}\,\mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) + \dfrac{\lambda}{4}[\mathrm{tr}(\mathbf{H}^\dagger \mathbf{H})]^2$

$W_\mu \equiv W_\mu^a \dfrac{\sigma^a}{2}, \quad (\sigma^2)^2 = 1, \quad \sigma^a \sigma^2 = -\sigma^2 (\sigma^a)^*$

$|\,(\partial_\mu - igW_\mu)\tilde{H}\,|^2 = [(\partial^\mu - igW^\mu)i\sigma^2 H^*]^\dagger (\partial_\mu - igW_\mu)i\sigma^2 H^* = H^\mathrm{T}\sigma^2(\overline{\partial^\mu} + igW^\mu)(\partial_\mu - igW_\mu)\sigma^2 H^*$

$\quad = H^\mathrm{T}(\overline{\partial^\mu} - igW^{\mu *})(\partial_\mu + igW_\mu^*)H^* = H^\dagger(\overline{\partial^\mu} + igW^\mu)(\partial_\mu - igW_\mu)H = |\,(\partial_\mu - igW_\mu)H\,|^2$

$\mathbf{D}_\mu \mathbf{H} \equiv (\partial_\mu - igW_\mu)\mathbf{H} = \begin{pmatrix} (\partial_\mu - igW_\mu)\tilde{H} & (\partial_\mu - igW_\mu)H \end{pmatrix}$

$\mathrm{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}] = |\,(\partial_\mu - igW_\mu)\tilde{H}\,|^2 + |\,(\partial_\mu - igW_\mu)H\,|^2 = 2\,|\,(\partial_\mu - igW_\mu)H\,|^2$

$W_\mu \to U_\mathrm{L} W_\mu U_\mathrm{L}^\dagger$

$\mathbf{D}_\mu \mathbf{H} \to (\partial_\mu - ig U_\mathrm{L} W_\mu U_\mathrm{L}^\dagger)U_\mathrm{L}\mathbf{H}U_\mathrm{R}^\dagger = U_\mathrm{L}\mathbf{D}_\mu\mathbf{H}U_\mathrm{R}^\dagger$

$\mathrm{tr}[(\mathbf{D}^\mu\mathbf{H})^\dagger \mathbf{D}_\mu\mathbf{H}] \to \mathrm{tr}(U_\mathrm{R}\mathbf{D}^\mu\mathbf{H}^\dagger U_\mathrm{L}^\dagger U_\mathrm{L}\mathbf{D}_\mu\mathbf{H}U_\mathrm{R}^\dagger) = \mathrm{tr}[(\mathbf{D}^\mu\mathbf{H})^\dagger\mathbf{D}_\mu\mathbf{H}]$

An $\mathrm{SU(2)_L \times SU(2)_R}$ global symmetry exists in the potential and the $\mathrm{SU(2)_L}$ gauge interaction of the Higgs field

$\langle H \rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \mathbf{H} \rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \to U_\mathrm{L}\langle\mathbf{H}\rangle U_\mathrm{R}^\dagger = \dfrac{1}{\sqrt{2}}U_\mathrm{L}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}U_\mathrm{R}^\dagger = \dfrac{v}{\sqrt{2}}U_\mathrm{L}U_\mathrm{R}^\dagger$

$U_\mathrm{L} = U_\mathrm{R} \quad \Rightarrow \quad U_\mathrm{L}\langle\mathbf{H}\rangle U_\mathrm{R}^\dagger = \dfrac{1}{\sqrt{2}}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \langle\mathbf{H}\rangle$

$\Rightarrow$ the vacuum is invariant under the $\mathrm{SU(2)_{L+R}}$ global symmetry (equal left and right rotations)

$\Rightarrow$ $\mathrm{SU(2)_L \times SU(2)_R} \to \underline{\mathrm{SU(2)_{L+R}}}$ custodial symmetry

$\Rightarrow$ W_μ^a transform as a triplet under $\mathrm{SU(2)_{L+R}}$

$\Rightarrow$ identical contributions to the masses of W_μ^a from electroweak symmetry breaking

$\Rightarrow$ the custodial symmetry protects the parameter $\rho = \dfrac{m_W^2}{m_Z^2 c_\mathrm{W}^2}$, and leads to $T = U = 0$

<u>Custodial symmetry violation in the SM</u>

1) $\mathrm{U(1)_Y}$ gauge interaction

$\mathrm{U(1)_Y}$ gauge transformation $H \to e^{iY_H\theta(x)}H, \quad \tilde{H} \to e^{-iY_H\theta(x)}\tilde{H}, \quad Y_H = \dfrac{1}{2}$

$\exp[-iY_H\theta(x)\sigma^3] = \sum_{k=0}^\infty \dfrac{1}{k!}[-iY_H\theta(x)\sigma^3]^k = \sum_{k=0}^\infty \dfrac{1}{(2k)!}[-iY_H\theta(x)]^{2k} + \sum_{k=1}^\infty \dfrac{1}{(2k+1)!}[-iY_H\theta(x)]^{2k+1}\sigma^3$

$\quad = \cos[Y_H\theta(x)] - i[Y_H\theta(x)]\sigma^3 = \begin{pmatrix} \cos[Y_H\theta(x)] - i\sin[Y_H\theta(x)] & \\ & \cos[Y_H\theta(x)] + i\sin[Y_H\theta(x)] \end{pmatrix} = \begin{pmatrix} e^{-iY_H\theta(x)} & \\ & e^{iY_H\theta(x)} \end{pmatrix}$

$\mathbf{H} \to \begin{pmatrix} e^{-iY_H\theta(x)}\tilde{H} & e^{iY_H\theta(x)}H \end{pmatrix} = \mathbf{H}\exp[-iY_H\theta(x)\sigma^3], \quad \dfrac{\sigma^3}{2}$ is just one $\mathrm{SU(2)_R}$ generator

$Y_H \neq 0$ violates custodial symmetry

2) Difference between Yukawa couplings of up-type and down-type fermions

$Q_\mathrm{L} \equiv \begin{pmatrix} t_\mathrm{L} \\ b_\mathrm{L}' \end{pmatrix} \to U_\mathrm{L}Q_\mathrm{L}, \quad \begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix} \to U_\mathrm{R}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix}, \quad \bar{Q}_\mathrm{L}\mathbf{H}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix} \to \bar{Q}_\mathrm{L}U_\mathrm{L}^\dagger U_\mathrm{L}\mathbf{H}U_\mathrm{R}^\dagger U_\mathrm{R}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix} = \bar{Q}_\mathrm{L}\mathbf{H}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix}$

$-y\bar{Q}_\mathrm{L}\mathbf{H}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix} = -y\bar{Q}_\mathrm{L}\begin{pmatrix} \tilde{H} & H \end{pmatrix}\begin{pmatrix} t_\mathrm{R} \\ b_\mathrm{R}' \end{pmatrix} = -y(\bar{Q}_\mathrm{L}\tilde{H}t_\mathrm{R} + \bar{Q}_\mathrm{L}Hb_\mathrm{R}')$

Yukawa couplings in the SM: $-y_t\bar{Q}_\mathrm{L}\tilde{H}t_\mathrm{R} - y_b\bar{Q}_\mathrm{L}Hb_\mathrm{R}' = -\bar{Q}_\mathrm{L}\mathbf{H}\begin{pmatrix} y_t t_\mathrm{R} \\ y_b b_\mathrm{R}' \end{pmatrix}$

$y_t \neq y_b$ violates custodial symmetry

<u>U(1)$_Y$ gauge symmetry</u>

$$\mathbf{H} \to \mathbf{H}\exp\left(-i\theta(x)\frac{\sigma^3}{2}\right), \quad B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu\theta$$

$$D_\mu H = \left(\partial_\mu - igW_\mu - \frac{i}{2}g'B_\mu\right)H, \quad D_\mu\tilde{H} = \left(\partial_\mu - igW_\mu + \frac{i}{2}g'B_\mu\right)\tilde{H}$$

$$D_\mu\mathbf{H} = \partial_\mu\mathbf{H} - igW_\mu\mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu$$

$$D_\mu\mathbf{H} \to \left[\partial_\mu\mathbf{H} + \mathbf{H}\left(-i\partial_\mu\theta\frac{\sigma^3}{2}\right) - igW_\mu\mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu + ig'\mathbf{H}\frac{1}{g'}\partial_\mu\theta\frac{\sigma^3}{2}\right]\exp\left(-i\theta(x)\frac{\sigma^3}{2}\right)$$

$$= D_\mu\mathbf{H}\exp\left(-i\theta(x)\frac{\sigma^3}{2}\right)$$

<u>$H^\dagger W_{\mu\nu}^a\sigma^a H B^{\mu\nu}$ operator</u>

$$W_{\mu\nu} = W_{\mu\nu}^a\frac{\sigma^a}{2} = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc}W_\mu^b W_\nu^c)\frac{\sigma^a}{2}$$

$SU(2)_L \times SU(2)_R$ global transformation:

$$\mathbf{H} \to U_L\mathbf{H}U_R^\dagger, \quad W_\mu \to U_L W_\mu U_L^\dagger, \quad W_{\mu\nu} \to U_L W_{\mu\nu}U_L^\dagger$$

$$\mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu} \to \mathrm{tr}\left(U_R\mathbf{H}^\dagger U_L^\dagger U_L W_{\mu\nu}U_L^\dagger U_L\mathbf{H}U_R^\dagger\right)B^{\mu\nu} = \mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu}$$

$$\tilde{H}^\dagger W_{\mu\nu}\tilde{H} = H^T\sigma^2 W_{\mu\nu}\sigma^2 H^* = -H^T W_{\mu\nu}^* H^* = -H^\dagger W_{\mu\nu}H$$

$$\mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu} = \mathrm{tr}\left[\begin{pmatrix}\tilde{H}^\dagger \\ H^\dagger\end{pmatrix}W_{\mu\nu}\begin{pmatrix}\tilde{H} & H\end{pmatrix}\right]B^{\mu\nu} = \mathrm{tr}\begin{pmatrix}\tilde{H}^\dagger W_{\mu\nu}\tilde{H} & \tilde{H}^\dagger W_{\mu\nu}H \\ H^\dagger W_{\mu\nu}\tilde{H} & H^\dagger W_{\mu\nu}H\end{pmatrix}B^{\mu\nu}$$

$$= (\tilde{H}^\dagger W_{\mu\nu}\tilde{H} + H^\dagger W_{\mu\nu}H)B^{\mu\nu} = 0$$

$\Rightarrow$ The operator $H^\dagger W_{\mu\nu}^a\sigma^a H B^{\mu\nu}$ does not respect the custodial symmetry

$\left\{\begin{array}{l}\text{However, scalar potential terms or Yukawa terms for EW multiplets that respect} \\ \text{the custodial symmetry, along with the U(1)$_Y$ gauge interaction that violates the} \\ \text{custodial symmetry, can contribute to this operator and hence to } S\end{array}\right\}$

Triplet in SU(2) as an example

Matrix notation

$U(x) = \exp[i\theta^a(x)t_{\mathrm{T}}^a], \quad U^\dagger = \exp(-i\theta^a t_{\mathrm{T}}^a)$

$\underline{D_\mu T \equiv (\partial_\mu - ig W_\mu^a t_{\mathrm{T}}^a)T}$

$T \to UT, \quad W_\mu^a t_{\mathrm{T}}^a \to U W_\mu^a t_{\mathrm{T}}^a U^\dagger + \dfrac{i}{g} U \partial_\mu U^\dagger$

$D_\mu T \to \left[\partial_\mu - ig\left(U W_\mu^a t_{\mathrm{T}}^a U^\dagger + \dfrac{i}{g} U \partial_\mu U^\dagger \right) \right] UT$

$\quad = U U^\dagger (\partial_\mu U) T + U \partial_\mu T - ig U W_\mu^a t_{\mathrm{T}}^a T + U(\partial_\mu U^\dagger) UT$

$\quad = U(\partial_\mu - ig W_\mu^a t_{\mathrm{T}}^a)T + U \partial_\mu (U^\dagger U) UT = U D_\mu T$

$\underline{D_\mu T \text{ transforms as } T}$

Infinitesimal transformation $\quad U \simeq 1 + i\theta^a t_{\mathrm{T}}^a$

$\delta T = i\theta^a t_{\mathrm{T}}^a T, \quad \delta(W_\mu^a t_{\mathrm{T}}^a) = i\theta^b[t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a] + \dfrac{1}{g}(\partial_\mu \theta^a)t_{\mathrm{T}}^a$

$\delta(D_\mu T) = (\partial_\mu - ig W_\mu^a t_{\mathrm{T}}^a)\delta T - ig\delta(W_\mu^a t_{\mathrm{T}}^a)T$

$\quad = (\partial_\mu - ig W_\mu^a t_{\mathrm{T}}^a)i\theta^b t_{\mathrm{T}}^b T - ig\left\{ i\theta^b[t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a] + \dfrac{1}{g}(\partial_\mu \theta^a)t_{\mathrm{T}}^a \right\}T$

$\quad = i(\partial_\mu \theta^a)t_{\mathrm{T}}^a T + i\theta^a t_{\mathrm{T}}^a \partial_\mu T + g\theta^b W_\mu^a t_{\mathrm{T}}^a t_{\mathrm{T}}^b T + g\theta^b[t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a]T - i(\partial_\mu \theta^a)t_{\mathrm{T}}^a T$

$\quad = i\theta^a t_{\mathrm{T}}^a \partial_\mu T + g\theta^b t_{\mathrm{T}}^b W_\mu^a t_{\mathrm{T}}^a T = i\theta^a t_{\mathrm{T}}^a (\partial_\mu - ig W_\mu^b t_{\mathrm{T}}^b)T = i\theta^a t_{\mathrm{T}}^a D_\mu T$

$D_\mu D_\nu T = \partial_\mu \partial_\nu T - ig(\partial_\mu W_\nu^a)t_{\mathrm{T}}^a T - ig W_\nu^a t_{\mathrm{T}}^a \partial_\mu T - ig W_\mu^a t_{\mathrm{T}}^a(\partial_\nu T - ig W_\nu^b t_{\mathrm{T}}^b T)$

$[D_\mu, D_\nu]T = -ig(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)t_{\mathrm{T}}^a T - g^2[W_\mu^a t_{\mathrm{T}}^a, W_\nu^b t_{\mathrm{T}}^b]T$

$W_{\mu\nu}^a t_{\mathrm{T}}^a T \equiv \dfrac{i}{g}[D_\mu, D_\nu]T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)t_{\mathrm{T}}^a T - ig[W_\mu^a t_{\mathrm{T}}^a, W_\nu^b t_{\mathrm{T}}^b]T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc}W_\mu^b W_\nu^c)t_{\mathrm{T}}^a T$

$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc}W_\mu^b W_\nu^c$

$[D_\mu, D_\nu]T \to U[D_\mu, D_\nu]T = U[D_\mu, D_\nu]U^\dagger UT \quad \Rightarrow \quad W_{\mu\nu}^a t_{\mathrm{T}}^a \to U W_{\mu\nu}^a t_{\mathrm{T}}^a U^\dagger$

$\mathrm{tr}(W_{\mu\nu}^a t_{\mathrm{T}}^a W^{b\mu\nu} t_{\mathrm{T}}^b)$ is a gauge invariant

<u>Tensor notation</u>

$$V^i_j(x) = \exp[i\theta^a(x)\tau^a]^i_j, \quad (V^\dagger)^i_j = \exp(-i\theta^a\tau^a)^i_j, \quad \tau^a \equiv \frac{\sigma^a}{2}$$

$$\underline{(D_\mu T)^i_j \equiv \partial_\mu T^i_j - ig(W_\mu)^i_k T^k_j + igT^i_k (W_\mu)^k_j}$$

[Ref: Ta-Pei Cheng & Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, Eq.(4.140)]

$$T^i_j \to V^i_k T^k_l (V^\dagger)^l_j, \quad (W_\mu)^i_j \to V^i_k (W_\mu)^k_l (V^\dagger)^l_j + \frac{i}{g} V^i_k \partial_\mu (V^\dagger)^k_j$$

Infinitesimal transformation $V^i_j \simeq \delta^i_j + i\theta^a(\tau^a)^i_j, \quad (V^\dagger)^i_j \simeq \delta^i_j - i\theta^a(\tau^a)^i_j$

$$\delta T^i_j = i\theta^a(\tau^a)^i_k T^k_j - i\theta^a T^i_l(\tau^a)^l_j = i\theta^a[(\tau^a)^i_k T^k_j - T^i_k(\tau^a)^k_j]$$

$$\delta(W_\mu)^i_j = i\theta^a[(\tau^a)^i_k(W_\mu)^k_j - (W_\mu)^i_k(\tau^a)^k_j] + \frac{1}{g}(\partial_\mu\theta^a)(\tau^a)^i_j$$

$$\delta(D_\mu T)^i_j = \delta(\partial_\mu T^i_j) - ig\delta(W_\mu)^i_l T^l_j + ig\delta(W_\mu)^l_j T^i_l - ig(W_\mu)^i_l \delta T^l_j + ig\delta T^i_l (W_\mu)^l_j$$

$$= i(\partial_\mu\theta^a)[(\tau^a)^i_k T^k_j - T^i_k(\tau^a)^k_j] + i\theta^a[(\tau^a)^i_k \partial_\mu T^k_j - (\partial_\mu T^i_k)(\tau^a)^k_j]$$

$$\quad + g\theta^a[(\tau^a)^i_k(W_\mu)^k_l - (W_\mu)^i_k(\tau^a)^k_l]T^l_j - i(\partial_\mu\theta^a)(\tau^a)^i_l T^l_j$$

$$\quad - g\theta^a T^i_l[(\tau^a)^l_k(W_\mu)^k_j - (W_\mu)^l_k(\tau^a)^k_j] + i(\partial_\mu\theta^a)T^i_l(\tau^a)^l_j$$

$$\quad + g(W_\mu)^i_l \theta^a[(\tau^a)^l_k T^k_j - T^l_k(\tau^a)^k_j] - g\theta^a[(\tau^a)^i_k T^k_l - T^i_k(\tau^a)^k_l](W_\mu)^l_j$$

$$= i\theta^a(\tau^a)^i_k \partial_\mu T^k_j + g\theta^a(\tau^a)^i_k(W_\mu)^k_l T^l_j - g\theta^a(\tau^a)^i_k T^l_l(W_\mu)^l_j$$

$$\quad - i\theta^a(\partial_\mu T^i_k)(\tau^a)^k_j - g\theta^a(W_\mu)^i_l T^l_k(\tau^a)^k_j + g\theta^a T^i_l(W_\mu)^l_k(\tau^a)^k_j$$

$$= i\theta^a(\tau^a)^i_k[\partial_\mu T^k_j - ig(W_\mu)^k_l T^l_j + igT^k_l(W_\mu)^l_j] - i\theta^a[\partial_\mu T^i_k - ig(W_\mu)^i_l T^l_k + igT^i_l(W_\mu)^l_k](\tau^a)^k_j$$

$$= i\theta^a[(\tau^a)^i_k(D_\mu T)^k_j - (D_\mu T)^i_k(\tau^a)^k_j]$$

$\underline{(D_\mu T)^i_j}$ transforms as T^i_j

$$(D_\mu D_\nu T)^i_j = \partial_\mu\partial_\nu T^i_j - ig\partial_\mu(W_\nu)^i_k T^k_j - ig(W_\nu)^i_k \partial_\mu T^k_j + ig(\partial_\mu T^i_k)(W_\nu)^k_j + igT^i_k \partial_\mu(W_\nu)^k_j$$

$$\quad\quad - ig(W_\mu)^i_l[\partial_\nu T^l_j - ig(W_\nu)^l_k T^k_j + igT^l_k(W_\nu)^k_j] + ig[\partial_\nu T^i_l - ig(W_\nu)^i_k T^k_l + igT^i_k(W_\nu)^k_l](W_\mu)^l_j$$

$$([D_\mu, D_\nu]T)^i_j = -ig[\partial_\mu(W_\nu)^i_k - \partial_\nu(W_\mu)^i_k]T^k_j + igT^i_k[\partial_\mu(W_\nu)^k_j - \partial_\nu(W_\mu)^k_j]$$

$$\quad\quad - g^2[(W_\mu)^i_l(W_\nu)^l_k - (W_\nu)^i_l(W_\mu)^l_k]T^k_j + g^2 T^i_k[(W_\mu)^k_l(W_\nu)^l_j - (W_\nu)^k_l(W_\mu)^l_j]$$

$$(W_{\mu\nu})^i_k T^k_j - T^i_k(W_{\mu\nu})^k_j \equiv \frac{i}{g}([D_\mu, D_\nu]T)^i_j = \{\partial_\mu(W_\nu)^i_k - \partial_\nu(W_\mu)^i_k - ig[(W_\mu)^i_l(W_\nu)^l_k - (W_\nu)^i_l(W_\mu)^l_k]\}T^k_j$$

$$\quad\quad - T^i_k\{\partial_\mu(W_\nu)^k_j - \partial_\nu(W_\mu)^k_j - ig[(W_\mu)^k_l(W_\nu)^l_j - (W_\nu)^k_l(W_\mu)^l_j]\}$$

$$(W_{\mu\nu})^i_k = \partial_\mu(W_\nu)^i_k - \partial_\nu(W_\mu)^i_k - ig[(W_\mu)^i_l(W_\nu)^l_k - (W_\nu)^i_l(W_\mu)^l_k]$$

$$(W_{\mu\nu})^i_k T^k_j - T^i_k(W_{\mu\nu})^k_j \to V^i_m[(W_{\mu\nu})^m_k T^k_n - T^m_k(W_{\mu\nu})^k_n](V^\dagger)^n_j$$

$$= V^i_m(W_{\mu\nu})^m_k(V^\dagger)^k_l V^l_p T^p_n(V^\dagger)^n_j - V^i_m T^m_k(V^\dagger)^k_l V^l_p(W_{\mu\nu})^p_n(V^\dagger)^n_j$$

$\Rightarrow \quad (W_{\mu\nu})^i_j$ transforms as T^i_j

$(W_{\mu\nu})^i_j(W^{\mu\nu})^j_i$ is a gauge invariant

<u>Gauge interactions</u>

For an SU(2) doublet D, $D_\mu D = (\partial_\mu - igW_\mu^a\tau^a)D$, or equivalently, $\underline{(D_\mu D)^i = \partial_\mu D^i - ig(W_\mu)^i_j D^j}$

$$\Rightarrow \quad (W_\mu)^i_j = W_\mu^a(\tau^a)^i_j = \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}$$

$W_\mu^+ = \sqrt{2}(W_\mu)^1_2, \quad W_\mu^- = \sqrt{2}(W_\mu)^2_1, \quad W_\mu^3 = 2(W_\mu)^1_1 = -2(W_\mu)^2_2$

For an SU(2) triplet T, $\underline{(D_\mu T)^i_j = \partial_\mu T^i_j - ig(W_\mu)^i_k T^k_j + igT^i_k (W_\mu)^k_j}$

$T^+ = -T^1_2, \quad T^- = -T^2_1, \quad T^0 = \sqrt{2}T^1_1 = -\sqrt{2}T^2_2$

$(T^+)^\dagger = -(T^\dagger)^2_1, \quad (T^-)^\dagger = -(T^\dagger)^1_2, \quad (T^0)^\dagger = \sqrt{2}(T^\dagger)^1_1 = -\sqrt{2}(T^\dagger)^2_2$

$(T^\dagger)^i_j \bar{\sigma}^\mu (W_\mu)^j_k T^k_i = (T^\dagger)^1_1 \bar{\sigma}^\mu (W_\mu)^1_1 T^1_1 + (T^\dagger)^1_1 \bar{\sigma}^\mu (W_\mu)^1_2 T^2_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu (W_\mu)^2_1 T^1_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu (W_\mu)^2_2 T^2_1$

$$\qquad + (T^\dagger)^2_1 \bar{\sigma}^\mu (W_\mu)^1_1 T^1_2 + (T^\dagger)^2_1 \bar{\sigma}^\mu (W_\mu)^1_2 T^2_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu (W_\mu)^2_1 T^1_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu (W_\mu)^2_2 T^2_2$$

$$= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^+ T^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^- T^0 - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^3 T^-$$

$$\quad + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^3 T^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^+ T^0 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^- T^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0$$

$$= \frac{1}{2}[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]$$

$(T^\dagger)^i_j \bar{\sigma}^\mu T^j_k (W_\mu)^k_i = (T^\dagger)^1_1 \bar{\sigma}^\mu T^1_1 (W_\mu)^1_1 + (T^\dagger)^1_1 \bar{\sigma}^\mu T^1_2 (W_\mu)^2_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu T^2_1 (W_\mu)^1_1 + (T^\dagger)^1_2 \bar{\sigma}^\mu T^2_2 (W_\mu)^2_1$

$$\qquad + (T^\dagger)^2_1 \bar{\sigma}^\mu T^1_1 (W_\mu)^1_2 + (T^\dagger)^2_1 \bar{\sigma}^\mu T^1_2 (W_\mu)^2_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu T^2_1 (W_\mu)^1_2 + (T^\dagger)^2_2 \bar{\sigma}^\mu T^2_2 (W_\mu)^2_2$$

$$= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^+ W_\mu^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^- W_\mu^3 + \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^0 W_\mu^-$$

$$\quad - \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^0 W_\mu^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^+ W_\mu^3 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^- W_\mu^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3$$

$$= \frac{1}{2}[-W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- + W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]$$

$\mathcal{L} \supset g[(T^\dagger)^i_j \bar{\sigma}^\mu (W_\mu)^j_k T^k_i - (T^\dagger)^i_j \bar{\sigma}^\mu T^j_k (W_\mu)^k_i]$

$$= g[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0$$

$$\qquad + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^-$$

$$\qquad - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]$$

For a general SU(2) multiplet $\psi^{i_1\cdots i_p}_{j_1\cdots j_q}$,

$$(D_\mu \psi)^{i_1\cdots i_p}_{j_1\cdots j_q} = \partial_\mu \psi^{i_1\cdots i_p}_{j_1\cdots j_q} - ig\left[\sum_{m=1}^{p}(W_\mu)^{i_m}_{k_m} \psi^{i_1\cdots k_m\cdots i_p}_{j_1\cdots j_q} - \sum_{n=1}^{q}\psi^{i_1\cdots i_p}_{j_1\cdots l_n\cdots j_q}(W_\mu)^{l_n}_{j_n}\right]$$

$\left[\begin{array}{l}\text{Note: in differential geometry, the gauge field } (W_\mu)^i_j \text{ is a connection form. The gauge connection defines} \\ \text{a principal bundle whose base space is the spacetime and structure group is the gauge group.}\end{array}\right]$