

量子场论讲义

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<http://yzhxxzy.github.io/cn/teaching.html>

更新日期：2018 年 9 月 3 日

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第 1 章 预备知识

1.1 量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由粒子组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon 方程**：

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\mathbf{x}, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi(\mathbf{x}, t). \quad (1.1)$$

它给出的自由粒子能量为

$$E = \pm \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}, \quad (1.2)$$

其中 \mathbf{p} 为粒子的动量， m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2 \leq E < \infty$ ；也可以为负，取值范围为 $-\infty < E \leq -mc^2$ 。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho = \frac{i\hbar}{2mc^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right), \quad (1.3)$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 不变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E < 0$ 的态都被填满而所有 $E > 0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E > 0$ 的电子跃迁到 $E < 0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的“反粒子”，称为**正电子** (positron)。正电子在 1932 年被 Carl Anderson 发现。

但是, Dirac 的空穴理论仍然面临一些困难, 比如, 为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场? 另一方面, Dirac 方程一开始作为描述单个粒子波函数的方程提出来, 但 Dirac 的解释却包含了无穷多个粒子。而且, 像光子和 π 介子这些不满足 Pauli 不相容原理的粒子, 空穴理论是不能成立的。此外, Dirac 方程只能描述自旋 1/2 的粒子, 不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难, 是否意味着处理这些问题的基础本身就不正确呢? 确实是这样的。量子力学的一条基本原理是: 观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而, 时间显然是一个观测量, 却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中, 描述系统的量子态时可以让态依赖于一个时间参数 t , 这是时间的概念进入量子力学的方式, 但并没有假定这个参数是某个厄米算符的本征值。另一方面, 粒子的空间位置 \mathbf{x} 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见, 在量子力学中, 对时间和空间的处理方式是完全不同的。而在狭义相对论中, Lorentz 对称性将两者混合起来。因此, 在结合量子力学与狭义相对论的过程中出现困难, 也是正常的。

那么, 如何在量子力学中平等地处理时间和空间呢? 一种途径是将时间提升为一个厄米算符, 但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数, 不再由厄米算符描写。这样, 我们可以在每个空间点 \mathbf{x} 处定义一个算符 $\hat{\phi}(\mathbf{x})$, 所有这些算符的集合称为量子场。在 Heisenberg 绘景中, 量子场算符也依赖于时间 t :

$$\hat{\phi}(\mathbf{x}, t) = e^{i\hat{H}t/\hbar} \hat{\phi}(\mathbf{x}) e^{-i\hat{H}t/\hbar}. \quad (1.4)$$

如此, 量子化的对象变成是由依赖于时空坐标的场组成的动力学系统, 这就是量子场论。这里的量子算符用 $\hat{}$ 符号标记, 为了简化记号, 后面将省略 $\hat{}$ 符号。

在量子场论中, 前面提到的困难都可以得到解决。现在, Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态, 包含粒子的态则是激发态, 激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子, 由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的, 不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x \rho$ 也可以用产生湮灭算符表达出来, 虽然它不一定是正定的, 但是它不再被解释为总概率, 而是被解释为正粒子数与反粒子数之差, 因而也不再出现负概率困难。

1.2 自然单位制

量子场论是结合量子力学和相对论的理论, 因而时常出现约化 Planck 常量 \hbar 和光速 c , 这一点可以从上一节的几个公式中看出来。于是, 为了简化表述, 通常采用自然单位制, 取

$$\hbar = c = 1. \quad (1.5)$$

从而, Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi(\mathbf{x}, t) = 0. \quad (1.6)$$

在自然单位制中，速度没有量纲 (dimension)；长度量纲与时间量纲相同，是能量量纲的倒数；能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22} \text{ MeV} \cdot \text{s}, \quad 1 = \hbar c = 1.973 \times 10^{-11} \text{ MeV} \cdot \text{cm}, \quad (1.7)$$

可得

$$1 \text{ s}^{-1} = 6.582 \times 10^{-22} \text{ MeV}, \quad 1 \text{ cm}^{-1} = 1.973 \times 10^{-11} \text{ MeV}. \quad (1.8)$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.036} \quad (1.9)$$

是没有量纲的，它的数值在任何单位制下都应该相同。因此，自然单位制不可能将 \hbar 、 c 、 ϵ_0 和 e 这四个常数同时归一化。在量子场论中，通常再取真空介电常数

$$\epsilon_0 = 1, \quad (1.10)$$

同时可得真空磁导率 $\mu_0 = 1/(\epsilon_0 c^2) = 1$ ，这样做其实是取了 Heaviside-Lorentz 单位制。从而，不同于 Gauss 单位制，Maxwell 方程组中不会出现无理数 4π ：

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \quad (1.11)$$

此处的单位制称为**有理化的**自然单位制。现在，精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$ ，而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的； 4π 因子会出现在 Coulomb 定律中，点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \quad (1.12)$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到**狭义相对论**，它的基本原理如下。

- (1) **光速不变原理**：在任意惯性参考系中，光速的大小不变。
- (2) **狭义相对性原理**：在任意惯性参考系中，物理定律具有相同的形式。

两个惯性参考系的 Descartes 坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动，则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \quad (1.13)$$

其中 Lorentz 因子 $\gamma \equiv (1 - \beta^2)^{-1/2}$ 。这种 Lorentz 变换称为沿 x 方向的**增速** (boost)。在此变换下，有

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2. \quad (1.14)$$

可见, $t^2 - x^2 - y^2 - z^2$ 在 Lorentz 变换下不变, 是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值, 这是 Lorentz 变换对应的对称性, 称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来, 可以构成 Minkowski 时空, 坐标记为

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \mathbf{x}), \quad \text{其中 } \mu = 0, 1, 2, 3. \quad (1.15)$$

上式中四种记法是等价的。 x^μ 是一个逆变 (contravariant) 的 Lorentz 四维矢量 (vector), “逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发, 可以定义 Lorentz 不变的内积¹

$$x^2 \equiv x \cdot x \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - |\mathbf{x}|^2. \quad (1.16)$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu} = g_{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.17)$$

可以把内积 (1.16) 简洁地写成

$$x^2 = g_{\mu\nu} x^\mu x^\nu. \quad (1.18)$$

这里采用了 **Einstein 求和约定**: 不写出求和符号, 重复的指标即表示求和。除非特别指出, 后面都默认使用这个约定。在上式中, 用同个字母表示的指标分别在上标和下标重复出现并求和, 这称为**缩并** (contraction), 是 Lorentz 不变量的特点。

为了进一步简化记号, 定义协变 (covariant) 的 Lorentz 四维矢量

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}). \quad (1.19)$$

“协变”指的是指标 μ 写在右下角。于是, 内积 x^2 的表达式 (1.18) 可以简化为

$$x^2 = x^\mu x_\mu. \quad (1.20)$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来, 变成协变矢量 x_μ 。从方阵的角度看, $g_{\mu\nu}$ 的逆为

$$g^{\mu\nu} = g^{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.21)$$

满足

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu, \quad (1.22)$$

¹这里的记号有些不一致, 第一个 x^2 是内积的记号, 而第二个 x^2 是第 2 个空间坐标。

其中 **Kronecker** 符号 δ^μ_ν 定义为

$$\delta^\mu_\nu = \delta_\mu^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \quad (1.23)$$

对于 Minkowski 度规, $g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同, 但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$, 对 μ 求和, 得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma_\nu x^\nu = x^\sigma, \quad (1.24)$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来, 变成逆变矢量 x^σ . 可见, 逆变矢量与协变矢量是一一对应的, 是对同一个 Lorentz 矢量的两种等价描述.

利用 Kronecker 符号的定义和 (1.22) 式, 可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \quad (1.25)$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \quad (1.26)$$

这两条式子表明, 度规也可以用来对度规自身的指标进行升降.

利用四维矢量的记号, 可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \quad (1.27)$$

其中

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (1.28)$$

注意: 在将 Λ^μ_ν 视作矩阵时, 偏左的指标 μ 表示行的编号, 偏右的指标 ν 表示列的编号. Λ^μ_ν 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变, 从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值. 我们可以将 Λ^μ_ν 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换, 称为 (齐次) **Lorentz 变换**, 使下式成立:

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \quad (1.29)$$

可见, Lorentz 变换 Λ^μ_ν 必须满足保度规条件

$$g_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta = g_{\alpha\beta}. \quad (1.30)$$

空间旋转变换保持 $|\mathbf{x}|^2$ 不变, 由 (1.16) 式可知, 这种变换也属于 Lorentz 变换. 例如, 绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \quad (1.31)$$

容易验证, 它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并, 可得

$$\Lambda_\nu^\gamma \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\alpha\beta} = \delta^\gamma_\beta, \quad (1.32)$$

其中

$$\Lambda_\nu^\gamma \equiv g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \quad (1.33)$$

可以看作是用度规对 Λ^μ_α 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu_\nu \equiv \Lambda^\mu_\nu, \quad (1.34)$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu_\rho \Lambda^\rho_\nu = \delta^\mu_\nu. \quad (1.35)$$

δ^μ_ν 也是一个 Lorentz 变换, 它使得 $x'^\mu = \delta^\mu_\nu x^\nu = x^\mu$, 即 x^μ 在这个变换下不变。可见, δ^μ_ν 是一个恒等变换。(1.35) 式表明, 对时空坐标矢量先作 Λ 变换, 再作 Λ^{-1} 变换, 得到的矢量还是原来的矢量。也就是说, 由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换, 也是一个 Lorentz 变换。在这些记号下, 协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu_\rho x^\rho = g_{\mu\nu} \Lambda^\nu_\rho g^{\rho\sigma} x_\sigma = \Lambda_\mu^\sigma x_\sigma = x_\sigma (\Lambda^{-1})^\sigma_\mu. \quad (1.36)$$

Λ^{-1} 既然是一个 Lorentz 变换, 必定满足保度规条件

$$g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta = g_{\alpha\beta}, \quad (1.37)$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma} = g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta g^{\alpha\rho} g^{\beta\sigma} = g^{\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \Lambda_\alpha^\mu \Lambda_\beta^\nu g^{\alpha\rho} g^{\beta\sigma} \\ &= g^{\gamma\delta} (g^{\alpha\rho} g_{\gamma\mu} \Lambda_\alpha^\mu) (g^{\beta\sigma} g_{\delta\nu} \Lambda_\beta^\nu) = g^{\gamma\delta} \Lambda^\rho_\gamma \Lambda^\sigma_\delta. \end{aligned} \quad (1.38)$$

这给出了保度规条件 (1.30) 的一个等价形式:

$$g^{\mu\nu} \Lambda^\alpha_\mu \Lambda^\beta_\nu = g^{\alpha\beta}. \quad (1.39)$$

将 Λ^μ_ν 视作矩阵 Λ , 则其转置矩阵 Λ^T 的分量满足 $(\Lambda^T)_\nu^\mu = \Lambda^\mu_\nu$, 由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = (\Lambda^T)_\alpha^\mu g_{\mu\nu} \Lambda^\nu_\beta, \quad (1.40)$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^T \mathbf{g} \Lambda. \quad (1.41)$$

取行列式得 $\det \mathbf{g} = \det \Lambda^T \cdot \det \mathbf{g} \cdot \det \Lambda = \det \mathbf{g} \cdot (\det \Lambda)^2$, 因此,

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \quad (1.42)$$

Lorentz 坐标变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$ 的 Jacobi 行列式为

$$\mathcal{J} = \det \left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} \right] = \det \Lambda, \quad (1.43)$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}| d^4x = |\det \Lambda| d^4x = d^4x. \quad (1.44)$$

可见, Minkowski 时空的体积元是 Lorentz 不变的。

$\det \Lambda$ 的值可以用来为 Lorentz 变换分类: $\det \Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换, $\det \Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外, 由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu} \Lambda^\mu_0 \Lambda^\nu_0 = (\Lambda^0_0)^2 - (\Lambda^i_0)^2, \quad (1.45)$$

则 $(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2 \geq 1$, 故有 $\Lambda^0_0 \geq +1$ 或 $\Lambda^0_0 \leq -1$ 。 $\Lambda^0_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换, $\Lambda^0_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上, 对称性由群论描述。对称变换的集合称为**群**, 群元素具有乘法, 满足下列四个条件。

- (1) 两个群元素的乘积即是两次对称变换相继作用, 乘法满足结合律。
- (2) 群中任意两个元素的乘积仍属于此群 (封闭性)。
- (3) 群中必有一个恒元 (对应于恒等变换), 它与任一元素的乘积仍为此元素。
- (4) 任一元素都可以在群中找到一个逆元 (对应于逆变换), 两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数 (如 β 、 θ 等) 来描述, 因而是一种连续变换, 所以 Lorentz 群是一个连续群, 参数的变化区域称为群空间。Lorentz 群的整个群空间不是连通的, 它有四个连通分支, 如图 1.1 所示, 分别是固有保时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \geq +1$)、固有反时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \leq -1$)、非固有保时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \geq +1$) 和非固有反时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \leq -1$), 四个分支之间彼此不连通。恒元 (即恒等变换) 在固有保时向分支里, 这个分支也称为**固有保时向 Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义宇称 (parity) 变换为

$$\mathcal{P}^\mu_\nu = (\mathcal{P}^{-1})^\mu_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.46)$$

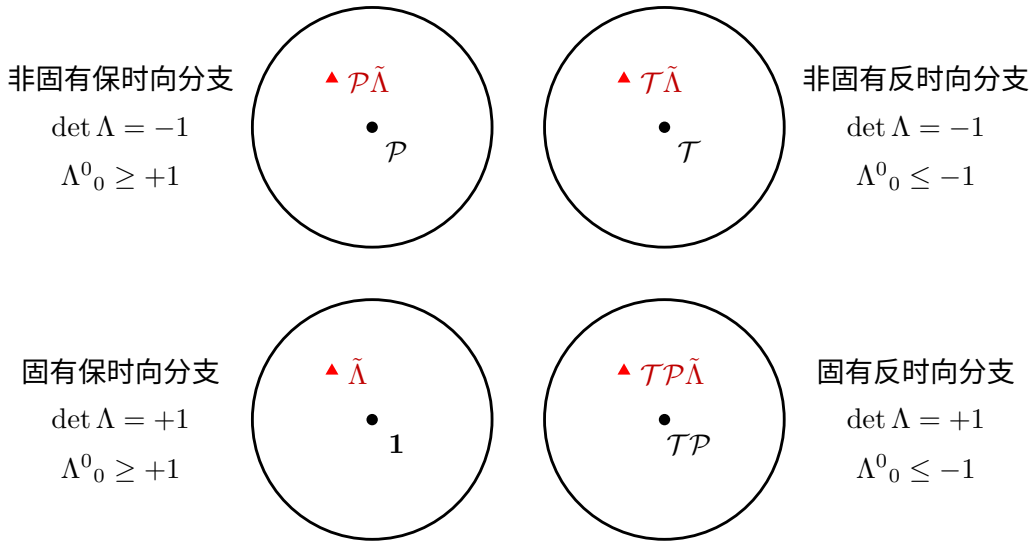


图 1.1: Lorentz 群的四个连通分支示意图。 1 、 \mathcal{P} 和 \mathcal{T} 分别代表恒等变换、宇称变换和时间反演变换， $\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的，亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \quad (1.47)$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素，乘上宇称变换或（和）时间反演变换，就可以到达 Lorentz 群的其它分支。

1.4 Lorentz 矢量

如果一些 $m \times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同，就可以用这些矩阵来表示这个群，这些矩阵构成了这个群的一个 m 维线性表示。利用群的线性表示，可以将对称变换视作矩阵，将变换作用的态视作列矩阵。

在上一节中，我们已经用矩阵的形式表示过 Lorentz 变换 $\Lambda^\mu{}_\nu$ ，可见， $\Lambda^\mu{}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**，因为 Lorentz 矢量 x^ν 可以看作是变换 $\Lambda^\mu{}_\nu$ 所作用的态。一般地，一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \quad (1.48)$$

类似于 (1.36) 式，逆变矢量 A^μ 对应的协变矢量 $A_\mu = g_{\mu\nu} A^\nu$ 在 Lorentz 变换下满足

$$A_\mu = A_\nu (\Lambda^{-1})^\nu{}_\mu. \quad (1.49)$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \quad (1.50)$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量:

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \quad (1.51)$$

这样的 Lorentz 不变量是 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负, Lorentz 矢量 A^μ 的自我内积不是正定的, 可以分为三类。

(1) 若 $A^2 > 0$, 则称 A^μ 为**类时**矢量。

(2) 若 $A^2 < 0$, 则称 A^μ 为**类空**矢量。

(3) 若 $A^2 = 0$, 则称 A^μ 为**类光**矢量。

由于 A^2 是 Lorentz 不变量, 不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中, 质点的能量 E 、动量 \mathbf{p} 和 (静止) 质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (1.52)$$

可以用 E 和 \mathbf{p} 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \quad (1.53)$$

称为**四维动量**, 它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \quad (1.54)$$

这是合理的, 因为质量 m 在狭义相对论中是一个 Lorentz 不变量。 p^μ 在 $m > 0$ 时是类时矢量, 在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla \right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla \right) = g^{\mu\nu} \partial_\nu, \quad (1.55)$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho^\nu = g^{\mu\nu}. \quad (1.56)$$

可见, 这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$ 时, 时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu_\nu \partial^\nu. \quad (1.57)$$

由上式、(1.56) 式和保度规条件 (1.39) 可得,

$$\partial'^\mu x'^\nu = \Lambda^\mu_\rho \partial^\rho (\Lambda^\nu_\sigma x^\sigma) = \Lambda^\mu_\rho \Lambda^\nu_\sigma \partial^\rho x^\sigma = \Lambda^\mu_\rho \Lambda^\nu_\sigma g^{\rho\sigma} = g^{\mu\nu}, \quad (1.58)$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的, 从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明, 时空导数的 Lorentz 变换形式与 Lorentz 矢量相同, 因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \quad (1.59)$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu} \partial'^\mu \partial'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma} \partial^\rho \partial^\sigma = \partial^2. \quad (1.60)$$

可见, ∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \quad (1.61)$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 不变性。

1.5 Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p + q$ 阶的 (p, q) 型 **Lorentz 张量** $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$ 应满足如下 Lorentz 变换规则:

$$T'^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \Lambda^{\mu_1}_{\rho_1} \cdots \Lambda^{\mu_p}_{\rho_p} T^{\rho_1 \cdots \rho_p}_{\sigma_1 \cdots \sigma_q} (\Lambda^{-1})^{\sigma_1}_{\nu_1} \cdots (\Lambda^{-1})^{\sigma_q}_{\nu_q}. \quad (1.62)$$

Lorentz 标量是 0 阶 Lorentz 张量, Lorentz 矢量是 1 阶 Lorentz 张量。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0, 2)$ 型 Lorentz 张量, 不过它在任何惯性系中不变, Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62), 可以验证, 如下表达式都是 Lorentz 标量 (亦即是 Lorentz 不变量):

$$g_{\mu\nu} T^{\mu\nu}, \quad T^{\mu\nu} A_\mu B_\nu, \quad T^{\mu\nu} T_{\mu\nu}, \quad g_{\mu\sigma} T^{\mu\nu}_\rho T^{\sigma\rho}_\nu. \quad (1.63)$$

实际上, 可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p, q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并, 可以得到一个 $(p-1, q-1)$ 型 Lorentz 张量。例如, 由

$$T'^{\mu\nu}_\mu = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}_\gamma (\Lambda^{-1})^\gamma_\mu = \Lambda^\nu_\beta T^{\alpha\beta}_\gamma \delta^\gamma_\alpha = \Lambda^\nu_\beta T^{\alpha\beta}_\alpha \quad (1.64)$$

可知, $T^{\mu\nu}_\mu$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita 符号**

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换,} \\ 0, & \text{其它情况。} \end{cases} \quad (1.65)$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是全反对称的, 即关于任意两个指标反对称, 如 $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\rho\nu\mu\sigma} = -\varepsilon^{\sigma\nu\rho\mu}$ 。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.66)$$

根据这些定义, $\varepsilon^{0123} = +1$, $\varepsilon_{0123} = -1$ 。从而,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 4!\varepsilon^{0123}\varepsilon_{0123} = -4!. \quad (1.67)$$

利用 Levi-Civita 符号可以把 $\det \Lambda$ 按照行列式定义写成

$$\det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.68)$$

对于固有 Lorentz 变换, $\det \Lambda = +1$, 有

$$\varepsilon^{0123} = \varepsilon^{0123} \det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.69)$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质, 可得

$$\varepsilon^{1023} = -\varepsilon^{0123} = -\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = \Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\beta\alpha\gamma\delta}. \quad (1.70)$$

依此类推, 可以证明

$$\varepsilon^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.71)$$

可见, 在固有 Lorentz 变换下, $\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量, 不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中, ρ 是电荷密度, \mathbf{J} 是电流密度, 它们可以组成一个 Lorentz 矢量 $J^{\mu} = (\rho, \mathbf{J})$, 从而, 电流连续性方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (1.72)$$

可以写成 Lorentz 不变的形式

$$\partial_{\mu}J^{\mu} = 0. \quad (1.73)$$

此外, 电场强度 \mathbf{E} 和磁感应强度 \mathbf{B} 可以用电势 Φ 和矢势 \mathbf{A} 表达为

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1.74)$$

这样, 方程

$$\nabla \cdot \mathbf{B} = 0 \quad (1.75)$$

是自动满足的。 Φ 和 \mathbf{A} 可以组成一个 Lorentz 矢量 $A^{\mu} = (\Phi, \mathbf{A})$, 称为四维矢势, 则 (1.74) 式的分量形式为

$$E^i = -\partial_i A^0 - \partial_0 A^i, \quad B^k = \varepsilon^{kij}\partial_i A^j, \quad i, j, k = 1, 2, 3. \quad (1.76)$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk} \equiv \varepsilon^{0ijk}, \quad (1.77)$$

因而 $\varepsilon^{123} = +1$ 。

引入电磁场的场强张量 (field strength tensor)

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (1.78)$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序，从上述定义可得

$$\begin{aligned} \partial^\rho F^{\mu\nu} &= \partial^\rho (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu \partial^\rho A^\nu - \partial^\mu \partial^\nu A^\rho + \partial^\nu \partial^\mu A^\rho - \partial^\nu \partial^\rho A^\mu \\ &= \partial^\mu F^{\rho\nu} + \partial^\nu F^{\mu\rho} = -\partial^\mu F^{\nu\rho} - \partial^\nu F^{\rho\mu}, \end{aligned} \quad (1.79)$$

即

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \quad (1.80)$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = -E^i, \quad (1.81)$$

可见， F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k} \varepsilon^{12k} = \varepsilon^{123} \varepsilon^{123} = 1$ 和 $\varepsilon^{12k} \varepsilon^{21k} = \varepsilon^{123} \varepsilon^{213} = -1$ ，依此类推，可以归纳出如下求和关系：

$$\varepsilon^{ijk} \varepsilon^{kmn} = \varepsilon^{ijk} \varepsilon^{mnk} = \delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}, \quad (1.82)$$

利用这个关系，可得

$$\varepsilon^{ijk} B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n = \delta^{im} \delta^{jn} \partial_m A^n - \delta^{in} \delta^{jm} \partial_m A^n = \partial_i A^j - \partial_j A^i, \quad (1.83)$$

从而，

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\varepsilon^{ijk} B^k, \quad (1.84)$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \quad (1.85)$$

Gauss 定律对应的方程

$$\nabla \cdot \mathbf{E} = \rho \quad (1.86)$$

等价于

$$J^0 = \rho = \partial_i E^i = -\partial_i F^{0i} = \partial_i F^{i0} = \partial_i F^{i0} + \partial_0 F^{00} = \partial_\mu F^{\mu 0}, \quad (1.87)$$

而 Ampère 定律对应的方程

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \quad (1.88)$$

等价于

$$J^i = \varepsilon^{ijk} \partial_j B^k - \partial_0 E^i = -\partial_j F^{ij} + \partial_0 F^{0i} = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \quad (1.89)$$

归纳起来, 有

$$\partial_\mu F^{\mu\nu} = J^\nu. \quad (1.90)$$

这个方程完全是用 Lorentz 张量写出来的, 它在不同惯性系中具有相同的形式, 即具有 **Lorentz 协变性**, 因而满足狭义相对性原理。

现在, Maxwell 方程组中还有一个方程没有讨论, 它是 Maxwell-Faraday 方程

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.91)$$

将它写成分量的形式, 得

$$\varepsilon^{kmn} \partial_m E^n = -\varepsilon^{kmn} \partial_m F^{0n} = \varepsilon^{kmn} \partial_m F^{n0} = -\partial_0 B^k, \quad (1.92)$$

从而

$$\partial_0 F^{ij} = -\varepsilon^{ijk} \partial_0 B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m F^{n0} = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial_m F^{n0} = \partial_i F^{j0} - \partial_j F^{i0}, \quad (1.93)$$

即

$$\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = 0. \quad (1.94)$$

这个方程与 Maxwell-Faraday 方程等价, 不过, 它只是前面得到的方程 (1.80) 取特定分量的形式。

利用四维 Levi-Civita 符号, 可以定义电磁场的对偶场强张量 (dual field strength tensor)

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (1.95)$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk} \varepsilon^{1jk} = \varepsilon^{123} \varepsilon^{123} + \varepsilon^{132} \varepsilon^{132} = 2$ 和 $\varepsilon^{1jk} \varepsilon^{2jk} = \varepsilon^{123} \varepsilon^{223} + \varepsilon^{132} \varepsilon^{232} = 0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk} \varepsilon^{ljk} = 2\delta^{il}, \quad (1.96)$$

利用这个关系, 可得

$$\begin{aligned} \tilde{F}^{0i} &= \frac{1}{2} \varepsilon^{0i\rho\sigma} F_{\rho\sigma} = \frac{1}{2} \varepsilon^{0ijk} F_{jk} = \frac{1}{2} \varepsilon^{0ijk} g_{j\mu} g_{k\nu} F^{\mu\nu} = \frac{1}{2} \varepsilon^{0ijk} g_{jm} g_{kn} F^{mn} = -\frac{1}{2} \varepsilon^{ijk} \delta^{jm} \delta^{kn} \varepsilon^{mnl} B^l \\ &= -\frac{1}{2} \varepsilon^{ijk} \varepsilon^{jkl} B^l = -\frac{1}{2} \varepsilon^{ijk} \varepsilon^{ljk} B^l = -\frac{1}{2} 2\delta^{il} B^l = -B^i, \end{aligned} \quad (1.97)$$

故 \tilde{F}^{0i} 对应于磁感应强度。另一方面,

$$\tilde{F}^{ij} = \frac{1}{2} \varepsilon^{ij\rho\sigma} F_{\rho\sigma} = \frac{1}{2} (\varepsilon^{ij0k} F_{0k} + \varepsilon^{ijk0} F_{k0}) = \varepsilon^{0ijk} F_{0k} = \varepsilon^{0ijk} g_{0\mu} g_{k\nu} F^{\mu\nu}$$

$$= \varepsilon^{ijk} g_{00} g_{kl} F^{0l} = -\varepsilon^{ijk} \delta^{kl} F^{0l} = -\varepsilon^{ijk} F^{0k} = \varepsilon^{ijk} E^k, \quad (1.98)$$

说明 \tilde{F}^{ij} 对应于电场强度。 $\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \quad (1.99)$$

由 \tilde{F}^{ij} 的定义, 有

$$\begin{aligned} \partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\rho\sigma} = -\frac{1}{2} \varepsilon^{\nu\mu\rho\sigma} \partial_\mu F_{\rho\sigma} = -\frac{1}{6} (\varepsilon^{\nu\mu\rho\sigma} \partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\sigma\mu\rho} \partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\rho\sigma\mu} \partial_\mu F_{\rho\sigma}) \\ &= -\frac{1}{6} (\varepsilon^{\nu\mu\rho\sigma} \partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\mu\rho\sigma} \partial_\rho F_{\sigma\mu} + \varepsilon^{\nu\mu\rho\sigma} \partial_\sigma F_{\mu\rho}) = -\frac{1}{6} \varepsilon^{\nu\mu\rho\sigma} (\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho}), \end{aligned} \quad (1.100)$$

因此, 方程 (1.80) 等价于

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \quad (1.101)$$

从这些讨论可以看到, 用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的, 而且方程的 Lorentz 协变性非常明确。

1.6 作用量原理

1.6.1 经典力学中的作用量原理

在经典力学中, 质点力学系统可以用拉格朗日量 (Lagrangian) 描述。对于具有 n 个自由度的系统, 可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i , 它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i = dq_i/dt$ 。拉格朗日量是广义坐标和广义速度的函数 $L(q_i, \dot{q}_i)$ 。拉格朗日量的时间积分

$$S = \int_{t_1}^{t_2} dt L[q_i(t), \dot{q}_i(t)] \quad (1.102)$$

称为作用量。

作用量原理指出, 作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零, 则有

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i, \quad (1.103)$$

即时间导数的变分等于变分的时间导数。从而可得

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L[q_i(t), \dot{q}_i(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right) \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] \end{aligned}$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2}, \quad (1.104)$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零, 即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$, 则上式最后一行第二项为零, 而 $\delta S = 0$ 等价于

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \quad (1.105)$$

这是 **Euler-Lagrange 方程**, 它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \quad (1.106)$$

反解上式表示的 n 个方程, 则可以用 q_i 和 p_i 将 \dot{q}_i 表达出来, 然后用 Legendre 变换定义**哈密顿量** (Hamiltonian)

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L, \quad (1.107)$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量, 变分为

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L = \int_{t_1}^{t_2} dt \delta (p_i \dot{q}_i - H) = \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \frac{d}{dt} \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left[\dot{q}_i \delta p_i + \frac{d}{dt} (p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] \\ &= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] + p_i \delta q_i \Big|_{t_1}^{t_2}. \end{aligned} \quad (1.108)$$

根据前面的假设, 上式最后一行第二项为零, 于是, $\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n. \quad (1.109)$$

这是 **Hamilton 正则运动方程**, 相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.105)。

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中, 场 $\phi(\mathbf{x}, t)$ 是系统的广义坐标, 每一个空间点 \mathbf{x} 都是一个自由度, 因此场论相当于具有无穷多自由度的质点力学。在局域场论中, 拉格朗日量 $L = \int d^3x \mathcal{L}(x)$, 其中 $\mathcal{L}(x)$ 称为**拉格朗日量密度** (下文将它简称为**拉氏量**)。 \mathcal{L} 是系统中 n 个场 $\phi_a(\mathbf{x}, t)$ ($a = 1, \dots, n$) 及其时空导数 $\partial_\mu \phi_a$ 的函数。现在, 作用量可以表达为

$$S = \int dt L = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \quad (1.110)$$

(1.44) 式告诉我们, 时空体积元 d^4x 是 Lorentz 不变的, 如果拉氏量 \mathcal{L} 也是 Lorentz 不变的, 则作用量 S 就是 Lorentz 不变的, 从而, 由作用量原理得到的运动方程满足狭义相对性原理。因此, 构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 \mathcal{L} , 即要求 \mathcal{L} 是一个 Lorentz 标量。

类似于前面质点力学的处理方式, 假设时空坐标的变分为零, 则对场的时空导数的变分等于场变分的时空导数, 即

$$\delta(\partial_\mu \phi_a) = \partial_\mu(\delta\phi_a). \quad (1.111)$$

于是, 利用分部积分可得

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta(\partial_\mu\phi_a) \right] = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \partial_\mu(\delta\phi_a) \right] \\ &= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right] - \left[\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \delta\phi_a \right\} \\ &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \delta\phi_a + \int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right]. \end{aligned} \quad (1.112)$$

上式最后一行第二项的积分项是关于时空坐标的全散度, 利用 Stokes 定理, 可以将它转化为积分区域边界面 \mathcal{S} 上的积分:

$$\int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right] = \int_{\mathcal{S}} dS_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a, \quad (1.113)$$

其中 dS_μ 是 \mathcal{S} 上的面元。进一步假设在边界面 \mathcal{S} 上 $\delta\phi_a = 0$, 则上式为零。我们通常讨论整个时空区域上的场, 从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零, 是很合理的。这样一来, $\delta S = 0$ 给出

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \quad (1.114)$$

这就是场的 *Euler-Lagrange* 方程, 它给出场的经典运动方程。

引入场的共轭动量密度 (conjugate momentum density)

$$\pi_a(\mathbf{x}, t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}, \quad (1.115)$$

则可以用 Legendre 变换将哈密顿量定义为

$$H \equiv \int d^3x \pi_a \dot{\phi}_a - L \equiv \int d^3x \mathcal{H}, \quad (1.116)$$

其中, 哈密顿量密度

$$\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a \dot{\phi}_a - \mathcal{L}. \quad (1.117)$$

作用量变分为

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} = \int d^4x \delta(\pi_a \dot{\phi}_a - \mathcal{H}) \\ &= \int d^4x \left[\dot{\phi}_a \delta\pi_a + \pi_a \delta\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\phi_a} \delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a} \delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)} \cdot \delta(\nabla\phi_a) \right] \end{aligned}$$

$$\begin{aligned}
&= \int d^4x \left[\dot{\phi}_a \delta \pi_a + \pi_a \frac{d}{dt} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a - \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \cdot \nabla (\delta \phi_a) \right] \\
&= \int d^4x \left\{ \dot{\phi}_a \delta \pi_a + \frac{d}{dt} (\pi_a \delta \phi_a) - \dot{\pi}_a \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a \right. \\
&\quad \left. - \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \delta \phi_a \right] + \left[\nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \right] \delta \phi_a \right\} \\
&= \int d^4x \left\{ \left(\dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \right) \delta \pi_a - \left[\dot{\pi}_a + \frac{\partial \mathcal{H}}{\partial \phi_a} - \nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \right] \delta \phi_a \right\} \\
&\quad + \int d^4x \frac{d}{dt} (\pi_a \delta \phi_a) - \int d^4x \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \delta \phi_a \right]. \tag{1.118}
\end{aligned}$$

与前面一样，假设在时空区域边界面上 $\delta \phi_a = 0$ ，则上式最后两行的两项均为零，于是， $\delta S = 0$ 给出场的正则运动方程

$$\dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi_a}, \quad \dot{\pi}_a = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)}. \tag{1.119}$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述，则它是一种连续变换，连续变换对应的对称性称为连续对称性。**Noether 定理**指出，如果一个系统具有某种不显含时间的连续对称性，就必然存在一种对应的守恒定律。Noether 定理首先是在经典物理中给出的，但实际上它对所有物理行为由作用量原理决定的系统都成立。因此，可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$S = \int_R d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{1.120}$$

考虑一个连续变换，使得

$$\phi_a(x) \rightarrow \phi'_a(x'), \tag{1.121}$$

其中已包含了坐标的变换

$$x^\mu \rightarrow x'^\mu, \tag{1.122}$$

它引起的拉氏量变换为

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x'). \tag{1.123}$$

记这个变换的无穷小变换形式为

$$\phi'_a(x') = \phi_a(x) + \delta \phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta \mathcal{L}, \tag{1.124}$$

如果在此变换下

$$\delta S = \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) = 0, \tag{1.125}$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4x' = |\mathcal{J}|d^4x, \quad \mathcal{J} = \det \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) \simeq \det \left[\delta^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu} \right], \quad (1.126)$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 \mathbf{A} 满足 $\det(\mathbf{A}) \ll 1$ ，则有如下表达式：

$$\det(\mathbf{1} + \mathbf{A}) \simeq 1 + \text{tr}(\mathbf{A}). \quad (1.127)$$

利用上式可以将 Jacobi 行列式 \mathcal{J} 化为

$$\mathcal{J} \simeq 1 + \text{tr} \left[\frac{\partial(\delta x^\mu)}{\partial x^\nu} \right] = 1 + \partial_\mu(\delta x^\mu), \quad (1.128)$$

从而，体积元的无穷小变换形式为

$$d^4x' \simeq [1 + \partial_\mu(\delta x^\mu)]d^4x. \quad (1.129)$$

作用量在此无穷小变换下的变分为

$$\begin{aligned} \delta S &= \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) \\ &= \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}'(x') + \int_R d^4x \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) \\ &\simeq \int_R d^4x [1 + \partial_\mu(\delta x^\mu)] \mathcal{L}'(x') - \int_R d^4x \mathcal{L}'(x') + \int_R d^4x \delta \mathcal{L} \\ &\simeq \int_R d^4x \mathcal{L}'(x') \partial_\mu(\delta x^\mu) + \int_R d^4x \delta \mathcal{L} \simeq \int_R d^4x [\delta \mathcal{L} + \mathcal{L}(x) \partial_\mu(\delta x^\mu)] \\ &= \int_R d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) + \mathcal{L} \partial_\mu(\delta x^\mu) \right]. \end{aligned} \quad (1.130)$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$ ，使得

$$\bar{\delta} \phi_a(x) = \phi'_a(x) - \phi_a(x). \quad (1.131)$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu \phi_a) = \partial_\mu(\bar{\delta} \phi_a), \quad (1.132)$$

δ 算符则不能。 $\delta \phi_a$ 与 $\bar{\delta} \phi_a$ 的关系为

$$\begin{aligned} \delta \phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta} \phi_a \\ &\simeq \bar{\delta} \phi_a + (\partial_\mu \phi'_a) \delta x^\mu \simeq \bar{\delta} \phi_a + (\partial_\mu \phi_a) \delta x^\mu, \end{aligned} \quad (1.133)$$

即

$$\bar{\delta} \phi = \delta \phi_a - (\partial_\mu \phi_a) \delta x^\mu. \quad (1.134)$$

同理,

$$\delta(\partial_\mu \phi_a) = \bar{\delta}(\partial_\mu \phi_a) + \partial_\nu(\partial_\mu \phi_a) \delta x^\nu = \partial_\mu(\bar{\delta} \phi_a) + \partial_\nu(\partial_\mu \phi_a) \delta x^\nu. \quad (1.135)$$

将 (1.133) 和 (1.135) 式代入 (1.130) 式, 得到

$$\begin{aligned} \delta S &= \int_R d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} [\bar{\delta} \phi_a + (\partial_\mu \phi_a) \delta x^\mu] + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} [\partial_\mu(\bar{\delta} \phi_a) + \partial_\nu(\partial_\mu \phi_a) \delta x^\nu] + \mathcal{L} \partial_\mu(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \bar{\delta} \phi_a + \frac{\partial \mathcal{L}}{\partial \phi_a} \frac{\partial \phi_a}{\partial x^\mu} \delta x^\mu + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \bar{\delta} \phi_a \right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \frac{\partial(\partial_\nu \phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu}(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] \bar{\delta} \phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a \right] \right. \\ &\quad \left. + \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \frac{\partial \phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \frac{\partial(\partial_\nu \phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu}(\delta x^\mu) \right] \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] \bar{\delta} \phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu \right] \right\}. \end{aligned} \quad (1.136)$$

第二步用到分部积分, 最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu}(\mathcal{L} \delta x^\mu) = \frac{\partial \mathcal{L}}{\partial \phi_a} \frac{\partial \phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_a)} \frac{\partial(\partial_\nu \phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu}(\delta x^\mu). \quad (1.137)$$

根据 Euler-Lagrange 方程 (1.114), (1.136) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的, $\delta S = 0$ 等价于第二项为零, 即

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu \right] = 0. \quad (1.138)$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu, \quad (1.139)$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \quad (1.140)$$

方程 (1.140) 左边对整个三维空间积分, 运用 Stokes 定理, 得

$$\int d^3x \partial_\mu j^\mu = \int d^3x \partial_0 j^0 + \int d^3x \partial_i j^i = \frac{d}{dt} \int d^3x j^0 + \int_S d\mathcal{S}_i j^i, \quad (1.141)$$

其中 $i = 1, 2, 3$ 。对于整个三维空间而言, 边界面 \mathcal{S} 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失, 从而, 在无穷远处 $j^i \rightarrow 0$, 所以上式最后一项为零。定义**守恒荷** (conserved charge)

$$Q \equiv \int d^3x j^0, \quad (1.142)$$

则由 (1.141) 和 (1.140) 式可得

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x j^0 = \int d^3x \partial_\mu j^\mu = 0. \quad (1.143)$$

可见, Q 不随时间变化, 是守恒的。

综上, 在场论中, 如果一个系统具有某种连续对称性, 则存在相应的守恒流 (1.139), 它满足守恒流方程 (1.140), 而守恒荷 (1.142) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2 时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^\mu = x^\mu - \varepsilon^\mu, \quad (1.144)$$

其中 ε^μ 是常数。要求场 ϕ_a 具有时空平移对称性, 则

$$\phi'_a(x') = \phi'_a(x - \varepsilon) = \phi_a(x). \quad (1.145)$$

现在, $\delta x^\mu = -\varepsilon^\mu$, 由 (1.134) 式可得

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) + \varepsilon^\mu\partial_\mu\phi_a = 0 + \varepsilon^\mu\partial_\mu\phi_a = \varepsilon^\rho\partial_\rho\phi_a, \quad (1.146)$$

代入到 Noether 守恒流表达式 (1.139), 得

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^\rho\partial_\rho\phi_a - \mathcal{L}\varepsilon^\mu = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] \varepsilon^\rho. \quad (1.147)$$

从而, $\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] = 0, \quad (1.148)$$

各项乘以 $g^{\rho\nu}$, 缩并, 得

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L} \right] = 0. \quad (1.149)$$

上式方括号部分是场的能动张量 (energy-momentum tensor)

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}, \quad (1.150)$$

它满足

$$\partial_\mu T^{\mu\nu} = 0. \quad (1.151)$$

因此, 对 $T^{0\nu}$ ($\nu = 0, 1, 2, 3$) 作全空间积分, 就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^0\phi_a - \mathcal{L}, \quad (1.152)$$

与 (1.117) 和 (1.115) 式比较, 可以看出 T^{00} 就是哈密顿量密度 \mathcal{H} 。 T^{00} 的全空间积分

$$H = \int d^3x T^{00} = \int d^3x \mathcal{H} \quad (1.153)$$

是场的哈密顿量，或者说**总能量**。 $T^{\mu\nu}$ 的 $0i$ 分量

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^i \phi_a = \pi_a \partial^i \phi_a \quad (1.154)$$

是场的动量密度，它的全空间积分

$$P^i = \int d^3x T^{0i} = \int d^3x \pi_a \partial^i \phi_a \quad (1.155)$$

是场的**总动量**。根据 (1.55) 式，上式也可以写成

$$\mathbf{P} = - \int d^3x \pi_a \nabla \phi_a. \quad (1.156)$$

H 和 P^i 都是守恒荷，可见，时间平移对称性对应于**能量守恒定律**，空间平移对称性对应于**动量守恒定律**。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad (1.157)$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30)，有

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\mu\nu} (\delta^\mu{}_\alpha + \omega^\mu{}_\alpha) (\delta^\nu{}_\beta + \omega^\nu{}_\beta) \simeq g_{\mu\nu} \delta^\mu{}_\alpha \delta^\nu{}_\beta + g_{\mu\nu} \delta^\mu{}_\alpha \omega^\nu{}_\beta + g_{\mu\nu} \omega^\mu{}_\alpha \delta^\nu{}_\beta \\ &= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{aligned} \quad (1.158)$$

可见，

$$\omega_{\mu\nu} \equiv g_{\mu\rho} \omega^\rho{}_\nu \quad (1.159)$$

关于两个指标反对称：

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (1.160)$$

因此， $\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31)，利用三角函数展开式 $\cos \theta = 1 + \mathcal{O}(\theta^2)$ 和 $\sin \theta = \theta + \mathcal{O}(\theta^3)$ ，可得

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho} \omega^\rho{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}. \quad (1.161)$$

对于沿 x 的增速变换 (1.28)，可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1} \beta, \quad (1.162)$$

再利用双曲函数公式 $\tanh \xi = \sinh \xi / \cosh \xi$ 和 $\cosh^2 \xi - \sinh^2 \xi = 1$ 得

$$\begin{aligned}\gamma &= (1 - \beta^2)^{-1/2} = (1 - \tanh^2 \xi)^{-1/2} = \left(\frac{\cosh^2 \xi - \sinh^2 \xi}{\cosh^2 \xi} \right)^{-1/2} = \cosh \xi, \\ \beta\gamma &= \tanh \xi \cosh \xi = \sinh \xi,\end{aligned}\quad (1.163)$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh \xi & -\sinh \xi & & \\ -\sinh \xi & \cosh \xi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$
 (1.164)

根据双曲函数展开式 $\cosh \xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh \xi = \xi + \mathcal{O}(\xi^3)$, 有

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho} \omega^\rho{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$
 (1.165)

在无穷小 Lorentz 变换 (1.157) 的作用下, 一般地, 场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2} \omega_{\mu\nu} (I^{\mu\nu})_{ab} \right] \phi_b(x) = \phi_a(x) - \frac{i}{2} \omega_{\mu\nu} (I^{\mu\nu})_{ab} \phi_b(x),$$
 (1.166)

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的生成元 (generator)。由于 $\omega_{\mu\nu}$ 是反对称的, 有

$$\omega_{\mu\nu} (I^{\mu\nu})_{ab} = \omega_{\nu\mu} (I^{\nu\mu})_{ab} = -\omega_{\mu\nu} (I^{\nu\mu})_{ab},$$
 (1.167)

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称:

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}.$$
 (1.168)

现在, $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, 而

$$\bar{\delta} \phi_a = \delta \phi_a - (\partial_\mu \phi_a) \delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu \phi_a) \delta x^\mu = -\frac{i}{2} \omega_{\nu\rho} (I^{\nu\rho})_{ab} \phi_b - (\partial_\nu \phi_a) \omega^\nu{}_\rho x^\rho, \quad (1.169)$$

故 Noether 流为

$$\begin{aligned}j^\mu &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu = -\frac{i}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (I^{\nu\rho})_{ab} \phi_b - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (\partial_\nu \phi_a) \omega^\nu{}_\rho x^\rho + \mathcal{L} \omega^\mu{}_\rho x^\rho \\ &= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (\partial_\nu \phi_a) - \delta^\mu{}_\nu \mathcal{L} \right] \omega^\nu{}_\rho x^\rho \\ &= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - T^\mu{}_\nu \omega^\nu{}_\rho x^\rho,\end{aligned}\quad (1.170)$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho} g_{\rho\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L} \quad (1.171)$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作：

$$T^\mu{}_\nu \omega^\nu{}_\rho = T^\mu{}_\nu \delta^\nu{}_\sigma \omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha} g_{\alpha\sigma} \omega^\sigma{}_\rho = T^{\mu\alpha} \omega_{\alpha\rho} = T^{\mu\nu} \omega_{\nu\rho}, \quad (1.172)$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned} T^\mu{}_\nu \omega^\nu{}_\rho x^\rho &= T^{\mu\nu} \omega_{\nu\rho} x^\rho = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\nu} \omega_{\rho\nu} x^\rho) = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\rho} \omega_{\nu\rho} x^\nu) \\ &= \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu). \end{aligned} \quad (1.173)$$

于是，Noether 流 (1.170) 可化为

$$j^\mu = \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) = \frac{1}{2} J^{\mu\nu\rho} \omega_{\nu\rho} \quad (1.174)$$

其中

$$J^{\mu\nu\rho} \equiv T^{\mu\rho} x^\nu - T^{\mu\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b. \quad (1.175)$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu J^{\mu\nu\rho} = 0, \quad (1.176)$$

守恒荷为

$$J^{\nu\rho} \equiv \int d^3x J^{0\nu\rho} = \int d^3x \left[T^{0\rho} x^\nu - T^{0\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b \right]. \quad (1.177)$$

易见 $J^{\nu\rho} = -J^{\rho\nu}$ ，因而一共有 6 个独立的守恒荷，满足 $dJ^{\nu\rho}/dt = 0$ 。

为明确物理含义，可将 $J^{\nu\rho}$ 分解成两项：

$$J^{\nu\rho} = L^{\nu\rho} + S^{\nu\rho}. \quad (1.178)$$

第一项为

$$\begin{aligned} L^{\nu\rho} &\equiv \int d^3x (T^{0\rho} x^\nu - T^{0\nu} x^\rho) \\ &= \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} \partial^\rho \phi_a - g^{0\rho} \mathcal{L} \right) x^\nu - \left(\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} \partial^\nu \phi_a - g^{0\nu} \mathcal{L} \right) x^\rho \right] \\ &= \int d^3x [(\pi_a \partial^\rho \phi_a - g^{0\rho} \mathcal{L}) x^\nu - (\pi_a \partial^\nu \phi_a - g^{0\nu} \mathcal{L}) x^\rho] \\ &= \int d^3x [\pi_a (x^\nu \partial^\rho - x^\rho \partial^\nu) \phi_a + (g^{0\nu} x^\rho - g^{0\rho} x^\nu) \mathcal{L}]. \end{aligned} \quad (1.179)$$

它的纯空间分量 L^{jk} 中只有 3 个是独立的，可以等价地定义成

$$L^i \equiv \frac{1}{2} \varepsilon^{ijk} L^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (x^j \partial^k - x^k \partial^j) \phi_a, \quad (1.180)$$

这是场的轨道角动量。第二项为

$$S^{\nu\rho} \equiv \int d^3x \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b = \int d^3x \pi_a (-i I^{\nu\rho})_{ab} \phi_b, \quad (1.181)$$

同样, 3 个独立的等价纯空间分量是

$$S^i \equiv \frac{1}{2} \varepsilon^{ijk} S^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (-i I^{jk})_{ab} \phi_b, \quad (1.182)$$

这是场的自旋角动量。因此, $J^{\nu\rho}$ 的纯空间分量等价于

$$J^i \equiv \frac{1}{2} \varepsilon^{ijk} J^{jk} = L^i + S^i, \quad (1.183)$$

这是场的总角动量。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 $SO(3)$, 而空间旋转对称性对应于角动量守恒定律。

另一方面, $L^{\nu\rho}$ 的 $i0$ 分量为

$$L^{i0} = \int d^3x (T^{00} x^i - T^{0i} x^0) = \int d^3x (x^i \mathcal{H} - x^0 \pi_a \partial^i \phi_a) = \int d^3x x^i \mathcal{H} - t P^i. \quad (1.184)$$

若 $dS^{i0}/dt = 0$, 则有 $dL^{i0}/dt = 0$, 从而

$$L^{i0}(t) = L^{i0}|_{t=0} = \int d^3x x^i \mathcal{H}(t=0), \quad (1.185)$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下, 能量密度相当于质量密度, 则 L^{i0} 是 $t=0$ 时刻的质心 (即质量中心, center of mass)。 L^{i0} 的守恒在经典力学中对应于质心运动守恒定律: 当没有外力存在时, 质心的加速度为零, 质心保持静止或作匀速直线运动。

1.7.4 U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu \phi^*) \partial_\mu \phi - m^2 \phi^* \phi. \quad (1.186)$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \quad (1.187)$$

其中 θ 是不依赖于 x^μ 的连续变换实参数, q 是一个常数。这里不包含坐标的变换。 $e^{iq\theta}$ 是个纯相位因子, 可以看成是一个 1 维幺正 (unitary) 矩阵, 形式为 $e^{iq\theta}$ 的所有变换组成的群称为 **U(1) 群**。 **整体** (global) 指的是变换参数不依赖于时空坐标。相应地, ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta} \phi^*(x). \quad (1.188)$$

容易看出, 由 (1.186) 式定义的 \mathcal{L} 在这种变换下不变, 即具有 U(1) 整体对称性。与前面叙述的两种对称性不同, 这里的对称性出现在由场组成的抽象空间中, 与时间和空间相对独立 ($\delta x^\mu = 0$), 因而是一种内部对称性。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \quad (1.189)$$

结合 $\delta x^\mu = 0$, 有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \quad (1.190)$$

于是, Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \bar{\delta}\phi^* = \partial^\mu \phi^* (iq\theta\phi) + \partial^\mu \phi (-iq\theta\phi^*) \\ &= iq\theta[(\partial^\mu \phi^*)\phi - (\partial^\mu \phi)\phi^*] = -q\theta\phi^* i \overleftrightarrow{\partial}^\mu \phi, \end{aligned} \quad (1.191)$$

其中, $\overleftrightarrow{\partial}^\mu$ 符号通过下式定义:

$$\phi^* \overleftrightarrow{\partial}^\mu \phi \equiv \phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi. \quad (1.192)$$

扔掉无穷小参数 $-\theta$, 定义

$$J^\mu \equiv q\phi^* i \overleftrightarrow{\partial}^\mu \phi, \quad (1.193)$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$, 相应的守恒荷为

$$Q = \int d^3x J^0 = q \int d^3x \phi^* i \overleftrightarrow{\partial}^0 \phi. \quad (1.194)$$

在实际情况下, q 是由 ϕ 场描述的粒子所携带的某种荷, 如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此, 一种 U(1) 整体对称性对应于一条荷数守恒定律, 比如, 电磁 U(1) 整体对称性就对应于**电荷守恒定律**。

第 2 章 标量场

本章讲述标量场的**正则量子化** (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1 简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \quad (2.1)$$

其中 m 是质量， ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 当成厄米算符，满足**正则对易关系**

$$[x, p] = xp - px = i. \quad (2.2)$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \quad (2.3)$$

a 称为**湮灭算符** (annihilation operator)， a^\dagger 称为**产生算符** (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\ &= \frac{1}{2}(-i[x, p] + i[p, x]) = -i[x, p] = 1. \end{aligned} \quad (2.4)$$

根据 (2.3) 式，可以反过来用 a 和 a^\dagger 表示 x 和 p ：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \quad (2.5)$$

从而，哈密顿量表示成

$$\begin{aligned} H &= -\frac{1}{2m} \frac{m\omega}{2}(a - a^\dagger)^2 + \frac{1}{2}m\omega^2 \frac{1}{2m\omega}(a + a^\dagger)^2 \\ &= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a). \end{aligned} \quad (2.6)$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$, 于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega \left(a^\dagger a + \frac{1}{2} \right) = \omega \left(N + \frac{1}{2} \right), \quad (2.7)$$

其中, $N \equiv a^\dagger a$ 是个厄米算符, 称为**粒子数算符**. N 还是个正定算符, 对于任意态 $|\psi\rangle$, N 的平均值非负:

$$\langle \psi | N | \psi \rangle = \langle \psi | a^\dagger a | \psi \rangle = \langle a\psi | a\psi \rangle \geq 0. \quad (2.8)$$

设 $|n\rangle$ 是 N 的本征态, 满足本征方程

$$N |n\rangle = n |n\rangle. \quad (2.9)$$

由 $n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle \geq 0$ 可知, n 是个非负实数. 利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B, \quad (2.10)$$

$$[A, BC] = ABC - BAC + BAC - BCA = [A, B]C + B[A, C], \quad (2.11)$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \quad (2.12)$$

从而, 有

$$Na^\dagger |n\rangle = ([N, a^\dagger] + a^\dagger N) |n\rangle = (a^\dagger + a^\dagger n) |n\rangle = (n+1)a^\dagger |n\rangle, \quad (2.13)$$

$$Na |n\rangle = ([N, a] + aN) |n\rangle = (-a + an) |n\rangle = (n-1)a |n\rangle. \quad (2.14)$$

可见, $a^\dagger |n\rangle$ 和 $a |n\rangle$ 都是 N 的本征态, 本征值分别为 $n+1$ 和 $n-1$, 也就是说,

$$a^\dagger |n\rangle = c_1 |n+1\rangle, \quad a |n\rangle = c_2 |n-1\rangle, \quad (2.15)$$

其中 c_1 和 c_2 是两个归一化常数. a^\dagger 将本征值为 n 的态变成本征值为 $n+1$ 的态, 因而也称为升算符 (raising operator); a 将本征值为 n 的态变成本征值为 $n-1$ 的态, 因而也称为降算符 (lowering operator). 为确定归一化常数的值, 可作如下计算:

$$n+1 = \langle n | (N+1) | n \rangle = \langle n | (a^\dagger a + 1) | n \rangle = \langle n | aa^\dagger | n \rangle = |c_1|^2 \langle n+1 | n+1 \rangle = |c_1|^2, \quad (2.16)$$

$$n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle = |c_2|^2 \langle n+1 | n+1 \rangle = |c_2|^2. \quad (2.17)$$

将 c_1 和 c_2 都取为实数, 则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$, 故

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \quad (2.18)$$

从 N 的某个本征态 $|n\rangle$ 出发, 用降算符 a 逐步操作, 可得本征值逐次减小的一系列本征态

$$a |n\rangle, a^2 |n\rangle, a^3 |n\rangle, \dots, \quad (2.19)$$

本征值分别为

$$n - 1, n - 2, n - 3, \dots \quad (2.20)$$

由于 $n \geq 0$ ，必定存在一个最小本征值 n_0 ，它的本征态 $|n_0\rangle$ 满足

$$a |n_0\rangle = 0. \quad (2.21)$$

于是，有

$$N |n_0\rangle = a^\dagger a |n_0\rangle = 0 = 0 |n_0\rangle, \quad (2.22)$$

可见， $n_0 = 0$ ，即

$$|n_0\rangle = |0\rangle. \quad (2.23)$$

反过来，从 $|0\rangle$ 出发，用升算符 a^\dagger 逐步操作，可得本征值逐次增加的一系列本征态

$$a^\dagger |0\rangle, (a^\dagger)^2 |0\rangle, (a^\dagger)^3 |0\rangle, \dots, \quad (2.24)$$

本征值分别为

$$1, 2, 3, \dots \quad (2.25)$$

综上，本征值 n 的取值是非负整数，是量子化的；本征态 $|n\rangle$ 可以用 a^\dagger 和 $|0\rangle$ 表示为

$$|n\rangle = c_3 (a^\dagger)^n |0\rangle. \quad (2.26)$$

为确定归一化常数 c_3 ，可作如下运算：

$$\begin{aligned} \langle n|n\rangle &= |c_3|^2 \langle 0| a^n (a^\dagger)^n |0\rangle = |c_3|^2 \langle 1| a^{n-1} (a^\dagger)^{n-1} |1\rangle = 1 \cdot 2 |c_3|^2 \langle 2| a^{n-2} (a^\dagger)^{n-2} |2\rangle = \dots \\ &= (n-1)! |c_3|^2 \langle n-1| a a^\dagger |n-1\rangle = n! |c_3|^2 \langle n|n\rangle, \end{aligned} \quad (2.27)$$

故 $|c_3|^2 = 1/n!$ 。取 c_3 为实数，可得 $c_3 = 1/\sqrt{n!}$ ，于是

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \quad (2.28)$$

从 (2.7) 式容易看出， $|n\rangle$ 也是 H 的本征态：

$$H |n\rangle = \omega \left(N + \frac{1}{2} \right) |n\rangle = \omega \left(n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle, \quad (2.29)$$

相应的能量本征值为

$$E_n = \omega \left(n + \frac{1}{2} \right). \quad (2.30)$$

基态 $|0\rangle$ 的能量本征值不是零，而是 $E_0 = \omega/2$ ，称为零点能 (zero-point energy)，这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态，将 $n > 0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态，每个声子具有一份能量 ω 。这样一来， n 表示声子的数目，故粒子数算符 N 描述的是声子数。 a^\dagger 的作用是产生一个声子，从而增加一份能量； a 的作用是湮灭一个声子，从而减少一份能量。这是将 a^\dagger 和 a 称为产生算符和湮灭算符的原因。

2.2 量子场论中的正则对易关系

在量子力学中, Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法, 它们之间可以通过一个幺正变换联系起来。在 Schrödinger 绘景中, 波函数 $\psi_S(t)$ 代表随时间演化的物理态, 而任意算符 O_S 不依赖于时间。在 Heisenberg 绘景中, 波函数 ψ_H 代表不随时间演化的物理态, 它与 $\psi_S(t)$ 的关系为

$$\psi_H = e^{iHt}\psi_S(t), \quad (2.31)$$

其中 H 是哈密顿量; 而算符 $O_H(t)$ 依赖于时间, 通过下式与 O_S 联系起来:

$$O_H(t) = e^{iHt}O_S(t)e^{-iHt}. \quad (2.32)$$

上一节的量子化可以认为是在 Schrödinger 绘景中实现的, 因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x_S, p_S] = i$, 它在 Heisenberg 绘景中的形式为

$$\begin{aligned} [x_H(t), p_H(t)] &= [e^{iHt}x_S e^{-iHt}, e^{iHt}p_S e^{-iHt}] = e^{iHt}x_S e^{-iHt} e^{iHt}p_S e^{-iHt} - e^{iHt}p_S e^{-iHt} e^{iHt}x_S e^{-iHt} \\ &= e^{iHt}x_S p_S e^{-iHt} - e^{iHt}p_S x_S e^{-iHt} = e^{iHt}[x_S, p_S]e^{-iHt} = e^{iHt}ie^{-iHt} = i. \end{aligned} \quad (2.33)$$

可见, 正则对易关系的形式不依赖于绘景。(2.33) 式是在同一时刻 t 成立的, 称为等时 (equal time) 对易关系。

将讨论推广到自由度为 n 的系统, 记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符, $p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响, 这些算符需要满足如下等时对易关系:

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [q_i(t), q_j(t)] = 0, \quad [p_i(t), p_j(t)] = 0. \quad (2.34)$$

1.1 节提到, 在量子场论中, 为了平等地处理时间和空间, 空间坐标 \mathbf{x} 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x}, t)$ 的参数。由于这里量子场作为算符是依赖于时间的, 使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行, 省略绘景的标志性下标 H。

场讨论的是无穷多自由度的系统, 每一个空间点 \mathbf{x} 上的 $\phi(\mathbf{x}, t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度, 我们可以先将空间离散化, 划分成 n 个小体积元 V_i , 然后再取 $V_i \rightarrow 0$ 的极限来得到 $n \rightarrow \infty$ 的结果。在体积元 V_i 中, 定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i} \int_{V_i} d^3x \phi(\mathbf{x}, t), \quad (2.35)$$

它是场 $\phi(\mathbf{x}, t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi, \partial_\mu \phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L}(\phi, \partial_\mu \phi), \quad (2.36)$$

当体积元取得足够小时, 它就成为 ϕ_i 和 $\partial_0 \phi_i$ 的函数 $\mathcal{L}_i(\phi_i, \partial_0 \phi_i)$ 。拉格朗日量可表达为

$$L = \int d^3x \mathcal{L} = \sum_i \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \mathcal{L}_i(\phi_i, \partial_0 \phi_i). \quad (2.37)$$

于是, 由 (1.106) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \delta_{ji} \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} = V_i \pi_i(t), \quad (2.38)$$

其中,

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]}. \quad (2.39)$$

现在, 等时对易关系变成

$$[\phi_i(t), \Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t), \phi_j(t)] = 0, \quad [\Pi_i(t), \Pi_j(t)] = 0. \quad (2.40)$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t), \pi_j(t)] = 0. \quad (2.41)$$

对于任意连续函数 $f(x)$, **Dirac δ 函数** $\delta(x)$ 使下式成立:

$$f(x) = \int dy f(y) \delta(x - y). \quad (2.42)$$

函数 $\delta(x)$ 只在 $x = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即

$$\delta(x) = \delta(-x), \quad (2.43)$$

而且满足

$$\int dx \delta(x) = 1. \quad (2.44)$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \quad (2.45)$$

则对于任意连续函数 $f(\mathbf{x})$, 下式成立:

$$f(\mathbf{x}) = \int d^3y f(\mathbf{y}) \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (2.46)$$

类似地, 函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x} = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$, 而且满足 $\int d^3x \delta^{(3)}(\mathbf{x}) = 1$ 。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值, 则它会满足

$$f_i = \sum_j f_j \delta_{ij} = \sum_j V_j f_j \frac{\delta_{ij}}{V_j}. \quad (2.47)$$

(2.46) 式是 (2.47) 式在 $V_i \rightarrow 0$ 时的极限。可见, 在 $V_i \rightarrow 0$ 极限下,

$$\frac{\delta_{ij}}{V_j} \rightarrow \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (2.48)$$

另一方面, 在此极限下, $\phi_i(t) \rightarrow \phi(\mathbf{x}, t)$, 而 $\pi_i(t)$ 变成由 (1.115) 式定义的共轭动量密度:

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial [\partial_0 \phi_i(t)]} \rightarrow \frac{\partial \mathcal{L}}{\partial [\partial_0 \phi(\mathbf{x}, t)]} = \pi(\mathbf{x}, t). \quad (2.49)$$

因此, 等时对易关系化为

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.50)$$

推广到包含若干个场 ϕ_a 的系统, 假设不同的场不会相互影响, 则有

$$[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0, \quad [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \quad (2.51)$$

这就是量子场论中的正则对易关系。此时, $\phi_a(\mathbf{x}, t)$ 和 $\pi_a(\mathbf{x}, t)$ 都是算符。

2.3 实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量, 就称它为**标量场**。在固有保时向 Lorentz 变换下, 若时空坐标的变换为 $x' = \Lambda x$, 则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \quad (2.52)$$

在本节中, 我们讨论**实标量场** $\phi(x)$, 它满足自共轭 (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \quad (2.53)$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的自由实标量场, 相应的 **Lorentz 不变拉氏量**可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m^2\phi^2. \quad (2.54)$$

注意到

$$\frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu \phi)\partial_\nu \phi = \frac{1}{2}[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2], \quad (2.55)$$

可得

$$\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi, \quad \frac{\partial \mathcal{L}}{\partial (\partial_i \phi)} = -\partial_i \phi = \partial^i \phi, \quad (2.56)$$

归纳起来, 有

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi. \quad (2.57)$$

因此, Euler-Lagrange 方程 (1.114) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi, \quad (2.58)$$

也就是说, $\phi(x)$ 满足 *Klein-Gordon* 方程

$$(\partial^2 + m^2)\phi(x) = 0. \quad (2.59)$$

2.3.1 平面波展开

设 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik \cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \quad (2.60)$$

则有

$$\partial^2 \varphi = \partial^\mu \partial_\mu \varphi = \partial^\mu (-ik_\mu \varphi) = -ik_\mu \partial^\mu \varphi = (-i)^2 k_\mu k^\mu = -k^2, \quad (2.61)$$

从而,

$$0 = (\partial^2 + m^2) \varphi = -(k^2 - m^2) \varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2] \varphi. \quad (2.62)$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$, 即 $k^0 = \pm E_{\mathbf{k}}$, 其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$. 因此, 有两种平面波解。

(1) $k^0 = E_{\mathbf{k}}$ 对应于正能解

$$\varphi_{\mathbf{k}}^{(+)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})]. \quad (2.63)$$

(2) $k^0 = -E_{\mathbf{k}}$ 对应于负能解

$$\varphi_{\mathbf{k}}^{(-)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})]. \quad (2.64)$$

从而, 场算符 $\phi(\mathbf{x}, t)$ 的通解可以写成如下形式:

$$\begin{aligned} \phi(\mathbf{x}, t) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} \varphi_{\mathbf{k}}^{(+)}(x) + \tilde{a}_{\mathbf{k}} \varphi_{\mathbf{k}}^{(-)}(x) \right] \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}} e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right], \end{aligned} \quad (2.65)$$

其中 $a_{\mathbf{k}}$ 和 $\tilde{a}_{\mathbf{k}}$ 是两个只依赖于 \mathbf{k} 的算符。这是一种 Fourier 变换, 把 $\phi(\mathbf{x}, t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭, 得

$$\begin{aligned} \phi^\dagger(\mathbf{x}, t) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right] \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} \right]. \end{aligned} \quad (2.66)$$

第二步利用了如下性质: 对整个三维动量空间进行积分时, 将积分项中的 \mathbf{k} 换成 $-\mathbf{k}$ 不会改变积分的结果。于是, 由自共轭条件 $\phi^\dagger(\mathbf{x}, t) = \phi(\mathbf{x}, t)$ 可得

$$\tilde{a}_{\mathbf{k}} = a_{-\mathbf{k}}^\dagger. \quad (2.67)$$

(注意: 由上式可以推出 $\tilde{a}_{\mathbf{k}}^\dagger = a_{-\mathbf{k}}$ 和 $\tilde{a}_{-\mathbf{k}}^\dagger = a_{\mathbf{k}}$ 。) 因而, 有

$$\phi(\mathbf{x}, t) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} \right]. \quad (2.68)$$

替换一下动量记号，可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.69)$$

其中， p^0 应该满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}, \quad (2.70)$$

而 $a_{\mathbf{p}}$ 是湮灭算符， $a_{\mathbf{p}}^\dagger$ 是产生算符。 $\phi(\mathbf{x}, t)$ 对应的共轭动量密度算符为

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (a_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.71)$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.72)$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x e^{i\mathbf{p} \cdot \mathbf{x}} = \int d^3x e^{-i\mathbf{p} \cdot \mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p}), \quad (2.73)$$

可得

$$\begin{aligned} \int d^3x e^{iq \cdot x} \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \end{aligned} \quad (2.74)$$

以及

$$\begin{aligned} \int d^3x e^{iq \cdot x} \partial_0 \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}). \end{aligned} \quad (2.75)$$

从而，有

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x e^{iq \cdot x} \partial_0 \phi - iq_0 \int d^3x e^{iq \cdot x} \phi = \int d^3x e^{iq \cdot x} (\partial_0 \phi - iq_0 \phi), \quad (2.76)$$

亦即

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\partial_0 \phi(x) - ip_0 \phi(x)]. \quad (2.77)$$

上式取厄米共轭，并使用自共轭条件 $\phi^\dagger = \phi$ ，得

$$a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} [\partial_0 \phi(x) + ip_0 \phi(x)]. \quad (2.78)$$

利用上面两个表达式和等时对易关系 (2.72)，可得

$$\begin{aligned} & [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.79)$$

最后一行中的 $\delta^{(3)}(\mathbf{p} - \mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = \mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$ ，则 $E_{\mathbf{p}} + E_{\mathbf{q}} = 2E_{\mathbf{p}} = \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}$ ，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (2.80)$$

类似地，

$$\begin{aligned} & [a_{\mathbf{p}}, a_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} (-iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} [-i(p_0 - q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{q}} - E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{q}} - E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}). \end{aligned} \quad (2.81)$$

最后一行中的 $\delta^{(3)}(\mathbf{p} + \mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = -\mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$ ，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \quad (2.82)$$

此外,

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}} a_{\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \quad (2.83)$$

综上, 产生湮灭算符满足如下对易关系:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \quad (2.84)$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.117), 实标量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = (\partial_0 \phi)^2 - \frac{1}{2} (\partial^\mu \phi) \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2]. \quad (2.85)$$

对全空间积分以得到哈密顿量:

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[(-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (-iq_0 a_{\mathbf{q}} e^{-iq \cdot x} + iq_0 a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + (i\mathbf{p} a_{\mathbf{p}} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + m^2 (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} e^{i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}}] \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} e^{i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t}] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \right. \\ &\quad \left. + (-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}) \right]. \quad (2.86) \end{aligned}$$

由 (2.70) 式可得 $-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 = 0$, 故上式最后两行方括号中第二项没有贡献。从而,

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} [2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p})] \end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2}, \quad (2.87)$$

其中第四步用到对易关系 (2.84)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。 $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 \mathbf{p} 的模式对应的粒子数密度算符 (动量空间中的密度), 相应的能量是 $E_{\mathbf{p}}$ 。在 (2.87) 式最后一行中, 第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.73) 式可得

$$(2\pi)^3 \delta^{(3)}(0) = \int d^3x = V, \quad (2.88)$$

其中 V 是进行积分的空间体积, 对于全空间而言是无穷大的。因此, (2.87) 式最后一行的第二项是一个无穷大 c 数, 是真空的零点能, 是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明, 一维简谐振子的零点能为 $E_0 = \omega/2$ 。这是自由度为 1 时的结果, 推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象, 这个零点能通常并不重要, 因为实验上只能测量两个能量之差。经过正则量子化之后, 实标量场的哈密顿量 H 是正定的, 不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \quad (2.89)$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3q E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \quad (2.90)$$

设 $|E\rangle$ 是 H 的本征态, 本征值为 E , 则

$$H |E\rangle = E |E\rangle. \quad (2.91)$$

从而, 有

$$H a_{\mathbf{p}}^\dagger |E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger |E\rangle. \quad (2.92)$$

$$H a_{\mathbf{p}} |E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p}} |E\rangle. \quad (2.93)$$

可见, 产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{k}}$, 湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{k}}$ 。

根据 (1.156) 式, 实标量场的总动量是

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi \nabla \phi = - \int d^3x (\partial_0 \phi) \nabla \phi \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[-p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} - p_0 \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right] \end{aligned}$$

$$\begin{aligned}
& + p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + p_0 \mathbf{q} a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p+q) \cdot x} \Big] \\
& = - \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - p_0 \mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} e^{-i(p_0 - q_0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} e^{i(p_0 - q_0)t} e^{i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} \right] \right. \\
& \quad \left. + p_0 \mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p_0 + q_0)t} e^{i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} \right] \right\} \\
& = - \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} e^{-i(p_0 - q_0)t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} e^{i(p_0 - q_0)t} \right] \right. \\
& \quad \left. + p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p_0 + q_0)t} \right] \right\} \\
& = - \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} (-E_{\mathbf{p}} \mathbf{p}) \left(a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) \\
& = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right). \tag{2.94}
\end{aligned}$$

先作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换, 再利用对易关系 (2.84), 可得

$$\begin{aligned}
& \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (-\mathbf{p}) \left(a_{-\mathbf{p}} a_{\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^{\dagger} a_{\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) \\
& = -\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right). \tag{2.95}
\end{aligned}$$

可见, (2.94) 式最后一行圆括号中最后两项没有贡献。从而,

$$\begin{aligned}
\mathbf{P} & = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} [2a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0)] \\
& = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{1}{2} \delta^{(3)}(0) \int d^3 p \mathbf{p}. \tag{2.96}
\end{aligned}$$

由于 $\int d^3 p \mathbf{p} = \int d^3 p (-\mathbf{p}) = -\int d^3 p \mathbf{p}$, 上式最后一行第二项没有贡献。于是,

$$\mathbf{P} = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}, \tag{2.97}$$

即总动量是所有动量模式所有粒子贡献的动量之和。

\mathbf{P} 与产生湮灭算符的对易子为

$$[\mathbf{P}, a_{\mathbf{p}}^{\dagger}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^{\dagger} [a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = \int d^3 q \mathbf{q} a_{\mathbf{q}}^{\dagger} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = \mathbf{p} a_{\mathbf{p}}^{\dagger}, \tag{2.98}$$

$$[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger}, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3 q \mathbf{q} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -\mathbf{p} a_{\mathbf{p}}. \tag{2.99}$$

2.3.4 粒子态

真空态 $|0\rangle$ 是能量最低的态, 对于任意动量 \mathbf{p} 对应的湮灭算符 $a_{\mathbf{p}}$, 满足

$$a_{\mathbf{p}} |0\rangle = 0, \tag{2.100}$$

归一化为

$$\langle 0|0\rangle = 1. \quad (2.101)$$

由哈密顿量的表达式 (2.87) 可得

$$H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = \delta^{(3)}(0) \int d^3p \frac{E_{\mathbf{p}}}{2}, \quad (2.102)$$

可见, 这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外, 由 (2.97) 式可知, $|0\rangle$ 的总动量本征值是零:

$$\mathbf{P}|0\rangle = 0|0\rangle, \quad (2.103)$$

即真空态不具有动量。

接着, 定义动量为 \mathbf{p} 的单粒子态为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger} |0\rangle. \quad (2.104)$$

从而, 利用 (2.89) 和 (2.98) 式可得

$$H|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} H + E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}\rangle, \quad (2.105)$$

$$\mathbf{P}|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{P} a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} \mathbf{P} + \mathbf{p} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{p} a_{\mathbf{p}}^{\dagger} |0\rangle = \mathbf{p} |\mathbf{p}\rangle. \quad (2.106)$$

可以看出, 相比于真空态 $|0\rangle$, 单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$, 也多了一份动量 \mathbf{p} 。因此, $|\mathbf{p}\rangle$ 描述的是一个动量为 \mathbf{p} 的粒子, 这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 满足狭义相对论中的能量-动量关系 (1.52), 而拉氏量 (2.54) 中的参数 m 就是粒子的质量。可以看出, 产生算符 $a_{\mathbf{p}}^{\dagger}$ 的作用是产生一个动量为 \mathbf{p} 的粒子。

此外, 可作如下计算:

$$a_{\mathbf{p}}|\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{q}}} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] |0\rangle = \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle. \quad (2.107)$$

如果 $\mathbf{p} \neq \mathbf{q}$, 则上式为零; 如果 $\mathbf{p} = \mathbf{q}$, 则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$ 。可见, 湮灭算符 $a_{\mathbf{p}}$ 的作用是湮灭一个动量为 \mathbf{p} 的粒子。

单粒子态的内积关系为

$$\begin{aligned} \langle \mathbf{q}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0| a_{\mathbf{q}} a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0| [a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] |0\rangle \\ &= 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.108)$$

上式是 *Lorentz* 不变的, 这是 (2.104) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明 若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i , 则如下等式成立:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \quad (2.109)$$

引入阶跃函数 (step function)

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (2.110)$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^μ 满足质壳条件 $p^2 - m^2 = 0$ 且能量为正 ($p^0 > 0$) 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned} \int d^4p \delta(p^2 - m^2) \theta(p^0) F(p) &= \int d^3p dp^0 \delta\left((p^0)^2 - |\mathbf{p}|^2 - m^2\right) \theta(p^0) F(p^0, \mathbf{p}) \\ &= \int d^3p \frac{1}{2\sqrt{|\mathbf{p}|^2 + m^2}} F\left(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}\right) = \int \frac{d^3p}{2E_{\mathbf{p}}} F\left(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}\right). \end{aligned} \quad (2.111)$$

这里第二步用到 (2.109) 式。可见,

$$\frac{d^3p}{2E_{\mathbf{p}}} \quad (2.112)$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$, 按照 δ 函数定义, 有

$$g(\mathbf{q}) = \int d^3p \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}) = \int \frac{d^3p}{2E_{\mathbf{p}}} 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}). \quad (2.113)$$

由于上式最左边和最右边都是 Lorentz 不变的,

$$2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad (2.114)$$

必定是 Lorentz 不变的。证毕。

进一步, 可以定义动量分别为 $\mathbf{p}_1, \dots, \mathbf{p}_n$ 的 n 个粒子对应的多粒子态为

$$|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \equiv \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle. \quad (2.115)$$

H 对它的作用给出

$$\begin{aligned} H |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} H a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} (a_{\mathbf{p}_1}^\dagger H + E_{\mathbf{p}_1} a_{\mathbf{p}_1}^\dagger) \cdots a_{\mathbf{p}_n}^\dagger |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger H a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + E_{\mathbf{p}_1} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger H \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \cdots = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger H |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= (E_{\text{vac}} + E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle, \end{aligned} \quad (2.116)$$

同理, \mathbf{P} 对它的作用给出

$$\mathbf{P} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle = (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle. \quad (2.117)$$

也就是说, 多粒子态 $|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.84) 可得

$$\begin{aligned}
 |\mathbf{p}_1, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\
 &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\
 &= |\mathbf{p}_1, \dots, \mathbf{p}_j, \dots, \mathbf{p}_i, \dots, \mathbf{p}_n\rangle.
 \end{aligned} \tag{2.118}$$

可以看出, 对调多粒子态中的任意两个粒子, 得到的态相同, 即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是**玻色子** (boson), 服从 Bose-Einstein 统计。得到这个结论的关键是两个产生算符相互对易。

双粒子态的内积关系为

$$\begin{aligned}
 \langle \mathbf{q}_1, \mathbf{q}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{q}_1} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger | 0 \rangle \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [\langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger a_{\mathbf{q}_1} a_{\mathbf{p}_2}^\dagger | 0 \rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [(2\pi)^3 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger | 0 \rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [(2\pi)^6 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) + (2\pi)^6 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2)] \\
 &= 4E_{\mathbf{p}_1} E_{\mathbf{p}_2} (2\pi)^6 [\delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) + \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2)].
 \end{aligned} \tag{2.119}$$

此外, 还可以定义动量均为 \mathbf{p} 的 n 个粒子对应的多粒子态为

$$|n_{\mathbf{p}}\rangle \equiv (2E_{\mathbf{p}})^{n_{\mathbf{p}}/2} (a_{\mathbf{p}}^\dagger)^{n_{\mathbf{p}}} |0\rangle, \tag{2.120}$$

则粒子数密度算符

$$N_{\mathbf{p}} \equiv a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \tag{2.121}$$

对它的作用为

$$\begin{aligned}
 N_{\mathbf{p}} |n_{\mathbf{q}}\rangle &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger [a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^2 a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-2} |0\rangle + 2(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= \cdots = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} a_{\mathbf{p}} |0\rangle + n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle.
 \end{aligned} \tag{2.122}$$

在动量空间对粒子数密度算符进行积分, 得到的是**粒子数算符**

$$N \equiv \int \frac{d^3 p}{(2\pi)^3} N_{\mathbf{p}} = \int \frac{d^3 p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{2.123}$$

由 (2.122) 式, 可得

$$\begin{aligned}
 N |n_{\mathbf{q}}\rangle &= \int \frac{d^3 p}{(2\pi)^3} N_{\mathbf{p}} |n_{\mathbf{q}}\rangle = \int \frac{d^3 p}{(2\pi)^3} n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = n_{\mathbf{q}} |n_{\mathbf{q}}\rangle.
 \end{aligned} \tag{2.124}$$

因此, $|n_{\mathbf{q}}\rangle$ 是 N 的本征态, 本征值为粒子数 $n_{\mathbf{q}}$ 。

更一般地, 可以定义多粒子态

$$|n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \quad (2.125)$$

来描述动量为 $\mathbf{p}_1, \dots, \mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}$ 个的情况。此时, 有

$$\begin{aligned} & N |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right. \\ &\quad \left. + n_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}-1} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] + n_{\mathbf{p}_1} |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= \dots = \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} a_{\mathbf{p}} |0\rangle \right] \\ &\quad + (n_{\mathbf{p}_1} + \dots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= (n_{\mathbf{p}_1} + \dots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle. \end{aligned} \quad (2.126)$$

可见, N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中, 我们讨论复标量场 $\phi(x)$, 它不满足自共轭条件 (2.53), 即

$$\phi^\dagger(x) \neq \phi(x). \quad (2.127)$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.186) 式的形式。不过, 由于 $\phi(x)$ 是量子场算符, 需要把那里的复共轭记号 $*$ 改成厄米共轭记号 \dagger , 故 Lorentz 不变拉氏量为

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m^2 \phi^\dagger \phi. \quad (2.128)$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量, 注意到

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^\dagger, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^\dagger, \quad (2.129)$$

则 Euler-Lagrange 方程 (1.114) 给出

$$(\partial^2 + m^2)\phi(x) = 0, \quad (\partial^2 + m^2)\phi^\dagger(x) = 0. \quad (2.130)$$

也就是说, $\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程.

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \quad (2.131)$$

从而, 拉氏量 (2.128) 化为

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}[\partial^\mu(\phi_1 - i\phi_2)]\partial_\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2 - \frac{1}{2}m^2\phi_2^2. \end{aligned} \quad (2.132)$$

与 (2.54) 式比较可知, 复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量.

2.4.1 平面波展开

对于复标量场, 我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开, 但不能够应用自共轭条件. 因此, 场算符 $\phi(\mathbf{x}, t)$ 的通解也具有 (2.65) 式的形式:

$$\begin{aligned} \phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}}e^{i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})}] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{-\mathbf{k}}e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}]. \end{aligned} \quad (2.133)$$

由于不满足自共轭条件 (2.53), 算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_{\mathbf{k}}$ 没有什么关系, 改记为

$$b_{\mathbf{k}}^\dagger = \tilde{a}_{-\mathbf{k}}, \quad (2.134)$$

则展开式变成

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + b_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}]. \quad (2.135)$$

替换一下动量记号, 可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.136)$$

其中, p^0 应该满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \quad (2.137)$$

取厄米共轭, 就得到 $\phi^\dagger(\mathbf{x}, t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b_{\mathbf{p}}e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.138)$$

现在, $a_{\mathbf{p}}$ 和 $b_{\mathbf{p}}$ 是两个相互独立的湮灭算符, 而 $a_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^\dagger = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (b_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.139)$$

$\phi^\dagger(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\dagger)} = \partial_0 \phi = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (a_{\mathbf{p}} e^{-ip \cdot x} - b_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.140)$$

根据 (2.51) 式, 等时对易关系为

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0, \\ [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0, \\ [\phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= [\phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t)] = [\phi(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.141)$$

2.4.2 产生湮灭算符的对易关系

由

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \phi &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} + b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + b_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger e^{2iq^0 t}) \end{aligned} \quad (2.142)$$

和

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \partial_0 \phi &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} - b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3 p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - b_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \end{aligned} \quad (2.143)$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3 x e^{iq \cdot x} \partial_0 \phi - iq_0 \int d^3 x e^{iq \cdot x} \phi = \int d^3 x e^{iq \cdot x} (\partial_0 \phi - iq_0 \phi). \quad (2.144)$$

于是,

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x e^{ip \cdot x} (\partial_0 \phi - ip_0 \phi), \quad a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x e^{-ip \cdot x} (\partial_0 \phi^\dagger + ip_0 \phi^\dagger). \quad (2.145)$$

从而, 有

$$\begin{aligned}
& [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{2.146}
\end{aligned}$$

以及

$$\begin{aligned}
& [a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] = 0. \tag{2.147}
\end{aligned}$$

另一方面, 由

$$\begin{aligned}
\int d^3x e^{iq \cdot x} \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q) \cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\
&= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}) \tag{2.148}
\end{aligned}$$

和

$$\begin{aligned}
\int d^3x e^{iq \cdot x} \partial_0 \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q) \cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\
&= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \tag{2.149}
\end{aligned}$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} b_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} b_{\mathbf{q}} = \int d^3x e^{iq \cdot x} \partial_0 \phi^\dagger - iq_0 \int d^3x e^{iq \cdot x} \phi^\dagger = \int d^3x e^{iq \cdot x} (\partial_0 \phi^\dagger - iq_0 \phi^\dagger). \tag{2.150}$$

于是,

$$b_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} (\partial_0 \phi^\dagger - ip_0 \phi^\dagger), \quad b_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} (\partial_0 \phi + ip_0 \phi). \quad (2.151)$$

从而, 有

$$\begin{aligned} & [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)]) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \end{aligned} \quad (2.152)$$

以及

$$\begin{aligned} & [b_{\mathbf{p}}, b_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.153)$$

此外, 还有

$$\begin{aligned} & [a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] = 0, \end{aligned} \quad (2.154)$$

以及

$$\begin{aligned} & [a_{\mathbf{p}}, b_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} (-iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [-i(p_0 - q_0)i\delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \quad (2.155)
\end{aligned}$$

归纳起来，产生湮灭算符的对易关系如下：

$$\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}] = 0, \\
[b_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p}}, b_{\mathbf{q}}] = [b_{\mathbf{p}}^{\dagger}, b_{\mathbf{q}}^{\dagger}] = 0, \\
[a_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] &= [b_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] = [a_{\mathbf{p}}, b_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, b_{\mathbf{q}}^{\dagger}] = 0.
\end{aligned} \quad (2.156)$$

这说明 $a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}}$ 与 $b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$ 是两套不同的产生湮灭算符，描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \quad [\phi^{\dagger}(x)]' = e^{-iq\theta} \phi^{\dagger}(x), \quad (2.157)$$

则拉氏量 (2.128) 不变。依照 1.7.4 小节的讨论，相应的守恒流为

$$J^{\mu} = q\phi^{\dagger} i \overleftrightarrow{\partial}^{\mu} \phi, \quad (2.158)$$

相应的守恒荷为

$$\begin{aligned}
Q &= q \int d^3x \phi^{\dagger} i \overleftrightarrow{\partial}^0 \phi = iq \int d^3x [\phi^{\dagger} \partial^0 \phi - (\partial^0 \phi^{\dagger}) \phi] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) \partial^0 (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right. \\
&\quad \left. - \partial^0 (b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (-ik^0) (a_{\mathbf{k}} e^{-ik\cdot x} - b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right. \\
&\quad \left. - (-ip^0) (b_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(ik^0 + ip^0) b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(p-k)\cdot x} + (-ik^0 - ip^0) a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(p-k)\cdot x} \right. \\
&\quad \left. + (-ik^0 + ip^0) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + (ik^0 - ip^0) a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(p+k)\cdot x} \right] \\
&= q \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[-(E_{\mathbf{k}} + E_{\mathbf{p}}) b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(p-k)\cdot x} + (E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(p-k)\cdot x} \right. \\
&\quad \left. + (E_{\mathbf{k}} - E_{\mathbf{p}}) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + (-E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(p+k)\cdot x} \right] \\
&= q \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left\{ (E_{\mathbf{k}} + E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} - \mathbf{k}) \left[-b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + (E_{\mathbf{k}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p} + \mathbf{k}) \left[b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} - a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right] \Big\} \\
& = q \int \frac{d^3 p}{(2\pi)^3} 2E_{\mathbf{p}} (-b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}) = q \int \frac{d^3 p}{(2\pi)^3} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger}). \quad (2.159)
\end{aligned}$$

利用对易关系 (2.156), 可得

$$Q = \int \frac{d^3 p}{(2\pi)^3} (q a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - q b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}) - (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3 p}{(2\pi)^3} q. \quad (2.160)$$

上式第二项是零点荷。在第一项的圆括号中, 粒子数密度算符 $a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$ 的系数是 q , 而粒子数密度算符 $b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}$ 的系数是 $-q$ 。可见, $a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q , 习惯上称为**正粒子**; 另一方面, $b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$, 习惯上称为**反粒子**。除去零点荷, 总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式类似, 但 Q/q 被解释为正粒子数与反粒子数之差, 因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的, 并且来自于一种内部对称性, 因而是一种**内部相加性量子数**。实际上, 反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 $U(1)$ 整体变换, 则自共轭条件 (2.53) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^{\dagger} = [e^{iq\theta}\phi(x)]^{\dagger} = e^{-iq\theta}\phi^{\dagger}(x) = e^{-iq\theta}\phi(x). \quad (2.161)$$

上式要求 $q = 0$ 。因此, 对实标量场不能进行非平庸的 $U(1)$ 整体变换。实际上, 自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数, 也就是说, 正粒子与反粒子是相同的, 实标量场描述的是一种纯中性粒子。

2.4.4 哈密顿量和总动量

根据 (1.117) 式, 复标量场的哈密顿量密度为

$$\begin{aligned}
\mathcal{H} &= \pi \partial_0 \phi + \pi^{\dagger} \partial_0 \phi^{\dagger} - \mathcal{L} = (\partial^0 \phi^{\dagger}) \partial_0 \phi + (\partial^0 \phi) \partial_0 \phi^{\dagger} - (\partial^{\mu} \phi^{\dagger}) \partial_{\mu} \phi + m^2 \phi^{\dagger} \phi \\
&= (\partial^0 \phi^{\dagger}) \partial_0 \phi + (\nabla \phi^{\dagger}) \cdot \nabla \phi + m^2 \phi^{\dagger} \phi. \quad (2.162)
\end{aligned}$$

于是, 哈密顿量可以写成

$$\begin{aligned}
H &= \int d^3 x \mathcal{H} = \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi + (\nabla \phi^{\dagger}) \cdot \nabla \phi + m^2 \phi^{\dagger} \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi + \nabla \cdot (\phi^{\dagger} \nabla \phi) - \phi^{\dagger} \nabla^2 \phi + m^2 \phi^{\dagger} \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi + \phi^{\dagger} (\partial^0 \partial_0 - \nabla^2 + m^2) \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi + \phi^{\dagger} (\partial^2 + m^2) \phi]. \quad (2.163)
\end{aligned}$$

上式第三步用了分部积分, 第四步扔掉了一个全散度, 最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.130) 消去。从而, 得到

$$H = \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi]$$

$$\begin{aligned}
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[\partial^0 (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \partial^0 \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right] \\
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left\{ (-ip^0) (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) (-iq_0) (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) [(-iq^0)(-iq_0) a_q e^{-iq \cdot x} + iq^0 iq_0 b_q^\dagger e^{iq \cdot x}] \right\} \\
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(p^0 q_0 + q^0 q_0) b_p b_q^\dagger e^{-i(p-q) \cdot x} + (p^0 q_0 + q^0 q_0) a_p^\dagger a_q e^{i(p-q) \cdot x} \right. \\
&\quad \left. + (-p^0 q_0 + q^0 q_0) b_p a_q e^{-i(p+q) \cdot x} + (-p^0 q_0 + q^0 q_0) a_p^\dagger b_q^\dagger e^{i(p+q) \cdot x} \right] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} E_q \left\{ (E_p + E_q) \delta^{(3)}(\mathbf{p} - \mathbf{q}) [b_p b_q^\dagger e^{-i(E_p - E_q)t} + a_p^\dagger a_q e^{i(E_p - E_q)t}] \right. \\
&\quad \left. + (E_q - E_p) \delta^{(3)}(\mathbf{p} + \mathbf{q}) [b_p a_q e^{-i(E_p + E_q)t} + a_p^\dagger b_q^\dagger e^{i(E_p + E_q)t}] \right\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} 2E_p^2 (b_p b_p^\dagger + a_p^\dagger a_p) = \int \frac{d^3p}{(2\pi)^3} E_p (b_p b_p^\dagger + a_p^\dagger a_p) \\
&= \int \frac{d^3p}{(2\pi)^3} E_p (a_p^\dagger a_p + b_p^\dagger b_p) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} E_p. \tag{2.164}
\end{aligned}$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 \mathbf{p} ，正粒子与反粒子具有相同的能量 E_p ，因而它们具有相同的质量 m 。

根据 (1.156) 式，复标量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x (\pi \nabla \phi + \pi^\dagger \nabla \phi^\dagger) = - \int d^3x [(\partial_0 \phi^\dagger) \nabla \phi + (\partial_0 \phi) \nabla \phi^\dagger] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[\partial_0 (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \nabla (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. + \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \nabla (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[-ip_0 (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) i\mathbf{q} (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - iq_0 (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) i\mathbf{p} (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(-E_p \mathbf{q} b_p b_q^\dagger - E_q \mathbf{p} b_q^\dagger b_p) e^{-i(p-q) \cdot x} \right. \\
&\quad + (-E_p \mathbf{q} a_p^\dagger a_q - E_q \mathbf{p} a_q a_p^\dagger) e^{i(p-q) \cdot x} \\
&\quad + (E_p \mathbf{q} b_p a_q + E_q \mathbf{p} a_q b_p) e^{-i(p+q) \cdot x} \\
&\quad \left. + (E_p \mathbf{q} a_p^\dagger b_q^\dagger + E_q \mathbf{p} b_q^\dagger a_p^\dagger) e^{i(p+q) \cdot x} \right] \\
&= - \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) [(-E_p \mathbf{q} b_p b_q^\dagger - E_q \mathbf{p} b_q^\dagger b_p) e^{-i(E_p - E_q)t} \right. \\
&\quad \left. + (-E_p \mathbf{q} a_p^\dagger a_q - E_q \mathbf{p} a_q a_p^\dagger) e^{i(E_p - E_q)t}] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[(E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \\
& \quad \left. + (E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^{\dagger} b_{\mathbf{q}}^{\dagger} + E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^{\dagger} a_{\mathbf{p}}^{\dagger}) e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \Big\} \\
& = - \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} \left[- E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) \right. \\
& \quad \left. - E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} a_{-\mathbf{p}} - a_{-\mathbf{p}} b_{\mathbf{p}}) e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}} \mathbf{p} (a_{\mathbf{p}}^{\dagger} b_{-\mathbf{p}}^{\dagger} - b_{-\mathbf{p}}^{\dagger} a_{\mathbf{p}}^{\dagger}) e^{2iE_{\mathbf{p}}t} \right] \\
& = \int \frac{d^3 p}{(2\pi)^3} \frac{\mathbf{p}}{2} (b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}) + \delta^{(3)}(0) \int d^3 p \mathbf{p} \\
& = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}). \tag{2.165}
\end{aligned}$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章 矢量场

3.1 量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换

$$|\Psi'\rangle = U(\Lambda) |\Psi\rangle, \quad (3.1)$$

其中 $U(\Lambda)$ 是一个线性幺正算符，描述量子 Lorentz 变换，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \quad (3.2)$$

先作 Lorentz 变换 Λ_1 ，再作 Lorentz 变换 Λ_2 ，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$ ，故以下同态 (homomorphic) 关系成立：

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \quad (3.3)$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(1) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \quad (3.4)$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \quad (3.5)$$

将无穷小 Lorentz 变换 (1.157) 记为 $\Lambda_\omega = 1 + \omega$ ，它诱导的无穷小幺正算符可表达为

$$U(1 + \omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \quad (3.6)$$

这里只展开到 ω 的一阶项。 $J^{\mu\nu}$ 是量子 Lorentz 变换的生成元算符¹。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的：

$$J^{\mu\nu} = -J^{\nu\mu}. \quad (3.7)$$

由 $U(1 + \omega)$ 的幺正性可得

$$1 = U^\dagger(1 + \omega)U(1 + \omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right] \left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \quad (3.8)$$

¹虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.177)。

最后一步忽略了 ω 的二阶项。可见, $J^{\mu\nu}$ 是厄米算符:

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \quad (3.9)$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda). \quad (3.10)$$

的左边和右边分别展开, 得

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U^{-1}(\Lambda) \left(1 - \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} \right) U(\Lambda) = 1 - \frac{i}{2} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda), \quad (3.11)$$

$$U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda) = U(\mathbf{1} + \Lambda^{-1}\omega\Lambda) = 1 - \frac{i}{2} (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu}. \quad (3.12)$$

因此, 有

$$\begin{aligned} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1}\omega\Lambda)^\alpha{}_\nu J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1})^\alpha{}_\beta \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} \\ &= g_{\mu\alpha} \Lambda_\beta{}^\alpha \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} = \Lambda^\beta{}_\mu \omega_{\beta\gamma} \Lambda^\gamma{}_\nu J^{\mu\nu} = \omega_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}, \end{aligned} \quad (3.13)$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立, 于是,

$$U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.14)$$

因此, $J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值 (expectation value) 与它在 $|\Psi\rangle$ 中的期待值有如下关系:

$$\langle \Psi' | J^{\mu\nu} | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \langle \Psi | J^{\rho\sigma} | \Psi \rangle. \quad (3.15)$$

也就是说, $U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换:

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.16)$$

可见, $J^{\mu\nu}$ 是一个 2 阶 Lorentz 张量。

接着, 考虑 Λ 的无穷小形式 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu$, 则

$$U(\Lambda) = 1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta}. \quad (3.17)$$

忽略二阶小量, (3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) &= \left(1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} \right) J^{\mu\nu} \left(1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta} \right) \\ &= J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\mu\nu} J^{\alpha\beta} + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\rho\sigma} [J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \quad (3.18)$$

右边为

$$\begin{aligned} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma} &= (\delta^\mu{}_\rho + \tilde{\omega}^\mu{}_\rho) (\delta^\nu{}_\sigma + \tilde{\omega}^\nu{}_\sigma) J^{\rho\sigma} = \delta^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} + \delta^\mu{}_\rho \tilde{\omega}^\nu{}_\sigma J^{\rho\sigma} + \tilde{\omega}^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^\nu{}_\sigma J^{\mu\sigma} + \tilde{\omega}^\mu{}_\rho J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma} g^{\nu\rho} J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho} g^{\mu\sigma} J^{\rho\nu} \end{aligned}$$

$$\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma} J^{\mu\rho} + g^{\mu\rho} J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma}), \tag{3.19}
\end{aligned}$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \\
&= i[g^{\nu\rho} J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma). \tag{3.20}
\end{aligned}$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群是一类特殊的连续群， n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie* 乘积，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$J^i \equiv \frac{1}{2}\varepsilon^{ijk} J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}$$

即

$$\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}$$

J^i 与 J^j 的对易关系为

$$\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij}, \tag{3.23}
\end{aligned}$$

第二、三步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.82) 式。由 (1.96) 式，有

$$J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}$$

从而推出

$$[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ ，写成分量的形式是 $L^i = \varepsilon^{ijk}x^jp^k$ ，从而，

$$\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^lp^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^lp^m = x^ip^j - x^jp^i. \tag{3.26}$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i, p^j] = i\delta^{ij}$ 可得

$$\begin{aligned}
 [L^i, L^j] &= \varepsilon^{ikl} \varepsilon^{jmn} [x^k p^l, x^m p^n] = \varepsilon^{ikl} \varepsilon^{jmn} \{x^k [p^l, x^m] p^n + x^m [x^k, p^n] p^l\} \\
 &= \varepsilon^{ikl} \varepsilon^{jmn} (-i\delta^{lm} x^k p^n + i\delta^{kn} x^m p^l) = i(-\varepsilon^{ikl} \varepsilon^{jln} x^k p^n + \varepsilon^{ikl} \varepsilon^{jmk} x^m p^l) \\
 &= i(\varepsilon^{ikl} \varepsilon^{jnl} x^k p^n - \varepsilon^{ilk} \varepsilon^{jmk} x^m p^l) = i[(\delta^{ij} \delta^{kn} - \delta^{in} \delta^{kj}) x^k p^n - (\delta^{ij} \delta^{lm} - \delta^{im} \delta^{lj}) x^m p^l] \\
 &= i[\delta^{ij} x^k p^k - x^j p^i - \delta^{ij} x^l p^l + x^i p^j] = i(x^i p^j - x^j p^i) = i\varepsilon^{ijk} L^k.
 \end{aligned} \tag{3.27}$$

可见, \mathbf{J} 与 \mathbf{L} 具有相同的对易关系, \mathbf{J} 也是一个角动量算符。实际上, \mathbf{J} 描述总角动量, 不止可以包含轨道角动量 \mathbf{L} , 也可以包含自旋角动量。

满足

$$O^T O = \mathbf{1} \tag{3.28}$$

的实方阵 O 称为实正交矩阵 (real orthogonal matrix)。对上式取行列式, 得

$$1 = \det O^T \cdot \det O = (\det O)^2. \tag{3.29}$$

可见, 实正交矩阵 O 的行列式为 $\det O = \pm 1$ 。由行列式为 +1 的 3 维实正交矩阵按照矩阵乘法构成的群, 称为空间旋转群 $\mathbf{SO}(3)$, 描述三维空间中的旋转变换。1.7.3 小节提到, $\mathbf{SO}(3)$ 群是 Lorentz 群的子群, J^i 可以看作 $\mathbf{SO}(3)$ 群的生成元算符, 而 (3.25) 式是 $\mathbf{SO}(3)$ 群的 Lie 代数关系。

另一方面, \mathbf{K} 是增速算符。 \mathbf{J} 与 \mathbf{K} 的对易关系为

$$\begin{aligned}
 [J^i, K^j] &= \frac{1}{2} \varepsilon^{ikl} [J^{kl}, J^{0j}] = \frac{i}{2} \varepsilon^{ikl} \{[g^{l0} J^{kj} - (k \leftrightarrow l)] - (0 \leftrightarrow j)\} \\
 &= i\varepsilon^{ikl} [g^{l0} J^{kj} - (0 \leftrightarrow j)] = i\varepsilon^{ikl} (g^{l0} J^{kj} - g^{lj} J^{k0}) = -i\varepsilon^{ikl} g^{lj} J^{k0} = i\varepsilon^{ikl} \delta^{lj} J^{k0} \\
 &= i\varepsilon^{ikj} J^{k0} = i\varepsilon^{ijk} J^{0k} = i\varepsilon^{ijk} K^k,
 \end{aligned} \tag{3.30}$$

而 \mathbf{K} 自身的对易关系为

$$\begin{aligned}
 [K^i, K^j] &= [J^{0i}, J^{0j}] = i(g^{i0} J^{0j} - g^{00} J^{ij} - g^{ij} J^{00} + g^{0j} J^{i0}) \\
 &= -i(g^{00} J^{ij} + g^{ij} J^{00}) = -iJ^{ij} = -i\varepsilon^{ijk} J^k.
 \end{aligned} \tag{3.31}$$

归纳起来, 有

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, K^j] = i\varepsilon^{ijk} K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k. \tag{3.32}$$

3.2 量子矢量场的 Lorentz 变换

3.2.1 Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 ω^α_β 可以转化为

$$\omega^\alpha_\beta = g^{\alpha\mu} \omega_{\mu\beta} = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\beta} - g^{\alpha\mu} \omega_{\beta\mu}) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\mu} \omega_{\nu\mu} \delta^\nu_\beta) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\nu} \omega_{\mu\nu} \delta^\mu_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta, \quad (3.33)$$

其中 $(\mathcal{J}^{\mu\nu})^\alpha_\beta$ 定义为

$$(\mathcal{J}^{\mu\nu})^\alpha_\beta \equiv i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta). \quad (3.34)$$

容易看出, $\mathcal{J}^{\mu\nu}$ 是反对称的:

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \quad (3.35)$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^\gamma_\beta = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\gamma}) = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha). \quad (3.36)$$

这样的话, 可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta = \delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta. \quad (3.37)$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha_\beta &= (\mathcal{J}^{\mu\nu})^\alpha_\gamma (\mathcal{J}^{\rho\sigma})^\gamma_\beta - (\mathcal{J}^{\rho\sigma})^\alpha_\gamma (\mathcal{J}^{\mu\nu})^\gamma_\beta \\ &= i^2(g^{\mu\alpha}\delta^\nu_\gamma - \delta^\mu_\gamma g^{\nu\alpha})(g^{\rho\gamma}\delta^\sigma_\beta - \delta^\rho_\beta g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}\delta^\nu_\gamma g^{\rho\gamma}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\nu_\gamma \delta^\rho_\beta g^{\sigma\gamma} + \delta^\mu_\gamma g^{\nu\alpha} g^{\rho\gamma}\delta^\sigma_\beta - \delta^\mu_\gamma g^{\nu\alpha} \delta^\rho_\beta g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}g^{\rho\nu}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\rho_\beta g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^\sigma_\beta - g^{\nu\alpha}\delta^\rho_\beta g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\nu\rho}g^{\mu\alpha}\delta^\sigma_\beta + g^{\mu\rho}g^{\nu\alpha}\delta^\sigma_\beta + g^{\nu\sigma}g^{\mu\alpha}\delta^\rho_\beta - g^{\mu\sigma}g^{\nu\alpha}\delta^\rho_\beta \\ &\quad - [-g^{\sigma\mu}g^{\rho\alpha}\delta^\nu_\beta + g^{\rho\mu}g^{\sigma\alpha}\delta^\nu_\beta + g^{\sigma\nu}g^{\rho\alpha}\delta^\mu_\beta - g^{\rho\nu}g^{\sigma\alpha}\delta^\mu_\beta] \\ &= g^{\nu\rho}(g^{\sigma\alpha}\delta^\mu_\beta - g^{\mu\alpha}\delta^\sigma_\beta) + g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma_\beta - g^{\sigma\alpha}\delta^\nu_\beta) + g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho_\beta - g^{\rho\alpha}\delta^\mu_\beta) + g^{\mu\sigma}(g^{\rho\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\rho_\beta) \\ &= -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^\alpha_\beta - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha_\beta \\ &= i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^\alpha_\beta - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta + g^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha_\beta], \end{aligned} \quad (3.38)$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}). \quad (3.39)$$

可见, $\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。 Λ^α_β 属于 Lorentz 群的矢量表示, 因而 $\mathcal{J}^{\mu\nu}$ 就是矢量表示的生成元。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \quad (3.40)$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^\omega = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \quad (3.41)$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 \mathbf{g} 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}^{\delta} g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}_{\gamma} g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}_{\beta}, \quad (3.42)$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \quad (3.43)$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1} \left[\sum_{n=0}^{\infty} \frac{(\omega^{\mathrm{T}})^n}{n!} \right] \mathbf{g} = \sum_{n=0}^{\infty} \frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \quad (3.44)$$

若两个同阶方阵 A 和 B 相互对易，即 $[A, B] = 0$ ，则二项式定理成立：

$$(A + B)^n = \sum_{j=0}^n \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.45)$$

阶乘的定义可以推广到负整数：对于整数 $m < 0$ ，定义

$$m! \rightarrow \infty, \quad \frac{1}{m!} \rightarrow 0. \quad (3.46)$$

从而，对于 $j > n$ ，有 $[(n-j)!]^{-1} \rightarrow 0$ 。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A + B)^n = \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.47)$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{n=0}^{\infty} \frac{B^{n-j}}{(n-j)!} = e^A e^B. \quad (3.48)$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^0 = \mathbf{1}. \quad (3.49)$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = \mathbf{g}, \quad (3.50)$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp \left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right) \quad (3.51)$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的有限变换。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu} = 0$ ，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

在正则量子化程序中, 标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符, 类似于 (3.16) 式, $\phi(x)$ 的固有保时向 Lorentz 变换关系 (2.52) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \quad (3.52)$$

上式表明, 变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$, (3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \quad (3.53)$$

相应地, $\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\phi(x)|\Psi'\rangle = \langle\Psi|U^{-1}(\Lambda)\phi(x)U(\Lambda)|\Psi\rangle = \langle\Psi|\phi(\Lambda^{-1}x)|\Psi\rangle. \quad (3.54)$$

另一方面, 由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的相应 Lorentz 变换形式为

$$\partial^\mu\phi'(x') = U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda) = \partial'^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial'^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \quad (3.55)$$

于是, 在固有保时向 Lorentz 变换下, 自由实标量场的拉氏量 (2.54) 的变换形式为

$$\begin{aligned} \mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial'^\mu\phi(x')\partial'_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda) \\ &= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial'^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\} \\ &= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)] \\ &= \mathcal{L}(x), \end{aligned} \quad (3.56)$$

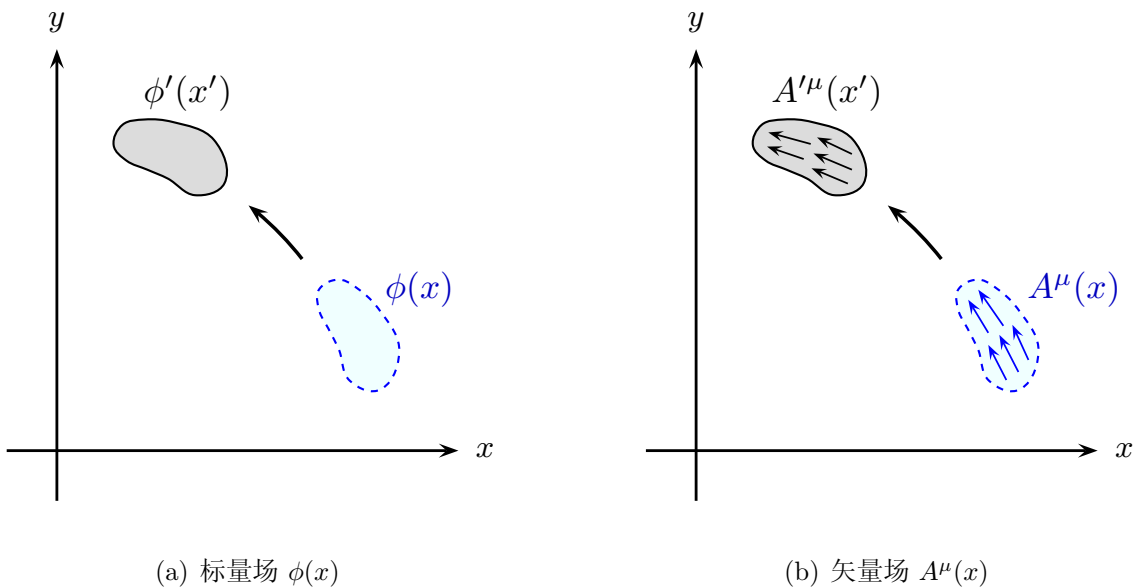


图 3.1: 在绕 z 轴空间旋转变换下, 标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而,

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(\Lambda^{-1}x). \quad (3.57)$$

可见, 拉氏量 (2.54) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, 可得

$$\begin{aligned} (\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu = g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta = g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) = g_{\nu\beta}g^{\mu\beta} + g^{\mu\beta}\omega_{\nu\beta} = \delta^\mu{}_\nu - g^{\mu\beta}\omega_{\beta\nu} \\ &= \delta^\mu{}_\nu - \omega^\mu{}_\nu, \end{aligned} \quad (3.58)$$

从而, 有

$$(\Lambda^{-1}x)^\mu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu. \quad (3.59)$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项, 得

$$\begin{aligned} \phi(\Lambda^{-1}x) &= \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \phi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \phi(x) = \phi(x) - \frac{1}{2}(\omega_{\mu\nu} x^\nu \partial^\mu + \omega_{\nu\mu} x^\mu \partial^\nu) \phi(x) \\ &= \phi(x) - \frac{1}{2}\omega_{\mu\nu}(x^\nu \partial^\mu - x^\mu \partial^\nu) \phi(x) = \phi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x) \\ &= \phi(x) - \frac{i}{2}\omega_{\mu\nu} i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x). \end{aligned} \quad (3.60)$$

根据 (3.6) 式, 将 (3.53) 式左边展开到 ω 的一阶项, 得

$$\begin{aligned} U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right) \\ &= \phi(x) - \frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}[\phi(x), J^{\mu\nu}]. \end{aligned} \quad (3.61)$$

两相比较, 给出

$$[\phi(x), J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi(x) = L^{\mu\nu}\phi(x), \quad (3.62)$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu} \equiv i(x^\mu \partial^\nu - x^\nu \partial^\mu). \quad (3.63)$$

对于空间分量 L^{ij} , 可以等价地定义

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{i}{2}\varepsilon^{ijk}(x^j \partial^k - x^k \partial^j) = \frac{i}{2}(\varepsilon^{ijk}x^j \partial^k - \varepsilon^{ikj}x^j \partial^k) = i\varepsilon^{ijk}x^j \partial^k, \quad (3.64)$$

写成空间矢量的形式是

$$\mathbf{L} = -i\mathbf{x} \times \nabla. \quad (3.65)$$

可见, \mathbf{L} 就是微分算符形式的轨道角动量算符。根据 (3.21) 式, (3.62) 式的空间分量部分可以改写为

$$[\phi(x), \mathbf{J}] = \mathbf{L}\phi(x). \quad (3.66)$$

上式表明, 总角动量算符 \mathbf{J} 生成了轨道角动量, 但没有生成自旋角动量。这说明标量场没有自旋, 对应于零自旋粒子。

3.2.3 量子矢量场的 Lorentz 变换形式

$\partial^\mu \phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有像 (3.55) 式那样的 Lorentz 变换形式, 即

$$A'^\mu(x') = U^{-1}(\Lambda) A^\mu(x') U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \quad (3.67)$$

或者写成

$$U^{-1}(\Lambda) A^\mu(x) U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \quad (3.68)$$

这就是量子矢量场的 Lorentz 变换形式。相应地, $A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle \Psi' | A^\mu(x) | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) A^\mu(x) U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\nu \langle \Psi | A^\nu(\Lambda^{-1}x) | \Psi \rangle. \quad (3.69)$$

对于固有保时向 Lorentz 变换, 根据矢量表示中的无穷小形式 (3.40), (3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] A^\nu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.70)$$

将上式与 (1.166) 式比较, 可以发现, 1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$ 。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出, 在 Lorentz 变换下, 除了矢量场的分布区域发生变化之外, 矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式, 在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项, 得

$$\begin{aligned} A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta \partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta} x^\beta \partial^\alpha A^\nu(x) \\ &= A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x). \end{aligned} \quad (3.71)$$

从而, (3.68) 式右边可展开为

$$\begin{aligned} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] \left[A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x) \right] \\ &= A^\mu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)]. \end{aligned} \quad (3.72)$$

另一方面, (3.68) 式左边的无穷小展开式为

$$\begin{aligned} U^{-1}(\Lambda) A^\mu(x) U(\Lambda) &= \left(1 + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} \right) A^\mu(x) \left(1 - \frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta} \right) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\alpha\beta} A^\mu(x) J^{\alpha\beta} + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} A^\mu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [A^\mu(x), J^{\rho\sigma}]. \end{aligned} \quad (3.73)$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.74)$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathcal{J}^{jk}, \quad \mathcal{J} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \quad (3.75)$$

再根据 (3.21) 和 (3.64) 式, (3.74) 式的空间分量部分可以改写为

$$[A^\mu(x), \mathbf{J}] = \mathbf{L} A^\mu(x) + (\mathcal{J})^\mu{}_\nu A^\nu(x). \quad (3.76)$$

上式表明, 总角动量算符 \mathbf{J} 不仅生成了轨道角动量, 还生成了由 \mathcal{J} 描述的自旋角动量。 \mathcal{J}^i 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \quad (3.77)$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \quad (3.78)$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \quad (3.79)$$

只关注空间分量, 可得

$$(\mathcal{J}^1 \mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2 \mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \quad (3.80)$$

因此, 有

$$(\mathcal{J}^2)^i{}_j = (\mathcal{J}^1 \mathcal{J}^1 + \mathcal{J}^2 \mathcal{J}^2 + \mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \quad (3.81)$$

根据量子力学的角动量理论, \mathcal{J}^2 的本征值为 $s(s+1)$, 即 $(\mathcal{J}^2)^i{}_j = s(s+1)\delta^i{}_j$, 其中 s 为自旋量子数。可见, 矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \quad (3.82)$$

经过量子化程序之后, 矢量场 $A^\mu(x)$ 应当描述自旋为 1 的粒子。

3.3 有质量矢量场的正则量子化

类似于电磁场, 对任意的矢量场 A^μ 可以定义反对称的场强张量

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (3.83)$$

对于一个自由的有质量的实矢量场 A^μ ，用场强张量可以将它的 **Lorentz** 不变拉氏量写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \quad (3.84)$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu] \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu. \end{aligned} \quad (3.85)$$

从而，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \quad (3.86)$$

Euler-Lagrange 方程 (1.114) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \quad (3.87)$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \quad (3.88)$$

上式称为 **Proca 方程**，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0. \quad (3.89)$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = m^2 \partial_\nu A^\nu. \quad (3.90)$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz 条件**

$$\partial_\mu A^\mu = 0. \quad (3.91)$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \partial^2 A^\nu. \quad (3.92)$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon* 方程

$$(\partial^2 + m^2)A^\mu(x) = 0. \quad (3.93)$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \quad (3.94)$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \quad (3.95)$$

由于 $\pi_0 = 0$ ，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。实际上，由于 Lorenz 条件 (3.91) 的存在， A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于其它 3 个分量的量。因此，正则量子化程序要求独立的正则变量满足等时对易关系

$$[A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = i\delta^i_j \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^i(\mathbf{x}, t), A^j(\mathbf{y}, t)] = [\pi_i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = 0. \quad (3.96)$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程，应该具有两个平面波解，即正能解 $\exp(-ip \cdot x)$ 和负能解 $\exp(ip \cdot x)$ 。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标，平面波展开式的系数也必须具有一个这样的指标。一般地，对于确定的动量 \mathbf{p} ，矢量场的正能解模式具有如下形式：

$$A^\mu(x; \mathbf{p}, \sigma) = e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (3.97)$$

这里的系数 $e^\mu(\mathbf{p}, \sigma)$ 是 Lorentz 矢量，称为**极化矢量** (polarization vector)，它依赖于动量 p ，而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而，可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点，一组极化矢量应当是线性独立且正交完备的。Lorentz 矢量空间是一个 4 维空间，因而这样的极化矢量应该有 4 个，包括 1 个类时的极化矢量 $e^\mu(\mathbf{p}, 0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 。在没有额外约束的情况下，我们要求这 4 个极化矢量是实的，而且满足 Lorentz 矢量空间中的正交归一关系

$$e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') = g_{\sigma\sigma'} \quad (3.98)$$

和完备性关系

$$\sum_{\sigma=0}^3 g_{\sigma\sigma'} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma') = g_{\mu\nu}. \quad (3.99)$$

上面这两个关系都是 Lorentz 协变的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^μ 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空**横向极化矢量**

$$e^\mu(\mathbf{p}, 1) = (0, \mathbf{e}(\mathbf{p}, 1)), \quad e^\mu(\mathbf{p}, 2) = (0, \mathbf{e}(\mathbf{p}, 2)). \quad (3.100)$$

此处，

$$\mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}| |\mathbf{p}_T|} (p^1 p^3, p^2 p^3, -|\mathbf{p}_T|^2), \quad \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|} (-p^2, p^1, 0), \quad (3.101)$$

其中

$$|\mathbf{p}_T| \equiv \sqrt{(p^1)^2 + (p^2)^2}. \quad (3.102)$$

“横向”指的是它们在三维空间中与 \mathbf{p} 垂直，即

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|}[(p^1)^2 p^3 + (p^2)^2 p^3 - p^3 |\mathbf{p}_T|^2] = 0, \quad (3.103)$$

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|}(-p^1 p^2 + p^2 p^1) = 0. \quad (3.104)$$

此外，存在如下关系：

$$\begin{aligned} \mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 1) &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} [(p^1)^2 (p^3)^2 + (p^2)^2 (p^3)^2 + |\mathbf{p}_T|^4] \\ &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 [(p^3)^2 + |\mathbf{p}_T|^2] = \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 |\mathbf{p}|^2 = 1, \end{aligned} \quad (3.105)$$

$$\mathbf{e}(\mathbf{p}, 2) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|^2} [(p^2)^2 + (p^1)^2] = \frac{1}{|\mathbf{p}_T|^2} |\mathbf{p}_T|^2 = 1, \quad (3.106)$$

$$\mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|^2} (-p^1 p^3 p^2 + p^2 p^3 p^1) = 0. \quad (3.107)$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = \delta_{ij}, \quad i, j = 1, 2. \quad (3.108)$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_\mu e^\mu(\mathbf{p}, 1) = p_\mu e^\mu(\mathbf{p}, 2) = 0, \quad (3.109)$$

和正交归一关系

$$e_\mu(\mathbf{p}, i) e^\mu(\mathbf{p}, j) = -\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = -\delta_{ij} = g_{ij}. \quad (3.110)$$

接着，要求第 3 个类空极化矢量 $e^\mu(\mathbf{p}, 3)$ 是纵向的，即在三维空间中与 \mathbf{p} 平行。这样还不能确定它的时间分量，为此，我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p}, 3) = 0$ ，而正交归一关系 (3.98) 将决定它的归一化。于是，纵向极化矢量的形式为

$$e^\mu(\mathbf{p}, 3) = \left(\frac{|\mathbf{p}|}{m}, \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} \right). \quad (3.111)$$

可以验证，它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p}, 3) = p^0 \frac{|\mathbf{p}|}{m} - \mathbf{p} \cdot \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} = \frac{p^0 |\mathbf{p}|}{m} - \frac{p^0 |\mathbf{p}|}{m} = 0, \quad (3.112)$$

和正交归一关系

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, 3) = \frac{|\mathbf{p}|}{m} \frac{|\mathbf{p}|}{m} - \frac{(p^0)^2 \mathbf{p} \cdot \mathbf{p}}{m^2 |\mathbf{p}|^2} = \frac{|\mathbf{p}|^2}{m^2} - \frac{(p^0)^2}{m^2} = -\frac{(p^0)^2 - |\mathbf{p}|^2}{m^2} = -1 = g_{33}; \quad (3.113)$$

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, i) = -\frac{p^0}{m|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.114)$$

最后，我们可以将类时极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p}, 0) = \frac{1}{m} p^\mu = \frac{1}{m} (p^0, \mathbf{p}). \quad (3.115)$$

它满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = \frac{p^2}{m^2} = 1 = g_{00}; \quad (3.116)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, i) = -\frac{1}{m} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2; \quad (3.117)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 3) = \frac{1}{m^2} p^0 |\mathbf{p}| - \frac{p^0}{m^2 |\mathbf{p}|} \mathbf{p} \cdot \mathbf{p} = 0. \quad (3.118)$$

不过，它不满足四维时空的横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = \frac{p^2}{m} = m. \quad (3.119)$$

可以验证，由 (3.100)、(3.101)、(3.111) 和 (3.115) 式定义这组极化矢量确实满足完备性关系 (3.99):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= \frac{1}{m^2} \begin{pmatrix} p^0 p^0 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^1 p^0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ -p^2 p^0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ -p^3 p^0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \\ & \quad - \frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{m^2} \begin{pmatrix} |\mathbf{p}|^2 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^0 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^3 \\ -p^0 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^3 \\ -p^0 p^3 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(p^1)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^1 p^3)^2 + (p^2)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{(p^2)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^2 p^3)^2 + (p^1)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} & \frac{(p^3)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{|\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2|\mathbf{p}_T|^2 + (p^1)^2(|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_T|^2} & -\frac{p^1p^2}{|\mathbf{p}|^2} + \frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{p^1p^3}{|\mathbf{p}|^2} + \frac{p^1p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^2}{|\mathbf{p}|^2} + \frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{(p^2)^2|\mathbf{p}_T|^2 + (p^2)^2(|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_T|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2} + \frac{p^2p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^3}{|\mathbf{p}|^2} + \frac{p^1p^3}{|\mathbf{p}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2} + \frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{(p^3)^2 + |\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \tag{3.120}
\end{aligned}$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91)，正能解模式 (3.97) 应满足

$$0 = \partial_\mu A^\mu(x; \mathbf{p}, \sigma) = -ip_\mu e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \tag{3.121}$$

即

$$p_\mu e^\mu(\mathbf{p}, \sigma) = 0. \tag{3.122}$$

也就是说，描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 不能用于描述有质量矢量场 A^μ 。这说明 A^μ 只有 3 个物理的极化状态，由类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 描述。根据完备性关系 (3.99)，这 3 个物理的极化矢量满足

$$-\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = \sum_{\sigma=1}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) = g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}, \tag{3.123}$$

即具有求和关系

$$\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{3.124}$$

通过如下线性组合，我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p, \lambda)$ ，其中 $\lambda = +, 0, -$ ：

$$\varepsilon^\mu(\mathbf{p}, \pm) \equiv \frac{1}{\sqrt{2}} [\mp e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)], \tag{3.125}$$

$$\varepsilon^\mu(\mathbf{p}, 0) \equiv e^\mu(\mathbf{p}, 3). \tag{3.126}$$

这样定义的 $\varepsilon^\mu(p, \pm)$ 是复的，而 $\varepsilon^\mu(p, 0)$ 是实的。它们都满足四维横向条件

$$p_\mu \varepsilon^\mu(\mathbf{p}, \lambda) = 0. \tag{3.127}$$

它们还满足

$$\begin{aligned}
\varepsilon_\mu^*(\mathbf{p}, \pm) \varepsilon^\mu(\mathbf{p}, \pm) &= \frac{1}{2} [\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)] [\mp e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)] \\
&= \frac{1}{2} e_\mu(\mathbf{p}, 1) e^\mu(\mathbf{p}, 1) + \frac{1}{2} e_\mu(\mathbf{p}, 2) e^\mu(\mathbf{p}, 2) = \frac{1}{2} (g_{11} + g_{22}) = -1, \tag{3.128}
\end{aligned}$$

$$\begin{aligned}\varepsilon_\mu^*(\mathbf{p}, \pm)\varepsilon^\mu(\mathbf{p}, \mp) &= \frac{1}{2}[\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)][\pm e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)] \\ &= -\frac{1}{2}e_\mu(\mathbf{p}, 1)e^\mu(\mathbf{p}, 1) + \frac{1}{2}e_\mu(\mathbf{p}, 2)e^\mu(\mathbf{p}, 2) = \frac{1}{2}(-g_{11} + g_{22}) = 0,\end{aligned}\quad (3.129)$$

$$\varepsilon_\mu^*(\mathbf{p}, 0)\varepsilon^\mu(\mathbf{p}, 0) = e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -1,\quad (3.130)$$

$$\varepsilon_\mu^*(\mathbf{p}, \pm)\varepsilon^\mu(\mathbf{p}, 0) = \frac{1}{2}[\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)]e^\mu(\mathbf{p}, 3) = 0,\quad (3.131)$$

即具有正交归一关系

$$\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon^\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}.\quad (3.132)$$

极化矢量求和关系则是

$$\begin{aligned}\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2}[e_\mu(p, 1) + ie_\mu(p, 2)][e_\nu(p, 1) - ie_\nu(p, 2)] \\ &\quad + \frac{1}{2}[-e_\mu(p, 1) + ie_\mu(p, 2)][-e_\nu(p, 1) - ie_\nu(p, 2)] + e_\mu(p, 3)e_\nu(p, 3) \\ &= e_\mu(p, 1)e_\nu(p, 1) + e_\mu(p, 2)e_\nu(p, 2) + e_\mu(p, 3)e_\nu(p, 3) \\ &= \sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma)e_\nu(\mathbf{p}, \sigma),\end{aligned}\quad (3.133)$$

与 (3.124) 式左边相等, 故

$$\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}.\quad (3.134)$$

四维横向条件 (3.127) 在上式中体现为

$$p^\nu \sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu + \frac{p_\mu p^2}{m^2} = -p_\mu + p_\mu = 0.\quad (3.135)$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 \mathbf{p} 的方向由 $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$ 表征, 于是, 在 Lorentz 群矢量表示中, 螺旋度定义为

$$\hat{\mathbf{p}} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & & \\ & 0 & -ip^3 & ip^2 \\ & ip^3 & 0 & -ip^1 \\ & -ip^2 & ip^1 & 0 \end{pmatrix}.\quad (3.136)$$

将 (3.101) 和 (3.111) 式代入 (3.125) 和 (3.126) 式, 得到 $\varepsilon^\mu(p, \lambda)$ 的列矢量形式为

$$\varepsilon^\mu(p, 0) = \frac{1}{m|\mathbf{p}|} \begin{pmatrix} |\mathbf{p}|^2 \\ p^0 p^1 \\ p^0 p^2 \\ p^0 p^3 \end{pmatrix}, \quad \varepsilon^\mu(p, +) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 + ip^2 |\mathbf{p}| \\ -p^2 p^3 - ip^1 |\mathbf{p}| \\ |\mathbf{p}_T|^2 \end{pmatrix},$$

$$\varepsilon^\mu(p, -) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ p^1 p^3 + ip^2 |\mathbf{p}| \\ p^2 p^3 - ip^1 |\mathbf{p}| \\ -|\mathbf{p}_T|^2 \end{pmatrix}. \quad (3.137)$$

从而, 可得

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, 0) = \frac{1}{m|\mathbf{p}|^2} \begin{pmatrix} 0 \\ -ip^3 p^0 p^2 + ip^2 p^0 p^3 \\ ip^3 p^0 p^1 - ip^1 p^0 p^3 \\ -ip^2 p^0 p^1 + ip^1 p^0 p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \varepsilon^\mu(p, 0), \quad (3.138)$$

$$\begin{aligned} (\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, +) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}_T|^2 \\ -ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}_T|^2 \\ ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| - ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = +\varepsilon^\mu(p, +), \end{aligned} \quad (3.139)$$

$$\begin{aligned} (\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, -) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}_T|^2 \\ ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}_T|^2 \\ -ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| + ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = -\varepsilon^\mu(p, -). \end{aligned} \quad (3.140)$$

归纳起来, 有

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, \lambda) = \lambda \varepsilon^\mu(p, \lambda). \quad (3.141)$$

上式说明极化矢量 $\varepsilon^\mu(p, \lambda)$ 是螺旋度的本征态, 本征值为 λ 。因此, $\varepsilon^\mu(p, \lambda)$ 描述动量为 \mathbf{p} 、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda = \pm 1$ 对应于两种**横向极化**, $\lambda = 0$ 对应于**纵向极化**。

有质量的实矢量场算符 $A^\mu(\mathbf{x}, t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态, 形式为

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right], \quad (3.142)$$

其中 $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ 。容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.143)$$

根据 (3.95) 式, 共轭动量密度为

$$\begin{aligned} \pi_i = -\partial_0 A_i + \partial_i A_0 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left\{ [ip_0 \varepsilon_i(\mathbf{p}, \lambda) - ip_i \varepsilon_0(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda} e^{-ip \cdot x} \right. \\ \left. + [-ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) + ip_i \varepsilon_0^*(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right\}, \end{aligned} \quad (3.144)$$

引入

$$\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \equiv \varepsilon_i(\mathbf{p}, \lambda) - \frac{p_i}{p_0} \varepsilon_0(\mathbf{p}, \lambda), \quad (3.145)$$

则有

$$p_0 \varepsilon_i(\mathbf{p}, \lambda) - p_i \varepsilon_0(\mathbf{p}, \lambda) = p_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda), \quad (3.146)$$

从而, 可以将共轭动量密度的平面波展开式写得更加紧凑:

$$\pi_i(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right]. \quad (3.147)$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned} & \int d^3 x e^{iq \cdot x} A^\mu \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} + \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq^0 t} \right] \end{aligned} \quad (3.148)$$

和

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \partial_0 A^\mu &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\ &\quad \left. - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} - \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq^0 t} \right], \end{aligned} \quad (3.149)$$

以及正交归一关系 (3.132), 可得

$$\begin{aligned}\varepsilon_\mu^*(\mathbf{q}, \lambda') \int d^3x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon_\mu(\mathbf{q}, \lambda') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} \\ &= -i\sqrt{2E_{\mathbf{q}}} \sum_{\lambda=\pm,0} \delta_{\lambda'\lambda} a_{\mathbf{q},\lambda} = -i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q},\lambda'}.\end{aligned}\quad (3.150)$$

由 Lorenz 条件 (3.91) 可得

$$\partial_0 A^0 = -\partial_i A^i, \quad (3.151)$$

根据 (3.95) 式, 有

$$\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \quad (3.152)$$

于是, 湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$\begin{aligned}a_{\mathbf{p},\lambda} &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \varepsilon_\mu^*(\mathbf{p}, \lambda) \int d^3x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu) \\ &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\varepsilon_0^*(\mathbf{p}, \lambda) \partial_0 A^0 + \varepsilon_i^*(\mathbf{p}, \lambda) \partial_0 A^i - ip_0 \varepsilon_\mu^*(\mathbf{p}, \lambda) A^\mu] \\ &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0 \\ &\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i].\end{aligned}\quad (3.153)$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$\begin{aligned}\int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0] &= \int d^3x [\varepsilon_0^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^i + \varepsilon_i^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^0] \\ &= \int d^3x [ip_i \varepsilon_0^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^0] \\ &= \int d^3x e^{ip \cdot x} [ip_i \varepsilon_0^*(\mathbf{p}, \lambda) A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0],\end{aligned}\quad (3.154)$$

从而, 有

$$\begin{aligned}a_{\mathbf{p},\lambda} &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [i\varepsilon_0^*(\mathbf{p}, \lambda) p_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0 \\ &\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i] \\ &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \{\varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - ip^\mu \varepsilon_\mu^*(\mathbf{p}, \lambda) A^0 - i[p_0 \varepsilon_i^*(\mathbf{p}, \lambda) - p_i \varepsilon_0^*(\mathbf{p}, \lambda)] A^i\}.\end{aligned}\quad (3.155)$$

再利用四维横向条件 (3.127) 和 (3.146) 式, 得到

$$a_{\mathbf{p},\lambda} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(x) - ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(x)]. \quad (3.156)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p},\lambda}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} [-\varepsilon^i(\mathbf{p}, \lambda) \pi_i(x) + ip_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda) A^i(x)]. \quad (3.157)$$

利用等时对易关系 (3.96), 可得湮灭算符与产生算符的对易关系为

$$\begin{aligned}
& [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left[-\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(\mathbf{x}, t) - ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(\mathbf{x}, t), \right. \\
&\quad \left. -\varepsilon^j(\mathbf{q}, \lambda') \pi_j(\mathbf{y}, t) + iq_0 \tilde{\varepsilon}_j(\mathbf{q}, \lambda') A^j(\mathbf{y}, t) \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left\{ -iq_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') [\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \right. \\
&\quad \left. + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') [A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] \right\} \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x}-\mathbf{y}) [-q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') \delta^j_i - p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') \delta^i_j] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} [-E_{\mathbf{q}} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{q}, \lambda') - E_{\mathbf{p}} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda')] \\
&= -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) [\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda')]. \tag{3.158}
\end{aligned}$$

根据定义式 (3.145)、四维横向条件 (3.127) 和正交归一关系 (3.132), 有

$$\begin{aligned}
\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') - \frac{1}{p_0} p_i \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') + \frac{1}{p_0} p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\
&= \varepsilon^{\mu*}(\mathbf{p}, \lambda) \varepsilon_\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}, \tag{3.159}
\end{aligned}$$

取复共轭, 可得

$$\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}. \tag{3.160}$$

于是,

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) (-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{3.161}$$

另一方面,

$$\begin{aligned}
& [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left[-\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(\mathbf{x}, t) - ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(\mathbf{x}, t), \right. \\
&\quad \left. -\varepsilon^{j*}(\mathbf{q}, \lambda') \pi_j(\mathbf{y}, t) - iq_0 \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') A^j(\mathbf{y}, t) \right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left\{ iq_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') [\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \right. \\
&\quad \left. + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{j*}(\mathbf{q}, \lambda') [A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] \right\} \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x}-\mathbf{y}) [q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') \delta^j_i - p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{j*}(\mathbf{q}, \lambda') \delta^i_j] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} [E_{\mathbf{q}} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(\mathbf{q}, \lambda') - E_{\mathbf{p}} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda')]
\end{aligned}$$

$$= -\frac{1}{2}(2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) e^{2iE_{\mathbf{p}}t} [\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda')]. \quad (3.162)$$

对四维横向条件 (3.127) 取复共轭, 得

$$p_\mu \varepsilon^{\mu*}(\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) + p_i \varepsilon^{i*}(\mathbf{p}, \lambda) = 0. \quad (3.163)$$

将上式中的 \mathbf{p} 替换成 $-\mathbf{p}$, 得

$$p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda) - p_i \varepsilon^{i*}(-\mathbf{p}, \lambda) = 0. \quad (3.164)$$

因此, 有

$$p_i \varepsilon^{i*}(\mathbf{p}, \lambda) = -p_0 \varepsilon^{0*}(\mathbf{p}, \lambda), \quad -p_i \varepsilon^{i*}(-\mathbf{p}, \lambda) = -p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda), \quad (3.165)$$

或者写成

$$\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(\mathbf{p}, \lambda), \quad -\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda). \quad (3.166)$$

从而, 可得

$$\begin{aligned} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \left[\varepsilon_i^*(-\mathbf{p}, \lambda) + \frac{p_i}{p_0} \varepsilon_0^*(-\mathbf{p}, \lambda) \right] \\ &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') + \frac{1}{p_0} p_i \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda') \\ &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') - \frac{1}{p_0} p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda') \\ &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') - \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda'), \end{aligned} \quad (3.167)$$

$$\begin{aligned} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') &= \left[\varepsilon_i^*(\mathbf{p}, \lambda) - \frac{p_i}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) \right] \varepsilon^{i*}(-\mathbf{p}, \lambda') \\ &= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) p_i \varepsilon^{i*}(-\mathbf{p}, \lambda') \\ &= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda') \\ &= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda) \varepsilon^{0*}(-\mathbf{p}, \lambda'). \end{aligned} \quad (3.168)$$

可见, $\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') = 0$, 故

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = 0. \quad (3.169)$$

综上, 产生湮灭算符的对易关系为

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = [a_{\mathbf{p}, \lambda}^\dagger, a_{\mathbf{q}, \lambda'}^\dagger] = 0. \quad (3.170)$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \quad (3.171)$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \quad (3.172)$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \quad (3.173)$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$, 因此,

$$A^0 = -\frac{1}{m^2} \partial_\mu F^{\mu 0} = -\frac{1}{m^2} \partial_i F^{i0} = -\frac{1}{m^2} \partial_i \pi^i = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.174)$$

从而, 可得

$$-\boldsymbol{\pi} \cdot \dot{\mathbf{A}} = \boldsymbol{\pi} \cdot (\boldsymbol{\pi} + \nabla A_0) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) - A_0 (\nabla \cdot \boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2. \quad (3.175)$$

另一方面,

$$\frac{1}{2} F_{0i} F^{0i} = \frac{1}{2} \pi_i \pi^i = -\frac{1}{2} \boldsymbol{\pi}^2. \quad (3.176)$$

利用 (1.82) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial_m A^n = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n = -\varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n, \quad (3.177)$$

从而,

$$\begin{aligned} \frac{1}{4} F_{ij} F^{ij} &= \frac{1}{4} F^{ij} F_{ij} = \frac{1}{4} \varepsilon^{ijk} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{ijl} \varepsilon^{lpq} \partial_p A^q = \frac{1}{4} 2 \delta^{kl} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{lpq} \partial_p A^q \\ &= \frac{1}{2} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{kpq} \partial_p A^q = \frac{1}{2} (\nabla \times \mathbf{A})^2. \end{aligned} \quad (3.178)$$

于是, 有

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} = -\frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2. \quad (3.179)$$

根据 (1.117) 式, 有质量矢量场的哈密顿量密度为

$$\begin{aligned} \mathcal{H} &= \pi_i \partial_0 A^i - \mathcal{L} = \pi_i \partial_0 A^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \\ &= -\boldsymbol{\pi} \cdot \dot{\mathbf{A}} - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) \\ &= \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} m^2 \mathbf{A}^2 \\ &= \frac{1}{2} \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2. \end{aligned} \quad (3.180)$$

上式最后一行第二项是一个全散度, 对全空间积分时它没有贡献。于是, 哈密顿量为

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x \left[\boldsymbol{\pi}^2 + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 + (\nabla \times \mathbf{A})^2 + m^2 \mathbf{A}^2 \right]. \quad (3.181)$$

下面逐项进行计算。

哈密顿量的第一项是

$$\begin{aligned}
& \frac{1}{2} \int d^3x \pi^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} (ip_0)(iq_0) \left[\tilde{\epsilon}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\epsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \cdot \left[\tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[-\tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad - \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + \tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 q_0}{(2\pi)^3 \sqrt{2E_p 2E_q}} \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_p} E_p^2 \left[\tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right. \\
&\quad \left. - \tilde{\epsilon}(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_p t} - \tilde{\epsilon}^*(\mathbf{p}, \lambda) \cdot \tilde{\epsilon}^*(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_p t} \right]. \quad (3.182)
\end{aligned}$$

第二项是

$$\begin{aligned}
& \frac{1}{2} \int d^3x \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \frac{(ip_0)(iq_0)}{m^2} \left[i\mathbf{p} \cdot \tilde{\epsilon}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + i\mathbf{p} \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \times \left[i\mathbf{q} \cdot \tilde{\epsilon}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} + i\mathbf{q} \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_p 2E_q} m^2} \left\{ -[\mathbf{p} \cdot \tilde{\epsilon}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad - [\mathbf{p} \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \cdot \tilde{\epsilon}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. - [\mathbf{p} \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right\} \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 q_0}{(2\pi)^3 \sqrt{2E_p 2E_q} m^2} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\epsilon}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + [\mathbf{p} \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right) \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\epsilon}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + [\mathbf{p} \cdot \tilde{\epsilon}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\epsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \frac{E_{\mathbf{p}}^2}{m^2} \left\{ [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger \right. \\
&\quad + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\quad \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \quad (3.183)
\end{aligned}$$

第三项是

$$\begin{aligned}
&\frac{1}{2} \int d^3x (\nabla \times \mathbf{A})^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[i\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - i\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \cdot \left[i\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - i\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. - [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right\} \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right) \right. \\
&\quad \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger \right. \\
&\quad + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} + [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\quad \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \quad (3.184)
\end{aligned}$$

第四项是

$$\begin{aligned}
&\frac{1}{2} \int d^3x m^2 \mathbf{A}^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \cdot \left[\boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q m^2}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} m^2 \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right. \\
&\quad \left. + \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.185)
\end{aligned}$$

综合起来，哈密顿量化为

$$\begin{aligned}
H = \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \left[f_1(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + f_1^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right. \\
\left. + f_2(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + f_2^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right], \quad (3.186)
\end{aligned}$$

其中，

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') \equiv E_{\mathbf{p}}^2 \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda') + \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] \\
+ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda'), \quad (3.187)
\end{aligned}$$

$$\begin{aligned}
f_2(\mathbf{p}, \lambda, \lambda') \equiv -E_{\mathbf{p}}^2 \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda') - \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] \\
+ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda'). \quad (3.188)
\end{aligned}$$

现在，我们计算 $f_1(\mathbf{p}, \lambda, \lambda')$ 。由 (3.145)、(3.166) 和 (3.132) 式，可得

$$\begin{aligned}
\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda') &= \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{\mathbf{p}}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right] \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon_0^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon_0(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= -\varepsilon_\mu(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') + \left(\frac{|\mathbf{p}|^2}{p_0^2} - 1 \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \delta_{\lambda\lambda'} - \frac{m^2}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \quad (3.189)
\end{aligned}$$

另一方面，

$$\begin{aligned}
&[\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] \\
&= \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right]
\end{aligned}$$

$$\begin{aligned}
&= [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(p_0^2 - 2|\mathbf{p}|^2 + \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \left[p_0^2 - |\mathbf{p}|^2 + \frac{|\mathbf{p}|^2}{p_0^2} (|\mathbf{p}|^2 - p_0^2) \right] \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(m^2 - m^2 \frac{|\mathbf{p}|^2}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \tag{3.190}
\end{aligned}$$

对于任意空间矢量 \mathbf{a} 和 \mathbf{b} , 利用 (1.82) 式, 有

$$\begin{aligned}
(\mathbf{p} \times \mathbf{a}) \cdot (\mathbf{p} \times \mathbf{b}) &= \varepsilon^{ijk} p^j a^k \varepsilon^{imn} p^m b^n = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}) p^j a^k p^m b^n \\
&= p^j a^k p^j b^k - p^j a^k p^k b^j = |\mathbf{p}|^2 \mathbf{a} \cdot \mathbf{b} - (\mathbf{p} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}), \tag{3.191}
\end{aligned}$$

从而, 可得

$$\begin{aligned}
[\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda'). \tag{3.192}
\end{aligned}$$

于是, (3.187) 式化为

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') &= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad + |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} + E_{\mathbf{p}}^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - E_{\mathbf{p}}^2 \varepsilon_\mu(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.193}
\end{aligned}$$

因此,

$$f_1(\mathbf{p}, \lambda, \lambda') = f_1^*(\mathbf{p}, \lambda, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.194}$$

接着, 我们计算 $f_2(\mathbf{p}, \lambda, \lambda')$ 。由 (3.145) 和 (3.166) 式, 可得

$$\begin{aligned}
\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda') &= \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\mathbf{p}}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right] \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{1}{E_{\mathbf{p}}^2} (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'). \tag{3.195}
\end{aligned}$$

另一方面,

$$\begin{aligned}
& [\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \tilde{\varepsilon}(-\mathbf{p}, \lambda')] \\
&= \left[\mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \varepsilon(-\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right] \\
&= [\mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda)][\mathbf{p} \cdot \varepsilon(-\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda)] \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \varepsilon(-\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= -p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^0(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \left(-p_0^2 + 2|\mathbf{p}|^2 - \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = -\frac{1}{E_{\mathbf{p}}^2} (E_{\mathbf{p}}^2 - |\mathbf{p}|^2)^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= -\frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'), \tag{3.196}
\end{aligned}$$

而

$$\begin{aligned}
[\mathbf{p} \times \varepsilon(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \varepsilon(-\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') - [\mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda)][\mathbf{p} \cdot \varepsilon(-\mathbf{p}, \lambda')] \\
&= |\mathbf{p}|^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') + p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\
&= |\mathbf{p}|^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'). \tag{3.197}
\end{aligned}$$

于是, (3.188) 式化为

$$\begin{aligned}
f_2(\mathbf{p}, \lambda, \lambda') &= -E_{\mathbf{p}}^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') - (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad + |\mathbf{p}|^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda') + m^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') \\
&= (-2E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 + E_{\mathbf{p}}^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = 0. \tag{3.198}
\end{aligned}$$

因此,

$$f_2(\mathbf{p}, \lambda, \lambda') = f_2^*(\mathbf{p}, \lambda, \lambda') = 0. \tag{3.199}$$

将 (3.194) 和 (3.199) 式代入 (3.186) 式, 再利用产生湮灭算符的对易关系 (3.170), 可得有质量矢量场的哈密顿量为

$$\begin{aligned}
H &= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right) = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} \right) \\
&= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{3}{2} E_{\mathbf{p}}. \tag{3.200}
\end{aligned}$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和, 第二项是零点能。

根据 (1.156) 式, 有质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi_i \nabla A^i \\
&= - \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \times \left[i\mathbf{q} \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - i\mathbf{q} \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 \mathbf{q}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[- \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 \mathbf{q}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
&\quad \left. + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.201)
\end{aligned}$$

由 (3.145) 和 (3.165) 式可得

$$\begin{aligned}
\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') &= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_i \varepsilon^i(-\mathbf{p}, \lambda') \\
&= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\
&= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'), \quad (3.202)
\end{aligned}$$

从而, 有

$$\begin{aligned}
&- \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda) \varepsilon^{0*}(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{-\mathbf{p}, \lambda'} a_{\mathbf{p}, \lambda} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{-\mathbf{p}, \lambda'}^\dagger a_{\mathbf{p}, \lambda}^\dagger e^{2iE_{\mathbf{p}}t} \right\}
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \quad (3.203)
\end{aligned}$$

上式第二步进行了 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换, 由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和, 这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.170)。留意到第一步与第三步的结果互为相反数, 可知上式为零。因此, (3.201) 式最后两行方括号中最后两项没有贡献。再利用 (3.160) 式, 可得

$$\begin{aligned}
\mathbf{P} &= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-\delta_{\lambda\lambda'} a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger - \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right] = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} \right] \\
&= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + \frac{3}{2} \delta^{(3)}(0) \int d^3p \mathbf{p} = \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}. \quad (3.204)
\end{aligned}$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m = 0$ 时, 由 (3.100) 和 (3.101) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 的形式不变, 但 (3.111) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 的良好定义。实际上, 在满足正确归一化的条件下, $m = 0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面, 由于无质量矢量粒子的动量 p^μ 的内积为 $p^2 = 0$, 也不能像 (3.115) 式那样将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 取为正比于 p^μ 的矢量, 否则将出现 $e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = 0$ 而不能得到正确的归一化。因此, 我们需要重新定义 $e^\mu(\mathbf{p}, 3)$ 和 $e^\mu(\mathbf{p}, 0)$ 。

在用 (3.100) 和 (3.101) 式定义 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 时, 我们已经选取了一个特定的惯性参考系。在这个参考系中, 可以定义一个类时单位矢量

$$n^\mu = (1, 0, 0, 0), \quad (3.205)$$

它的 Lorentz 不变内积是

$$n^2 = 1. \quad (3.206)$$

然后, 将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 在此参考系中的形式就取为 n^μ , 即

$$e^\mu(\mathbf{p}, 0) = n^\mu. \quad (3.207)$$

$e^\mu(\mathbf{p}, 0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面, 纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式:

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n) n^\mu}{p \cdot n}. \quad (3.208)$$

$p^2 = (p^0)^2 - |\mathbf{p}|^2 = 0$ 表明

$$p^0 = |\mathbf{p}|, \quad (3.209)$$

从而, $e^\mu(\mathbf{p}, 3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n} = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right). \quad (3.210)$$

这样定义的 $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 0) = n^2 = 1, \quad e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|^2} = -1; \quad (3.211)$$

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 1) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 2) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 3) = 0; \quad (3.212)$$

$$e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, i) = -\frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.213)$$

此外, 可以验证, 由 (3.100)、(3.101)、(3.207) 和 (3.208) 式定义的这组极化矢量确实满足完备性关系 (3.99):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0)e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1)e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2)e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3)e_\nu(\mathbf{p}, 3) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \\ & \quad - \frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ 0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ 0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2 (p^3)^2 + (p^2)^2 |\mathbf{p}|^2 + (p^1)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{(p^2)^2 (p^3)^2 + (p^1)^2 |\mathbf{p}|^2 + (p^2)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} & -\frac{|\mathbf{p}_T|^2 + (p^3)^2}{|\mathbf{p}|^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \quad (3.214) \end{aligned}$$

不过, $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 都不满足四维横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = p \cdot n = p^0 = |\mathbf{p}|, \quad p_\mu e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|} = -|\mathbf{p}| = -p \cdot n. \quad (3.215)$$

横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 具有求和关系

$$\begin{aligned} -\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) &= \sum_{\sigma=1}^2 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - g_{33} e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu - (p \cdot n) n_\mu}{p \cdot n} \frac{p_\nu - (p \cdot n) n_\nu}{p \cdot n} \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu p_\nu - (p \cdot n) p_\mu n_\nu - (p \cdot n) p_\nu n_\mu + (p \cdot n)^2 n_\mu n_\nu}{(p \cdot n)^2} \\ &= g_{\mu\nu} + \frac{p_\mu p_\nu}{(p \cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}, \end{aligned} \quad (3.216)$$

即

$$\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.217)$$

根据 (3.125) 式, 作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 满足

$$\begin{aligned} \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2} [e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &\quad + \frac{1}{2} [-e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [-e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &= e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) + e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) = \sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma), \end{aligned} \quad (3.218)$$

因而具有求和关系

$$\sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.219)$$

四维横向条件 $p_\mu \varepsilon^\mu(\mathbf{p}, \pm) = 0$ 在上式中体现为

$$p^\nu \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu - \frac{p_\mu p^2}{(p \cdot n)^2} + \frac{p_\mu (p \cdot n) + p^2 n_\mu}{p \cdot n} = -p_\mu + p_\mu = 0. \quad (3.220)$$

3.4.2 无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中, 令参数 $m = 0$, 就得到自由无质量实矢量场 $A^\mu(x)$ 的拉氏量

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (3.221)$$

其中 $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ 。同理, 令 Proca 方程中 $m = 0$, 就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu} = 0. \quad (3.222)$$

根据 1.5 节的讨论, 这个方程就是无源的 **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子, **光子**是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作**规范变换** (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x), \quad (3.223)$$

其中, 作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数, 依赖于时空坐标, 因而这样的变换是**局域** (local) 变换。在此规范变换下, 场强张量不变:

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu [A^\nu(x) + \partial^\nu \chi(x)] - \partial^\nu [A^\mu(x) + \partial^\mu \chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu \partial^\nu \chi(x) - \partial^\nu \partial^\mu \chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \quad (3.224)$$

因而, 拉氏量 (3.221) 和无源 Maxwell 方程 (3.222) 都不会改变, 这称为**规范对称性** (gauge symmetry)。

在经典电动力学中, 这种对称性广为人知, 它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定, 因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$, 也就是说, 不依赖于**规范**的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算, 常常将规范固定下来, 使得计算过程依赖于选取的规范, 不过, 最后得出的可观测量必须是**规范不变** (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**, 规范条件为

$$\partial_\mu A^\mu = 0. \quad (3.225)$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同, 但是, 在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件, 而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$, 令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2 \chi(x) = -\partial_\mu A^\mu(x), \quad (3.226)$$

那么, 作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件:

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2 \chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \quad (3.227)$$

但是, 经过这种变换之后, 矢量场仍然没有被唯一地确定: 对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$, 取满足齐次波动方程

$$\partial^2 \tilde{\chi}(x) = 0 \quad (3.228)$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换, 都能得到满足 Lorenz 规范条件的另一个矢量场 $A'^\mu(x)$ 。可见, 存在无穷多个规范等价的矢量场, 它们描述相同的物理, 而且全都满足 Lorenz 规范条件 (3.225)。

矢量场 $A^\mu(x)$ 有 4 个分量，因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是，上述规范等价性表明， $A^\mu(x)$ 并没有 3 个独立的自由度，否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上，无质量矢量场 $A^\mu(x)$ 只具有 2 个独立的自由度，也就是说，有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论：电磁波具有 2 种独立的极化态，以螺旋度 λ 来表征的话，就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有质量矢量场 $A^\mu(x)$ 的量子化程序时，由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度，因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难，因为 Proca 方程要求 $A^0(x)$ 不是独立变量，而是由 (3.174) 式决定的：

$$A^0 = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.229)$$

于是，以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的，自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时，(3.229) 式显然不能成立。因此，对于无质量矢量场，最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度，可以在拉氏量中增加一个不会影响最终物理结果的项：

$$\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.230)$$

其中 ξ 是一个可以自由选取的实参数。可以看出，在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.225) 的情况下，由 (3.230) 式定义的 \mathcal{L}_1 等价于由 (3.221) 式定义的 \mathcal{L} 。新增的项 $-\frac{1}{2} \xi (\partial_\mu A^\mu)^2$ 破坏了规范对称性，相当于把规范固定下来，因而称为规范固定项 (gauge-fixing term)。可以将 \mathcal{L}_1 展开为

$$\mathcal{L}_1 = -\frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu + \frac{1}{2} (\partial_\nu A_\mu) \partial^\mu A^\nu - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.231)$$

从而， A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_1}{\partial (\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi (\partial_\nu A^\nu) \frac{\partial (\partial_\sigma A^\sigma)}{\partial (\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0} \partial_\nu A^\nu, \quad (3.232)$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi \partial_\mu A^\mu. \quad (3.233)$$

因此， $\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0 。

现在，正则量子化程序要求算符 A^μ 和 π_μ 满足如下等时对易关系：

$$[A^\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = i\delta^\mu_\nu \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] = [\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = 0. \quad (3.234)$$

但是，这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子，利用 (3.233) 式，可得

$$[A^0(\mathbf{x}, t), \partial_\mu A^\mu(\mathbf{y}, t)] = -\frac{1}{\xi} [A^0(\mathbf{x}, t), \pi_0(\mathbf{y}, t)] = -\frac{i}{\xi} \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (3.235)$$

上式在 $\mathbf{x} = \mathbf{y}$ 处非零, 因而必有 $\partial_\mu A^\mu \neq 0$ 。所以, A^μ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.225)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$, 但对于量子场 $A^\mu(x)$ 来说限制太强了, 下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - \xi g^{\mu\nu}(\partial_\rho A^\rho), \quad \frac{\partial \mathcal{L}_1}{\partial A_\nu} = 0, \quad (3.236)$$

可得, 与 \mathcal{L}_1 对应的 Euler-Lagrange 方程为

$$0 = \partial_\mu \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu \partial_\mu A^\mu - \xi g^{\mu\nu} \partial_\mu(\partial_\rho A^\rho) = -\partial^2 A^\nu + (1 - \xi) \partial^\nu(\partial_\rho A^\rho), \quad (3.237)$$

即

$$\partial^2 A^\mu - (1 - \xi) \partial^\mu(\partial_\nu A^\nu) = 0. \quad (3.238)$$

若取 $\xi = 1$, 则上式化为 **d'Alembert 方程**

$$\partial^2 A^\mu(x) = 0, \quad (3.239)$$

可以看作无质量情况下的 Klein-Gordon 方程。可见, 将规范固定参数取为

$$\xi = 1 \quad (3.240)$$

将有利于简化计算, 这种取法称为 **Feynman 规范**, 本节后续计算采用这个规范。在 Feynman 规范下, 拉氏量化为

$$\begin{aligned} \mathcal{L}_1 &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu) \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu) - \frac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu - \frac{1}{2}\partial^\mu(A_\mu\partial_\nu A^\nu) + \frac{1}{2}A_\mu\partial^\mu\partial_\nu A^\nu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu - A_\mu\partial_\nu A^\nu). \end{aligned} \quad (3.241)$$

上式最后一行第二项是一个全散度, 它不会影响作用量和运动方程, 可以舍弃。因此, 可以采用更加简化的拉氏量

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \quad (3.242)$$

此时, 共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_2}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu. \quad (3.243)$$

对于 d'Alembert 方程 (3.239), 平面波解的正能解和负能解分别正比于 $\exp(-ip \cdot x)$ 和 $\exp(ip \cdot x)$, 其中

$$p^0 = E_{\mathbf{p}} = |\mathbf{p}|. \quad (3.244)$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p}, \sigma)$, 可以对无质量矢量场 $A^\mu(\mathbf{x}, t)$ 作如下平面波展开:

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}). \quad (3.245)$$

容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.246)$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x}, t) = -\partial_0 A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e_\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}). \quad (3.247)$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned} \int d^3x e^{iq \cdot x} A^\mu &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}] \end{aligned} \quad (3.248)$$

和

$$\begin{aligned} &\int d^3x e^{iq \cdot x} \partial_0 A^\mu \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}], \end{aligned} \quad (3.249)$$

以及正交归一关系 (3.98), 可得

$$\begin{aligned} e_\mu(\mathbf{q}, \sigma') \int d^3x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= e_\mu(\mathbf{q}, \sigma') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} \\ &= -i\sqrt{2E_{\mathbf{q}}} \sum_{\sigma=0}^3 \delta_{\sigma'\sigma} a_{\mathbf{q};\sigma} = -i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q};\sigma'}. \end{aligned} \quad (3.250)$$

于是, 有

$$a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} e_\mu(\mathbf{p}, \sigma) \int d^3x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu). \quad (3.251)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p};\sigma}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} e_\mu(\mathbf{p}, \sigma) \int d^3x e^{-ip \cdot x} (\partial_0 A^\mu + ip_0 A^\mu). \quad (3.252)$$

根据等时对易关系 (3.234)，湮灭算符与产生算符的对易关系为

$$\begin{aligned}
[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times [\partial_0 A^{\mu}(\mathbf{x}, t) - ip_0 A^{\mu}(\mathbf{x}, t), \partial_0 A^{\nu}(\mathbf{y}, t) + iq_0 A^{\nu}(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times [-\pi^{\mu}(\mathbf{x}, t) - ip_0 A^{\mu}(\mathbf{x}, t), -\pi^{\nu}(\mathbf{y}, t) + iq_0 A^{\nu}(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times \{-iq_0 [\pi^{\mu}(\mathbf{x}, t), A^{\nu}(\mathbf{y}, t)] + ip_0 [A^{\mu}(\mathbf{x}, t), \pi^{\nu}(\mathbf{y}, t)]\} \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} [-(p_0 + q_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= -(2\pi)^3 e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{p}, \sigma') \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{3.253}
\end{aligned}$$

最后一步用到正交归一关系 (3.98)。另一方面，两个湮灭算符之间的对易关系为

$$\begin{aligned}
[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times [\partial_0 A^{\mu}(\mathbf{x}, t) - ip_0 A^{\mu}(\mathbf{x}, t), \partial_0 A^{\nu}(\mathbf{y}, t) - iq_0 A^{\nu}(\mathbf{y}, t)] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times [-\pi^{\mu}(\mathbf{x}, t) - ip_0 A^{\mu}(\mathbf{x}, t), -\pi^{\nu}(\mathbf{y}, t) - iq_0 A^{\nu}(\mathbf{y}, t)] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&\quad \times \{iq_0 [\pi^{\mu}(\mathbf{x}, t), A^{\nu}(\mathbf{y}, t)] + ip_0 [A^{\mu}(\mathbf{x}, t), \pi^{\nu}(\mathbf{y}, t)]\} \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [(q_0 - p_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \tag{3.254}
\end{aligned}$$

归纳起来，产生湮灭算符的对易关系为

$$[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^\dagger] = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a_{\mathbf{p};\sigma}^\dagger, a_{\mathbf{q};\sigma'}^\dagger] = 0. \tag{3.255}$$

3.4.4 哈密顿量和总动量

根据 (1.117)、(3.243) 和 (3.242) 式, 无质量矢量场的哈密顿量密度是

$$\begin{aligned}\mathcal{H} &= \pi_\mu \partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu) \partial^0 A^\mu + \frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu \\ &= -\frac{1}{2} (\partial_0 A_\mu) \partial^0 A^\mu + \frac{1}{2} (\partial_i A_\mu) \partial^i A^\mu = -\frac{1}{2} [\pi_\mu \pi^\mu + (\nabla A_\mu) \cdot (\nabla A^\mu)].\end{aligned}\quad (3.256)$$

于是, 哈密顿量表达为

$$\begin{aligned}H &= \int d^3x \mathcal{H} = -\frac{1}{2} \int d^3x [\pi_\mu \pi^\mu + (\nabla A_\mu) \cdot (\nabla A^\mu)] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left[(ip_0)(iq_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) (a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + (i\mathbf{p} a_{\mathbf{p};\sigma} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p-q) \cdot x} \right. \\ &\quad \left. + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + (p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) \\ &\quad \times \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\ &\quad \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 + |\mathbf{p}|^2) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \right. \\ &\quad \left. - e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 - |\mathbf{p}|^2) (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \\ &= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} g_{\sigma\sigma'} (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}.\end{aligned}\quad (3.257)$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负, 与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规, 时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论, 将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态, 满足

$$a_{\mathbf{p};\sigma} |0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = 2\delta^{(3)}(0) \int d^3p E_{\mathbf{p}}. \quad (3.258)$$

动量为 p 、极化态为 σ 的单粒子态定义为

$$|\mathbf{p}; \sigma\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p};\sigma}^\dagger |0\rangle. \quad (3.259)$$

从而, 由

$$\begin{aligned} [H, a_{\mathbf{p};\sigma}^\dagger] &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) [a_{\mathbf{q};\sigma'}^\dagger a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^\dagger [a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^\dagger] \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^\dagger (2\pi)^3 (-g_{\sigma'\sigma}) \delta^{(3)}(\mathbf{q} - \mathbf{p}) \\ &= E_{\mathbf{p}} \sum_{\sigma'=0}^3 g_{\sigma'\sigma} g_{\sigma'\sigma} a_{\mathbf{p};\sigma'}^\dagger = E_{\mathbf{p}} a_{\mathbf{p};\sigma}^\dagger \end{aligned} \quad (3.260)$$

可得

$$\begin{aligned} H|\mathbf{p}; \sigma\rangle &= \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p};\sigma}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger H) |0\rangle \\ &= \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} + E_{\text{vac}}) a_{\mathbf{p};\sigma}^\dagger |0\rangle = (E_{\mathbf{p}} + E_{\text{vac}}) |\mathbf{p}; \sigma\rangle. \end{aligned} \quad (3.261)$$

这似乎是一个正常的结果, 说明单粒子态 $|\mathbf{p}; \sigma\rangle$ 比真空多了一份能量 $E_{\mathbf{p}}$ 。

利用产生湮灭算符的对易关系 (3.255), 可以计算单粒子态的内积:

$$\begin{aligned} \langle \mathbf{q}; \sigma' | \mathbf{p}; \sigma \rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | a_{\mathbf{q};\sigma'} a_{\mathbf{p};\sigma}^\dagger | 0 \rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | [a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} - (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q})] | 0 \rangle \\ &= -\sqrt{2E_{\mathbf{p}}} g_{\sigma\sigma'} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (3.262)$$

于是, 有

$$\langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -\sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0), \quad \langle \mathbf{p}; i | \mathbf{p}; i \rangle = \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0), \quad i = 1, 2, 3. \quad (3.263)$$

上式表明, 单粒子态 $|\mathbf{p}; 0\rangle$ 的自我内积是负的, 从而导致它的能量期待值也是负的:

$$\langle \mathbf{p}; 0 | H | \mathbf{p}; 0 \rangle = (E_{\mathbf{p}} + E_{\text{vac}}) \langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -(E_{\mathbf{p}} + E_{\text{vac}}) \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0) < 0. \quad (3.264)$$

这个负能量结果在物理上看起来是不可接受的, 它的根源在于不定度规。

不过, 如前所述, 无质量矢量场只有 2 种独立的极化态, 对应于 2 种横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$, 纵向极化和类时极化都应该是非物理的。选取一定的规范条件, 应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.225) 与正则量子化程序不相容, 我们不能直接使用这个条件, 而需要将它转换到物理 Hilbert 空间中的态的期待值上, 要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle \Psi | \partial_\mu A^\mu(x) | \Psi \rangle = 0. \quad (3.265)$$

上式称为弱 **Lorenz** 规范条件。

$A^\mu(x)$ 的平面波展开式 (3.245) 可以分解成正能解和负能解两个部分：

$$A^\mu(x) = A^{(+)\mu}(x) + A^{(-)\mu}(x). \quad (3.266)$$

其中，正能解部分为

$$A^{(+)\mu}(\mathbf{x}, t) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x}, \quad (3.267)$$

上式的厄米共轭即是负能解部分

$$A^{(-)\mu}(\mathbf{x}, t) \equiv [A^{(+)\mu}(\mathbf{x}, t)]^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}. \quad (3.268)$$

如果要求

$$\partial_\mu A^{(+)\mu}(x) |\Psi\rangle = 0 \quad (3.269)$$

对任意物理态 $|\Psi\rangle$ 成立，则伴随有

$$\langle\Psi| \partial_\mu A^{(-)\mu}(x) = \langle\Psi| [\partial_\mu A^{(+)\mu}(x)]^\dagger = 0, \quad (3.270)$$

从而，弱 Lorenz 规范条件 (3.265) 得到满足：

$$\langle\Psi| \partial_\mu A^\mu(x) |\Psi\rangle = \langle\Psi| \partial_\mu A^{(+)\mu}(x) |\Psi\rangle + \langle\Psi| \partial_\mu A^{(-)\mu}(x) |\Psi\rangle = 0. \quad (3.271)$$

利用 (3.109) 和 (3.215) 式，规范条件 (3.269) 可化为

$$\begin{aligned} 0 &= \partial_\mu A^{(+)\mu}(x) |\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} [p_\mu e^\mu(\mathbf{p}, 0) a_{\mathbf{p};0} + p_\mu e^\mu(\mathbf{p}, 1) a_{\mathbf{p};1} + p_\mu e^\mu(\mathbf{p}, 2) a_{\mathbf{p};2} + p_\mu e^\mu(\mathbf{p}, 3) a_{\mathbf{p};3}] |\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} p \cdot n (a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle. \end{aligned} \quad (3.272)$$

这意味着

$$(a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle = 0 \quad (3.273)$$

对任意物理态 $|\Psi\rangle$ 和任意动量 \mathbf{p} 成立。从而，也有

$$\langle\Psi| (a_{\mathbf{p};0}^\dagger - a_{\mathbf{p};3}^\dagger) = 0. \quad (3.274)$$

于是，

$$\langle\Psi| a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} |\Psi\rangle = \langle\Psi| a_{\mathbf{p};3}^\dagger a_{\mathbf{p};3} |\Psi\rangle. \quad (3.275)$$

这样一来，根据 (3.257) 式计算， $|\Psi\rangle$ 的能量期待值为

$$\langle\Psi| H |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\Psi| \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) |\Psi\rangle + E_{\text{vac}} \langle\Psi| \Psi\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle. \quad (3.276)$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.156) 式可得无质量矢量场的总动量为

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi_\mu \nabla A^\mu \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times (ip_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \\ &= \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left[-a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \right. \\ &\quad \left. - e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \right]. \quad (3.277) \end{aligned}$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换，可得

$$\begin{aligned} &- \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) (a_{-\mathbf{p};\sigma'} a_{\mathbf{p};\sigma} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p};\sigma'}^\dagger a_{\mathbf{p};\sigma}^\dagger e^{2iE_{\mathbf{p}}t}) \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}). \quad (3.278) \end{aligned}$$

可以看出，上式为零。于是，总动量化为

$$\begin{aligned} \mathbf{P} &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} g_{\sigma\sigma'} (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) \end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + \delta^{(3)}(0) \int d^3p \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right). \tag{3.279}
\end{aligned}$$

根据 (3.275) 式, 物理态 $|\Psi\rangle$ 的动量期待值为

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \langle \Psi | \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle. \tag{3.280}$$

同样, 只有两种物理的横向极化才对动量有净贡献。

通过线性组合, 可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$a_{\mathbf{p},\pm} \equiv \frac{1}{\sqrt{2}} (\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}), \tag{3.281}$$

相应的产生算符可以通过取厄米共轭得到。反过来, 有

$$a_{\mathbf{p};1} = -\frac{1}{\sqrt{2}} (a_{\mathbf{p},+} - a_{\mathbf{p},-}), \quad a_{\mathbf{p};2} = -\frac{i}{\sqrt{2}} (a_{\mathbf{p},+} + a_{\mathbf{p},-}). \tag{3.282}$$

利用对易关系 (3.255), 可得

$$\begin{aligned}
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = \frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \pm a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = -\frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = 0, \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0, \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \pm a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0.
\end{aligned} \tag{3.283}$$

于是, 这组产生湮灭算符的对易关系可以整理为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0, \quad \lambda, \lambda' = \pm. \tag{3.284}$$

根据 (3.125) 式, 可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 表示 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$:

$$e^\mu(\mathbf{p}, 1) = -\frac{1}{\sqrt{2}} [\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)], \quad e^\mu(\mathbf{p}, 2) = \frac{i}{\sqrt{2}} [\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)]. \tag{3.285}$$

从而, 有

$$\begin{aligned}
\sum_{\sigma=1}^2 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p}, 1) a_{\mathbf{p};1} + e^\mu(\mathbf{p}, 2) a_{\mathbf{p};2} \\
&= \frac{1}{2} [\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)] (a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2} [\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)] (a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\
&= \varepsilon^\mu(\mathbf{p}, +) a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p}, -) a_{\mathbf{p},-} = \sum_{\lambda=\pm} \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda},
\end{aligned} \tag{3.286}$$

取厄米共轭, 得

$$\sum_{\sigma=1}^2 e^{\mu}(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^{\dagger} = \sum_{\lambda=\pm} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger}. \quad (3.287)$$

于是, 可以把 $A^{\mu}(x)$ 的平面波展开式 (3.245) 改写成

$$\begin{aligned} A^{\mu}(\mathbf{x}, t) = & \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0,3} e^{\mu}(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}) \\ & + \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} [\varepsilon^{\mu}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x}], \end{aligned} \quad (3.288)$$

第一行对应于非物理极化态, 第二行对应于两种物理的螺旋度本征极化态。可见, (3.281) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda = \pm$ 对应的湮灭算符。

此外, 由 (3.282) 式可得

$$\begin{aligned} \sum_{\sigma=1}^2 a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{p};\sigma} &= a_{\mathbf{p};1}^{\dagger} a_{\mathbf{p};1} + a_{\mathbf{p};2}^{\dagger} a_{\mathbf{p};2} = \frac{1}{2} (a_{\mathbf{p},+}^{\dagger} - a_{\mathbf{p},-}^{\dagger}) (a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2} (a_{\mathbf{p},+}^{\dagger} + a_{\mathbf{p},-}^{\dagger}) (a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= a_{\mathbf{p},+}^{\dagger} a_{\mathbf{p},+} + a_{\mathbf{p},-}^{\dagger} a_{\mathbf{p},-} = \sum_{\lambda=\pm} a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda}, \end{aligned} \quad (3.289)$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3 p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle, \quad (3.290)$$

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda} | \Psi \rangle. \quad (3.291)$$

附录 A 英汉对照

Annihilation operator: 湮灭算符	Lowering operator: 降算符
Antichronous: 反时向	Metric: 度规
Axial vector: 轴矢量	Mode: 模式
Boost: 增速	Orthochronous: 保时向
Boson: 玻色子	Parity: 宇称
Canonical quantization: 正则量子化	Phonon: 声子
Conjugate momentum density: 共轭动量密度	Picture: 绘景
Conserved charge: 守恒荷	Plane-wave solution: 平面波解
Conserved current: 守恒流	Polarization vector: 极化矢量
Contraction: 缩并	Positron: 正电子
Contravariant vector: 逆变矢量	Proper: 固有的
Covariant vector: 协变矢量	Pseudoscalar: 赝标量
Creation operator: 产生算符	Raising operator: 升算符
Electron: 电子	Real orthogonal matrix: 实正交矩阵
Energy-momentum tensor: 能动张量	Scalar: 标量
Expectation value: 期待值	Self-conjugate: 自共轭
Fermion: 费米子	Simple harmonic oscillator: 简谐振子
Field strength tensor: 场强张量	Space inversion: 空间反射
Gauge-fixing term: 规范固定项	Spinor: 旋量
Gauge invariant: 规范不变量	Spinor bilinear: 旋量双线性型
Gauge symmetry: 规范对称性	Spinor representation: 旋量表示
Gauge transformation: 规范变换	Step function: 阶跃函数
Generalized coordinate: 广义坐标	Tensor: 张量
Generator: 生成元	Time reversal: 时间反演
Global: 整体	Unitary: 幺正
Helicity: 螺旋度	Vacuum: 真空
Hermitian conjugate: 厄米共轭	Vector: 矢量
Hermitian operator: 厄米算符	Zero-point energy: 零点能
Homomorphic: 同态	
Improper: 非固有的	
Local: 局域	