

量子场论讲义

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<http://yzhxxzy.github.io/cn/teaching.html>

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第 1 章 预备知识

1.1 量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由粒子组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon 方程**：

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\mathbf{x}, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi(\mathbf{x}, t). \quad (1.1)$$

它给出的自由粒子能量为

$$E = \pm \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}, \quad (1.2)$$

其中 \mathbf{p} 为粒子的动量， m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2 \leq E < \infty$ ；也可以为负，取值范围为 $-\infty < E \leq -mc^2$ 。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho = \frac{i\hbar}{2mc^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right), \quad (1.3)$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 协变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E < 0$ 的态都被填满而所有 $E > 0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E > 0$ 的电子跃迁到 $E < 0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的“反粒子 (anti-particle)”，称为**正电子 (positron)**。正电子在 1932 年被 Carl Anderson 发现。

但是, Dirac 的空穴理论仍然面临一些困难, 比如, 为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场? 另一方面, Dirac 方程一开始作为描述单个粒子波函数的方程提出来, 但 Dirac 的解释却包含了无穷多个粒子。而且, 像光子和 π 介子这些不满足 Pauli 不相容原理的粒子, 空穴理论是不能成立的。此外, Dirac 方程只能描述自旋 1/2 的粒子, 不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难, 是否意味着处理这些问题的基础本身就不正确呢? 确实是这样的。量子力学的一条基本原理是: 观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而, 时间显然是一个观测量, 却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中, 描述系统的量子态时可以让态依赖于一个时间参数 t , 这是时间的概念进入量子力学的方式, 但并没有假定这个参数是某个厄米算符的本征值。另一方面, 粒子的空间位置 \mathbf{x} 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见, 在量子力学中, 对时间和空间的处理方式是完全不同的。而在狭义相对论中, Lorentz 对称性将两者混合起来。因此, 在结合量子力学与狭义相对论的过程中出现困难, 也是正常的。

那么, 如何在量子力学中平等地处理时间和空间呢? 一种途径是将时间提升为一个厄米算符, 但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数, 不再由厄米算符描写。这样, 我们可以在每个空间点 \mathbf{x} 处定义一个算符 $\hat{\phi}(\mathbf{x})$, 所有这些算符的集合称为量子场。在 Heisenberg 绘景中, 量子场算符也依赖于时间 t :

$$\hat{\phi}(\mathbf{x}, t) = e^{i\hat{H}t/\hbar} \hat{\phi}(\mathbf{x}) e^{-i\hat{H}t/\hbar}. \quad (1.4)$$

如此, 量子化的对象变成是由依赖于时空坐标的场组成的动力学系统, 这就是量子场论。这里的量子算符用 $\hat{}$ 符号标记, 为了简化记号, 后面将省略 $\hat{}$ 符号。

在量子场论中, 前面提到的困难都可以得到解决。现在, Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态, 包含粒子的态则是激发态, 激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子, 由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的, 不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x \rho$ 也可以用产生湮灭算符表达出来, 虽然它不一定是正定的, 但是它不再被解释为总概率, 而是被解释为正粒子数与反粒子数之差, 因而也不再出现负概率困难。

1.2 自然单位制

量子场论是结合量子力学和相对论的理论, 因而时常出现约化 Planck 常量 \hbar 和光速 c , 这一点可以从上一节的几个公式中看出来。于是, 为了简化表述, 通常采用自然单位制, 取

$$\hbar = c = 1. \quad (1.5)$$

从而, Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi(\mathbf{x}, t) = 0. \quad (1.6)$$

在自然单位制中, 速度没有量纲 (dimension); 长度量纲与时间量纲相同, 是能量量纲的倒数; 能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22} \text{ MeV} \cdot \text{s}, \quad 1 = \hbar c = 1.973 \times 10^{-11} \text{ MeV} \cdot \text{cm}, \quad (1.7)$$

可得

$$1 \text{ s}^{-1} = 6.582 \times 10^{-22} \text{ MeV}, \quad 1 \text{ cm}^{-1} = 1.973 \times 10^{-11} \text{ MeV}. \quad (1.8)$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.036} \quad (1.9)$$

是没有量纲的, 它的数值在任何单位制下都应该相同。因此, 自然单位制不可能将 \hbar 、 c 、 ϵ_0 和 e 这四个常数同时归一化。在量子场论中, 通常再取真空介电常数

$$\epsilon_0 = 1, \quad (1.10)$$

同时可得真空磁导率 $\mu_0 = 1/(\epsilon_0 c^2) = 1$, 这样做其实是取了 Heaviside-Lorentz 单位制。从而, 不同于 Gauss 单位制, Maxwell 方程组中不会出现无理数 4π :

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \quad (1.11)$$

此处的单位制称为**有理化的**自然单位制。现在, 精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$, 而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的; 4π 因子会出现在 Coulomb 定律中, 点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \quad (1.12)$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到**狭义相对论**, 它的基本原理如下。

- (1) **光速不变原理**: 在任意惯性参考系中, 光速的大小不变。
- (2) **狭义相对性原理**: 在任意惯性参考系中, 物理定律具有相同的形式。

两个惯性参考系的 Descartes 坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动, 则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \quad (1.13)$$

其中 Lorentz 因子 $\gamma \equiv (1 - \beta^2)^{-1/2}$. 这种 Lorentz 变换称为沿 x 方向的**增速** (boost)。在此变换下, 有

$$t'^2 - x'^2 - y'^2 - z'^2 = \gamma^2(t - \beta x)^2 - \gamma^2(x - \beta t)^2 - y^2 - z^2$$

$$= \frac{1}{1-\beta^2}(t^2 + \beta^2 x^2 - 2\beta xt - x^2 - \beta^2 t^2 + 2\beta xt) - y^2 - z^2 = t^2 - x^2 - y^2 - z^2. \quad (1.14)$$

可见, $t^2 - x^2 - y^2 - z^2$ 在 Lorentz 变换下不变, 是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值, 这是 Lorentz 变换对应的对称性, 称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来, 可以构成 Minkowski 时空, 坐标记为

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \mathbf{x}), \quad \text{其中 } \mu = 0, 1, 2, 3. \quad (1.15)$$

上式中四种记法是等价的。 x^μ 是一个逆变 (contravariant) 的 Lorentz 四维矢量 (vector), “逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发, 可以定义 Lorentz 不变的内积¹

$$x^2 \equiv x \cdot x \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - |\mathbf{x}|^2. \quad (1.16)$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu} = g_{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.17)$$

可以把内积 (1.16) 简洁地写成

$$x^2 = g_{\mu\nu} x^\mu x^\nu. \quad (1.18)$$

这里采用了 **Einstein 求和约定**: 不写出求和符号, 重复的指标即表示求和。除非特别指出, 后面都默认使用这个约定。在上式中, 用同个字母表示的指标分别在上标和下标重复出现并求和, 这称为**缩并** (contraction), 是 Lorentz 不变量的特点。

为了进一步简化记号, 定义协变 (covariant) 的 Lorentz 四维矢量

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}). \quad (1.19)$$

“协变”指的是指标 μ 写在右下角。于是, 内积 x^2 的表达式 (1.18) 可以简化为

$$x^2 = x^\mu x_\mu. \quad (1.20)$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来, 变成协变矢量 x_μ 。从方阵的角度看, $g_{\mu\nu}$ 的逆为

$$g^{\mu\nu} = g^{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.21)$$

¹这里的记号有些不一致, 第一个 x^2 是内积的记号, 而第二个 x^2 是第 2 个空间坐标。

满足

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu, \quad (1.22)$$

其中 **Kronecker 符号** $\delta^\mu{}_\nu$ 定义为

$$\delta^\mu{}_\nu = \delta_\mu{}^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \quad (1.23)$$

对于 Minkowski 度规, $g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同, 但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$, 对 μ 求和, 得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma{}_\nu x^\nu = x^\sigma, \quad (1.24)$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来, 变成逆变矢量 x^σ . 可见, 逆变矢量与协变矢量是一一对应的, 是对同一个 Lorentz 矢量的两种等价描述.

利用 Kronecker 符号的定义和 (1.22) 式, 可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu{}_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \quad (1.25)$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho{}_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \quad (1.26)$$

这两条式子表明, 度规也可以用来对度规自身的指标进行升降.

利用四维矢量的记号, 可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.27)$$

其中

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (1.28)$$

注意: 在将 $\Lambda^\mu{}_\nu$ 视作矩阵时, 偏左的指标 μ 表示行的编号, 偏右的指标 ν 表示列的编号. $\Lambda^\mu{}_\nu$ 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变, 从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值. 我们可以将 $\Lambda^\mu{}_\nu$ 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换, 称为 (齐次) **Lorentz 变换**, 使下式成立:

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \quad (1.29)$$

可见, Lorentz 变换 $\Lambda^\mu{}_\nu$ 必须满足**保度规条件**

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}. \quad (1.30)$$

空间旋转变换保持 $|\mathbf{x}|^2$ 不变, 由 (1.16) 式可知, 这种变换也属于 Lorentz 变换. 例如, 绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \quad (1.31)$$

容易验证, 它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并, 可得

$$\Lambda_\nu^\gamma \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\alpha\beta} = \delta^\gamma_\beta, \quad (1.32)$$

其中

$$\Lambda_\nu^\gamma \equiv g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \quad (1.33)$$

可以看作是用度规对 Λ^μ_α 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu_\nu \equiv \Lambda^\mu_\nu, \quad (1.34)$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu_\rho \Lambda^\rho_\nu = \delta^\mu_\nu. \quad (1.35)$$

δ^μ_ν 也是一个 Lorentz 变换, 它使得 $x'^\mu = \delta^\mu_\nu x^\nu = x^\mu$, 即 x^μ 在这个变换下不变。可见, δ^μ_ν 是一个恒等变换。(1.35) 式表明, 对时空坐标矢量先作 Λ 变换, 再作 Λ^{-1} 变换, 得到的矢量还是原来的矢量。也就是说, 由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换, 也是一个 Lorentz 变换。在这些记号下, 协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu_\rho x^\rho = g_{\mu\nu} \Lambda^\nu_\rho g^{\rho\sigma} x_\sigma = \Lambda_\mu^\sigma x_\sigma = x_\sigma (\Lambda^{-1})^\sigma_\mu. \quad (1.36)$$

Λ^{-1} 既然是一个 Lorentz 变换, 必定满足保度规条件

$$g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta = g_{\alpha\beta}, \quad (1.37)$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma} = g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta g^{\alpha\rho} g^{\beta\sigma} = g^{\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \Lambda_\alpha^\mu \Lambda_\beta^\nu g^{\alpha\rho} g^{\beta\sigma} \\ &= g^{\gamma\delta} (g^{\alpha\rho} g_{\gamma\mu} \Lambda_\alpha^\mu) (g^{\beta\sigma} g_{\delta\nu} \Lambda_\beta^\nu) = g^{\gamma\delta} \Lambda^\rho_\gamma \Lambda^\sigma_\delta. \end{aligned} \quad (1.38)$$

这给出了保度规条件 (1.30) 的一个等价形式:

$$g^{\mu\nu} \Lambda^\alpha_\mu \Lambda^\beta_\nu = g^{\alpha\beta}. \quad (1.39)$$

将 Λ^μ_ν 视作矩阵 Λ , 则其转置矩阵 Λ^T 的分量满足 $(\Lambda^T)_\nu^\mu = \Lambda^\mu_\nu$, 由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = (\Lambda^T)_\alpha^\mu g_{\mu\nu} \Lambda^\nu_\beta, \quad (1.40)$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^T \mathbf{g} \Lambda. \quad (1.41)$$

取行列式得 $\det \mathbf{g} = \det \Lambda^T \cdot \det \mathbf{g} \cdot \det \Lambda = \det \mathbf{g} \cdot (\det \Lambda)^2$, 因此,

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \quad (1.42)$$

Lorentz 坐标变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$ 的 Jacobi 行列式为

$$\mathcal{J} = \det \left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} \right] = \det \Lambda, \quad (1.43)$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}| d^4x = |\det \Lambda| d^4x = d^4x. \quad (1.44)$$

可见, Minkowski 时空的体积元是 Lorentz 不变的。

$\det \Lambda$ 的值可以用来为 Lorentz 变换分类: $\det \Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换, $\det \Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外, 由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu} \Lambda^\mu_0 \Lambda^\nu_0 = (\Lambda^0_0)^2 - (\Lambda^i_0)^2, \quad (1.45)$$

则 $(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2 \geq 1$, 故有 $\Lambda^0_0 \geq +1$ 或 $\Lambda^0_0 \leq -1$ 。 $\Lambda^0_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换, $\Lambda^0_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上, 对称性由群论描述。对称变换的集合称为**群**, 群元素具有乘法, 满足下列四个条件。

- (1) 两个群元素的乘积即是两次对称变换相继作用, 乘法满足结合律。
- (2) 群中任意两个元素的乘积仍属于此群 (封闭性)。
- (3) 群中必有一个恒元 (对应于恒等变换), 它与任一元素的乘积仍为此元素。
- (4) 任一元素都可以在群中找到一个逆元 (对应于逆变换), 两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数 (如 β 、 θ 等) 来描述, 因而是一种连续变换, 所以 Lorentz 群是一个连续群, 参数的变化区域称为群空间。Lorentz 群的整个群空间不是连通的, 它有四个连通分支, 如图 1.1 所示, 分别是固有保时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \geq +1$)、固有反时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \leq -1$)、非固有保时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \geq +1$) 和非固有反时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \leq -1$), 四个分支之间彼此不连通。恒元 (即恒等变换) 在固有保时向分支里, 这个分支也称为**固有保时向 Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义宇称 (parity) 变换为

$$\mathcal{P}^\mu_\nu = (\mathcal{P}^{-1})^\mu_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.46)$$

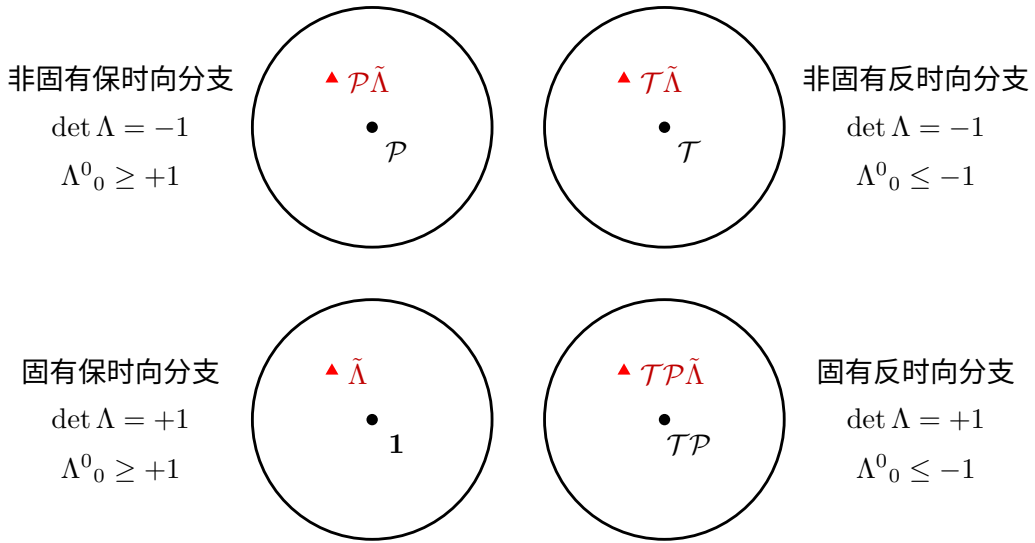


图 1.1: Lorentz 群的四个连通分支示意图。1、 \mathcal{P} 和 \mathcal{T} 分别代表恒等变换、宇称变换和时间反演变换， $\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的，亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \quad (1.47)$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素，乘上宇称变换或（和）时间反演变换，就可以到达 Lorentz 群的其它分支。

1.4 Lorentz 矢量

如果一些 $m \times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同，就可以用这些矩阵来表示这个群，这些矩阵构成了这个群的一个 m 维线性表示。利用群的线性表示，可以将对称变换视作矩阵，将变换作用的态视作列矩阵。

在上一节中，我们已经用矩阵的形式表示过 Lorentz 变换 $\Lambda^\mu{}_\nu$ ，可见， $\Lambda^\mu{}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**，因为 Lorentz 矢量 x^ν 可以看作是变换 $\Lambda^\mu{}_\nu$ 所作用的态。一般地，一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \quad (1.48)$$

类似于 (1.36) 式，逆变矢量 A^μ 对应的协变矢量 $A_\mu = g_{\mu\nu} A^\nu$ 在 Lorentz 变换下满足

$$A_\mu = A_\nu (\Lambda^{-1})^\nu{}_\mu. \quad (1.49)$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \quad (1.50)$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量:

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \quad (1.51)$$

Lorentz 不变量也称为 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负, Lorentz 矢量 A^μ 的自我内积的符号不是确定的, 可以分为三类。

(1) 若 $A^2 > 0$, 则称 A^μ 为**类时**矢量。

(2) 若 $A^2 < 0$, 则称 A^μ 为**类空**矢量。

(3) 若 $A^2 = 0$, 则称 A^μ 为**类光**矢量。

由于 A^2 是 Lorentz 不变量, 不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中, 质点的能量 E 、动量 \mathbf{p} 和 (静止) 质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (1.52)$$

可以用 E 和 \mathbf{p} 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \quad (1.53)$$

称为**四维动量**, 它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \quad (1.54)$$

这是合理的, 因为质量 m 在狭义相对论中是一个 Lorentz 不变量。 p^μ 在 $m > 0$ 时是类时矢量, 在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla \right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla \right) = g^{\mu\nu} \partial_\nu, \quad (1.55)$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho^\nu = g^{\mu\nu}. \quad (1.56)$$

可见, 这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$ 时, 时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu_\nu \partial^\nu. \quad (1.57)$$

由上式、(1.56) 式和保度规条件 (1.39) 可得,

$$\partial'^\mu x'^\nu = \Lambda^\mu_\rho \partial^\rho (\Lambda^\nu_\sigma x^\sigma) = \Lambda^\mu_\rho \Lambda^\nu_\sigma \partial^\rho x^\sigma = \Lambda^\mu_\rho \Lambda^\nu_\sigma g^{\rho\sigma} = g^{\mu\nu}, \quad (1.58)$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的，从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明，时空导数的 Lorentz 变换形式与 Lorentz 矢量相同，因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \quad (1.59)$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu} \partial'^\mu \partial'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma} \partial^\rho \partial^\sigma = \partial^2. \quad (1.60)$$

可见， ∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \quad (1.61)$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 协变性。

1.5 Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p + q$ 阶的 (p, q) 型 **Lorentz 张量** $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}$ 具有 p 个逆变指标和 q 个协变指标，并满足如下 Lorentz 变换规则：

$$T'^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} = \Lambda^{\mu_1}{}_{\rho_1} \cdots \Lambda^{\mu_p}{}_{\rho_p} T^{\rho_1 \cdots \rho_p}{}_{\sigma_1 \cdots \sigma_q} (\Lambda^{-1})^{\sigma_1}{}_{\nu_1} \cdots (\Lambda^{-1})^{\sigma_q}{}_{\nu_q}. \quad (1.62)$$

这里的逆变指标和协变指标统称为 *Lorentz 指标*。Lorentz 标量是 0 阶 Lorentz 张量，不具有 Lorentz 指标；Lorentz 矢量是 1 阶 Lorentz 张量，具有 1 个 Lorentz 指标。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0, 2)$ 型 Lorentz 张量，不过它在任何惯性系中不变，Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62)，可以验证，如下表达式都是 Lorentz 标量 (亦即 Lorentz 不变量)：

$$g_{\mu\nu} T^{\mu\nu}, \quad T^{\mu\nu} A_\mu B_\nu, \quad T^{\mu\nu} T_{\mu\nu}, \quad g_{\mu\sigma} T^{\mu\nu}{}_\rho T^{\sigma\rho}{}_\nu. \quad (1.63)$$

实际上，可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p, q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并，可以得到一个 $(p-1, q-1)$ 型 Lorentz 张量。例如，由

$$T'^{\mu\nu}{}_\mu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma (\Lambda^{-1})^\gamma{}_\mu = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma \delta^\gamma{}_\alpha = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\alpha \quad (1.64)$$

可知， $T^{\mu\nu}{}_\mu$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita** 符号

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换,} \\ 0, & \text{其它情况.} \end{cases} \quad (1.65)$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是全反对称的, 即关于任意两个指标反对称, 如 $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\rho\nu\mu\sigma} = -\varepsilon^{\sigma\nu\rho\mu}$ 。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.66)$$

$\varepsilon_{\mu\nu\rho\sigma}$ 也是全反对称的, 如

$$\varepsilon_{\nu\mu\rho\sigma} = g_{\nu\alpha}g_{\mu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta} = g_{\mu\beta}g_{\nu\alpha}g_{\rho\gamma}g_{\sigma\delta}(-\varepsilon^{\beta\alpha\gamma\delta}) = -\varepsilon_{\mu\nu\rho\sigma}. \quad (1.67)$$

根据这些定义,

$$\varepsilon^{0123} = +1, \quad \varepsilon_{0123} = -1. \quad (1.68)$$

从而,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 4!\varepsilon^{0123}\varepsilon_{0123} = -4!. \quad (1.69)$$

利用 Levi-Civita 符号可以把 $\det \Lambda$ 按照行列式定义写成

$$\det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.70)$$

对于固有 Lorentz 变换, $\det \Lambda = +1$, 有

$$\varepsilon^{0123} = \varepsilon^{0123}\det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.71)$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质, 可得

$$\varepsilon^{1023} = -\varepsilon^{0123} = -\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = \Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\beta\alpha\gamma\delta}. \quad (1.72)$$

依此类推, 可以证明

$$\varepsilon^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.73)$$

可见, 在固有 Lorentz 变换下, $\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量, 不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中, ρ 是电荷密度, \mathbf{J} 是电流密度, 它们可以组成一个 Lorentz 矢量 $J^{\mu} = (\rho, \mathbf{J})$, 从而, 电流连续性方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (1.74)$$

可以写成 Lorentz 协变的形式

$$\partial_{\mu}J^{\mu} = 0. \quad (1.75)$$

此外, 电场强度 \mathbf{E} 和磁感应强度 \mathbf{B} 可以用电势 Φ 和矢势 \mathbf{A} 表达为

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1.76)$$

这样, 方程

$$\nabla \cdot \mathbf{B} = 0 \quad (1.77)$$

是自动满足的。 Φ 和 \mathbf{A} 可以组成一个 Lorentz 矢量 $A^\mu = (\Phi, \mathbf{A})$, 称为四维矢势, 则 (1.76) 式的分量形式为

$$E^i = -\partial_i A^0 - \partial_0 A^i, \quad B^k = \varepsilon^{kij} \partial_i A^j, \quad i, j, k = 1, 2, 3. \quad (1.78)$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk} \equiv \varepsilon^{0ijk}, \quad (1.79)$$

因而 $\varepsilon^{123} = +1$ 。

引入电磁场的场强张量 (field strength tensor)

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (1.80)$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序, 从上述定义可得

$$\begin{aligned} \partial^\rho F^{\mu\nu} &= \partial^\rho (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu \partial^\rho A^\nu - \partial^\mu \partial^\nu A^\rho + \partial^\nu \partial^\mu A^\rho - \partial^\nu \partial^\rho A^\mu \\ &= \partial^\mu F^{\rho\nu} + \partial^\nu F^{\mu\rho} = -\partial^\mu F^{\nu\rho} - \partial^\nu F^{\rho\mu}, \end{aligned} \quad (1.81)$$

即

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \quad (1.82)$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = -E^i, \quad (1.83)$$

可见, F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k} \varepsilon^{12k} = \varepsilon^{123} \varepsilon^{123} = 1$ 和 $\varepsilon^{12k} \varepsilon^{21k} = \varepsilon^{123} \varepsilon^{213} = -1$, 依此类推, 可以归纳出如下求和关系:

$$\varepsilon^{ijk} \varepsilon^{kmn} = \varepsilon^{ijk} \varepsilon^{mnk} = \delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}, \quad (1.84)$$

利用这个关系, 可得

$$\varepsilon^{ijk} B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n = \delta^{im} \delta^{jn} \partial_m A^n - \delta^{in} \delta^{jm} \partial_m A^n = \partial_i A^j - \partial_j A^i, \quad (1.85)$$

从而,

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\varepsilon^{ijk} B^k, \quad (1.86)$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \quad (1.87)$$

Gauss 定律对应的方程

$$\nabla \cdot \mathbf{E} = \rho \quad (1.88)$$

等价于

$$J^0 = \rho = \partial_i E^i = -\partial_i F^{0i} = \partial_i F^{i0} = \partial_i F^{i0} + \partial_0 F^{00} = \partial_\mu F^{\mu 0}, \quad (1.89)$$

而 Ampère 定律对应的方程

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \quad (1.90)$$

等价于

$$J^i = \varepsilon^{ijk} \partial_j B^k - \partial_0 E^i = -\partial_j F^{ij} + \partial_0 F^{0i} = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \quad (1.91)$$

归纳起来, 有

$$\partial_\mu F^{\mu\nu} = J^\nu. \quad (1.92)$$

这个方程完全是用 Lorentz 张量写出来的, 它在不同惯性系中具有相同的形式, 即具有 **Lorentz 协变性**, 因而满足狭义相对性原理。

现在, Maxwell 方程组中还有一个方程没有讨论, 它是 Maxwell-Faraday 方程

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.93)$$

将它写成分量的形式, 得

$$\varepsilon^{kmn} \partial_m E^n = -\varepsilon^{kmn} \partial_m F^{0n} = \varepsilon^{kmn} \partial_m F^{n0} = -\partial_0 B^k, \quad (1.94)$$

从而

$$\partial_0 F^{ij} = -\varepsilon^{ijk} \partial_0 B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m F^{n0} = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial_m F^{n0} = \partial_i F^{j0} - \partial_j F^{i0}, \quad (1.95)$$

即

$$\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = 0. \quad (1.96)$$

这个方程与 Maxwell-Faraday 方程等价, 不过, 它只是前面得到的方程 (1.82) 取特定分量的形式。

利用四维 Levi-Civita 符号, 可以定义电磁场的对偶场强张量 (dual field strength tensor)

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (1.97)$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk}\varepsilon^{1jk} = \varepsilon^{123}\varepsilon^{123} + \varepsilon^{132}\varepsilon^{132} = 2$ 和 $\varepsilon^{1jk}\varepsilon^{2jk} = \varepsilon^{123}\varepsilon^{223} + \varepsilon^{132}\varepsilon^{232} = 0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk}\varepsilon^{ljk} = 2\delta^{il}, \quad (1.98)$$

利用这个关系, 可得

$$\begin{aligned} \tilde{F}^{0i} &= \frac{1}{2}\varepsilon^{0i\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\varepsilon^{0ijk}g_{j\mu}g_{k\nu}F^{\mu\nu} = \frac{1}{2}\varepsilon^{0ijk}g_{jm}g_{kn}F^{mn} = -\frac{1}{2}\varepsilon^{ijk}\delta^{jm}\delta^{kn}\varepsilon^{mnl}B^l \\ &= -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{jkl}B^l = -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}B^l = -\frac{1}{2}2\delta^{il}B^l = -B^i, \end{aligned} \quad (1.99)$$

故 \tilde{F}^{0i} 对应于磁感应强度。另一方面,

$$\begin{aligned} \tilde{F}^{ij} &= \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(\varepsilon^{ij0k}F_{0k} + \varepsilon^{ijk0}F_{k0}) = \varepsilon^{0ijk}F_{0k} = \varepsilon^{0ijk}g_{0\mu}g_{k\nu}F^{\mu\nu} \\ &= \varepsilon^{ijk}g_{00}g_{kl}F^{0l} = -\varepsilon^{ijk}\delta^{kl}F^{0l} = -\varepsilon^{ijk}F^{0k} = \varepsilon^{ijk}E^k, \end{aligned} \quad (1.100)$$

说明 \tilde{F}^{ij} 对应于电场强度。 $\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \quad (1.101)$$

由 $\tilde{F}^{\mu\nu}$ 的定义, 有

$$\begin{aligned} \partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\sigma\mu\rho}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\rho\sigma\mu}\partial_\mu F_{\rho\sigma}) \\ &= -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\mu\rho\sigma}\partial_\rho F_{\sigma\mu} + \varepsilon^{\nu\mu\rho\sigma}\partial_\sigma F_{\mu\rho}) = -\frac{1}{6}\varepsilon^{\nu\mu\rho\sigma}(\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho}), \end{aligned} \quad (1.102)$$

因此, 方程 (1.82) 等价于

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \quad (1.103)$$

从这些讨论可以看到, 用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的, 而且方程的 Lorentz 协变性非常明确。

1.6 作用量原理

1.6.1 经典力学中的作用量原理

在经典力学中, 质点力学系统可以用拉格朗日量 (Lagrangian) 描述。对于具有 n 个自由度的系统, 可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i , 它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i = dq_i/dt$ 。拉格朗日量是广义坐标和广义速度的函数 $L(q_i, \dot{q}_i)$ 。拉格朗日量的时间积分

$$S = \int_{t_1}^{t_2} dt L[q_i(t), \dot{q}_i(t)] \quad (1.104)$$

称为作用量。

作用量原理指出，作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零，则有

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i, \quad (1.105)$$

即时间导数的变分等于变分的时间导数。从而可得

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L[q_i(t), \dot{q}_i(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right) \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] \\ &= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \left. \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2}, \end{aligned} \quad (1.106)$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零，即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$ ，则上式最后一行第二项为零，而 $\delta S = 0$ 等价于

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \quad (1.107)$$

这是 **Euler-Lagrange 方程**，它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \quad (1.108)$$

反解上式表示的 n 个方程，则可以用 q_i 和 p_i 将 \dot{q}_i 表达出来，然后用 Legendre 变换定义**哈密顿量** (Hamiltonian)

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L, \quad (1.109)$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量，变分为

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L = \int_{t_1}^{t_2} dt \delta (p_i \dot{q}_i - H) = \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \frac{d}{dt} \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left[\dot{q}_i \delta p_i + \frac{d}{dt} (p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] \\ &= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] + p_i \delta q_i \Big|_{t_1}^{t_2}. \end{aligned} \quad (1.110)$$

根据前面的假设，上式最后一行第二项为零，于是， $\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n. \quad (1.111)$$

这是 **Hamilton 正则运动方程**，相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.107)。

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中，场 $\phi(\mathbf{x}, t)$ 是系统的广义坐标，每一个空间点 \mathbf{x} 都是一个自由度，因此场论相当于具有无穷多自由度的质点力学。在局域场论中，拉格朗日量 $L = \int d^3x \mathcal{L}(x)$ ，其中 $\mathcal{L}(x)$ 称为拉格朗日量密度（下文将它简称为拉氏量）。 \mathcal{L} 是系统中 n 个场 $\phi_a(\mathbf{x}, t)$ ($a = 1, \dots, n$) 及其时空导数 $\partial_\mu \phi_a$ 的函数。现在，作用量可以表达为

$$S = \int dt L = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \quad (1.112)$$

(1.44) 式告诉我们，时空体积元 d^4x 是 Lorentz 不变的，如果拉氏量 \mathcal{L} 也是 Lorentz 不变的，则作用量 S 就是 Lorentz 不变的，从而，由作用量原理得到的运动方程满足狭义相对性原理。因此，构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 \mathcal{L} ，即要求 \mathcal{L} 是一个 Lorentz 标量。

类似于前面质点力学的处理方式，假设时空坐标的变分为零，则对场的时空导数的变分等于场变分的时空导数，即

$$\delta(\partial_\mu \phi_a) = \partial_\mu(\delta\phi_a). \quad (1.113)$$

于是，利用分部积分可得

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu(\delta\phi_a) \right] \\ &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right] - \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] \delta\phi_a \right\} \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] \delta\phi_a + \int d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right]. \end{aligned} \quad (1.114)$$

上式最后一行第二项的积分项是关于时空坐标的全散度，利用 Stokes 定理，可以将它转化为积分区域边界面 \mathcal{S} 上的积分：

$$\int d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right] = \int_{\mathcal{S}} d\mathcal{S}_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a, \quad (1.115)$$

其中 $d\mathcal{S}_\mu$ 是 \mathcal{S} 上的面元。进一步假设在边界面 \mathcal{S} 上 $\delta\phi_a = 0$ ，则上式为零。我们通常讨论整个时空区域上的场，从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零，是很合理的。这样一来， $\delta S = 0$ 给出

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \quad (1.116)$$

这就是场的 *Euler-Lagrange* 方程，它给出场的经典运动方程。

引入场的共轭动量密度 (conjugate momentum density)

$$\pi_a(\mathbf{x}, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}, \quad (1.117)$$

则可以用 Legendre 变换将哈密顿量定义为

$$H \equiv \int d^3x \pi_a \dot{\phi}_a - L \equiv \int d^3x \mathcal{H}, \quad (1.118)$$

其中，哈密顿量密度

$$\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a \dot{\phi}_a - \mathcal{L}. \quad (1.119)$$

作用量变分为

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} = \int d^4x \delta(\pi_a \dot{\phi}_a - \mathcal{H}) \\ &= \int d^4x \left[\dot{\phi}_a \delta \pi_a + \pi_a \delta \dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a - \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \cdot \delta(\nabla\phi_a) \right] \\ &= \int d^4x \left[\dot{\phi}_a \delta \pi_a + \pi_a \frac{d}{dt} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a - \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \cdot \nabla(\delta \phi_a) \right] \\ &= \int d^4x \left\{ \dot{\phi}_a \delta \pi_a + \frac{d}{dt}(\pi_a \delta \phi_a) - \dot{\pi}_a \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a \right. \\ &\quad \left. - \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \delta \phi_a \right] + \left[\nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \right] \delta \phi_a \right\} \\ &= \int d^4x \left\{ \left(\dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \right) \delta \pi_a - \left[\dot{\pi}_a + \frac{\partial \mathcal{H}}{\partial \phi_a} - \nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \right] \delta \phi_a \right\} \\ &\quad + \int d^4x \frac{d}{dt}(\pi_a \delta \phi_a) - \int d^4x \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \delta \phi_a \right]. \end{aligned} \quad (1.120)$$

与前面一样，假设在时空区域边界面上 $\delta \phi_a = 0$ ，则上式最后两行的两项均为零，于是， $\delta S = 0$ 给出场的正则运动方程

$$\dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi_a}, \quad \dot{\pi}_a = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)}. \quad (1.121)$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述，则它是一种连续变换，连续变换对应的对称性称为连续对称性。**Noether 定理**指出，如果一个系统具有某种不显含时间的连续对称性，就必然存在一种对应的守恒定律。Noether 定理首先是在经典物理中给出的，但实际上它对所有物理行为由作用量原理决定的系统都成立。因此，可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$S = \int_R d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \quad (1.122)$$

考虑一个连续变换，使得

$$\phi_a(x) \rightarrow \phi'_a(x'), \quad (1.123)$$

其中已包含了坐标的变换

$$x^\mu \rightarrow x'^\mu, \quad (1.124)$$

它引起的拉氏量变换为

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x'). \quad (1.125)$$

记这个变换的无穷小变换形式为

$$\phi'_a(x') = \phi_a(x) + \delta\phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}, \quad (1.126)$$

如果在此变换下

$$\delta S = \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) = 0, \quad (1.127)$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4x' = |\mathcal{J}| d^4x, \quad \mathcal{J} = \det \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) \simeq \det \left[\delta^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu} \right], \quad (1.128)$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 \mathbf{A} 满足 $\det(\mathbf{A}) \ll 1$ ，则有如下表达式：

$$\det(\mathbf{1} + \mathbf{A}) \simeq 1 + \text{tr}(\mathbf{A}). \quad (1.129)$$

利用上式可以将 Jacobi 行列式 \mathcal{J} 化为

$$\mathcal{J} \simeq 1 + \text{tr} \left[\frac{\partial(\delta x^\mu)}{\partial x^\nu} \right] = 1 + \partial_\mu(\delta x^\mu), \quad (1.130)$$

从而，体积元的无穷小变换形式为

$$d^4x' \simeq [1 + \partial_\mu(\delta x^\mu)] d^4x. \quad (1.131)$$

作用量在此无穷小变换下的变分为

$$\begin{aligned} \delta S &= \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) \\ &= \int_{R'} d^4x' \mathcal{L}'(x') - \int_R d^4x \mathcal{L}'(x') + \int_R d^4x \mathcal{L}'(x') - \int_R d^4x \mathcal{L}(x) \\ &\simeq \int_R d^4x [1 + \partial_\mu(\delta x^\mu)] \mathcal{L}'(x') - \int_R d^4x \mathcal{L}'(x') + \int_R d^4x \delta\mathcal{L} \\ &\simeq \int_R d^4x \mathcal{L}'(x') \partial_\mu(\delta x^\mu) + \int_R d^4x \delta\mathcal{L} \simeq \int_R d^4x [\delta\mathcal{L} + \mathcal{L}(x) \partial_\mu(\delta x^\mu)] \\ &= \int_R d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta(\partial_\mu\phi_a) + \mathcal{L} \partial_\mu(\delta x^\mu) \right]. \end{aligned} \quad (1.132)$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$ ，使得

$$\bar{\delta}\phi_a(x) = \phi'_a(x) - \phi_a(x). \quad (1.133)$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu\phi_a) = \partial_\mu(\bar{\delta}\phi_a), \quad (1.134)$$

δ 算符则不能。 $\delta\phi_a$ 与 $\bar{\delta}\phi_a$ 的关系为

$$\begin{aligned}\delta\phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta}\phi_a \\ &\simeq \bar{\delta}\phi_a + (\partial_\mu\phi'_a)\delta x^\mu \simeq \bar{\delta}\phi + (\partial_\mu\phi_a)\delta x^\mu,\end{aligned}\quad (1.135)$$

即

$$\bar{\delta}\phi = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu. \quad (1.136)$$

同理,

$$\delta(\partial_\mu\phi_a) = \bar{\delta}(\partial_\mu\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu = \partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu. \quad (1.137)$$

将 (1.135) 和 (1.137) 式代入 (1.132) 式, 得到

$$\begin{aligned}\delta S &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} [\bar{\delta}\phi_a + (\partial_\mu\phi_a)\delta x^\mu] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} [\partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu] + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} \bar{\delta}\phi_a + \frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a \right) - \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) \bar{\delta}\phi_a \right. \\ &\quad \left. + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \bar{\delta}\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a \right] \right. \\ &\quad \left. + \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu) \right] \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \bar{\delta}\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu \right] \right\}. \quad (1.138)\end{aligned}$$

第二步用到分部积分, 最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu) = \frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu). \quad (1.139)$$

根据 Euler-Lagrange 方程 (1.116), (1.138) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的, $\delta S = 0$ 等价于第二项为零, 即

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu \right] = 0. \quad (1.140)$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu, \quad (1.141)$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \quad (1.142)$$

方程 (1.142) 左边对整个三维空间积分, 运用 Stokes 定理, 得

$$\int d^3x \partial_\mu j^\mu = \int d^3x \partial_0 j^0 + \int d^3x \partial_i j^i = \frac{d}{dt} \int d^3x j^0 + \int_S d\mathcal{S}_i j^i, \quad (1.143)$$

其中 $i = 1, 2, 3$ 。对于整个三维空间而言，边界面 \mathcal{S} 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失，从而，在无穷远处 $j^i \rightarrow 0$ ，所以上式最后一项为零。定义守恒荷 (conserved charge)

$$Q \equiv \int d^3x j^0, \quad (1.144)$$

则由 (1.143) 和 (1.142) 式可得

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x j^0 = \int d^3x \partial_\mu j^\mu = 0. \quad (1.145)$$

可见， Q 不随时间变化，是守恒的。

综上，在场论中，如果一个系统具有某种连续对称性，则存在相应的守恒流 (1.141)，它满足守恒流方程 (1.142)，而守恒荷 (1.144) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2 时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^\mu = x^\mu - \varepsilon^\mu, \quad (1.146)$$

其中 ε^μ 是常数。要求场 ϕ_a 具有时空平移对称性，则

$$\phi'_a(x') = \phi'_a(x - \varepsilon) = \phi_a(x). \quad (1.147)$$

现在， $\delta x^\mu = -\varepsilon^\mu$ ，由 (1.136) 式可得

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) + \varepsilon^\mu\partial_\mu\phi_a = 0 + \varepsilon^\mu\partial_\mu\phi_a = \varepsilon^\rho\partial_\rho\phi_a, \quad (1.148)$$

代入到 Noether 守恒流表达式 (1.141)，得

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^\rho\partial_\rho\phi_a - \mathcal{L}\varepsilon^\mu = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] \varepsilon^\rho. \quad (1.149)$$

从而， $\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] = 0, \quad (1.150)$$

各项乘以 $g^{\rho\nu}$ ，缩并，得

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L} \right] = 0. \quad (1.151)$$

上式方括号部分是场的能动张量 (energy-momentum tensor)

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}, \quad (1.152)$$

它满足

$$\partial_\mu T^{\mu\nu} = 0. \quad (1.153)$$

因此，对 $T^{0\nu}$ ($\nu = 0, 1, 2, 3$) 作全空间积分，就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^0 \phi_a - \mathcal{L}, \quad (1.154)$$

与 (1.119) 和 (1.117) 式比较，可以看出 T^{00} 就是哈密顿量密度 \mathcal{H} 。 T^{00} 的全空间积分

$$H = \int d^3x T^{00} = \int d^3x \mathcal{H} \quad (1.155)$$

是场的哈密顿量，或者说总能量。 $T^{\mu\nu}$ 的 0i 分量

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^i \phi_a = \pi_a \partial^i \phi_a \quad (1.156)$$

是场的动量密度，它的全空间积分

$$P^i = \int d^3x T^{0i} = \int d^3x \pi_a \partial^i \phi_a \quad (1.157)$$

是场的总动量。根据 (1.55) 式，上式也可以写成

$$\mathbf{P} = - \int d^3x \pi_a \nabla \phi_a. \quad (1.158)$$

H 和 P^i 都是守恒荷，可见，时间平移对称性对应于能量守恒定律，空间平移对称性对应于动量守恒定律。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad (1.159)$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30)，有

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\mu\nu} (\delta^\mu{}_\alpha + \omega^\mu{}_\alpha) (\delta^\nu{}_\beta + \omega^\nu{}_\beta) \simeq g_{\mu\nu} \delta^\mu{}_\alpha \delta^\nu{}_\beta + g_{\mu\nu} \delta^\mu{}_\alpha \omega^\nu{}_\beta + g_{\mu\nu} \omega^\mu{}_\alpha \delta^\nu{}_\beta \\ &= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{aligned} \quad (1.160)$$

可见，

$$\omega_{\mu\nu} \equiv g_{\mu\rho} \omega^\rho{}_\nu \quad (1.161)$$

关于两个指标反对称：

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (1.162)$$

因此， $\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31)，利用三角函数展开式 $\cos\theta = 1 + \mathcal{O}(\theta^2)$ 和 $\sin\theta = \theta + \mathcal{O}(\theta^3)$ ，可得

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}. \quad (1.163)$$

对于沿 x 的增速变换 (1.28)，可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1}\beta, \quad (1.164)$$

再利用双曲函数公式 $\tanh\xi = \sinh\xi/\cosh\xi$ 和 $\cosh^2\xi - \sinh^2\xi = 1$ 得

$$\begin{aligned} \gamma &= (1 - \beta^2)^{-1/2} = (1 - \tanh^2\xi)^{-1/2} = \left(\frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi} \right)^{-1/2} = \cosh\xi, \\ \beta\gamma &= \tanh\xi \cosh\xi = \sinh\xi, \end{aligned} \quad (1.165)$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\xi & -\sinh\xi & & \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (1.166)$$

根据双曲函数展开式 $\cosh\xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh\xi = \xi + \mathcal{O}(\xi^3)$ ，有

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \quad (1.167)$$

在无穷小 Lorentz 变换 (1.159) 的作用下，一般地，场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab} \right] \phi_b(x) = \phi_a(x) - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x), \quad (1.168)$$

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的生成元 (generator)。由于 $\omega_{\mu\nu}$ 是反对称的，有

$$\omega_{\mu\nu}(I^{\mu\nu})_{ab} = \omega_{\nu\mu}(I^{\nu\mu})_{ab} = -\omega_{\mu\nu}(I^{\nu\mu})_{ab}, \quad (1.169)$$

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称：

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}. \quad (1.170)$$

现在， $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ ，而

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu\phi_a)\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}(I^{\nu\rho})_{ab}\phi_b - (\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho, \quad (1.171)$$

故 Noether 流为

$$\begin{aligned}
 j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu = -\frac{i}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (I^{\nu\rho})_{ab} \phi_b - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (\partial_\nu \phi_a) \omega^\nu{}_\rho x^\rho + \mathcal{L} \omega^\mu{}_\rho x^\rho \\
 &= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (\partial_\nu \phi_a) - \delta^\mu{}_\nu \mathcal{L} \right] \omega^\nu{}_\rho x^\rho \\
 &= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - T^\mu{}_\nu \omega^\nu{}_\rho x^\rho,
 \end{aligned} \tag{1.172}$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho} g_{\rho\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L} \tag{1.173}$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作：

$$T^\mu{}_\nu \omega^\nu{}_\rho = T^\mu{}_\nu \delta^\nu{}_\sigma \omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha} g_{\alpha\sigma} \omega^\sigma{}_\rho = T^{\mu\alpha} \omega_{\alpha\rho} = T^{\mu\nu} \omega_{\nu\rho}, \tag{1.174}$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned}
 T^\mu{}_\nu \omega^\nu{}_\rho x^\rho &= T^{\mu\nu} \omega_{\nu\rho} x^\rho = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\nu} \omega_{\rho\nu} x^\rho) = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\rho} \omega_{\nu\rho} x^\nu) \\
 &= \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu).
 \end{aligned} \tag{1.175}$$

于是，Noether 流 (1.172) 可化为

$$j^\mu = \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) = \frac{1}{2} J^{\mu\nu\rho} \omega_{\nu\rho} \tag{1.176}$$

其中

$$J^{\mu\nu\rho} \equiv T^{\mu\rho} x^\nu - T^{\mu\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b. \tag{1.177}$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu J^{\mu\nu\rho} = 0, \tag{1.178}$$

守恒荷为

$$J^{\nu\rho} \equiv \int d^3x J^{0\nu\rho} = \int d^3x \left[T^{0\rho} x^\nu - T^{0\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b \right]. \tag{1.179}$$

易见 $J^{\nu\rho} = -J^{\rho\nu}$ ，因而一共有 6 个独立的守恒荷，满足 $dJ^{\nu\rho}/dt = 0$ 。

为明确物理含义，可将 $J^{\nu\rho}$ 分解成两项：

$$J^{\nu\rho} = L^{\nu\rho} + S^{\nu\rho}. \tag{1.180}$$

第一项为

$$\begin{aligned}
 L^{\nu\rho} &\equiv \int d^3x (T^{0\rho} x^\nu - T^{0\nu} x^\rho) \\
 &= \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^\rho \phi_a - g^{0\rho} \mathcal{L} \right) x^\nu - \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^\nu \phi_a - g^{0\nu} \mathcal{L} \right) x^\rho \right]
 \end{aligned}$$

$$\begin{aligned}
&= \int d^3x [(\pi_a \partial^\rho \phi_a - g^{0\rho} \mathcal{L})x^\nu - (\pi_a \partial^\nu \phi_a - g^{0\nu} \mathcal{L})x^\rho] \\
&= \int d^3x [\pi_a (x^\nu \partial^\rho - x^\rho \partial^\nu) \phi_a + (g^{0\nu} x^\rho - g^{0\rho} x^\nu) \mathcal{L}].
\end{aligned} \tag{1.181}$$

它的纯空间分量 L^{jk} 中只有 3 个是独立的，可以等价地定义成

$$L^i \equiv \frac{1}{2} \varepsilon^{ijk} L^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (x^j \partial^k - x^k \partial^j) \phi_a, \tag{1.182}$$

这是场的**轨道角动量**。第二项为

$$S^{\nu\rho} \equiv \int d^3x \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b = \int d^3x \pi_a (-i I^{\nu\rho})_{ab} \phi_b, \tag{1.183}$$

同样，3 个独立的等价纯空间分量是

$$S^i \equiv \frac{1}{2} \varepsilon^{ijk} S^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (-i I^{jk})_{ab} \phi_b, \tag{1.184}$$

这是场的**自旋角动量**。因此， $J^{\nu\rho}$ 的纯空间分量等价于

$$J^i \equiv \frac{1}{2} \varepsilon^{ijk} J^{jk} = L^i + S^i, \tag{1.185}$$

这是场的**总角动量**。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 $SO(3)$ ，而空间旋转对称性对应于**角动量守恒定律**。

另一方面， $L^{\nu\rho}$ 的 $i0$ 分量为

$$L^{i0} = \int d^3x (T^{00} x^i - T^{0i} x^0) = \int d^3x (x^i \mathcal{H} - x^0 \pi_a \partial^i \phi_a) = \int d^3x x^i \mathcal{H} - t P^i. \tag{1.186}$$

若 $dS^{i0}/dt = 0$ ，则有 $dL^{i0}/dt = 0$ ，从而

$$L^{i0}(t) = L^{i0}|_{t=0} = \int d^3x x^i \mathcal{H}(t=0), \tag{1.187}$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下，能量密度相当于质量密度，则 L^{i0} 是 $t=0$ 时刻的质心（即质量中心，center of mass）。 L^{i0} 的守恒在经典力学中对应于**质心运动守恒定律**：当没有外力存在时，质心的加速度为零，质心保持静止或作匀速直线运动。

1.7.4 U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu \phi^*) \partial_\mu \phi - m^2 \phi^* \phi. \tag{1.188}$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \tag{1.189}$$

其中 θ 是不依赖于 x^μ 的连续变换实参数, q 是一个常数。这里不包含坐标的变换。 $e^{iq\theta}$ 是个纯相位因子, 可以看成是一个 1 维幺正 (unitary) 矩阵, 形式为 $e^{iq\theta}$ 的所有变换组成的群称为 **U(1) 群**。**整体** (global) 指的是变换参数不依赖于时空坐标。相应地, ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta} \phi^*(x). \quad (1.190)$$

容易看出, 由 (1.188) 式定义的 \mathcal{L} 在这种变换下不变, 即具有 U(1) 整体对称性。与前面叙述的两种对称性不同, 这里的对称性出现在由场组成的抽象空间中, 与时间和空间相对独立 ($\delta x^\mu = 0$), 因而是一种**内部对称性**。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \quad (1.191)$$

结合 $\delta x^\mu = 0$, 有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \quad (1.192)$$

于是, Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \bar{\delta}\phi^* = \partial^\mu \phi^* (iq\theta\phi) + \partial^\mu \phi (-iq\theta\phi^*) \\ &= iq\theta [(\partial^\mu \phi^*)\phi - (\partial^\mu \phi)\phi^*] = -q\theta \phi^* i \overleftrightarrow{\partial}^\mu \phi, \end{aligned} \quad (1.193)$$

其中, $\overleftrightarrow{\partial}^\mu$ 符号通过下式定义:

$$\phi^* \overleftrightarrow{\partial}^\mu \phi \equiv \phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi. \quad (1.194)$$

扔掉无穷小参数 $-\theta$, 定义

$$J^\mu \equiv q\phi^* i \overleftrightarrow{\partial}^\mu \phi, \quad (1.195)$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$, 相应的守恒荷为

$$Q = \int d^3x J^0 = q \int d^3x \phi^* i \overleftrightarrow{\partial}^0 \phi. \quad (1.196)$$

在实际情况下, q 是由 ϕ 场描述的粒子所携带的某种荷, 如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此, 一种 U(1) 整体对称性对应于一条荷数守恒定律, 比如, 电磁 U(1) 整体对称性就对应于**电荷守恒定律**。

第 2 章 标量场

本章讲述标量场的**正则量子化** (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1 简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \quad (2.1)$$

其中 m 是质量， ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 看作厄米算符，满足**正则对易关系**

$$[x, p] = xp - px = i. \quad (2.2)$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \quad (2.3)$$

a 称为**湮灭算符** (annihilation operator)， a^\dagger 称为**产生算符** (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\ &= \frac{1}{2}(-i[x, p] + i[p, x]) = -i[x, p] = 1. \end{aligned} \quad (2.4)$$

根据 (2.3) 式，可以反过来用 a 和 a^\dagger 表示 x 和 p ：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \quad (2.5)$$

从而，哈密顿量表示成

$$\begin{aligned} H &= -\frac{1}{2m} \frac{m\omega}{2}(a - a^\dagger)^2 + \frac{1}{2}m\omega^2 \frac{1}{2m\omega}(a + a^\dagger)^2 \\ &= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a). \end{aligned} \quad (2.6)$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$, 于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega \left(a^\dagger a + \frac{1}{2} \right) = \omega \left(N + \frac{1}{2} \right), \quad (2.7)$$

其中, $N \equiv a^\dagger a$ 是个厄米算符, 称为**粒子数算符**. N 还是个**正定算符**, 对于任意量子态 $|\psi\rangle$, N 的期待值 (expectation value) 非负:

$$\langle \psi | N | \psi \rangle = \langle \psi | a^\dagger a | \psi \rangle = \langle a\psi | a\psi \rangle \geq 0. \quad (2.8)$$

设 $|n\rangle$ 是 N 的本征态, 满足本征方程

$$N |n\rangle = n |n\rangle. \quad (2.9)$$

由 $n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle \geq 0$ 可知, n 是个非负实数. 利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B, \quad (2.10)$$

$$[A, BC] = ABC - BAC + BAC - BCA = [A, B]C + B[A, C], \quad (2.11)$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \quad (2.12)$$

从而, 有

$$Na^\dagger |n\rangle = ([N, a^\dagger] + a^\dagger N) |n\rangle = (a^\dagger + a^\dagger n) |n\rangle = (n+1)a^\dagger |n\rangle, \quad (2.13)$$

$$Na |n\rangle = ([N, a] + aN) |n\rangle = (-a + an) |n\rangle = (n-1)a |n\rangle. \quad (2.14)$$

可见, $a^\dagger |n\rangle$ 和 $a |n\rangle$ 都是 N 的本征态, 本征值分别为 $n+1$ 和 $n-1$, 也就是说,

$$a^\dagger |n\rangle = c_1 |n+1\rangle, \quad a |n\rangle = c_2 |n-1\rangle, \quad (2.15)$$

其中 c_1 和 c_2 是两个归一化常数. a^\dagger 将本征值为 n 的态变成本征值为 $n+1$ 的态, 因而也称为**升算符** (raising operator); a 将本征值为 n 的态变成本征值为 $n-1$ 的态, 因而也称为**降算符** (lowering operator). 为确定归一化常数的值, 可作如下计算:

$$n+1 = \langle n | (N+1) | n \rangle = \langle n | (a^\dagger a + 1) | n \rangle = \langle n | aa^\dagger | n \rangle = |c_1|^2 \langle n+1 | n+1 \rangle = |c_1|^2, \quad (2.16)$$

$$n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle = |c_2|^2 \langle n-1 | n-1 \rangle = |c_2|^2. \quad (2.17)$$

将 c_1 和 c_2 都取为实数, 则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$, 故

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \quad (2.18)$$

从 N 的某个本征态 $|n\rangle$ 出发, 用降算符 a 逐步操作, 可得本征值逐次减小的一系列本征态

$$a |n\rangle, a^2 |n\rangle, a^3 |n\rangle, \dots, \quad (2.19)$$

本征值分别为

$$n-1, n-2, n-3, \dots \quad (2.20)$$

由于 $n \geq 0$, 必定存在一个最小本征值 n_0 , 它的本征态 $|n_0\rangle$ 满足

$$a|n_0\rangle = 0. \quad (2.21)$$

于是, 有

$$N|n_0\rangle = a^\dagger a|n_0\rangle = 0 = 0|n_0\rangle, \quad (2.22)$$

可见, $n_0 = 0$, 即

$$|n_0\rangle = |0\rangle. \quad (2.23)$$

反过来, 从 $|0\rangle$ 出发, 用升算符 a^\dagger 逐步操作, 可得本征值逐次增加的一系列本征态

$$a^\dagger|0\rangle, (a^\dagger)^2|0\rangle, (a^\dagger)^3|0\rangle, \dots, \quad (2.24)$$

本征值分别为

$$1, 2, 3, \dots \quad (2.25)$$

综上, 本征值 n 的取值是非负整数, 是量子化的; 本征态 $|n\rangle$ 可以用 a^\dagger 和 $|0\rangle$ 表示为

$$|n\rangle = c_3(a^\dagger)^n|0\rangle. \quad (2.26)$$

为确定归一化常数 c_3 , 可作如下运算:

$$\begin{aligned} \langle n|n\rangle &= |c_3|^2 \langle 0|a^n(a^\dagger)^n|0\rangle = |c_3|^2 \langle 1|a^{n-1}(a^\dagger)^{n-1}|1\rangle = 1 \cdot 2 |c_3|^2 \langle 2|a^{n-2}(a^\dagger)^{n-2}|2\rangle = \dots \\ &= (n-1)! |c_3|^2 \langle n-1|aa^\dagger|n-1\rangle = n! |c_3|^2 \langle n|n\rangle, \end{aligned} \quad (2.27)$$

故 $|c_3|^2 = 1/n!$ 。取 c_3 为实数, 可得 $c_3 = 1/\sqrt{n!}$, 于是

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle. \quad (2.28)$$

从 (2.7) 式容易看出, $|n\rangle$ 也是 H 的本征态:

$$H|n\rangle = \omega \left(N + \frac{1}{2} \right) |n\rangle = \omega \left(n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle, \quad (2.29)$$

相应的能量本征值为

$$E_n = \omega \left(n + \frac{1}{2} \right). \quad (2.30)$$

基态 $|0\rangle$ 的能量本征值不是零, 而是 $E_0 = \omega/2$, 称为零点能 (zero-point energy), 这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态, 将 $n > 0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态, 每个声子具有一份能量 ω 。这样一来, n 表示声子的数目, 故粒子数算符 N 描述的是声子数。 a^\dagger 的作用是产生一个声子, 从而增加一份能量; a 的作用是湮灭一个声子, 从而减少一份能量。这是将 a^\dagger 和 a 称为产生算符和湮灭算符的原因。

2.2 量子场论中的正则对易关系

在量子力学中, Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法, 它们之间可以通过一个幺正变换联系起来。在 Schrödinger 绘景中, 波函数 $\psi_S(t)$ 代表随时间演化的物理态, 而任意算符 O_S 不依赖于时间。在 Heisenberg 绘景中, 波函数 ψ_H 代表不随时间演化的物理态, 它与 $\psi_S(t)$ 的关系为

$$\psi_H = e^{iHt}\psi_S(t), \quad (2.31)$$

其中 H 是哈密顿量; 而算符 $O_H(t)$ 依赖于时间, 通过下式与 O_S 联系起来:

$$O_H(t) = e^{iHt}O_S(t)e^{-iHt}. \quad (2.32)$$

上一节的量子化可以认为是在 Schrödinger 绘景中实现的, 因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x_S, p_S] = i$, 它在 Heisenberg 绘景中的形式为

$$\begin{aligned} [x_H(t), p_H(t)] &= [e^{iHt}x_S e^{-iHt}, e^{iHt}p_S e^{-iHt}] = e^{iHt}x_S e^{-iHt} e^{iHt}p_S e^{-iHt} - e^{iHt}p_S e^{-iHt} e^{iHt}x_S e^{-iHt} \\ &= e^{iHt}x_S p_S e^{-iHt} - e^{iHt}p_S x_S e^{-iHt} = e^{iHt}[x_S, p_S]e^{-iHt} = e^{iHt}ie^{-iHt} = i. \end{aligned} \quad (2.33)$$

可见, 正则对易关系的形式不依赖于绘景。(2.33) 式是在同一时刻 t 成立的, 称为等时 (equal time) 对易关系。

将讨论推广到自由度为 n 的系统, 记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符, $p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响, 这些算符需要满足如下等时对易关系:

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [q_i(t), q_j(t)] = 0, \quad [p_i(t), p_j(t)] = 0. \quad (2.34)$$

1.1 节提到, 在量子场论中, 为了平等地处理时间和空间, 空间坐标 \mathbf{x} 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x}, t)$ 的参数。由于这里量子场作为算符是依赖于时间的, 使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行, 省略绘景的标志性下标 H。

场讨论的是无穷多自由度的系统, 每一个空间点 \mathbf{x} 上的 $\phi(\mathbf{x}, t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度, 我们可以先将空间离散化, 划分成 n 个小体积元 V_i , 然后再取 $V_i \rightarrow 0$ 的极限来得到 $n \rightarrow \infty$ 的结果。在体积元 V_i 中, 定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i} \int_{V_i} d^3x \phi(\mathbf{x}, t), \quad (2.35)$$

它是场 $\phi(\mathbf{x}, t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi, \partial_\mu \phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L}(\phi, \partial_\mu \phi), \quad (2.36)$$

当体积元取得足够小时, 它就成为 ϕ_i 和 $\partial_0 \phi_i$ 的函数 $\mathcal{L}_i(\phi_i, \partial_0 \phi_i)$ 。拉格朗日量可表达为

$$L = \int d^3x \mathcal{L} = \sum_i \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \mathcal{L}_i(\phi_i, \partial_0 \phi_i). \quad (2.37)$$

于是, 由 (1.108) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \delta_{ji} \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} = V_i \pi_i(t), \quad (2.38)$$

其中,

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]}. \quad (2.39)$$

现在, 等时对易关系变成

$$[\phi_i(t), \Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t), \phi_j(t)] = 0, \quad [\Pi_i(t), \Pi_j(t)] = 0. \quad (2.40)$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t), \pi_j(t)] = 0. \quad (2.41)$$

对于任意连续函数 $f(x)$, **Dirac δ 函数** $\delta(x)$ 使下式成立:

$$f(x) = \int dy f(y) \delta(x - y). \quad (2.42)$$

函数 $\delta(x)$ 只在 $x = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即

$$\delta(x) = \delta(-x), \quad (2.43)$$

而且满足

$$\int dx \delta(x) = 1. \quad (2.44)$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \quad (2.45)$$

则对于任意连续函数 $f(\mathbf{x})$, 下式成立:

$$f(\mathbf{x}) = \int d^3y f(\mathbf{y}) \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (2.46)$$

类似地, 函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x} = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$, 而且满足 $\int d^3x \delta^{(3)}(\mathbf{x}) = 1$ 。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值, 则它会满足

$$f_i = \sum_j f_j \delta_{ij} = \sum_j V_j f_j \frac{\delta_{ij}}{V_j}. \quad (2.47)$$

(2.46) 式是 (2.47) 式在 $V_i \rightarrow 0$ 时的极限。可见, 在 $V_i \rightarrow 0$ 极限下,

$$\frac{\delta_{ij}}{V_j} \rightarrow \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (2.48)$$

另一方面, 在此极限下, $\phi_i(t) \rightarrow \phi(\mathbf{x}, t)$, 而 $\pi_i(t)$ 变成由 (1.117) 式定义的共轭动量密度:

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial [\partial_0 \phi_i(t)]} \rightarrow \frac{\partial \mathcal{L}}{\partial [\partial_0 \phi(\mathbf{x}, t)]} = \pi(\mathbf{x}, t). \quad (2.49)$$

因此, 等时对易关系化为

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.50)$$

推广到包含若干个场 ϕ_a 的系统, 假设不同的场不会相互影响, 则有

$$[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0, \quad [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \quad (2.51)$$

这就是量子场论中的正则对易关系。此时, $\phi_a(\mathbf{x}, t)$ 和 $\pi_a(\mathbf{x}, t)$ 都是算符。

2.3 实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量, 就称它为**标量场**。在固有保时向 Lorentz 变换下, 若时空坐标的变换为 $x' = \Lambda x$, 则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \quad (2.52)$$

在本节中, 我们讨论**实标量场** $\phi(x)$, 它满足自共轭 (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \quad (2.53)$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的自由实标量场, 相应的 **Lorentz 不变拉氏量**可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m^2\phi^2. \quad (2.54)$$

注意到

$$\frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu \phi)\partial_\nu \phi = \frac{1}{2}[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2], \quad (2.55)$$

可得

$$\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi, \quad \frac{\partial \mathcal{L}}{\partial (\partial_i \phi)} = -\partial_i \phi = \partial^i \phi, \quad (2.56)$$

归纳起来, 有

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi. \quad (2.57)$$

因此, Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi, \quad (2.58)$$

也就是说, $\phi(x)$ 满足 *Klein-Gordon* 方程

$$(\partial^2 + m^2)\phi(x) = 0. \quad (2.59)$$

2.3.1 平面波展开

设 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik \cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \quad (2.60)$$

则有

$$\partial^2 \varphi = \partial^\mu \partial_\mu \varphi = \partial^\mu (-ik_\mu \varphi) = -ik_\mu \partial^\mu \varphi = (-i)^2 k_\mu k^\mu \varphi = -k^2 \varphi, \quad (2.61)$$

从而,

$$0 = (\partial^2 + m^2) \varphi = -(k^2 - m^2) \varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2] \varphi. \quad (2.62)$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$, 即 $k^0 = \pm E_{\mathbf{k}}$, 其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$. 因此, 有两种平面波解。

(1) $k^0 = E_{\mathbf{k}}$ 对应于正能解

$$\varphi_{\mathbf{k}}^{(+)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})]. \quad (2.63)$$

(2) $k^0 = -E_{\mathbf{k}}$ 对应于负能解

$$\varphi_{\mathbf{k}}^{(-)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})]. \quad (2.64)$$

从而, 场算符 $\phi(\mathbf{x}, t)$ 的通解可以写成如下形式:

$$\begin{aligned} \phi(\mathbf{x}, t) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} \varphi_{\mathbf{k}}^{(+)}(x) + \tilde{a}_{\mathbf{k}} \varphi_{\mathbf{k}}^{(-)}(x) \right] \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}} e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right], \end{aligned} \quad (2.65)$$

其中 $a_{\mathbf{k}}$ 和 $\tilde{a}_{\mathbf{k}}$ 是两个只依赖于 \mathbf{k} 的算符。这是一种 Fourier 变换, 把 $\phi(\mathbf{x}, t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭, 得

$$\begin{aligned} \phi^\dagger(\mathbf{x}, t) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right] \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} \right]. \end{aligned} \quad (2.66)$$

第二步利用了如下性质: 对整个三维动量空间进行积分时, 将积分项中的 \mathbf{k} 换成 $-\mathbf{k}$ 不会改变积分的结果。于是, 由自共轭条件 $\phi^\dagger(\mathbf{x}, t) = \phi(\mathbf{x}, t)$ 可得

$$\tilde{a}_{\mathbf{k}} = a_{-\mathbf{k}}^\dagger. \quad (2.67)$$

(注意: 由上式可以推出 $\tilde{a}_{\mathbf{k}}^\dagger = a_{-\mathbf{k}}$ 和 $\tilde{a}_{-\mathbf{k}}^\dagger = a_{\mathbf{k}}$ 。) 因而, 有

$$\phi(\mathbf{x}, t) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})} + a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})} \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} \right]. \quad (2.68)$$

替换一下动量记号，可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.69)$$

其中， p^0 是正的，满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}, \quad (2.70)$$

而 $a_{\mathbf{p}}$ 是湮灭算符， $a_{\mathbf{p}}^\dagger$ 是产生算符。 $\phi(\mathbf{x}, t)$ 对应的共轭动量密度算符为

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.71)$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.72)$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x e^{i\mathbf{p} \cdot \mathbf{x}} = \int d^3x e^{-i\mathbf{p} \cdot \mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p}), \quad (2.73)$$

可得

$$\begin{aligned} \int d^3x e^{iq \cdot x} \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \end{aligned} \quad (2.74)$$

以及

$$\begin{aligned} \int d^3x e^{iq \cdot x} \partial_0 \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}). \end{aligned} \quad (2.75)$$

从而，有

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x e^{iq \cdot x} \partial_0 \phi - iq_0 \int d^3x e^{iq \cdot x} \phi = \int d^3x e^{iq \cdot x} (\partial_0 \phi - iq_0 \phi), \quad (2.76)$$

亦即

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\partial_0 \phi(x) - ip_0 \phi(x)]. \quad (2.77)$$

上式取厄米共轭，并使用自共轭条件 $\phi^\dagger = \phi$ ，得

$$a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} [\partial_0 \phi(x) + ip_0 \phi(x)]. \quad (2.78)$$

利用上面两个表达式和等时对易关系 (2.72)，可得

$$\begin{aligned} & [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.79)$$

在以上计算过程中， $x^0 = y^0 = t$ 。上式最后一行中的 $\delta^{(3)}(\mathbf{p} - \mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = \mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$ ，则 $E_{\mathbf{p}} + E_{\mathbf{q}} = 2E_{\mathbf{p}} = \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}$ ，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (2.80)$$

类似地，

$$\begin{aligned} & [a_{\mathbf{p}}, a_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} (-iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} [-i(p_0 - q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{q}} - E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{q}} - E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}). \end{aligned} \quad (2.81)$$

最后一行中的 $\delta^{(3)}(\mathbf{p} + \mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = -\mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$ ，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \quad (2.82)$$

此外,

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}} a_{\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \quad (2.83)$$

综上, 产生湮灭算符满足如下对易关系:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \quad (2.84)$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.119), 实标量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = (\partial_0 \phi)^2 - \frac{1}{2} (\partial^\mu \phi) \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2]. \quad (2.85)$$

对全空间积分以得到哈密顿量:

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[(-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (-iq_0 a_{\mathbf{q}} e^{-iq \cdot x} + iq_0 a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + (i\mathbf{p} a_{\mathbf{p}} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + m^2 (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} e^{i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}}] \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} e^{i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t}] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \right. \\ &\quad \left. + (-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}) \right]. \quad (2.86) \end{aligned}$$

由 (2.70) 式可得 $-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 = 0$, 故上式最后两行方括号中第二项没有贡献。从而,

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} [2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p})] \end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2}, \quad (2.87)$$

其中第四步用到对易关系 (2.84)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。 $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 \mathbf{p} 的模式对应的粒子数密度算符 (动量空间中的密度), 相应的能量是 $E_{\mathbf{p}}$ 。在 (2.87) 式最后一行中, 第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.73) 式可得

$$(2\pi)^3 \delta^{(3)}(0) = \int d^3x = V, \quad (2.88)$$

其中 V 是进行积分的空间体积, 对于全空间而言是无穷大的。因此, (2.87) 式最后一行的第二项是一个无穷大 c 数, 是真空的零点能, 是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明, 一维简谐振子的零点能为 $E_0 = \omega/2$ 。这是自由度为 1 时的结果, 推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象, 这个零点能通常并不重要, 因为实验上只能测量两个能量之差。经过正则量子化之后, 实标量场的哈密顿量 H 是正定的, 不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \quad (2.89)$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3q E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \quad (2.90)$$

设 $|E\rangle$ 是 H 的本征态, 本征值为 E , 则

$$H |E\rangle = E |E\rangle. \quad (2.91)$$

从而, 有

$$H a_{\mathbf{p}}^\dagger |E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger |E\rangle. \quad (2.92)$$

$$H a_{\mathbf{p}} |E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p}} |E\rangle. \quad (2.93)$$

可见, 当 $a_{\mathbf{p}}^\dagger |E\rangle \neq 0$ 时, 产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{p}}$; 当 $a_{\mathbf{p}} |E\rangle \neq 0$ 时, 湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{p}}$ 。

根据 (1.158) 式, 实标量场的总动量是

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi \nabla \phi = - \int d^3x (\partial_0 \phi) \nabla \phi \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[-p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} - p_0 \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right] \end{aligned}$$

$$\begin{aligned}
& + p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + p_0 \mathbf{q} a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p+q) \cdot x} \Big] \\
& = - \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - p_0 \mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} e^{-i(p_0 - q_0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} e^{i(p_0 - q_0)t} e^{i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} \right] \right. \\
& \quad \left. + p_0 \mathbf{q} \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p_0 + q_0)t} e^{i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} \right] \right\} \\
& = - \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} e^{-i(p_0 - q_0)t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} e^{i(p_0 - q_0)t} \right] \right. \\
& \quad \left. + p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} e^{i(p_0 + q_0)t} \right] \right\} \\
& = - \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} (-E_{\mathbf{p}} \mathbf{p}) \left(a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) \\
& = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right). \tag{2.94}
\end{aligned}$$

先作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换, 再利用对易关系 (2.84), 可得

$$\begin{aligned}
& \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (-\mathbf{p}) \left(a_{-\mathbf{p}} a_{\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^{\dagger} a_{\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right) \\
& = -\frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^{\dagger} a_{-\mathbf{p}}^{\dagger} e^{2iE_{\mathbf{p}}t} \right). \tag{2.95}
\end{aligned}$$

可见, (2.94) 式最后一行圆括号中最后两项没有贡献。从而,

$$\begin{aligned}
\mathbf{P} & = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} [2a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(0)] \\
& = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{1}{2} \delta^{(3)}(0) \int d^3 p \mathbf{p}. \tag{2.96}
\end{aligned}$$

由于 $\int d^3 p \mathbf{p} = \int d^3 p (-\mathbf{p}) = -\int d^3 p \mathbf{p}$, 上式最后一行第二项没有贡献。于是,

$$\mathbf{P} = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}, \tag{2.97}$$

即总动量是所有动量模式所有粒子贡献的动量之和。

\mathbf{P} 与产生湮灭算符的对易子为

$$[\mathbf{P}, a_{\mathbf{p}}^{\dagger}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^{\dagger} [a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = \int d^3 q \mathbf{q} a_{\mathbf{q}}^{\dagger} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = \mathbf{p} a_{\mathbf{p}}^{\dagger}, \tag{2.98}$$

$$[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^{\dagger}, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3 q \mathbf{q} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -\mathbf{p} a_{\mathbf{p}}. \tag{2.99}$$

2.3.4 粒子态

真空态 $|0\rangle$ 是能量最低的态, 对于任意动量 \mathbf{p} 对应的湮灭算符 $a_{\mathbf{p}}$, 满足

$$a_{\mathbf{p}} |0\rangle = 0, \tag{2.100}$$

归一化为

$$\langle 0|0\rangle = 1. \quad (2.101)$$

由哈密顿量的表达式 (2.87) 可得

$$H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = \delta^{(3)}(0) \int d^3p \frac{E_{\mathbf{p}}}{2}, \quad (2.102)$$

可见, 这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外, 由 (2.97) 式可知, $|0\rangle$ 的总动量本征值是零:

$$\mathbf{P}|0\rangle = 0|0\rangle, \quad (2.103)$$

即真空态不具有动量。

接着, 定义动量为 \mathbf{p} 的单粒子态为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger} |0\rangle. \quad (2.104)$$

从而, 利用 (2.89) 和 (2.98) 式可得

$$H|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} H + E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}\rangle, \quad (2.105)$$

$$\mathbf{P}|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{P} a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} \mathbf{P} + \mathbf{p} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{p} a_{\mathbf{p}}^{\dagger} |0\rangle = \mathbf{p} |\mathbf{p}\rangle. \quad (2.106)$$

可以看出, 相比于真空态 $|0\rangle$, 单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$, 也多了一份动量 \mathbf{p} 。因此, $|\mathbf{p}\rangle$ 描述的是一个动量为 \mathbf{p} 的粒子, 这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 满足狭义相对论中的能量-动量关系 (1.52), 而拉氏量 (2.54) 中的参数 m 就是粒子的质量。可以看出, 产生算符 $a_{\mathbf{p}}^{\dagger}$ 的作用是产生一个动量为 \mathbf{p} 的粒子。

此外, 可作如下计算:

$$a_{\mathbf{p}}|\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{q}}} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] |0\rangle = \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle. \quad (2.107)$$

如果 $\mathbf{p} \neq \mathbf{q}$, 则上式为零; 如果 $\mathbf{p} = \mathbf{q}$, 则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$ 。可见, 湮灭算符 $a_{\mathbf{p}}$ 的作用是减少一个动量为 \mathbf{p} 的粒子。

单粒子态的内积关系为

$$\begin{aligned} \langle \mathbf{q}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}} \langle 0| a_{\mathbf{q}} a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}} \langle 0| [a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] |0\rangle \\ &= 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.108)$$

上式是 *Lorentz* 不变的, 这是 (2.104) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明 若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i , 则如下等式成立:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \quad (2.109)$$

引入阶跃函数 (step function)

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (2.110)$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^μ 满足质壳条件 $p^2 - m^2 = 0$ 且能量为正 ($p^0 > 0$) 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned} \int d^4p \delta(p^2 - m^2) \theta(p^0) F(p) &= \int d^3p dp^0 \delta\left((p^0)^2 - |\mathbf{p}|^2 - m^2\right) \theta(p^0) F(p^0, \mathbf{p}) \\ &= \int d^3p \frac{1}{2\sqrt{|\mathbf{p}|^2 + m^2}} F\left(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}\right) = \int \frac{d^3p}{2E_{\mathbf{p}}} F\left(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}\right). \end{aligned} \quad (2.111)$$

这里第二步用到 (2.109) 式。可见,

$$\frac{d^3p}{2E_{\mathbf{p}}} \quad (2.112)$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$, 按照 δ 函数定义, 有

$$g(\mathbf{q}) = \int d^3p \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}) = \int \frac{d^3p}{2E_{\mathbf{p}}} 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}). \quad (2.113)$$

由于上式最左边和最右边都是 Lorentz 不变的,

$$2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad (2.114)$$

必定是 Lorentz 不变的。证毕。

进一步, 可以定义动量分别为 $\mathbf{p}_1, \dots, \mathbf{p}_n$ 的 n 个粒子对应的多粒子态为

$$|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \equiv \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle. \quad (2.115)$$

H 对它的作用给出

$$\begin{aligned} H |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} H a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} (a_{\mathbf{p}_1}^\dagger H + E_{\mathbf{p}_1} a_{\mathbf{p}_1}^\dagger) \cdots a_{\mathbf{p}_n}^\dagger |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger H a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + E_{\mathbf{p}_1} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger H \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \cdots = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger H |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= (E_{\text{vac}} + E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle, \end{aligned} \quad (2.116)$$

同理, \mathbf{P} 对它的作用给出

$$\mathbf{P} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle = (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle. \quad (2.117)$$

也就是说, 多粒子态 $|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.84) 可得

$$\begin{aligned}
 |\mathbf{p}_1, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\
 &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\
 &= |\mathbf{p}_1, \dots, \mathbf{p}_j, \dots, \mathbf{p}_i, \dots, \mathbf{p}_n\rangle.
 \end{aligned} \tag{2.118}$$

可以看出, 对调多粒子态中的任意两个粒子, 得到的态相同, 即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是**玻色子** (boson), 服从 Bose-Einstein 统计。得到这个结论的关键在于两个产生算符相互对易。

双粒子态的内积关系为

$$\begin{aligned}
 \langle \mathbf{q}_1, \mathbf{q}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{q}_1} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger | 0 \rangle \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle + \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger a_{\mathbf{q}_1} a_{\mathbf{p}_2}^\dagger | 0 \rangle] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger | 0 \rangle] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} [(2\pi)^6 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) + (2\pi)^6 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2)] \\
 &= 4E_{\mathbf{p}_1} E_{\mathbf{p}_2} (2\pi)^6 [\delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) + \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1)].
 \end{aligned} \tag{2.119}$$

此外, 还可以定义动量均为 \mathbf{p} 的 n 个粒子对应的多粒子态为

$$|n_{\mathbf{p}}\rangle \equiv (2E_{\mathbf{p}})^{n_{\mathbf{p}}/2} (a_{\mathbf{p}}^\dagger)^{n_{\mathbf{p}}} |0\rangle, \tag{2.120}$$

则粒子数密度算符

$$N_{\mathbf{p}} \equiv a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \tag{2.121}$$

对它的作用为

$$\begin{aligned}
 N_{\mathbf{p}} |n_{\mathbf{q}}\rangle &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger [a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^2 a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-2} |0\rangle + 2(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= \cdots = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} a_{\mathbf{p}} |0\rangle + n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle.
 \end{aligned} \tag{2.122}$$

在动量空间对粒子数密度算符进行积分, 得到的是**粒子数算符**

$$N \equiv \int \frac{d^3p}{(2\pi)^3} N_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{2.123}$$

由 (2.122) 式, 可得

$$\begin{aligned}
 N |n_{\mathbf{q}}\rangle &= \int \frac{d^3p}{(2\pi)^3} N_{\mathbf{p}} |n_{\mathbf{q}}\rangle = \int \frac{d^3p}{(2\pi)^3} n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = n_{\mathbf{q}} |n_{\mathbf{q}}\rangle.
 \end{aligned} \tag{2.124}$$

因此, $|n_{\mathbf{q}}\rangle$ 是 N 的本征态, 本征值为粒子数 $n_{\mathbf{q}}$ 。

更一般地, 可以定义多粒子态

$$|n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \quad (2.125)$$

来描述动量为 $\mathbf{p}_1, \dots, \mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}$ 个的情况。此时, 有

$$\begin{aligned} & N |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right. \\ &\quad \left. + n_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}-1} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] + n_{\mathbf{p}_1} |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= \dots = \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \dots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} a_{\mathbf{p}} |0\rangle \right] \\ &\quad + (n_{\mathbf{p}_1} + \dots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle \\ &= (n_{\mathbf{p}_1} + \dots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \dots, n_{\mathbf{p}_m}\rangle. \end{aligned} \quad (2.126)$$

可见, N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中, 我们讨论复标量场 $\phi(x)$, 它不满足自共轭条件 (2.53), 即

$$\phi^\dagger(x) \neq \phi(x). \quad (2.127)$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.188) 式的形式。不过, 由于 $\phi(x)$ 是量子场算符, 需要把那里的复共轭记号 $*$ 改成厄米共轭记号 \dagger , 故 Lorentz 不变拉氏量为

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m^2 \phi^\dagger \phi. \quad (2.128)$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量, 注意到

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^\dagger, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^\dagger, \quad (2.129)$$

则 Euler-Lagrange 方程 (1.116) 给出

$$(\partial^2 + m^2)\phi(x) = 0, \quad (\partial^2 + m^2)\phi^\dagger(x) = 0. \quad (2.130)$$

也就是说, $\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程.

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \quad (2.131)$$

从而, 拉氏量 (2.128) 化为

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}[\partial^\mu(\phi_1 - i\phi_2)]\partial_\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2 - \frac{1}{2}m^2\phi_2^2. \end{aligned} \quad (2.132)$$

与 (2.54) 式比较可知, 复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量.

2.4.1 平面波展开

对于复标量场, 我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开, 但不能够应用自共轭条件. 因此, 场算符 $\phi(\mathbf{x}, t)$ 的通解也具有 (2.65) 式的形式:

$$\begin{aligned} \phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}}e^{i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})}] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{-\mathbf{k}}e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}]. \end{aligned} \quad (2.133)$$

由于不满足自共轭条件 (2.53), 算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_{\mathbf{k}}$ 没有什么关系, 改记为

$$b_{\mathbf{k}}^\dagger = \tilde{a}_{-\mathbf{k}}, \quad (2.134)$$

则展开式变成

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + b_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})}]. \quad (2.135)$$

替换一下动量记号, 可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.136)$$

其中, p^0 应该满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \quad (2.137)$$

取厄米共轭, 就得到 $\phi^\dagger(\mathbf{x}, t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b_{\mathbf{p}}e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.138)$$

现在, $a_{\mathbf{p}}$ 和 $b_{\mathbf{p}}$ 是两个相互独立的湮灭算符, 而 $a_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^\dagger = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (b_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad (2.139)$$

$\phi^\dagger(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\dagger)} = \partial_0 \phi = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (a_{\mathbf{p}} e^{-ip \cdot x} - b_{\mathbf{p}}^\dagger e^{ip \cdot x}). \quad (2.140)$$

根据 (2.51) 式, 等时对易关系为

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0, \\ [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0, \\ [\phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= [\phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t)] = [\phi(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.141)$$

2.4.2 产生湮灭算符的对易关系

由

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \phi &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} + b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger e^{2iq^0 t}) \end{aligned} \quad (2.142)$$

和

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \partial_0 \phi &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} - b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3 p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - b_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \end{aligned} \quad (2.143)$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3 x e^{iq \cdot x} \partial_0 \phi - iq_0 \int d^3 x e^{iq \cdot x} \phi = \int d^3 x e^{iq \cdot x} (\partial_0 \phi - iq_0 \phi). \quad (2.144)$$

于是,

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x e^{ip \cdot x} (\partial_0 \phi - ip_0 \phi), \quad a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x e^{-ip \cdot x} (\partial_0 \phi^\dagger + ip_0 \phi^\dagger). \quad (2.145)$$

从而, 有

$$\begin{aligned}
& [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{2.146}
\end{aligned}$$

以及

$$\begin{aligned}
& [a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] = 0. \tag{2.147}
\end{aligned}$$

另一方面, 由

$$\begin{aligned}
\int d^3x e^{iq \cdot x} \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q) \cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\
&= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}) \tag{2.148}
\end{aligned}$$

和

$$\begin{aligned}
\int d^3x e^{iq \cdot x} \partial_0 \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q) \cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\
&= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0 + q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \tag{2.149}
\end{aligned}$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} b_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} b_{\mathbf{q}} = \int d^3x e^{iq \cdot x} \partial_0 \phi^\dagger - iq_0 \int d^3x e^{iq \cdot x} \phi^\dagger = \int d^3x e^{iq \cdot x} (\partial_0 \phi^\dagger - iq_0 \phi^\dagger). \tag{2.150}$$

于是,

$$b_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} (\partial_0 \phi^\dagger - ip_0 \phi^\dagger), \quad b_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} (\partial_0 \phi + ip_0 \phi). \quad (2.151)$$

从而, 有

$$\begin{aligned} & [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)]) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \end{aligned} \quad (2.152)$$

以及

$$\begin{aligned} & [b_{\mathbf{p}}, b_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.153)$$

此外, 还有

$$\begin{aligned} & [a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] = 0, \end{aligned} \quad (2.154)$$

以及

$$\begin{aligned} & [a_{\mathbf{p}}, b_{\mathbf{q}}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} (-iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [-i(p_0 - q_0)i\delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \quad (2.155)
\end{aligned}$$

归纳起来，产生湮灭算符的对易关系如下：

$$\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}] = 0, \\
[b_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p}}, b_{\mathbf{q}}] = [b_{\mathbf{p}}^{\dagger}, b_{\mathbf{q}}^{\dagger}] = 0, \\
[a_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}] &= [b_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] = [a_{\mathbf{p}}, b_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, b_{\mathbf{q}}^{\dagger}] = 0.
\end{aligned} \quad (2.156)$$

这说明 $a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}}$ 与 $b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$ 是两套不同的产生湮灭算符，描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \quad [\phi^{\dagger}(x)]' = e^{-iq\theta} \phi^{\dagger}(x), \quad (2.157)$$

则拉氏量 (2.128) 不变。依照 1.7.4 小节的讨论，相应的守恒流为

$$J^{\mu} = q\phi^{\dagger} i \overleftrightarrow{\partial}^{\mu} \phi, \quad (2.158)$$

相应的守恒荷为

$$\begin{aligned}
Q &= q \int d^3x \phi^{\dagger} i \overleftrightarrow{\partial}^0 \phi = iq \int d^3x [\phi^{\dagger} \partial^0 \phi - (\partial^0 \phi^{\dagger}) \phi] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) \partial^0 (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right. \\
&\quad \left. - \partial^0 (b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (-ik^0) (a_{\mathbf{k}} e^{-ik\cdot x} - b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right. \\
&\quad \left. - (-ip^0) (b_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) (a_{\mathbf{k}} e^{-ik\cdot x} + b_{\mathbf{k}}^{\dagger} e^{ik\cdot x}) \right] \\
&= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[(ik^0 + ip^0) b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(p-k)\cdot x} + (-ik^0 - ip^0) a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(p-k)\cdot x} \right. \\
&\quad \left. + (-ik^0 + ip^0) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + (ik^0 - ip^0) a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(p+k)\cdot x} \right] \\
&= q \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[-(E_{\mathbf{k}} + E_{\mathbf{p}}) b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(p-k)\cdot x} + (E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(p-k)\cdot x} \right. \\
&\quad \left. + (E_{\mathbf{k}} - E_{\mathbf{p}}) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k)\cdot x} + (-E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(p+k)\cdot x} \right] \\
&= q \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left\{ (E_{\mathbf{k}} + E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} - \mathbf{k}) \left[-b_{\mathbf{p}} b_{\mathbf{k}}^{\dagger} e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + (E_{\mathbf{k}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p} + \mathbf{k}) \left[b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} - a_{\mathbf{p}}^{\dagger} b_{\mathbf{k}}^{\dagger} e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right] \Big\} \\
& = q \int \frac{d^3 p}{(2\pi)^3} 2E_{\mathbf{p}} (-b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}) = q \int \frac{d^3 p}{(2\pi)^3} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger}). \quad (2.159)
\end{aligned}$$

利用对易关系 (2.156), 可得

$$Q = \int \frac{d^3 p}{(2\pi)^3} (q a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - q b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}) - (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3 p}{(2\pi)^3} q. \quad (2.160)$$

上式第二项是零点荷。在第一项的圆括号中, 粒子数密度算符 $a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$ 的系数是 q , 而粒子数密度算符 $b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}$ 的系数是 $-q$ 。可见, $a_{\mathbf{p}}^{\dagger}, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q , 习惯上称为**正粒子**; 另一方面, $b_{\mathbf{p}}^{\dagger}, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$, 习惯上称为**反粒子**。除去零点荷, 总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式类似, 但 Q/q 被解释为正粒子数与反粒子数之差, 因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的, 并且来自于一种内部对称性, 因而是一种**内部相加性量子数**。实际上, 反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 $U(1)$ 整体变换, 则自共轭条件 (2.53) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^{\dagger} = [e^{iq\theta}\phi(x)]^{\dagger} = e^{-iq\theta}\phi^{\dagger}(x) = e^{-iq\theta}\phi(x). \quad (2.161)$$

上式要求 $q = 0$ 。因此, 对实标量场不能进行非平庸的 $U(1)$ 整体变换。实际上, 自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数, 也就是说, 正粒子与反粒子是相同的, 实标量场描述的是一种纯中性粒子。

2.4.4 哈密顿量和总动量

根据 (1.119) 式, 复标量场的哈密顿量密度为

$$\begin{aligned}
\mathcal{H} &= \pi \partial_0 \phi + \pi^{\dagger} \partial_0 \phi^{\dagger} - \mathcal{L} = (\partial^0 \phi^{\dagger}) \partial_0 \phi + (\partial^0 \phi) \partial_0 \phi^{\dagger} - (\partial^{\mu} \phi^{\dagger}) \partial_{\mu} \phi + m^2 \phi^{\dagger} \phi \\
&= (\partial^0 \phi^{\dagger}) \partial_0 \phi + (\nabla \phi^{\dagger}) \cdot \nabla \phi + m^2 \phi^{\dagger} \phi. \quad (2.162)
\end{aligned}$$

于是, 哈密顿量可以写成

$$\begin{aligned}
H &= \int d^3 x \mathcal{H} = \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi + (\nabla \phi^{\dagger}) \cdot \nabla \phi + m^2 \phi^{\dagger} \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi + \nabla \cdot (\phi^{\dagger} \nabla \phi) - \phi^{\dagger} \nabla^2 \phi + m^2 \phi^{\dagger} \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi + \phi^{\dagger} (\partial^0 \partial_0 - \nabla^2 + m^2) \phi] \\
&= \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi + \phi^{\dagger} (\partial^2 + m^2) \phi]. \quad (2.163)
\end{aligned}$$

上式第三步用了分部积分, 第四步扔掉了一个全散度, 最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.130) 消去。从而, 得到

$$H = \int d^3 x [(\partial^0 \phi^{\dagger}) \partial_0 \phi - \phi^{\dagger} \partial^0 \partial_0 \phi]$$

$$\begin{aligned}
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[\partial^0 (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \partial^0 \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right] \\
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left\{ (-ip^0) (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) (-iq_0) (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) [(-iq^0)(-iq_0) a_q e^{-iq \cdot x} + iq^0 iq_0 b_q^\dagger e^{iq \cdot x}] \right\} \\
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(p^0 q_0 + q^0 q_0) b_p b_q^\dagger e^{-i(p-q) \cdot x} + (p^0 q_0 + q^0 q_0) a_p^\dagger a_q e^{i(p-q) \cdot x} \right. \\
&\quad \left. + (-p^0 q_0 + q^0 q_0) b_p a_q e^{-i(p+q) \cdot x} + (-p^0 q_0 + q^0 q_0) a_p^\dagger b_q^\dagger e^{i(p+q) \cdot x} \right] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} E_q \left\{ (E_p + E_q) \delta^{(3)}(\mathbf{p} - \mathbf{q}) [b_p b_q^\dagger e^{-i(E_p - E_q)t} + a_p^\dagger a_q e^{i(E_p - E_q)t}] \right. \\
&\quad \left. + (E_q - E_p) \delta^{(3)}(\mathbf{p} + \mathbf{q}) [b_p a_q e^{-i(E_p + E_q)t} + a_p^\dagger b_q^\dagger e^{i(E_p + E_q)t}] \right\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} 2E_p^2 (b_p b_p^\dagger + a_p^\dagger a_p) = \int \frac{d^3p}{(2\pi)^3} E_p (b_p b_p^\dagger + a_p^\dagger a_p) \\
&= \int \frac{d^3p}{(2\pi)^3} E_p (a_p^\dagger a_p + b_p^\dagger b_p) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} E_p. \tag{2.164}
\end{aligned}$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 \mathbf{p} ，正粒子与反粒子具有相同的能量 E_p ，因而它们具有相同的质量 m 。

根据 (1.158) 式，复标量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x (\pi \nabla \phi + \pi^\dagger \nabla \phi^\dagger) = - \int d^3x [(\partial_0 \phi^\dagger) \nabla \phi + (\partial_0 \phi) \nabla \phi^\dagger] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[\partial_0 (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \nabla (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. + \partial_0 (a_q e^{-iq \cdot x} + b_q^\dagger e^{iq \cdot x}) \nabla (b_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[-ip_0 (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) i\mathbf{q} (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) \right. \\
&\quad \left. - iq_0 (a_q e^{-iq \cdot x} - b_q^\dagger e^{iq \cdot x}) i\mathbf{p} (b_p e^{-ip \cdot x} - a_p^\dagger e^{ip \cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(-E_p \mathbf{q} b_p b_q^\dagger - E_q \mathbf{p} b_q^\dagger b_p) e^{-i(p-q) \cdot x} \right. \\
&\quad + (-E_p \mathbf{q} a_p^\dagger a_q - E_q \mathbf{p} a_q a_p^\dagger) e^{i(p-q) \cdot x} \\
&\quad + (E_p \mathbf{q} b_p a_q + E_q \mathbf{p} a_q b_p) e^{-i(p+q) \cdot x} \\
&\quad \left. + (E_p \mathbf{q} a_p^\dagger b_q^\dagger + E_q \mathbf{p} b_q^\dagger a_p^\dagger) e^{i(p+q) \cdot x} \right] \\
&= - \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) [(-E_p \mathbf{q} b_p b_q^\dagger - E_q \mathbf{p} b_q^\dagger b_p) e^{-i(E_p - E_q)t} \right. \\
&\quad \left. + (-E_p \mathbf{q} a_p^\dagger a_q - E_q \mathbf{p} a_q a_p^\dagger) e^{i(E_p - E_q)t}] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[(E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \\
& \quad \left. + (E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^{\dagger} b_{\mathbf{q}}^{\dagger} + E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^{\dagger} a_{\mathbf{p}}^{\dagger}) e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \Big\} \\
& = - \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} \left[- E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) \right. \\
& \quad \left. - E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} a_{-\mathbf{p}} - a_{-\mathbf{p}} b_{\mathbf{p}}) e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}} \mathbf{p} (a_{\mathbf{p}}^{\dagger} b_{-\mathbf{p}}^{\dagger} - b_{-\mathbf{p}}^{\dagger} a_{\mathbf{p}}^{\dagger}) e^{2iE_{\mathbf{p}}t} \right] \\
& = \int \frac{d^3 p}{(2\pi)^3} \frac{\mathbf{p}}{2} (b_{\mathbf{p}} b_{\mathbf{p}}^{\dagger} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}) + \delta^{(3)}(0) \int d^3 p \mathbf{p} \\
& = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}}). \tag{2.165}
\end{aligned}$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章 矢量场

3.1 量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换

$$|\Psi'\rangle = U(\Lambda) |\Psi\rangle, \quad (3.1)$$

其中 $U(\Lambda)$ 是一个线性幺正算符，描述量子 Lorentz 变换，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \quad (3.2)$$

先作 Lorentz 变换 Λ_1 ，再作 Lorentz 变换 Λ_2 ，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$ ，故以下同态 (homomorphic) 关系成立：

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \quad (3.3)$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(1) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \quad (3.4)$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \quad (3.5)$$

将无穷小 Lorentz 变换 (1.159) 记为 $\Lambda_\omega = 1 + \omega$ ，它诱导的无穷小幺正算符可表达为

$$U(1 + \omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \quad (3.6)$$

这里只展开到 ω 的一阶项。 $J^{\mu\nu}$ 是量子 Lorentz 变换的生成元算符¹。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的：

$$J^{\mu\nu} = -J^{\nu\mu}. \quad (3.7)$$

由 $U(1 + \omega)$ 的幺正性可得

$$1 = U^\dagger(1 + \omega)U(1 + \omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right] \left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \quad (3.8)$$

¹虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.179)。

最后一步忽略了 ω 的二阶项。可见, $J^{\mu\nu}$ 是厄米算符:

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \quad (3.9)$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda). \quad (3.10)$$

的左边和右边分别展开, 得

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U^{-1}(\Lambda) \left(1 - \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} \right) U(\Lambda) = 1 - \frac{i}{2} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda), \quad (3.11)$$

$$U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda) = U(\mathbf{1} + \Lambda^{-1}\omega\Lambda) = 1 - \frac{i}{2} (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu}. \quad (3.12)$$

因此, 有

$$\begin{aligned} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1}\omega\Lambda)^\alpha{}_\nu J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1})^\alpha{}_\beta \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} \\ &= g_{\mu\alpha} \Lambda_\beta{}^\alpha \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} = \Lambda^\beta{}_\mu \omega_{\beta\gamma} \Lambda^\gamma{}_\nu J^{\mu\nu} = \omega_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}, \end{aligned} \quad (3.13)$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立, 于是,

$$U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.14)$$

因此, $J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值与它在 $|\Psi\rangle$ 中的期待值有如下关系:

$$\langle \Psi' | J^{\mu\nu} | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \langle \Psi | J^{\rho\sigma} | \Psi \rangle. \quad (3.15)$$

也就是说, $U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换:

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.16)$$

可见, $J^{\mu\nu}$ 是一个 2 阶 Lorentz 张量。

接着, 考虑 Λ 的无穷小形式 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu$, 则

$$U(\Lambda) = 1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta}. \quad (3.17)$$

忽略二阶小量, (3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) &= \left(1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} \right) J^{\mu\nu} \left(1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta} \right) \\ &= J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\mu\nu} J^{\alpha\beta} + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\rho\sigma} [J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \quad (3.18)$$

右边为

$$\begin{aligned} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma} &= (\delta^\mu{}_\rho + \tilde{\omega}^\mu{}_\rho) (\delta^\nu{}_\sigma + \tilde{\omega}^\nu{}_\sigma) J^{\rho\sigma} = \delta^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} + \delta^\mu{}_\rho \tilde{\omega}^\nu{}_\sigma J^{\rho\sigma} + \tilde{\omega}^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^\nu{}_\sigma J^{\mu\sigma} + \tilde{\omega}^\mu{}_\rho J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma} g^{\nu\rho} J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho} g^{\mu\sigma} J^{\rho\nu} \end{aligned}$$

$$\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma} J^{\mu\rho} + g^{\mu\rho} J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma}), \tag{3.19}
\end{aligned}$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \\
&= i[g^{\nu\rho} J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma). \tag{3.20}
\end{aligned}$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群 是一类特殊的连续群， n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie* 乘积，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$J^i \equiv \frac{1}{2}\varepsilon^{ijk} J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}$$

即

$$\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}$$

J^i 与 J^j 的对易关系为

$$\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij}, \tag{3.23}
\end{aligned}$$

第二、三步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.84) 式。由 (1.98) 式，有

$$J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}$$

从而推出

$$[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ ，写成分量的形式是 $L^i = \varepsilon^{ijk}x^jp^k$ ，从而，

$$\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^lp^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^lp^m = x^ip^j - x^jp^i. \tag{3.26}$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i, p^j] = i\delta^{ij}$ 可得

$$\begin{aligned}
 [L^i, L^j] &= \varepsilon^{ikl} \varepsilon^{jmn} [x^k p^l, x^m p^n] = \varepsilon^{ikl} \varepsilon^{jmn} \{x^k [p^l, x^m] p^n + x^m [x^k, p^n] p^l\} \\
 &= \varepsilon^{ikl} \varepsilon^{jmn} (-i\delta^{lm} x^k p^n + i\delta^{kn} x^m p^l) = i(-\varepsilon^{ikl} \varepsilon^{jln} x^k p^n + \varepsilon^{ikl} \varepsilon^{jmk} x^m p^l) \\
 &= i(\varepsilon^{ikl} \varepsilon^{jnl} x^k p^n - \varepsilon^{ilk} \varepsilon^{jmk} x^m p^l) = i[(\delta^{ij} \delta^{kn} - \delta^{in} \delta^{kj}) x^k p^n - (\delta^{ij} \delta^{lm} - \delta^{im} \delta^{lj}) x^m p^l] \\
 &= i[\delta^{ij} x^k p^k - x^j p^i - \delta^{ij} x^l p^l + x^i p^j] = i(x^i p^j - x^j p^i) = i\varepsilon^{ijk} L^k.
 \end{aligned} \tag{3.27}$$

可见, \mathbf{J} 与 \mathbf{L} 具有相同的对易关系, \mathbf{J} 也是一个角动量算符。实际上, \mathbf{J} 描述总角动量, 不止可以包含轨道角动量 \mathbf{L} , 也可以包含自旋角动量。

满足

$$O^T O = \mathbf{1} \tag{3.28}$$

的实方阵 O 称为实正交矩阵 (real orthogonal matrix)。对上式取行列式, 得

$$1 = \det O^T \cdot \det O = (\det O)^2. \tag{3.29}$$

可见, 实正交矩阵 O 的行列式为 $\det O = \pm 1$ 。由行列式为 +1 的 3 维实正交矩阵按照矩阵乘法构成的群, 称为空间旋转群 $\mathbf{SO}(3)$, 描述三维空间中的旋转变换。1.7.3 小节提到, $\mathbf{SO}(3)$ 群是 Lorentz 群的子群, J^i 可以看作 $\mathbf{SO}(3)$ 群的生成元算符, 而 (3.25) 式是 $\mathbf{SO}(3)$ 群的 Lie 代数关系。

另一方面, \mathbf{K} 是增速算符。 \mathbf{J} 与 \mathbf{K} 的对易关系为

$$\begin{aligned}
 [J^i, K^j] &= \frac{1}{2} \varepsilon^{ikl} [J^{kl}, J^{0j}] = \frac{i}{2} \varepsilon^{ikl} \{[g^{l0} J^{kj} - (k \leftrightarrow l)] - (0 \leftrightarrow j)\} \\
 &= i\varepsilon^{ikl} [g^{l0} J^{kj} - (0 \leftrightarrow j)] = i\varepsilon^{ikl} (g^{l0} J^{kj} - g^{lj} J^{k0}) = -i\varepsilon^{ikl} g^{lj} J^{k0} = i\varepsilon^{ikl} \delta^{lj} J^{k0} \\
 &= i\varepsilon^{ikj} J^{k0} = i\varepsilon^{ijk} J^{0k} = i\varepsilon^{ijk} K^k,
 \end{aligned} \tag{3.30}$$

而 \mathbf{K} 自身的对易关系为

$$\begin{aligned}
 [K^i, K^j] &= [J^{0i}, J^{0j}] = i(g^{i0} J^{0j} - g^{00} J^{ij} - g^{ij} J^{00} + g^{0j} J^{i0}) \\
 &= -i(g^{00} J^{ij} + g^{ij} J^{00}) = -iJ^{ij} = -i\varepsilon^{ijk} J^k.
 \end{aligned} \tag{3.31}$$

归纳起来, 有

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, K^j] = i\varepsilon^{ijk} K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k. \tag{3.32}$$

3.2 量子矢量场的 Lorentz 变换

3.2.1 Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 ω^α_β 可以转化为

$$\omega^\alpha_\beta = g^{\alpha\mu} \omega_{\mu\beta} = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\beta} - g^{\alpha\mu} \omega_{\beta\mu}) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\mu} \omega_{\nu\mu} \delta^\nu_\beta) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\nu} \omega_{\mu\nu} \delta^\mu_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta, \quad (3.33)$$

其中 $(\mathcal{J}^{\mu\nu})^\alpha_\beta$ 定义为

$$(\mathcal{J}^{\mu\nu})^\alpha_\beta \equiv i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta). \quad (3.34)$$

容易看出, $\mathcal{J}^{\mu\nu}$ 是反对称的:

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \quad (3.35)$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^\gamma_\beta = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\gamma}) = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha). \quad (3.36)$$

这样的话, 可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta = \delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta. \quad (3.37)$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha_\beta &= (\mathcal{J}^{\mu\nu})^\alpha_\gamma (\mathcal{J}^{\rho\sigma})^\gamma_\beta - (\mathcal{J}^{\rho\sigma})^\alpha_\gamma (\mathcal{J}^{\mu\nu})^\gamma_\beta \\ &= i^2(g^{\mu\alpha}\delta^\nu_\gamma - \delta^\mu_\gamma g^{\nu\alpha})(g^{\rho\gamma}\delta^\sigma_\beta - \delta^\rho_\beta g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}\delta^\nu_\gamma g^{\rho\gamma}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\nu_\gamma \delta^\rho_\beta g^{\sigma\gamma} + \delta^\mu_\gamma g^{\nu\alpha} g^{\rho\gamma}\delta^\sigma_\beta - \delta^\mu_\gamma g^{\nu\alpha} \delta^\rho_\beta g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}g^{\rho\nu}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\rho_\beta g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^\sigma_\beta - g^{\nu\alpha}\delta^\rho_\beta g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\nu\rho}g^{\mu\alpha}\delta^\sigma_\beta + g^{\mu\rho}g^{\nu\alpha}\delta^\sigma_\beta + g^{\nu\sigma}g^{\mu\alpha}\delta^\rho_\beta - g^{\mu\sigma}g^{\nu\alpha}\delta^\rho_\beta \\ &\quad - [-g^{\sigma\mu}g^{\rho\alpha}\delta^\nu_\beta + g^{\rho\mu}g^{\sigma\alpha}\delta^\nu_\beta + g^{\sigma\nu}g^{\rho\alpha}\delta^\mu_\beta - g^{\rho\nu}g^{\sigma\alpha}\delta^\mu_\beta] \\ &= g^{\nu\rho}(g^{\sigma\alpha}\delta^\mu_\beta - g^{\mu\alpha}\delta^\sigma_\beta) + g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma_\beta - g^{\sigma\alpha}\delta^\nu_\beta) + g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho_\beta - g^{\rho\alpha}\delta^\mu_\beta) + g^{\mu\sigma}(g^{\rho\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\rho_\beta) \\ &= -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^\alpha_\beta - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha_\beta \\ &= i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^\alpha_\beta - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta + g^{\mu\sigma}(\mathcal{J}^{\nu\rho})^\alpha_\beta], \end{aligned} \quad (3.38)$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}). \quad (3.39)$$

可见, $\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。 Λ^α_β 属于 Lorentz 群的矢量表示, 因而 $\mathcal{J}^{\mu\nu}$ 就是矢量表示的生成元。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \quad (3.40)$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^\omega = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \quad (3.41)$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 \mathbf{g} 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}^{\delta} g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}_{\gamma} g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}_{\beta}, \quad (3.42)$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \quad (3.43)$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1} \left[\sum_{n=0}^{\infty} \frac{(\omega^{\mathrm{T}})^n}{n!} \right] \mathbf{g} = \sum_{n=0}^{\infty} \frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \quad (3.44)$$

若两个同阶方阵 A 和 B 相互对易，即 $[A, B] = 0$ ，则二项式定理成立：

$$(A + B)^n = \sum_{j=0}^n \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.45)$$

阶乘的定义可以推广到负整数：对于整数 $m < 0$ ，定义

$$m! \rightarrow \infty, \quad \frac{1}{m!} \rightarrow 0. \quad (3.46)$$

从而，对于 $j > n$ ，有 $[(n-j)!]^{-1} \rightarrow 0$ 。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A + B)^n = \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.47)$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{n=0}^{\infty} \frac{B^{n-j}}{(n-j)!} = e^A e^B. \quad (3.48)$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^0 = \mathbf{1}. \quad (3.49)$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = \mathbf{g}, \quad (3.50)$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp \left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right) \quad (3.51)$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的有限变换。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu} = 0$ ，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

在正则量子化程序中, 标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符, 类似于 (3.16) 式, $\phi(x)$ 的固有保时向 Lorentz 变换关系 (2.52) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \quad (3.52)$$

上式表明, 变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$, (3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \quad (3.53)$$

相应地, $\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\phi(x)|\Psi'\rangle = \langle\Psi|U^{-1}(\Lambda)\phi(x)U(\Lambda)|\Psi\rangle = \langle\Psi|\phi(\Lambda^{-1}x)|\Psi\rangle. \quad (3.54)$$

另一方面, 由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的相应 Lorentz 变换形式为

$$\partial^\mu\phi'(x') = U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda) = \partial'^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial'^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \quad (3.55)$$

于是, 在固有保时向 Lorentz 变换下, 自由实标量场的拉氏量 (2.54) 的变换形式为

$$\begin{aligned} \mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial'^\mu\phi(x')\partial'_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda) \\ &= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial'^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\} \\ &= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)] \\ &= \mathcal{L}(x), \end{aligned} \quad (3.56)$$

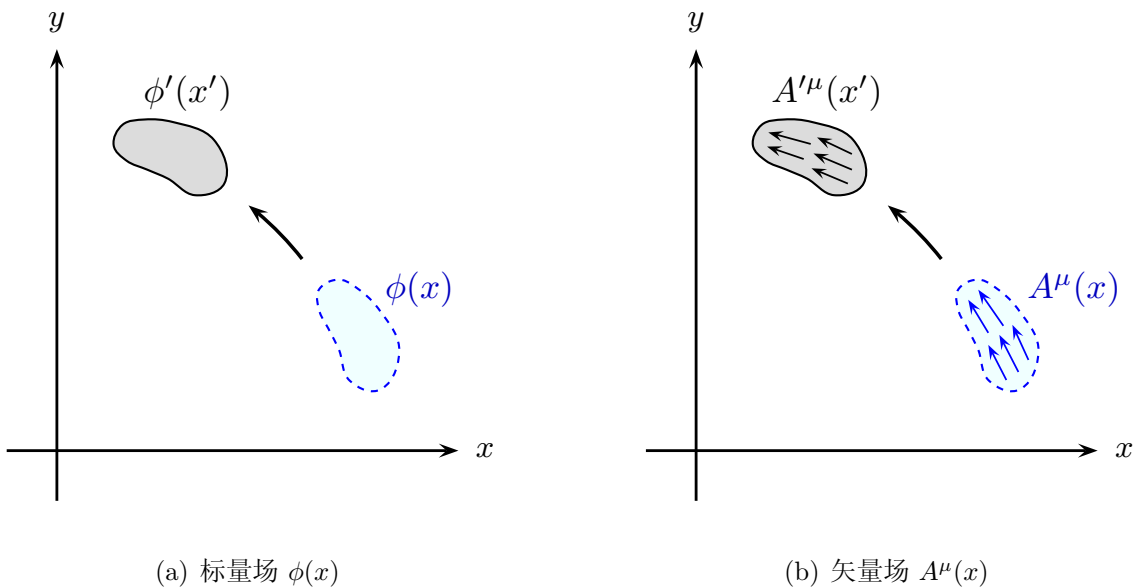


图 3.1: 在绕 z 轴空间旋转变换下, 标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而,

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(\Lambda^{-1}x). \quad (3.57)$$

可见, 拉氏量 (2.54) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, 可得

$$\begin{aligned} (\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu = g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta = g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) = g_{\nu\beta}g^{\mu\beta} + g^{\mu\beta}\omega_{\nu\beta} = \delta^\mu{}_\nu - g^{\mu\beta}\omega_{\beta\nu} \\ &= \delta^\mu{}_\nu - \omega^\mu{}_\nu, \end{aligned} \quad (3.58)$$

从而, 有

$$(\Lambda^{-1}x)^\mu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu. \quad (3.59)$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项, 得

$$\begin{aligned} \phi(\Lambda^{-1}x) &= \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \phi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \phi(x) = \phi(x) - \frac{1}{2}(\omega_{\mu\nu} x^\nu \partial^\mu + \omega_{\nu\mu} x^\mu \partial^\nu) \phi(x) \\ &= \phi(x) - \frac{1}{2}\omega_{\mu\nu}(x^\nu \partial^\mu - x^\mu \partial^\nu) \phi(x) = \phi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x) \\ &= \phi(x) - \frac{i}{2}\omega_{\mu\nu} i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x). \end{aligned} \quad (3.60)$$

根据 (3.6) 式, 将 (3.53) 式左边展开到 ω 的一阶项, 得

$$\begin{aligned} U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right) \\ &= \phi(x) - \frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}[\phi(x), J^{\mu\nu}]. \end{aligned} \quad (3.61)$$

两相比较, 给出

$$[\phi(x), J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi(x) = L^{\mu\nu}\phi(x), \quad (3.62)$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu} \equiv i(x^\mu \partial^\nu - x^\nu \partial^\mu). \quad (3.63)$$

对于空间分量 L^{ij} , 可以等价地定义

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{i}{2}\varepsilon^{ijk}(x^j \partial^k - x^k \partial^j) = \frac{i}{2}(\varepsilon^{ijk}x^j \partial^k - \varepsilon^{ikj}x^j \partial^k) = i\varepsilon^{ijk}x^j \partial^k, \quad (3.64)$$

写成空间矢量的形式是

$$\mathbf{L} = -i\mathbf{x} \times \nabla. \quad (3.65)$$

可见, \mathbf{L} 就是微分算符形式的轨道角动量算符。根据 (3.21) 式, (3.62) 式的纯空间分量部分可以改写为

$$[\phi(x), \mathbf{J}] = \mathbf{L}\phi(x). \quad (3.66)$$

上式表明, 总角动量算符 \mathbf{J} 生成了轨道角动量, 但没有生成自旋角动量。这说明标量场没有自旋, 对应于零自旋粒子。

3.2.3 量子矢量场的 Lorentz 变换形式

$\partial^\mu \phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有像 (3.55) 式那样的 Lorentz 变换形式, 即

$$A'^\mu(x') = U^{-1}(\Lambda) A^\mu(x') U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \quad (3.67)$$

或者写成

$$U^{-1}(\Lambda) A^\mu(x) U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \quad (3.68)$$

这就是量子矢量场的 Lorentz 变换形式。相应地, $A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle \Psi' | A^\mu(x) | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) A^\mu(x) U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\nu \langle \Psi | A^\nu(\Lambda^{-1}x) | \Psi \rangle. \quad (3.69)$$

对于固有保时向 Lorentz 变换, 根据矢量表示中的无穷小形式 (3.40), (3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] A^\nu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.70)$$

将上式与 (1.168) 式比较, 可以发现, 1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$ 。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出, 在 Lorentz 变换下, 除了矢量场的分布区域发生变化之外, 矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式, 在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项, 得

$$\begin{aligned} A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta \partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta} x^\beta \partial^\alpha A^\nu(x) \\ &= A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x). \end{aligned} \quad (3.71)$$

从而, (3.68) 式右边可展开为

$$\begin{aligned} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] \left[A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x) \right] \\ &= A^\mu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)]. \end{aligned} \quad (3.72)$$

另一方面, (3.68) 式左边的无穷小展开式为

$$\begin{aligned} U^{-1}(\Lambda) A^\mu(x) U(\Lambda) &= \left(1 + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} \right) A^\mu(x) \left(1 - \frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta} \right) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\alpha\beta} A^\mu(x) J^{\alpha\beta} + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} A^\mu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [A^\mu(x), J^{\rho\sigma}]. \end{aligned} \quad (3.73)$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.74)$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathcal{J}^{jk}, \quad \mathcal{J} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \quad (3.75)$$

再根据 (3.21) 和 (3.64) 式, (3.74) 式的纯空间分量部分可以改写为

$$[A^\mu(x), \mathbf{J}] = \mathbf{L} A^\mu(x) + (\mathcal{J})^\mu{}_\nu A^\nu(x). \quad (3.76)$$

上式表明, 总角动量算符 \mathbf{J} 不仅生成了轨道角动量, 还生成了由 \mathcal{J} 描述的自旋角动量。 \mathcal{J}^i 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \quad (3.77)$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \quad (3.78)$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \quad (3.79)$$

只关注空间分量, 可得

$$(\mathcal{J}^1 \mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2 \mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \quad (3.80)$$

因此, 有

$$(\mathcal{J}^2)^i{}_j = (\mathcal{J}^1 \mathcal{J}^1 + \mathcal{J}^2 \mathcal{J}^2 + \mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \quad (3.81)$$

根据量子力学的角动量理论, \mathcal{J}^2 的本征值为 $s(s+1)$, 即 $(\mathcal{J}^2)^i{}_j = s(s+1)\delta^i{}_j$, 其中 s 为自旋量子数。可见, 矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \quad (3.82)$$

经过量子化程序之后, 矢量场 $A^\mu(x)$ 应当描述自旋为 1 的粒子。

3.3 有质量矢量场的正则量子化

类似于电磁场, 对任意的矢量场 A^μ 可以定义反对称的场强张量

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (3.83)$$

对于一个自由的有质量的实矢量场 A^μ ，用场强张量可以将它的 **Lorentz** 不变拉氏量写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \quad (3.84)$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu] \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu. \end{aligned} \quad (3.85)$$

从而，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \quad (3.86)$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \quad (3.87)$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \quad (3.88)$$

上式称为 **Proca 方程**，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0. \quad (3.89)$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = m^2 \partial_\nu A^\nu. \quad (3.90)$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz 条件**

$$\partial_\mu A^\mu = 0. \quad (3.91)$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \partial^2 A^\nu. \quad (3.92)$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon* 方程

$$(\partial^2 + m^2)A^\mu(x) = 0. \quad (3.93)$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \quad (3.94)$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \quad (3.95)$$

由于 $\pi_0 = 0$ ，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。实际上，由于 Lorenz 条件 (3.91) 的存在， A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于其它 3 个分量的量。因此，正则量子化程序要求独立的正则变量满足等时对易关系

$$[A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = i\delta^i_j \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^i(\mathbf{x}, t), A^j(\mathbf{y}, t)] = [\pi_i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = 0. \quad (3.96)$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程，应该具有两个平面波解，即正能解 $\exp(-ip \cdot x)$ 和负能解 $\exp(ip \cdot x)$ 。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标，平面波展开式的系数也必须具有一个这样的指标。一般地，对于确定的动量 \mathbf{p} ，矢量场的正能解模式具有如下形式：

$$A^\mu(x; \mathbf{p}, \sigma) = e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (3.97)$$

这里的系数 $e^\mu(\mathbf{p}, \sigma)$ 是 Lorentz 矢量，称为**极化矢量** (polarization vector)，它依赖于动量 p ，而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而，可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点，一组极化矢量应当是线性独立且正交完备的。Lorentz 矢量空间是一个 4 维空间，因而这样的极化矢量应该有 4 个，包括 1 个类时的极化矢量 $e^\mu(\mathbf{p}, 0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 。在没有额外约束的情况下，我们要求这 4 个极化矢量是实的，而且满足 Lorentz 矢量空间中的正交归一关系

$$e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') = g_{\sigma\sigma'} \quad (3.98)$$

和完备性关系

$$\sum_{\sigma=0}^3 g_{\sigma\sigma'} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma') = g_{\mu\nu}. \quad (3.99)$$

上面这两个关系都是 Lorentz 协变的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^μ 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空**横向极化矢量**

$$e^\mu(\mathbf{p}, 1) = (0, \mathbf{e}(\mathbf{p}, 1)), \quad e^\mu(\mathbf{p}, 2) = (0, \mathbf{e}(\mathbf{p}, 2)). \quad (3.100)$$

此处，

$$\mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}| |\mathbf{p}_T|} (p^1 p^3, p^2 p^3, -|\mathbf{p}_T|^2), \quad \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|} (-p^2, p^1, 0), \quad (3.101)$$

其中

$$|\mathbf{p}_T| \equiv \sqrt{(p^1)^2 + (p^2)^2}. \quad (3.102)$$

“横向”指的是它们在三维空间中与 \mathbf{p} 垂直，即

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|}[(p^1)^2 p^3 + (p^2)^2 p^3 - p^3 |\mathbf{p}_T|^2] = 0, \quad (3.103)$$

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|}(-p^1 p^2 + p^2 p^1) = 0. \quad (3.104)$$

此外，存在如下关系：

$$\begin{aligned} \mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 1) &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} [(p^1)^2 (p^3)^2 + (p^2)^2 (p^3)^2 + |\mathbf{p}_T|^4] \\ &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 [(p^3)^2 + |\mathbf{p}_T|^2] = \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 |\mathbf{p}|^2 = 1, \end{aligned} \quad (3.105)$$

$$\mathbf{e}(\mathbf{p}, 2) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|^2} [(p^2)^2 + (p^1)^2] = \frac{1}{|\mathbf{p}_T|^2} |\mathbf{p}_T|^2 = 1, \quad (3.106)$$

$$\mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|^2} (-p^1 p^3 p^2 + p^2 p^3 p^1) = 0. \quad (3.107)$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = \delta_{ij}, \quad i, j = 1, 2. \quad (3.108)$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_\mu e^\mu(\mathbf{p}, 1) = p_\mu e^\mu(\mathbf{p}, 2) = 0, \quad (3.109)$$

和正交归一关系

$$e_\mu(\mathbf{p}, i) e^\mu(\mathbf{p}, j) = -\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = -\delta_{ij} = g_{ij}. \quad (3.110)$$

接着，要求第 3 个类空极化矢量 $e^\mu(\mathbf{p}, 3)$ 是纵向的，即在三维空间中与 \mathbf{p} 平行。这样还不能确定它的时间分量，为此，我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p}, 3) = 0$ ，而正交归一关系 (3.98) 将决定它的归一化。于是，纵向极化矢量的形式为

$$e^\mu(\mathbf{p}, 3) = \left(\frac{|\mathbf{p}|}{m}, \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} \right). \quad (3.111)$$

可以验证，它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p}, 3) = p^0 \frac{|\mathbf{p}|}{m} - \mathbf{p} \cdot \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} = \frac{p^0 |\mathbf{p}|}{m} - \frac{p^0 |\mathbf{p}|}{m} = 0, \quad (3.112)$$

和正交归一关系

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, 3) = \frac{|\mathbf{p}|}{m} \frac{|\mathbf{p}|}{m} - \frac{(p^0)^2 \mathbf{p} \cdot \mathbf{p}}{m^2 |\mathbf{p}|^2} = \frac{|\mathbf{p}|^2}{m^2} - \frac{(p^0)^2}{m^2} = -\frac{(p^0)^2 - |\mathbf{p}|^2}{m^2} = -1 = g_{33}; \quad (3.113)$$

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, i) = -\frac{p^0}{m|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.114)$$

最后，我们可以将类时极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p}, 0) = \frac{1}{m} p^\mu = \frac{1}{m} (p^0, \mathbf{p}). \quad (3.115)$$

它满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = \frac{p^2}{m^2} = 1 = g_{00}; \quad (3.116)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, i) = -\frac{1}{m} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2; \quad (3.117)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 3) = \frac{1}{m^2} p^0 |\mathbf{p}| - \frac{p^0}{m^2 |\mathbf{p}|} \mathbf{p} \cdot \mathbf{p} = 0. \quad (3.118)$$

不过，它不满足四维时空的横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = \frac{p^2}{m} = m. \quad (3.119)$$

可以验证，由 (3.100)、(3.101)、(3.111) 和 (3.115) 式定义的一组极化矢量确实满足完备性关系 (3.99):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= \frac{1}{m^2} \begin{pmatrix} p^0 p^0 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^1 p^0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ -p^2 p^0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ -p^3 p^0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \\ & \quad - \frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{m^2} \begin{pmatrix} |\mathbf{p}|^2 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^0 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^3 \\ -p^0 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^3 \\ -p^0 p^3 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(p^1)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^1 p^3)^2 + (p^2)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{(p^2)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^2 p^3)^2 + (p^1)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} & \frac{(p^3)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{|\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2|\mathbf{p}_T|^2 + (p^1)^2(|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_T|^2} & -\frac{p^1p^2}{|\mathbf{p}|^2} + \frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{p^1p^3}{|\mathbf{p}|^2} + \frac{p^1p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^2}{|\mathbf{p}|^2} + \frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{(p^2)^2|\mathbf{p}_T|^2 + (p^2)^2(|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_T|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2} + \frac{p^2p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^3}{|\mathbf{p}|^2} + \frac{p^1p^3}{|\mathbf{p}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2} + \frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{(p^3)^2 + |\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \tag{3.120}
\end{aligned}$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91)，正能解模式 (3.97) 应满足

$$0 = \partial_\mu A^\mu(x; \mathbf{p}, \sigma) = -ip_\mu e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \tag{3.121}$$

即

$$p_\mu e^\mu(\mathbf{p}, \sigma) = 0. \tag{3.122}$$

也就是说，描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 不能用于描述有质量矢量场 A^μ 。这说明 A^μ 只有 3 个物理的极化状态，由类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 描述。根据完备性关系 (3.99)，这 3 个物理的极化矢量满足

$$-\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = \sum_{\sigma=1}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) = g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}, \tag{3.123}$$

即具有求和关系

$$\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{3.124}$$

通过如下线性组合，我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p, \lambda)$ ，其中 $\lambda = +, 0, -$ ：

$$\varepsilon^\mu(\mathbf{p}, \pm) \equiv \frac{1}{\sqrt{2}} [\mp e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)], \tag{3.125}$$

$$\varepsilon^\mu(\mathbf{p}, 0) \equiv e^\mu(\mathbf{p}, 3). \tag{3.126}$$

这样定义的 $\varepsilon^\mu(p, \pm)$ 是复的，而 $\varepsilon^\mu(p, 0)$ 是实的。它们都满足四维横向条件

$$p_\mu \varepsilon^\mu(\mathbf{p}, \lambda) = 0. \tag{3.127}$$

它们还满足

$$\begin{aligned}
\varepsilon_\mu^*(\mathbf{p}, \pm) \varepsilon^\mu(\mathbf{p}, \pm) &= \frac{1}{2} [\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)] [\mp e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)] \\
&= \frac{1}{2} e_\mu(\mathbf{p}, 1) e^\mu(\mathbf{p}, 1) + \frac{1}{2} e_\mu(\mathbf{p}, 2) e^\mu(\mathbf{p}, 2) = \frac{1}{2} (g_{11} + g_{22}) = -1, \tag{3.128}
\end{aligned}$$

$$\begin{aligned}\varepsilon_\mu^*(\mathbf{p}, \pm)\varepsilon^\mu(\mathbf{p}, \mp) &= \frac{1}{2}[\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)][\pm e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)] \\ &= -\frac{1}{2}e_\mu(\mathbf{p}, 1)e^\mu(\mathbf{p}, 1) + \frac{1}{2}e_\mu(\mathbf{p}, 2)e^\mu(\mathbf{p}, 2) = \frac{1}{2}(-g_{11} + g_{22}) = 0,\end{aligned}\quad (3.129)$$

$$\varepsilon_\mu^*(\mathbf{p}, 0)\varepsilon^\mu(\mathbf{p}, 0) = e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -1, \quad (3.130)$$

$$\varepsilon_\mu^*(\mathbf{p}, \pm)\varepsilon^\mu(\mathbf{p}, 0) = \frac{1}{2}[\mp e_\mu(\mathbf{p}, 1) + ie_\mu(\mathbf{p}, 2)]e^\mu(\mathbf{p}, 3) = 0, \quad (3.131)$$

即具有正交归一关系

$$\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon^\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}. \quad (3.132)$$

极化矢量求和关系则是

$$\begin{aligned}\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2}[e_\mu(p, 1) + ie_\mu(p, 2)][e_\nu(p, 1) - ie_\nu(p, 2)] \\ &\quad + \frac{1}{2}[-e_\mu(p, 1) + ie_\mu(p, 2)][-e_\nu(p, 1) - ie_\nu(p, 2)] + e_\mu(p, 3)e_\nu(p, 3) \\ &= e_\mu(p, 1)e_\nu(p, 1) + e_\mu(p, 2)e_\nu(p, 2) + e_\mu(p, 3)e_\nu(p, 3) \\ &= \sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma)e_\nu(\mathbf{p}, \sigma),\end{aligned}\quad (3.133)$$

与 (3.124) 式左边相等, 故

$$\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \quad (3.134)$$

四维横向条件 (3.127) 在上式中体现为

$$p^\nu \sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p}, \lambda)\varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu + \frac{p_\mu p^2}{m^2} = -p_\mu + p_\mu = 0. \quad (3.135)$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 \mathbf{p} 的方向由 $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$ 表征, 于是, 在 Lorentz 群矢量表示中, 螺旋度矩阵定义为

$$\hat{\mathbf{p}} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & & \\ & 0 & -ip^3 & ip^2 \\ & ip^3 & 0 & -ip^1 \\ & -ip^2 & ip^1 & 0 \end{pmatrix}. \quad (3.136)$$

将 (3.101) 和 (3.111) 式代入 (3.125) 和 (3.126) 式, 得到 $\varepsilon^\mu(p, \lambda)$ 的列矢量形式为

$$\varepsilon^\mu(p, 0) = \frac{1}{m|\mathbf{p}|} \begin{pmatrix} |\mathbf{p}|^2 \\ p^0 p^1 \\ p^0 p^2 \\ p^0 p^3 \end{pmatrix}, \quad \varepsilon^\mu(p, +) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 + ip^2 |\mathbf{p}| \\ -p^2 p^3 - ip^1 |\mathbf{p}| \\ |\mathbf{p}_T|^2 \end{pmatrix},$$

$$\varepsilon^\mu(p, -) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ p^1 p^3 + ip^2 |\mathbf{p}| \\ p^2 p^3 - ip^1 |\mathbf{p}| \\ -|\mathbf{p}_T|^2 \end{pmatrix}. \quad (3.137)$$

从而, 可得

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, 0) = \frac{1}{m|\mathbf{p}|^2} \begin{pmatrix} 0 \\ -ip^3 p^0 p^2 + ip^2 p^0 p^3 \\ ip^3 p^0 p^1 - ip^1 p^0 p^3 \\ -ip^2 p^0 p^1 + ip^1 p^0 p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \varepsilon^\mu(p, 0), \quad (3.138)$$

$$\begin{aligned} (\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, +) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}_T|^2 \\ -ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}_T|^2 \\ ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| - ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = +\varepsilon^\mu(p, +), \end{aligned} \quad (3.139)$$

$$\begin{aligned} (\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, -) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}_T|^2 \\ ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}_T|^2 \\ -ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| + ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = -\varepsilon^\mu(p, -). \end{aligned} \quad (3.140)$$

归纳起来, 有

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\mathcal{J}}) \varepsilon^\mu(p, \lambda) = \lambda \varepsilon^\mu(p, \lambda). \quad (3.141)$$

上式说明极化矢量 $\varepsilon^\mu(p, \lambda)$ 是螺旋度的本征态, 本征值为 λ 。因此, $\varepsilon^\mu(p, \lambda)$ 描述动量为 \mathbf{p} 、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda = \pm 1$ 对应于两种**横向极化**, $\lambda = 0$ 对应于**纵向极化**。

有质量的实矢量场算符 $A^\mu(\mathbf{x}, t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态, 形式为

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right], \quad (3.142)$$

其中 $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ 。容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.143)$$

根据 (3.95) 式, 共轭动量密度为

$$\begin{aligned} \pi_i = -\partial_0 A_i + \partial_i A_0 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left\{ [ip_0 \varepsilon_i(\mathbf{p}, \lambda) - ip_i \varepsilon_0(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda} e^{-ip \cdot x} \right. \\ \left. + [-ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) + ip_i \varepsilon_0^*(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right\}, \end{aligned} \quad (3.144)$$

引入

$$\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \equiv \varepsilon_i(\mathbf{p}, \lambda) - \frac{p_i}{p_0} \varepsilon_0(\mathbf{p}, \lambda), \quad (3.145)$$

则有

$$p_0 \varepsilon_i(\mathbf{p}, \lambda) - p_i \varepsilon_0(\mathbf{p}, \lambda) = p_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda), \quad (3.146)$$

从而, 可以将共轭动量密度的平面波展开式写得更加紧凑:

$$\pi_i(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right]. \quad (3.147)$$

易见, 它也满足自共轭条件

$$[\pi_i(\mathbf{x}, t)]^\dagger = \pi_i(\mathbf{x}, t). \quad (3.148)$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned} & \int d^3 x e^{iq \cdot x} A^\mu \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} + \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq^0 t} \right] \end{aligned} \quad (3.149)$$

和

$$\begin{aligned} \int d^3 x e^{iq \cdot x} \partial_0 A^\mu &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\ &\quad \left. - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \end{aligned}$$

$$= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} - \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq_0 t} \right], \quad (3.150)$$

以及正交归一关系 (3.132), 可得

$$\begin{aligned} \varepsilon_\mu^*(\mathbf{q}, \lambda') \int d^3x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon_\mu(\mathbf{q}, \lambda') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} \\ &= -i\sqrt{2E_{\mathbf{q}}} \sum_{\lambda=\pm,0} (-\delta_{\lambda'\lambda}) a_{\mathbf{q},\lambda} = i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q},\lambda'}. \end{aligned} \quad (3.151)$$

由 Lorenz 条件 (3.91) 可得

$$\partial_0 A^0 = -\partial_i A^i, \quad (3.152)$$

根据 (3.95) 式, 有

$$\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \quad (3.153)$$

于是, 湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$\begin{aligned} a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \varepsilon_\mu^*(\mathbf{p}, \lambda) \int d^3x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu) \\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\varepsilon_0^*(\mathbf{p}, \lambda) \partial_0 A^0 + \varepsilon_i^*(\mathbf{p}, \lambda) \partial_0 A^i - ip_0 \varepsilon_\mu^*(\mathbf{p}, \lambda) A^\mu] \\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0 \\ &\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i]. \end{aligned} \quad (3.154)$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$\begin{aligned} \int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0] &= \int d^3x [\varepsilon_0^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^i + \varepsilon_i^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^0] \\ &= \int d^3x [ip_i \varepsilon_0^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^0] \\ &= \int d^3x e^{ip \cdot x} [i\varepsilon_0^*(\mathbf{p}, \lambda) p_i A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0], \end{aligned} \quad (3.155)$$

从而, 有

$$\begin{aligned} a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [i\varepsilon_0^*(\mathbf{p}, \lambda) p_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0 \\ &\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i] \\ &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \{ \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - ip^\mu \varepsilon_\mu^*(\mathbf{p}, \lambda) A^0 - i[p_0 \varepsilon_i^*(\mathbf{p}, \lambda) - p_i \varepsilon_0^*(\mathbf{p}, \lambda)] A^i \}. \end{aligned} \quad (3.156)$$

再利用四维横向条件 (3.127) 和 (3.146) 式, 得到

$$a_{\mathbf{p},\lambda} = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(x) - ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(x)]$$

$$= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(x) + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(x)]. \quad (3.157)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p}, \lambda}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} [\varepsilon^i(\mathbf{p}, \lambda) \pi_i(x) - ip_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda) A^i(x)]. \quad (3.158)$$

利用等时对易关系 (3.96), 可得湮灭算符与产生算符的对易关系为

$$\begin{aligned} & [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(\mathbf{x}, t) + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(\mathbf{x}, t), \\ & \quad \varepsilon^j(\mathbf{q}, \lambda') \pi_j(\mathbf{y}, t) - iq_0 \tilde{\varepsilon}_j(\mathbf{q}, \lambda') A^j(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \{ -iq_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') [\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \\ & \quad + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') [A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] \} \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \delta^{(3)}(\mathbf{x} - \mathbf{y}) [-q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') \delta^j_i - p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') \delta^i_j] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} [-E_{\mathbf{q}} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{q}, \lambda') - E_{\mathbf{p}} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda')] \\ &= -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) [\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda')]. \end{aligned} \quad (3.159)$$

根据定义式 (3.145)、四维横向条件 (3.127) 和正交归一关系 (3.132), 有

$$\begin{aligned} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') - \frac{1}{p_0} p_i \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\ &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') + \frac{1}{p_0} p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\ &= \varepsilon^{\mu*}(\mathbf{p}, \lambda) \varepsilon_\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}, \end{aligned} \quad (3.160)$$

取复共轭, 可得

$$\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}. \quad (3.161)$$

于是,

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] = -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (3.162)$$

另一方面,

$$\begin{aligned} & [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] \\ &= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(\mathbf{x}, t) + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(\mathbf{x}, t), \\ & \quad \varepsilon^{j*}(\mathbf{q}, \lambda') \pi_j(\mathbf{y}, t) + iq_0 \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') A^j(\mathbf{y}, t)] \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \{ i q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') [\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \\
&\quad + i p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{j*}(\mathbf{q}, \lambda') [A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] \} \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x} - \mathbf{y}) [q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j^*(\mathbf{q}, \lambda') \delta^j_i - p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{j*}(\mathbf{q}, \lambda') \delta^i_j] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} [E_{\mathbf{q}} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(\mathbf{q}, \lambda') - E_{\mathbf{p}} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda')] \\
&= -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) e^{2iE_{\mathbf{p}}t} [\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda')]. \tag{3.163}
\end{aligned}$$

对四维横向条件 (3.127) 取复共轭, 得

$$p_\mu \varepsilon^{\mu*}(\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) + p_i \varepsilon^{i*}(\mathbf{p}, \lambda) = 0. \tag{3.164}$$

将上式中的 \mathbf{p} 替换成 $-\mathbf{p}$, 得

$$p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda) - p_i \varepsilon^{i*}(-\mathbf{p}, \lambda) = 0. \tag{3.165}$$

因此, 有

$$p_i \varepsilon^{i*}(\mathbf{p}, \lambda) = -p_0 \varepsilon^{0*}(\mathbf{p}, \lambda), \quad -p_i \varepsilon^{i*}(-\mathbf{p}, \lambda) = -p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda), \tag{3.166}$$

或者写成

$$\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(\mathbf{p}, \lambda), \quad -\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda) = p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda). \tag{3.167}$$

从而, 可得

$$\begin{aligned}
\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \left[\varepsilon_i^*(-\mathbf{p}, \lambda) + \frac{p_i}{p_0} \varepsilon_0^*(-\mathbf{p}, \lambda) \right] \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') + \frac{1}{p_0} p_i \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda') \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') - \frac{1}{p_0} p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda') \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i^*(-\mathbf{p}, \lambda') - \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0^*(-\mathbf{p}, \lambda'), \tag{3.168} \\
\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') &= \left[\varepsilon_i^*(\mathbf{p}, \lambda) - \frac{p_i}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) \right] \varepsilon^{i*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) p_i \varepsilon^{i*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda) \varepsilon^{0*}(-\mathbf{p}, \lambda'). \tag{3.169}
\end{aligned}$$

可见, $\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') = 0$, 故

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = 0. \tag{3.170}$$

综上, 产生湮灭算符的对易关系为

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = [a_{\mathbf{p}, \lambda}^\dagger, a_{\mathbf{q}, \lambda'}^\dagger] = 0. \tag{3.171}$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \quad (3.172)$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \quad (3.173)$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \quad (3.174)$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$, 因此,

$$A^0 = -\frac{1}{m^2} \partial_\mu F^{\mu 0} = -\frac{1}{m^2} \partial_i F^{i0} = -\frac{1}{m^2} \partial_i \pi^i = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.175)$$

从而, 可得

$$-\boldsymbol{\pi} \cdot \dot{\mathbf{A}} = \boldsymbol{\pi} \cdot (\boldsymbol{\pi} + \nabla A_0) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) - A_0 (\nabla \cdot \boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2. \quad (3.176)$$

另一方面,

$$\frac{1}{2} F_{0i} F^{0i} = \frac{1}{2} \pi_i \pi^i = -\frac{1}{2} \boldsymbol{\pi}^2. \quad (3.177)$$

利用 (1.84) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial_m A^n = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n = -\varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n, \quad (3.178)$$

从而,

$$\begin{aligned} \frac{1}{4} F_{ij} F^{ij} &= \frac{1}{4} F^{ij} F^{ij} = \frac{1}{4} \varepsilon^{ijk} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{ijl} \varepsilon^{lpq} \partial_p A^q = \frac{1}{4} 2 \delta^{kl} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{lpq} \partial_p A^q \\ &= \frac{1}{2} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{kpq} \partial_p A^q = \frac{1}{2} (\nabla \times \mathbf{A})^2. \end{aligned} \quad (3.179)$$

于是, 有

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} = -\frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2. \quad (3.180)$$

根据 (1.119) 式, 有质量矢量场的哈密顿量密度为

$$\begin{aligned} \mathcal{H} &= \pi_i \partial_0 A^i - \mathcal{L} = \pi_i \partial_0 A^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \\ &= -\boldsymbol{\pi} \cdot \dot{\mathbf{A}} - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) \\ &= \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} m^2 \mathbf{A}^2 \\ &= \frac{1}{2} \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2. \end{aligned} \quad (3.181)$$

上式最后一行第二项是一个全散度，对全空间积分时它没有贡献。于是，哈密顿量为

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x \left[\boldsymbol{\pi}^2 + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 + (\nabla \times \mathbf{A})^2 + m^2 \mathbf{A}^2 \right]. \quad (3.182)$$

下面逐项进行计算。

哈密顿量的第一项是

$$\begin{aligned} & \frac{1}{2} \int d^3x \boldsymbol{\pi}^2 \\ &= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0)(iq_0) \left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\ & \quad \cdot \left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\ &= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\ & \quad - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} + \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\ & \quad \left. + \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 q_0}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\ & \quad \left. \left. + \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right] \right. \\ & \quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\ & \quad \left. \left. + \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} E_{\mathbf{p}}^2 \left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\ & \quad \left. - \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} - \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.183) \end{aligned}$$

第二项是

$$\begin{aligned} & \frac{1}{2} \int d^3x \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 \\ &= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \frac{(ip_0)(iq_0)}{m^2} \left[i\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + i\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\ & \quad \times \left[i\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} + i\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\ &= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \frac{1}{m^2} \left\{ -[\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\ & \quad - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\ & \quad \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right\} \\ &= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 q_0}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \frac{1}{m^2} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \\
& + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \\
& \quad \left. + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \Big\} \\
= & \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \frac{E_{\mathbf{p}}^2}{m^2} \left\{ [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger \right. \\
& + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \\
& \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \tag{3.184}
\end{aligned}$$

第三项是

$$\begin{aligned}
& \frac{1}{2} \int d^3 x (\nabla \times \mathbf{A})^2 \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[i\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} - i\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
& \cdot \left[i\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - i\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
& + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\
& \left. - [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right\} \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
& + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \\
& \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
& \left. \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \right\} \\
= & \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger \right. \\
& + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} + [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \\
& \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \tag{3.185}
\end{aligned}$$

第四项是

$$\begin{aligned}
& \frac{1}{2} \int d^3 x m^2 \mathbf{A}^2 \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
& \cdot \left[\boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right.
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} + \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(\mathbf{p}+\mathbf{q}) \cdot \mathbf{x}} \\
& + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(\mathbf{p}+\mathbf{q}) \cdot \mathbf{x}} \Big] \\
= & \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q m^2}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
& \left. \left. + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
& \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
& \left. \left. + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
= & \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} m^2 \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
& \left. + \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.186)
\end{aligned}$$

综合起来，哈密顿量化为

$$\begin{aligned}
H = & \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \left[f_1(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + f_1^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
& \left. + f_2(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + f_2^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right], \quad (3.187)
\end{aligned}$$

其中，

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') \equiv & E_{\mathbf{p}}^2 \tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda') + \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda')] \\
& + [\mathbf{p} \times \varepsilon(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \varepsilon^*(\mathbf{p}, \lambda')] + m^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda'), \quad (3.188)
\end{aligned}$$

$$\begin{aligned}
f_2(\mathbf{p}, \lambda, \lambda') \equiv & -E_{\mathbf{p}}^2 \tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}(-\mathbf{p}, \lambda') - \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\varepsilon}(-\mathbf{p}, \lambda')] \\
& + [\mathbf{p} \times \varepsilon(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \varepsilon(-\mathbf{p}, \lambda')] + m^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda'). \quad (3.189)
\end{aligned}$$

现在，我们计算 $f_1(\mathbf{p}, \lambda, \lambda')$ 。由 (3.145)、(3.167) 和 (3.132) 式，可得

$$\begin{aligned}
\tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda') & = \left[\varepsilon(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\varepsilon^*(\mathbf{p}, \lambda') - \frac{\mathbf{p}}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right] \\
= & \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
= & \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon_0^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon_0(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
= & -\varepsilon_\mu(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') + \left(\frac{|\mathbf{p}|^2}{p_0^2} - 1 \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
= & \delta_{\lambda \lambda'} - \frac{m^2}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \quad (3.190)
\end{aligned}$$

另一方面，

$$[\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda')]$$

$$\begin{aligned}
&= \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right] \\
&= [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(p_0^2 - 2|\mathbf{p}|^2 + \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \left[p_0^2 - |\mathbf{p}|^2 + \frac{|\mathbf{p}|^2}{p_0^2} (|\mathbf{p}|^2 - p_0^2) \right] \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(m^2 - m^2 \frac{|\mathbf{p}|^2}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \tag{3.191}
\end{aligned}$$

对于任意空间矢量 \mathbf{a} 和 \mathbf{b} , 利用 (1.84) 式, 有

$$\begin{aligned}
(\mathbf{p} \times \mathbf{a}) \cdot (\mathbf{p} \times \mathbf{b}) &= \varepsilon^{ijk} p^j a^k \varepsilon^{imn} p^m b^n = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}) p^j a^k p^m b^n \\
&= p^j a^k p^j b^k - p^j a^k p^k b^j = |\mathbf{p}|^2 \mathbf{a} \cdot \mathbf{b} - (\mathbf{p} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}), \tag{3.192}
\end{aligned}$$

从而, 可得

$$\begin{aligned}
[\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda'). \tag{3.193}
\end{aligned}$$

于是, (3.188) 式化为

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') &= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad + |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} + E_{\mathbf{p}}^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - E_{\mathbf{p}}^2 \varepsilon_{\mu}(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.194}
\end{aligned}$$

因此,

$$f_1(\mathbf{p}, \lambda, \lambda') = f_1^*(\mathbf{p}, \lambda, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.195}$$

接着, 我们计算 $f_2(\mathbf{p}, \lambda, \lambda')$ 。由 (3.145) 和 (3.167) 式, 可得

$$\begin{aligned}
\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda') &= \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\mathbf{p}}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right] \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda')
\end{aligned}$$

$$= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{1}{E_{\mathbf{p}}^2} (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'). \quad (3.196)$$

另一方面,

$$\begin{aligned} & [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] \\ &= \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right] \\ &= [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \varepsilon_0(-\mathbf{p}, \lambda') \\ &\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\ &= -p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\ &\quad + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^0(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\ &= \left(-p_0^2 + 2|\mathbf{p}|^2 - \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = -\frac{1}{E_{\mathbf{p}}^2} (E_{\mathbf{p}}^2 - |\mathbf{p}|^2)^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\ &= -\frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'), \end{aligned} \quad (3.197)$$

而

$$\begin{aligned} [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] \\ &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\ &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'). \end{aligned} \quad (3.198)$$

于是, (3.189) 式化为

$$\begin{aligned} f_2(\mathbf{p}, \lambda, \lambda') &= -E_{\mathbf{p}}^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\ &\quad + |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda') + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') \\ &= (-2E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 + E_{\mathbf{p}}^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = 0. \end{aligned} \quad (3.199)$$

因此,

$$f_2(\mathbf{p}, \lambda, \lambda') = f_2^*(\mathbf{p}, \lambda, \lambda') = 0. \quad (3.200)$$

将 (3.195) 和 (3.200) 式代入 (3.187) 式, 再利用产生湮灭算符的对易关系 (3.171), 可得有质量矢量场的哈密顿量为

$$\begin{aligned} H &= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right) = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} \right) \\ &= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{3}{2} E_{\mathbf{p}}. \end{aligned} \quad (3.201)$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和，第二项是零点能。

根据 (1.158) 式，有质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi_i \nabla A^i \\
&= - \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \times \left[i\mathbf{q} \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - i\mathbf{q} \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 \mathbf{q}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[- \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 \mathbf{q}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right. \\
&\quad \left. + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.202)
\end{aligned}$$

由 (3.145) 和 (3.166) 式可得

$$\begin{aligned}
\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') &= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_i \varepsilon^i(-\mathbf{p}, \lambda') \\
&= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\
&= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'), \quad (3.203)
\end{aligned}$$

从而，有

$$\begin{aligned}
&- \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda) \varepsilon^{0*}(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{-\mathbf{p},\lambda'} a_{\mathbf{p},\lambda} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{-\mathbf{p},\lambda'}^\dagger a_{\mathbf{p},\lambda}^\dagger e^{2iE_{\mathbf{p}}t} \right\}
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \quad (3.204)
\end{aligned}$$

上式第二步进行了 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换, 由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和, 这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.171)。留意到第一步与第三步的结果互为相反数, 可知上式为零。因此, (3.202) 式最后两行方括号中最后两项没有贡献。再利用 (3.161) 式, 可得

$$\begin{aligned}
\mathbf{P} &= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-\delta_{\lambda\lambda'} a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger - \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right] = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} \right] \\
&= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + \frac{3}{2} \delta^{(3)}(0) \int d^3p \mathbf{p} = \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}. \quad (3.205)
\end{aligned}$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m = 0$ 时, 由 (3.100) 和 (3.101) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 的形式不变, 但 (3.111) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 的良好定义。实际上, 在满足正确归一化的条件下, $m = 0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面, 由于无质量矢量粒子的动量 p^μ 的内积为 $p^2 = 0$, 也不能像 (3.115) 式那样将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 取为正比于 p^μ 的矢量, 否则将出现 $e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = 0$ 而不能得到正确的归一化。因此, 我们需要重新定义 $e^\mu(\mathbf{p}, 3)$ 和 $e^\mu(\mathbf{p}, 0)$ 。

在用 (3.100) 和 (3.101) 式定义 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 时, 我们已经选取了一个特定的惯性参考系。在这个参考系中, 可以定义一个类时单位矢量

$$n^\mu = (1, 0, 0, 0), \quad (3.206)$$

它的 Lorentz 不变内积是

$$n^2 = 1. \quad (3.207)$$

然后, 将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 在此参考系中的形式就取为 n^μ , 即

$$e^\mu(\mathbf{p}, 0) = n^\mu. \quad (3.208)$$

$e^\mu(\mathbf{p}, 0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面, 纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式:

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n) n^\mu}{p \cdot n}. \quad (3.209)$$

$p^2 = (p^0)^2 - |\mathbf{p}|^2 = 0$ 表明

$$p^0 = |\mathbf{p}|, \quad (3.210)$$

从而, $e^\mu(\mathbf{p}, 3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n} = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right). \quad (3.211)$$

这样定义的 $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 0) = n^2 = 1, \quad e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|^2} = -1; \quad (3.212)$$

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 1) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 2) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 3) = 0; \quad (3.213)$$

$$e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, i) = -\frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.214)$$

此外, 可以验证, 由 (3.100)、(3.101)、(3.208) 和 (3.209) 式定义的这组极化矢量确实满足完备性关系 (3.99):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \\ &\quad - \frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ 0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ 0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2 (p^3)^2 + (p^2)^2 |\mathbf{p}|^2 + (p^1)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{(p^2)^2 (p^3)^2 + (p^1)^2 |\mathbf{p}|^2 + (p^2)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} & -\frac{|\mathbf{p}_T|^2 + (p^3)^2}{|\mathbf{p}|^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \quad (3.215) \end{aligned}$$

不过, $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 都不满足四维横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = p \cdot n = p^0 = |\mathbf{p}|, \quad p_\mu e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|} = -|\mathbf{p}| = -p \cdot n. \quad (3.216)$$

横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 具有求和关系

$$\begin{aligned} -\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) &= \sum_{\sigma=1}^2 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - g_{33} e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu - (p \cdot n) n_\mu}{p \cdot n} \frac{p_\nu - (p \cdot n) n_\nu}{p \cdot n} \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu p_\nu - (p \cdot n) p_\mu n_\nu - (p \cdot n) p_\nu n_\mu + (p \cdot n)^2 n_\mu n_\nu}{(p \cdot n)^2} \\ &= g_{\mu\nu} + \frac{p_\mu p_\nu}{(p \cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}, \end{aligned} \quad (3.217)$$

即

$$\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.218)$$

根据 (3.125) 式, 作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 满足

$$\begin{aligned} \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2} [e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &\quad + \frac{1}{2} [-e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [-e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &= e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) + e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) = \sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma), \end{aligned} \quad (3.219)$$

因而具有求和关系

$$\sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.220)$$

四维横向条件 $p_\mu \varepsilon^\mu(\mathbf{p}, \pm) = 0$ 在上式中体现为

$$p^\nu \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu - \frac{p_\mu p^2}{(p \cdot n)^2} + \frac{p_\mu (p \cdot n) + p^2 n_\mu}{p \cdot n} = -p_\mu + p_\mu = 0. \quad (3.221)$$

3.4.2 无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中, 令参数 $m = 0$, 就得到自由无质量实矢量场 $A^\mu(x)$ 的拉氏量

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (3.222)$$

其中 $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ 。同理, 令 Proca 方程中 $m = 0$, 就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu} = 0. \quad (3.223)$$

根据 1.5 节的讨论, 这个方程就是无源的 **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子, **光子**是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作**规范变换** (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x), \quad (3.224)$$

其中, 作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数, 依赖于时空坐标, 因而这样的变换是**局域** (local) 变换。在此规范变换下, 场强张量不变:

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu [A^\nu(x) + \partial^\nu \chi(x)] - \partial^\nu [A^\mu(x) + \partial^\mu \chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu \partial^\nu \chi(x) - \partial^\nu \partial^\mu \chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \quad (3.225)$$

因而, 拉氏量 (3.222) 和无源 Maxwell 方程 (3.223) 都不会改变, 这称为**规范对称性** (gauge symmetry)。

在经典电动力学中, 这种对称性广为人知, 它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定, 因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$, 也就是说, 不依赖于**规范**的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算, 常常将规范固定下来, 使得计算过程依赖于选取的规范, 不过, 最后得出的可观测量必须是**规范不变** (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**, 规范条件为

$$\partial_\mu A^\mu = 0. \quad (3.226)$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同, 但是, 在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件, 而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$, 令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2 \chi(x) = -\partial_\mu A^\mu(x), \quad (3.227)$$

那么, 作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件:

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2 \chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \quad (3.228)$$

但是, 经过这种变换之后, 矢量场仍然没有被唯一地确定: 对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$, 取满足齐次波动方程

$$\partial^2 \tilde{\chi}(x) = 0 \quad (3.229)$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换, 都能得到满足 Lorenz 规范条件的另一个矢量场 $A'^\mu(x)$ 。可见, 存在无穷多个规范等价的矢量场, 它们描述相同的物理, 而且全都满足 Lorenz 规范条件 (3.226)。

矢量场 $A^\mu(x)$ 有 4 个分量，因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是，上述规范等价性表明， $A^\mu(x)$ 并没有 3 个独立的自由度，否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上，无质量矢量场 $A^\mu(x)$ 只具有 2 个独立的自由度，也就是说，有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论：电磁波具有 2 种独立的极化态，以螺旋度 λ 来表征的话，就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有质量矢量场 $A^\mu(x)$ 的量子化程序时，由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度，因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难，因为 Proca 方程要求 $A^0(x)$ 不是独立变量，而是由 (3.175) 式决定的：

$$A^0 = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.230)$$

于是，以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的，自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时，(3.230) 式显然不能成立。因此，对于无质量矢量场，最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度，可以在拉氏量中增加一个不会影响最终物理结果的项：

$$\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.231)$$

其中 ξ 是一个可以自由选取的实参数。可以看出，在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.226) 的情况下，由 (3.231) 式定义的 \mathcal{L}_1 等价于由 (3.222) 式定义的 \mathcal{L} 。新增的项 $-\frac{1}{2} \xi (\partial_\mu A^\mu)^2$ 破坏了规范对称性，相当于把规范固定下来，因而称为规范固定项 (gauge-fixing term)。可以将 \mathcal{L}_1 展开为

$$\mathcal{L}_1 = -\frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu + \frac{1}{2} (\partial_\nu A_\mu) \partial^\mu A^\nu - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.232)$$

从而， A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_1}{\partial (\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi (\partial_\nu A^\nu) \frac{\partial (\partial_\sigma A^\sigma)}{\partial (\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0} \partial_\nu A^\nu, \quad (3.233)$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi \partial_\mu A^\mu. \quad (3.234)$$

因此， $\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0 。

现在，正则量子化程序要求算符 A^μ 和 π_μ 满足如下等时对易关系：

$$[A^\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = i\delta^\mu_\nu \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] = [\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = 0. \quad (3.235)$$

但是，这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子，利用 (3.234) 式，可得

$$[A^0(\mathbf{x}, t), \partial_\mu A^\mu(\mathbf{y}, t)] = -\frac{1}{\xi} [A^0(\mathbf{x}, t), \pi_0(\mathbf{y}, t)] = -\frac{i}{\xi} \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (3.236)$$

上式在 $\mathbf{x} = \mathbf{y}$ 处非零, 因而必有 $\partial_\mu A^\mu \neq 0$ 。所以, A^μ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.226)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$, 但对于量子场 $A^\mu(x)$ 来说限制太强了, 下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - \xi g^{\mu\nu}(\partial_\rho A^\rho), \quad \frac{\partial \mathcal{L}_1}{\partial A_\nu} = 0, \quad (3.237)$$

可得, 与 \mathcal{L}_1 对应的 Euler-Lagrange 方程为

$$0 = \partial_\mu \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu \partial_\mu A^\mu - \xi g^{\mu\nu} \partial_\mu(\partial_\rho A^\rho) = -\partial^2 A^\nu + (1 - \xi) \partial^\nu(\partial_\rho A^\rho), \quad (3.238)$$

即

$$\partial^2 A^\mu - (1 - \xi) \partial^\mu(\partial_\nu A^\nu) = 0. \quad (3.239)$$

若取 $\xi = 1$, 则上式化为 **d'Alembert 方程**

$$\partial^2 A^\mu(x) = 0, \quad (3.240)$$

可以看作无质量情况下的 Klein-Gordon 方程。可见, 将规范固定参数取为

$$\xi = 1 \quad (3.241)$$

将有利于简化计算, 这种取法称为 **Feynman 规范**, 本节后续计算采用这个规范。在 Feynman 规范下, 拉氏量化为

$$\begin{aligned} \mathcal{L}_1 &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu) \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu) - \frac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu - \frac{1}{2}\partial^\mu(A_\mu\partial_\nu A^\nu) + \frac{1}{2}A_\mu\partial^\mu\partial_\nu A^\nu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu - A_\mu\partial_\nu A^\nu). \end{aligned} \quad (3.242)$$

上式最后一行第二项是一个全散度, 它不会影响作用量和运动方程, 可以舍弃。因此, 可以采用更加简化的拉氏量

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \quad (3.243)$$

此时, 共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_2}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu. \quad (3.244)$$

对于 d'Alembert 方程 (3.240), 平面波解的正能解和负能解分别正比于 $\exp(-ip \cdot x)$ 和 $\exp(ip \cdot x)$, 其中

$$p^0 = E_{\mathbf{p}} = |\mathbf{p}|. \quad (3.245)$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p}, \sigma)$, 可以对无质量矢量场 $A^\mu(\mathbf{x}, t)$ 作如下平面波展开:

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}). \quad (3.246)$$

容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.247)$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x}, t) = -\partial_0 A_\mu = \int \frac{d^3 p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e_\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}), \quad (3.248)$$

它也满足自共轭条件

$$[\pi_\mu(\mathbf{x}, t)]^\dagger = \pi_\mu(\mathbf{x}, t). \quad (3.249)$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned} \int d^3 x e^{iq \cdot x} A^\mu &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}] \end{aligned} \quad (3.250)$$

和

$$\begin{aligned} &\int d^3 x e^{iq \cdot x} \partial_0 A^\mu \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3 x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}], \end{aligned} \quad (3.251)$$

以及正交归一关系 (3.98), 可得

$$\begin{aligned} e_\mu(\mathbf{q}, \sigma') \int d^3 x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= e_\mu(\mathbf{q}, \sigma') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} \\ &= -i\sqrt{2E_{\mathbf{q}}} \sum_{\sigma=0}^3 g_{\sigma'\sigma} a_{\mathbf{q};\sigma} = -i\sqrt{2E_{\mathbf{q}}} g_{\sigma'\sigma} a_{\mathbf{q};\sigma}. \end{aligned} \quad (3.252)$$

注意, 虽然上式出现了重复的指标 σ' , 但此处不需要对 σ' 求和。于是, 有

$$a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) \int d^3 x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu). \quad (3.253)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p};\sigma}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) \int d^3x e^{-ip \cdot x} (\partial_0 A^\mu + ip_0 A^\mu). \quad (3.254)$$

根据等时对易关系 (3.235), 湮灭算符与产生算符的对易关系为

$$\begin{aligned} [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^\dagger] &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times [\partial_0 A^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), \partial_0 A^\nu(\mathbf{y}, t) + iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times [-\pi^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), -\pi^\nu(\mathbf{y}, t) + iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times \{-iq_0 [\pi^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] + ip_0 [A^\mu(\mathbf{x}, t), \pi^\nu(\mathbf{y}, t)]\} \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [-(p_0 + q_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} \\ &= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (3.255)$$

倒数第二步用到正交归一关系 (3.98)。另一方面, 两个湮灭算符之间的对易关系为

$$\begin{aligned} [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\ &\quad \times [\partial_0 A^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), \partial_0 A^\nu(\mathbf{y}, t) - iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\ &\quad \times [-\pi^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), -\pi^\nu(\mathbf{y}, t) - iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\ &\quad \times \{iq_0 [\pi^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] + ip_0 [A^\mu(\mathbf{x}, t), \pi^\nu(\mathbf{y}, t)]\} \\ &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [(q_0 - p_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} \\ &= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \end{aligned} \quad (3.256)$$

归纳起来, 产生湮灭算符的对易关系为

$$[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^\dagger] = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a_{\mathbf{p};\sigma}^\dagger, a_{\mathbf{q};\sigma'}^\dagger] = 0. \quad (3.257)$$

3.4.4 哈密顿量和总动量

根据 (1.119)、(3.244) 和 (3.243) 式, 无质量矢量场的哈密顿量密度是

$$\begin{aligned}\mathcal{H} &= \pi_\mu \partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu) \partial^0 A^\mu + \frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu \\ &= -\frac{1}{2} (\partial_0 A_\mu) \partial^0 A^\mu + \frac{1}{2} (\partial_i A_\mu) \partial^i A^\mu = -\frac{1}{2} [\pi_\mu \pi^\mu + (\nabla A_\mu) \cdot (\nabla A^\mu)].\end{aligned}\quad (3.258)$$

于是, 哈密顿量表达为

$$\begin{aligned}H &= \int d^3x \mathcal{H} = -\frac{1}{2} \int d^3x [\pi_\mu \pi^\mu + (\nabla A_\mu) \cdot (\nabla A^\mu)] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left[(ip_0)(iq_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) (a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + (i\mathbf{p} a_{\mathbf{p};\sigma} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p-q) \cdot x} \right. \\ &\quad \left. + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + (p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) \\ &\quad \times \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\ &\quad \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 + |\mathbf{p}|^2) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \right. \\ &\quad \left. - e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 - |\mathbf{p}|^2) (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \right] \\ &= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \\ &= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} g_{\sigma\sigma'} (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}.\end{aligned}\quad (3.259)$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负, 与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规, 时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论, 将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态, 满足

$$a_{\mathbf{p};\sigma} |0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad H |0\rangle = E_{\text{vac}} |0\rangle, \quad E_{\text{vac}} = 2\delta^{(3)}(0) \int d^3p E_{\mathbf{p}}. \quad (3.260)$$

动量为 \mathbf{p} 、极化态为 σ 的单粒子态定义为

$$|\mathbf{p}; \sigma\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p};\sigma}^{\dagger} |0\rangle. \quad (3.261)$$

从而, 由

$$\begin{aligned} [H, a_{\mathbf{p};\sigma}^{\dagger}] &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) [a_{\mathbf{q};\sigma'}^{\dagger} a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^{\dagger}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^{\dagger} [a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^{\dagger}] \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^{\dagger} (2\pi)^3 (-g_{\sigma'\sigma}) \delta^{(3)}(\mathbf{q} - \mathbf{p}) \\ &= E_{\mathbf{p}} \sum_{\sigma'=0}^3 g_{\sigma'\sigma'} g_{\sigma'\sigma} a_{\mathbf{p};\sigma'}^{\dagger} = E_{\mathbf{p}} a_{\mathbf{p};\sigma}^{\dagger} \end{aligned} \quad (3.262)$$

可得

$$\begin{aligned} H |\mathbf{p}; \sigma\rangle &= \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p};\sigma}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} a_{\mathbf{p};\sigma}^{\dagger} + a_{\mathbf{p};\sigma}^{\dagger} H) |0\rangle \\ &= \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} + E_{\text{vac}}) a_{\mathbf{p};\sigma}^{\dagger} |0\rangle = (E_{\mathbf{p}} + E_{\text{vac}}) |\mathbf{p}; \sigma\rangle. \end{aligned} \quad (3.263)$$

这似乎是一个正常的结果, 说明单粒子态 $|\mathbf{p}; \sigma\rangle$ 比真空多了一份能量 $E_{\mathbf{p}}$ 。

利用产生湮灭算符的对易关系 (3.257), 可以计算单粒子态的内积:

$$\begin{aligned} \langle \mathbf{q}; \sigma' | \mathbf{p}; \sigma \rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | a_{\mathbf{q};\sigma'} a_{\mathbf{p};\sigma}^{\dagger} | 0 \rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | [a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{q};\sigma'} - (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q})] | 0 \rangle \\ &= -\sqrt{2E_{\mathbf{p}}} g_{\sigma\sigma'} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (3.264)$$

于是, 有

$$\langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -\sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0), \quad \langle \mathbf{p}; i | \mathbf{p}; i \rangle = \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0), \quad i = 1, 2, 3. \quad (3.265)$$

上式表明, 单粒子态 $|\mathbf{p}; 0\rangle$ 的自我内积是负的, 从而导致它的能量期待值也是负的:

$$\langle \mathbf{p}; 0 | H | \mathbf{p}; 0 \rangle = (E_{\mathbf{p}} + E_{\text{vac}}) \langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -(E_{\mathbf{p}} + E_{\text{vac}}) \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(0) < 0. \quad (3.266)$$

这个负能量结果在物理上看起来是不可接受的, 它的根源在于不定度规。

不过, 如前所述, 无质量矢量场只有 2 种独立的极化态, 对应于 2 种横向极化矢量 $e^{\mu}(\mathbf{p}, 1)$ 和 $e^{\mu}(\mathbf{p}, 2)$, 纵向极化和类时极化都应该是非物理的。选取一定的规范条件, 应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.226) 与正则量子化程序不相容, 我们不能直接使用这个条件, 而需要将它转换到物理 Hilbert 空间中的态的期待值上, 要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle \Psi | \partial_{\mu} A^{\mu}(x) | \Psi \rangle = 0. \quad (3.267)$$

上式称为弱 **Lorenz** 规范条件。

$A^\mu(x)$ 的平面波展开式 (3.246) 可以分解成正能解和负能解两个部分：

$$A^\mu(x) = A^{(+)\mu}(x) + A^{(-)\mu}(x). \quad (3.268)$$

其中，正能解部分为

$$A^{(+)\mu}(\mathbf{x}, t) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x}, \quad (3.269)$$

上式的厄米共轭即是负能解部分

$$A^{(-)\mu}(\mathbf{x}, t) \equiv [A^{(+)\mu}(\mathbf{x}, t)]^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}. \quad (3.270)$$

如果要求

$$\partial_\mu A^{(+)\mu}(x) |\Psi\rangle = 0 \quad (3.271)$$

对任意物理态 $|\Psi\rangle$ 成立，则伴随有

$$\langle\Psi| \partial_\mu A^{(-)\mu}(x) = \langle\Psi| [\partial_\mu A^{(+)\mu}(x)]^\dagger = 0, \quad (3.272)$$

从而，弱 Lorenz 规范条件 (3.267) 得到满足：

$$\langle\Psi| \partial_\mu A^\mu(x) |\Psi\rangle = \langle\Psi| \partial_\mu A^{(+)\mu}(x) |\Psi\rangle + \langle\Psi| \partial_\mu A^{(-)\mu}(x) |\Psi\rangle = 0. \quad (3.273)$$

利用 (3.109) 和 (3.216) 式，规范条件 (3.271) 可化为

$$\begin{aligned} 0 &= \partial_\mu A^{(+)\mu}(x) |\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} [p_\mu e^\mu(\mathbf{p}, 0) a_{\mathbf{p};0} + p_\mu e^\mu(\mathbf{p}, 1) a_{\mathbf{p};1} + p_\mu e^\mu(\mathbf{p}, 2) a_{\mathbf{p};2} + p_\mu e^\mu(\mathbf{p}, 3) a_{\mathbf{p};3}] |\Psi\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} p \cdot n (a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle. \end{aligned} \quad (3.274)$$

这意味着

$$(a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle = 0 \quad (3.275)$$

对任意物理态 $|\Psi\rangle$ 和任意动量 \mathbf{p} 成立。从而，也有

$$\langle\Psi| (a_{\mathbf{p};0}^\dagger - a_{\mathbf{p};3}^\dagger) = 0. \quad (3.276)$$

于是，

$$\langle\Psi| a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} |\Psi\rangle = \langle\Psi| a_{\mathbf{p};3}^\dagger a_{\mathbf{p};3} |\Psi\rangle. \quad (3.277)$$

这样一来，根据 (3.259) 式计算， $|\Psi\rangle$ 的能量期待值为

$$\langle\Psi| H |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\Psi| \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) |\Psi\rangle + E_{\text{vac}} \langle\Psi| \Psi\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle. \quad (3.278)$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.158) 式可得无质量矢量场的总动量为

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi_\mu \nabla A^\mu \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times (ip_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \\ &= \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left[-a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\ &\quad \times \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \right. \\ &\quad \left. - e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \right]. \quad (3.279) \end{aligned}$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换，可得

$$\begin{aligned} &- \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) (a_{-\mathbf{p};\sigma'} a_{\mathbf{p};\sigma} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p};\sigma'}^\dagger a_{\mathbf{p};\sigma}^\dagger e^{2iE_{\mathbf{p}}t}) \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t}). \quad (3.280) \end{aligned}$$

可以看出，上式为零。于是，总动量化为

$$\begin{aligned} \mathbf{P} &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} g_{\sigma\sigma'} (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) \end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + \delta^{(3)}(0) \int d^3p \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right). \tag{3.281}
\end{aligned}$$

根据 (3.277) 式, 物理态 $|\Psi\rangle$ 的动量期待值为

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \langle \Psi | \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle. \tag{3.282}$$

同样, 只有两种物理的横向极化才对动量有净贡献。

通过线性组合, 可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$a_{\mathbf{p},\pm} \equiv \frac{1}{\sqrt{2}} (\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}), \tag{3.283}$$

相应的产生算符可以通过取厄米共轭得到。反过来, 有

$$a_{\mathbf{p};1} = -\frac{1}{\sqrt{2}} (a_{\mathbf{p},+} - a_{\mathbf{p},-}), \quad a_{\mathbf{p};2} = -\frac{i}{\sqrt{2}} (a_{\mathbf{p},+} + a_{\mathbf{p},-}). \tag{3.284}$$

利用对易关系 (3.257), 可得

$$\begin{aligned}
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = \frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \pm a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = -\frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = 0, \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0, \\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \pm a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0. \tag{3.285}
\end{aligned}$$

于是, 这组产生湮灭算符的对易关系可以整理为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0, \quad \lambda, \lambda' = \pm. \tag{3.286}$$

根据 (3.125) 式, 可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 表示 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$:

$$e^\mu(\mathbf{p}, 1) = -\frac{1}{\sqrt{2}} [\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)], \quad e^\mu(\mathbf{p}, 2) = \frac{i}{\sqrt{2}} [\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)]. \tag{3.287}$$

从而, 有

$$\begin{aligned}
\sum_{\sigma=1}^2 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p}, 1) a_{\mathbf{p};1} + e^\mu(\mathbf{p}, 2) a_{\mathbf{p};2} \\
&= \frac{1}{2} [\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)] (a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2} [\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)] (a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\
&= \varepsilon^\mu(\mathbf{p}, +) a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p}, -) a_{\mathbf{p},-} = \sum_{\lambda=\pm} \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \tag{3.288}
\end{aligned}$$

取厄米共轭, 得

$$\sum_{\sigma=1}^2 e^{\mu}(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^{\dagger} = \sum_{\lambda=\pm} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger}. \quad (3.289)$$

于是, 可以把 $A^{\mu}(x)$ 的平面波展开式 (3.246) 改写成

$$\begin{aligned} A^{\mu}(\mathbf{x}, t) = & \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0,3} e^{\mu}(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}) \\ & + \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\varepsilon^{\mu}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right], \end{aligned} \quad (3.290)$$

第一行对应于非物理极化态, 第二行对应于两种物理的螺旋度本征极化态。可见, (3.283) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda = \pm$ 对应的湮灭算符。

此外, 由 (3.284) 式可得

$$\begin{aligned} \sum_{\sigma=1}^2 a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{p};\sigma} &= a_{\mathbf{p};1}^{\dagger} a_{\mathbf{p};1} + a_{\mathbf{p};2}^{\dagger} a_{\mathbf{p};2} = \frac{1}{2} (a_{\mathbf{p},+}^{\dagger} - a_{\mathbf{p},-}^{\dagger}) (a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2} (a_{\mathbf{p},+}^{\dagger} + a_{\mathbf{p},-}^{\dagger}) (a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= a_{\mathbf{p},+}^{\dagger} a_{\mathbf{p},+} + a_{\mathbf{p},-}^{\dagger} a_{\mathbf{p},-} = \sum_{\lambda=\pm} a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda}, \end{aligned} \quad (3.291)$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3 p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle, \quad (3.292)$$

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^{\dagger} a_{\mathbf{p},\lambda} | \Psi \rangle. \quad (3.293)$$

第 4 章 旋量场

4.1 Lorentz 群的旋量表示

旋量表示 (spinor representation) 是 Lorentz 群的一个线性表示, 它在物理上扮演着非常重要的角色, P. A. M. Dirac 在 1928 年首次将它引入到描述电子的理论中。3.1 节提到, Lorentz 群的线性表示可以通过构造满足 Lorentz 代数关系 (3.20) 的生成元矩阵来得到, 下面我们就用这样的方式来建立旋量表示。

首先, 我们假设能够找到一组满足如下反对易关系的 $N \times N$ 矩阵 γ^μ ($\mu = 0, 1, 2, 3$):

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1} = 2g^{\mu\nu}. \quad (4.1)$$

最后一步是一种简写, 省略了 $N \times N$ 单位矩阵 $\mathbf{1}$ 。这样的 γ^μ 称为 **Dirac 矩阵**。然后, 以 Dirac 矩阵的对易子定义另一组 $N \times N$ 矩阵

$$S^{\mu\nu} \equiv \frac{i}{4} [\gamma^\mu, \gamma^\nu]. \quad (4.2)$$

显然, $S^{\mu\nu}$ 关于 μ 和 ν 反对称:

$$S^{\mu\nu} = -S^{\nu\mu}. \quad (4.3)$$

因而 $S^{\mu\nu}$ 的独立分量有 6 个。

利用对易子公式

$$[AB, C] = ABC + ACB - ACB - CAB = A\{B, C\} - \{A, C\}B, \quad (4.4)$$

可得

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \frac{i}{4} [\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu, \gamma^\rho] = \frac{i}{4} [\gamma^\mu \gamma^\nu - (2g^{\nu\mu} - \gamma^\mu \gamma^\nu), \gamma^\rho] = \frac{i}{2} [\gamma^\mu \gamma^\nu, \gamma^\rho] - \frac{i}{2} [g^{\nu\mu}, \gamma^\rho] \\ &= \frac{i}{2} [\gamma^\mu \gamma^\nu, \gamma^\rho] = \frac{i}{2} (\gamma^\mu \{\gamma^\nu, \gamma^\rho\} - \{\gamma^\mu, \gamma^\rho\} \gamma^\nu) = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}). \end{aligned} \quad (4.5)$$

从而, 根据对易子公式 (2.11), 有

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \frac{i}{4} [S^{\mu\nu}, \gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho] = \frac{i}{4} ([S^{\mu\nu}, \gamma^\rho \gamma^\sigma] - [S^{\mu\nu}, \gamma^\sigma \gamma^\rho]) \\ &= \frac{i}{4} ([S^{\mu\nu}, \gamma^\rho] \gamma^\sigma + \gamma^\rho [S^{\mu\nu}, \gamma^\sigma] - [S^{\mu\nu}, \gamma^\sigma] \gamma^\rho - \gamma^\sigma [S^{\mu\nu}, \gamma^\rho]) \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{4} [i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})\gamma^\sigma + i\gamma^\rho(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma}) \\
&\quad - i(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma})\gamma^\rho - i\gamma^\sigma(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})] \\
&= \frac{i^2}{4} (\gamma^\mu \gamma^\sigma g^{\nu\rho} - \gamma^\nu \gamma^\sigma g^{\mu\rho} + \gamma^\rho \gamma^\mu g^{\nu\sigma} - \gamma^\rho \gamma^\nu g^{\mu\sigma} \\
&\quad - \gamma^\mu \gamma^\rho g^{\nu\sigma} + \gamma^\nu \gamma^\rho g^{\mu\sigma} - \gamma^\sigma \gamma^\mu g^{\nu\rho} + \gamma^\sigma \gamma^\nu g^{\mu\rho}) \\
&= \frac{i^2}{4} [g^{\nu\rho}(\gamma^\mu \gamma^\sigma - \gamma^\sigma \gamma^\mu) - g^{\mu\rho}(\gamma^\nu \gamma^\sigma - \gamma^\sigma \gamma^\nu) - g^{\nu\sigma}(\gamma^\mu \gamma^\rho - \gamma^\rho \gamma^\mu) + g^{\mu\sigma}(\gamma^\nu \gamma^\rho - \gamma^\rho \gamma^\nu)] \\
&= i(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho}). \tag{4.6}
\end{aligned}$$

可见, $S^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20), 因而是 Lorentz 群某个线性表示的生成元。以 $S^{\mu\nu}$ 生成的线性表示就是**旋量表示**。

根据 (3.2.1) 小节的讨论, 一组变换参数 $\omega_{\mu\nu}$ 在 Lorentz 群的矢量表示中可以生成固有保时向的有限变换 (3.51):

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^X, \quad X \equiv -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \tag{4.7}$$

类似地, 这组参数在旋量表示中生成了固有保时向的有限变换

$$D(\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = e^Y, \quad Y \equiv -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}. \tag{4.8}$$

这样定义的 $D(\Lambda)$ 是旋量表示中的 Lorentz 变换矩阵, 对于任意的 Lorentz 变换 Λ_1 和 Λ_2 , 满足同态关系

$$D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1). \tag{4.9}$$

由 (3.48) 式可得

$$e^{-Y}e^Y = e^{-Y+Y} = e^0 = \mathbf{1}, \tag{4.10}$$

故 $D(\Lambda)$ 的逆矩阵为

$$D(\Lambda^{-1}) = D^{-1}(\Lambda) = e^{-Y} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{4.11}$$

这里先来介绍一些将会用到的对易子公式。以如下方式定义 B 与 A 的多重对易子 $[B, A^{(n)}]$:

$$\begin{aligned}
[B, A^{(0)}] &= B, \quad [B, A^{(1)}] = [B, A] = [[B, A^{(0)}], A] \\
[B, A^{(2)}] &= [[B, A], A] = [[B, A^{(1)}], A], \quad \dots, \quad [B, A^{(n)}] = [[B, A^{(n-1)}], A].
\end{aligned} \tag{4.12}$$

于是, 下式成立:

$$BA^k = \sum_{n=0}^k \frac{k!}{(k-n)!n!} A^{k-n} [B, A^{(n)}]. \tag{4.13}$$

下面用数学归纳法证明这个等式。

证明 当 $k=0$ 和 $k=1$ 时, (4.13) 式明显成立:

$$BA^0 = B = [B, A^{(0)}] = \frac{0!}{(0-0)!0!} A^{0-0} [B, A^{(0)}], \tag{4.14}$$

$$BA^1 = BA = AB + [B, A] = \frac{1!}{(1-0)!0!} A^{1-0}[B, A^{(0)}] + \frac{1!}{(1-1)!1!} A^{1-1}[B, A^{(1)}]. \quad (4.15)$$

假设 $k = m$ 时 (4.13) 式成立, 则有

$$\begin{aligned} BA^{m+1} &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n}[B, A^{(n)}]A = \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n}(A[B, A^{(n)}] + [[B, A^{(n)}], A]) \\ &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m+1-n}[B, A^{(n)}] + \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n}[B, A^{(n+1)}] \\ &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m+1-n}[B, A^{(n)}] + \sum_{j=1}^{m+1} \frac{m!}{(m-j+1)!(j-1)!} A^{m-j+1}[B, A^{(j)}] \\ &= \frac{m!}{(m-0)!0!} A^{m+1}[B, A^{(0)}] + \sum_{n=1}^m \left[\frac{m!}{(m-n)!n!} + \frac{m!}{(m-n+1)!(n-1)!} \right] A^{m+1-n}[B, A^{(n)}] \\ &\quad + \frac{m!}{[m-(m+1)+1]![(m+1)-1]!} A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= A^{m+1}[B, A^{(0)}] + \sum_{n=1}^m \left[\frac{m!}{(m-n)!n!} + \frac{n}{m-n+1} \frac{m!}{(m-n)!n!} \right] A^{m+1-n}[B, A^{(n)}] \\ &\quad + A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= \frac{(m+1)!}{[(m+1)-0]!0!} A^{m+1}[B, A^{(0)}] + \sum_{n=1}^m \frac{(m+1)!}{(m-n+1)!n!} A^{m+1-n}[B, A^{(n)}] \\ &\quad + \frac{(m+1)!}{[(m+1)-(m+1)]!(m+1)!} A^{m-(m+1)+1}[B, A^{(m+1)}] \\ &= \sum_{n=0}^{m+1} \frac{(m+1)!}{[(m+1)-n]!n!} A^{(m+1)-n}[B, A^{(n)}], \end{aligned} \quad (4.16)$$

即 $k = m+1$ 时 (4.13) 式也成立。于是, (4.13) 式对任意非负整数 k 成立。证毕。

根据推广的阶乘定义 (3.46) 可以将 (4.13) 式右边的级数化成无穷级数:

$$BA^k = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} A^{k-n}[B, A^{(n)}]. \quad (4.17)$$

利用上式, 可得

$$\begin{aligned} e^{-A} B e^A &= e^{-A} \sum_{k=0}^{\infty} \frac{1}{k!} B A^k = e^{-A} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} A^{k-n}[B, A^{(n)}] \\ &= e^{-A} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{(k-n)!} A^{k-n}[B, A^{(n)}] = e^{-A} \sum_{n=0}^{\infty} \frac{1}{n!} e^A [B, A^{(n)}] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [B, A^{(n)}]. \end{aligned} \quad (4.18)$$

现在, 我们继续讨论 Lorentz 群的旋量表示。由 (4.5) 和 (3.34) 式可得

$$[\gamma^\mu, S^{\rho\sigma}] = -[S^{\rho\sigma}, \gamma^\mu] = [S^{\sigma\rho}, \gamma^\mu] = i(\gamma^\sigma g^{\rho\mu} - \gamma^\rho g^{\sigma\mu}) = i(g^{\rho\mu} \delta^\sigma_\nu - g^{\sigma\mu} \delta^\rho_\nu) \gamma^\nu = (\mathcal{J}^{\rho\sigma})^\mu_\nu \gamma^\nu. \quad (4.19)$$

从而, 有

$$\begin{aligned}
 [\gamma^\mu, Y^{(1)}] &= [\gamma^\mu, Y] = -\frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] = -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu = X^\mu{}_\nu\gamma^\nu, \\
 [\gamma^\mu, Y^{(2)}] &= [[\gamma^\mu, Y^{(1)}], Y] = X^\mu{}_\nu[\gamma^\nu, Y] = X^\mu{}_\nu X^\nu{}_\rho\gamma^\rho = (X^2)^\mu{}_\nu\gamma^\nu, \\
 &\dots \\
 [\gamma^\mu, Y^{(n)}] &= (X^n)^\mu{}_\nu\gamma^\nu.
 \end{aligned} \tag{4.20}$$

于是, 利用 (4.18) 式可以推出

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = e^{-Y}\gamma^\mu e^Y = \sum_{n=0}^{\infty} \frac{1}{n!}[\gamma^\mu, Y^{(n)}] = \sum_{n=0}^{\infty} \frac{1}{n!}(X^n)^\mu{}_\nu\gamma^\nu = (e^X)^\mu{}_\nu\gamma^\nu, \tag{4.21}$$

即

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{4.22}$$

上式是 γ^μ 在旋量表示中的 Lorentz 变换关系, 它说明 γ^μ 是一个 Lorentz 矢量。 $N \times N$ 单位矩阵 $\mathbf{1}$ 满足

$$D^{-1}(\Lambda)\mathbf{1}D(\Lambda) = \mathbf{1}, \tag{4.23}$$

因而 $\mathbf{1}$ 是一个 Lorentz 标量。生成元 $S^{\mu\nu}$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)S^{\mu\nu}D(\Lambda) = \frac{i}{4}[D^{-1}(\Lambda)\gamma^\mu D(\Lambda), D^{-1}(\Lambda)\gamma^\nu D(\Lambda)] = \frac{i}{4}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma[\gamma^\rho, \gamma^\sigma] = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma S^{\rho\sigma}, \tag{4.24}$$

可见, $S^{\mu\nu}$ 是一个 2 阶反对称 Lorentz 张量。

$S^{\mu\nu}$ 是用 2 个 Dirac 矩阵的乘积构造出来的反对称张量, 类似地, 我们也可以用 3 个 Dirac 矩阵的乘积来构造一个 3 阶全反对称张量

$$\Gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \equiv \frac{1}{3!}(\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho). \tag{4.25}$$

上式第二步中的中括号表示对 μ, ν, ρ 三个指标作全反对称操作: 在偶次置换前面加上正号, 奇次置换前面加上负号, 然后对所有置换求和并除以置换方式的数目。 $\Gamma^{\mu\nu\rho}$ 的 Lorentz 变换形式是

$$D^{-1}(\Lambda)\Gamma^{\mu\nu\rho}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Gamma^{\alpha\beta\gamma}. \tag{4.26}$$

由全反对称性可知, $\Gamma^{\mu\nu\rho}$ 的独立分量只有 4 个, 可取为 Γ^{012} 、 Γ^{023} 、 Γ^{013} 和 Γ^{123} 。从 Dirac 矩阵的反对易关系 (4.1) 可知, 当 $\mu \neq \nu$ 时, γ^μ 与 γ^ν 是反对易的, 即

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad \mu \neq \nu. \tag{4.27}$$

于是, 从定义式 (4.25) 可得

$$\Gamma^{012} = \gamma^0\gamma^1\gamma^2, \quad \Gamma^{023} = \gamma^0\gamma^2\gamma^3, \quad \Gamma^{013} = \gamma^0\gamma^1\gamma^3, \quad \Gamma^{123} = \gamma^1\gamma^2\gamma^3. \tag{4.28}$$

更进一步, 可以用 4 个 Dirac 矩阵的乘积来构造一个 4 阶全反对称张量

$$\begin{aligned}
 \Gamma^{\mu\nu\rho\sigma} &\equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \\
 &\equiv \frac{1}{4!}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho + \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho - \gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma \\
 &\quad - \gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu + \gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu + \gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho \\
 &\quad + \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - \gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu \\
 &\quad - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu). \quad (4.29)
 \end{aligned}$$

从而, $\Gamma^{\mu\nu\rho\sigma}$ 具有如下性质:

$$\Gamma^{\mu\nu\rho\sigma} = \begin{cases} +\Gamma^{0123}, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -\Gamma^{0123}, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换,} \\ 0, & \text{其它情况.} \end{cases} \quad (4.30)$$

可见, 它只有 1 个独立分量, 可取为

$$\Gamma^{0123} = \gamma^0\gamma^1\gamma^2\gamma^3. \quad (4.31)$$

结合四维 Levi-Civita 符号的定义 (1.65), 可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \Gamma^{0123} = \varepsilon^{\mu\nu\rho\sigma} \gamma^0\gamma^1\gamma^2\gamma^3 \quad (4.32)$$

受到四维时空的维度限制, 我们不能以同样的方式定义高于 4 阶的全反对称张量。现在, 我们拥有一组矩阵

$$\{\mathbf{1}, \gamma^\mu, S^{\mu\nu}, \Gamma^{\mu\nu\rho}, \Gamma^{\mu\nu\rho\sigma}\}, \quad (4.33)$$

它们各自的独立分量个数之和为 $1 + 4 + 6 + 4 + 1 = 16$ 。利用反对易关系 (4.1), 可以将任意多个 Dirac 矩阵的乘积转化为集合 (4.33) 中的矩阵与度规张量乘积的线性组合。例如,

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\gamma^\mu\gamma^\nu - \frac{1}{2}\gamma^\nu\gamma^\mu + g^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + g^{\mu\nu} = -2iS^{\mu\nu} + g^{\mu\nu}. \quad (4.34)$$

又如,

$$\begin{aligned}
 \gamma^\mu\gamma^\nu\gamma^\rho &= \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
 &= \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{4}\gamma^\nu\gamma^\mu\gamma^\rho + \frac{1}{2}g^{\mu\nu}\gamma^\rho - \frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu + \frac{1}{4}\gamma^\rho\gamma^\mu\gamma^\nu - \frac{1}{2}g^{\mu\rho}\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
 &= \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{8}\gamma^\nu\gamma^\mu\gamma^\rho + \frac{1}{8}\gamma^\nu\gamma^\rho\gamma^\mu - \frac{1}{4}g^{\rho\mu}\gamma^\nu + \frac{1}{2}g^{\mu\nu}\gamma^\rho - \frac{1}{4}\gamma^\mu\gamma^\rho\gamma^\nu \\
 &\quad + \frac{1}{8}\gamma^\rho\gamma^\mu\gamma^\nu - \frac{1}{8}\gamma^\rho\gamma^\nu\gamma^\mu + \frac{1}{4}g^{\mu\nu}\gamma^\rho - \frac{1}{2}g^{\mu\rho}\gamma^\nu + g^{\rho\nu}\gamma^\mu \\
 &= \frac{3!}{8}\Gamma^{\mu\nu\rho} + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{8}\gamma^\mu\gamma^\rho\gamma^\nu - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu \\
 &= \frac{3}{4}\Gamma^{\mu\nu\rho} + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho + \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{4}g^{\rho\nu}\gamma^\mu - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu
 \end{aligned}$$

$$= \frac{3}{4}\Gamma^{\mu\nu\rho} + \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{3}{4}g^{\rho\mu}\gamma^\nu + \frac{3}{4}g^{\mu\nu}\gamma^\rho + \frac{3}{4}g^{\rho\nu}\gamma^\mu, \quad (4.35)$$

故

$$\gamma^\mu\gamma^\nu\gamma^\rho = \Gamma^{\mu\nu\rho} - g^{\rho\mu}\gamma^\nu + g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu. \quad (4.36)$$

因此, 对于由 Dirac 矩阵乘积的线性组合构造的矩阵, 集合 (4.33) 构成一组完备的基底。

定义 γ^μ 对应的协变矢量

$$\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu, \quad (4.37)$$

则

$$\gamma_0 = \gamma^0, \quad \gamma_i = -\gamma^i, \quad i = 1, 2, 3. \quad (4.38)$$

由反对易关系 (4.1) 有

$$(\gamma^0)^2 = \frac{1}{2}\{\gamma^0, \gamma^0\} = g^{00} = \mathbf{1}, \quad (\gamma^i)^2 = \frac{1}{2}\{\gamma^i, \gamma^i\} = g^{ii} = -\mathbf{1}. \quad (4.39)$$

我们约定 γ^0 是厄米矩阵, γ^i 是反厄米矩阵, 即

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \quad (4.40)$$

则可得

$$(\gamma^0)^\dagger\gamma^0 = (\gamma^0)^2 = \mathbf{1}, \quad (\gamma^i)^\dagger\gamma^i = -(\gamma^i)^2 = \mathbf{1}. \quad (4.41)$$

可见, 在此约定下, γ^0 和 γ^i 都是幺正矩阵。

这里引入一个新的矩阵

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (4.42)$$

从 (4.27) 式可得

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{cases} +\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -\gamma^0\gamma^1\gamma^2\gamma^3, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换.} \end{cases} \quad (4.43)$$

这种置换性质与四维 Levi-Civita 符号 (1.65) 相同, 因而置换操作带来的符号在 $\varepsilon_{\mu\nu\rho\sigma}$ 与 $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ 的乘积中相互抵消, 如

$$\varepsilon_{1023}\gamma^1\gamma^0\gamma^2\gamma^3 = -\varepsilon_{0123}(-\gamma^0\gamma^1\gamma^2\gamma^3) = \varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3. \quad (4.44)$$

由此可得

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \quad (4.45)$$

对于固有保时向 Lorentz 变换 (4.7), 用度规对 (1.73) 式升降指标, 有

$$\varepsilon_{\mu\nu\rho\sigma} = \Lambda_\mu^\alpha\Lambda_\nu^\beta\Lambda_\rho^\gamma\Lambda_\sigma^\delta\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}(\Lambda^{-1})^\alpha_\mu(\Lambda^{-1})^\beta_\nu(\Lambda^{-1})^\gamma_\rho(\Lambda^{-1})^\delta_\sigma. \quad (4.46)$$

于是, γ^5 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)\gamma^5D(\Lambda) = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu_\alpha\Lambda^\nu_\beta\Lambda^\rho_\gamma\Lambda^\sigma_\delta\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$$

$$\begin{aligned}
&= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}(\Lambda^{-1})^\kappa{}_\mu(\Lambda^{-1})^\lambda{}_\nu(\Lambda^{-1})^\tau{}_\rho(\Lambda^{-1})^\varepsilon{}_\sigma\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \\
&= -\frac{i}{4!}\varepsilon_{\kappa\lambda\tau\varepsilon}\delta^\kappa{}_\alpha\delta^\lambda{}_\beta\delta^\tau{}_\gamma\delta^\varepsilon{}_\delta\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = -\frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = \gamma^5.
\end{aligned} \quad (4.47)$$

可见, γ^5 是一个 Lorentz 标量。 γ^5 的平方为

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0 = -(-\mathbf{1})^3 = \mathbf{1}. \quad (4.48)$$

根据约定 (4.40), γ^5 是厄米矩阵:

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \quad (4.49)$$

γ^5 与 γ^μ 反对易:

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu - \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu) = 0. \quad (4.50)$$

由 (4.32) 式可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 = -i\varepsilon^{\mu\nu\rho\sigma}\gamma^5. \quad (4.51)$$

可见, $\Gamma^{\mu\nu\rho\sigma}$ 正比于 γ^5 。此外, 由 (4.28) 式有

$$\Gamma^{012} = \gamma^0\gamma^1\gamma^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^3\gamma^5 = i\gamma_3\gamma^5 = i\varepsilon^{0123}\gamma_3\gamma^5, \quad (4.52)$$

$$\Gamma^{023} = \gamma^0\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^1\gamma^5 = i\gamma_1\gamma^5 = i\varepsilon^{0231}\gamma_1\gamma^5, \quad (4.53)$$

$$\Gamma^{013} = \gamma^0\gamma^1\gamma^3 = -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = -\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^2\gamma^5 = -i\gamma_2\gamma^5 = i\varepsilon^{0132}\gamma_2\gamma^5, \quad (4.54)$$

$$\Gamma^{123} = \gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5 = -i\gamma_0\gamma^5 = i\varepsilon^{1230}\gamma_0\gamma^5. \quad (4.55)$$

综合起来, 得

$$\Gamma^{\mu\nu\rho} = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5. \quad (4.56)$$

根据上式, $\Gamma^{\mu\nu\rho}$ 可以写成 $\gamma^\mu\gamma^5$ 的 4 个独立分量的线性组合。 $\gamma^\mu\gamma^5$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)\gamma^\mu\gamma^5D(\Lambda) = D^{-1}(\Lambda)\gamma^\mu D(\Lambda)D^{-1}(\Lambda)\gamma^5D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu\gamma^5, \quad (4.57)$$

因而它是一个 Lorentz 矢量。再引入

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = 2S^{\mu\nu}, \quad (4.58)$$

它正比于 $S^{\mu\nu}$, 所以也是一个 2 阶反对称 Lorentz 张量:

$$D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\sigma^{\alpha\beta}. \quad (4.59)$$

于是, 我们可以用 γ^5 、 $\gamma^\mu\gamma^5$ 和 $\sigma^{\mu\nu}$ 分别代替集合 (4.33) 中的 $\Gamma^{\mu\nu\rho\sigma}$ 、 $\Gamma^{\mu\nu\rho}$ 和 $S^{\mu\nu}$ 作为基底, 从而得到另一组完备的矩阵基底

$$\{\mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}, \quad (4.60)$$

它们各自的独立分量个数之和仍是 16。

依照约定 (4.40), γ^0 即是厄米的又是幺正的, 我们可以用它定义一个幺正变换矩阵 β :

$$\beta^{-1} = \beta^\dagger = \beta \equiv \gamma^0. \quad (4.61)$$

从而, 有

$$\beta^{-1}\gamma^0\beta = \gamma^0\gamma^0\gamma^0 = +\gamma^0, \quad \beta^{-1}\gamma^i\beta = \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i. \quad (4.62)$$

根据宇称变换 \mathcal{P} 的定义 (1.46), 可以将这两个式子合写为

$$\beta^{-1}\gamma^\mu\beta = \mathcal{P}^\mu{}_\nu\gamma^\nu. \quad (4.63)$$

这表明 β 相当于旋量表示中的宇称变换矩阵 $D(\mathcal{P})$, 它是非固有保时向的, 上式就是 γ^μ 的宇称变换形式。(4.62) 式说明 γ^0 是宇称本征态, 本征值为 $+$, 即具有**偶宇称**; γ^i 也是宇称本征态, 本征值为 $-$, 即具有**奇宇称**。虽然单位矩阵 $\mathbf{1}$ 与 γ_5 都是 Lorentz 标量, 但它们的宇称是不同的:

$$\beta^{-1}\mathbf{1}\beta = +\mathbf{1}, \quad \beta^{-1}\gamma^5\beta = \gamma^0\gamma^5\gamma^0 = -\gamma^5\gamma^0\gamma^0 = -\gamma^5. \quad (4.64)$$

像 γ_5 这样具有奇宇称的 Lorentz 标量, 称为**赝标量** (pseudoscalar)。此外, $\gamma^\mu\gamma^5$ 的宇称变换形式是

$$\beta^{-1}\gamma^\mu\gamma^5\beta = \beta^{-1}\gamma^\mu\beta\beta^{-1}\gamma^5\beta = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \quad (4.65)$$

即

$$\beta^{-1}\gamma^0\gamma^5\beta = -\gamma^0\gamma^5, \quad \beta^{-1}\gamma^i\gamma^5\beta = +\gamma^i\gamma^5. \quad (4.66)$$

可以看出, 虽然 $\gamma^\mu\gamma^5$ 也是 Lorentz 矢量, 但它的分量的宇称性质与 γ^μ 相反。宇称变换性质像 $\gamma^\mu\gamma^5$ 这样的 Lorentz 矢量称为**轴矢量** (axial vector)。最后, $\sigma^{\mu\nu}$ 的宇称变换形式为

$$\beta^{-1}\sigma^{\mu\nu}\beta = \frac{i}{2}[\beta^{-1}\gamma^\mu\beta, \beta^{-1}\gamma^\nu\beta] = \frac{i}{2}\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta[\gamma^\alpha, \gamma^\beta] = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\sigma^{\alpha\beta}, \quad (4.67)$$

即

$$\beta^{-1}\sigma^{0i}\beta = \mathcal{P}^0{}_\alpha\mathcal{P}^i{}_\beta\sigma^{\alpha\beta} = -\sigma^{0i}, \quad \beta^{-1}\sigma^{ij}\beta = \mathcal{P}^i{}_\alpha\mathcal{P}^j{}_\beta\sigma^{\alpha\beta} = +\sigma^{ij}. \quad (4.68)$$

可见, 基底集合 (4.60) 是由标量 $\mathbf{1}$ 、赝标量 γ^5 、矢量 γ^μ 、轴矢量 $\gamma^\mu\gamma^5$ 和 2 阶反对称张量 $\sigma^{\mu\nu}$ 组成的, 综合考虑固有保时向 Lorentz 变换和宇称变换, 则这些基底的变换性质各不相同, 因而它们彼此之间是相互独立的, 总共有 16 个独立而完备的基底。由于独立的 $N \times N$ 矩阵最多有 N^2 个, 为了得到 16 个这样的基底, 需要 $N \geq 4$ 。我们考虑最简单的情况, 将 Dirac 矩阵取为 4×4 矩阵。

4.2 Dirac 旋量场

在 Lorentz 群的旋量表示中, 被变换矩阵 $D(\Lambda)$ 作用的态称为 **Dirac 旋量** (spinor)。由于 $D(\Lambda)$ 是 4×4 矩阵, 一个 Dirac 旋量 ψ_a 应当具有 4 个分量 ($a = 1, 2, 3, 4$), 相应的 Lorentz 变

换形式为

$$\psi'_a = D_{ab}(\Lambda)\psi_b. \quad (4.69)$$

隐去旋量指标 a 和 b ，上式化为

$$\psi' = D(\Lambda)\psi. \quad (4.70)$$

我们可以将 ψ 和 ψ' 看作列矢量，而上式右边的乘积就是线性代数中矩阵与列矢量的乘积。

进一步，如果 ψ_a 依赖于时空坐标 x^μ ，它就成为 **Dirac 旋量场** $\psi_a(x)$ 。类似于 (3.67) 式，量子 Dirac 旋量场的 Lorentz 变换形式是

$$\psi'_a(x') = U^{-1}(\Lambda)\psi_a(x')U(\Lambda) = D_{ab}(\Lambda)\psi_b(x). \quad (4.71)$$

对于固有保时向 Lorentz 变换，由 (4.8) 式可得 $D_{ab}(\Lambda)$ 的无穷小形式为

$$D_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}, \quad (4.72)$$

于是，(4.71) 式的无穷小形式是

$$\psi'_a(x') = \psi_a(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}\psi_b(x). \quad (4.73)$$

将上式与 (1.168) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在旋量表示中对应于 $S^{\mu\nu}$ 。隐去旋量指标，则 (4.71) 式化为

$$\psi'(x') = U^{-1}(\Lambda)\psi(x')U(\Lambda) = D(\Lambda)\psi(x), \quad (4.74)$$

也可以写成

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x). \quad (4.75)$$

对于无穷小变换，根据 (3.59) 式，将 $\psi(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned} \psi(\Lambda^{-1}x) &= \psi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \psi(x) = \psi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \psi(x) = \psi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\psi(x) \\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu} i(x^\mu \partial^\nu - x^\nu \partial^\mu)\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu} L^{\mu\nu} \psi(x), \end{aligned} \quad (4.76)$$

其中 $L^{\mu\nu}$ 是 (3.63) 式定义微分算符。从而，(4.75) 式右边展开到 ω 一阶项的形式为

$$D(\Lambda)\psi(\Lambda^{-1}x) = \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \left[\psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x)\right] = \psi(x) - \frac{i}{2}\omega_{\mu\nu}(L^{\mu\nu} + S^{\mu\nu})\psi(x). \quad (4.77)$$

另一方面，根据 (3.6) 式可以将 (4.75) 式左边展开为

$$\begin{aligned} U^{-1}(\Lambda)\psi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\psi(x)\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) \\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu}\psi(x)J^{\mu\nu} + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu}[\psi(x), J^{\mu\nu}]. \end{aligned} \quad (4.78)$$

两相比较，得到

$$[\psi(x), J^{\mu\nu}] = (L^{\mu\nu} + S^{\mu\nu})\psi(x). \quad (4.79)$$

$S^{\mu\nu}$ 的空间分量等价于三维矢量

$$S^i \equiv \frac{1}{2}\varepsilon^{ijk}S^{jk}, \quad \mathbf{S} = (S^{23}, S^{31}, S^{12}). \quad (4.80)$$

再根据 (3.21) 和 (3.64) 式, (4.79) 式的纯空间分量部分可以改写为

$$[\psi(x), \mathbf{J}] = (\mathbf{L} + \mathbf{S})\psi(x). \quad (4.81)$$

上式表明, 除了轨道角动量 \mathbf{L} , 总角动量算符 \mathbf{J} 还生成了由 \mathbf{S} 描述的自旋角动量。

描述半整数自旋经常用到 3 个 2×2 的 **Pauli 矩阵**

$$\sigma^1 \equiv \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \quad (4.82)$$

它们都是厄米又么正的:

$$(\sigma^i)^{-1} = (\sigma^i)^\dagger = \sigma^i. \quad (4.83)$$

Pauli 矩阵的两两乘积为

$$\begin{aligned} (\sigma^1)^2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad (\sigma^2)^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\ (\sigma^3)^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \sigma^1\sigma^2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\sigma^3, \\ \sigma^2\sigma^1 &= \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} -i & \\ & i \end{pmatrix} = -i\sigma^3, \quad \sigma^2\sigma^3 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma^1, \\ \sigma^3\sigma^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} & -i \\ -i & \end{pmatrix} = -i\sigma^1, \quad \sigma^3\sigma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma^2, \\ \sigma^1\sigma^3 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i\sigma^2. \end{aligned} \quad (4.84)$$

归纳起来, 有

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k. \quad (4.85)$$

从而可得

$$[\sigma^i, \sigma^j] = i\varepsilon^{ijk}\sigma^k - i\varepsilon^{jik}\sigma^k = 2i\varepsilon^{ijk}\sigma^k, \quad (4.86)$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} + i\varepsilon^{ijk}\sigma^k + i\varepsilon^{jik}\sigma^k = 2\delta^{ij}. \quad (4.87)$$

利用 Pauli 矩阵可以将 Dirac 矩阵表示成 2×2 分块形式:

$$\gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}, \quad (4.88)$$

其中 $\mathbf{1}$ 表示 2×2 单位矩阵。容易验证, 这样表示的 Dirac 矩阵符合约定 (4.40), 而且满足反对易关系 (4.1):

$$\{\gamma^0, \gamma^0\} = 2 \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} = 2 \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{00}, \quad (4.89)$$

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} = 0 = 2g^{0i}, \quad (4.90)$$

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} -\sigma^i \sigma^j - \sigma^j \sigma^i & \\ & -\sigma^i \sigma^j - \sigma^j \sigma^i \end{pmatrix} = -2\delta^{ij} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{ij}. \quad (4.91)$$

实际上, Dirac 矩阵有多种表示方式, (4.88) 式这种表示方式称为 **Weyl 表象**, 也称为**手征表象** (chiral representation)。Dirac 矩阵的所有表示方式都是等价的, 彼此可以通过相似变换联系起来。

在 Weyl 表象中, 由 (4.86) 式可得 $S^{\mu\nu}$ 的空间分量为

$$\begin{aligned} S^{ij} &= \frac{i}{4} [\gamma^i, \gamma^j] = \frac{i}{4} \begin{pmatrix} -\sigma^i \sigma^j + \sigma^j \sigma^i & \\ & -\sigma^i \sigma^j + \sigma^j \sigma^i \end{pmatrix} \\ &= \frac{i}{4} \begin{pmatrix} -2i\varepsilon^{ijk} \sigma^k & \\ & -2i\varepsilon^{ijk} \sigma^k \end{pmatrix} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}, \end{aligned} \quad (4.92)$$

从 Pauli 矩阵的厄米性可知, S^{ij} 是厄米矩阵:

$$(S^{ij})^\dagger = S^{ij}. \quad (4.93)$$

由 (1.98) 式可得

$$S^i = \frac{1}{2} \varepsilon^{ijk} S^{jk} = \frac{1}{4} \varepsilon^{ijk} \varepsilon^{jkl} \begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{4} 2\delta^{il} \begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}. \quad (4.94)$$

于是, 自旋角动量矩阵的平方为

$$\mathbf{S}^2 = S^i S^i = \frac{1}{4} \begin{pmatrix} \sigma^i \sigma^i & \\ & \sigma^i \sigma^i \end{pmatrix} = \frac{3}{4} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \frac{1}{2} \left(\frac{1}{2} + 1 \right) = s(s+1). \quad (4.95)$$

可见, Dirac 旋量场 $\psi(x)$ 的自旋量子数是

$$s = \frac{1}{2}. \quad (4.96)$$

经过量子化程序之后, $\psi(x)$ 应当描述**自旋为 1/2** 的粒子。

4.3 Dirac 方程

为了写下 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量, 我们需要结合两个旋量场来得到 Lorentz 标量。在 Weyl 表象中, $S^{\mu\nu}$ 的 $0i$ 分量为

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{i}{2}\gamma^0\gamma^i = \frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}. \quad (4.97)$$

由 Pauli 矩阵的厄米性可得

$$(S^{0i})^\dagger = -\frac{i}{2}\begin{pmatrix} -(\sigma^i)^\dagger & \\ & (\sigma^i)^\dagger \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} = -S^{0i}. \quad (4.98)$$

可见, S^{0i} 不是厄米矩阵。于是, 当 $\omega_{0i} \neq 0$ 时,

$$D^\dagger(\Lambda) = \left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \right]^\dagger = \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right] \neq \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = D^{-1}(\Lambda), \quad (4.99)$$

即 $D(\Lambda)$ 不是么正矩阵。因此, 一般地, $\psi^\dagger(x)\psi(x)$ 不是 Lorentz 标量:

$$\psi'^\dagger(x')\psi'(x') = \psi^\dagger(x)D^\dagger(\Lambda)D(\Lambda)\psi(x) \neq \psi^\dagger(x)\psi(x). \quad (4.100)$$

根据约定 (4.40), 可得

$$(\gamma^0)^\dagger\gamma^0 = \gamma^0\gamma^0, \quad (\gamma^i)^\dagger\gamma^0 = -\gamma^i\gamma^0 = \gamma^0\gamma^i. \quad (4.101)$$

这两条式子可以合起来写成

$$(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu. \quad (4.102)$$

从而, 有

$$(S^{\mu\nu})^\dagger\gamma^0 = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger\gamma^0 = -\frac{i}{4}[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0 = -\frac{i}{4}\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) = \gamma^0 S^{\mu\nu}. \quad (4.103)$$

于是, 可得

$$\begin{aligned} D^\dagger(\Lambda)\gamma^0 &= \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\gamma^0 = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]^n \gamma^0 = \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n \\ &= \gamma^0 \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \gamma^0 D^{-1}(\Lambda). \end{aligned} \quad (4.104)$$

根据上式, 定义

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0, \quad (4.105)$$

则它的 Lorentz 变换形式为

$$\bar{\psi}'(x') = \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)D^\dagger(\Lambda)\gamma^0 = \psi^\dagger(x)\gamma^0 D^{-1}(\Lambda) = \bar{\psi}(x)D^{-1}(\Lambda). \quad (4.106)$$

这样一来, $\bar{\psi}(x)\psi(x)$ 就是一个 Lorentz 标量:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)D(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x). \quad (4.107)$$

$\bar{\psi}(x)\psi(x)$ 这种形式的量属于旋量双线性型 (spinor bilinear), 我们可以使用 $\bar{\psi}(x)$ 构造一些 Lorentz 协变的其它旋量双线性型. $\bar{\psi}(x)i\gamma^5\psi(x)$ 是一个 Lorentz 标量:

$$\bar{\psi}'(x')i\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)i\gamma^5D(\Lambda)\psi(x) = \bar{\psi}(x)i\gamma^5\psi(x). \quad (4.108)$$

$\bar{\psi}(x)\gamma^\mu\psi(x)$ 和 $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ 都是 Lorentz 矢量:

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x), \quad (4.109)$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu\gamma^5D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x). \quad (4.110)$$

$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ 是一个 2 阶反对称 Lorentz 张量:

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda)\psi(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x) \quad (4.111)$$

如果将 $\psi(x)$ 看作旋量空间中的列矢量, 则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 都是行矢量, 因而这些旋量双线性型都只是旋量空间中的 1×1 矩阵, 也就是数. 由 γ^0 和 γ^5 的厄米性及 (4.102) 式可知, 这些旋量双线性型都是厄米的, 或者说, 都是实数:

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \quad (4.112)$$

$$(\bar{\psi}i\gamma^5\psi)^\dagger = -i\psi^\dagger\gamma^5\gamma^0\psi = i\psi^\dagger\gamma^0\gamma^5\psi = \bar{\psi}i\gamma^5\psi, \quad (4.113)$$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = \psi^\dagger(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \quad (4.114)$$

$$(\bar{\psi}\gamma^\mu\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad (4.115)$$

$$(\bar{\psi}\sigma^{\mu\nu}\psi)^\dagger = -\frac{i}{2}\psi^\dagger[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0\psi = -\frac{i}{2}\psi^\dagger\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\psi = \bar{\psi}\sigma^{\mu\nu}\psi. \quad (4.116)$$

此外, 包含时空导数的旋量双线性型 $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ 是 Lorentz 标量:

$$\begin{aligned} \bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x) &= \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) = \bar{\psi}(x)\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) \\ &= \bar{\psi}(x)\delta^\nu{}_\rho\gamma^\rho\partial_\nu\psi(x) = \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x). \end{aligned} \quad (4.117)$$

利用 $\bar{\psi}\gamma^\mu\partial_\mu\psi$ 和 $\bar{\psi}\psi$ 可以写下自由 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \quad (4.118)$$

于是, 有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}. \quad (4.119)$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi}. \quad (4.120)$$

对上式取厄米共轭, 得到

$$0 = -i(\gamma^\mu)^\dagger \partial_\mu (\psi^\dagger \gamma^0)^\dagger + m(\psi^\dagger \gamma^0)^\dagger = -i(\gamma^\mu)^\dagger \gamma^0 \partial_\mu \psi + m\gamma^0 \psi = -\gamma^0 (i\gamma^\mu \partial_\mu - m)\psi, \quad (4.121)$$

即

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (4.122)$$

上式就是 **Dirac 方程**, 标明旋量指标的形式为

$$[i(\gamma^\mu)_{ab} \partial_\mu - m\delta_{ab}]\psi_b(x) = 0. \quad (4.123)$$

可以验证, Dirac 方程具有 Lorentz 协变性:

$$\begin{aligned} (i\gamma^\mu \partial'_\mu - m)\psi'(x') &= [i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]D(\Lambda)\psi(x) = D(\Lambda)[iD^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\psi(x) \\ &= D(\Lambda)[i\Lambda^\mu{}_\rho \gamma^\rho (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\psi(x) = D(\Lambda)(i\delta^\nu{}_\rho \gamma^\rho \partial_\nu - m)\psi(x) \\ &= D(\Lambda)(i\gamma^\nu \partial_\nu - m)\psi(x) = 0. \end{aligned} \quad (4.124)$$

对 Dirac 方程 (4.122) 左边乘以 $(-i\gamma^\mu \partial_\mu - m)$, 利用反对易关系 (4.1), 可得

$$\begin{aligned} 0 &= (-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi = (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = \left[\frac{1}{2} \gamma^\mu \gamma^\nu (\partial_\mu \partial_\nu + \partial_\nu \partial_\mu) + m^2 \right] \psi \\ &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi = (g^{\mu\nu} \partial_\mu \partial_\nu + m^2)\psi = (\partial^2 + m^2)\psi. \end{aligned} \quad (4.125)$$

也就是说, 自由的 Dirac 旋量场 $\psi(x)$ 满足 *Klein-Gordon* 方程

$$(\partial^2 + m^2)\psi(x) = 0. \quad (4.126)$$

由 (4.92) 和 (4.97) 式可以看出, 旋量表示的生成元在 Weyl 表象中都是分块对角的, 因而它可以分解为两个 2 维表示的直和。相应地, 可以把具有 4 个分量的 Dirac 旋量场 ψ 分解为两个二分量旋量 φ_L 和 φ_R :

$$\psi = \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix}. \quad (4.127)$$

这样的二分量旋量称为 **Weyl 旋量**, 其中, φ_L 称为**左手** (left-handed) Weyl 旋量, φ_R 称为**右手** (right-handed) Weyl 旋量。

用 2×2 单位矩阵和 Pauli 矩阵定义

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma}), \quad (4.128)$$

那么, Weyl 表象中的 Dirac 矩阵 (4.88) 可以简洁地表示成

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}. \quad (4.129)$$

从而, Dirac 方程 (4.122) 化为

$$0 = (i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} = \begin{pmatrix} i\sigma^\mu \partial_\mu \varphi_R - m\varphi_L \\ i\bar{\sigma}^\mu \partial_\mu \varphi_L - m\varphi_R \end{pmatrix}, \quad (4.130)$$

即

$$\begin{cases} i\bar{\sigma}^\mu \partial_\mu \varphi_L - m\varphi_R = 0, \\ i\sigma^\mu \partial_\mu \varphi_R - m\varphi_L = 0. \end{cases} \quad (4.131)$$

这是一组相互耦合的方程。如果 $m = 0$, 方程组中的两个方程就变得相互独立了:

$$i\bar{\sigma}^\mu \partial_\mu \varphi_L = 0, \quad i\sigma^\mu \partial_\mu \varphi_R = 0. \quad (4.132)$$

这两个独立的方程称为 **Weyl 方程**。可见, 非零质量 m 的存在将左手和右手 Weyl 旋量耦合起来。

4.4 Dirac 旋量场的平面波展开

4.4.1 平面波解的一般形式

本小节讨论与表象选取无关。

对于确定的动量 \mathbf{p} , 我们假设 Dirac 方程具有如下形式的平面波解:

$$\psi_a(x; \mathbf{k}) = w_a(k^0, \mathbf{k})e^{-ik \cdot x}. \quad (4.133)$$

其中, 系数 $w_a(k^0, \mathbf{k})$ 是四分量旋量, 带着一个旋量指标 a 。隐去旋量指标, 将这个平面波解代入到 Dirac 方程 (4.122) 中, 可得

$$0 = (i\gamma^\mu \partial_\mu - m)\psi(x; \mathbf{k}) = (\gamma^\mu k_\mu - m)w(k^0, \mathbf{k})e^{-ik \cdot x} = (k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)w(k^0, \mathbf{k})e^{-ik \cdot x}. \quad (4.134)$$

因此, 有

$$(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)w(k^0, \mathbf{k}) = 0. \quad (4.135)$$

对上式左乘 γ^0 , 可得

$$[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]w(k^0, \mathbf{k}) = 0. \quad (4.136)$$

通过移项, 上式化为

$$[\gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) + m\gamma^0]w(k^0, \mathbf{k}) = k^0w_a(k^0, \mathbf{k}). \quad (4.137)$$

这是一个本征值方程, 它具有非平庸解的条件是特征多项式 $\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]$ 为零, 即

$$\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = 0. \quad (4.138)$$

这个方程的根给出 k^0 的本征值, 相应的非平庸解是本征矢量。

方程 (4.138) 可化为

$$0 = \det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = \det[\gamma^0(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det(\gamma^0) \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m). \quad (4.139)$$

由 (4.39) 式可得 $[\det(\gamma^0)]^2 = \det(\gamma^0\gamma^0) = \det(\mathbf{1}) = 4$, 故 $\det(\gamma^0) \neq 0$ 。因而方程 (4.138) 等价于

$$\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = 0. \quad (4.140)$$

利用 (4.48) 式, 上式左边可化为

$$\begin{aligned} \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) &= \det[(\gamma^5)^2(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det[\gamma^5(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)\gamma^5] \\ &= \det[(\gamma^5)^2(-k^0\gamma^0 + \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det[-(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m]. \end{aligned} \quad (4.141)$$

这里第二步用到行列式性质

$$\det(AB) = \det(BA), \quad (4.142)$$

第三步用到 γ^5 与 γ^μ 反对易的性质 (4.50)。由反对易关系 (4.1) 有

$$(k_\mu\gamma^\mu)^2 = k_\mu k_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} k_\mu k_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = k_\mu k_\nu g^{\mu\nu} = k^2 = (k^0)^2 - |\mathbf{k}|^2. \quad (4.143)$$

从而, 可得

$$\begin{aligned} [\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2 &= \det[(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \det[-(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \\ &= \det(k_\mu\gamma^\mu - m) \det(-k_\mu\gamma^\mu - m) = \det[(k_\mu\gamma^\mu - m)(-k_\mu\gamma^\mu - m)] \\ &= \det[-(k_\mu\gamma^\mu)^2 + m^2] = \det\{[-(k^0)^2 + |\mathbf{k}|^2 + m^2]\mathbf{1}\} \\ &= [-(k^0)^2 + |\mathbf{k}|^2 + m^2]^4 = [E_{\mathbf{k}}^2 - (k^0)^2]^4, \end{aligned} \quad (4.144)$$

其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$ 。于是, 方程 (4.140) 化为

$$0 = \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = [E_{\mathbf{k}}^2 - (k^0)^2]^2 = (E_{\mathbf{k}} + k^0)^2 (E_{\mathbf{k}} - k^0)^2. \quad (4.145)$$

这个方程有 2 个根 $k^0 = \pm E_{\mathbf{k}}$; 这 2 个根都是 2 重根, 各自对应于 2 个独立的本征矢量, 共有 4 个线性无关的本征矢量。

(1) $k^0 = E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.146)$$

因而平面波解中有 2 个正能解, 形式为

$$w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2. \quad (4.147)$$

(2) $k^0 = -E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.148)$$

因而平面波解中有 2 个负能解, 形式为

$$w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2. \quad (4.149)$$

可以将这 4 个本征矢量的正交归一关系取为

$$\begin{aligned} w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma') &= 2E_{\mathbf{k}} \delta_{\sigma\sigma'}, & w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma') &= 2E_{\mathbf{k}} \delta_{\sigma\sigma'}, \\ w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma') = 0. \end{aligned} \quad (4.150)$$

按如下定义引入四分量旋量 $u(\mathbf{k}; \sigma)$ 和 $v(\mathbf{k}; \sigma)$:

$$u(\mathbf{k}; \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad v(-\mathbf{k}; \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.151)$$

第二个定义式等价于

$$v(\mathbf{k}; \sigma) = w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}; \sigma). \quad (4.152)$$

于是, Dirac 方程的正能解和负能解可以分别写作

$$\psi^{(+)}(x; \mathbf{k}; \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = u(\mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})], \quad (4.153)$$

$$\psi^{(-)}(x; \mathbf{k}; \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = v(\mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})]. \quad (4.154)$$

替换一下动量记号, 可得

$$\psi^{(+)}(x; \mathbf{p}; \sigma) = u(\mathbf{p}; \sigma) e^{-ip \cdot x}, \quad \psi^{(-)}(x; \mathbf{p}; \sigma) = v(\mathbf{p}; \sigma) e^{ip \cdot x}, \quad p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.155)$$

从而, Dirac 旋量场算符 $\psi(\mathbf{x}, t)$ 的平面波展开式可写作

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=1}^2 [\psi^{(+)}(x; \mathbf{p}; \sigma) a_{\mathbf{p};\sigma} + \psi^{(-)}(x; \mathbf{p}; \sigma) b_{\mathbf{p};\sigma}^{\dagger}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=1}^2 [u(\mathbf{p}; \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x} + v(\mathbf{p}; \sigma) b_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}]. \end{aligned} \quad (4.156)$$

其中, $a_{\mathbf{p};\sigma}$ 是湮灭算符, $b_{\mathbf{p};\sigma}^{\dagger}$ 是产生算符。一般地, $a_{\mathbf{p};\sigma} \neq b_{\mathbf{p};\sigma}$ 。

旋量系数 $u(\mathbf{p}; \sigma)$ 和 $v(\mathbf{p}; \sigma)$ 的正交归一关系为

$$u^{\dagger}(\mathbf{p}; \sigma) u(\mathbf{p}; \sigma') = w^{(+)\dagger}(E_{\mathbf{p}}, \mathbf{p}; \sigma) w^{(+)}(E_{\mathbf{p}}, \mathbf{p}; \sigma') = 2E_{\mathbf{p}} \delta_{\sigma\sigma'}, \quad (4.157)$$

$$v^{\dagger}(\mathbf{p}; \sigma) v(\mathbf{p}; \sigma') = w^{(-)\dagger}(-E_{\mathbf{p}}, -\mathbf{p}; \sigma) w^{(-)}(-E_{\mathbf{p}}, -\mathbf{p}; \sigma') = 2E_{\mathbf{p}} \delta_{\sigma\sigma'}, \quad (4.158)$$

$$u^{\dagger}(\mathbf{p}; \sigma) v(-\mathbf{p}; \sigma') = w^{(+)\dagger}(E_{\mathbf{p}}, \mathbf{p}; \sigma) w^{(-)}(-E_{\mathbf{p}}, \mathbf{p}; \sigma') = 0. \quad (4.159)$$

4.4.2 Weyl 表象中的平面波解

本小节在 Weyl 表象中讨论 Dirac 方程的平面波解。

Dirac 旋量场描述自旋为 1/2 的粒子, 因而粒子的自旋在动量方向上的投影有 2 种取值, +1/2 和 -1/2, 归一化后对应于 2 种螺旋度 $\lambda = \pm$ 。类似于矢量场的情况, Dirac 旋量场所描述的粒子的状态可以用螺旋度本征值 λ 来表征。因此, 无论是平面波解的正能解还是负能解, 都能够以 2 种螺旋度本征态作为 2 个独立的本征矢量。

按照这个思路，可以把正能解的 2 个本征矢量记作

$$\psi^{(+)}(x; \mathbf{p}, \lambda) = u(\mathbf{p}, \lambda)e^{-ip \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.160)$$

根据 Dirac 方程 (4.122)，有

$$0 = (i\gamma^\mu \partial_\mu - m)\psi^{(+)}(x; \mathbf{p}, \lambda) = (p_\mu \gamma^\mu - m)u(\mathbf{p}, \lambda)e^{-ip \cdot x}, \quad (4.161)$$

即

$$(\not{p} - m)u(\mathbf{p}, \lambda) = 0, \quad (4.162)$$

其中， \not{p} 的定义为

$$\not{p} \equiv p_\mu \gamma^\mu. \quad (4.163)$$

这种斜线记号称为 **Dirac 斜线** (slash)，是 R. Feynman 引进的。

将四分量旋量 $u(\mathbf{p}, \lambda)$ 分解为两个二分量旋量 $f_\lambda(\mathbf{p})$ 和 $g_\lambda(\mathbf{p})$ ，

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.164)$$

那么，根据 Weyl 表象中的 Dirac 矩阵表达式 (4.129)，方程 (4.162) 化为

$$0 = (\not{p} - m)u(\mathbf{p}, \lambda) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix} \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu f_\lambda(\mathbf{p}) - m g_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.165)$$

即

$$(p \cdot \sigma)g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) = 0, \quad (4.166)$$

$$(p \cdot \bar{\sigma})f_\lambda(\mathbf{p}) - m g_\lambda(\mathbf{p}) = 0. \quad (4.167)$$

将 (4.129) 式代入反对易关系 (4.1)，可得

$$2g^{\mu\nu} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}, \quad (4.168)$$

故

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu}, \quad (4.169)$$

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu}. \quad (4.170)$$

因而，有

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2} p_\mu p_\nu 2g^{\mu\nu} = p^2. \quad (4.171)$$

由方程 (4.167) 可得

$$g_\lambda(\mathbf{p}) = \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p}). \quad (4.172)$$

将上式代入到由方程 (4.166) 得出的关系中, 有

$$f_\lambda(\mathbf{p}) = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{m} g_\lambda(\mathbf{p}) = \frac{1}{m^2} (\mathbf{p} \cdot \boldsymbol{\sigma})(\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}) f_\lambda(\mathbf{p}) = \frac{p^2}{m^2} f_\lambda(\mathbf{p}) = f_\lambda(\mathbf{p}). \quad (4.173)$$

可见, 关系式 (4.172) 是自洽的。这样的话, 只要选取合适的 $f_\lambda(\mathbf{p})$, 然后由 (4.164) 和 (4.172) 式得到

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}}{m} f_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.174)$$

就可以满足方程 (4.162)。

在 Weyl 表象中, 根据 (4.94) 式, 自旋角动量矩阵 \mathbf{S} 在动量 \mathbf{p} 方向上的投影为

$$\hat{\mathbf{p}} \cdot \mathbf{S} = \frac{1}{2} \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}. \quad (4.175)$$

归一化后, 得到螺旋度矩阵

$$2\hat{\mathbf{p}} \cdot \mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}. \quad (4.176)$$

上式的两个分块相同, 因此, 左手和右手 Weyl 旋量对应的螺旋度矩阵是相同的, 都是

$$\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}. \quad (4.177)$$

引入作为螺旋度本征态的二分量旋量 $\xi_\lambda(\mathbf{p})$, 满足

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda \xi_\lambda(\mathbf{p}), \quad \lambda = \pm. \quad (4.178)$$

我们要求 $\xi_\lambda(\mathbf{p})$ 具有正交归一关系

$$\xi_\lambda^\dagger(\mathbf{p}) \xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'} \quad (4.179)$$

和完备性关系

$$\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}. \quad (4.180)$$

此外, 由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 可得

$$(\mathbf{p} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda |\mathbf{p}| \xi_\lambda(\mathbf{p}) \quad (4.181)$$

我们将 $\xi_\lambda(\mathbf{p})$ 称为螺旋态。在实际应用中, 可以把螺旋态 $\xi_\lambda(\mathbf{p})$ 取为如下形式:

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}| + p^3 \end{pmatrix}. \quad (4.182)$$

可以验证, 它们确实是 $\lambda = \pm$ 的本征态:

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_+(\mathbf{p}) = \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix}$$

$$\begin{aligned}
&= \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3(|\mathbf{p}| + p^3) + (p^1 - ip^2)(p^1 + ip^2) \\ (p^1 + ip^2)(|\mathbf{p}| + p^3) - p^3(p^1 + ip^2) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3|\mathbf{p}| + |\mathbf{p}|^2 \\ (p^1 + ip^2)|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3 + |\mathbf{p}| \\ p^1 + ip^2 \end{pmatrix} \\
&= +\xi_+(\mathbf{p}), \tag{4.183}
\end{aligned}$$

$$\begin{aligned}
(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_-(\mathbf{p}) &= \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix} \begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}| + p^3 \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} -p^3(p^1 - ip^2) + (p^1 - ip^2)(|\mathbf{p}| + p^3) \\ (p^1 + ip^2)(-p^1 + ip^2) - p^3(|\mathbf{p}| + p^3) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} (p^1 - ip^2)|\mathbf{p}| \\ -|\mathbf{p}|^2 - p^3|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^1 - ip^2 \\ -|\mathbf{p}| - p^3 \end{pmatrix} \\
&= -\xi_-(\mathbf{p}). \tag{4.184}
\end{aligned}$$

而且，满足正交归一关系：

$$\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_+(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} \begin{pmatrix} |\mathbf{p}| + p^3 & p^1 - ip^2 \\ p^1 + ip^2 \end{pmatrix} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix} \\
&= \frac{(|\mathbf{p}| + p^3)^2 + |p^1 + ip^2|^2}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} = \frac{2|\mathbf{p}|^2 + 2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} = 1, \tag{4.185}
\end{aligned}$$

$$\begin{aligned}
\xi_-^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} \begin{pmatrix} -p^1 - ip^2 & |\mathbf{p}| + p^3 \\ |\mathbf{p}| + p^3 \end{pmatrix} \begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}| + p^3 \end{pmatrix} \\
&= \frac{|-p^1 + ip^2|^2 + (|\mathbf{p}| + p^3)^2}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} = \frac{2|\mathbf{p}|^2 + 2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} = 1, \tag{4.186}
\end{aligned}$$

$$\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} \begin{pmatrix} |\mathbf{p}| + p^3 & p^1 - ip^2 \\ p^1 + ip^2 \end{pmatrix} \begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}| + p^3 \end{pmatrix} \\
&= \frac{-(|\mathbf{p}| + p^3)(p^1 - ip^2) + (p^1 + ip^2)(|\mathbf{p}| + p^3)}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} = 0. \tag{4.187}
\end{aligned}$$

也满足完备性关系：

$$\begin{aligned}
\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) &= \xi_+(\mathbf{p})\xi_+^\dagger(\mathbf{p}) + \xi_-(\mathbf{p})\xi_-^\dagger(\mathbf{p}) \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} \begin{pmatrix} (|\mathbf{p}| + p^3)^2 + |-p^1 + ip^2|^2 & (|\mathbf{p}| + p^3)(p^1 - ip^2) + (|\mathbf{p}| + p^3)(-p^1 + ip^2) \\ (|\mathbf{p}| + p^3)(p^1 + ip^2) + (|\mathbf{p}| + p^3)(-p^1 - ip^2) & |p^1 + ip^2|^2 + (|\mathbf{p}| + p^3)^2 \end{pmatrix} \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}| + p^3)} \begin{pmatrix} 2|\mathbf{p}|^2 + 2p^3|\mathbf{p}| & 0 \\ 0 & 2|\mathbf{p}|^2 + 2p^3|\mathbf{p}| \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}. \tag{4.188}
\end{aligned}$$

当 $p^3 = -|\mathbf{p}|$ 时，(4.182) 式失去良好的定义，此时我们可以将螺旋态取成

$$\xi_+(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{4.189}$$

现在, 将 $f_\lambda(\mathbf{p})$ 取为

$$f_\lambda(\mathbf{p}) = C_\lambda \xi_\lambda(\mathbf{p}), \quad (4.190)$$

其中 C_λ 是常数。从而, 利用 (4.181) 式, (4.174) 式可化为

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix}. \quad (4.191)$$

再取

$$C_\lambda = \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \quad (4.192)$$

则由

$$\sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} = \sqrt{E_{\mathbf{p}}^2 - \lambda^2|\mathbf{p}|^2} = \sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2} = m, \quad (4.193)$$

有

$$\begin{aligned} C_\lambda \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} &= \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} = \frac{\sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}}{m} \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} \\ &= \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}. \end{aligned} \quad (4.194)$$

于是, 得到 $u(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \xi_\lambda(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \xi_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.195)$$

其中, $\omega_\lambda(\mathbf{p})$ 定义为

$$\omega_\lambda(\mathbf{p}) \equiv \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}, \quad (4.196)$$

它是关于 \mathbf{p} 的偶函数:

$$\omega_\lambda(-\mathbf{p}) = \omega_\lambda(\mathbf{p}). \quad (4.197)$$

这样的话, 根据 (4.176) 式, $u(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 λ :

$$(2\hat{\mathbf{p}} \cdot \mathbf{S})u(\mathbf{p}, \lambda) = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda u(\mathbf{p}, \lambda). \quad (4.198)$$

另一方面, 可以把负能解的 2 个本征矢量记作

$$\psi^{(-)}(x; \mathbf{p}, \lambda) = v(\mathbf{p}, \lambda) e^{ip \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.199)$$

根据 Dirac 方程 (4.122), 有

$$0 = (i\gamma^\mu \partial_\mu - m)\psi^{(-)}(x; \mathbf{p}, \lambda) = (-p_\mu \gamma^\mu - m)v(\mathbf{p}, \lambda) e^{ip \cdot x}, \quad (4.200)$$

即

$$(\not{p} + m)v(\mathbf{p}, \lambda) = 0. \quad (4.201)$$

同样, 将四分量旋量 $v(\mathbf{p}, \lambda)$ 分解为两个二分量旋量 $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$,

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.202)$$

则有

$$0 = (\not{p} + m)v(\mathbf{p}, \lambda) = \begin{pmatrix} m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & m \end{pmatrix} \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.203)$$

即

$$(p \cdot \sigma) \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = 0, \quad (4.204)$$

$$(p \cdot \bar{\sigma}) \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) = 0. \quad (4.205)$$

由方程 (4.205) 可得

$$\tilde{g}_\lambda(\mathbf{p}) = -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}). \quad (4.206)$$

将上式代入到由方程 (4.204) 得出的关系中, 根据 (4.171) 式, 有

$$\tilde{f}_\lambda(\mathbf{p}) = -\frac{p \cdot \sigma}{m} \tilde{g}_\lambda(\mathbf{p}) = \frac{1}{m^2} (p \cdot \sigma) (p \cdot \bar{\sigma}) \tilde{f}_\lambda(\mathbf{p}) = \frac{p^2}{m^2} \tilde{f}_\lambda(\mathbf{p}) = \tilde{f}_\lambda(\mathbf{p}). \quad (4.207)$$

可见, 关系式 (4.206) 是自洽的。

现在, 将 $\tilde{f}_\lambda(\mathbf{p})$ 取为

$$\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p}), \quad (4.208)$$

其中 \tilde{C}_λ 是常数。在这里, 我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$, 而非 $\xi_\lambda(\mathbf{p})$ 。这种取法的原因将在 4.5.4 小节中说明, 现在姑且接受这种选择。从而, 有

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}. \quad (4.209)$$

再取

$$\tilde{C}_\lambda = -\lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}, \quad (4.210)$$

则由

$$\begin{aligned} -\tilde{C}_\lambda \frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} &= \lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} = \lambda \frac{\sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}}{m} \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} \\ &= \lambda \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \end{aligned} \quad (4.211)$$

可得 $v(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} -\lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \\ \lambda \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}. \quad (4.212)$$

这样一来, $v(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 $-\lambda$:

$$(2\hat{\mathbf{p}} \cdot \mathbf{S})v(\mathbf{p}, \lambda) = \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda v(\mathbf{p}, \lambda). \quad (4.213)$$

根据 $\xi_{\lambda}(\mathbf{p})$ 的正交归一关系 (4.179), 可以验证, $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 满足 (4.157) 和 (4.158) 式表示的正交归一关系:

$$\begin{aligned} u^{\dagger}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) & \omega_{\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})\delta_{\lambda\lambda'} + \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &= [\omega_{-\lambda}^2(\mathbf{p}) + \omega_{\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_{\mathbf{p}} - \lambda|\mathbf{p}|) + (E_{\mathbf{p}} + \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \end{aligned} \quad (4.214)$$

$$\begin{aligned} v^{\dagger}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^{\dagger}(\mathbf{p}) & \lambda \omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda' \omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda' \omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda\lambda'[\omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_{-\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) = \lambda\lambda'[\omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &= \lambda^2[\omega_{\lambda}^2(\mathbf{p}) + \omega_{-\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_{\mathbf{p}} + \lambda|\mathbf{p}|) + (E_{\mathbf{p}} - \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \end{aligned} \quad (4.215)$$

依照螺旋态的本征值方程 (4.178), 可得

$$(-\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = -\lambda \xi_{-\lambda}(-\mathbf{p}), \quad (4.216)$$

从而, 有

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = \lambda \xi_{-\lambda}(-\mathbf{p}). \quad (4.217)$$

可见, $\xi_{-\lambda}(-\mathbf{p})$ 与 $\xi_{\lambda}(\mathbf{p})$ 服从相同的本征值方程, 这意味着 $\xi_{-\lambda}(-\mathbf{p}) \propto \xi_{\lambda}(\mathbf{p})$, 故

$$\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \propto \xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}. \quad (4.218)$$

于是, (4.159) 式表示的正交关系也成立:

$$\begin{aligned} u^{\dagger}(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) & \omega_{\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda' \omega_{\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\ \lambda' \omega_{-\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \end{pmatrix} \\ &= \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(-\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda'}(-\mathbf{p})]\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\ &\propto \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &\propto \lambda[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{\lambda\lambda'} = 0. \end{aligned} \quad (4.219)$$

整理一下, 旋量系数 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 满足如下正交归一关系:

$$u^{\dagger}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = v^{\dagger}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \quad u^{\dagger}(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda') = v^{\dagger}(-\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0. \quad (4.220)$$

此外, 由 (4.193) 式有

$$\omega_{\lambda}(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} = m. \quad (4.221)$$

从而, 利用

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda) &= u^\dagger(\mathbf{p}, \lambda)\gamma^0 = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix},\end{aligned}\quad (4.222)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda) &= v^\dagger(\mathbf{p}, \lambda)\gamma^0 = \begin{pmatrix} -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix},\end{aligned}\quad (4.223)$$

可得

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= [\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = 2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} = 2m\delta_{\lambda\lambda'},\end{aligned}\quad (4.224)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda'\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda'\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= -\lambda\lambda'[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = -2\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} \\ &= -2m\delta_{\lambda\lambda'},\end{aligned}\quad (4.225)$$

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda'\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda'\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ &= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda, -\lambda'} \\ &= -\lambda[-\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})]\delta_{\lambda, -\lambda'} = 0,\end{aligned}\quad (4.226)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{-\lambda, \lambda'} \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{-\lambda, \lambda'} = 0.\end{aligned}\quad (4.227)$$

整理一下, 有

$$\bar{u}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 2m\delta_{\lambda\lambda'}, \quad \bar{v}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = -2m\delta_{\lambda\lambda'}, \quad \bar{u}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = \bar{v}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0.\quad (4.228)$$

另一方面, 利用等式

$$(p \cdot \bar{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} + \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}),\quad (4.229)$$

$$(p \cdot \sigma)\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} - \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p}),\quad (4.230)$$

以及 (4.221) 式和 $\xi_\lambda(\mathbf{p})$ 的完备性关系 (4.180), 可得

$$\begin{aligned}
\sum_{\lambda=\pm} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p}) \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm} \begin{pmatrix} m \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & (p \cdot \sigma) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \\ (p \cdot \bar{\sigma}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & m \xi_\lambda^\dagger(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} \\
&= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = p_\mu \gamma^\mu + m.
\end{aligned} \tag{4.231}$$

通过等式

$$(p \cdot \bar{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} - \lambda |\mathbf{p}|) \xi_{-\lambda}(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \tag{4.232}$$

$$(p \cdot \sigma) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} + \lambda |\mathbf{p}|) \xi_{-\lambda}(\mathbf{p}) = \omega_\lambda^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}), \tag{4.233}$$

则可以得到

$$\begin{aligned}
\sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm} \begin{pmatrix} -\lambda^2 \omega_\lambda(\mathbf{p}) \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda^2 \omega_\lambda^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \\ \lambda^2 \omega_{-\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda^2 \omega_{-\lambda}(\mathbf{p}) \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \sum_{\lambda=\pm} \begin{pmatrix} -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & (p \cdot \sigma) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \\ (p \cdot \bar{\sigma}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
&= \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = p_\mu \gamma^\mu - m.
\end{aligned} \tag{4.234}$$

整理一下, 有如下螺旋度求和关系, 或者说, 自旋求和关系:

$$\sum_{\lambda=\pm} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) = \not{p} + m, \quad \sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) = \not{p} - m. \tag{4.235}$$

用 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 可以把 Dirac 旋量场算符 $\psi(\mathbf{x}, t)$ 的平面波展开式写作

$$\begin{aligned}
\psi(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\psi^{(+)}(x; \mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} + \psi^{(-)}(x; \mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger \right] \\
&= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right].
\end{aligned} \tag{4.236}$$

从而, 有

$$\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right], \tag{4.237}$$

$$\bar{\psi}(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\bar{u}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + \bar{v}(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]. \tag{4.238}$$

4.4.3 哈密顿量和产生湮灭算符

根据 (4.119) 式, $\psi(x)$ 对应的共轭动量密度是

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \quad (4.239)$$

它的平面波展开式为

$$\pi(\mathbf{x}, t) = i\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]. \quad (4.240)$$

自由运动的旋量场 $\psi(x)$ 满足 Dirac 方程 (4.122), 相应地, 拉氏量 (4.118) 化为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (4.241)$$

于是, 根据 (1.119) 式, 自由 Dirac 旋量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \psi - \mathcal{L} = \pi \partial_0 \psi = i\psi^\dagger \partial_0 \psi. \quad (4.242)$$

从而, 哈密顿量为

$$\begin{aligned} H &= \int d^3 x \mathcal{H} = \int d^3 x \psi^\dagger i \partial_0 \psi \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right] \\ &\quad \times \left[q_0 u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - q_0 v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} q_0 \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} \right. \\ &\quad - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} - u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} E_{\mathbf{q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right. \right. \\ &\quad \left. \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[-u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \right. \\ &\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \right\} \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} E_{\mathbf{p}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right. \\ &\quad \left. - u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} E_{\mathbf{p}} \left(2E_{\mathbf{p}} \delta_{\lambda \lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - 2E_{\mathbf{p}} \delta_{\lambda \lambda'} b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right) \end{aligned}$$

$$= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right). \quad (4.243)$$

倒数第二步用到正交归一关系 (4.220)。

另一方面，利用正交归一关系 (4.220)，可得

$$\begin{aligned} & \int d^3x e^{ip \cdot x} u^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t) \\ &= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\ & \quad \left. + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda'} + u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}} \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda'}) = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p},\lambda}. \end{aligned} \quad (4.244)$$

从而，湮灭算符 $a_{\mathbf{p},\lambda}$ 和产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 可以表示为

$$a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} u^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad a_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} \psi^\dagger(\mathbf{x}, t) u(\mathbf{p}, \lambda). \quad (4.245)$$

同理，可以推出

$$\begin{aligned} & \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t) \\ &= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right] \\ &= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right. \\ & \quad \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p},\lambda'}^\dagger \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}} \delta_{\lambda\lambda'} b_{\mathbf{p},\lambda'}^\dagger) = \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p},\lambda}^\dagger. \end{aligned} \quad (4.246)$$

于是，产生算符 $b_{\mathbf{p},\lambda}^\dagger$ 和湮灭算符 $b_{\mathbf{p},\lambda}$ 可以表示为

$$b_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \psi^\dagger(\mathbf{x}, t) v(\mathbf{p}, \lambda). \quad (4.247)$$

4.5 Dirac 旋量场的正则量子化

4.5.1 用等时对易关系量子化 Dirac 旋量场的困难

在标量场和矢量场的正则量子化程序中，我们先假设场算符与其共轭动量密度算符满足等时对易关系 (2.51)，然后推导出产生湮灭算符的对易关系，再通过计算给出正定的哈密顿量（对于无质量矢量场，需要用弱 Lorenz 规范条件来得到正定的哈密顿量期待值），从而说明在量子场论中使用正则量子化方法是合理的。在本小节中，我们将尝试用类似的等时对易关系对 Dirac 旋量场进行量子化，不过，我们会发现这种方法并不能给出正定的哈密顿量，因而是不可行的。

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \quad (4.248)$$

这里已经将旋量指标明显地写出来。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = 0. \quad (4.249)$$

接下来，我们计算产生湮灭算符的对易关系。由 (4.245) 式和正交归一关系 (4.220)，可得

$$\begin{aligned} [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] u_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} u_a^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') \\ &= \frac{1}{2E_{\mathbf{p}}} u_a^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.250)$$

另外，有

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) u_b^\dagger(\mathbf{q}, \lambda') [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = 0. \quad (4.251)$$

由 (4.247) 式和正交归一关系 (4.220)，可得

$$\begin{aligned} [b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} v_b^\dagger(\mathbf{q}, \lambda') [\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] v_a(\mathbf{p}, \lambda) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} v_b^\dagger(\mathbf{q}, \lambda') v_a(\mathbf{p}, \lambda) (-\delta_{ba}) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= -\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} v^\dagger(\mathbf{q}, \lambda') v(\mathbf{p}, \lambda) \\ &= -\frac{1}{2E_{\mathbf{p}}} v^\dagger(\mathbf{p}, \lambda') v(\mathbf{p}, \lambda) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.252)$$

注意, 这个结果非同寻常地多了一个负号。此外, 还有

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} v_b^\dagger(\mathbf{q}, \lambda') [\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] v_a(\mathbf{p}, \lambda) = 0, \quad (4.253)$$

$$[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) v_b^\dagger(\mathbf{q}, \lambda') [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = 0, \quad (4.254)$$

以及

$$\begin{aligned} [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] v_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \\ &= \frac{1}{2E_{\mathbf{p}}} e^{2iE_{\mathbf{p}}t} u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \end{aligned} \quad (4.255)$$

上式最后一步用到正交归一关系 (4.220)。

整理起来, 通过等时对易关系 (4.248) 得到的产生湮灭算符对易关系为

$$\begin{aligned} [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0, \\ [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] &= -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0, \\ [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] &= [b_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0. \end{aligned} \quad (4.256)$$

利用这样的对易关系, 可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right) \\ &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) + (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \end{aligned} \quad (4.257)$$

上式最后一行第二项是零点能。在第一项中由 $a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为正, 但由 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为负。从而, 粒子数密度 $b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}$ 越大, 场的总能量越少, 这显然是非物理的。因此, 用正则对易关系 (4.248) 对 Dirac 旋量场进行量子化是不可行的。

4.5.2 用等时反对易关系量子化 Dirac 旋量场

从 (4.257) 式的计算过程可以看出, 如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}^\dagger$ 位置的同时可以改变圆括号中第二项的符号, 就可以得到正定的哈密顿量。这意味着我们需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b_{\mathbf{p},\lambda}^\dagger$ 的对易关系, 而是反对易关系。为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b_{\mathbf{p},\lambda}^\dagger$ 的反对易关系, 则需要舍弃等时对易关系 (4.248), 代之以等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} = i\delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} = 0. \quad (4.258)$$

采用反对易关系进行量子化的方法称为 **Jordan-Wigner 量子化**。根据 (4.239) 式, 这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = 0. \quad (4.259)$$

接下来, 我们计算产生湮灭算符的反对易关系。计算过程与上一小节类似, 只是我们要将 (4.250) 至 (4.255) 式中表示对易的方括号改成表示反对易的花括号。因此, 可得

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) \{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} u_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \end{aligned} \quad (4.260)$$

和

$$\{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) u_b^\dagger(\mathbf{q}, \lambda') \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = 0. \quad (4.261)$$

另外, 有

$$\begin{aligned} \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') \{\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} v_a(\mathbf{p}, \lambda) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') v_a(\mathbf{p}, \lambda) \delta_{ba}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} v^\dagger(\mathbf{q}, \lambda') v(\mathbf{p}, \lambda) \\ &= \frac{1}{2E_{\mathbf{p}}} v^\dagger(\mathbf{p}, \lambda') v(\mathbf{p}, \lambda) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.262)$$

与 (4.252) 式不同, 上式的结果具有正常的符号。此外, 还有

$$\{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') \{\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} v_a(\mathbf{p}, \lambda) = 0, \quad (4.263)$$

$$\{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) v_b^\dagger(\mathbf{q}, \lambda') \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = 0, \quad (4.264)$$

以及

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) \{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} v_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) = 0. \end{aligned} \quad (4.265)$$

整理起来, 通过等时反对易关系 (4.258) 得到的产生湮灭算符反对易关系为

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}\} = \{a_{\mathbf{p}, \lambda}^\dagger, a_{\mathbf{q}, \lambda'}^\dagger\} = 0, \\ \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \{b_{\mathbf{p}, \lambda}^\dagger, b_{\mathbf{q}, \lambda'}^\dagger\} = 0, \\ \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= \{b_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} = \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \{a_{\mathbf{p}, \lambda}^\dagger, b_{\mathbf{q}, \lambda'}^\dagger\} = 0. \end{aligned} \quad (4.266)$$

$a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 各自描述一种粒子。利用这样的反对易关系, 可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right) \\ &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) - (2\pi)^3 \delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \end{aligned} \quad (4.267)$$

上式最后一行第二项是零点能。第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和, 它是正定的。可见, 用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的。

利用 (4.4) 式和反对易关系 (4.266), 可得哈密顿量 H 与产生湮灭算符的对易子为

$$\begin{aligned} [H, a_{\mathbf{p},\lambda}^\dagger] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} a_{\mathbf{p},\lambda}^\dagger \right] \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(a_{\mathbf{q},\lambda'}^\dagger \{a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} - \{a_{\mathbf{q},\lambda'}^\dagger, a_{\mathbf{p},\lambda}^\dagger\} a_{\mathbf{q},\lambda'} \right. \\ &\quad \left. + b_{\mathbf{q},\lambda'}^\dagger \{b_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} - \{b_{\mathbf{q},\lambda'}^\dagger, a_{\mathbf{p},\lambda}^\dagger\} b_{\mathbf{q},\lambda'} \right) \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q},\lambda'}^\dagger \{a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} \\ &= \sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'}^\dagger \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger, \end{aligned} \quad (4.268)$$

$$\begin{aligned} [H, a_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} a_{\mathbf{p},\lambda} \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(-\{a_{\mathbf{q},\lambda'}^\dagger, a_{\mathbf{p},\lambda}\} a_{\mathbf{q},\lambda'} \right) \\ &= -\sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p},\lambda}, \end{aligned} \quad (4.269)$$

$$\begin{aligned} [H, b_{\mathbf{p},\lambda}^\dagger] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} b_{\mathbf{p},\lambda}^\dagger \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} b_{\mathbf{q},\lambda'}^\dagger \{b_{\mathbf{q},\lambda'}, b_{\mathbf{p},\lambda}^\dagger\} \\ &= \sum_{\lambda'} \int d^3q E_{\mathbf{q}} b_{\mathbf{q},\lambda'}^\dagger \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger, \end{aligned} \quad (4.270)$$

$$\begin{aligned} [H, b_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} b_{\mathbf{p},\lambda} \right] = \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(-\{b_{\mathbf{q},\lambda'}^\dagger, b_{\mathbf{p},\lambda}\} b_{\mathbf{q},\lambda'} \right) \\ &= -\sum_{\lambda'} \int d^3q E_{\mathbf{q}} b_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} b_{\mathbf{p},\lambda}. \end{aligned} \quad (4.271)$$

设 $|E\rangle$ 是 H 的本征态, 本征值为 E , 则

$$H |E\rangle = E |E\rangle. \quad (4.272)$$

从而, 可得

$$\begin{aligned} H a_{\mathbf{p},\lambda}^\dagger |E\rangle &= (a_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p},\lambda}^\dagger |E\rangle, \\ H a_{\mathbf{p},\lambda} |E\rangle &= (a_{\mathbf{p},\lambda} H - E_{\mathbf{p}} a_{\mathbf{p},\lambda}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p},\lambda} |E\rangle, \end{aligned}$$

$$\begin{aligned}
Hb_{\mathbf{p},\lambda}^\dagger |E\rangle &= (b_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) b_{\mathbf{p},\lambda}^\dagger |E\rangle, \\
Hb_{\mathbf{p},\lambda} |E\rangle &= (b_{\mathbf{p},\lambda} H - E_{\mathbf{p}} b_{\mathbf{p},\lambda}) |E\rangle = (E - E_{\mathbf{p}}) b_{\mathbf{p},\lambda} |E\rangle.
\end{aligned} \tag{4.273}$$

可见, 当 $a_{\mathbf{p},\lambda}^\dagger |E\rangle$ 和 $b_{\mathbf{p},\lambda}^\dagger |E\rangle$ 不为零时, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 和 $b_{\mathbf{p},\lambda}^\dagger$ 的作用都是使能量本征值增加 $E_{\mathbf{p}}$; 当 $a_{\mathbf{p},\lambda} |E\rangle$ 和 $b_{\mathbf{p},\lambda} |E\rangle$ 不为零时, 湮灭算符 $a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的作用都是使能量本征值减少 $E_{\mathbf{p}}$ 。

根据 (1.158) 式, Dirac 旋量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi \nabla \psi = \int d^3x \psi^\dagger (-i \nabla) \psi \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip \cdot x} \right] \\
&\quad \times \left[\mathbf{q} u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - \mathbf{q} v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \mathbf{q} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} \right. \\
&\quad - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} - u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \\
&\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \mathbf{q} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right. \right. \\
&\quad \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right] \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[- u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{q},\lambda'}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[\mathbf{p} u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} - \mathbf{p} v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda'}^\dagger \right. \\
&\quad \left. + \mathbf{p} u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} - \mathbf{p} v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \mathbf{p} \left(2E_{\mathbf{p}} \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} - 2E_{\mathbf{p}} \delta_{\lambda\lambda'} b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda'}^\dagger \right) \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right) \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) - 2\delta^{(3)}(0) \int d^3p \mathbf{p} \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right). \tag{4.274}
\end{aligned}$$

倒数第四步用到正交归一关系 (4.220), 倒数第二步用到反对易关系 (4.266)。总动量是所有动量模式所有螺旋度所有粒子贡献的动量之和。

4.5.3 U(1) 整体对称性

类似于复标量场，Dirac 旋量场也具有 U(1) 整体对称性。对 Dirac 旋量场 $\psi(x)$ 作 U(1) 整体变换

$$\psi'(x) = e^{iq\theta}\psi(x), \quad (4.275)$$

则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = \psi^\dagger(x)e^{-iq\theta}, \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{-iq\theta}. \quad (4.276)$$

在此变换下，拉氏量 (4.118) 不变：

$$\begin{aligned} \mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi'(x) \\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x). \end{aligned} \quad (4.277)$$

容易验证，4.3 节中列举的旋量双线性型都在这种 U(1) 整体变换下不变。因此，用这些旋量双线性型构造的拉氏量都具有 U(1) 整体对称性。

U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + iq\theta\psi(x). \quad (4.278)$$

由于 $\delta x^\mu = 0$ ，根据 (1.136) 式可得

$$\bar{\delta}\psi = \delta\psi = iq\theta\psi. \quad (4.279)$$

按照 (1.141) 式，相应的 Noether 守恒流为

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar{\delta}\psi = i\bar{\psi}\gamma^\mu(iq\theta\psi) = -q\theta\bar{\psi}\gamma^\mu\psi. \quad (4.280)$$

扔掉无穷小参数 $-\theta$ ，定义

$$J^\mu \equiv q\bar{\psi}\gamma^\mu\psi, \quad (4.281)$$

则 Noether 定理给出

$$\partial_\mu J^\mu = 0. \quad (4.282)$$

相应的守恒荷为

$$\begin{aligned} Q &= \int d^3x J^0 = q \int d^3x \bar{\psi}\gamma^0\psi = q \int d^3x \psi^\dagger\psi \\ &= q \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right] \\ &\quad \times \left[u(\mathbf{k}, \lambda') a_{\mathbf{k}, \lambda'} e^{-ik \cdot x} + v(\mathbf{k}, \lambda') b_{\mathbf{k}, \lambda'}^\dagger e^{ik \cdot x} \right] \\ &= q \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{k}, \lambda'} e^{i(p-k) \cdot x} \right. \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{k}, \lambda'}^\dagger e^{-i(p-k) \cdot x} \right. \\ &\quad \left. + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{k}, \lambda'} e^{i(p-k) \cdot x} \right. \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{k}, \lambda'}^\dagger e^{-i(p-k) \cdot x} \right] \end{aligned}$$

$$\begin{aligned}
& +v^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{k},\lambda'}^\dagger e^{-i(p-k)\cdot x} + u^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{k},\lambda'}^\dagger e^{i(p+k)\cdot x} \\
& +v^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(p+k)\cdot x} \Big] \\
& = q \sum_{\lambda\lambda'} \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left\{ \delta^{(3)}(\mathbf{p}-\mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right. \right. \\
& \quad \left. \left. +v^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{k},\lambda'}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right] \right. \\
& \quad \left. +\delta^{(3)}(\mathbf{p}+\mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{k},\lambda'}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right. \right. \\
& \quad \left. \left. +v^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right] \right\} \\
& = q \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[u^\dagger(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda')a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} + v^\dagger(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda'}^\dagger \right. \\
& \quad \left. +u^\dagger(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda)u(-\mathbf{p}, \lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} \right] \\
& = q \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} + 2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda'}^\dagger \right) \\
& = q \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}^\dagger \right) \\
& = \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \left(q a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - q b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) + 2\delta^{(3)}(0) \int d^3p q. \tag{4.283}
\end{aligned}$$

上式第二项是零点荷。从第一项的形式可以看出，由 $a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 描述的粒子是**正粒子**，具有的荷为 q ；由 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 描述的粒子是**反粒子**，具有的荷为 $-q$ 。除去零点荷，总荷是所有动量模式所有螺旋度所有正反粒子贡献的荷之和。

4.5.4 粒子态

对于自由的 Dirac 旋量场，真空态定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$a_{\mathbf{p},\lambda}|0\rangle = b_{\mathbf{p},\lambda}|0\rangle = 0, \tag{4.284}$$

满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = -2\delta^{(3)}(0) \int d^3p E_{\mathbf{p}}. \tag{4.285}$$

动量为 \mathbf{p} 、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$|\mathbf{p}, \lambda, +\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad |\mathbf{p}, \lambda, -\rangle \equiv \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p},\lambda}^\dagger |0\rangle. \tag{4.286}$$

根据 (4.268) 和 (4.270) 式，有

$$\begin{aligned}
H|\mathbf{p}, \lambda, +\rangle &= \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p},\lambda}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger) |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) a_{\mathbf{p},\lambda}^\dagger |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}, \lambda, +\rangle, \\
H|\mathbf{p}, \lambda, -\rangle &= \sqrt{2E_{\mathbf{p}}} H b_{\mathbf{p},\lambda}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (b_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger) |0\rangle
\end{aligned} \tag{4.287}$$

$$= \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) b_{\mathbf{p},\lambda}^{\dagger} |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}, \lambda, -\rangle. \quad (4.288)$$

可见, $|\mathbf{p}, \lambda, +\rangle$ 和 $|\mathbf{p}, \lambda, -\rangle$ 都比真空态多了一份能量 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ 。

将 $\psi(x)$ 的平面波解 (4.236) 代入 (4.81) 式左边, 得

$$[\psi(x), \mathbf{J}] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left\{ u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, \mathbf{J}] e^{-ip \cdot x} + v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, \mathbf{J}] e^{ip \cdot x} \right\}, \quad (4.289)$$

代入右边, 得

$$\begin{aligned} (\mathbf{L} + \mathbf{S})\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} (-i\mathbf{x} \times \nabla + \mathbf{S}) \left[u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[(\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right]. \end{aligned} \quad (4.290)$$

可见, 对于动量模式 \mathbf{p} 和螺旋度 λ , 有

$$u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, \mathbf{J}] = (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \quad v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, \mathbf{J}] = (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.291)$$

根据 (4.198) 和 (4.213) 式, $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态, 因而

$$u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = 2\hat{\mathbf{p}} \cdot (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} = (2\hat{\mathbf{p}} \cdot \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} = \lambda u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \quad (4.292)$$

$$v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = 2\hat{\mathbf{p}} \cdot (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} = (2\hat{\mathbf{p}} \cdot \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} = -\lambda v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.293)$$

故

$$[a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = \lambda a_{\mathbf{p},\lambda}, \quad [b_{\mathbf{p},\lambda}^{\dagger}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = -\lambda b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.294)$$

由于 \mathbf{J} 是厄米算符, 对第一式取厄米共轭可得

$$\lambda a_{\mathbf{p},\lambda}^{\dagger} = [a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}]^{\dagger} = (2\hat{\mathbf{p}} \cdot \mathbf{J}) a_{\mathbf{p},\lambda}^{\dagger} - a_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) = [2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p},\lambda}^{\dagger}]. \quad (4.295)$$

于是, 有

$$[2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p},\lambda}^{\dagger}] = \lambda a_{\mathbf{p},\lambda}^{\dagger}, \quad [2\hat{\mathbf{p}} \cdot \mathbf{J}, b_{\mathbf{p},\lambda}^{\dagger}] = \lambda b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.296)$$

\mathbf{J} 是总角动量算符, 真空态 $|0\rangle$ 不具有角动量, 所以满足

$$\mathbf{J} |0\rangle = 0. \quad (4.297)$$

由此, 可得

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) a_{\mathbf{p},\lambda}^{\dagger} |0\rangle = [a_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda a_{\mathbf{p},\lambda}^{\dagger}] |0\rangle = \lambda a_{\mathbf{p},\lambda}^{\dagger} |0\rangle, \quad (4.298)$$

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) b_{\mathbf{p},\lambda}^{\dagger} |0\rangle = [b_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda b_{\mathbf{p},\lambda}^{\dagger}] |0\rangle = \lambda b_{\mathbf{p},\lambda}^{\dagger} |0\rangle. \quad (4.299)$$

在没有轨道角动量的情况下, $2\hat{\mathbf{p}} \cdot \mathbf{J}$ 是螺旋度算符。因此, 上面两式说明 $|\mathbf{p}, \lambda, +\rangle$ 和 $|\mathbf{p}, \lambda, -\rangle$ 都是螺旋度本征态, 本征值为 λ :

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) |\mathbf{p}, \lambda, \pm\rangle = \lambda |\mathbf{p}, \lambda, \pm\rangle. \quad (4.300)$$

这正是我们所期望的。

以上讨论表明, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 的作用是产生一个动量为 \mathbf{p} 、螺旋度为 λ 的正粒子, 另一个产生算符 $b_{\mathbf{p},\lambda}^\dagger$ 的作用是产生一个动量为 \mathbf{p} 、螺旋度为 λ 的反粒子。正粒子和反粒子具有相同的质量 m 。

在 (4.208) 式中, 我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$, 使得 $v(\mathbf{p}, \lambda)$ 的螺旋度本征值为 $-\lambda$, 从而得到 $b_{\mathbf{p},\lambda}^\dagger |0\rangle$ 的螺旋度本征值为 λ 的结果。如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$, 依照上述推导, $b_{\mathbf{p},\lambda}^\dagger |0\rangle$ 的螺旋度本征值就会变成 $-\lambda$; 也就是说, $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 将描述螺旋度为 $-\lambda$ 的反粒子。这不符合我们的记号, 因此, 我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 (4.208) 式的形式。

由反对易关系 (4.266), 可得

$$\begin{aligned} a_{\mathbf{p},\lambda} |\mathbf{q}, \lambda', +\rangle &= \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{p},\lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle, \end{aligned} \quad (4.301)$$

$$\begin{aligned} b_{\mathbf{p},\lambda} |\mathbf{q}, \lambda', -\rangle &= \sqrt{2E_{\mathbf{q}}} b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{p},\lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle. \end{aligned} \quad (4.302)$$

可以看出, 湮灭算符 $a_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 \mathbf{p} 、螺旋度为 λ 的正粒子, 湮灭算符 $b_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 \mathbf{p} 、螺旋度为 λ 的反粒子。

将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle \equiv \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{p}_3} E_{\mathbf{p}_4}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle. \quad (4.303)$$

根据反对易关系 (4.266), 有

$$\begin{aligned} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle &= -a_{\mathbf{p}_2, \lambda_2}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle = -a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle \\ &= -a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger |0\rangle = -b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle \\ &= -a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle = -b_{\mathbf{p}_4, \lambda_4}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger |0\rangle. \end{aligned} \quad (4.304)$$

从而, 可得

$$\begin{aligned} |\mathbf{p}_2, \lambda_2, +; \mathbf{p}_1, \lambda_1, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_4, \lambda_4, -; \mathbf{p}_3, \lambda_3, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_1, \lambda_1, +; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_4, \lambda_4, -; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_4, \lambda_4, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_1, \lambda_1, +\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle. \end{aligned} \quad (4.305)$$

也就是说, 交换任意两个粒子, 得到的态相差一个负号, 故多粒子态对于全同粒子交换是反对称的。这说明旋量场描述的粒子是费米子 (fermion), 服从 Fermi-Dirac 统计。得到这个结论的

关键在于两个产生算符相互反对易。对于两个相同的产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 或 $b_{\mathbf{p},\lambda}^\dagger$ ，反对易关系导致

$$a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle = -a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle = -b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad (4.306)$$

故

$$a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle = 0, \quad b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle = 0. \quad (4.307)$$

这说明在没有其它自由度的情况下，不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态，这就是 **Pauli 不相容原理**。

在第 2 章和第 3 章中，我们分别讨论了自旋为 0 的标量场和自旋为 1 的矢量场，合适的处理方式是通过反对易关系对它们进行量子化，因而它们都描述玻色子。另一方面，在本章中，我们需要采用反对易关系才能对自旋为 1/2 的旋量场进行合适的量子化，因而旋量场描述的粒子是费米子。实际上，这样的状况是普遍的，存在**自旋-统计定理**：整数自旋的物理场必须用对易关系进行量子化，对应的粒子是玻色子；半整数自旋的物理场必须用反对易关系进行量子化，对应的粒子是费米子。

将两个正费米子组成的双粒子态记为

$$|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +\rangle \equiv \sqrt{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle, \quad (4.308)$$

则双粒子态的内积关系是

$$\begin{aligned} & \langle \mathbf{q}_1, \lambda'_1, +; \mathbf{q}_2, \lambda'_2, + | \mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, + \rangle \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{q}_1, \lambda'_1} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\ & \quad \left. - \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{q}_1, \lambda'_1} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right] \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\ & \quad \left. - (2\pi)^3 \delta_{\lambda_2 \lambda'_1} \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_1, \lambda_1}^\dagger | 0 \rangle \right] \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^6 \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) \right. \\ & \quad \left. - (2\pi)^6 \delta_{\lambda_2 \lambda'_1} \delta_{\lambda_1 \lambda'_2} \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \right] \\ &= 4E_{\mathbf{p}_1} E_{\mathbf{p}_2} (2\pi)^6 \left[\delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) \right. \\ & \quad \left. - \delta_{\lambda_1 \lambda'_2} \delta_{\lambda_2 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \right]. \quad (4.309) \end{aligned}$$

上式最后两行方括号中第二项前面有一个负号，由产生湮灭算符的反对易关系引起。这是双费米子态内积关系与双玻色子态内积关系 (2.119) 在形式上的不同之处。

附录 A 英汉对照

Annihilation operator: 湮灭算符	Hermitian operator: 厄米算符
Antichronous: 反时向	Homomorphic: 同态
Anti-particle: 反粒子	Improper: 非固有
Axial vector: 轴矢量	Lagrangian: 拉格朗日量
Boost: 增速	Left-handed: 左手
Boson: 玻色子	Local: 局域
Canonical quantization: 正则量子化	Lowering operator: 降算符
Chiral representation: 手征表象	Metric: 度规
Conjugate momentum density: 共轭动量密度	Mode: 模式
Conserved charge: 守恒荷	Orthochronous: 保时向
Conserved current: 守恒流	Parity: 宇称
Contraction: 缩并	Phonon: 声子
Contravariant vector: 逆变矢量	Picture: 绘景
Covariant vector: 协变矢量	Plane-wave solution: 平面波解
Creation operator: 产生算符	Polarization vector: 极化矢量
Dirac slash: Dirac 斜线	Positron: 正电子
Electron: 电子	Proper: 固有
Energy-momentum tensor: 能动张量	Pseudoscalar: 赝标量
Expectation value: 期待值	Raising operator: 升算符
Fermion: 费米子	Right-handed: 右手
Field strength tensor: 场强张量	Real orthogonal matrix: 实正交矩阵
Gauge-fixing term: 规范固定项	Scalar: 标量
Gauge invariant: 规范不变量	Self-conjugate: 自共轭
Gauge symmetry: 规范对称性	Simple harmonic oscillator: 简谐振子
Gauge transformation: 规范变换	Space inversion: 空间反射
Generalized coordinate: 广义坐标	Spinor: 旋量
Generator: 生成元	Spinor bilinear: 旋量双线性型
Global: 整体	Spinor representation: 旋量表示
Hamiltonian: 哈密顿量	Step function: 阶跃函数
Helicity: 螺旋度	Tensor: 张量
Hermitian conjugate: 厄米共轭	Time reversal: 时间反演

Unitary: 幺正

Vacuum: 真空

Vector: 矢量

Zero-point energy: 零点能