

量子场论讲义

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第 1 章 预备知识

1.1 量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由粒子组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon** 方程：

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\mathbf{x}, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi(\mathbf{x}, t). \quad (1.1)$$

它给出的自由粒子能量为

$$E = \pm \sqrt{|\mathbf{p}|^2 c^2 + m^2 c^4}, \quad (1.2)$$

其中 \mathbf{p} 为粒子的动量， m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2 \leq E < \infty$ ；也可以为负，取值范围为 $-\infty < E \leq -mc^2$ 。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho = \frac{i\hbar}{2mc^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right), \quad (1.3)$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 协变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E < 0$ 的态都被填满而所有 $E > 0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E > 0$ 的电子跃迁到 $E < 0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的“反粒子 (anti-particle)”，称为**正电子 (positron)**。正电子在 1932 年被 Carl Anderson 发现。

但是, Dirac 的空穴理论仍然面临一些困难, 比如, 为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场? 另一方面, Dirac 方程一开始作为描述单个粒子波函数的方程提出来, 但 Dirac 的解释却包含了无穷多个粒子。而且, 像光子和 π 介子这些不满足 Pauli 不相容原理的粒子, 空穴理论是不能成立的。此外, Dirac 方程只能描述自旋 1/2 的粒子, 不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难, 是否意味着处理这些问题的基础本身就不正确呢? 确实是这样的。量子力学的一条基本原理是: 观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而, 时间显然是一个观测量, 却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中, 描述系统的量子态时可以让态依赖于一个时间参数 t , 这是时间的概念进入量子力学的方式, 但并没有假定这个参数是某个厄米算符的本征值。另一方面, 粒子的空间位置 \mathbf{x} 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见, 在量子力学中, 对时间和空间的处理方式是完全不同的。而在狭义相对论中, Lorentz 对称性将两者混合起来。因此, 在结合量子力学与狭义相对论的过程中出现困难, 也是正常的。

那么, 如何在量子力学中平等地处理时间和空间呢? 一种途径是将时间提升为一个厄米算符, 但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数, 不再由厄米算符描写。这样, 我们可以在每个空间点 \mathbf{x} 处定义一个算符 $\hat{\phi}(\mathbf{x})$, 所有这些算符的集合称为量子场。在 Heisenberg 绘景中, 量子场算符还依赖于时间 t ,

$$\hat{\phi}(\mathbf{x}, t) = e^{i\hat{H}t/\hbar} \hat{\phi}(\mathbf{x}) e^{-i\hat{H}t/\hbar}. \quad (1.4)$$

如此, 量子化的对象变成是由依赖于时空坐标的场组成的动力学系统, 这就是量子场论。这里的量子算符用 $\hat{}$ 符号标记, 为了简化记号, 后面将省略 $\hat{}$ 符号。

在量子场论中, 前面提到的困难都可以得到解决。现在, Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态, 包含粒子的态则是激发态, 激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子, 由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的, 不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x \rho$ 也可以用产生湮灭算符表达出来, 虽然它不一定是正定的, 但是它不再被解释为总概率, 而是被解释为正粒子数与反粒子数之差, 因而也不再出现负概率困难。

1.2 自然单位制

量子场论是结合量子力学和相对论的理论, 因而时常出现约化 Planck 常量 \hbar 和光速 c , 这一点可以从上一节的几个公式中看出来。于是, 为了简化表述, 通常采用自然单位制, 取

$$\hbar = c = 1. \quad (1.5)$$

从而, Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi(\mathbf{x}, t) = 0. \quad (1.6)$$

在自然单位制中, 速度没有量纲 (dimension); 长度量纲与时间量纲相同, 是能量量纲的倒数; 能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22} \text{ MeV} \cdot \text{s}, \quad 1 = \hbar c = 1.973 \times 10^{-11} \text{ MeV} \cdot \text{cm}, \quad (1.7)$$

可得

$$1 \text{ s}^{-1} = 6.582 \times 10^{-22} \text{ MeV}, \quad 1 \text{ cm}^{-1} = 1.973 \times 10^{-11} \text{ MeV}. \quad (1.8)$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.036} \quad (1.9)$$

是没有量纲的, 它的数值在任何单位制下都应该相同。因此, 自然单位制不可能将 \hbar 、 c 、 ϵ_0 和 e 这四个常数同时归一化。在量子场论中, 通常再取真空介电常数

$$\epsilon_0 = 1, \quad (1.10)$$

同时可得真空磁导率 $\mu_0 = 1/(\epsilon_0 c^2) = 1$, 这样做其实是取了 Heaviside-Lorentz 单位制。从而, 不同于 Gauss 单位制, Maxwell 方程组中不会出现无理数 4π ,

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \quad (1.11)$$

此处的单位制称为有理化的自然单位制。现在, 精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$, 而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的; 4π 因子会出现在 Coulomb 定律中, 点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \quad (1.12)$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到狭义相对论, 它的基本原理如下。

- (1) 光速不变原理: 在任意惯性参考系中, 光速的大小不变。
- (2) 狭义相对性原理: 在任意惯性参考系中, 物理定律具有相同的形式。

两个惯性参考系的直角坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动, 则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \quad (1.13)$$

其中 Lorentz 因子 $\gamma \equiv (1 - \beta^2)^{-1/2}$. 这种 Lorentz 变换称为沿 x 方向的增速 (boost)。在此变换下, 有

$$t'^2 - x'^2 - y'^2 - z'^2 = \gamma^2(t - \beta x)^2 - \gamma^2(x - \beta t)^2 - y^2 - z^2$$

$$= \frac{1}{1-\beta^2}(t^2 + \beta^2 x^2 - 2\beta xt - x^2 - \beta^2 t^2 + 2\beta xt) - y^2 - z^2 = t^2 - x^2 - y^2 - z^2. \quad (1.14)$$

可见, $t^2 - x^2 - y^2 - z^2$ 在 Lorentz 变换下不变, 是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值, 这是 Lorentz 变换对应的对称性, 称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来, 可以构成 Minkowski 时空, 坐标记为

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \mathbf{x}), \quad \text{其中 } \mu = 0, 1, 2, 3. \quad (1.15)$$

上式中四种记法是等价的。 x^μ 是一个逆变 (contravariant) 的 Lorentz 四维矢量 (vector), “逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发, 可以定义 Lorentz 不变的内积¹

$$x^2 \equiv x \cdot x \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - |\mathbf{x}|^2. \quad (1.16)$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu} = g_{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.17)$$

可以把内积 (1.16) 简洁地写成

$$x^2 = g_{\mu\nu} x^\mu x^\nu. \quad (1.18)$$

这里采用了 **Einstein 求和约定**: 不写出求和符号, 重复的指标即表示求和。除非特别指出, 后面都默认使用这个约定。在上式中, 用同个字母表示的指标分别在上标和下标重复出现并求和, 这称为缩并 (contraction), 是 Lorentz 不变量的特点。

为了进一步简化记号, 定义协变 (covariant) 的 Lorentz 四维矢量

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}). \quad (1.19)$$

“协变”指的是指标 μ 写在右下角。于是, 内积 x^2 的表达式 (1.18) 可以简化为

$$x^2 = x^\mu x_\mu. \quad (1.20)$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来, 变成协变矢量 x_μ 。从方阵的角度看, $g_{\mu\nu}$ 的逆为

$$g^{\mu\nu} = g^{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.21)$$

¹这里的记号有些不一致, 第一个 x^2 是内积的记号, 而第二个 x^2 是第 2 个空间坐标。

满足

$$g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu, \quad (1.22)$$

其中 Kronecker 符号 $\delta^\mu{}_\nu$ 定义为

$$\delta^\mu{}_\nu = \delta_\mu{}^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \quad (1.23)$$

对于 Minkowski 度规, $g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同, 但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$, 对 μ 求和, 得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma{}_\nu x^\nu = x^\sigma, \quad (1.24)$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来, 变成逆变矢量 x^σ . 可见, 逆变矢量与协变矢量是一一对应的, 是对同一个 Lorentz 矢量的两种等价描述.

利用 Kronecker 符号的定义和 (1.22) 式, 可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu{}_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \quad (1.25)$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho{}_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \quad (1.26)$$

这两条式子表明, 度规也可以用来对度规自身的指标进行升降.

利用四维矢量的记号, 可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (1.27)$$

其中

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (1.28)$$

注意: 在将 $\Lambda^\mu{}_\nu$ 视作矩阵时, 偏左的指标 μ 表示行的编号, 偏右的指标 ν 表示列的编号. $\Lambda^\mu{}_\nu$ 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变, 从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值. 我们可以将 $\Lambda^\mu{}_\nu$ 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换, 称为 (齐次) **Lorentz 变换**, 使下式成立:

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \quad (1.29)$$

可见, Lorentz 变换 $\Lambda^\mu{}_\nu$ 必须满足保度规条件

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}. \quad (1.30)$$

空间旋转变换保持 $|\mathbf{x}|^2$ 不变, 由 (1.16) 式可知, 这种变换也属于 Lorentz 变换. 例如, 绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \quad (1.31)$$

容易验证, 它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并, 可得

$$\Lambda_\nu^\gamma \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = g^{\gamma\alpha} g_{\alpha\beta} = \delta^\gamma_\beta, \quad (1.32)$$

其中

$$\Lambda_\nu^\gamma \equiv g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu_\alpha \quad (1.33)$$

可以看作是用度规对 Λ^μ_α 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu_\nu \equiv \Lambda_\nu^\mu, \quad (1.34)$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu_\rho \Lambda^\rho_\nu = \delta^\mu_\nu. \quad (1.35)$$

δ^μ_ν 也是一个 Lorentz 变换, 它使得 $x'^\mu = \delta^\mu_\nu x^\nu = x^\mu$, 即 x^μ 在这个变换下不变。可见, δ^μ_ν 是一个恒等变换。(1.35) 式表明, 对时空坐标矢量先作 Λ 变换, 再作 Λ^{-1} 变换, 得到的矢量还是原来的矢量。也就是说, 由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换, 也是一个 Lorentz 变换。在这些记号下, 协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu_\rho x^\rho = g_{\mu\nu} \Lambda^\nu_\rho g^{\rho\sigma} x_\sigma = \Lambda_\mu^\sigma x_\sigma = x_\sigma (\Lambda^{-1})^\sigma_\mu. \quad (1.36)$$

Λ^{-1} 既然是一个 Lorentz 变换, 必定满足保度规条件

$$g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta = g_{\alpha\beta}, \quad (1.37)$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma} = g_{\mu\nu} (\Lambda^{-1})^\mu_\alpha (\Lambda^{-1})^\nu_\beta g^{\alpha\rho} g^{\beta\sigma} = g^{\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \Lambda_\alpha^\mu \Lambda_\beta^\nu g^{\alpha\rho} g^{\beta\sigma} \\ &= g^{\gamma\delta} (g^{\alpha\rho} g_{\gamma\mu} \Lambda_\alpha^\mu) (g^{\beta\sigma} g_{\delta\nu} \Lambda_\beta^\nu) = g^{\gamma\delta} \Lambda^\rho_\gamma \Lambda^\sigma_\delta. \end{aligned} \quad (1.38)$$

这给出了保度规条件 (1.30) 的一个等价形式:

$$g^{\mu\nu} \Lambda^\alpha_\mu \Lambda^\beta_\nu = g^{\alpha\beta}. \quad (1.39)$$

将 Λ^μ_ν 视作矩阵 Λ , 则其转置矩阵 Λ^T 的分量满足 $(\Lambda^T)_\nu^\mu = \Lambda^\mu_\nu$, 由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = (\Lambda^T)_\alpha^\mu g_{\mu\nu} \Lambda^\nu_\beta, \quad (1.40)$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^T \mathbf{g} \Lambda. \quad (1.41)$$

取行列式得 $\det \mathbf{g} = \det \Lambda^T \cdot \det \mathbf{g} \cdot \det \Lambda = \det \mathbf{g} \cdot (\det \Lambda)^2$, 因此,

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \quad (1.42)$$

Lorentz 坐标变换 $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ 的 Jacobi 行列式为

$$\mathcal{J} = \det \left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} \right] = \det \Lambda, \quad (1.43)$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}| d^4x = |\det \Lambda| d^4x = d^4x. \quad (1.44)$$

可见, Minkowski 时空的体积元是 Lorentz 不变的。

$\det \Lambda$ 的值可以用来为 Lorentz 变换分类: $\det \Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换, $\det \Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外, 由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu} \Lambda^{\mu}_0 \Lambda^{\nu}_0 = (\Lambda^0_0)^2 - (\Lambda^i_0)^2, \quad (1.45)$$

则 $(\Lambda^0_0)^2 = 1 + (\Lambda^i_0)^2 \geq 1$, 故有 $\Lambda^0_0 \geq +1$ 或 $\Lambda^0_0 \leq -1$ 。 $\Lambda^0_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换, $\Lambda^0_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上, 对称性由群论描述。对称变换的集合称为群, 群元素具有乘法, 满足下列四个条件。

- (1) 两个群元素的乘积即是两次对称变换相继作用, 乘法满足结合律。
- (2) 群中任意两个元素的乘积仍属于此群 (封闭性)。
- (3) 群中必有一个恒元 (对应于恒等变换), 它与任一元素的乘积仍为此元素。
- (4) 任一元素都可以在群中找到一个逆元 (对应于逆变换), 两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数 (如 β 、 θ 等) 来描述, 因而是一种连续变换, 所以 Lorentz 群是一个连续群, 参数的变化区域称为群空间。Lorentz 群的整个群空间不是连通的, 它有四个连通分支, 如图 1.1 所示, 分别是固有保时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \geq +1$)、固有反时向分支 ($\det \Lambda = +1$ 且 $\Lambda^0_0 \leq -1$)、非固有保时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \geq +1$) 和非固有反时向分支 ($\det \Lambda = -1$ 且 $\Lambda^0_0 \leq -1$), 四个分支之间彼此不连通。恒元 (即恒等变换) 在固有保时向分支里, 这个分支也称为固有保时向 **Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义宇称 (parity) 变换为

$$\mathcal{P}^{\mu}_{\nu} = (\mathcal{P}^{-1})^{\mu}_{\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.46)$$

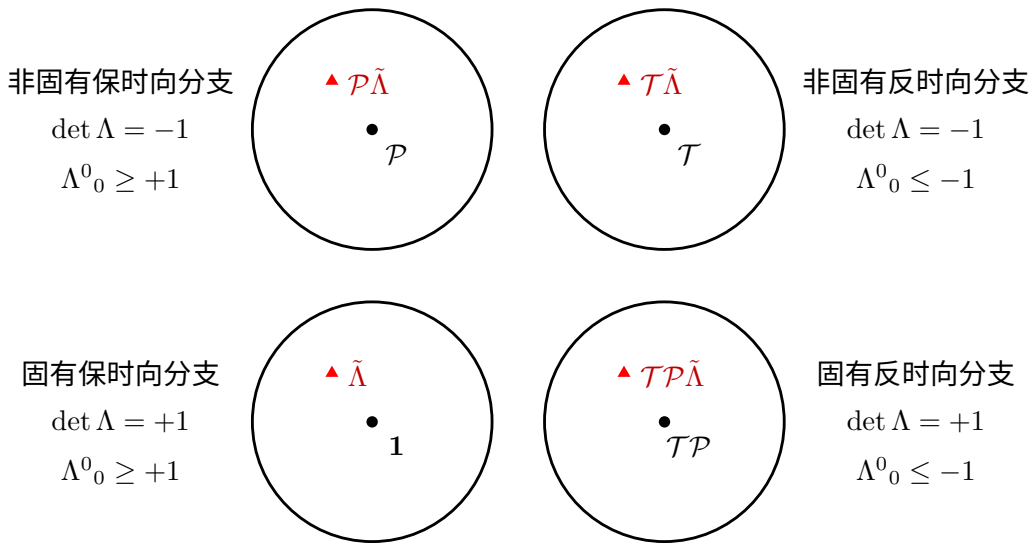


图 1.1: Lorentz 群的四个连通分支示意图。1、 \mathcal{P} 和 \mathcal{T} 分别代表恒等变换、宇称变换和时间反演变换, $\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的, 亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \quad (1.47)$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素, 乘上宇称变换或 (和) 时间反演变换, 就可以到达 Lorentz 群的其它分支。

1.4 Lorentz 矢量

如果一些 $m \times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同, 就可以用这些矩阵来表示这个群, 这些矩阵构成了这个群的一个 m 维线性表示。利用群的线性表示, 可以将对称变换视作矩阵, 将变换作用的态视作列矩阵。

在上一节中, 我们已经用矩阵的形式表示过 Lorentz 变换 $\Lambda^\mu{}_\nu$, 可见, $\Lambda^\mu{}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**, 因为 Lorentz 矢量 x^ν 可以看作是变换 $\Lambda^\mu{}_\nu$ 所作用的态。一般地, 一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \quad (1.48)$$

类似于 (1.36) 式, 逆变矢量 A^μ 对应的协变矢量 $A_\mu = g_{\mu\nu} A^\nu$ 在 Lorentz 变换下满足

$$A_\mu = A_\nu (\Lambda^{-1})^\nu{}_\mu. \quad (1.49)$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \quad (1.50)$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量:

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \quad (1.51)$$

Lorentz 不变量也称为 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负, Lorentz 矢量 A^μ 的自我内积的符号不是确定的, 可以分为三类。

- (1) 若 $A^2 > 0$, 则称 A^μ 为类时矢量。
- (2) 若 $A^2 < 0$, 则称 A^μ 为类空矢量。
- (3) 若 $A^2 = 0$, 则称 A^μ 为类光矢量。

由于 A^2 是 Lorentz 不变量, 不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中, 质点的能量 E 、动量 \mathbf{p} 和 (静止) 质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (1.52)$$

可以用 E 和 \mathbf{p} 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \quad (1.53)$$

称为四维动量, 它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \quad (1.54)$$

这是合理的, 因为质量 m 在狭义相对论中是一个 Lorentz 不变量。 p^μ 在 $m > 0$ 时是类时矢量, 在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla \right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla \right) = g^{\mu\nu} \partial_\nu, \quad (1.55)$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho^\nu = g^{\mu\nu}. \quad (1.56)$$

可见, 这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu_\nu x^\nu$ 时, 时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu_\nu \partial^\nu. \quad (1.57)$$

由上式、(1.56) 式和保度规条件 (1.39) 可得

$$\partial'^\mu x'^\nu = \Lambda^\mu_\rho \partial^\rho (\Lambda^\nu_\sigma x^\sigma) = \Lambda^\mu_\rho \Lambda^\nu_\sigma \partial^\rho x^\sigma = \Lambda^\mu_\rho \Lambda^\nu_\sigma g^{\rho\sigma} = g^{\mu\nu}, \quad (1.58)$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的, 从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明, 时空导数的 Lorentz 变换形式与 Lorentz 矢量相同, 因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \quad (1.59)$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu} \partial'^\mu \partial'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma} \partial^\rho \partial^\sigma = \partial^2. \quad (1.60)$$

可见, ∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \quad (1.61)$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 协变性。

1.5 Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p + q$ 阶的 (p, q) 型 **Lorentz 张量** $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$ 具有 p 个逆变指标和 q 个协变指标, 并满足如下 Lorentz 变换规则:

$$T'^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \Lambda^{\mu_1}_{\rho_1} \cdots \Lambda^{\mu_p}_{\rho_p} T^{\rho_1 \cdots \rho_p}_{\sigma_1 \cdots \sigma_q} (\Lambda^{-1})^{\sigma_1}_{\nu_1} \cdots (\Lambda^{-1})^{\sigma_q}_{\nu_q}. \quad (1.62)$$

这里的逆变指标和协变指标统称为 *Lorentz 指标*。Lorentz 标量是 0 阶 Lorentz 张量, 不具有 Lorentz 指标; Lorentz 矢量是 1 阶 Lorentz 张量, 具有 1 个 Lorentz 指标。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0, 2)$ 型 Lorentz 张量, 不过它在任何惯性系中不变, Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62), 可以验证, 如下表达式都是 Lorentz 标量 (亦即 Lorentz 不变量):

$$g_{\mu\nu} T^{\mu\nu}, \quad T^{\mu\nu} A_\mu B_\nu, \quad T^{\mu\nu} T_{\mu\nu}, \quad g_{\mu\sigma} T^{\mu\nu} T^{\sigma\rho}_{\nu}. \quad (1.63)$$

实际上, 可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p, q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并, 可以得到一个 $(p-1, q-1)$ 型 Lorentz 张量。例如, 由

$$T'^{\mu\nu}_{\mu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}_\gamma (\Lambda^{-1})^\gamma_\mu = \Lambda^\nu_\beta T^{\alpha\beta}_\gamma \delta^\gamma_\alpha = \Lambda^\nu_\beta T^{\alpha\beta}_\alpha \quad (1.64)$$

可知, $T^{\mu\nu}_{\mu}$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita 符号**

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -1, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换,} \\ 0, & \text{其它情况。} \end{cases} \quad (1.65)$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是全反对称的, 即关于任意两个指标反对称, 如 $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\rho\nu\mu\sigma} = -\varepsilon^{\sigma\nu\rho\mu}$ 。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.66)$$

$\varepsilon_{\mu\nu\rho\sigma}$ 也是全反对称的, 如

$$\varepsilon_{\nu\mu\rho\sigma} = g_{\nu\alpha}g_{\mu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta} = g_{\mu\beta}g_{\nu\alpha}g_{\rho\gamma}g_{\sigma\delta}(-\varepsilon^{\beta\alpha\gamma\delta}) = -\varepsilon_{\mu\nu\rho\sigma}. \quad (1.67)$$

根据这些定义,

$$\varepsilon^{0123} = +1, \quad \varepsilon_{0123} = -1. \quad (1.68)$$

从而,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 4!\varepsilon^{0123}\varepsilon_{0123} = -4!. \quad (1.69)$$

利用 Levi-Civita 符号可以把 $\det \Lambda$ 按照行列式定义写成

$$\det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.70)$$

对于固有 Lorentz 变换, $\det \Lambda = +1$, 有

$$\varepsilon^{0123} = \varepsilon^{0123}\det \Lambda = \Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.71)$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质, 可得

$$\varepsilon^{1023} = -\varepsilon^{0123} = -\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = -\Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\alpha\beta\gamma\delta} = \Lambda^1_{\beta}\Lambda^0_{\alpha}\Lambda^2_{\gamma}\Lambda^3_{\delta}\varepsilon^{\beta\alpha\gamma\delta}. \quad (1.72)$$

依此类推, 可以证明

$$\varepsilon^{\mu\nu\rho\sigma} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}\Lambda^{\sigma}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}. \quad (1.73)$$

可见, 在固有 Lorentz 变换下, $\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量, 不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中, ρ 是电荷密度, \mathbf{J} 是电流密度, 它们可以组成一个 Lorentz 矢量 $J^{\mu} = (\rho, \mathbf{J})$, 从而, 电流连续性方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (1.74)$$

可以写成 Lorentz 协变的形式

$$\partial_{\mu}J^{\mu} = 0. \quad (1.75)$$

此外, 电场强度 \mathbf{E} 和磁感应强度 \mathbf{B} 可以用电势 Φ 和矢势 \mathbf{A} 表达为

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (1.76)$$

这样, 方程

$$\nabla \cdot \mathbf{B} = 0 \quad (1.77)$$

是自动满足的。 Φ 和 \mathbf{A} 可以组成一个 Lorentz 矢量 $A^\mu = (\Phi, \mathbf{A})$, 称为四维矢势, 则 (1.76) 式的分量形式为

$$E^i = -\partial_i A^0 - \partial_0 A^i, \quad B^k = \varepsilon^{kij} \partial_i A^j, \quad i, j, k = 1, 2, 3. \quad (1.78)$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk} \equiv \varepsilon^{0ijk}, \quad (1.79)$$

因而 $\varepsilon^{123} = +1$ 。

引入电磁场的场强张量 (field strength tensor)

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (1.80)$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序, 从上述定义可得

$$\begin{aligned} \partial^\rho F^{\mu\nu} &= \partial^\rho (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu \partial^\rho A^\nu - \partial^\mu \partial^\nu A^\rho + \partial^\nu \partial^\mu A^\rho - \partial^\nu \partial^\rho A^\mu \\ &= \partial^\mu F^{\rho\nu} + \partial^\nu F^{\mu\rho} = -\partial^\mu F^{\nu\rho} - \partial^\nu F^{\rho\mu}, \end{aligned} \quad (1.81)$$

即

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \quad (1.82)$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = -E^i, \quad (1.83)$$

可见, F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k} \varepsilon^{12k} = \varepsilon^{123} \varepsilon^{123} = 1$ 和 $\varepsilon^{12k} \varepsilon^{21k} = \varepsilon^{123} \varepsilon^{213} = -1$, 依此类推, 可以归纳出如下求和关系:

$$\varepsilon^{ijk} \varepsilon^{kmn} = \varepsilon^{ijk} \varepsilon^{mnk} = \delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}, \quad (1.84)$$

利用这个关系, 可得

$$\varepsilon^{ijk} B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n = \delta^{im} \delta^{jn} \partial_m A^n - \delta^{in} \delta^{jm} \partial_m A^n = \partial_i A^j - \partial_j A^i, \quad (1.85)$$

从而,

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\varepsilon^{ijk} B^k, \quad (1.86)$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \quad (1.87)$$

Gauss 定律对应的方程

$$\nabla \cdot \mathbf{E} = \rho \quad (1.88)$$

等价于

$$J^0 = \rho = \partial_i E^i = -\partial_i F^{0i} = \partial_i F^{i0} = \partial_i F^{i0} + \partial_0 F^{00} = \partial_\mu F^{\mu 0}, \quad (1.89)$$

而 Ampère 定律对应的方程

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \quad (1.90)$$

等价于

$$J^i = \varepsilon^{ijk} \partial_j B^k - \partial_0 E^i = -\partial_j F^{ij} + \partial_0 F^{0i} = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \quad (1.91)$$

归纳起来, 有

$$\partial_\mu F^{\mu\nu} = J^\nu. \quad (1.92)$$

这个方程完全是用 Lorentz 张量写出来的, 它在不同惯性系中具有相同的形式, 即具有 **Lorentz** 协变性, 因而满足狭义相对性原理。

现在, Maxwell 方程组中还有一个方程没有讨论, 它是 Maxwell-Faraday 方程

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.93)$$

将它写成分量的形式, 得

$$\varepsilon^{kmn} \partial_m E^n = -\varepsilon^{kmn} \partial_m F^{0n} = \varepsilon^{kmn} \partial_m F^{n0} = -\partial_0 B^k, \quad (1.94)$$

从而

$$\partial_0 F^{ij} = -\varepsilon^{ijk} \partial_0 B^k = \varepsilon^{ijk} \varepsilon^{kmn} \partial_m F^{n0} = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial_m F^{n0} = \partial_i F^{j0} - \partial_j F^{i0}, \quad (1.95)$$

即

$$\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = 0. \quad (1.96)$$

这个方程与 Maxwell-Faraday 方程等价, 不过, 它只是前面得到的方程 (1.82) 取特定分量的形式。

利用四维 Levi-Civita 符号, 可以定义电磁场的对偶场强张量 (dual field strength tensor)

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (1.97)$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk} \varepsilon^{1jk} = \varepsilon^{123} \varepsilon^{123} + \varepsilon^{132} \varepsilon^{132} = 2$ 和 $\varepsilon^{1jk} \varepsilon^{2jk} = \varepsilon^{123} \varepsilon^{223} + \varepsilon^{132} \varepsilon^{232} = 0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk} \varepsilon^{ljk} = 2\delta^{il}, \quad (1.98)$$

利用这个关系, 可得

$$\begin{aligned} \tilde{F}^{0i} &= \frac{1}{2} \varepsilon^{0i\rho\sigma} F_{\rho\sigma} = \frac{1}{2} \varepsilon^{0ijk} F_{jk} = \frac{1}{2} \varepsilon^{0ijk} g_{j\mu} g_{k\nu} F^{\mu\nu} = \frac{1}{2} \varepsilon^{0ijk} g_{jm} g_{kn} F^{mn} = -\frac{1}{2} \varepsilon^{ijk} \delta^{jm} \delta^{kn} \varepsilon^{mnl} B^l \\ &= -\frac{1}{2} \varepsilon^{ijk} \varepsilon^{jkl} B^l = -\frac{1}{2} \varepsilon^{ijk} \varepsilon^{ljk} B^l = -\frac{1}{2} 2\delta^{il} B^l = -B^i, \end{aligned} \quad (1.99)$$

故 \tilde{F}^{0i} 对应于磁感应强度。另一方面,

$$\begin{aligned}\tilde{F}^{ij} &= \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(\varepsilon^{ij0k}F_{0k} + \varepsilon^{ijk0}F_{k0}) = \varepsilon^{0ijk}F_{0k} = \varepsilon^{0ijk}g_{0\mu}g_{k\nu}F^{\mu\nu} \\ &= \varepsilon^{ijk}g_{00}g_{kl}F^{0l} = -\varepsilon^{ijk}\delta^{kl}F^{0l} = -\varepsilon^{ijk}F^{0k} = \varepsilon^{ijk}E^k,\end{aligned}\quad (1.100)$$

说明 \tilde{F}^{ij} 对应于电场强度。 $\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}.\quad (1.101)$$

由 $\tilde{F}^{\mu\nu}$ 的定义, 有

$$\begin{aligned}\partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\sigma\mu\rho}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\rho\sigma\mu}\partial_\mu F_{\rho\sigma}) \\ &= -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\mu\rho\sigma}\partial_\rho F_{\sigma\mu} + \varepsilon^{\nu\mu\rho\sigma}\partial_\sigma F_{\mu\rho}) = -\frac{1}{6}\varepsilon^{\nu\mu\rho\sigma}(\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho}),\end{aligned}\quad (1.102)$$

因此, 方程 (1.82) 等价于

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.\quad (1.103)$$

从这些讨论可以看到, 用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的, 而且方程的 Lorentz 协变性非常明确。

1.6 作用量原理

1.6.1 经典力学中的作用量原理

在经典力学中, 质点力学系统可以用拉格朗日量 (Lagrangian) 描述。对于具有 n 个自由度的系统, 可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i , 它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i = dq_i/dt$ 。拉格朗日量是广义坐标和广义速度的函数 $L(q_i, \dot{q}_i)$ 。拉格朗日量的时间积分

$$S = \int_{t_1}^{t_2} dt L[q_i(t), \dot{q}_i(t)]\quad (1.104)$$

称为作用量。

作用量原理指出, 作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零, 则有

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i,\quad (1.105)$$

即时间导数的变分等于变分的时间导数。从而可得

$$\delta S = \int_{t_1}^{t_2} dt \delta L[q_i(t), \dot{q}_i(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right)$$

$$\begin{aligned}
&= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] \\
&= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2},
\end{aligned} \tag{1.106}$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零, 即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$, 则上式最后一行第二项为零, 而 $\delta S = 0$ 等价于

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n. \tag{1.107}$$

这是 **Euler-Lagrange** 方程, 它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \tag{1.108}$$

反解上式表示的 n 个方程, 则可以用 q_i 和 p_i 将 \dot{q}_i 表达出来, 然后用 Legendre 变换定义哈密顿量 (Hamiltonian)

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L, \tag{1.109}$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量, 变分为

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt \delta L = \int_{t_1}^{t_2} dt \delta(p_i \dot{q}_i - H) = \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\
&= \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \frac{d}{dt} \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\
&= \int_{t_1}^{t_2} dt \left[\dot{q}_i \delta p_i + \frac{d}{dt} (p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] \\
&= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] + p_i \delta q_i \Big|_{t_1}^{t_2}.
\end{aligned} \tag{1.110}$$

根据前面的假设, 上式最后一行第二项为零, 于是, $\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n. \tag{1.111}$$

这是 **Hamilton** 正则运动方程, 相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.107)。

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中, 场 $\phi(\mathbf{x}, t)$ 是系统的广义坐标, 每一个空间点 \mathbf{x} 都是一个自由度, 因此场论相当于具有无穷多自由度的质点力学。在局域场论中, 拉格朗日量 $L = \int d^3x \mathcal{L}(x)$, 其中 $\mathcal{L}(x)$ 称为拉格朗日量密度 (下文将它简称为拉氏量)。 \mathcal{L} 是系统中 n 个场 $\phi_a(\mathbf{x}, t)$ ($a = 1, \dots, n$) 及其时空导数 $\partial_\mu \phi_a$ 的函数。现在, 作用量可以表达为

$$S = \int dt L = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{1.112}$$

(1.44) 式告诉我们, 时空体积元 d^4x 是 Lorentz 不变的, 如果拉氏量 \mathcal{L} 也是 Lorentz 不变的, 则作用量 S 就是 Lorentz 不变的, 从而, 由作用量原理得到的运动方程满足狭义相对性原理。因此, 构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 \mathcal{L} , 即要求 \mathcal{L} 是一个 Lorentz 标量。

类似于前面质点力学的处理方式, 假设时空坐标的变分为零, 则对场的时空导数的变分等于场变分的时空导数, 即

$$\delta(\partial_\mu \phi_a) = \partial_\mu(\delta\phi_a). \quad (1.113)$$

于是, 利用分部积分可得

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta(\partial_\mu\phi_a) \right] = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \partial_\mu(\delta\phi_a) \right] \\ &= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} \delta\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right] - \left[\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \delta\phi_a \right\} \\ &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \delta\phi_a + \int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right]. \end{aligned} \quad (1.114)$$

上式最后一行第二项的积分项是关于时空坐标的全散度, 利用 Stokes 定理, 可以将它转化为积分区域边界面 \mathcal{S} 上的积分:

$$\int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a \right] = \int_{\mathcal{S}} dS_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \delta\phi_a, \quad (1.115)$$

其中 dS_μ 是 \mathcal{S} 上的面元。进一步假设在边界面 \mathcal{S} 上 $\delta\phi_a = 0$, 则上式为零。我们通常讨论整个时空区域上的场, 从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零, 是很合理的。这样一来, $\delta S = 0$ 给出

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \quad (1.116)$$

这就是场的 *Euler-Lagrange* 方程, 它给出场的经典运动方程。

引入场的共轭动量密度 (conjugate momentum density)

$$\pi_a(\mathbf{x}, t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}, \quad (1.117)$$

则可以用 Legendre 变换将哈密顿量定义为

$$H \equiv \int d^3x \pi_a \dot{\phi}_a - L \equiv \int d^3x \mathcal{H}, \quad (1.118)$$

其中, 哈密顿量密度

$$\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a \dot{\phi}_a - \mathcal{L}. \quad (1.119)$$

作用量变分为

$$\delta S = \int d^4x \delta\mathcal{L} = \int d^4x \delta(\pi_a \dot{\phi}_a - \mathcal{H})$$

$$\begin{aligned}
&= \int d^4x \left[\dot{\phi}_a \delta \pi_a + \pi_a \delta \dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a - \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \cdot \delta (\nabla \phi_a) \right] \\
&= \int d^4x \left[\dot{\phi}_a \delta \pi_a + \pi_a \frac{d}{dt} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a - \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \cdot \nabla (\delta \phi_a) \right] \\
&= \int d^4x \left\{ \dot{\phi}_a \delta \pi_a + \frac{d}{dt} (\pi_a \delta \phi_a) - \dot{\pi}_a \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \phi_a} \delta \phi_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \delta \pi_a \right. \\
&\quad \left. - \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \delta \phi_a \right] + \left[\nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \right] \delta \phi_a \right\} \\
&= \int d^4x \left\{ \left(\dot{\phi}_a - \frac{\partial \mathcal{H}}{\partial \pi_a} \right) \delta \pi_a - \left[\dot{\pi}_a + \frac{\partial \mathcal{H}}{\partial \phi_a} - \nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \right] \delta \phi_a \right\} \\
&\quad + \int d^4x \frac{d}{dt} (\pi_a \delta \phi_a) - \int d^4x \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)} \delta \phi_a \right]. \tag{1.120}
\end{aligned}$$

与前面一样, 假设在时空区域边界面上 $\delta \phi_a = 0$, 则上式最后两行的两项均为零, 于是, $\delta S = 0$ 给出场的正则运动方程

$$\dot{\phi}_a = \frac{\partial \mathcal{H}}{\partial \pi_a}, \quad \dot{\pi}_a = -\frac{\partial \mathcal{H}}{\partial \phi_a} + \nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \phi_a)}. \tag{1.121}$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述, 则它是一种连续变换, 连续变换对应的对称性称为连续对称性。Noether 定理指出, 如果一个系统具有某种不显含时间的连续对称性, 就必然存在一种对应的守恒定律。Noether 定理首先是在经典物理中给出的, 但实际上它对所有物理行为由作用量原理决定的系统都成立。因此, 可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$S = \int_R d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{1.122}$$

考虑一个连续变换, 使得

$$\phi_a(x) \rightarrow \phi'_a(x'), \tag{1.123}$$

其中已包含了坐标的变换

$$x^\mu \rightarrow x'^\mu, \tag{1.124}$$

它引起的拉氏量变换为

$$\mathcal{L}(x) \rightarrow \mathcal{L}'(x'). \tag{1.125}$$

记这个变换的无穷小变换形式为

$$\phi'_a(x') = \phi_a(x) + \delta \phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta \mathcal{L}, \tag{1.126}$$

如果在此变换下

$$\delta S = \int_{R'} d^4 x' \mathcal{L}'(x') - \int_R d^4 x \mathcal{L}(x) = 0, \quad (1.127)$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4 x' = |\mathcal{J}| d^4 x, \quad \mathcal{J} = \det \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) \simeq \det \left[\delta^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu} \right], \quad (1.128)$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 \mathbf{A} 满足 $\det(\mathbf{A}) \ll 1$ ，则有如下表达式：

$$\det(\mathbf{1} + \mathbf{A}) \simeq 1 + \text{tr}(\mathbf{A}). \quad (1.129)$$

利用上式可以将 Jacobi 行列式 \mathcal{J} 化为

$$\mathcal{J} \simeq 1 + \text{tr} \left[\frac{\partial(\delta x^\mu)}{\partial x^\nu} \right] = 1 + \partial_\mu(\delta x^\mu), \quad (1.130)$$

从而，体积元的无穷小变换形式为

$$d^4 x' \simeq [1 + \partial_\mu(\delta x^\mu)] d^4 x. \quad (1.131)$$

作用量在此无穷小变换下的变分为

$$\begin{aligned} \delta S &= \int_{R'} d^4 x' \mathcal{L}'(x') - \int_R d^4 x \mathcal{L}(x) \\ &= \int_{R'} d^4 x' \mathcal{L}'(x') - \int_R d^4 x \mathcal{L}'(x') + \int_R d^4 x \mathcal{L}'(x') - \int_R d^4 x \mathcal{L}(x) \\ &\simeq \int_R d^4 x [1 + \partial_\mu(\delta x^\mu)] \mathcal{L}'(x') - \int_R d^4 x \mathcal{L}'(x') + \int_R d^4 x \delta \mathcal{L} \\ &\simeq \int_R d^4 x \mathcal{L}'(x') \partial_\mu(\delta x^\mu) + \int_R d^4 x \delta \mathcal{L} \simeq \int_R d^4 x [\delta \mathcal{L} + \mathcal{L}(x) \partial_\mu(\delta x^\mu)] \\ &= \int_R d^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) + \mathcal{L} \partial_\mu(\delta x^\mu) \right]. \end{aligned} \quad (1.132)$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$ ，使得

$$\bar{\delta} \phi_a(x) = \phi'_a(x) - \phi_a(x). \quad (1.133)$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu \phi_a) = \partial_\mu(\bar{\delta} \phi_a), \quad (1.134)$$

δ 算符则不能。 $\delta \phi_a$ 与 $\bar{\delta} \phi_a$ 的关系为

$$\begin{aligned} \delta \phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta} \phi_a \\ &\simeq \bar{\delta} \phi_a + (\partial_\mu \phi'_a) \delta x^\mu \simeq \bar{\delta} \phi_a + (\partial_\mu \phi_a) \delta x^\mu, \end{aligned} \quad (1.135)$$

即

$$\bar{\delta}\phi = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu. \quad (1.136)$$

同理,

$$\delta(\partial_\mu\phi_a) = \bar{\delta}(\partial_\mu\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu = \partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu. \quad (1.137)$$

将 (1.135) 和 (1.137) 式代入 (1.132) 式, 得到

$$\begin{aligned} \delta S &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} [\bar{\delta}\phi_a + (\partial_\mu\phi_a)\delta x^\mu] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} [\partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu] + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} \bar{\delta}\phi_a + \frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a \right) - \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) \bar{\delta}\phi_a \right. \\ &\quad \left. + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu) \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \bar{\delta}\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a \right] \right. \\ &\quad \left. + \left[\frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu) \right] \right\} \\ &= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right] \bar{\delta}\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu \right] \right\}. \end{aligned} \quad (1.138)$$

第二步用到分部积分, 最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu} (\mathcal{L} \delta x^\mu) = \frac{\partial\mathcal{L}}{\partial\phi_a} \frac{\partial\phi_a}{\partial x^\mu} \delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)} \frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu} \delta x^\mu + \mathcal{L} \frac{\partial}{\partial x^\mu} (\delta x^\mu). \quad (1.139)$$

根据 Euler-Lagrange 方程 (1.116), (1.138) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的, $\delta S = 0$ 等价于第二项为零, 即

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu \right] = 0. \quad (1.140)$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \bar{\delta}\phi_a + \mathcal{L} \delta x^\mu, \quad (1.141)$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \quad (1.142)$$

方程 (1.142) 左边对整个三维空间积分, 运用 Stokes 定理, 得

$$\int d^3x \partial_\mu j^\mu = \int d^3x \partial_0 j^0 + \int d^3x \partial_i j^i = \frac{d}{dt} \int d^3x j^0 + \int_S dS_i j^i, \quad (1.143)$$

其中 $i = 1, 2, 3$ 。对于整个三维空间而言, 边界面 S 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失, 从而, 在无穷远处 $j^i \rightarrow 0$, 所以上式最后一项为零。定义守恒荷 (conserved charge)

$$Q \equiv \int d^3x j^0, \quad (1.144)$$

则由 (1.143) 和 (1.142) 式可得

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x j^0 = \int d^3x \partial_\mu j^\mu = 0. \quad (1.145)$$

可见, Q 不随时间变化, 是守恒的。

综上, 在场论中, 如果一个系统具有某种连续对称性, 则存在相应的守恒流 (1.141), 它满足守恒流方程 (1.142), 而守恒荷 (1.144) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2 时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^\mu = x^\mu - \varepsilon^\mu, \quad (1.146)$$

其中 ε^μ 是常数。要求场 ϕ_a 具有时空平移对称性, 则

$$\phi'_a(x') = \phi'_a(x - \varepsilon) = \phi_a(x). \quad (1.147)$$

现在, $\delta x^\mu = -\varepsilon^\mu$, 由 (1.136) 式可得

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) + \varepsilon^\mu\partial_\mu\phi_a = 0 + \varepsilon^\mu\partial_\mu\phi_a = \varepsilon^\rho\partial_\rho\phi_a, \quad (1.148)$$

代入到 Noether 守恒流表达式 (1.141), 得

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^\rho\partial_\rho\phi_a - \mathcal{L}\varepsilon^\mu = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] \varepsilon^\rho. \quad (1.149)$$

从而, $\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^\mu{}_\rho\mathcal{L} \right] = 0, \quad (1.150)$$

各项乘以 $g^{\rho\nu}$, 缩并, 得

$$\partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L} \right] = 0. \quad (1.151)$$

上式方括号部分是场的能动张量 (energy-momentum tensor)

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}, \quad (1.152)$$

它满足

$$\partial_\mu T^{\mu\nu} = 0. \quad (1.153)$$

因此, 对 $T^{0\nu}$ ($\nu = 0, 1, 2, 3$) 作全空间积分, 就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^0\phi_a - \mathcal{L}, \quad (1.154)$$

与 (1.119) 和 (1.117) 式比较, 可以看出 T^{00} 就是哈密顿量密度 \mathcal{H} 。 T^{00} 的全空间积分

$$H = \int d^3x T^{00} = \int d^3x \mathcal{H} \quad (1.155)$$

是场的哈密顿量, 或者说总能量。 $T^{\mu\nu}$ 的 $0i$ 分量

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^i \phi_a = \pi_a \partial^i \phi_a \quad (1.156)$$

是场的动量密度, 它的全空间积分

$$P^i = \int d^3x T^{0i} = \int d^3x \pi_a \partial^i \phi_a \quad (1.157)$$

是场的总动量。根据 (1.55) 式, 上式也可以写成

$$\mathbf{P} = - \int d^3x \pi_a \nabla \phi_a. \quad (1.158)$$

H 和 P^i 都是守恒荷, 可见, 时间平移对称性对应于能量守恒定律, 空间平移对称性对应于动量守恒定律。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad (1.159)$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30), 有

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\mu\nu} (\delta^\mu{}_\alpha + \omega^\mu{}_\alpha) (\delta^\nu{}_\beta + \omega^\nu{}_\beta) \simeq g_{\mu\nu} \delta^\mu{}_\alpha \delta^\nu{}_\beta + g_{\mu\nu} \delta^\mu{}_\alpha \omega^\nu{}_\beta + g_{\mu\nu} \omega^\mu{}_\alpha \delta^\nu{}_\beta \\ &= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{aligned} \quad (1.160)$$

可见,

$$\omega_{\mu\nu} \equiv g_{\mu\rho} \omega^\rho{}_\nu \quad (1.161)$$

关于两个指标反对称:

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \quad (1.162)$$

因此, $\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31), 利用三角函数展开式 $\cos \theta = 1 + \mathcal{O}(\theta^2)$ 和 $\sin \theta = \theta + \mathcal{O}(\theta^3)$, 可得

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho} \omega^\rho{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}. \quad (1.163)$$

对于沿 x 的增速变换 (1.28), 可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1}\beta, \quad (1.164)$$

再利用双曲函数公式 $\tanh \xi = \sinh \xi / \cosh \xi$ 和 $\cosh^2 \xi - \sinh^2 \xi = 1$ 得

$$\begin{aligned} \gamma &= (1 - \beta^2)^{-1/2} = (1 - \tanh^2 \xi)^{-1/2} = \left(\frac{\cosh^2 \xi - \sinh^2 \xi}{\cosh^2 \xi} \right)^{-1/2} = \cosh \xi, \\ \beta\gamma &= \tanh \xi \cosh \xi = \sinh \xi, \end{aligned} \quad (1.165)$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh \xi & -\sinh \xi & & \\ -\sinh \xi & \cosh \xi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (1.166)$$

根据双曲函数展开式 $\cosh \xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh \xi = \xi + \mathcal{O}(\xi^3)$, 有

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho} \omega^\rho{}_\nu = \begin{pmatrix} 0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \quad (1.167)$$

在无穷小 Lorentz 变换 (1.159) 的作用下, 一般地, 场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2} \omega_{\mu\nu} (I^{\mu\nu})_{ab} \right] \phi_b(x) = \phi_a(x) - \frac{i}{2} \omega_{\mu\nu} (I^{\mu\nu})_{ab} \phi_b(x), \quad (1.168)$$

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的生成元 (generator)。由于 $\omega_{\mu\nu}$ 是反对称的, 有

$$\omega_{\mu\nu} (I^{\mu\nu})_{ab} = \omega_{\nu\mu} (I^{\nu\mu})_{ab} = -\omega_{\mu\nu} (I^{\nu\mu})_{ab}, \quad (1.169)$$

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称:

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}. \quad (1.170)$$

现在, $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, 而

$$\bar{\delta} \phi_a = \delta \phi_a - (\partial_\mu \phi_a) \delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu \phi_a) \delta x^\mu = -\frac{i}{2} \omega_{\nu\rho} (I^{\nu\rho})_{ab} \phi_b - (\partial_\nu \phi_a) \omega^\nu{}_\rho x^\rho, \quad (1.171)$$

故 Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \bar{\delta} \phi_a + \mathcal{L} \delta x^\mu = -\frac{i}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (I^{\nu\rho})_{ab} \phi_b - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (\partial_\nu \phi_a) \omega^\nu{}_\rho x^\rho + \mathcal{L} \omega^\mu{}_\rho x^\rho \\ &= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (\partial_\nu \phi_a) - \delta^\mu{}_\nu \mathcal{L} \right] \omega^\nu{}_\rho x^\rho \end{aligned}$$

$$= \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - T^\mu{}_\nu \omega^\nu{}_\rho x^\rho, \quad (1.172)$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho} g_{\rho\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu{}_\nu \mathcal{L} \quad (1.173)$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作：

$$T^\mu{}_\nu \omega^\nu{}_\rho = T^\mu{}_\nu \delta^\nu{}_\sigma \omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha} g_{\alpha\sigma} \omega^\sigma{}_\rho = T^{\mu\alpha} \omega_{\alpha\rho} = T^{\mu\nu} \omega_{\nu\rho}, \quad (1.174)$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned} T^\mu{}_\nu \omega^\nu{}_\rho x^\rho &= T^{\mu\nu} \omega_{\nu\rho} x^\rho = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\nu} \omega_{\rho\nu} x^\rho) = \frac{1}{2} (T^{\mu\nu} \omega_{\nu\rho} x^\rho - T^{\mu\rho} \omega_{\nu\rho} x^\nu) \\ &= \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu). \end{aligned} \quad (1.175)$$

于是, Noether 流 (1.172) 可化为

$$j^\mu = \frac{1}{2} \omega_{\nu\rho} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b - \frac{1}{2} \omega_{\nu\rho} (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) = \frac{1}{2} \mathbb{J}^{\mu\nu\rho} \omega_{\nu\rho} \quad (1.176)$$

其中

$$\mathbb{J}^{\mu\nu\rho} \equiv T^{\mu\rho} x^\nu - T^{\mu\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b. \quad (1.177)$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu \mathbb{J}^{\mu\nu\rho} = 0, \quad (1.178)$$

守恒荷为

$$\mathbb{J}^{\nu\rho} \equiv \int d^3x J^{0\nu\rho} = \int d^3x \left[T^{0\rho} x^\nu - T^{0\nu} x^\rho + \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} (-i I^{\nu\rho})_{ab} \phi_b \right]. \quad (1.179)$$

易见 $\mathbb{J}^{\nu\rho} = -\mathbb{J}^{\rho\nu}$, 因而一共有 6 个独立的守恒荷, 满足 $d\mathbb{J}^{\nu\rho}/dt = 0$ 。

为明确物理含义, 可将 $\mathbb{J}^{\nu\rho}$ 分解成两项:

$$\mathbb{J}^{\nu\rho} = \mathbb{L}^{\nu\rho} + \mathbb{S}^{\nu\rho}. \quad (1.180)$$

第一项为

$$\begin{aligned} \mathbb{L}^{\nu\rho} &\equiv \int d^3x (T^{0\rho} x^\nu - T^{0\nu} x^\rho) \\ &= \int d^3x \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} \partial^\rho \phi_a - g^{0\rho} \mathcal{L} \right) x^\nu - \left(\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} \partial^\nu \phi_a - g^{0\nu} \mathcal{L} \right) x^\rho \right] \\ &= \int d^3x [(\pi_a \partial^\rho \phi_a - g^{0\rho} \mathcal{L}) x^\nu - (\pi_a \partial^\nu \phi_a - g^{0\nu} \mathcal{L}) x^\rho] \\ &= \int d^3x [\pi_a (x^\nu \partial^\rho - x^\rho \partial^\nu) \phi_a + (g^{0\nu} x^\rho - g^{0\rho} x^\nu) \mathcal{L}]. \end{aligned} \quad (1.181)$$

它的纯空间分量 \mathbb{L}^{jk} 中只有 3 个是独立的, 可以等价地定义成

$$\mathbb{L}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathbb{L}^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (x^j \partial^k - x^k \partial^j) \phi_a, \quad (1.182)$$

这是场的轨道角动量。第二项为

$$\mathbb{S}^{\nu\rho} \equiv \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_a)} (-iI^{\nu\rho})_{ab} \phi_b = \int d^3x \pi_a (-iI^{\nu\rho})_{ab} \phi_b, \quad (1.183)$$

同样, 3 个独立的等价纯空间分量是

$$\mathbb{S}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathbb{S}^{jk} = \frac{1}{2} \varepsilon^{ijk} \int d^3x \pi_a (-iI^{jk})_{ab} \phi_b, \quad (1.184)$$

这是场的自旋角动量。因此, $\mathbb{J}^{\nu\rho}$ 的纯空间分量等价于

$$\mathbb{J}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathbb{J}^{jk} = \mathbb{L}^i + \mathbb{S}^i, \quad (1.185)$$

这是场的总角动量。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 $\text{SO}(3)$, 而空间旋转对称性对应于角动量守恒定律。

另一方面, $\mathbb{L}^{\nu\rho}$ 的 $i0$ 分量为

$$\mathbb{L}^{i0} = \int d^3x (T^{00}x^i - T^{0i}x^0) = \int d^3x (x^i \mathcal{H} - x^0 \pi_a \partial^i \phi_a) = \int d^3x x^i \mathcal{H} - tP^i. \quad (1.186)$$

若 $d\mathbb{S}^{i0}/dt = 0$, 则有 $d\mathbb{L}^{i0}/dt = 0$, 从而

$$\mathbb{L}^{i0}(t) = \mathbb{L}^{i0}|_{t=0} = \int d^3x x^i \mathcal{H}(t=0), \quad (1.187)$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下, 能量密度相当于质量密度, 则 \mathbb{L}^{i0} 是 $t=0$ 时刻的质心 (即质量中心, center of mass)。 \mathbb{L}^{i0} 的守恒在经典力学中对应于质心运动守恒定律: 当没有外力存在时, 质心的加速度为零, 质心保持静止或作匀速直线运动。

1.7.4 U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu \phi^*) \partial_\mu \phi - m^2 \phi^* \phi. \quad (1.188)$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \quad (1.189)$$

其中 θ 是不依赖于 x^μ 的连续变换实参数, q 是一个常数。这里不包含坐标的变换。 $e^{iq\theta}$ 是个纯相位因子, 可以看成是一个 1 维幺正 (unitary) 矩阵, 形式为 $e^{iq\theta}$ 的所有变换组成的群称为 U(1) 群。整体 (global) 指的是变换参数不依赖于时空坐标。相应地, ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta} \phi^*(x). \quad (1.190)$$

容易看出, 由 (1.188) 式定义的 \mathcal{L} 在这种变换下不变, 即具有 U(1) 整体对称性。与前面叙述的两种对称性不同, 这里的对称性出现在由场组成的抽象空间中, 与时间和空间相对独立 ($\delta x^\mu = 0$), 因而是一种内部对称性。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \quad (1.191)$$

结合 $\delta x^\mu = 0$, 有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \quad (1.192)$$

于是, Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \bar{\delta}\phi^* = \partial^\mu \phi^* (iq\theta\phi) + \partial^\mu \phi (-iq\theta\phi^*) \\ &= iq\theta [(\partial^\mu \phi^*)\phi - (\partial^\mu \phi)\phi^*] = -q\theta \phi^* i \overleftrightarrow{\partial}^\mu \phi, \end{aligned} \quad (1.193)$$

其中, $\overleftrightarrow{\partial}^\mu$ 符号通过下式定义:

$$\phi^* \overleftrightarrow{\partial}^\mu \phi \equiv \phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi. \quad (1.194)$$

扔掉无穷小参数 $-\theta$, 定义

$$J^\mu \equiv q\phi^* i \overleftrightarrow{\partial}^\mu \phi, \quad (1.195)$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$, 相应的守恒荷为

$$Q = \int d^3x J^0 = q \int d^3x \phi^* i \overleftrightarrow{\partial}^0 \phi. \quad (1.196)$$

在实际情况下, q 是由 ϕ 场描述的粒子所携带的某种荷, 如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此, 一种 U(1) 整体对称性对应于一条荷数守恒定律, 比如, 电磁 U(1) 整体对称性就对应于电荷守恒定律。

第 2 章 标量场

本章讲述标量场的正则量子化 (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1 简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \quad (2.1)$$

其中 m 是质量， ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 看作厄米算符，满足正则对易关系

$$[x, p] = xp - px = i. \quad (2.2)$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \quad (2.3)$$

a 称为湮灭算符 (annihilation operator)， a^\dagger 称为产生算符 (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\ &= \frac{1}{2}(-i[x, p] + i[p, x]) = -i[x, p] = 1. \end{aligned} \quad (2.4)$$

根据 (2.3) 式，可以反过来用 a 和 a^\dagger 表示 x 和 p ：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \quad (2.5)$$

从而，哈密顿量表示成

$$\begin{aligned} H &= -\frac{1}{2m} \frac{m\omega}{2} (a - a^\dagger)^2 + \frac{1}{2} m\omega^2 \frac{1}{2m\omega} (a + a^\dagger)^2 \\ &= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a). \end{aligned} \quad (2.6)$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$, 于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega \left(a^\dagger a + \frac{1}{2} \right) = \omega \left(N + \frac{1}{2} \right), \quad (2.7)$$

其中, $N \equiv a^\dagger a$ 是个厄米算符, 称为粒子数算符。 N 还是个正定算符, 对于任意量子态 $|\psi\rangle$, N 的期待值 (expectation value) 非负:

$$\langle \psi | N | \psi \rangle = \langle \psi | a^\dagger a | \psi \rangle = \langle a\psi | a\psi \rangle \geq 0. \quad (2.8)$$

设 $|n\rangle$ 是 N 的本征态, 归一化为 $\langle n | n \rangle = 1$ 。它满足本征方程

$$N |n\rangle = n |n\rangle. \quad (2.9)$$

由 $n = \langle n | n | n \rangle = \langle n | N | n \rangle \geq 0$ 可知, 本征值 n 是个非负实数。利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B, \quad (2.10)$$

$$[A, BC] = ABC - BAC + BAC - BCA = [A, B]C + B[A, C], \quad (2.11)$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \quad (2.12)$$

从而, 有

$$Na^\dagger |n\rangle = ([N, a^\dagger] + a^\dagger N) |n\rangle = (a^\dagger + a^\dagger n) |n\rangle = (n+1)a^\dagger |n\rangle, \quad (2.13)$$

$$Na |n\rangle = ([N, a] + aN) |n\rangle = (-a + an) |n\rangle = (n-1)a |n\rangle. \quad (2.14)$$

可见, $a^\dagger |n\rangle$ 和 $a |n\rangle$ 都是 N 的本征态, 本征值分别为 $n+1$ 和 $n-1$, 也就是说,

$$a^\dagger |n\rangle = c_1 |n+1\rangle, \quad a |n\rangle = c_2 |n-1\rangle, \quad (2.15)$$

其中 c_1 和 c_2 是两个归一化常数。 a^\dagger 将本征值为 n 的态变成本征值为 $n+1$ 的态, 因而也称为升算符 (raising operator); a 将本征值为 n 的态变成本征值为 $n-1$ 的态, 因而也称为降算符 (lowering operator)。为确定归一化常数的值, 可作如下计算:

$$n+1 = \langle n | (N+1) | n \rangle = \langle n | (a^\dagger a + 1) | n \rangle = \langle n | aa^\dagger | n \rangle = |c_1|^2 \langle n+1 | n+1 \rangle = |c_1|^2, \quad (2.16)$$

$$n = \langle n | N | n \rangle = \langle n | a^\dagger a | n \rangle = |c_2|^2 \langle n-1 | n-1 \rangle = |c_2|^2. \quad (2.17)$$

将 c_1 和 c_2 都取为实数, 则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$, 故

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle. \quad (2.18)$$

从 N 的某个本征态 $|n\rangle$ 出发, 用降算符 a 逐步操作, 可得本征值逐次减小的一系列本征态

$$a |n\rangle, a^2 |n\rangle, a^3 |n\rangle, \dots, \quad (2.19)$$

本征值分别为

$$n - 1, n - 2, n - 3, \dots \quad (2.20)$$

由于 $n \geq 0$, 必定存在一个最小本征值 n_0 , 它的本征态 $|n_0\rangle$ 满足

$$a |n_0\rangle = 0. \quad (2.21)$$

于是, 有

$$N |n_0\rangle = a^\dagger a |n_0\rangle = 0 = 0 |n_0\rangle, \quad (2.22)$$

可见, $n_0 = 0$, 即

$$|n_0\rangle = |0\rangle. \quad (2.23)$$

反过来, 从 $|0\rangle$ 出发, 用升算符 a^\dagger 逐步操作, 可得本征值逐次增加的一系列本征态

$$a^\dagger |0\rangle, (a^\dagger)^2 |0\rangle, (a^\dagger)^3 |0\rangle, \dots, \quad (2.24)$$

本征值分别为

$$1, 2, 3, \dots \quad (2.25)$$

综上, 本征值 n 的取值是非负整数, 是量子化的; 本征态 $|n\rangle$ 可以用 a^\dagger 和 $|0\rangle$ 表示为

$$|n\rangle = c_3 (a^\dagger)^n |0\rangle. \quad (2.26)$$

为确定归一化常数 c_3 , 可作如下运算:

$$\begin{aligned} \langle n|n\rangle &= |c_3|^2 \langle 0| a^n (a^\dagger)^n |0\rangle = |c_3|^2 \langle 1| a^{n-1} (a^\dagger)^{n-1} |1\rangle = 1 \cdot 2 |c_3|^2 \langle 2| a^{n-2} (a^\dagger)^{n-2} |2\rangle = \dots \\ &= (n-1)! |c_3|^2 \langle n-1| a a^\dagger |n-1\rangle = n! |c_3|^2 \langle n|n\rangle, \end{aligned} \quad (2.27)$$

故 $|c_3|^2 = 1/n!$ 。取 c_3 为实数, 可得 $c_3 = 1/\sqrt{n!}$, 于是

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \quad (2.28)$$

从 (2.7) 式容易看出, $|n\rangle$ 也是 H 的本征态:

$$H |n\rangle = \omega \left(N + \frac{1}{2} \right) |n\rangle = \omega \left(n + \frac{1}{2} \right) |n\rangle = E_n |n\rangle, \quad (2.29)$$

相应的能量本征值为

$$E_n = \omega \left(n + \frac{1}{2} \right). \quad (2.30)$$

基态 $|0\rangle$ 的能量本征值不是零, 而是 $E_0 = \omega/2$, 称为零点能 (zero-point energy), 这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态, 将 $n > 0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态, 每个声子具有一份能量 ω 。这样一来, n 表示声子的数目, 故粒子数算符 N 描述的是声子数。 a^\dagger 的作用是产生一个声子, 从而增加一份能量; a 的作用是湮灭一个声子, 从而减少一份能量。这是将 a^\dagger 和 a 称为产生算符和湮灭算符的原因。

2.2 量子场论中的正则对易关系

在量子力学中, 当系统的哈密顿量 H 不含时间时, Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法, 它们之间可以通过含时的么正变换联系起来。

在 Schrödinger 绘景中, 态矢 $|\Psi(t)\rangle^S$ 代表随时间演化的物理态, 而算符 O^S 不依赖于时间。 $|\Psi(t)\rangle^S$ 与 $t = 0$ 时刻态矢 $|\Psi(0)\rangle^S$ 通过么正变换 e^{-iHt} 联系起来:

$$|\Psi(t)\rangle^S = e^{-iHt}|\Psi(0)\rangle^S. \quad (2.31)$$

由于 H 不含时, 有

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle^S = i\frac{\partial e^{-iHt}}{\partial t}|\Psi(0)\rangle^S = He^{-iHt}|\Psi(0)\rangle^S = H|\Psi(t)\rangle^S, \quad (2.32)$$

这就是 Schrödinger 方程。可见, (2.31) 式确实是 Schrödinger 方程的解。

在 Heisenberg 绘景中, 态矢 $|\Psi\rangle^H$ 定义为

$$|\Psi\rangle^H = e^{iHt}|\Psi(t)\rangle^S = |\Psi(0)\rangle^S, \quad (2.33)$$

它不随时间演化:

$$i\frac{\partial}{\partial t}|\Psi\rangle^H = 0. \quad (2.34)$$

而算符 $O^H(t)$ 依赖于时间, 通过一个含时的相似变换与 O^S 联系起来,

$$O^H(t) = e^{iHt}O^S e^{-iHt}. \quad (2.35)$$

从而, 有

$${}^H\langle\Psi|O^H(t)|\Psi\rangle^H = {}^H\langle\Psi|e^{iHt}O^S e^{-iHt}|\Psi\rangle^H = {}^S\langle\Psi(t)|O^S|\Psi(t)\rangle^S, \quad (2.36)$$

可见, 在两种绘景中, 力学量在态上的平均值相同。因此, 两种绘景描述相同的物理。

上一节的量子化可以认为是在 Schrödinger 绘景中实现的, 因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x^S, p^S] = i$, 它在 Heisenberg 绘景中的形式为

$$\begin{aligned} [x^H(t), p^H(t)] &= [e^{iHt}x^S e^{-iHt}, e^{iHt}p^S e^{-iHt}] = e^{iHt}x^S e^{-iHt}e^{iHt}p^S e^{-iHt} - e^{iHt}p^S e^{-iHt}e^{iHt}x^S e^{-iHt} \\ &= e^{iHt}x^S p^S e^{-iHt} - e^{iHt}p^S x^S e^{-iHt} = e^{iHt}[x^S, p^S]e^{-iHt} = e^{iHt}ie^{-iHt} = i. \end{aligned} \quad (2.37)$$

可见, 正则对易关系的形式不依赖于绘景。(2.37) 式是在同一时刻 t 成立的, 称为等时 (equal time) 对易关系。

将讨论推广到自由度为 n 的系统, 记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符, $p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响, 这些算符需要满足如下等时对易关系:

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [q_i(t), q_j(t)] = 0, \quad [p_i(t), p_j(t)] = 0. \quad (2.38)$$

1.1 节提到, 在量子场论中, 为了平等地处理时间和空间, 空间坐标 \mathbf{x} 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x}, t)$ 的参数。由于这里量子场作为算符是依赖于时间的, 使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行, 省略绘景的标志性上标 H。

场论讨论的是无穷多自由度的系统, 每一个空间点 \mathbf{x} 上的 $\phi(\mathbf{x}, t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度, 我们可以先将空间离散化, 划分成 n 个小体积元 V_i , 然后再取 $V_i \rightarrow 0$ 的极限来得到 $n \rightarrow \infty$ 的结果。在体积元 V_i 中, 定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i} \int_{V_i} d^3x \phi(\mathbf{x}, t), \quad (2.39)$$

它是场 $\phi(\mathbf{x}, t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi, \partial_\mu \phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L}(\phi, \partial_\mu \phi), \quad (2.40)$$

当体积元取得足够小时, 它就成为 ϕ_i 和 $\partial_0 \phi_i$ 的函数 $\mathcal{L}_i(\phi_i, \partial_0 \phi_i)$ 。拉格朗日量可表达为

$$L = \int d^3x \mathcal{L} = \sum_i \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \frac{1}{V_i} \int_{V_i} d^3x \mathcal{L} = \sum_i V_i \mathcal{L}_i(\phi_i, \partial_0 \phi_i). \quad (2.41)$$

于是, 由 (1.108) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0 \phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0 \phi_i(t)]} = \sum_j V_j \delta_{ji} \frac{\partial \mathcal{L}_i}{\partial[\partial_0 \phi_i(t)]} = V_i \pi_i(t), \quad (2.42)$$

其中,

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0 \phi_i(t)]}. \quad (2.43)$$

现在, 等时对易关系变成

$$[\phi_i(t), \Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t), \phi_j(t)] = 0, \quad [\Pi_i(t), \Pi_j(t)] = 0. \quad (2.44)$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t), \pi_j(t)] = 0. \quad (2.45)$$

对于任意连续函数 $f(x)$, Dirac δ 函数 $\delta(x)$ 使下式成立:

$$f(x) = \int dy f(y) \delta(x - y). \quad (2.46)$$

函数 $\delta(x)$ 只在 $x = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即

$$\delta(x) = \delta(-x), \quad (2.47)$$

而且满足

$$\int dx \delta(x) = 1, \quad (2.48)$$

$$f(x)\delta(x-y) = f(y)\delta(x-y), \quad (2.49)$$

$$x\delta(x) = 0. \quad (2.50)$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \quad (2.51)$$

则对于任意连续函数 $f(\mathbf{x})$, 下式成立:

$$f(\mathbf{x}) = \int d^3y f(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.52)$$

类似地, 函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x} = 0$ 处非零, 是关于 \mathbf{x} 的偶函数, 即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$, 而且满足 $\int d^3x \delta^{(3)}(\mathbf{x}) = 1$ 。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值, 则它会满足

$$f_i = \sum_j f_j \delta_{ij} = \sum_j V_j f_j \frac{\delta_{ij}}{V_j}. \quad (2.53)$$

(2.52) 式是 (2.53) 式在 $V_i \rightarrow 0$ 时的极限。可见, 在 $V_i \rightarrow 0$ 极限下,

$$\frac{\delta_{ij}}{V_j} \rightarrow \delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.54)$$

另一方面, 在此极限下, $\phi_i(t) \rightarrow \phi(\mathbf{x}, t)$, 而 $\pi_i(t)$ 变成由 (1.117) 式定义的共轭动量密度:

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial [\partial_0 \phi_i(t)]} \rightarrow \frac{\partial \mathcal{L}}{\partial [\partial_0 \phi(\mathbf{x}, t)]} = \pi(\mathbf{x}, t). \quad (2.55)$$

因此, 等时对易关系化为

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.56)$$

推广到包含若干个场 ϕ_a 的系统, 假设不同的场不会相互影响, 则有

$$[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0, \quad [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \quad (2.57)$$

这就是量子场论中的正则对易关系。此时, $\phi_a(\mathbf{x}, t)$ 和 $\pi_a(\mathbf{x}, t)$ 都是算符。

2.3 实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量, 就称它为标量场。在固有保时向 Lorentz 变换下, 若时空坐标的变换为 $x' = \Lambda x$, 则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \quad (2.58)$$

在本节中, 我们讨论实标量场 $\phi(x)$, 它满足自共轭 (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \quad (2.59)$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的自由实标量场，相应的 Lorentz 不变拉氏量可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m^2\phi^2. \quad (2.60)$$

注意到

$$\frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu \phi)\partial_\nu \phi = \frac{1}{2}[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2], \quad (2.61)$$

可得

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} = -\partial_i \phi = \partial^i \phi, \quad (2.62)$$

归纳起来，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi. \quad (2.63)$$

因此，Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi, \quad (2.64)$$

也就是说， $\phi(x)$ 满足 Klein-Gordon 方程

$$(\partial^2 + m^2)\phi(x) = 0. \quad (2.65)$$

2.3.1 平面波展开

设 $\phi(x)$ 满足的 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik \cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \quad (2.66)$$

则有

$$\partial^2 \varphi = \partial^\mu \partial_\mu \varphi = \partial^\mu (-ik_\mu \varphi) = -ik_\mu \partial^\mu \varphi = (-i)^2 k_\mu k^\mu \varphi = -k^2 \varphi, \quad (2.67)$$

从而，

$$0 = (\partial^2 + m^2)\varphi = -(k^2 - m^2)\varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2]\varphi. \quad (2.68)$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$ ，即 $k^0 = \pm E_{\mathbf{k}}$ ，其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$ 。因此，有两种平面波解。

(1) $k^0 = E_{\mathbf{k}}$ 对应于正能解

$$\varphi_{\mathbf{k}}^{(+)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[-i(E_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x})]. \quad (2.69)$$

(2) $k^0 = -E_{\mathbf{k}}$ 对应于负能解

$$\varphi_{\mathbf{k}}^{(-)}(x) = \exp[-i(k^0 x^0 - \mathbf{k} \cdot \mathbf{x})] = \exp[i(E_{\mathbf{k}} t + \mathbf{k} \cdot \mathbf{x})]. \quad (2.70)$$

从而, 场算符 $\phi(\mathbf{x}, t)$ 的一般解可以写成如下形式:

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} \varphi_{\mathbf{k}}^{(+)}(x) + \tilde{a}_{\mathbf{k}} \varphi_{\mathbf{k}}^{(-)}(x) \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}} e^{i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})} \right],\end{aligned}\quad (2.71)$$

其中 $a_{\mathbf{k}}$ 和 $\tilde{a}_{\mathbf{k}}$ 是两个只依赖于 \mathbf{k} 的算符。这是一种 Fourier 变换, 把 $\phi(\mathbf{x}, t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭, 得

$$\begin{aligned}\phi^\dagger(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})} \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})} + \tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} \right].\end{aligned}\quad (2.72)$$

第二步利用了如下性质: 对整个三维动量空间进行积分时, 将积分项中的 \mathbf{k} 换成 $-\mathbf{k}$ 不会改变积分的结果。于是, 自共轭条件 $\phi^\dagger(\mathbf{x}, t) = \phi(\mathbf{x}, t)$ 意味着

$$\tilde{a}_{\mathbf{k}} = a_{-\mathbf{k}}^\dagger. \quad (2.73)$$

(注意, 由上式可以推出 $\tilde{a}_{\mathbf{k}}^\dagger = a_{-\mathbf{k}}$ 和 $\tilde{a}_{-\mathbf{k}}^\dagger = a_{\mathbf{k}}$ 。) 因而有

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + a_{-\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})} \right] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} + a_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})} \right].\end{aligned}\quad (2.74)$$

替换一下动量记号, 可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right), \quad (2.75)$$

其中, p^0 是正的, 满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}, \quad (2.76)$$

而 $a_{\mathbf{p}}$ 是湮灭算符, $a_{\mathbf{p}}^\dagger$ 是产生算符。 $\phi(\mathbf{x}, t)$ 对应的共轭动量密度算符为

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right). \quad (2.77)$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (2.78)$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p}), \quad (2.79)$$

可得

$$\begin{aligned} \int d^3x e^{iq\cdot x} \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q)\cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0t}), \end{aligned} \quad (2.80)$$

以及

$$\begin{aligned} \int d^3x e^{iq\cdot x} \partial_0 \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q)\cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}] \\ &= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0t}). \end{aligned} \quad (2.81)$$

从而, 有

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x e^{iq\cdot x} \partial_0 \phi - iq_0 \int d^3x e^{iq\cdot x} \phi = \int d^3x e^{iq\cdot x} (\partial_0 \phi - iq_0 \phi), \quad (2.82)$$

亦即

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip\cdot x} [\partial_0 \phi(x) - ip_0 \phi(x)]. \quad (2.83)$$

上式取厄米共轭, 并使用自共轭条件 $\phi^\dagger = \phi$, 得

$$a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip\cdot x} [\partial_0 \phi(x) + ip_0 \phi(x)]. \quad (2.84)$$

利用上面两个表达式和等时对易关系 (2.78), 可得

$$\begin{aligned} &[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip\cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq\cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p\cdot x - q\cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} [-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{2.85}$$

在以上计算过程中, $x^0 = y^0 = t$ 。根据 δ 函数的性质 (2.49), 有

$$\frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \frac{E_{\mathbf{p}}+E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}}}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{2.86}$$

故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{2.87}$$

类似地,

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip\cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq\cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p\cdot x + q\cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} (-iq_0[\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p}+\mathbf{q}).
\end{aligned} \tag{2.88}$$

根据 δ 函数的性质 (2.49), 有

$$\frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) = \frac{E_{\mathbf{p}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{p}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0, \tag{2.89}$$

故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \tag{2.90}$$

此外,

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}} a_{\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \tag{2.91}$$

综上, 产生湮灭算符满足如下对易关系:

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \tag{2.92}$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.119), 实标量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} = (\partial_0 \phi)^2 - \frac{1}{2} (\partial^\mu \phi) \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 = \frac{1}{2} [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2]. \quad (2.93)$$

对全空间积分以得到哈密顿量:

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x [(\partial_0 \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (-iq_0 a_{\mathbf{q}} e^{-iq \cdot x} + iq_0 a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + (i\mathbf{p} a_{\mathbf{p}} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. + m^2 (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \frac{1}{2} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left\{ (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} e^{i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}}] \right. \\ &\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} e^{i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) (p_0 q_0 + \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0 - q_0)t}] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + m^2) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0 + q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0 + q_0)t}] \right\} \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left[(E_p^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \right. \\ &\quad \left. + (-E_p^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_p t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_p t}) \right]. \quad (2.94) \end{aligned}$$

由 (2.76) 式可得 $-E_p^2 + |\mathbf{p}|^2 + m^2 = 0$, 故上式最后两行方括号中第二项没有贡献。从而,

$$\begin{aligned} H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} (E_p^2 + |\mathbf{p}|^2 + m^2) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} 2E_p^2 (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p [2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p})] \\ &= \int \frac{d^3p}{(2\pi)^3} E_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2}, \quad (2.95) \end{aligned}$$

其中第四步用到对易关系 (2.92)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。 $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 \mathbf{p} 的模式对应的粒子数密度算符 (动量空间中的密度), 相应的能量是 E_p 。在 (2.95) 式最后一行中, 第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.79) 式可得

$$(2\pi)^3 \delta^{(3)}(\mathbf{0}) = \int d^3x = V, \quad (2.96)$$

其中 V 是进行积分的空间体积, 对于全空间而言是无穷大的。因此, (2.95) 式最后一行的第二项是一个无穷大 c 数, 是真空的零点能, 是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明, 一维简谐振子的零点能为 $E_0 = \omega/2$ 。这是自由度为 1 时的结果, 推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象, 这个零点能通常并不重要, 因为实验上只能测量两个能量之差。经过正则量子化之后, 实标量场的哈密顿量 H 是正定的, 不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \quad (2.97)$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3q E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \quad (2.98)$$

设 $|E\rangle$ 是 H 的本征态, 本征值为 E , 则

$$H |E\rangle = E |E\rangle. \quad (2.99)$$

从而, 有

$$H a_{\mathbf{p}}^\dagger |E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger |E\rangle. \quad (2.100)$$

$$H a_{\mathbf{p}} |E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p}} |E\rangle. \quad (2.101)$$

可见, 当 $a_{\mathbf{p}}^\dagger |E\rangle \neq 0$ 时, 产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{p}}$; 当 $a_{\mathbf{p}} |E\rangle \neq 0$ 时, 湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{p}}$ 。

根据 (1.158) 式, 实标量场的总动量是

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi \nabla \phi = - \int d^3x (\partial_0 \phi) \nabla \phi \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (-ip_0 a_{\mathbf{p}} e^{-ip \cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q}} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q}}^\dagger e^{iq \cdot x}) \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[-p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q) \cdot x} - p_0 \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right. \\ &\quad \left. + p_0 \mathbf{q} a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + p_0 \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q) \cdot x} \right] \\ &= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ -p_0 \mathbf{q} [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} e^{-i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t} e^{i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}}] \right. \\ &\quad \left. + p_0 \mathbf{q} [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} e^{-i(\mathbf{p}+\mathbf{q}) \cdot \mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t} e^{i(\mathbf{p}+\mathbf{q}) \cdot \mathbf{x}}] \right\} \\ &= - \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ -p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} - \mathbf{q}) [a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t}] \right. \\ &\quad \left. + p_0 \mathbf{q} \delta^{(3)}(\mathbf{p} + \mathbf{q}) [a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}] \right\} \\ &= - \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (-E_{\mathbf{p}} \mathbf{p}) (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}) \end{aligned}$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t} \right). \quad (2.102)$$

先作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换, 再利用对易关系 (2.92), 可得

$$\begin{aligned} \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t} \right) &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} (-\mathbf{p}) \left(a_{-\mathbf{p}} a_{\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t} \right) \\ &= -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t} \right). \end{aligned} \quad (2.103)$$

可见, (2.102) 式最后一行圆括号中最后两项没有贡献。从而,

$$\begin{aligned} \mathbf{P} &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} [2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{0})] \\ &= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \delta^{(3)}(\mathbf{0}) \int d^3p \mathbf{p}. \end{aligned} \quad (2.104)$$

由于 $\int d^3p \mathbf{p} = \int d^3p (-\mathbf{p}) = -\int d^3p \mathbf{p}$, 上式最后一行第二项没有贡献。于是,

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \quad (2.105)$$

即总动量是所有动量模式所有粒子贡献的动量之和。

\mathbf{P} 与产生湮灭算符的对易子为

$$[\mathbf{P}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q \mathbf{q} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q} - \mathbf{p}) = \mathbf{p} a_{\mathbf{p}}^\dagger, \quad (2.106)$$

$$[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} \mathbf{q} [a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = - \int d^3q \mathbf{q} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -\mathbf{p} a_{\mathbf{p}}. \quad (2.107)$$

2.3.4 粒子态

真空态 $|0\rangle$ 是能量最低的态, 对于任意动量 \mathbf{p} 对应的湮灭算符 $a_{\mathbf{p}}$, 满足

$$a_{\mathbf{p}} |0\rangle = 0, \quad (2.108)$$

归一化为

$$\langle 0|0\rangle = 1. \quad (2.109)$$

由哈密顿量的表达式 (2.95) 可得

$$H |0\rangle = E_{\text{vac}} |0\rangle, \quad E_{\text{vac}} = \delta^{(3)}(\mathbf{0}) \int d^3p \frac{E_{\mathbf{p}}}{2}, \quad (2.110)$$

可见, 这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外, 由 (2.105) 式可知, $|0\rangle$ 的总动量本征值是零:

$$\mathbf{P} |0\rangle = \mathbf{0} |0\rangle, \quad (2.111)$$

即真空态不具有动量。

接着, 定义动量为 \mathbf{p} 的单粒子态为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger} |0\rangle. \quad (2.112)$$

从而, 利用 (2.97) 和 (2.106) 式可得

$$H |\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} H + E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) a_{\mathbf{p}}^{\dagger} |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}\rangle, \quad (2.113)$$

$$\mathbf{P} |\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{P} a_{\mathbf{p}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p}}^{\dagger} \mathbf{P} + \mathbf{p} a_{\mathbf{p}}^{\dagger}) |0\rangle = \sqrt{2E_{\mathbf{p}}} \mathbf{p} a_{\mathbf{p}}^{\dagger} |0\rangle = \mathbf{p} |\mathbf{p}\rangle. \quad (2.114)$$

可以看出, 相比于真空态 $|0\rangle$, 单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$, 也多了一份动量 \mathbf{p} 。因此, $|\mathbf{p}\rangle$ 描述的是一个动量为 \mathbf{p} 的粒子, 这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 满足狭义相对论中的能量-动量关系 (1.52), 而拉氏量 (2.60) 中的参数 m 就是粒子的质量。可以看出, 产生算符 $a_{\mathbf{p}}^{\dagger}$ 的作用是产生一个动量为 \mathbf{p} 的粒子。

此外, 可作如下计算:

$$a_{\mathbf{p}} |\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} |0\rangle = \sqrt{2E_{\mathbf{q}}} [a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] |0\rangle = \sqrt{2E_{\mathbf{p}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle. \quad (2.115)$$

如果 $\mathbf{p} \neq \mathbf{q}$, 则上式为零; 如果 $\mathbf{p} = \mathbf{q}$, 则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$ 。可见, 湮灭算符 $a_{\mathbf{p}}$ 的作用是减少一个动量为 \mathbf{p} 的粒子。

单粒子态的内积关系为

$$\begin{aligned} \langle \mathbf{q} | \mathbf{p} \rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | a_{\mathbf{q}} a_{\mathbf{p}}^{\dagger} | 0 \rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | [a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] | 0 \rangle \\ &= 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (2.116)$$

上式是 Lorentz 不变的, 这是 (2.112) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明 若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i , 则如下等式成立:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \quad (2.117)$$

引入阶跃函数 (step function)

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (2.118)$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^{μ} 满足质壳条件 $p^2 - m^2 = 0$ 且能量为正 ($p^0 > 0$) 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned} \int d^4p \delta(p^2 - m^2) \theta(p^0) F(p) &= \int d^3p dp^0 \delta((p^0)^2 - |\mathbf{p}|^2 - m^2) \theta(p^0) F(p^0, \mathbf{p}) \\ &= \int d^3p \frac{1}{2\sqrt{|\mathbf{p}|^2 + m^2}} F(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}) = \int \frac{d^3p}{2E_{\mathbf{p}}} F(\sqrt{|\mathbf{p}|^2 + m^2}, \mathbf{p}). \end{aligned} \quad (2.119)$$

这里第二步用到 (2.117) 式。可见,

$$\frac{d^3p}{2E_{\mathbf{p}}} \quad (2.120)$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$, 按照 δ 函数定义, 有

$$g(\mathbf{q}) = \int d^3p \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}) = \int \frac{d^3p}{2E_{\mathbf{p}}} 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) g(\mathbf{p}). \quad (2.121)$$

由于上式最左边和最右边都是 Lorentz 不变的,

$$2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \quad (2.122)$$

必定是 Lorentz 不变的。证毕。

进一步, 可以定义动量分别为 $\mathbf{p}_1, \dots, \mathbf{p}_n$ 的 n 个粒子对应的多粒子态为

$$|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \equiv \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle. \quad (2.123)$$

H 对它的作用给出

$$\begin{aligned} H |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} H a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} (a_{\mathbf{p}_1}^\dagger H + E_{\mathbf{p}_1} a_{\mathbf{p}_1}^\dagger) \cdots a_{\mathbf{p}_n}^\dagger |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger H a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + E_{\mathbf{p}_1} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger H \cdots a_{\mathbf{p}_n}^\dagger |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= \cdots = \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger \cdots a_{\mathbf{p}_n}^\dagger H |0\rangle + (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle \\ &= (E_{\text{vac}} + E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \cdots + E_{\mathbf{p}_n}) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle, \end{aligned} \quad (2.124)$$

同理, \mathbf{P} 对它的作用给出

$$\mathbf{P} |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle = (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n) |\mathbf{p}_1, \dots, \mathbf{p}_n\rangle. \quad (2.125)$$

也就是说, 多粒子态 $|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.92) 可得

$$\begin{aligned} |\mathbf{p}_1, \dots, \mathbf{p}_i, \dots, \mathbf{p}_j, \dots, \mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\ &= \sqrt{2E_{\mathbf{p}_1}} \cdots \sqrt{2E_{\mathbf{p}_n}} a_{\mathbf{p}_1}^\dagger \cdots a_{\mathbf{p}_j}^\dagger \cdots a_{\mathbf{p}_i}^\dagger \cdots a_{\mathbf{p}_n}^\dagger |0\rangle \\ &= |\mathbf{p}_1, \dots, \mathbf{p}_j, \dots, \mathbf{p}_i, \dots, \mathbf{p}_n\rangle. \end{aligned} \quad (2.126)$$

可以看出, 对调多粒子态中的任意两个粒子, 得到的态相同, 即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是玻色子 (boson), 服从 Bose-Einstein 统计。得到这个结论的关键在于两个产生算符相互对易。

双粒子态的内积关系为

$$\begin{aligned}
 \langle \mathbf{q}_1, \mathbf{q}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{q}_1} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger | 0 \rangle \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle + \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger a_{\mathbf{q}_1} a_{\mathbf{p}_2}^\dagger | 0 \rangle \right] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_2}^\dagger | 0 \rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2} a_{\mathbf{p}_1}^\dagger | 0 \rangle \right] \\
 &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^6 \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) + (2\pi)^6 \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \right] \\
 &= 4E_{\mathbf{p}_1} E_{\mathbf{p}_2} (2\pi)^6 \left[\delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) + \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \right]. \quad (2.127)
 \end{aligned}$$

此外，还可以定义动量均为 \mathbf{p} 的 n 个粒子对应的多粒子态为

$$|n_{\mathbf{p}}\rangle \equiv (2E_{\mathbf{p}})^{n_{\mathbf{p}}/2} (a_{\mathbf{p}}^\dagger)^{n_{\mathbf{p}}} |0\rangle, \quad (2.128)$$

则粒子数密度算符

$$N_{\mathbf{p}} \equiv a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \quad (2.129)$$

对它的作用为

$$\begin{aligned}
 N_{\mathbf{p}} |n_{\mathbf{q}}\rangle &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger [a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})] (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^2 a_{\mathbf{p}} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-2} |0\rangle + 2(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= \cdots = (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} a_{\mathbf{p}} |0\rangle + n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle. \quad (2.130)
 \end{aligned}$$

在动量空间对粒子数密度算符进行积分，得到的是粒子数算符

$$N \equiv \int \frac{d^3 p}{(2\pi)^3} N_{\mathbf{p}} = \int \frac{d^3 p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (2.131)$$

由 (2.130) 式，可得

$$\begin{aligned}
 N |n_{\mathbf{q}}\rangle &= \int \frac{d^3 p}{(2\pi)^3} N_{\mathbf{p}} |n_{\mathbf{q}}\rangle = \int \frac{d^3 p}{(2\pi)^3} n_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}-1} |0\rangle \\
 &= n_{\mathbf{q}} (2E_{\mathbf{q}})^{n_{\mathbf{q}}/2} (a_{\mathbf{q}}^\dagger)^{n_{\mathbf{q}}} |0\rangle = n_{\mathbf{q}} |n_{\mathbf{q}}\rangle. \quad (2.132)
 \end{aligned}$$

因此， $|n_{\mathbf{q}}\rangle$ 是 N 的本征态，本征值为粒子数 $n_{\mathbf{q}}$ 。

更一般地，可以定义多粒子态

$$|n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \quad (2.133)$$

来描述动量为 $\mathbf{p}_1, \cdots, \mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}$ 个的情况。此时，有

$$N |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] a_{\mathbf{p}}^\dagger a_{\mathbf{p}} (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right. \\
&\quad \left. + n_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}-1} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} a_{\mathbf{p}} (a_{\mathbf{p}_2}^\dagger)^{n_{\mathbf{p}_2}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} |0\rangle \right] + n_{\mathbf{p}_1} |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \\
&= \cdots = \int \frac{d^3p}{(2\pi)^3} \left[\prod_{i=1}^m (2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2} \right] \left[a_{\mathbf{p}}^\dagger (a_{\mathbf{p}_1}^\dagger)^{n_{\mathbf{p}_1}} \cdots (a_{\mathbf{p}_i}^\dagger)^{n_{\mathbf{p}_i}} a_{\mathbf{p}} |0\rangle \right] \\
&\quad + (n_{\mathbf{p}_1} + \cdots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle \\
&= (n_{\mathbf{p}_1} + \cdots + n_{\mathbf{p}_m}) |n_{\mathbf{p}_1}, \cdots, n_{\mathbf{p}_m}\rangle.
\end{aligned} \tag{2.134}$$

可见, N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中, 我们讨论复标量场 $\phi(x)$, 它不满足自共轭条件 (2.59), 即

$$\phi^\dagger(x) \neq \phi(x). \tag{2.135}$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.188) 式的形式。不过, 由于 $\phi(x)$ 是量子场算符, 需要把那里的复共轭记号 $*$ 改成厄米共轭记号 \dagger , 故 Lorentz 不变拉氏量为

$$\mathcal{L} = (\partial^\mu \phi^\dagger) \partial_\mu \phi - m^2 \phi^\dagger \phi. \tag{2.136}$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量, 注意到

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi^\dagger} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^\dagger, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^\dagger, \tag{2.137}$$

则 Euler-Lagrange 方程 (1.116) 给出

$$(\partial^2 + m^2)\phi(x) = 0, \quad (\partial^2 + m^2)\phi^\dagger(x) = 0. \tag{2.138}$$

也就是说, $\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程。

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{2.139}$$

从而, 拉氏量 (2.136) 化为

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[\partial^\mu(\phi_1 - i\phi_2)]\partial_\mu(\phi_1 + i\phi_2) - \frac{1}{2}m^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \\ &= \frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2 - \frac{1}{2}m^2\phi_2^2.\end{aligned}\quad (2.140)$$

与 (2.60) 式比较可知, 复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量。

2.4.1 平面波展开

对于复标量场, 我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开, 但不能够应用自共轭条件。因此, 场算符 $\phi(\mathbf{x}, t)$ 的一般解也具有 (2.71) 式的形式:

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \tilde{a}_{\mathbf{k}} e^{i(E_{\mathbf{k}}t + \mathbf{k}\cdot\mathbf{x})}] \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + \tilde{a}_{-\mathbf{k}} e^{i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}].\end{aligned}\quad (2.141)$$

由于不满足自共轭条件 (2.59), 算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_{\mathbf{k}}$ 没有什么关系, 改记为

$$b_{\mathbf{k}}^\dagger = \tilde{a}_{-\mathbf{k}}, \quad (2.142)$$

则展开式变成

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} [a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + b_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}]. \quad (2.143)$$

替换一下动量记号, 可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip\cdot x} + b_{\mathbf{p}}^\dagger e^{ip\cdot x}), \quad (2.144)$$

其中, p^0 应该满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \quad (2.145)$$

取厄米共轭, 就得到 $\phi^\dagger(\mathbf{x}, t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}). \quad (2.146)$$

现在, $a_{\mathbf{p}}$ 和 $b_{\mathbf{p}}$ 是两个相互独立的湮灭算符, 而 $a_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (b_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}), \quad (2.147)$$

$\phi^\dagger(\mathbf{x}, t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^\dagger)} = \partial_0\phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) (a_{\mathbf{p}} e^{-ip\cdot x} - b_{\mathbf{p}}^\dagger e^{ip\cdot x}). \quad (2.148)$$

根据 (2.57) 式, 等时对易关系为

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0, \\ [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0, \\ [\phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] &= [\phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t)] = [\phi(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.149)$$

2.4.2 产生湮灭算符的对易关系

由

$$\begin{aligned} \int d^3x e^{iq \cdot x} \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} + b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} + b_{-\mathbf{q}}^\dagger e^{2iq^0t}) \end{aligned} \quad (2.150)$$

和

$$\begin{aligned} \int d^3x e^{iq \cdot x} \partial_0 \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [a_{\mathbf{p}} e^{-i(p-q) \cdot x} - b_{\mathbf{p}}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (a_{\mathbf{q}} - b_{-\mathbf{q}}^\dagger e^{2iq^0t}), \end{aligned} \quad (2.151)$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x e^{iq \cdot x} \partial_0 \phi - iq_0 \int d^3x e^{iq \cdot x} \phi = \int d^3x e^{iq \cdot x} (\partial_0 \phi - iq_0 \phi). \quad (2.152)$$

于是,

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} (\partial_0 \phi - ip_0 \phi), \quad a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} (\partial_0 \phi^\dagger + ip_0 \phi^\dagger). \quad (2.153)$$

从而, 有

$$\begin{aligned} [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) + iq_0 \phi^\dagger(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} [-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \tag{2.154}
\end{aligned}$$

以及

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip\cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq\cdot y} \{\partial_0 \phi(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p\cdot x+q\cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) - iq_0 \phi(\mathbf{y}, t)] = 0. \tag{2.155}
\end{aligned}$$

另一方面, 由

$$\begin{aligned}
\int d^3x e^{iq\cdot x} \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q)\cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}] \\
&= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}) \tag{2.156}
\end{aligned}$$

和

$$\begin{aligned}
\int d^3x e^{iq\cdot x} \partial_0 \phi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x [b_{\mathbf{p}} e^{-i(p-q)\cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}] \\
&= \int d^3p \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} [b_{\mathbf{p}} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} (b_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}), \tag{2.157}
\end{aligned}$$

可得

$$-i\sqrt{2E_{\mathbf{q}}} b_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} b_{\mathbf{q}} = \int d^3x e^{iq\cdot x} \partial_0 \phi^\dagger - iq_0 \int d^3x e^{iq\cdot x} \phi^\dagger = \int d^3x e^{iq\cdot x} (\partial_0 \phi^\dagger - iq_0 \phi^\dagger). \tag{2.158}$$

于是,

$$b_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip\cdot x} (\partial_0 \phi^\dagger - ip_0 \phi^\dagger), \quad b_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip\cdot x} (\partial_0 \phi + ip_0 \phi). \tag{2.159}$$

从而, 有

$$\begin{aligned}
&[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip\cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{-iq\cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (iq_0 [\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] - ip_0 [\phi^\dagger(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t)]) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [-i(p_0 + q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
&= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{2.160}
\end{aligned}$$

以及

$$\begin{aligned}
&[b_{\mathbf{p}}, b_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi^\dagger(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi(\mathbf{x}, t) - ip_0 \phi^\dagger(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] = 0. \tag{2.161}
\end{aligned}$$

此外, 还有

$$\begin{aligned}
&[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{-iq \cdot y} \{\partial_0 \phi(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)\}] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi^\dagger(\mathbf{y}, t) + iq_0 \phi(\mathbf{y}, t)] = 0, \tag{2.162}
\end{aligned}$$

以及

$$\begin{aligned}
&[a_{\mathbf{p}}, b_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y [e^{ip \cdot x} \{\partial_0 \phi(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t)\}, e^{iq \cdot y} \{\partial_0 \phi^\dagger(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)\}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [\pi^\dagger(\mathbf{x}, t) - ip_0 \phi(\mathbf{x}, t), \pi(\mathbf{y}, t) - iq_0 \phi^\dagger(\mathbf{y}, t)] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} (-iq_0 [\pi^\dagger(\mathbf{x}, t), \phi^\dagger(\mathbf{y}, t)] - ip_0 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)]) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} [-i(p_0 - q_0) i \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} = \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \tag{2.163}
\end{aligned}$$

归纳起来, 产生湮灭算符的对易关系如下:

$$\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), & [a_{\mathbf{p}}, a_{\mathbf{q}}] &= [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \\
[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), & [b_{\mathbf{p}}, b_{\mathbf{q}}] &= [b_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0,
\end{aligned} \tag{2.164}$$

$$[a_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = [b_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = [a_{\mathbf{p}}, b_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0.$$

这说明 $a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$ 与 $b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$ 是两套不同的产生湮灭算符, 描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta} \phi(x), \quad [\phi^\dagger(x)]' = e^{-iq\theta} \phi^\dagger(x), \quad (2.165)$$

则拉氏量 (2.136) 不变。依照 1.7.4 小节的讨论, 相应的守恒流为

$$J^\mu = q\phi^\dagger i \overleftrightarrow{\partial}^\mu \phi, \quad (2.166)$$

相应的守恒荷为

$$\begin{aligned} Q &= q \int d^3x \phi^\dagger i \overleftrightarrow{\partial}^0 \phi = iq \int d^3x [\phi^\dagger \partial^0 \phi - (\partial^0 \phi^\dagger) \phi] \\ &= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \partial^0 (a_{\mathbf{k}} e^{-ik \cdot x} + b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \right. \\ &\quad \left. - \partial^0 (b_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (a_{\mathbf{k}} e^{-ik \cdot x} + b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \right] \\ &= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{k}}}} \left[(b_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (-ik^0) (a_{\mathbf{k}} e^{-ik \cdot x} - b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \right. \\ &\quad \left. - (-ip^0) (b_{\mathbf{p}} e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (a_{\mathbf{k}} e^{-ik \cdot x} + b_{\mathbf{k}}^\dagger e^{ik \cdot x}) \right] \\ &= iq \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{k}}}} \left[(ik^0 + ip^0) b_{\mathbf{p}} b_{\mathbf{k}}^\dagger e^{-i(p-k) \cdot x} + (-ik^0 - ip^0) a_{\mathbf{p}}^\dagger a_{\mathbf{k}} e^{i(p-k) \cdot x} \right. \\ &\quad \left. + (-ik^0 + ip^0) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k) \cdot x} + (ik^0 - ip^0) a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(p+k) \cdot x} \right] \\ &= q \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{k}}}} \left[-(E_{\mathbf{k}} + E_{\mathbf{p}}) b_{\mathbf{p}} b_{\mathbf{k}}^\dagger e^{-i(p-k) \cdot x} + (E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger a_{\mathbf{k}} e^{i(p-k) \cdot x} \right. \\ &\quad \left. + (E_{\mathbf{k}} - E_{\mathbf{p}}) b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(p+k) \cdot x} + (-E_{\mathbf{k}} + E_{\mathbf{p}}) a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(p+k) \cdot x} \right] \\ &= q \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{k}}}} \left\{ (E_{\mathbf{k}} + E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} - \mathbf{k}) \left[-b_{\mathbf{p}} b_{\mathbf{k}}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{k}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{k}} e^{i(E_{\mathbf{p}} - E_{\mathbf{k}})t} \right] \right. \\ &\quad \left. + (E_{\mathbf{k}} - E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} + \mathbf{k}) \left[b_{\mathbf{p}} a_{\mathbf{k}} e^{-i(E_{\mathbf{p}} + E_{\mathbf{k}})t} - a_{\mathbf{p}}^\dagger b_{\mathbf{k}}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{k}})t} \right] \right\} \\ &= q \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} 2E_{\mathbf{p}} (-b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = q \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{\mathbf{p}} b_{\mathbf{p}}^\dagger). \end{aligned} \quad (2.167)$$

利用对易关系 (2.164), 可得

$$Q = \int \frac{d^3p}{(2\pi)^3} (q a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - q b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) - (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} q. \quad (2.168)$$

上式第二项是零点荷。在第一项的圆括号中, 粒子数密度算符 $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 的系数是 q , 而粒子数密度算符 $b_{\mathbf{p}}^\dagger b_{\mathbf{p}}$ 的系数是 $-q$ 。可见, $a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q , 习惯上称为正粒子; 另一方面,

$b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$, 习惯上称为反粒子。除去零点荷, 总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式的全空间积分类似, 但 Q/q 被解释为正粒子数与反粒子数之差, 可正可负, 因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的, 并且来自于一种内部对称性, 因而是一种内部相加性量子数。实际上, 反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 $U(1)$ 整体变换, 则自共轭条件 (2.59) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^\dagger = [e^{iq\theta}\phi(x)]^\dagger = e^{-iq\theta}\phi^\dagger(x) = e^{-iq\theta}\phi(x). \quad (2.169)$$

上式要求 $q = 0$ 。因此, 对实标量场不能进行非平庸的 $U(1)$ 整体变换。实际上, 自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数, 也就是说, 正粒子与反粒子是相同的, 实标量场描述的是一种纯中性粒子。

2.4.4 哈密顿量和总动量

根据 (1.119) 式, 复标量场的哈密顿量密度为

$$\begin{aligned} \mathcal{H} &= \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L} = (\partial^0\phi^\dagger)\partial_0\phi + (\partial^0\phi)\partial_0\phi^\dagger - (\partial^\mu\phi^\dagger)\partial_\mu\phi + m^2\phi^\dagger\phi \\ &= (\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger) \cdot \nabla\phi + m^2\phi^\dagger\phi. \end{aligned} \quad (2.170)$$

于是, 哈密顿量可以写成

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \int d^3x [(\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger) \cdot \nabla\phi + m^2\phi^\dagger\phi] \\ &= \int d^3x [(\partial^0\phi^\dagger)\partial_0\phi + \nabla \cdot (\phi^\dagger\nabla\phi) - \phi^\dagger\nabla^2\phi + m^2\phi^\dagger\phi] \\ &= \int d^3x [(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^0\partial_0 - \nabla^2 + m^2)\phi] \\ &= \int d^3x [(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^2 + m^2)\phi]. \end{aligned} \quad (2.171)$$

上式第三步用了分部积分, 第四步扔掉了一个全散度, 最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.138) 消去。从而, 得到

$$\begin{aligned} H &= \int d^3x [(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi] \\ &= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \left[\partial^0 (b_{\mathbf{p}}e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \partial_0 (a_{\mathbf{q}}e^{-iq \cdot x} + b_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. - (b_{\mathbf{p}}e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \partial^0\partial_0 (a_{\mathbf{q}}e^{-iq \cdot x} + b_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right] \\ &= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \left\{ (-ip^0) (b_{\mathbf{p}}e^{-ip \cdot x} - a_{\mathbf{p}}^\dagger e^{ip \cdot x}) (-iq_0) (a_{\mathbf{q}}e^{-iq \cdot x} - b_{\mathbf{q}}^\dagger e^{iq \cdot x}) \right. \\ &\quad \left. - (b_{\mathbf{p}}e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) [(-iq^0)(-iq_0)a_{\mathbf{q}}e^{-iq \cdot x} + iq^0iq_0b_{\mathbf{q}}^\dagger e^{iq \cdot x}] \right\} \end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(p^0 q_0 + q^0 q_0) b_{\mathbf{p}} b_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p^0 q_0 + q^0 q_0) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \right. \\
&\quad \left. + (-p^0 q_0 + q^0 q_0) b_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + (-p^0 q_0 + q^0 q_0) a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x} \right] \\
&= \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} E_{\mathbf{q}} \left\{ (E_{\mathbf{p}} + E_{\mathbf{q}}) \delta^{(3)}(\mathbf{p} - \mathbf{q}) [b_{\mathbf{p}} b_{\mathbf{q}}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t}] \right. \\
&\quad \left. + (E_{\mathbf{q}} - E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} + \mathbf{q}) [b_{\mathbf{p}} a_{\mathbf{q}} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t}] \right\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} 2E_p^2 (b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}. \tag{2.172}
\end{aligned}$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 \mathbf{p} ，正粒子与反粒子具有相同的能量 $E_{\mathbf{p}}$ ，因而它们具有相同的质量 m 。

根据 (1.158) 式，复标量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x (\pi \nabla \phi + \pi^\dagger \nabla \phi^\dagger) = - \int d^3x [(\partial_0 \phi^\dagger) \nabla \phi + (\partial_0 \phi) \nabla \phi^\dagger] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[\partial_0 (b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}) \nabla (a_{\mathbf{q}} e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}) \right. \\
&\quad \left. + \partial_0 (a_{\mathbf{q}} e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}) \nabla (b_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[-ip_0 (b_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}) i\mathbf{q} (a_{\mathbf{q}} e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}) \right. \\
&\quad \left. - iq_0 (a_{\mathbf{q}} e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}) i\mathbf{p} (b_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}) \right] \\
&= - \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \left[(-E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} b_{\mathbf{q}}^\dagger - E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^\dagger b_{\mathbf{p}}) e^{-i(p-q)\cdot x} \right. \\
&\quad + (-E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} a_{\mathbf{p}}^\dagger) e^{i(p-q)\cdot x} \\
&\quad + (E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} b_{\mathbf{p}}) e^{-i(p+q)\cdot x} \\
&\quad \left. + (E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger) e^{i(p+q)\cdot x} \right] \\
&= - \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_p 2E_q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[(-E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} b_{\mathbf{q}}^\dagger - E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^\dagger b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right. \right. \\
&\quad \left. + (-E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} a_{\mathbf{p}}^\dagger) e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right] \\
&\quad + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[(E_{\mathbf{p}} \mathbf{q} b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}} \mathbf{p} a_{\mathbf{q}} b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \\
&\quad \left. + (E_{\mathbf{p}} \mathbf{q} a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}} \mathbf{p} b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger) e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \left. \right\} \\
&= - \int \frac{d^3p}{(2\pi)^3 2E_p} \left[-E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger) \right. \\
&\quad \left. - E_{\mathbf{p}} \mathbf{p} (b_{\mathbf{p}} a_{-\mathbf{p}} - a_{-\mathbf{p}} b_{\mathbf{p}}) e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}} \mathbf{p} (a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger - b_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger) e^{2iE_{\mathbf{p}}t} \right]
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} (b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger) = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) + \delta^{(3)}(\mathbf{0}) \int d^3p \mathbf{p} \\
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}). \tag{2.173}
\end{aligned}$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章 矢量场

3.1 量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性么正变换

$$|\Psi'\rangle = U(\Lambda) |\Psi\rangle, \quad (3.1)$$

其中 $U(\Lambda)$ 是一个线性么正算符，描述量子 Lorentz 变换，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \quad (3.2)$$

先作 Lorentz 变换 Λ_1 ，再作 Lorentz 变换 Λ_2 ，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$ ，故以下同态 (homomorphic) 关系成立：

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \quad (3.3)$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(\mathbf{1}) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \quad (3.4)$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \quad (3.5)$$

将无穷小 Lorentz 变换 (1.159) 记为 $\Lambda_\omega = \mathbf{1} + \omega$ ，它诱导的无穷小么正算符可表达为

$$U(\mathbf{1} + \omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \quad (3.6)$$

这里只展开到 ω 的一阶项。 $J^{\mu\nu}$ 是量子 Lorentz 变换的生成元算符¹。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的：

$$J^{\mu\nu} = -J^{\nu\mu}. \quad (3.7)$$

由 $U(\mathbf{1} + \omega)$ 的么正性可得

$$1 = U^\dagger(\mathbf{1} + \omega)U(\mathbf{1} + \omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right] \left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \quad (3.8)$$

¹虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.179)。

最后一步忽略了 ω 的二阶项。可见, $J^{\mu\nu}$ 是厄米算符:

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \quad (3.9)$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda). \quad (3.10)$$

的左边和右边分别展开, 得

$$U^{-1}(\Lambda)U(\mathbf{1} + \omega)U(\Lambda) = U^{-1}(\Lambda) \left(1 - \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} \right) U(\Lambda) = 1 - \frac{i}{2} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda), \quad (3.11)$$

$$U(\Lambda^{-1}(\mathbf{1} + \omega)\Lambda) = U(\mathbf{1} + \Lambda^{-1}\omega\Lambda) = 1 - \frac{i}{2} (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu}. \quad (3.12)$$

因此, 有

$$\begin{aligned} U^{-1}(\Lambda) \omega_{\mu\nu} J^{\mu\nu} U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu} J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1}\omega\Lambda)^\alpha{}_\nu J^{\mu\nu} = g_{\mu\alpha} (\Lambda^{-1})^\alpha{}_\beta \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} \\ &= g_{\mu\alpha} \Lambda_\beta{}^\alpha \omega^\beta{}_\gamma \Lambda^\gamma{}_\nu J^{\mu\nu} = \Lambda^\beta{}_\mu \omega_{\beta\gamma} \Lambda^\gamma{}_\nu J^{\mu\nu} = \omega_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}, \end{aligned} \quad (3.13)$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立, 于是,

$$U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.14)$$

因此, $J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值与它在 $|\Psi\rangle$ 中的期待值有如下关系:

$$\langle \Psi' | J^{\mu\nu} | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \langle \Psi | J^{\rho\sigma} | \Psi \rangle. \quad (3.15)$$

也就是说, $U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换:

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}. \quad (3.16)$$

可见, $J^{\mu\nu}$ 是一个 2 阶 Lorentz 张量。

接着, 考虑 Λ 的无穷小形式 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu$, 则

$$U(\Lambda) = 1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta}. \quad (3.17)$$

忽略二阶小量, (3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda) J^{\mu\nu} U(\Lambda) &= \left(1 + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} \right) J^{\mu\nu} \left(1 - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\alpha\beta} \right) \\ &= J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\alpha\beta} J^{\mu\nu} J^{\alpha\beta} + \frac{i}{2} \tilde{\omega}_{\gamma\delta} J^{\gamma\delta} J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2} \tilde{\omega}_{\rho\sigma} [J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \quad (3.18)$$

右边为

$$\begin{aligned} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma} &= (\delta^\mu{}_\rho + \tilde{\omega}^\mu{}_\rho) (\delta^\nu{}_\sigma + \tilde{\omega}^\nu{}_\sigma) J^{\rho\sigma} = \delta^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} + \delta^\mu{}_\rho \tilde{\omega}^\nu{}_\sigma J^{\rho\sigma} + \tilde{\omega}^\mu{}_\rho \delta^\nu{}_\sigma J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^\nu{}_\sigma J^{\mu\sigma} + \tilde{\omega}^\mu{}_\rho J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma} g^{\nu\rho} J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho} g^{\mu\sigma} J^{\rho\nu} \end{aligned}$$

$$\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma} J^{\mu\rho} + g^{\mu\rho} J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho} J^{\mu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma}), \tag{3.19}
\end{aligned}$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}) \\
&= i[g^{\nu\rho} J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma). \tag{3.20}
\end{aligned}$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群是一类特殊的连续群， n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie* 乘积，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$J^i \equiv \frac{1}{2}\varepsilon^{ijk} J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}$$

即

$$\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}$$

J^i 与 J^j 的对易关系为

$$\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij}, \tag{3.23}
\end{aligned}$$

第三、四步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.84) 式。由 (1.98) 式，有

$$J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}$$

从而推出

$$[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ ，写成分量的形式是 $L^i = \varepsilon^{ijk}x^jp^k$ ，从而，

$$\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^lp^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^lp^m = x^ip^j - x^jp^i. \tag{3.26}$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i, p^j] = i\delta^{ij}$ 可得

$$\begin{aligned}
 [L^i, L^j] &= \varepsilon^{ikl} \varepsilon^{jmn} [x^k p^l, x^m p^n] = \varepsilon^{ikl} \varepsilon^{jmn} \{x^k [p^l, x^m] p^n + x^m [x^k, p^n] p^l\} \\
 &= \varepsilon^{ikl} \varepsilon^{jmn} (-i\delta^{lm} x^k p^n + i\delta^{kn} x^m p^l) = i(-\varepsilon^{ikl} \varepsilon^{jln} x^k p^n + \varepsilon^{ikl} \varepsilon^{jmk} x^m p^l) \\
 &= i(\varepsilon^{ikl} \varepsilon^{jnl} x^k p^n - \varepsilon^{ilk} \varepsilon^{jmk} x^m p^l) = i[(\delta^{ij} \delta^{kn} - \delta^{in} \delta^{kj}) x^k p^n - (\delta^{ij} \delta^{lm} - \delta^{im} \delta^{lj}) x^m p^l] \\
 &= i[\delta^{ij} x^k p^k - x^j p^i - \delta^{ij} x^l p^l + x^i p^j] = i(x^i p^j - x^j p^i) = i\varepsilon^{ijk} L^k.
 \end{aligned} \tag{3.27}$$

可见, \mathbf{J} 与 \mathbf{L} 具有相同的对易关系, \mathbf{J} 也是一个角动量算符。实际上, \mathbf{J} 描述总角动量, 不止可以包含轨道角动量 \mathbf{L} , 也可以包含自旋角动量。

满足

$$O^T O = \mathbf{1} \tag{3.28}$$

的实方阵 O 称为实正交矩阵 (real orthogonal matrix)。对上式取行列式, 得

$$1 = \det O^T \cdot \det O = (\det O)^2. \tag{3.29}$$

可见, 实正交矩阵 O 的行列式为 $\det O = \pm 1$ 。由行列式为 $+1$ 的 3 维实正交矩阵按照矩阵乘法构成的群, 称为空间旋转群 $\mathbf{SO}(3)$, 描述三维空间中的旋转变换。1.7.3 小节提到, $\mathbf{SO}(3)$ 群是 Lorentz 群的子群, J^i 可以看作 $\mathbf{SO}(3)$ 群的生成元算符, 而 (3.25) 式是 $\mathbf{SO}(3)$ 群的 Lie 代数关系。

另一方面, \mathbf{K} 是增速算符。 \mathbf{J} 与 \mathbf{K} 的对易关系为

$$\begin{aligned}
 [J^i, K^j] &= \frac{1}{2} \varepsilon^{ikl} [J^{kl}, J^{0j}] = \frac{i}{2} \varepsilon^{ikl} \{[g^{l0} J^{kj} - (k \leftrightarrow l)] - (0 \leftrightarrow j)\} \\
 &= i\varepsilon^{ikl} [g^{l0} J^{kj} - (0 \leftrightarrow j)] = i\varepsilon^{ikl} (g^{l0} J^{kj} - g^{lj} J^{k0}) = -i\varepsilon^{ikl} g^{lj} J^{k0} = i\varepsilon^{ikl} \delta^{lj} J^{k0} \\
 &= i\varepsilon^{ikj} J^{k0} = i\varepsilon^{ijk} J^{0k} = i\varepsilon^{ijk} K^k,
 \end{aligned} \tag{3.30}$$

而 \mathbf{K} 自身的对易关系为

$$\begin{aligned}
 [K^i, K^j] &= [J^{0i}, J^{0j}] = i(g^{i0} J^{0j} - g^{00} J^{ij} - g^{ij} J^{00} + g^{0j} J^{i0}) \\
 &= -i(g^{00} J^{ij} + g^{ij} J^{00}) = -iJ^{ij} = -i\varepsilon^{ijk} J^k.
 \end{aligned} \tag{3.31}$$

归纳起来, 有

$$[J^i, J^j] = i\varepsilon^{ijk} J^k, \quad [J^i, K^j] = i\varepsilon^{ijk} K^k, \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k. \tag{3.32}$$

3.2 量子矢量场的 Lorentz 变换

3.2.1 Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 ω^α_β 可以转化为

$$\omega^\alpha_\beta = g^{\alpha\mu} \omega_{\mu\beta} = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\beta} - g^{\alpha\mu} \omega_{\beta\mu}) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\mu} \omega_{\nu\mu} \delta^\nu_\beta) = \frac{1}{2} (g^{\alpha\mu} \omega_{\mu\nu} \delta^\nu_\beta - g^{\alpha\nu} \omega_{\mu\nu} \delta^\mu_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta, \quad (3.33)$$

其中 $(\mathcal{J}^{\mu\nu})^\alpha_\beta$ 定义为

$$(\mathcal{J}^{\mu\nu})^\alpha_\beta \equiv i(g^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta). \quad (3.34)$$

容易看出, $\mathcal{J}^{\mu\nu}$ 是反对称的:

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \quad (3.35)$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^\gamma_\beta = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^\nu_\beta - \delta^\mu_\beta g^{\nu\gamma}) = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha). \quad (3.36)$$

这样的话, 可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^\alpha_\beta = \delta^\alpha_\beta + \omega^\alpha_\beta = \delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha_\beta. \quad (3.37)$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha_\beta &= (\mathcal{J}^{\mu\nu})^\alpha_\gamma(\mathcal{J}^{\rho\sigma})^\gamma_\beta - (\mathcal{J}^{\rho\sigma})^\alpha_\gamma(\mathcal{J}^{\mu\nu})^\gamma_\beta \\ &= i^2(g^{\mu\alpha}\delta^\nu_\gamma - \delta^\mu_\gamma g^{\nu\alpha})(g^{\rho\gamma}\delta^\sigma_\beta - \delta^\rho_\beta g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}\delta^\nu_\gamma g^{\rho\gamma}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\nu_\gamma \delta^\rho_\beta g^{\sigma\gamma} + \delta^\mu_\gamma g^{\nu\alpha} g^{\rho\gamma}\delta^\sigma_\beta - \delta^\mu_\gamma g^{\nu\alpha} \delta^\rho_\beta g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\mu\alpha}g^{\rho\nu}\delta^\sigma_\beta + g^{\mu\alpha}\delta^\rho_\beta g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^\sigma_\beta - g^{\nu\alpha}\delta^\rho_\beta g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\ &= -g^{\nu\rho}g^{\mu\alpha}\delta^\sigma_\beta + g^{\mu\rho}g^{\nu\alpha}\delta^\sigma_\beta + g^{\nu\sigma}g^{\mu\alpha}\delta^\rho_\beta - g^{\mu\sigma}g^{\nu\alpha}\delta^\rho_\beta \\ &\quad - [-g^{\sigma\mu}g^{\rho\alpha}\delta^\nu_\beta + g^{\rho\mu}g^{\sigma\alpha}\delta^\nu_\beta + g^{\sigma\nu}g^{\rho\alpha}\delta^\mu_\beta - g^{\rho\nu}g^{\sigma\alpha}\delta^\mu_\beta] \\ &= g^{\nu\rho}(g^{\sigma\alpha}\delta^\mu_\beta - g^{\mu\alpha}\delta^\sigma_\beta) + g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma_\beta - g^{\sigma\alpha}\delta^\nu_\beta) + g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho_\beta - g^{\rho\alpha}\delta^\mu_\beta) + g^{\mu\sigma}(g^{\rho\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\rho_\beta) \\ &= -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^\alpha_\beta - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha_\beta \\ &= i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^\alpha_\beta - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha_\beta - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha_\beta + g^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha_\beta], \end{aligned} \quad (3.38)$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\rho\nu}). \quad (3.39)$$

可见, $\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。 Λ^α_β 属于 Lorentz 群的矢量表示, 因而 $\mathcal{J}^{\mu\nu}$ 就是矢量表示的生成元。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \quad (3.40)$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^\omega = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \quad (3.41)$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 \mathbf{g} 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}^{\delta} g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}_{\gamma} g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}_{\beta}, \quad (3.42)$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \quad (3.43)$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1} \left[\sum_{n=0}^{\infty} \frac{(\omega^{\mathrm{T}})^n}{n!} \right] \mathbf{g} = \sum_{n=0}^{\infty} \frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \quad (3.44)$$

若两个同阶方阵 A 和 B 相互对易，即 $[A, B] = 0$ ，则二项式定理成立：

$$(A + B)^n = \sum_{j=0}^n \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.45)$$

阶乘的定义可以推广到负整数：对于整数 $m < 0$ ，定义

$$m! \rightarrow \infty, \quad \frac{1}{m!} \rightarrow 0. \quad (3.46)$$

从而，对于 $j > n$ ，有 $[(n-j)!]^{-1} \rightarrow 0$ 。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A + B)^n = \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j}. \quad (3.47)$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} A^j B^{n-j} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{n=0}^{\infty} \frac{B^{n-j}}{(n-j)!} = e^A e^B. \quad (3.48)$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^0 = \mathbf{1}. \quad (3.49)$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\Lambda = \mathbf{g}, \quad (3.50)$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp \left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right) \quad (3.51)$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的有限变换。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu} = 0$ ，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

在正则量子化程序中, 标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符, 类似于 (3.16) 式, $\phi(x)$ 的固有保时向 Lorentz 变换关系 (2.58) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \quad (3.52)$$

上式表明, 变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$, (3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \quad (3.53)$$

相应地, $\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\phi(x)|\Psi'\rangle = \langle\Psi|U^{-1}(\Lambda)\phi(x)U(\Lambda)|\Psi\rangle = \langle\Psi|\phi(\Lambda^{-1}x)|\Psi\rangle. \quad (3.54)$$

另一方面, 由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的相应 Lorentz 变换形式为

$$\partial^\mu\phi'(x') = U^{-1}(\Lambda)\partial^\mu\phi(x')U(\Lambda) = \partial^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \quad (3.55)$$

于是, 在固有保时向 Lorentz 变换下, 自由实标量场的拉氏量 (2.60) 的变换形式为

$$\begin{aligned} \mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial^\mu\phi(x')\partial_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda) \\ &= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\} \\ &= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)] \\ &= \mathcal{L}(x), \end{aligned} \quad (3.56)$$

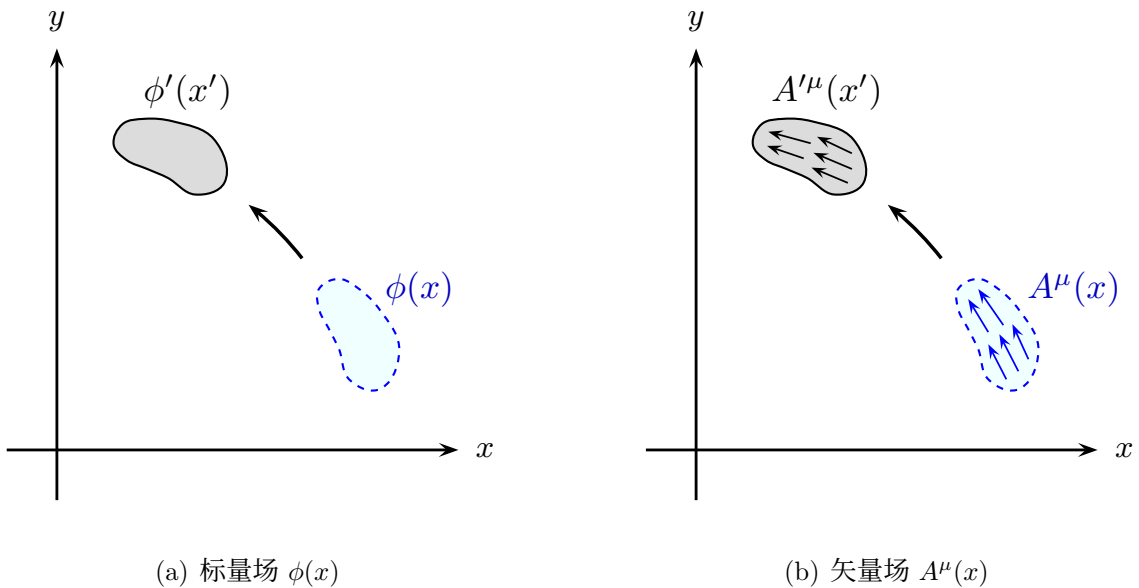


图 3.1: 在绕 z 轴空间旋转变换下, 标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而,

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda) = \mathcal{L}(\Lambda^{-1}x). \quad (3.57)$$

可见, 拉氏量 (2.60) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, 可得

$$\begin{aligned} (\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu = g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta = g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta + \omega^\alpha{}_\beta) = g_{\nu\beta}g^{\mu\beta} + g^{\mu\beta}\omega_{\nu\beta} = \delta^\mu{}_\nu - g^{\mu\beta}\omega_{\beta\nu} \\ &= \delta^\mu{}_\nu - \omega^\mu{}_\nu, \end{aligned} \quad (3.58)$$

从而, 有

$$(\Lambda^{-1}x)^\mu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu. \quad (3.59)$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项, 得

$$\begin{aligned} \phi(\Lambda^{-1}x) &= \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = \phi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \phi(x) = \phi(x) - \frac{1}{2}(\omega_{\mu\nu} x^\nu \partial^\mu + \omega_{\nu\mu} x^\mu \partial^\nu) \phi(x) \\ &= \phi(x) - \frac{1}{2}\omega_{\mu\nu}(x^\nu \partial^\mu - x^\mu \partial^\nu) \phi(x) = \phi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x) \\ &= \phi(x) - \frac{i}{2}\omega_{\mu\nu} i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi(x). \end{aligned} \quad (3.60)$$

根据 (3.6) 式, 将 (3.53) 式左边展开到 ω 的一阶项, 得

$$\begin{aligned} U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right) \\ &= \phi(x) - \frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x) = \phi(x) - \frac{i}{2}\omega_{\mu\nu}[\phi(x), J^{\mu\nu}]. \end{aligned} \quad (3.61)$$

两相比较, 给出

$$[\phi(x), J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi(x) = L^{\mu\nu}\phi(x), \quad (3.62)$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu} \equiv i(x^\mu \partial^\nu - x^\nu \partial^\mu). \quad (3.63)$$

对于空间分量 L^{ij} , 可以等价地定义

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{i}{2}\varepsilon^{ijk}(x^j \partial^k - x^k \partial^j) = \frac{i}{2}(\varepsilon^{ijk}x^j \partial^k - \varepsilon^{ikj}x^j \partial^k) = i\varepsilon^{ijk}x^j \partial^k, \quad (3.64)$$

写成空间矢量的形式是

$$\mathbf{L} = -i\mathbf{x} \times \nabla. \quad (3.65)$$

可见, \mathbf{L} 就是微分算符形式的轨道角动量算符。根据 (3.21) 式, (3.62) 式的纯空间分量部分可以改写为

$$[\phi(x), \mathbf{J}] = \mathbf{L}\phi(x). \quad (3.66)$$

上式表明, 总角动量算符 \mathbf{J} 生成了轨道角动量, 但没有生成自旋角动量。这说明标量场没有自旋, 对应于零自旋粒子。

3.2.3 量子矢量场的 Lorentz 变换形式

$\partial^\mu \phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有像 (3.55) 式那样的 Lorentz 变换形式, 即

$$A'^\mu(x') = U^{-1}(\Lambda) A^\mu(x') U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \quad (3.67)$$

或者写成

$$U^{-1}(\Lambda) A^\mu(x) U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \quad (3.68)$$

这就是量子矢量场的 Lorentz 变换形式。相应地, $A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle \Psi' | A^\mu(x) | \Psi' \rangle = \langle \Psi | U^{-1}(\Lambda) A^\mu(x) U(\Lambda) | \Psi \rangle = \Lambda^\mu{}_\nu \langle \Psi | A^\nu(\Lambda^{-1}x) | \Psi \rangle. \quad (3.69)$$

对于固有保时向 Lorentz 变换, 根据矢量表示中的无穷小形式 (3.40), (3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] A^\nu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.70)$$

将上式与 (1.168) 式比较, 可以发现, 1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$ 。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出, 在 Lorentz 变换下, 除了矢量场的分布区域发生变化之外, 矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式, 在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项, 得

$$\begin{aligned} A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta \partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta} x^\beta \partial^\alpha A^\nu(x) \\ &= A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x). \end{aligned} \quad (3.71)$$

从而, (3.68) 式右边可展开为

$$\begin{aligned} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \right] \left[A^\nu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\nu(x) \right] \\ &= A^\mu(x) + \frac{1}{2} \omega_{\alpha\beta} (x^\alpha \partial^\beta - x^\beta \partial^\alpha) A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)]. \end{aligned} \quad (3.72)$$

另一方面, (3.68) 式左边的无穷小展开式为

$$\begin{aligned} U^{-1}(\Lambda) A^\mu(x) U(\Lambda) &= \left(1 + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} \right) A^\mu(x) \left(1 - \frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta} \right) \\ &= A^\mu(x) - \frac{i}{2} \omega_{\alpha\beta} A^\mu(x) J^{\alpha\beta} + \frac{i}{2} \omega_{\gamma\delta} J^{\gamma\delta} A^\mu(x) = A^\mu(x) - \frac{i}{2} \omega_{\rho\sigma} [A^\mu(x), J^{\rho\sigma}]. \end{aligned} \quad (3.73)$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \quad (3.74)$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2} \varepsilon^{ijk} \mathcal{J}^{jk}, \quad \mathcal{J} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \quad (3.75)$$

再根据 (3.21) 和 (3.64) 式, (3.74) 式的纯空间分量部分可以改写为

$$[A^\mu(x), \mathbf{J}] = \mathbf{L} A^\mu(x) + (\mathcal{J})^\mu{}_\nu A^\nu(x). \quad (3.76)$$

上式表明, 总角动量算符 \mathbf{J} 不仅生成了轨道角动量, 还生成了由 \mathcal{J} 描述的自旋角动量。 \mathcal{J}^i 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \quad (3.77)$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ -i & & & 0 \end{pmatrix}, \quad (3.78)$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \quad (3.79)$$

只关注空间分量, 可得

$$(\mathcal{J}^1 \mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2 \mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \quad (3.80)$$

因此, 有

$$(\mathcal{J}^2)^i{}_j = (\mathcal{J}^1 \mathcal{J}^1 + \mathcal{J}^2 \mathcal{J}^2 + \mathcal{J}^3 \mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \quad (3.81)$$

根据量子力学的角动量理论, \mathcal{J}^2 的本征值为 $s(s+1)$, 即 $(\mathcal{J}^2)^i{}_j = s(s+1)\delta^i{}_j$, 其中 s 为自旋量子数。可见, 矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \quad (3.82)$$

经过量子化程序之后, 矢量场 $A^\mu(x)$ 应当描述自旋为 1 的粒子。

3.3 有质量矢量场的正则量子化

类似于电磁场, 对任意的矢量场 A^μ 可以定义反对称的场强张量

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (3.83)$$

对于一个自由的有质量的实矢量场 A^μ ，用场强张量可以将它的 **Lorentz** 不变拉氏量写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \quad (3.84)$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu] \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu. \end{aligned} \quad (3.85)$$

从而，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \quad (3.86)$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \quad (3.87)$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \quad (3.88)$$

上式称为 **Proca** 方程，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0. \quad (3.89)$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = m^2 \partial_\nu A^\nu. \quad (3.90)$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz** 条件

$$\partial_\mu A^\mu = 0. \quad (3.91)$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \partial^2 A^\nu. \quad (3.92)$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon* 方程

$$(\partial^2 + m^2)A^\mu(x) = 0. \quad (3.93)$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \quad (3.94)$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \quad (3.95)$$

由于 $\pi_0 = 0$, 它不能作为与 A^0 对应的正则共轭场, 因而不能为 A^0 构造正则对易关系。实际上, 由于 Lorenz 条件 (3.91) 的存在, A^μ 只有 3 个独立分量, 我们可以将 A^0 视作依赖于其它 3 个分量的量。因此, 正则量子化程序要求独立的正则变量满足等时对易关系

$$[A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = i\delta^i_j \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^i(\mathbf{x}, t), A^j(\mathbf{y}, t)] = [\pi_i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] = 0. \quad (3.96)$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程, 应该具有两个平面波解, 即正能解 $\exp(-ip \cdot x)$ 和负能解 $\exp(ip \cdot x)$ 。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标, 平面波展开式的系数也必须具有一个这样的指标。一般地, 对于确定的动量 \mathbf{p} , 矢量场的正能解模式具有如下形式:

$$A^\mu(x; \mathbf{p}, \sigma) = e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (3.97)$$

这里的系数 $e^\mu(\mathbf{p}, \sigma)$ 是 Lorentz 矢量, 称为极化矢量 (polarization vector), 它依赖于动量 p , 而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底, 从而, 可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点, 一组极化矢量应当是线性独立且正交完备的。Lorentz 矢量空间是一个 4 维空间, 因而这样的极化矢量应该有 4 个, 包括 1 个类时的极化矢量 $e^\mu(\mathbf{p}, 0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 。

在没有额外约束的情况下, 我们要求这 4 个极化矢量是实的, 而且满足 Lorentz 矢量空间中的正交归一关系

$$e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') = g_{\sigma\sigma'}. \quad (3.98)$$

进一步, 要求这组极化矢量是完备的, 也就是说, 任意依赖于 \mathbf{p} 的 Lorentz 矢量 $V_\mu(\mathbf{p})$ 能够以它们为基底展开成

$$V_\mu(\mathbf{p}) = \sum_{\sigma=0}^3 v_\sigma(\mathbf{p}) e_\mu(\mathbf{p}, \sigma). \quad (3.99)$$

根据正交归一关系 (3.98), 可得

$$g_{\sigma\sigma'} e_\mu(\mathbf{p}, \sigma) V^\mu(\mathbf{p}) = g_{\sigma\sigma'} e_\mu(\mathbf{p}, \sigma) \sum_{\sigma'=0}^3 v_{\sigma'}(\mathbf{p}) e^\mu(\mathbf{p}, \sigma') = g_{\sigma\sigma'} \sum_{\sigma'=0}^3 v_{\sigma'}(\mathbf{p}) g_{\sigma\sigma'} = g_{\sigma\sigma}^2 v_\sigma(\mathbf{p}). \quad (3.100)$$

由于 $g_{\sigma\sigma}^2 = 1$, 上式化为

$$v_\sigma(\mathbf{p}) = g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) V^\mu(\mathbf{p}). \quad (3.101)$$

这是展开系数 $v_\sigma(\mathbf{p})$ 的计算公式。将它代回 (3.99) 式, 有

$$g_{\mu\nu} V^\nu(\mathbf{p}) = V_\mu(\mathbf{p}) = \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\nu(\mathbf{p}, \sigma) V^\nu(\mathbf{p}) e_\mu(\mathbf{p}, \sigma) = \left[\sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \right] V^\nu(\mathbf{p}). \quad (3.102)$$

比较上式最左边和最右边，即得

$$\sum_{\sigma=0}^3 g_{\sigma\sigma} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{p}, \sigma) = g_{\mu\nu}. \quad (3.103)$$

这就是完备性关系。正交归一关系 (3.98) 和完备性关系 (3.103) 都是 Lorentz 协变的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^{μ} 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空横向极化矢量

$$e^{\mu}(\mathbf{p}, 1) = (0, \mathbf{e}(\mathbf{p}, 1)), \quad e^{\mu}(\mathbf{p}, 2) = (0, \mathbf{e}(\mathbf{p}, 2)). \quad (3.104)$$

此处，

$$\mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|} (p^1 p^3, p^2 p^3, -|\mathbf{p}_T|^2), \quad \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|} (-p^2, p^1, 0), \quad (3.105)$$

其中

$$|\mathbf{p}_T| \equiv \sqrt{(p^1)^2 + (p^2)^2}. \quad (3.106)$$

“横向”指的是它们在三维空间中与 \mathbf{p} 垂直，即

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 1) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|} [(p^1)^2 p^3 + (p^2)^2 p^3 - p^3 |\mathbf{p}_T|^2] = 0, \quad (3.107)$$

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|} (-p^1 p^2 + p^2 p^1) = 0. \quad (3.108)$$

此外，存在如下关系：

$$\begin{aligned} \mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 1) &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} [(p^1)^2 (p^3)^2 + (p^2)^2 (p^3)^2 + |\mathbf{p}_T|^4] \\ &= \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 [(p^3)^2 + |\mathbf{p}_T|^2] = \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} |\mathbf{p}_T|^2 |\mathbf{p}|^2 = 1, \end{aligned} \quad (3.109)$$

$$\mathbf{e}(\mathbf{p}, 2) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}_T|^2} [(p^2)^2 + (p^1)^2] = \frac{1}{|\mathbf{p}_T|^2} |\mathbf{p}_T|^2 = 1, \quad (3.110)$$

$$\mathbf{e}(\mathbf{p}, 1) \cdot \mathbf{e}(\mathbf{p}, 2) = \frac{1}{|\mathbf{p}||\mathbf{p}_T|^2} (-p^1 p^3 p^2 + p^2 p^3 p^1) = 0. \quad (3.111)$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = \delta_{ij}, \quad i, j = 1, 2. \quad (3.112)$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_{\mu} e^{\mu}(\mathbf{p}, 1) = p_{\mu} e^{\mu}(\mathbf{p}, 2) = 0, \quad (3.113)$$

和正交归一关系

$$e_{\mu}(\mathbf{p}, i) e^{\mu}(\mathbf{p}, j) = -\mathbf{e}(\mathbf{p}, i) \cdot \mathbf{e}(\mathbf{p}, j) = -\delta_{ij} = g_{ij}. \quad (3.114)$$

接着, 要求第 3 个类空极化矢量 $e^\mu(\mathbf{p}, 3)$ 是纵向的, 即在三维空间中与 \mathbf{p} 平行。这样还不能确定它的时间分量, 为此, 我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p}, 3) = 0$, 而正交归一关系 (3.98) 将决定它的归一化。于是, 纵向极化矢量的形式为

$$e^\mu(\mathbf{p}, 3) = \left(\frac{|\mathbf{p}|}{m}, \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} \right). \quad (3.115)$$

可以验证, 它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p}, 3) = p^0 \frac{|\mathbf{p}|}{m} - \mathbf{p} \cdot \frac{p^0 \mathbf{p}}{m|\mathbf{p}|} = \frac{p^0 |\mathbf{p}|}{m} - \frac{p^0 |\mathbf{p}|}{m} = 0, \quad (3.116)$$

和正交归一关系

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, 3) = \frac{|\mathbf{p}|}{m} \frac{|\mathbf{p}|}{m} - \frac{(p^0)^2 \mathbf{p} \cdot \mathbf{p}}{m^2 |\mathbf{p}|^2} = \frac{|\mathbf{p}|^2}{m^2} - \frac{(p^0)^2}{m^2} = -\frac{(p^0)^2 - |\mathbf{p}|^2}{m^2} = -1 = g_{33}; \quad (3.117)$$

$$e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, i) = -\frac{p^0}{m|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.118)$$

最后, 我们可以将类时极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p}, 0) = \frac{1}{m} p^\mu = \frac{1}{m} (p^0, \mathbf{p}). \quad (3.119)$$

它满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = \frac{p^2}{m^2} = 1 = g_{00}; \quad (3.120)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, i) = -\frac{1}{m} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2; \quad (3.121)$$

$$e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 3) = \frac{1}{m^2} p^0 |\mathbf{p}| - \frac{p^0}{m^2 |\mathbf{p}|} \mathbf{p} \cdot \mathbf{p} = 0. \quad (3.122)$$

不过, 它不满足四维时空的横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = \frac{p^2}{m} = m. \quad (3.123)$$

可以验证, 由 (3.104)、(3.105)、(3.115) 和 (3.119) 式定义的这组极化矢量确实满足完备性关系 (3.103):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= \frac{1}{m^2} \begin{pmatrix} p^0 p^0 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^1 p^0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ -p^2 p^0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ -p^3 p^0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{m^2} \begin{pmatrix} |\mathbf{p}|^2 & -p^0 p^1 & -p^0 p^2 & -p^0 p^3 \\ -p^0 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^1 p^3 \\ -p^0 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^2 p^3 \\ -p^0 p^3 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2} p^3 p^3 \end{pmatrix} \\
& = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(p^1)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^1 p^3)^2 + (p^2)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2]}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{(p^2)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{(p^2 p^3)^2 + (p^1)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] + \frac{p^2 p^3}{|\mathbf{p}|^2} & \frac{(p^3)^2}{m^2} \left[1 - \frac{(p^0)^2}{|\mathbf{p}|^2} \right] - \frac{|\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \\
& = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2 |\mathbf{p}_T|^2 + (p^1)^2 (|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^2)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{p^1 p^2}{|\mathbf{p}|^2} + \frac{p^1 p^2}{|\mathbf{p}|^2} & -\frac{p^1 p^3}{|\mathbf{p}|^2} + \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1 p^2}{|\mathbf{p}|^2} + \frac{p^1 p^2}{|\mathbf{p}|^2} & -\frac{(p^2)^2 |\mathbf{p}_T|^2 + (p^2)^2 (|\mathbf{p}|^2 - |\mathbf{p}_T|^2) + (p^1)^2 |\mathbf{p}|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{p^2 p^3}{|\mathbf{p}|^2} + \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1 p^3}{|\mathbf{p}|^2} + \frac{p^1 p^3}{|\mathbf{p}|^2} & -\frac{p^2 p^3}{|\mathbf{p}|^2} + \frac{p^2 p^3}{|\mathbf{p}|^2} & -\frac{(p^3)^2 + |\mathbf{p}_T|^2}{|\mathbf{p}|^2} \end{pmatrix} \\
& = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \tag{3.124}
\end{aligned}$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91), 正能解模式 (3.97) 应满足

$$0 = \partial_\mu A^\mu(x; \mathbf{p}, \sigma) = -ip_\mu e^\mu(\mathbf{p}, \sigma) \exp(-ip \cdot x), \tag{3.125}$$

即

$$p_\mu e^\mu(\mathbf{p}, \sigma) = 0. \tag{3.126}$$

也就是说, 描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此, 类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 不能用于描述有质量矢量场 A^μ 。这说明 A^μ 只有 3 个物理的极化状态, 由类空的极化矢量 $e^\mu(\mathbf{p}, 1)$ 、 $e^\mu(\mathbf{p}, 2)$ 和 $e^\mu(\mathbf{p}, 3)$ 描述。根据完备性关系 (3.103), 这 3 个物理的极化矢量满足

$$-\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = \sum_{\sigma=1}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) = g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}, \tag{3.127}$$

即具有求和关系

$$\sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{3.128}$$

通过如下线性组合, 我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p, \lambda)$, 其中 $\lambda = +, 0, -$:

$$\varepsilon^\mu(\mathbf{p}, \pm) \equiv \frac{1}{\sqrt{2}} [\mp e^\mu(\mathbf{p}, 1) - ie^\mu(\mathbf{p}, 2)], \tag{3.129}$$

$$\varepsilon^\mu(\mathbf{p}, 0) \equiv e^\mu(\mathbf{p}, 3). \quad (3.130)$$

这样定义的 $\varepsilon^\mu(p, \pm)$ 是复的, 而 $\varepsilon^\mu(p, 0)$ 是实的。它们都满足四维横向条件

$$p_\mu \varepsilon^\mu(\mathbf{p}, \lambda) = 0. \quad (3.131)$$

它们还满足

$$\begin{aligned} \varepsilon_\mu^*(\mathbf{p}, \pm) \varepsilon^\mu(\mathbf{p}, \pm) &= \frac{1}{2} [\mp e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [\mp e^\mu(\mathbf{p}, 1) - i e^\mu(\mathbf{p}, 2)] \\ &= \frac{1}{2} e_\mu(\mathbf{p}, 1) e^\mu(\mathbf{p}, 1) + \frac{1}{2} e_\mu(\mathbf{p}, 2) e^\mu(\mathbf{p}, 2) = \frac{1}{2} (g_{11} + g_{22}) = -1, \end{aligned} \quad (3.132)$$

$$\begin{aligned} \varepsilon_\mu^*(\mathbf{p}, \pm) \varepsilon^\mu(\mathbf{p}, \mp) &= \frac{1}{2} [\mp e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [\pm e^\mu(\mathbf{p}, 1) - i e^\mu(\mathbf{p}, 2)] \\ &= -\frac{1}{2} e_\mu(\mathbf{p}, 1) e^\mu(\mathbf{p}, 1) + \frac{1}{2} e_\mu(\mathbf{p}, 2) e^\mu(\mathbf{p}, 2) = \frac{1}{2} (-g_{11} + g_{22}) = 0, \end{aligned} \quad (3.133)$$

$$\varepsilon_\mu^*(\mathbf{p}, 0) \varepsilon^\mu(\mathbf{p}, 0) = e_\mu(\mathbf{p}, 3) e^\mu(\mathbf{p}, 3) = -1, \quad (3.134)$$

$$\varepsilon_\mu^*(\mathbf{p}, \pm) \varepsilon^\mu(\mathbf{p}, 0) = \frac{1}{2} [\mp e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] e^\mu(\mathbf{p}, 3) = 0, \quad (3.135)$$

即具有正交归一关系

$$\varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon^\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}. \quad (3.136)$$

极化矢量求和关系则是

$$\begin{aligned} \sum_{\lambda=\pm,0} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2} [e_\mu(p, 1) + i e_\mu(p, 2)] [e_\nu(p, 1) - i e_\nu(p, 2)] \\ &\quad + \frac{1}{2} [-e_\mu(p, 1) + i e_\mu(p, 2)] [-e_\nu(p, 1) - i e_\nu(p, 2)] + e_\mu(p, 3) e_\nu(p, 3) \\ &= e_\mu(p, 1) e_\nu(p, 1) + e_\mu(p, 2) e_\nu(p, 2) + e_\mu(p, 3) e_\nu(p, 3) \\ &= \sum_{\sigma=1}^3 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma), \end{aligned} \quad (3.137)$$

与 (3.128) 式左边相等, 故

$$\sum_{\lambda=\pm,0} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \quad (3.138)$$

四维横向条件 (3.131) 在上式中体现为

$$p^\nu \sum_{\lambda=\pm,0} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu + \frac{p_\mu p^2}{m^2} = -p_\mu + p_\mu = 0. \quad (3.139)$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 \mathbf{p} 的方向由 $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$ 表征, 于是, 在 Lorentz 群矢量表示中, 螺旋度矩阵定义为

$$\hat{\mathbf{p}} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathcal{J} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} 0 & & & \\ & 0 & -ip^3 & ip^2 \\ & ip^3 & 0 & -ip^1 \\ & -ip^2 & ip^1 & 0 \end{pmatrix}. \quad (3.140)$$

这里已经使用了 \mathcal{J} 的矩阵表达式 (3.77)、(3.78) 和 (3.79)。将 (3.105) 和 (3.115) 式代入 (3.129) 和 (3.130) 式, 得到 $\varepsilon^\mu(p, \lambda)$ 的列矢量形式为

$$\begin{aligned}\varepsilon^\mu(p, 0) &= \frac{1}{m|\mathbf{p}|} \begin{pmatrix} |\mathbf{p}|^2 \\ p^0 p^1 \\ p^0 p^2 \\ p^0 p^3 \end{pmatrix}, \quad \varepsilon^\mu(p, +) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 + ip^2 |\mathbf{p}| \\ -p^2 p^3 - ip^1 |\mathbf{p}| \\ |\mathbf{p}_T|^2 \end{pmatrix}, \\ \varepsilon^\mu(p, -) &= \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_T|} \begin{pmatrix} 0 \\ p^1 p^3 + ip^2 |\mathbf{p}| \\ p^2 p^3 - ip^1 |\mathbf{p}| \\ -|\mathbf{p}_T|^2 \end{pmatrix}.\end{aligned}\quad (3.141)$$

从而, 可得

$$(\hat{\mathbf{p}} \cdot \mathcal{J})\varepsilon^\mu(p, 0) = \frac{1}{m|\mathbf{p}|^2} \begin{pmatrix} 0 \\ -ip^3 p^0 p^2 + ip^2 p^0 p^3 \\ ip^3 p^0 p^1 - ip^1 p^0 p^3 \\ -ip^2 p^0 p^1 + ip^1 p^0 p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \varepsilon^\mu(p, 0), \quad (3.142)$$

$$\begin{aligned}(\hat{\mathbf{p}} \cdot \mathcal{J})\varepsilon^\mu(p, +) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}_T|^2 \\ -ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}_T|^2 \\ ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| - ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| + ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| - ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = +\varepsilon^\mu(p, +),\end{aligned}\quad (3.143)$$

$$\begin{aligned}(\hat{\mathbf{p}} \cdot \mathcal{J})\varepsilon^\mu(p, -) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -ip^2(p^3)^2 - p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}_T|^2 \\ ip^1(p^3)^2 - p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}_T|^2 \\ -ip^1 p^2 p^3 + (p^2)^2 |\mathbf{p}| + ip^1 p^2 p^3 + (p^1)^2 |\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_T|} \begin{pmatrix} 0 \\ -p^1 p^3 |\mathbf{p}| - ip^2 |\mathbf{p}|^2 \\ -p^2 p^3 |\mathbf{p}| + ip^1 |\mathbf{p}|^2 \\ |\mathbf{p}_T|^2 |\mathbf{p}| \end{pmatrix} = -\varepsilon^\mu(p, -).\end{aligned}\quad (3.144)$$

归纳起来, 有

$$(\hat{\mathbf{p}} \cdot \mathcal{J})\varepsilon^\mu(p, \lambda) = \lambda \varepsilon^\mu(p, \lambda). \quad (3.145)$$

上式说明极化矢量 $\varepsilon^\mu(p, \lambda)$ 是螺旋度的本征态, 本征值为 λ 。因此, $\varepsilon^\mu(p, \lambda)$ 描述动量为 \mathbf{p} 、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda = \pm 1$ 对应于两种横向极化, $\lambda = 0$ 对应于纵向极化。

有质量的实矢量场算符 $A^\mu(\mathbf{x}, t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态, 形式为

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right], \quad (3.146)$$

其中 $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ 。容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.147)$$

根据 (3.95) 式, 共轭动量密度为

$$\begin{aligned} \pi_i = -\partial_0 A_i + \partial_i A_0 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left\{ [ip_0 \varepsilon_i(\mathbf{p}, \lambda) - ip_i \varepsilon_0(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda} e^{-ip \cdot x} \right. \\ \left. + [-ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) + ip_i \varepsilon_0^*(\mathbf{p}, \lambda)] a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right\}, \end{aligned} \quad (3.148)$$

引入

$$\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \equiv \varepsilon_i(\mathbf{p}, \lambda) - \frac{p_i}{p_0} \varepsilon_0(\mathbf{p}, \lambda), \quad (3.149)$$

则有

$$p_0 \varepsilon_i(\mathbf{p}, \lambda) - p_i \varepsilon_0(\mathbf{p}, \lambda) = p_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda), \quad (3.150)$$

从而, 可以将共轭动量密度的平面波展开式写得更加紧凑:

$$\pi_i(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right]. \quad (3.151)$$

易见, 它也满足自共轭条件

$$[\pi_i(\mathbf{x}, t)]^\dagger = \pi_i(\mathbf{x}, t). \quad (3.152)$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned} & \int d^3x e^{iq \cdot x} A^\mu \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} + \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq^0 t} \right] \end{aligned} \quad (3.153)$$

和

$$\int d^3x e^{iq \cdot x} \partial_0 A^\mu = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p-q) \cdot x} - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p+q) \cdot x} \right]$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\
&\quad \left. - \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} - \varepsilon^{\mu*}(-\mathbf{q}, \lambda) a_{-\mathbf{q},\lambda}^\dagger e^{2iq^0t} \right], \tag{3.154}
\end{aligned}$$

以及正交归一关系 (3.136), 可得

$$\begin{aligned}
\varepsilon_\mu^*(\mathbf{q}, \lambda') \int d^3x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon_\mu^*(\mathbf{q}, \lambda') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{q}, \lambda) a_{\mathbf{q},\lambda} \\
&= -i\sqrt{2E_{\mathbf{q}}} \sum_{\lambda=\pm,0} (-\delta_{\lambda'\lambda}) a_{\mathbf{q},\lambda} = i\sqrt{2E_{\mathbf{q}}} a_{\mathbf{q},\lambda'}. \tag{3.155}
\end{aligned}$$

由 Lorenz 条件 (3.91) 可得

$$\partial_0 A^0 = -\partial_i A^i, \tag{3.156}$$

根据 (3.95) 式, 有

$$\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \tag{3.157}$$

于是, 湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \varepsilon_\mu^*(\mathbf{p}, \lambda) \int d^3x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu) \\
&= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\varepsilon_0^*(\mathbf{p}, \lambda) \partial_0 A^0 + \varepsilon_i^*(\mathbf{p}, \lambda) \partial_0 A^i - ip_0 \varepsilon_\mu^*(\mathbf{p}, \lambda) A^\mu] \\
&= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0 \\
&\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i]. \tag{3.158}
\end{aligned}$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$\begin{aligned}
\int d^3x e^{ip \cdot x} [-\varepsilon_0^*(\mathbf{p}, \lambda) \partial_i A^i - \varepsilon_i^*(\mathbf{p}, \lambda) \partial_i A^0] &= \int d^3x [\varepsilon_0^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^i + \varepsilon_i^*(\mathbf{p}, \lambda) (\partial_i e^{ip \cdot x}) A^0] \\
&= \int d^3x [ip_i \varepsilon_0^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) e^{ip \cdot x} A^0] \\
&= \int d^3x e^{ip \cdot x} [i\varepsilon_0^*(\mathbf{p}, \lambda) p_i A^i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0], \tag{3.159}
\end{aligned}$$

从而, 有

$$\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [i\varepsilon_0^*(\mathbf{p}, \lambda) p_i A^i + \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i + ip_i \varepsilon_i^*(\mathbf{p}, \lambda) A^0 \\
&\quad - ip_0 \varepsilon_0^*(\mathbf{p}, \lambda) A^0 - ip_0 \varepsilon_i^*(\mathbf{p}, \lambda) A^i] \\
&= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \{ \varepsilon_i^*(\mathbf{p}, \lambda) \pi_i - ip^\mu \varepsilon_\mu^*(\mathbf{p}, \lambda) A^0 - i[p_0 \varepsilon_i^*(\mathbf{p}, \lambda) - p_i \varepsilon_0^*(\mathbf{p}, \lambda)] A^i \}. \tag{3.160}
\end{aligned}$$

再利用四维横向条件 (3.131) 和 (3.150) 式, 得到

$$\begin{aligned} a_{\mathbf{p},\lambda} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [-\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(x) - ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(x)] \\ &= \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} [\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(x) + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(x)]. \end{aligned} \quad (3.161)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p},\lambda}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} [\varepsilon^i(\mathbf{p}, \lambda) \pi_i(x) - ip_0 \tilde{\varepsilon}_i(\mathbf{p}, \lambda) A^i(x)]. \quad (3.162)$$

利用等时对易关系 (3.96), 可得湮灭算符与产生算符的对易关系为

$$\begin{aligned} &[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [\varepsilon^{i*}(\mathbf{p}, \lambda) \pi_i(\mathbf{x}, t) + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) A^i(\mathbf{x}, t), \\ &\quad \varepsilon^j(\mathbf{q}, \lambda') \pi_j(\mathbf{y}, t) - iq_0 \tilde{\varepsilon}_j(\mathbf{q}, \lambda') A^j(\mathbf{y}, t)] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \{ -iq_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') [\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \\ &\quad + ip_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') [A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)] \} \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \delta^{(3)}(\mathbf{x} - \mathbf{y}) [-q_0 \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_j(\mathbf{q}, \lambda') \delta_i^j - p_0 \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^j(\mathbf{q}, \lambda') \delta_j^i] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} [-E_{\mathbf{q}} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{q}, \lambda') - E_{\mathbf{p}} \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda')] \\ &= -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) [\varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda')]. \end{aligned} \quad (3.163)$$

根据定义式 (3.149)、四维横向条件 (3.131) 和正交归一关系 (3.136), 有

$$\begin{aligned} \varepsilon^{i*}(\mathbf{p}, \lambda) \tilde{\varepsilon}_i(\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') - \frac{1}{p_0} p_i \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\ &= \varepsilon^{i*}(\mathbf{p}, \lambda) \varepsilon_i(\mathbf{p}, \lambda') + \frac{1}{p_0} p_0 \varepsilon^{0*}(\mathbf{p}, \lambda) \varepsilon_0(\mathbf{p}, \lambda') \\ &= \varepsilon^{\mu*}(\mathbf{p}, \lambda) \varepsilon_\mu(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}, \end{aligned} \quad (3.164)$$

取复共轭, 可得

$$\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') = -\delta_{\lambda\lambda'}. \quad (3.165)$$

于是,

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = -\frac{1}{2} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) (-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (3.166)$$

另一方面,

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}]$$

$$\begin{aligned}
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [\varepsilon^{i*}(\mathbf{p}, \lambda)\pi_i(\mathbf{x}, t) + ip_0\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)A^i(\mathbf{x}, t), \\
&\quad \varepsilon^{j*}(\mathbf{q}, \lambda')\pi_j(\mathbf{y}, t) + iq_0\tilde{\varepsilon}_j^*(\mathbf{q}, \lambda')A^j(\mathbf{y}, t)] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \{iq_0\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_j^*(\mathbf{q}, \lambda')[\pi_i(\mathbf{x}, t), A^j(\mathbf{y}, t)] \\
&\quad + ip_0\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{j*}(\mathbf{q}, \lambda')[A^i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)]\} \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \delta^{(3)}(\mathbf{x} - \mathbf{y}) [q_0\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_j^*(\mathbf{q}, \lambda')\delta^j_i - p_0\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{j*}(\mathbf{q}, \lambda')\delta^i_j] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(p^0+q^0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} [E_{\mathbf{q}}\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_i^*(\mathbf{q}, \lambda') - E_{\mathbf{p}}\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(\mathbf{q}, \lambda')] \\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p} + \mathbf{q})e^{2iE_{\mathbf{p}}t} [\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda')]. \tag{3.167}
\end{aligned}$$

对四维横向条件 (3.131) 取复共轭, 得

$$p_\mu\varepsilon^{\mu*}(\mathbf{p}, \lambda) = p_0\varepsilon^{0*}(\mathbf{p}, \lambda) + p_i\varepsilon^{i*}(\mathbf{p}, \lambda) = 0. \tag{3.168}$$

将上式中的 \mathbf{p} 替换成 $-\mathbf{p}$, 得

$$p_0\varepsilon^{0*}(-\mathbf{p}, \lambda) - p_i\varepsilon^{i*}(-\mathbf{p}, \lambda) = 0. \tag{3.169}$$

因此, 有

$$p_i\varepsilon^{i*}(\mathbf{p}, \lambda) = -p_0\varepsilon^{0*}(\mathbf{p}, \lambda), \quad -p_i\varepsilon^{i*}(-\mathbf{p}, \lambda) = -p_0\varepsilon^{0*}(-\mathbf{p}, \lambda), \tag{3.170}$$

或者写成

$$\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) = p_0\varepsilon^{0*}(\mathbf{p}, \lambda), \quad -\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda) = p_0\varepsilon^{0*}(-\mathbf{p}, \lambda). \tag{3.171}$$

从而, 可得

$$\begin{aligned}
\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') &= \varepsilon^{i*}(\mathbf{p}, \lambda) \left[\varepsilon_i^*(-\mathbf{p}, \lambda) + \frac{p_i}{p_0}\varepsilon_0^*(-\mathbf{p}, \lambda) \right] \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda)\varepsilon_i^*(-\mathbf{p}, \lambda') + \frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p}, \lambda)\varepsilon_0^*(-\mathbf{p}, \lambda') \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda)\varepsilon_i^*(-\mathbf{p}, \lambda') - \frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p}, \lambda)\varepsilon_0^*(-\mathbf{p}, \lambda') \\
&= \varepsilon^{i*}(\mathbf{p}, \lambda)\varepsilon_i^*(-\mathbf{p}, \lambda') - \varepsilon^{0*}(\mathbf{p}, \lambda)\varepsilon_0^*(-\mathbf{p}, \lambda'), \tag{3.172}
\end{aligned}$$

$$\begin{aligned}
\tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda') &= \left[\varepsilon_i^*(\mathbf{p}, \lambda) - \frac{p_i}{p_0}\varepsilon_0^*(\mathbf{p}, \lambda) \right] \varepsilon^{i*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0}\varepsilon_0^*(\mathbf{p}, \lambda)p_i\varepsilon^{i*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda') - \frac{1}{p_0}\varepsilon_0^*(\mathbf{p}, \lambda)p_0\varepsilon^{0*}(-\mathbf{p}, \lambda') \\
&= \varepsilon_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda)\varepsilon^{0*}(-\mathbf{p}, \lambda'). \tag{3.173}
\end{aligned}$$

可见, $\varepsilon^{i*}(\mathbf{p}, \lambda)\tilde{\varepsilon}_i^*(-\mathbf{p}, \lambda') - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda)\varepsilon^{i*}(-\mathbf{p}, \lambda') = 0$, 故

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = 0. \tag{3.174}$$

综上, 产生湮灭算符的对易关系为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0. \quad (3.175)$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \quad (3.176)$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \quad (3.177)$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \quad (3.178)$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$, 因此,

$$A^0 = -\frac{1}{m^2} \partial_\mu F^{\mu 0} = -\frac{1}{m^2} \partial_i F^{i0} = -\frac{1}{m^2} \partial_i \pi^i = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.179)$$

从而, 可得

$$-\boldsymbol{\pi} \cdot \dot{\mathbf{A}} = \boldsymbol{\pi} \cdot (\boldsymbol{\pi} + \nabla A_0) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) - A_0 (\nabla \cdot \boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2. \quad (3.180)$$

另一方面,

$$\frac{1}{2} F_{0i} F^{0i} = \frac{1}{2} \pi_i \pi^i = -\frac{1}{2} \boldsymbol{\pi}^2. \quad (3.181)$$

利用 (1.84) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) \partial^m A^n = \varepsilon^{ijk} \varepsilon^{kmn} \partial^m A^n = -\varepsilon^{ijk} \varepsilon^{kmn} \partial_m A^n, \quad (3.182)$$

从而,

$$\begin{aligned} \frac{1}{4} F_{ij} F^{ij} &= \frac{1}{4} F^{ij} F^{ij} = \frac{1}{4} \varepsilon^{ijk} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{ijl} \varepsilon^{lpq} \partial_p A^q = \frac{1}{4} 2 \delta^{kl} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{lpq} \partial_p A^q \\ &= \frac{1}{2} \varepsilon^{kmn} (\partial_m A^n) \varepsilon^{kpq} \partial_p A^q = \frac{1}{2} (\nabla \times \mathbf{A})^2. \end{aligned} \quad (3.183)$$

于是, 有

$$\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} = -\frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2. \quad (3.184)$$

根据 (1.119) 式, 有质量矢量场的哈密顿量密度为

$$\begin{aligned} \mathcal{H} &= \pi_i \partial_0 A^i - \mathcal{L} = \pi_i \partial_0 A^i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \\ &= -\boldsymbol{\pi} \cdot \dot{\mathbf{A}} - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) \end{aligned}$$

$$\begin{aligned}
&= \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 - \frac{1}{2} \boldsymbol{\pi}^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 - \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} m^2 \mathbf{A}^2 \\
&= \frac{1}{2} \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2.
\end{aligned} \tag{3.185}$$

上式最后一行第二项是一个全散度，对全空间积分时它没有贡献。于是，哈密顿量为

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x \left[\boldsymbol{\pi}^2 + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 + (\nabla \times \mathbf{A})^2 + m^2 \mathbf{A}^2 \right]. \tag{3.186}$$

下面逐项进行计算。

哈密顿量的第一项是

$$\begin{aligned}
&\frac{1}{2} \int d^3x \boldsymbol{\pi}^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0)(iq_0) \left[\tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} - \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \cdot \left[\tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[-\tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad - \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} + \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 q_0}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} E_{\mathbf{p}}^2 \left[\tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
&\quad \left. - \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} - \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\epsilon}}^*(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \tag{3.187}
\end{aligned}$$

第二项是

$$\begin{aligned}
&\frac{1}{2} \int d^3x \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 \\
&= \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \frac{(ip_0)(iq_0)}{m^2} \left[i\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + i\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
&\quad \times \left[i\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} + i\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 q_0}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \frac{1}{m^2} \left\{ -[\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \right. \\
&\quad \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{p}, \lambda)][\mathbf{q} \cdot \tilde{\boldsymbol{\epsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right\}
\end{aligned}$$

$$\begin{aligned}
& - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \Big\} \\
& = \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q p_0 q_0}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}} m^2} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
& \quad \left. \left. + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right) \right. \\
& \quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
& \quad \left. \left. + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{q} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \right\} \\
& = \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \frac{E_{\mathbf{p}}^2}{m^2} \left\{ [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger \right. \\
& \quad + [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \\
& \quad \left. - [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \tag{3.188}
\end{aligned}$$

第三项是

$$\begin{aligned}
& \frac{1}{2} \int d^3 x (\nabla \times \mathbf{A})^2 \\
& = \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[i\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} - i\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right] \\
& \quad \cdot \left[i\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - i\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
& = \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
& \quad + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} - [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\
& \quad \left. - [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right\} \\
& = \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
& \quad \left. \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right) \right. \\
& \quad \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left([\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
& \quad \left. \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{q} \times \boldsymbol{\varepsilon}^*(\mathbf{q}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right) \right\} \\
& = \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \left\{ [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger \right. \\
& \quad + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} + [\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \\
& \quad \left. + [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(-\mathbf{p}, \lambda')] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \tag{3.189}
\end{aligned}$$

第四项是

$$\begin{aligned}
& \frac{1}{2} \int d^3 x m^2 \mathbf{A}^2 \\
& = \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right]
\end{aligned}$$

$$\begin{aligned}
& \cdot \left[\varepsilon(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} + \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3x d^3p d^3q m^2}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\
&\quad + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} + \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \\
&\quad \left. + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= \frac{1}{2} \sum_{\lambda \lambda'} \int \frac{d^3p d^3q m^2}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\
&\quad \left. \left. + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\
&\quad \left. \left. + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= \sum_{\lambda \lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} m^2 \left[\varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
&\quad \left. + \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + \varepsilon^*(\mathbf{p}, \lambda) \cdot \varepsilon^*(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.190)
\end{aligned}$$

综合起来，哈密顿量化为

$$\begin{aligned}
H = \sum_{\lambda \lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} & \left[f_1(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger + f_1^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right. \\
& \left. + f_2(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} + f_2^*(\mathbf{p}, \lambda, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right], \quad (3.191)
\end{aligned}$$

其中，

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') &\equiv E_{\mathbf{p}}^2 \tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda') + \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda')] \\
&\quad + [\mathbf{p} \times \varepsilon(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \varepsilon^*(\mathbf{p}, \lambda')] + m^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda'), \quad (3.192)
\end{aligned}$$

$$\begin{aligned}
f_2(\mathbf{p}, \lambda, \lambda') &\equiv -E_{\mathbf{p}}^2 \tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}(-\mathbf{p}, \lambda') - \frac{E_{\mathbf{p}}^2}{m^2} [\mathbf{p} \cdot \tilde{\varepsilon}(\mathbf{p}, \lambda)] [\mathbf{p} \cdot \tilde{\varepsilon}(-\mathbf{p}, \lambda')] \\
&\quad + [\mathbf{p} \times \varepsilon(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \varepsilon(-\mathbf{p}, \lambda')] + m^2 \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon(-\mathbf{p}, \lambda'). \quad (3.193)
\end{aligned}$$

现在，我们计算 $f_1(\mathbf{p}, \lambda, \lambda')$ 。由 (3.149)、(3.171) 和 (3.136) 式，可得

$$\begin{aligned}
\tilde{\varepsilon}(\mathbf{p}, \lambda) \cdot \tilde{\varepsilon}^*(\mathbf{p}, \lambda') &= \left[\varepsilon(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\varepsilon^*(\mathbf{p}, \lambda') - \frac{\mathbf{p}}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right] \\
&= \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \varepsilon(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \varepsilon(\mathbf{p}, \lambda) \cdot \varepsilon^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon_0^*(\mathbf{p}, \lambda') - \frac{\varepsilon_0^*(\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon_0(\mathbf{p}, \lambda) + \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= -\varepsilon_\mu(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') + \left(\frac{|\mathbf{p}|^2}{p_0^2} - 1 \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \delta_{\lambda \lambda'} - \frac{m^2}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \quad (3.194)
\end{aligned}$$

另一方面,

$$\begin{aligned}
& [\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p}, \lambda')] \\
&= \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0^*(\mathbf{p}, \lambda') \right] \\
&= [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(p_0^2 - 2|\mathbf{p}|^2 + \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \left[p_0^2 - |\mathbf{p}|^2 + \frac{|\mathbf{p}|^2}{p_0^2} (|\mathbf{p}|^2 - p_0^2) \right] \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&= \left(m^2 - m^2 \frac{|\mathbf{p}|^2}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') = \frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda'). \tag{3.195}
\end{aligned}$$

对于任意空间矢量 \mathbf{a} 和 \mathbf{b} , 利用 (1.84) 式, 有

$$\begin{aligned}
(\mathbf{p} \times \mathbf{a}) \cdot (\mathbf{p} \times \mathbf{b}) &= \varepsilon^{ijk} p^j a^k \varepsilon^{imn} p^m b^n = (\delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}) p^j a^k p^m b^n \\
&= p^j a^k p^j b^k - p^j a^k p^k b^j = |\mathbf{p}|^2 \mathbf{a} \cdot \mathbf{b} - (\mathbf{p} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}), \tag{3.196}
\end{aligned}$$

从而, 可得

$$\begin{aligned}
[\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda')] \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda'). \tag{3.197}
\end{aligned}$$

于是, (3.192) 式化为

$$\begin{aligned}
f_1(\mathbf{p}, \lambda, \lambda') &= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0^*(\mathbf{p}, \lambda') \\
&\quad + |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} + E_{\mathbf{p}}^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}^*(\mathbf{p}, \lambda') - E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^{0*}(\mathbf{p}, \lambda') \\
&= E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} - E_{\mathbf{p}}^2 \varepsilon_{\mu}(\mathbf{p}, \lambda) \varepsilon^{\mu*}(\mathbf{p}, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.198}
\end{aligned}$$

因此,

$$f_1(\mathbf{p}, \lambda, \lambda') = f_1^*(\mathbf{p}, \lambda, \lambda') = 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'}. \tag{3.199}$$

接着, 我们计算 $f_2(\mathbf{p}, \lambda, \lambda')$ 。由 (3.149) 和 (3.171) 式, 可得

$$\tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda) \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda') = \left[\boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{\mathbf{p}}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \cdot \left[\boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\mathbf{p}}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right]$$

$$\begin{aligned}
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} \mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') + \frac{\varepsilon_0(-\mathbf{p}, \lambda')}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{1}{E_{\mathbf{p}}^2} (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'). \tag{3.200}
\end{aligned}$$

另一方面,

$$\begin{aligned}
&[\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \tilde{\boldsymbol{\varepsilon}}(-\mathbf{p}, \lambda')] \\
&= \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) \right] \left[\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(-\mathbf{p}, \lambda') \right] \\
&= [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] + \frac{|\mathbf{p}|^2}{p_0} [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) [\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= -p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(-\mathbf{p}, \lambda') + \frac{|\mathbf{p}|^2}{p_0} p_0 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad + \frac{|\mathbf{p}|^2}{p_0} \varepsilon_0(\mathbf{p}, \lambda) p_0 \varepsilon^0(-\mathbf{p}, \lambda') - \frac{|\mathbf{p}|^4}{p_0^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= \left(-p_0^2 + 2|\mathbf{p}|^2 - \frac{|\mathbf{p}|^4}{p_0^2} \right) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = -\frac{1}{E_{\mathbf{p}}^2} (E_{\mathbf{p}}^2 - |\mathbf{p}|^2)^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&= -\frac{m^4}{E_{\mathbf{p}}^2} \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda'), \tag{3.201}
\end{aligned}$$

而

$$\begin{aligned}
[\mathbf{p} \times \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)] \cdot [\mathbf{p} \times \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] &= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - [\mathbf{p} \cdot \boldsymbol{\varepsilon}(\mathbf{p}, \lambda)][\mathbf{p} \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda')] \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + p_0 \varepsilon^0(\mathbf{p}, \lambda) p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\
&= |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'). \tag{3.202}
\end{aligned}$$

于是, (3.193) 式化为

$$\begin{aligned}
f_2(\mathbf{p}, \lambda, \lambda') &= -E_{\mathbf{p}}^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') - (2E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') + m^2 \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') \\
&\quad + |\mathbf{p}|^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') + E_{\mathbf{p}}^2 \varepsilon^0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda') + m^2 \boldsymbol{\varepsilon}(\mathbf{p}, \lambda) \cdot \boldsymbol{\varepsilon}(-\mathbf{p}, \lambda') \\
&= (-2E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 + E_{\mathbf{p}}^2) \varepsilon_0(\mathbf{p}, \lambda) \varepsilon_0(-\mathbf{p}, \lambda') = 0. \tag{3.203}
\end{aligned}$$

因此,

$$f_2(\mathbf{p}, \lambda, \lambda') = f_2^*(\mathbf{p}, \lambda, \lambda') = 0. \tag{3.204}$$

将 (3.199) 和 (3.204) 式代入 (3.191) 式, 再利用产生湮灭算符的对易关系 (3.175), 可得有质量矢量场的哈密顿量为

$$H = \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \delta_{\lambda\lambda'} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right) = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \left(a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}^\dagger + a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} \right)$$

$$= \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} \frac{3}{2} E_{\mathbf{p}}. \quad (3.205)$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和，第二项是零点能。

根据 (1.158) 式，有质量矢量场的总动量为

$$\begin{aligned} \mathbf{P} &= - \int d^3x \pi_i \nabla A^i \\ &= - \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} (ip_0) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} \right] \\ &\quad \times \left[i\mathbf{q} \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - i\mathbf{q} \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\ &= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q p_0 \mathbf{q}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[- \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right. \\ &\quad - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \\ &\quad \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q p_0 \mathbf{q}}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left\{ - \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger e^{-i(p_0 - q_0)t} \right. \right. \\ &\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p_0 - q_0)t} \right] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0 + q_0)t} \right. \right. \\ &\quad \left. \left. + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\ &= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^{i*}(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'}^\dagger + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^i(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} \right. \\ &\quad \left. + \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right]. \quad (3.206) \end{aligned}$$

由 (3.149) 和 (3.170) 式可得

$$\begin{aligned} \tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') &= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_i \varepsilon^i(-\mathbf{p}, \lambda') \\ &= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \frac{\varepsilon_0(\mathbf{p}, \lambda)}{p_0} p_0 \varepsilon^0(-\mathbf{p}, \lambda') \\ &= \varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda'), \end{aligned} \quad (3.207)$$

从而，有

$$\begin{aligned} &- \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right] \\ &= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(\mathbf{p}, \lambda) \varepsilon^i(-\mathbf{p}, \lambda') - \varepsilon_0(\mathbf{p}, \lambda) \varepsilon^0(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\ &\quad \left. + [\varepsilon_i^*(\mathbf{p}, \lambda) \varepsilon^{i*}(-\mathbf{p}, \lambda') - \varepsilon_0^*(\mathbf{p}, \lambda) \varepsilon^{0*}(-\mathbf{p}, \lambda')] a_{\mathbf{p},\lambda}^\dagger a_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{-\mathbf{p}, \lambda'} a_{\mathbf{p}, \lambda} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{-\mathbf{p}, \lambda'}^\dagger a_{\mathbf{p}, \lambda}^\dagger e^{2iE_{\mathbf{p}}t} \right\} \\
&= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left\{ [\varepsilon_i(-\mathbf{p}, \lambda') \varepsilon^i(\mathbf{p}, \lambda) - \varepsilon_0(-\mathbf{p}, \lambda') \varepsilon^0(\mathbf{p}, \lambda)] a_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \right. \\
&\quad \left. + [\varepsilon_i^*(-\mathbf{p}, \lambda') \varepsilon^{i*}(\mathbf{p}, \lambda) - \varepsilon_0^*(-\mathbf{p}, \lambda') \varepsilon^{0*}(\mathbf{p}, \lambda)] a_{\mathbf{p}, \lambda}^\dagger a_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right\}. \quad (3.208)
\end{aligned}$$

上式第二步进行了 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换, 由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和, 这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.175)。留意到第一步与第三步的结果互为相反数, 可知上式为零。因此, (3.206) 式最后两行方括号中最后两项没有贡献。再利用 (3.165) 式, 可得

$$\begin{aligned}
\mathbf{P} &= - \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-\delta_{\lambda\lambda'} a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda'}^\dagger - \delta_{\lambda\lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} \right] = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[a_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda}^\dagger + a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} \right] \\
&= \sum_{\lambda=\pm, 0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda} + \frac{3}{2} \delta^{(3)}(\mathbf{0}) \int d^3p \mathbf{p} = \sum_{\lambda=\pm, 0} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda}. \quad (3.209)
\end{aligned}$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m = 0$ 时, 由 (3.104) 和 (3.105) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 的形式不变, 但 (3.115) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 的良好定义。实际上, 在满足正确归一化的条件下, $m = 0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面, 由于无质量矢量粒子的动量 p^μ 的内积为 $p^2 = 0$, 也不能像 (3.119) 式那样将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 取为正比于 p^μ 的矢量, 否则将出现 $e_\mu(\mathbf{p}, 0) e^\mu(\mathbf{p}, 0) = 0$ 而不能得到正确的归一化。因此, 我们需要重新定义 $e^\mu(\mathbf{p}, 3)$ 和 $e^\mu(\mathbf{p}, 0)$ 。

在用 (3.104) 和 (3.105) 式定义 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 时, 我们已经选取了一个特定的惯性参考系。在这个参考系中, 可以定义一个类时单位矢量

$$n^\mu = (1, 0, 0, 0), \quad (3.210)$$

它的 Lorentz 不变内积是

$$n^2 = 1. \quad (3.211)$$

然后, 将类时极化矢量 $e^\mu(\mathbf{p}, 0)$ 在此参考系中的形式就取为 n^μ , 即

$$e^\mu(\mathbf{p}, 0) = n^\mu. \quad (3.212)$$

$e^\mu(\mathbf{p}, 0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面, 纵向极化矢量 $e^\mu(\mathbf{p}, 3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式:

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n}. \quad (3.213)$$

$p^2 = (p^0)^2 - |\mathbf{p}|^2 = 0$ 表明

$$p^0 = |\mathbf{p}|, \quad (3.214)$$

从而, $e^\mu(\mathbf{p}, 3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n} = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right). \quad (3.215)$$

这样定义的 $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 满足正交归一关系 (3.98):

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 0) = n^2 = 1, \quad e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|^2} = -1; \quad (3.216)$$

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 1) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 2) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 3) = 0; \quad (3.217)$$

$$e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, i) = -\frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \quad (3.218)$$

此外, 可以验证, 由 (3.104)、(3.105)、(3.212) 和 (3.213) 式定义的这组极化矢量确实满足完备性关系 (3.103):

$$\begin{aligned} & \sum_{\sigma=0}^3 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\ &= e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^3 p^1 p^3 & p^1 p^3 p^2 p^3 & -p^1 p^3 |\mathbf{p}_T|^2 \\ 0 & p^2 p^3 p^1 p^3 & p^2 p^3 p^2 p^3 & -p^2 p^3 |\mathbf{p}_T|^2 \\ 0 & -|\mathbf{p}_T|^2 p^1 p^3 & -|\mathbf{p}_T|^2 p^2 p^3 & |\mathbf{p}_T|^4 \end{pmatrix} \\ &\quad - \frac{1}{|\mathbf{p}_T|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2 p^2 & -p^2 p^1 & 0 \\ 0 & -p^1 p^2 & p^1 p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1 p^1 & p^1 p^2 & p^1 p^3 \\ 0 & p^2 p^1 & p^2 p^2 & p^2 p^3 \\ 0 & p^3 p^1 & p^3 p^2 & p^3 p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2 (p^3)^2 + (p^2)^2 |\mathbf{p}|^2 + (p^1)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1 p^2 [(p^3)^2 - |\mathbf{p}|^2] + p^1 p^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & -\frac{(p^2)^2 (p^3)^2 + (p^1)^2 |\mathbf{p}|^2 + (p^2)^2 |\mathbf{p}_T|^2}{|\mathbf{p}|^2 |\mathbf{p}_T|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1 p^3}{|\mathbf{p}|^2} - \frac{p^1 p^3}{|\mathbf{p}|^2} & \frac{p^2 p^3}{|\mathbf{p}|^2} - \frac{p^2 p^3}{|\mathbf{p}|^2} & -\frac{|\mathbf{p}_T|^2 + (p^3)^2}{|\mathbf{p}|^2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}. \quad (3.219)$$

不过, $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 都不满足四维横向条件:

$$p_\mu e^\mu(\mathbf{p}, 0) = p \cdot n = p^0 = |\mathbf{p}|, \quad p_\mu e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|} = -|\mathbf{p}| = -p \cdot n. \quad (3.220)$$

横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$ 具有求和关系

$$\begin{aligned} -\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) &= \sum_{\sigma=1}^2 g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = g_{\mu\nu} - g_{00} e_\mu(\mathbf{p}, 0) e_\nu(\mathbf{p}, 0) - g_{33} e_\mu(\mathbf{p}, 3) e_\nu(\mathbf{p}, 3) \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu - (p \cdot n) n_\mu}{p \cdot n} \frac{p_\nu - (p \cdot n) n_\nu}{p \cdot n} \\ &= g_{\mu\nu} - n_\mu n_\nu + \frac{p_\mu p_\nu - (p \cdot n) p_\mu n_\nu - (p \cdot n) p_\nu n_\mu + (p \cdot n)^2 n_\mu n_\nu}{(p \cdot n)^2} \\ &= g_{\mu\nu} + \frac{p_\mu p_\nu}{(p \cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}, \end{aligned} \quad (3.221)$$

即

$$\sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.222)$$

根据 (3.129) 式, 作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 满足

$$\begin{aligned} \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) &= \frac{1}{2} [e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &\quad + \frac{1}{2} [-e_\mu(\mathbf{p}, 1) + i e_\mu(\mathbf{p}, 2)] [-e_\nu(\mathbf{p}, 1) - i e_\nu(\mathbf{p}, 2)] \\ &= e_\mu(\mathbf{p}, 1) e_\nu(\mathbf{p}, 1) + e_\mu(\mathbf{p}, 2) e_\nu(\mathbf{p}, 2) = \sum_{\sigma=1}^2 e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma), \end{aligned} \quad (3.223)$$

因而具有求和关系

$$\sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p \cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n}. \quad (3.224)$$

四维横向条件 $p_\mu \varepsilon^\mu(\mathbf{p}, \pm) = 0$ 在上式中体现为

$$p^\nu \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = -p_\mu - \frac{p_\mu p^2}{(p \cdot n)^2} + \frac{p_\mu (p \cdot n) + p^2 n_\mu}{p \cdot n} = -p_\mu + p_\mu = 0. \quad (3.225)$$

3.4.2 无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中, 令参数 $m = 0$, 就得到自由无质量实矢量场 $A^\mu(x)$ 的拉氏量

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (3.226)$$

其中 $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ 。同样, 令 Proca 方程中 $m = 0$, 就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu} = 0. \quad (3.227)$$

根据 1.5 节的讨论, 这个方程就是无源的 **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子, 光子是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作规范变换 (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x), \quad (3.228)$$

其中, 作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数, 依赖于时空坐标, 因而这样的变换是局域 (local) 变换。在此规范变换下, 场强张量不变:

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu [A^\nu(x) + \partial^\nu \chi(x)] - \partial^\nu [A^\mu(x) + \partial^\mu \chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu \partial^\nu \chi(x) - \partial^\nu \partial^\mu \chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \quad (3.229)$$

因而, 拉氏量 (3.226) 和无源 Maxwell 方程 (3.227) 都不会改变, 这称为规范对称性 (gauge symmetry)。

在经典电动力学中, 这种对称性广为人知, 它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定, 因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$, 也就是说, 不依赖于规范的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算, 常常将规范固定下来, 使得计算过程依赖于选取的规范, 不过, 最后得出的可观测量必须是规范不变 (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**, 规范条件为

$$\partial_\mu A^\mu = 0. \quad (3.230)$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同, 但是, 在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件, 而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$, 令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2 \chi(x) = -\partial_\mu A^\mu(x), \quad (3.231)$$

那么, 作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件:

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2 \chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \quad (3.232)$$

但是, 经过这种变换之后, 矢量场仍然没有被唯一地确定: 对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$, 取满足齐次波动方程

$$\partial^2 \tilde{\chi}(x) = 0 \quad (3.233)$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换, 都能得到满足 Lorenz 规范条件的另一个矢量场 $\tilde{A}^\mu(x)$ 。可见, 存在无穷多个规范等价的矢量场, 它们描述相同的物理, 而且全都满足 Lorenz 规范条件 (3.230)。

矢量场 $A^\mu(x)$ 有 4 个分量, 因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是, 上述规范等价性表明, $A^\mu(x)$ 并没有 3 个独立的自由度, 否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上, 无质量矢量场 $A^\mu(x)$ 只具有 2 个独立的自由度, 也就是说, 有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论: 电磁波具有 2 种独立的极化态, 以螺旋度 λ 来表征的话, 就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有质量矢量场 $A^\mu(x)$ 的量子化程序时, 由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度, 因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难, 因为 Proca 方程要求 $A^0(x)$ 不是独立变量, 而是由 (3.179) 式决定的:

$$A^0 = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}. \quad (3.234)$$

于是, 以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的, 自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时, (3.234) 式显然不能成立。因此, 对于无质量矢量场, 最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度, 可以在拉氏量中增加一个不会影响最终物理结果的项:

$$\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.235)$$

其中 ξ 是一个可以自由选取的实参数。可以看出, 在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.230) 的情况下, 由 (3.235) 式定义的 \mathcal{L}_1 等价于由 (3.226) 式定义的 \mathcal{L} 。新增的项 $-\frac{1}{2} \xi (\partial_\mu A^\mu)^2$ 破坏了规范对称性, 相当于把规范固定下来, 因而称为规范固定项 (gauge-fixing term)。可以将 \mathcal{L}_1 展开为

$$\mathcal{L}_1 = -\frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu + \frac{1}{2} (\partial_\nu A_\mu) \partial^\mu A^\nu - \frac{1}{2} \xi (\partial_\mu A^\mu)^2, \quad (3.236)$$

从而, A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_1}{\partial (\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi (\partial_\nu A^\nu) \frac{\partial (\partial_\sigma A^\sigma)}{\partial (\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0} \partial_\nu A^\nu, \quad (3.237)$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi \partial_\mu A^\mu. \quad (3.238)$$

因此, $\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0 。

现在, 正则量子化程序要求算符 A^μ 和 π_μ 满足如下等时对易关系:

$$[A^\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = i\delta^\mu_\nu \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [A^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] = [\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{y}, t)] = 0. \quad (3.239)$$

但是, 这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子, 利用 (3.238) 式, 可得

$$[A^0(\mathbf{x}, t), \partial_\mu A^\mu(\mathbf{y}, t)] = -\frac{1}{\xi} [A^0(\mathbf{x}, t), \pi_0(\mathbf{y}, t)] = -\frac{i}{\xi} \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (3.240)$$

上式在 $\mathbf{x} = \mathbf{y}$ 处非零, 因而必有 $\partial_\mu A^\mu \neq 0$ 。所以, A^μ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.230)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$, 但对于量子场 $A^\mu(x)$ 来说限制太强了, 下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - \xi g^{\mu\nu}(\partial_\rho A^\rho), \quad \frac{\partial \mathcal{L}_1}{\partial A_\nu} = 0, \quad (3.241)$$

可得, 与 \mathcal{L}_1 对应的 Euler-Lagrange 方程为

$$0 = \partial_\mu \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu \partial_\mu A^\mu - \xi g^{\mu\nu} \partial_\mu(\partial_\rho A^\rho) = -\partial^2 A^\nu + (1 - \xi) \partial^\nu(\partial_\rho A^\rho), \quad (3.242)$$

即

$$\partial^2 A^\mu - (1 - \xi) \partial^\mu(\partial_\nu A^\nu) = 0. \quad (3.243)$$

若取 $\xi = 1$, 则上式化为 **d'Alembert 方程**

$$\partial^2 A^\mu(x) = 0, \quad (3.244)$$

可以看作无质量情况下的 Klein-Gordon 方程。可见, 将规范固定参数取为

$$\xi = 1 \quad (3.245)$$

将有利于简化计算, 这种取法称为 **Feynman 规范**, 本节后续计算采用这个规范。在 Feynman 规范下, 拉氏量化为

$$\begin{aligned} \mathcal{L}_1 &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu) \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu) - \frac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu - \frac{1}{2}\partial^\mu(A_\mu\partial_\nu A^\nu) + \frac{1}{2}A_\mu\partial^\mu\partial_\nu A^\nu \\ &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu - A^\mu\partial_\nu A^\nu). \end{aligned} \quad (3.246)$$

上式最后一行第二项是一个全散度, 它不会影响作用量和运动方程, 可以舍弃。因此, 可以采用更加简化的拉氏量

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \quad (3.247)$$

此时, 共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_2}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu. \quad (3.248)$$

对于 d'Alembert 方程 (3.244), 平面波解的正能解和负能解分别正比于 $\exp(-ip \cdot x)$ 和 $\exp(ip \cdot x)$, 其中

$$p^0 = E_{\mathbf{p}} = |\mathbf{p}|. \quad (3.249)$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p}, \sigma)$, 可以对无质量矢量场 $A^\mu(\mathbf{x}, t)$ 作如下平面波展开:

$$A^\mu(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}). \quad (3.250)$$

容易验证, 这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x}, t)]^\dagger = A^\mu(\mathbf{x}, t). \quad (3.251)$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x}, t) = -\partial_0 A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e_\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}), \quad (3.252)$$

它也满足自共轭条件

$$[\pi_\mu(\mathbf{x}, t)]^\dagger = \pi_\mu(\mathbf{x}, t). \quad (3.253)$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned} \int d^3x e^{iq \cdot x} A^\mu &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) + a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}] \end{aligned} \quad (3.254)$$

和

$$\begin{aligned} &\int d^3x e^{iq \cdot x} \partial_0 A^\mu \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{i(p+q) \cdot x}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) [a_{\mathbf{p};\sigma} e^{-i(p^0-q^0)t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) - a_{\mathbf{p};\sigma}^\dagger e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p}+\mathbf{q})] \\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 [e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q}, \sigma) a_{-\mathbf{q};\sigma}^\dagger e^{2iq^0 t}], \end{aligned} \quad (3.255)$$

以及正交归一关系 (3.98), 可得

$$\begin{aligned} e_\mu(\mathbf{q}, \sigma') \int d^3x e^{iq \cdot x} (\partial_0 A^\mu - iq_0 A^\mu) &= e_\mu(\mathbf{q}, \sigma') \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{q}, \sigma) a_{\mathbf{q};\sigma} \\ &= -i\sqrt{2E_{\mathbf{q}}} \sum_{\sigma=0}^3 g_{\sigma'\sigma} a_{\mathbf{q};\sigma} = -i\sqrt{2E_{\mathbf{q}}} g_{\sigma'\sigma'} a_{\mathbf{q};\sigma'}. \end{aligned} \quad (3.256)$$

注意, 虽然上式出现了重复的指标 σ' , 但此处不需要对 σ' 求和。于是, 有

$$a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) \int d^3x e^{ip \cdot x} (\partial_0 A^\mu - ip_0 A^\mu). \quad (3.257)$$

对上式取厄米共轭, 得

$$a_{\mathbf{p};\sigma}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) \int d^3x e^{-ip \cdot x} (\partial_0 A^\mu + ip_0 A^\mu). \quad (3.258)$$

根据等时对易关系 (3.239), 湮灭算符与产生算符的对易关系为

$$\begin{aligned} [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^\dagger] &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times [\partial_0 A^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), \partial_0 A^\nu(\mathbf{y}, t) + iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times [-\pi^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), -\pi^\nu(\mathbf{y}, t) + iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} \\ &\quad \times \{-iq_0 [\pi^\mu(\mathbf{x}, t), A^\nu(\mathbf{y}, t)] + ip_0 [A^\mu(\mathbf{x}, t), \pi^\nu(\mathbf{y}, t)]\} \\ &= \frac{g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} [-(p_0 + q_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\ &= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 - q^0)t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} \\ &= -\frac{E_{\mathbf{p}} + E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= -(2\pi)^3 g_{\sigma\sigma} g_{\sigma'\sigma'} g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (3.259)$$

倒数第二步用到正交归一关系 (3.98)。另一方面, 两个湮灭算符之间的对易关系为

$$\begin{aligned} [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\ &\quad \times [\partial_0 A^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), \partial_0 A^\nu(\mathbf{y}, t) - iq_0 A^\nu(\mathbf{y}, t)] \\ &= \frac{-g_{\sigma\sigma} g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\ &\quad \times [-\pi^\mu(\mathbf{x}, t) - ip_0 A^\mu(\mathbf{x}, t), -\pi^\nu(\mathbf{y}, t) - iq_0 A^\nu(\mathbf{y}, t)] \end{aligned}$$

$$\begin{aligned}
&= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \\
&\quad \times \{iq_0 [\pi^{\mu}(\mathbf{x}, t), A^{\nu}(\mathbf{y}, t)] + ip_0 [A^{\mu}(\mathbf{x}, t), \pi^{\nu}(\mathbf{y}, t)]\} \\
&= \frac{-g_{\sigma\sigma}g_{\sigma'\sigma'}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e_{\nu}(\mathbf{q}, \sigma') \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} [(q_0 - p_0) g^{\mu\nu} \delta^{(3)}(\mathbf{x} - \mathbf{y})] \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} g_{\sigma\sigma}g_{\sigma'\sigma'} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') \int d^3x e^{i(p^0 + q^0)t} e^{-i(\mathbf{p} + \mathbf{q}) \cdot \mathbf{x}} \\
&= \frac{E_{\mathbf{p}} - E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} g_{\sigma\sigma}g_{\sigma'\sigma'} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \quad (3.260)
\end{aligned}$$

归纳起来, 产生湮灭算符的对易关系为

$$[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^{\dagger}] = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a_{\mathbf{p};\sigma}^{\dagger}, a_{\mathbf{q};\sigma'}^{\dagger}] = 0. \quad (3.261)$$

3.4.4 哈密顿量和总动量

根据 (1.119)、(3.248) 和 (3.247) 式, 无质量矢量场的哈密顿量密度是

$$\begin{aligned}
\mathcal{H} &= \pi_{\mu} \partial^0 A^{\mu} - \mathcal{L}_2 = -(\partial_0 A_{\mu}) \partial^0 A^{\mu} + \frac{1}{2} (\partial_{\mu} A_{\nu}) \partial^{\mu} A^{\nu} \\
&= -\frac{1}{2} (\partial_0 A_{\mu}) \partial^0 A^{\mu} + \frac{1}{2} (\partial_i A_{\mu}) \partial^i A^{\mu} = -\frac{1}{2} [\pi_{\mu} \pi^{\mu} + (\nabla A_{\mu}) \cdot (\nabla A^{\mu})]. \quad (3.262)
\end{aligned}$$

于是, 哈密顿量表达为

$$\begin{aligned}
H &= \int d^3x \mathcal{H} = -\frac{1}{2} \int d^3x [\pi_{\mu} \pi^{\mu} + (\nabla A_{\mu}) \cdot (\nabla A^{\mu})] \\
&= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') \\
&\quad \times \left[(ip_0)(iq_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}) (a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - a_{\mathbf{q};\sigma'}^{\dagger} e^{iq \cdot x}) \right. \\
&\quad \left. + (i\mathbf{p} a_{\mathbf{p};\sigma} e^{-ip \cdot x} - i\mathbf{p} a_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}) \cdot (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^{\dagger} e^{iq \cdot x}) \right] \\
&= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') \left[(p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^{\dagger} e^{-i(p-q) \cdot x} \right. \\
&\quad \left. + (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + (p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} \right. \\
&\quad \left. + (-p_0 q_0 - \mathbf{p} \cdot \mathbf{q}) a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{q};\sigma'}^{\dagger} e^{i(p+q) \cdot x} \right] \\
&= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{q}, \sigma') (p_0 q_0 + \mathbf{p} \cdot \mathbf{q}) \\
&\quad \times \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^{\dagger} e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. - \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{q};\sigma'}^{\dagger} e^{i(p_0 + q_0)t} \right] \right\} \\
&= -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[e_{\mu}(\mathbf{p}, \sigma) e^{\mu}(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 + |\mathbf{p}|^2) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^{\dagger} + a_{\mathbf{p};\sigma}^{\dagger} a_{\mathbf{p};\sigma'}) \right]
\end{aligned}$$

$$\begin{aligned}
& - e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (E_{\mathbf{p}}^2 - |\mathbf{p}|^2) \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t} \right) \Big] \\
& = -\frac{1}{2} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') \left(a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'} \right) \\
& = -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} g_{\sigma\sigma'} \left(a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'} \right) = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) \left(a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) \\
& = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\
& = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \tag{3.263}
\end{aligned}$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负，与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规，时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论，将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态，满足

$$a_{\mathbf{p};\sigma} |0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}} |0\rangle, \quad E_{\text{vac}} = 2\delta^{(3)}(\mathbf{0}) \int d^3p E_{\mathbf{p}}. \tag{3.264}$$

动量为 \mathbf{p} 、极化态为 σ 的单粒子态定义为

$$|\mathbf{p}; \sigma\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p};\sigma}^\dagger |0\rangle. \tag{3.265}$$

从而，由

$$\begin{aligned}
[H, a_{\mathbf{p};\sigma}^\dagger] &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) [a_{\mathbf{q};\sigma'}^\dagger a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^\dagger [a_{\mathbf{q};\sigma'}, a_{\mathbf{p};\sigma}^\dagger] \\
&= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \sum_{\sigma'=0}^3 (-g_{\sigma'\sigma'}) a_{\mathbf{q};\sigma'}^\dagger (2\pi)^3 (-g_{\sigma'\sigma}) \delta^{(3)}(\mathbf{q} - \mathbf{p}) \\
&= E_{\mathbf{p}} \sum_{\sigma'=0}^3 g_{\sigma'\sigma'} g_{\sigma'\sigma} a_{\mathbf{p};\sigma'}^\dagger = E_{\mathbf{p}} a_{\mathbf{p};\sigma}^\dagger \tag{3.266}
\end{aligned}$$

可得

$$\begin{aligned}
H|\mathbf{p}; \sigma\rangle &= \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p};\sigma}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger H) |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} (E_{\mathbf{p}} + E_{\text{vac}}) a_{\mathbf{p};\sigma}^\dagger |0\rangle = (E_{\mathbf{p}} + E_{\text{vac}}) |\mathbf{p}; \sigma\rangle. \tag{3.267}
\end{aligned}$$

这似乎是一个正常的结果，说明单粒子态 $|\mathbf{p}; \sigma\rangle$ 比真空多了一份能量 $E_{\mathbf{p}}$ 。

利用产生湮灭算符的对易关系 (3.261)，可以计算单粒子态的内积：

$$\begin{aligned}
\langle \mathbf{q}; \sigma' | \mathbf{p}; \sigma \rangle &= \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | a_{\mathbf{q};\sigma'} a_{\mathbf{p};\sigma}^\dagger | 0 \rangle = \sqrt{2E_{\mathbf{q}} 2E_{\mathbf{p}}} \langle 0 | [a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} - (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q})] | 0 \rangle \\
&= -2E_{\mathbf{p}} (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{3.268}
\end{aligned}$$

于是, 有

$$\langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{0}), \quad \langle \mathbf{p}; i | \mathbf{p}; i \rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{0}), \quad i = 1, 2, 3. \quad (3.269)$$

上式表明, 单粒子态 $|\mathbf{p}; 0\rangle$ 的自我内积是负的, 从而导致它的能量期待值也是负的:

$$\langle \mathbf{p}; 0 | H | \mathbf{p}; 0 \rangle = (E_{\mathbf{p}} + E_{\text{vac}}) \langle \mathbf{p}; 0 | \mathbf{p}; 0 \rangle = -2E_{\mathbf{p}}(E_{\mathbf{p}} + E_{\text{vac}})(2\pi)^3 \delta^{(3)}(\mathbf{0}) < 0. \quad (3.270)$$

这个负能量结果在物理上看起来是不可接受的, 它的根源在于不定度规。

不过, 如前所述, 无质量矢量场只有 2 种独立的极化态, 对应于 2 种横向极化矢量 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$, 纵向极化和类时极化都应该是非物理的。选取一定的规范条件, 应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.230) 与正则量子化程序不相容, 我们不能直接使用这个条件, 而需要将它转换到物理 Hilbert 空间中的态的期待值上, 要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle \Psi | \partial_\mu A^\mu(x) | \Psi \rangle = 0. \quad (3.271)$$

上式称为弱 Lorenz 规范条件。

$A^\mu(x)$ 的平面波展开式 (3.250) 可以分解成正能解和负能解两个部分:

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x). \quad (3.272)$$

其中, 正能解部分为

$$A^{\mu(+)}(\mathbf{x}, t) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x}, \quad (3.273)$$

上式的厄米共轭即是负能解部分

$$A^{\mu(-)}(\mathbf{x}, t) \equiv [A^{\mu(+)}(\mathbf{x}, t)]^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}. \quad (3.274)$$

如果要求

$$\partial_\mu A^{\mu(+)}(x) | \Psi \rangle = 0 \quad (3.275)$$

对任意物理态 $|\Psi\rangle$ 成立, 则伴随有

$$\langle \Psi | \partial_\mu A^{\mu(-)}(x) = \langle \Psi | [\partial_\mu A^{\mu(+)}(x)]^\dagger = 0, \quad (3.276)$$

从而, 弱 Lorenz 规范条件 (3.271) 得到满足:

$$\langle \Psi | \partial_\mu A^\mu(x) | \Psi \rangle = \langle \Psi | \partial_\mu A^{\mu(+)}(x) | \Psi \rangle + \langle \Psi | \partial_\mu A^{\mu(-)}(x) | \Psi \rangle = 0. \quad (3.277)$$

利用 (3.113) 和 (3.220) 式, 规范条件 (3.275) 可化为

$$0 = \partial_\mu A^{\mu(+)}(x) | \Psi \rangle$$

$$\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} [p_\mu e^\mu(\mathbf{p}, 0)a_{\mathbf{p};0} + p_\mu e^\mu(\mathbf{p}, 1)a_{\mathbf{p};1} + p_\mu e^\mu(\mathbf{p}, 2)a_{\mathbf{p};2} + p_\mu e^\mu(\mathbf{p}, 3)a_{\mathbf{p};3}] |\Psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{-ie^{-ip \cdot x}}{\sqrt{2E_{\mathbf{p}}}} p \cdot n (a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle.
\end{aligned} \tag{3.278}$$

这意味着

$$(a_{\mathbf{p};0} - a_{\mathbf{p};3}) |\Psi\rangle = 0 \tag{3.279}$$

对任意物理态 $|\Psi\rangle$ 和任意动量 \mathbf{p} 成立。从而，也有

$$\langle \Psi | (a_{\mathbf{p};0}^\dagger - a_{\mathbf{p};3}^\dagger) = 0. \tag{3.280}$$

于是，

$$\langle \Psi | a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} |\Psi\rangle = \langle \Psi | a_{\mathbf{p};3}^\dagger a_{\mathbf{p};3} |\Psi\rangle. \tag{3.281}$$

这样一来，根据 (3.263) 式计算， $|\Psi\rangle$ 的能量期待值为

$$\begin{aligned}
\langle \Psi | H | \Psi \rangle &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle \Psi | \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle.
\end{aligned} \tag{3.282}$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.158) 式可得无质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi_\mu \nabla A^\mu \\
&= - \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\
&\quad \times (ip_0) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} - a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) (i\mathbf{q} a_{\mathbf{q};\sigma'} e^{-iq \cdot x} - i\mathbf{q} a_{\mathbf{q};\sigma'}^\dagger e^{iq \cdot x}) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\
&\quad \times \left[-a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p-q) \cdot x} - a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p-q) \cdot x} + a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q) \cdot x} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p+q) \cdot x} \right] \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} p_0 \mathbf{q} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{q}, \sigma') \\
&\quad \times \left\{ -\delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'}^\dagger e^{-i(p_0 - q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'} e^{i(p_0 - q_0)t} \right] \right. \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0 + q_0)t} + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{q};\sigma'}^\dagger e^{i(p_0 + q_0)t} \right] \right\} \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \left[-e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'}) \right]
\end{aligned}$$

$$- e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t} \right) \Big]. \quad (3.283)$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p} \rightarrow -\mathbf{p}$ 的替换和 $\sigma \leftrightarrow \sigma'$ 的互换, 可得

$$\begin{aligned} & - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(-\mathbf{p}, \sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t} \right) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) \left(a_{-\mathbf{p};\sigma'} a_{\mathbf{p};\sigma} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p};\sigma'}^\dagger a_{\mathbf{p};\sigma}^\dagger e^{2iE_{\mathbf{p}}t} \right) \\ &= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(-\mathbf{p}, \sigma') e^\mu(\mathbf{p}, \sigma) \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p};\sigma}^\dagger a_{-\mathbf{p};\sigma'}^\dagger e^{2iE_{\mathbf{p}}t} \right). \end{aligned} \quad (3.284)$$

可以看出, 上式为零。于是, 总动量化为

$$\begin{aligned} \mathbf{P} &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} e_\mu(\mathbf{p}, \sigma) e^\mu(\mathbf{p}, \sigma') \left(a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'} \right) \\ &= - \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} g_{\sigma\sigma'} \left(a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma'} \right) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma}) (a_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}^\dagger + a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) \\ &= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=0}^3 (-g_{\sigma\sigma} a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma}) + \delta^{(3)}(\mathbf{0}) \int d^3p \frac{\mathbf{p}}{2} \sum_{\sigma=0}^3 (-g_{\sigma\sigma})^2 \\ &= \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right). \end{aligned} \quad (3.285)$$

根据 (3.281) 式, 物理态 $|\Psi\rangle$ 的动量期待值为

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \langle \Psi | \left(-a_{\mathbf{p};0}^\dagger a_{\mathbf{p};0} + \sum_{\sigma=1}^3 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} \right) | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\sigma=1}^2 \langle \Psi | a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} | \Psi \rangle. \quad (3.286)$$

同样, 只有两种物理的横向极化才对动量有净贡献。

通过线性组合, 可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$a_{\mathbf{p};\pm} \equiv \frac{1}{\sqrt{2}} (\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}), \quad (3.287)$$

相应的产生算符可以通过取厄米共轭得到。反过来, 有

$$a_{\mathbf{p};1} = -\frac{1}{\sqrt{2}} (a_{\mathbf{p};+} - a_{\mathbf{p};-}), \quad a_{\mathbf{p};2} = -\frac{i}{\sqrt{2}} (a_{\mathbf{p};+} + a_{\mathbf{p};-}). \quad (3.288)$$

利用对易关系 (3.261), 可得

$$\begin{aligned} [a_{\mathbf{p};\pm}, a_{\mathbf{q};\pm}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = \frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \\ [a_{\mathbf{p};\pm}, a_{\mathbf{q};\mp}^\dagger] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \pm a_{\mathbf{q};1}^\dagger - i a_{\mathbf{q};2}^\dagger] = -\frac{1}{2} [a_{\mathbf{p};1}, a_{\mathbf{q};1}^\dagger] + \frac{1}{2} [a_{\mathbf{p};2}, a_{\mathbf{q};2}^\dagger] = 0, \\ [a_{\mathbf{p};\pm}, a_{\mathbf{q};\pm}] &= \frac{1}{2} [\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}, \mp a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0, \end{aligned}$$

$$[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] = \frac{1}{2}[\mp a_{\mathbf{p},1} + ia_{\mathbf{p},2}, \pm a_{\mathbf{q},1} + ia_{\mathbf{q},2}] = 0. \quad (3.289)$$

于是, 这组产生湮灭算符的对易关系可以整理为

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0, \quad \lambda, \lambda' = \pm. \quad (3.290)$$

根据 (3.129) 式, 可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$ 表示 $e^\mu(\mathbf{p}, 1)$ 和 $e^\mu(\mathbf{p}, 2)$:

$$e^\mu(\mathbf{p}, 1) = -\frac{1}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)], \quad e^\mu(\mathbf{p}, 2) = \frac{i}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)]. \quad (3.291)$$

从而, 有

$$\begin{aligned} \sum_{\sigma=1}^2 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p}, 1) a_{\mathbf{p};1} + e^\mu(\mathbf{p}, 2) a_{\mathbf{p};2} \\ &= \frac{1}{2}[\varepsilon^\mu(\mathbf{p}, +) - \varepsilon^\mu(\mathbf{p}, -)](a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}[\varepsilon^\mu(\mathbf{p}, +) + \varepsilon^\mu(\mathbf{p}, -)](a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= \varepsilon^\mu(\mathbf{p}, +) a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p}, -) a_{\mathbf{p},-} = \sum_{\lambda=\pm} \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \end{aligned} \quad (3.292)$$

取厄米共轭, 得

$$\sum_{\sigma=1}^2 e^\mu(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^\dagger = \sum_{\lambda=\pm} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger. \quad (3.293)$$

于是, 可以把 $A^\mu(x)$ 的平面波展开式 (3.250) 改写成

$$\begin{aligned} A^\mu(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0,3} e^\mu(\mathbf{p}, \sigma) (a_{\mathbf{p};\sigma} e^{-ip \cdot x} + a_{\mathbf{p};\sigma}^\dagger e^{ip \cdot x}) \\ &\quad + \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} [\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}], \end{aligned} \quad (3.294)$$

第一行对应于非物理极化态, 第二行对应于两种物理的螺旋度本征极化态。可见, (3.287) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda = \pm$ 对应的湮灭算符。

此外, 由 (3.288) 式可得

$$\begin{aligned} \sum_{\sigma=1}^2 a_{\mathbf{p};\sigma}^\dagger a_{\mathbf{p};\sigma} &= a_{\mathbf{p};1}^\dagger a_{\mathbf{p};1} + a_{\mathbf{p};2}^\dagger a_{\mathbf{p};2} = \frac{1}{2}(a_{\mathbf{p},+}^\dagger - a_{\mathbf{p},-}^\dagger)(a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}(a_{\mathbf{p},+}^\dagger + a_{\mathbf{p},-}^\dagger)(a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= a_{\mathbf{p},+}^\dagger a_{\mathbf{p},+} + a_{\mathbf{p},-}^\dagger a_{\mathbf{p},-} = \sum_{\lambda=\pm} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}, \end{aligned} \quad (3.295)$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle \Psi | H | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} | \Psi \rangle + E_{\text{vac}} \langle \Psi | \Psi \rangle, \quad (3.296)$$

$$\langle \Psi | \mathbf{P} | \Psi \rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\lambda=\pm} \langle \Psi | a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} | \Psi \rangle. \quad (3.297)$$

第 4 章 旋量场

4.1 Lorentz 群的旋量表示

旋量表示 (spinor representation) 是 Lorentz 群的一个线性表示, 它在物理上扮演着非常重要的角色, Dirac 在 1928 年首次将它引入到描述电子的理论中。3.1 节提到, Lorentz 群的线性表示可以通过构造满足 Lorentz 代数关系 (3.20) 的生成元矩阵来得到, 下面我们就用这样的方式来建立旋量表示。

首先, 我们假设能够找到一组满足如下反对易关系的 $N \times N$ 矩阵 γ^μ ($\mu = 0, 1, 2, 3$):

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1} = 2g^{\mu\nu}. \quad (4.1)$$

最后一步是一种简写, 省略了 $N \times N$ 单位矩阵 $\mathbf{1}$ 。这样的 γ^μ 称为 **Dirac 矩阵**。当 $\mu \neq \nu$ 时, γ^μ 与 γ^ν 是反对易的, 即

$$\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu, \quad \mu \neq \nu. \quad (4.2)$$

当 $\mu = \nu$ 时, 有

$$(\gamma^0)^2 = \frac{1}{2}\{\gamma^0, \gamma^0\} = g^{00} = \mathbf{1}, \quad (\gamma^i)^2 = \frac{1}{2}\{\gamma^i, \gamma^i\} = g^{ii} = -\mathbf{1}. \quad (4.3)$$

我们约定 γ^0 是厄米矩阵, γ^i 是反厄米矩阵, 即

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \quad (4.4)$$

则可得

$$(\gamma^0)^\dagger \gamma^0 = (\gamma^0)^2 = \mathbf{1}, \quad (\gamma^i)^\dagger \gamma^i = -(\gamma^i)^2 = \mathbf{1}. \quad (4.5)$$

可见, 在此约定下, γ^0 和 γ^i 都是么正矩阵。

然后, 以 Dirac 矩阵的对易子定义另一组 $N \times N$ 矩阵

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \quad (4.6)$$

显然, $S^{\mu\nu}$ 关于 μ 和 ν 反对称:

$$S^{\mu\nu} = -S^{\nu\mu}. \quad (4.7)$$

因而 $S^{\mu\nu}$ 的独立分量有 6 个。

利用对易子公式

$$[AB, C] = ABC + ACB - ACB - CAB = A\{B, C\} - \{A, C\}B, \quad (4.8)$$

可得

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \frac{i}{4}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho] = \frac{i}{4}[\gamma^\mu\gamma^\nu - (2g^{\nu\mu} - \gamma^\mu\gamma^\nu), \gamma^\rho] = \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] - \frac{i}{2}[g^{\nu\mu}, \gamma^\rho] \\ &= \frac{i}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] = \frac{i}{2}(\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \{\gamma^\mu, \gamma^\rho\}\gamma^\nu) = i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}). \end{aligned} \quad (4.9)$$

从而, 根据对易子公式 (2.11), 有

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \frac{i}{4}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] = \frac{i}{4}([S^{\mu\nu}, \gamma^\rho\gamma^\sigma] - [S^{\mu\nu}, \gamma^\sigma\gamma^\rho]) \\ &= \frac{i}{4}([S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \gamma^\rho[S^{\mu\nu}, \gamma^\sigma] - [S^{\mu\nu}, \gamma^\sigma]\gamma^\rho - \gamma^\sigma[S^{\mu\nu}, \gamma^\rho]) \\ &= \frac{i}{4}[i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})\gamma^\sigma + i\gamma^\rho(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma}) \\ &\quad - i(\gamma^\mu g^{\nu\sigma} - \gamma^\nu g^{\mu\sigma})\gamma^\rho - i\gamma^\sigma(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})] \\ &= \frac{i^2}{4}(\gamma^\mu\gamma^\sigma g^{\nu\rho} - \gamma^\nu\gamma^\sigma g^{\mu\rho} + \gamma^\rho\gamma^\mu g^{\nu\sigma} - \gamma^\rho\gamma^\nu g^{\mu\sigma} \\ &\quad - \gamma^\mu\gamma^\rho g^{\nu\sigma} + \gamma^\nu\gamma^\rho g^{\mu\sigma} - \gamma^\sigma\gamma^\mu g^{\nu\rho} + \gamma^\sigma\gamma^\nu g^{\mu\rho}) \\ &= \frac{i^2}{4}[g^{\nu\rho}(\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu) - g^{\mu\rho}(\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu) - g^{\nu\sigma}(\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu) + g^{\mu\sigma}(\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu)] \\ &= i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}). \end{aligned} \quad (4.10)$$

可见, $S^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20), 因而是 Lorentz 群某个线性表示的生成元。以 $S^{\mu\nu}$ 生成的线性表示就是旋量表示。

根据 (3.2.1) 小节的讨论, 一组变换参数 $\omega_{\mu\nu}$ 在 Lorentz 群的矢量表示中可以生成固有保时向的有限变换 (3.51):

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^X, \quad X \equiv -\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}. \quad (4.11)$$

类似地, 这组参数在旋量表示中生成了固有保时向的有限变换

$$D(\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = e^Y, \quad Y \equiv -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}. \quad (4.12)$$

这样定义的 $D(\Lambda)$ 是旋量表示中的 Lorentz 变换矩阵, 对于任意的 Lorentz 变换 Λ_1 和 Λ_2 , 满足同态关系

$$D(\Lambda_2\Lambda_1) = D(\Lambda_2)D(\Lambda_1). \quad (4.13)$$

由 (3.48) 式可得

$$e^{-Y}e^Y = e^{-Y+Y} = e^0 = \mathbf{1}, \quad (4.14)$$

故 $D(\Lambda)$ 的逆矩阵为

$$D(\Lambda^{-1}) = D^{-1}(\Lambda) = e^{-Y} = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \quad (4.15)$$

这里先来介绍一些将会用到的对易子公式。以如下方式定义 B 与 A 的多重对易子 $[B, A^{(n)}]$:

$$\begin{aligned} [B, A^{(0)}] &= B, \quad [B, A^{(1)}] = [B, A] = [[B, A^{(0)}], A] \\ [B, A^{(2)}] &= [[B, A], A] = [[B, A^{(1)}], A], \quad \dots, \quad [B, A^{(n)}] = [[B, A^{(n-1)}], A]. \end{aligned} \quad (4.16)$$

于是, 下式成立:

$$BA^k = \sum_{n=0}^k \frac{k!}{(k-n)!n!} A^{k-n} [B, A^{(n)}]. \quad (4.17)$$

下面用数学归纳法证明这个等式。

证明 当 $k=0$ 和 $k=1$ 时, (4.17) 式明显成立:

$$BA^0 = B = [B, A^{(0)}] = \frac{0!}{(0-0)!0!} A^{0-0} [B, A^{(0)}], \quad (4.18)$$

$$BA^1 = BA = AB + [B, A] = \frac{1!}{(1-0)!0!} A^{1-0} [B, A^{(0)}] + \frac{1!}{(1-1)!1!} A^{1-1} [B, A^{(1)}]. \quad (4.19)$$

假设 $k=m$ 时 (4.17) 式成立, 则有

$$\begin{aligned} BA^{m+1} &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n} [B, A^{(n)}] A = \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n} (A[B, A^{(n)}] + [[B, A^{(n)}], A]) \\ &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m+1-n} [B, A^{(n)}] + \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m-n} [B, A^{(n+1)}] \\ &= \sum_{n=0}^m \frac{m!}{(m-n)!n!} A^{m+1-n} [B, A^{(n)}] + \sum_{j=1}^{m+1} \frac{m!}{(m-j+1)!(j-1)!} A^{m-j+1} [B, A^{(j)}] \\ &= \frac{m!}{(m-0)!0!} A^{m+1} [B, A^{(0)}] + \sum_{n=1}^m \left[\frac{m!}{(m-n)!n!} + \frac{m!}{(m-n+1)!(n-1)!} \right] A^{m+1-n} [B, A^{(n)}] \\ &\quad + \frac{m!}{[m-(m+1)+1]![(m+1)-1]!} A^{m-(m+1)+1} [B, A^{(m+1)}] \\ &= A^{m+1} [B, A^{(0)}] + \sum_{n=1}^m \left[\frac{m!}{(m-n)!n!} + \frac{n}{m-n+1} \frac{m!}{(m-n)!n!} \right] A^{m+1-n} [B, A^{(n)}] \\ &\quad + A^{m-(m+1)+1} [B, A^{(m+1)}] \\ &= \frac{(m+1)!}{[(m+1)-0]!0!} A^{m+1} [B, A^{(0)}] + \sum_{n=1}^m \frac{(m+1)!}{(m-n+1)!n!} A^{m+1-n} [B, A^{(n)}] \\ &\quad + \frac{(m+1)!}{[(m+1)-(m+1)]!(m+1)!} A^{m-(m+1)+1} [B, A^{(m+1)}] \\ &= \sum_{n=0}^{m+1} \frac{(m+1)!}{[(m+1)-n]!n!} A^{(m+1)-n} [B, A^{(n)}], \end{aligned} \quad (4.20)$$

即 $k = m + 1$ 时 (4.17) 式也成立。于是, (4.17) 式对任意非负整数 k 成立。证毕。

根据推广的阶乘定义 (3.46) 可以将 (4.17) 式右边的级数化成无穷级数:

$$BA^k = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} A^{k-n} [B, A^{(n)}]. \quad (4.21)$$

利用上式, 可得

$$\begin{aligned} e^{-A} B e^A &= e^{-A} \sum_{k=0}^{\infty} \frac{1}{k!} B A^k = e^{-A} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} A^{k-n} [B, A^{(n)}] \\ &= e^{-A} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{(k-n)!} A^{k-n} [B, A^{(n)}] = e^{-A} \sum_{n=0}^{\infty} \frac{1}{n!} e^A [B, A^{(n)}] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [B, A^{(n)}]. \end{aligned} \quad (4.22)$$

现在, 我们继续讨论 Lorentz 群的旋量表示。由 (4.9) 和 (3.34) 式可得

$$[\gamma^\mu, S^{\rho\sigma}] = -[S^{\rho\sigma}, \gamma^\mu] = [S^{\sigma\rho}, \gamma^\mu] = i(\gamma^\sigma g^{\rho\mu} - \gamma^\rho g^{\sigma\mu}) = i(g^{\rho\mu} \delta^\sigma_\nu - g^{\sigma\mu} \delta^\rho_\nu) \gamma^\nu = (\mathcal{J}^{\rho\sigma})^\mu_\nu \gamma^\nu. \quad (4.23)$$

从而, 有

$$\begin{aligned} [\gamma^\mu, Y^{(1)}] &= [\gamma^\mu, Y] = -\frac{i}{2} \omega_{\rho\sigma} [\gamma^\mu, S^{\rho\sigma}] = -\frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu_\nu \gamma^\nu = X^\mu_\nu \gamma^\nu, \\ [\gamma^\mu, Y^{(2)}] &= [[\gamma^\mu, Y^{(1)}], Y] = X^\mu_\nu [\gamma^\nu, Y] = X^\mu_\nu X^\nu_\rho \gamma^\rho = (X^2)^\mu_\nu \gamma^\nu, \\ &\dots \\ [\gamma^\mu, Y^{(n)}] &= (X^n)^\mu_\nu \gamma^\nu. \end{aligned} \quad (4.24)$$

于是, 利用 (4.22) 式可以推出

$$D^{-1}(\Lambda) \gamma^\mu D(\Lambda) = e^{-Y} \gamma^\mu e^Y = \sum_{n=0}^{\infty} \frac{1}{n!} [\gamma^\mu, Y^{(n)}] = \sum_{n=0}^{\infty} \frac{1}{n!} (X^n)^\mu_\nu \gamma^\nu = (e^X)^\mu_\nu \gamma^\nu, \quad (4.25)$$

即

$$D^{-1}(\Lambda) \gamma^\mu D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu. \quad (4.26)$$

上式是 γ^μ 在旋量表示中的 Lorentz 变换关系, 它说明 γ^μ 是一个 Lorentz 矢量。相应的协变矢量为

$$\gamma_\mu \equiv g_{\mu\nu} \gamma^\nu, \quad (4.27)$$

从而,

$$\gamma_0 = \gamma^0, \quad \gamma_i = -\gamma^i, \quad i = 1, 2, 3. \quad (4.28)$$

$N \times N$ 单位矩阵 $\mathbf{1}$ 满足

$$D^{-1}(\Lambda) \mathbf{1} D(\Lambda) = \mathbf{1}, \quad (4.29)$$

因而 $\mathbf{1}$ 是一个 Lorentz 标量。生成元 $S^{\mu\nu}$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda)S^{\mu\nu}D(\Lambda) = \frac{i}{4}[D^{-1}(\Lambda)\gamma^\mu D(\Lambda), D^{-1}(\Lambda)\gamma^\nu D(\Lambda)] = \frac{i}{4}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma[\gamma^\rho, \gamma^\sigma] = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma S^{\rho\sigma}, \quad (4.30)$$

可见, $S^{\mu\nu}$ 是一个 2 阶反对称 Lorentz 张量。

$S^{\mu\nu}$ 是用 2 个 Dirac 矩阵的乘积构造出来的反对称张量, 类似地, 我们也可以用 3 个 Dirac 矩阵的乘积来构造一个 3 阶全反对称张量

$$\Gamma^{\mu\nu\rho} \equiv \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \equiv \frac{1}{3!}(\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho). \quad (4.31)$$

上式第二步中的中括号表示对 μ, ν, ρ 三个指标作全反对称操作: 在偶次置换前面加上正号, 奇次置换前面加上负号, 然后对所有置换求和并除以置换方式的数目。 $\Gamma^{\mu\nu\rho}$ 的 Lorentz 变换形式是

$$D^{-1}(\Lambda)\Gamma^{\mu\nu\rho}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Gamma^{\alpha\beta\gamma}. \quad (4.32)$$

由全反对称性可知, $\Gamma^{\mu\nu\rho}$ 的独立分量只有 4 个, 可取为 Γ^{012} 、 Γ^{023} 、 Γ^{013} 和 Γ^{123} 。根据 (4.2) 式和定义式 (4.31), 可得

$$\Gamma^{012} = \gamma^0\gamma^1\gamma^2, \quad \Gamma^{023} = \gamma^0\gamma^2\gamma^3, \quad \Gamma^{013} = \gamma^0\gamma^1\gamma^3, \quad \Gamma^{123} = \gamma^1\gamma^2\gamma^3. \quad (4.33)$$

更进一步, 可以用 4 个 Dirac 矩阵的乘积来构造一个 4 阶全反对称张量

$$\begin{aligned} \Gamma^{\mu\nu\rho\sigma} &\equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} \\ &\equiv \frac{1}{4!}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho + \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho - \gamma^\mu\gamma^\sigma\gamma^\rho\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma \\ &\quad - \gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\sigma\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\sigma\gamma^\nu\gamma^\rho\gamma^\mu + \gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu + \gamma^\sigma\gamma^\nu\gamma^\mu\gamma^\rho \\ &\quad + \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - \gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu \\ &\quad - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu). \end{aligned} \quad (4.34)$$

从而, $\Gamma^{\mu\nu\rho\sigma}$ 具有如下性质:

$$\Gamma^{\mu\nu\rho\sigma} = \begin{cases} +\Gamma^{0123}, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -\Gamma^{0123}, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换,} \\ 0, & \text{其它情况.} \end{cases} \quad (4.35)$$

可见, 它只有 1 个独立分量, 可取为

$$\Gamma^{0123} = \gamma^0\gamma^1\gamma^2\gamma^3. \quad (4.36)$$

结合四维 Levi-Civita 符号的定义 (1.65), 可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}\Gamma^{0123} = \varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 \quad (4.37)$$

受到四维时空的维度限制, 我们不能以同样的方式定义高于 4 阶的全反对称张量。现在, 我们拥有一组矩阵

$$\{1, \gamma^\mu, S^{\mu\nu}, \Gamma^{\mu\nu\rho}, \Gamma^{\mu\nu\rho\sigma}\}, \quad (4.38)$$

它们各自的独立分量个数之和为 $1 + 4 + 6 + 4 + 1 = 16$ 。利用反对易关系 (4.1), 可以将任意多个 Dirac 矩阵的乘积转化为集合 (4.38) 中的矩阵与度规张量乘积的线性组合。例如,

$$\gamma^\mu \gamma^\nu = \frac{1}{2} \gamma^\mu \gamma^\nu - \frac{1}{2} \gamma^\nu \gamma^\mu + g^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] + g^{\mu\nu} = -2iS^{\mu\nu} + g^{\mu\nu}. \quad (4.39)$$

又如,

$$\begin{aligned} \gamma^\mu \gamma^\nu \gamma^\rho &= \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho + \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho = \frac{1}{2} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{2} \gamma^\mu \gamma^\rho \gamma^\nu + g^{\rho\nu} \gamma^\mu \\ &= \frac{1}{4} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{4} \gamma^\nu \gamma^\mu \gamma^\rho + \frac{1}{2} g^{\mu\nu} \gamma^\rho - \frac{1}{4} \gamma^\mu \gamma^\rho \gamma^\nu + \frac{1}{4} \gamma^\rho \gamma^\mu \gamma^\nu - \frac{1}{2} g^{\mu\rho} \gamma^\nu + g^{\rho\nu} \gamma^\mu \\ &= \frac{1}{4} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{8} \gamma^\nu \gamma^\mu \gamma^\rho + \frac{1}{8} \gamma^\nu \gamma^\rho \gamma^\mu - \frac{1}{4} g^{\rho\mu} \gamma^\nu + \frac{1}{2} g^{\mu\nu} \gamma^\rho - \frac{1}{4} \gamma^\mu \gamma^\rho \gamma^\nu \\ &\quad + \frac{1}{8} \gamma^\rho \gamma^\mu \gamma^\nu - \frac{1}{8} \gamma^\rho \gamma^\nu \gamma^\mu + \frac{1}{4} g^{\mu\nu} \gamma^\rho - \frac{1}{2} g^{\mu\rho} \gamma^\nu + g^{\rho\nu} \gamma^\mu \\ &= \frac{3!}{8} \Gamma^{\mu\nu\rho} + \frac{1}{8} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{8} \gamma^\mu \gamma^\rho \gamma^\nu - \frac{3}{4} g^{\rho\mu} \gamma^\nu + \frac{3}{4} g^{\mu\nu} \gamma^\rho + g^{\rho\nu} \gamma^\mu \\ &= \frac{3}{4} \Gamma^{\mu\nu\rho} + \frac{1}{8} \gamma^\mu \gamma^\nu \gamma^\rho + \frac{1}{8} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{1}{4} g^{\rho\nu} \gamma^\mu - \frac{3}{4} g^{\rho\mu} \gamma^\nu + \frac{3}{4} g^{\mu\nu} \gamma^\rho + g^{\rho\nu} \gamma^\mu \\ &= \frac{3}{4} \Gamma^{\mu\nu\rho} + \frac{1}{4} \gamma^\mu \gamma^\nu \gamma^\rho - \frac{3}{4} g^{\rho\mu} \gamma^\nu + \frac{3}{4} g^{\mu\nu} \gamma^\rho + \frac{3}{4} g^{\rho\nu} \gamma^\mu, \end{aligned} \quad (4.40)$$

故

$$\gamma^\mu \gamma^\nu \gamma^\rho = \Gamma^{\mu\nu\rho} - g^{\rho\mu} \gamma^\nu + g^{\mu\nu} \gamma^\rho + g^{\rho\nu} \gamma^\mu. \quad (4.41)$$

因此, 对于由 Dirac 矩阵乘积的线性组合构造的矩阵, 集合 (4.38) 构成一组完备的基底。

这里引入一个新的矩阵

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (4.42)$$

从 (4.2) 式可得

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = \begin{cases} +\gamma^0 \gamma^1 \gamma^2 \gamma^3, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的偶次置换,} \\ -\gamma^0 \gamma^1 \gamma^2 \gamma^3, & (\mu, \nu, \rho, \sigma) \text{ 是 } (0, 1, 2, 3) \text{ 的奇次置换.} \end{cases} \quad (4.43)$$

这种置换性质与四维 Levi-Civita 符号 (1.65) 相同, 因而置换操作带来的符号在 $\varepsilon_{\mu\nu\rho\sigma}$ 与 $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$ 的乘积中相互抵消, 如

$$\varepsilon_{1023} \gamma^1 \gamma^0 \gamma^2 \gamma^3 = -\varepsilon_{0123} (-\gamma^0 \gamma^1 \gamma^2 \gamma^3) = \varepsilon_{0123} \gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (4.44)$$

由此可得

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\varepsilon_{0123} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma. \quad (4.45)$$

对于固有保时向 Lorentz 变换 (4.11), 用度规对 (1.73) 式升降指标, 有

$$\varepsilon_{\mu\nu\rho\sigma} = \Lambda_\mu^\alpha \Lambda_\nu^\beta \Lambda_\rho^\gamma \Lambda_\sigma^\delta \varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta} (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\Lambda^{-1})^\gamma_\rho (\Lambda^{-1})^\delta_\sigma. \quad (4.46)$$

于是, γ^5 的 Lorentz 变换形式为

$$\begin{aligned} D^{-1}(\Lambda) \gamma^5 D(\Lambda) &= -\frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \\ &= -\frac{i}{4!} \varepsilon_{\kappa\lambda\tau\varepsilon} (\Lambda^{-1})^\kappa_\mu (\Lambda^{-1})^\lambda_\nu (\Lambda^{-1})^\tau_\rho (\Lambda^{-1})^\varepsilon_\sigma \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \\ &= -\frac{i}{4!} \varepsilon_{\kappa\lambda\tau\varepsilon} \delta^\kappa_\alpha \delta^\lambda_\beta \delta^\tau_\gamma \delta^\varepsilon_\delta \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = -\frac{i}{4!} \varepsilon_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = \gamma^5. \end{aligned} \quad (4.47)$$

可见, γ^5 是一个 Lorentz 标量。 γ^5 的平方为

$$(\gamma^5)^2 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^3 \gamma^2 \gamma^1 \gamma^0 = -(-\mathbf{1})^3 = \mathbf{1}. \quad (4.48)$$

根据约定 (4.4), γ^5 是厄米矩阵:

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger (\gamma^2)^\dagger (\gamma^1)^\dagger (\gamma^0)^\dagger = i\gamma^3 \gamma^2 \gamma^1 \gamma^0 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5. \quad (4.49)$$

γ^5 与 γ^μ 反对易:

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu + \gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3) = i(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu - \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu) = 0. \quad (4.50)$$

由 (4.37) 式可得

$$\Gamma^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\varepsilon^{\mu\nu\rho\sigma} \gamma^5. \quad (4.51)$$

可见, $\Gamma^{\mu\nu\rho\sigma}$ 正比于 γ^5 。此外, 由 (4.33) 式有

$$\Gamma^{012} = \gamma^0 \gamma^1 \gamma^2 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^3 = \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^3 \gamma^5 = i\gamma_3 \gamma^5 = i\varepsilon^{0123} \gamma_3 \gamma^5, \quad (4.52)$$

$$\Gamma^{023} = \gamma^0 \gamma^2 \gamma^3 = -\gamma^0 \gamma^1 \gamma^1 \gamma^2 \gamma^3 = \gamma^1 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^1 \gamma^5 = i\gamma_1 \gamma^5 = i\varepsilon^{0231} \gamma_1 \gamma^5, \quad (4.53)$$

$$\Gamma^{013} = \gamma^0 \gamma^1 \gamma^3 = -\gamma^0 \gamma^1 \gamma^2 \gamma^2 \gamma^3 = -\gamma^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = i\gamma^2 \gamma^5 = -i\gamma_2 \gamma^5 = i\varepsilon^{0132} \gamma_2 \gamma^5, \quad (4.54)$$

$$\Gamma^{123} = \gamma^1 \gamma^2 \gamma^3 = \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\gamma^0 \gamma^5 = -i\gamma_0 \gamma^5 = i\varepsilon^{1230} \gamma_0 \gamma^5. \quad (4.55)$$

综合起来, 得

$$\Gamma^{\mu\nu\rho} = i\varepsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma^5. \quad (4.56)$$

根据上式, $\Gamma^{\mu\nu\rho}$ 可以写成 $\gamma^\mu \gamma^5$ 的 4 个独立分量的线性组合。 $\gamma^\mu \gamma^5$ 的 Lorentz 变换形式为

$$D^{-1}(\Lambda) \gamma^\mu \gamma^5 D(\Lambda) = D^{-1}(\Lambda) \gamma^\mu D(\Lambda) D^{-1}(\Lambda) \gamma^5 D(\Lambda) = \Lambda^\mu_\nu \gamma^\nu \gamma^5, \quad (4.57)$$

因而它是一个 Lorentz 矢量。再引入

$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu] = 2S^{\mu\nu}, \quad (4.58)$$

它正比于 $S^{\mu\nu}$ ，所以也是一个 2 阶反对称 Lorentz 张量：

$$D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda) = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\sigma^{\alpha\beta}. \quad (4.59)$$

于是，我们可以用 γ^5 、 $\gamma^\mu\gamma^5$ 和 $\sigma^{\mu\nu}$ 分别代替集合 (4.38) 中的 $\Gamma^{\mu\nu\rho\sigma}$ 、 $\Gamma^{\mu\nu\rho}$ 和 $S^{\mu\nu}$ 作为基底，从而得到另一组完备的矩阵基底

$$\{\mathbf{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}, \quad (4.60)$$

它们各自的独立分量个数之和仍是 16。

依照约定 (4.4)， γ^0 即是厄米的又是么正的，我们可以用它定义一个么正变换矩阵 β ：

$$\beta^{-1} = \beta^\dagger = \beta \equiv \gamma^0. \quad (4.61)$$

从而，有

$$\beta^{-1}\gamma^0\beta = \gamma^0\gamma^0\gamma^0 = +\gamma^0, \quad \beta^{-1}\gamma^i\beta = \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i. \quad (4.62)$$

根据宇称变换 \mathcal{P} 的定义 (1.46)，可以将这两个式子合写为

$$\beta^{-1}\gamma^\mu\beta = \mathcal{P}^\mu{}_\nu\gamma^\nu. \quad (4.63)$$

这表明 β 相当于旋量表示中的宇称变换矩阵 $D(\mathcal{P})$ ，它是非固有保时向的，上式就是 γ^μ 的宇称变换形式。(4.62) 式说明 γ^0 是宇称本征态，本征值为 $+$ ，即具有偶宇称； γ^i 也是宇称本征态，本征值为 $-$ ，即具有奇宇称。虽然单位矩阵 $\mathbf{1}$ 与 γ_5 都是 Lorentz 标量，但它们的宇称是不同的：

$$\beta^{-1}\mathbf{1}\beta = +\mathbf{1}, \quad \beta^{-1}\gamma^5\beta = \gamma^0\gamma^5\gamma^0 = -\gamma^5\gamma^0\gamma^0 = -\gamma^5. \quad (4.64)$$

像 γ^5 这样具有奇宇称的 Lorentz 标量，称为赝标量 (pseudoscalar)。此外， $\gamma^\mu\gamma^5$ 的宇称变换形式是

$$\beta^{-1}\gamma^\mu\gamma^5\beta = \beta^{-1}\gamma^\mu\beta\beta^{-1}\gamma^5\beta = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \quad (4.65)$$

即

$$\beta^{-1}\gamma^0\gamma^5\beta = -\gamma^0\gamma^5, \quad \beta^{-1}\gamma^i\gamma^5\beta = +\gamma^i\gamma^5. \quad (4.66)$$

可以看出，虽然 $\gamma^\mu\gamma^5$ 也是 Lorentz 矢量，但它的分量的宇称性质与 γ^μ 相反。宇称变换性质像 $\gamma^\mu\gamma^5$ 这样的 Lorentz 矢量称为轴矢量 (axial vector)。最后， $\sigma^{\mu\nu}$ 的宇称变换形式为

$$\beta^{-1}\sigma^{\mu\nu}\beta = \frac{i}{2}[\beta^{-1}\gamma^\mu\beta, \beta^{-1}\gamma^\nu\beta] = \frac{i}{2}\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta[\gamma^\alpha, \gamma^\beta] = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\sigma^{\alpha\beta}, \quad (4.67)$$

即

$$\beta^{-1}\sigma^{0i}\beta = \mathcal{P}^0{}_\alpha\mathcal{P}^i{}_\beta\sigma^{\alpha\beta} = -\sigma^{0i}, \quad \beta^{-1}\sigma^{ij}\beta = \mathcal{P}^i{}_\alpha\mathcal{P}^j{}_\beta\sigma^{\alpha\beta} = +\sigma^{ij}. \quad (4.68)$$

可见，基底集合 (4.60) 是由标量 $\mathbf{1}$ 、赝标量 γ^5 、矢量 γ^μ 、轴矢量 $\gamma^\mu\gamma^5$ 和 2 阶反对称张量 $\sigma^{\mu\nu}$ 组成的，综合考虑固有保时向 Lorentz 变换和宇称变换，则这些基底的变换性质各不相同，因而它们彼此之间是相互独立的，总共有 16 个独立而完备的基底。由于独立的 $N \times N$ 矩阵最多有 N^2 个，为了得到 16 个这样的基底，需要 $N \geq 4$ 。我们考虑最简单的情况，将 Dirac 矩阵取为 4×4 矩阵。

4.2 Dirac 旋量场

在 Lorentz 群的旋量表示中, 被变换矩阵 $D(\Lambda)$ 作用的态称为 **Dirac 旋量** (spinor)。由于 $D(\Lambda)$ 是 4×4 矩阵, 一个 Dirac 旋量 ψ_a 应当具有 4 个分量 ($a = 1, 2, 3, 4$), 相应的 Lorentz 变换形式为

$$\psi'_a = D_{ab}(\Lambda)\psi_b. \quad (4.69)$$

隐去旋量指标 a 和 b , 上式化为

$$\psi' = D(\Lambda)\psi. \quad (4.70)$$

我们可以将 ψ 和 ψ' 看作列矢量, 而上式右边的乘积就是线性代数中矩阵与列矢量的乘积。

进一步, 如果 ψ_a 依赖于时空坐标 x^μ , 它就成为 **Dirac 旋量场** $\psi_a(x)$ 。类似于 (3.67) 式, 量子 Dirac 旋量场的 Lorentz 变换形式是

$$\psi'_a(x') = U^{-1}(\Lambda)\psi_a(x')U(\Lambda) = D_{ab}(\Lambda)\psi_b(x). \quad (4.71)$$

对于固有保时向 Lorentz 变换, 由 (4.12) 式可得 $D_{ab}(\Lambda)$ 的无穷小形式为

$$D_{ab}(\Lambda) = \delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}, \quad (4.72)$$

于是, (4.71) 式的无穷小形式是

$$\psi'_a(x') = \psi_a(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})_{ab}\psi_b(x). \quad (4.73)$$

将上式与 (1.168) 式比较, 可以发现, 1.7.3 小节中的 $I^{\mu\nu}$ 在旋量表示中对应于 $S^{\mu\nu}$ 。隐去旋量指标, 则 (4.71) 式化为

$$\psi'(x') = U^{-1}(\Lambda)\psi(x')U(\Lambda) = D(\Lambda)\psi(x), \quad (4.74)$$

也可以写成

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x). \quad (4.75)$$

对于无穷小变换, 根据 (3.59) 式, 将 $\psi(\Lambda^{-1}x)$ 展开到 ω 的一阶项, 得

$$\begin{aligned} \psi(\Lambda^{-1}x) &= \psi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \psi(x) = \psi(x) - \omega_{\mu\nu} x^\nu \partial^\mu \psi(x) = \psi(x) + \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\psi(x) \\ &= \psi(x) - \frac{i}{2}\omega_{\mu\nu} i(x^\mu \partial^\nu - x^\nu \partial^\mu)\psi(x) = \psi(x) - \frac{i}{2}\omega_{\mu\nu} L^{\mu\nu} \psi(x), \end{aligned} \quad (4.76)$$

其中 $L^{\mu\nu}$ 是 (3.63) 式定义微分算符。从而, (4.75) 式右边展开到 ω 一阶项的形式为

$$D(\Lambda)\psi(\Lambda^{-1}x) = \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \left[\psi(x) - \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi(x)\right] = \psi(x) - \frac{i}{2}\omega_{\mu\nu}(L^{\mu\nu} + S^{\mu\nu})\psi(x). \quad (4.77)$$

另一方面, 根据 (3.6) 式可以将 (4.75) 式左边展开为

$$U^{-1}(\Lambda)\psi(x)U(\Lambda) = \left(1 + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma}\right)\psi(x)\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)$$

$$= \psi(x) - \frac{i}{2} \omega_{\mu\nu} \psi(x) J^{\mu\nu} + \frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma} \psi(x) = \psi(x) - \frac{i}{2} \omega_{\mu\nu} [\psi(x), J^{\mu\nu}]. \quad (4.78)$$

两相比较, 得到

$$[\psi(x), J^{\mu\nu}] = (L^{\mu\nu} + S^{\mu\nu})\psi(x). \quad (4.79)$$

$S^{\mu\nu}$ 的空间分量等价于三维矢量

$$S^i \equiv \frac{1}{2} \varepsilon^{ijk} S^{jk}, \quad \mathbf{S} = (S^{23}, S^{31}, S^{12}). \quad (4.80)$$

再根据 (3.21) 和 (3.64) 式, (4.79) 式的纯空间分量部分可以改写为

$$[\psi(x), \mathbf{J}] = (\mathbf{L} + \mathbf{S})\psi(x). \quad (4.81)$$

上式表明, 除了轨道角动量 \mathbf{L} , 总角动量算符 \mathbf{J} 还生成了由 \mathbf{S} 描述的自旋角动量。

描述半整数自旋经常用到 3 个 2×2 的 **Pauli 矩阵**

$$\sigma^1 \equiv \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \quad (4.82)$$

它们都是既厄米又么正的:

$$(\sigma^i)^{-1} = (\sigma^i)^\dagger = \sigma^i. \quad (4.83)$$

Pauli 矩阵的两两乘积为

$$\begin{aligned} (\sigma^1)^2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad (\sigma^2)^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\ (\sigma^3)^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \sigma^1 \sigma^2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\sigma^3, \\ \sigma^2 \sigma^1 &= \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} -i & \\ & i \end{pmatrix} = -i\sigma^3, \quad \sigma^2 \sigma^3 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma^1, \\ \sigma^3 \sigma^2 &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} & -i \\ -i & \end{pmatrix} = -i\sigma^1, \quad \sigma^3 \sigma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma^2, \\ \sigma^1 \sigma^3 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -i\sigma^2. \end{aligned} \quad (4.84)$$

归纳起来, 有

$$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk} \sigma^k. \quad (4.85)$$

从而可得

$$[\sigma^i, \sigma^j] = i\varepsilon^{ijk} \sigma^k - i\varepsilon^{jik} \sigma^k = 2i\varepsilon^{ijk} \sigma^k, \quad (4.86)$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} + i\varepsilon^{ijk} \sigma^k + i\varepsilon^{jik} \sigma^k = 2\delta^{ij}. \quad (4.87)$$

利用 Pauli 矩阵可以将 Dirac 矩阵表示成 2×2 分块形式:

$$\gamma^0 = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}, \quad (4.88)$$

其中 $\mathbf{1}$ 表示 2×2 单位矩阵。容易验证, 这样表示的 Dirac 矩阵符合约定 (4.4), 而且满足反对易关系 (4.1):

$$\{\gamma^0, \gamma^0\} = 2 \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} = 2 \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{00}, \quad (4.89)$$

$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} = 0 = 2g^{0i}, \quad (4.90)$$

$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} -\sigma^i \sigma^j - \sigma^j \sigma^i & \\ & -\sigma^i \sigma^j - \sigma^j \sigma^i \end{pmatrix} = -2\delta^{ij} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = 2g^{ij}. \quad (4.91)$$

实际上, Dirac 矩阵有多种表示方式, (4.88) 式这种表示方式称为 **Weyl 表象**, 也称为手征表象 (chiral representation)。Dirac 矩阵的所有表示方式都是等价的, 彼此可以通过相似变换联系起来。

在 Weyl 表象中, 由 (4.86) 式可得 $S^{\mu\nu}$ 的空间分量为

$$\begin{aligned} S^{ij} &= \frac{i}{4} [\gamma^i, \gamma^j] = \frac{i}{4} \begin{pmatrix} -\sigma^i \sigma^j + \sigma^j \sigma^i & \\ & -\sigma^i \sigma^j + \sigma^j \sigma^i \end{pmatrix} \\ &= \frac{i}{4} \begin{pmatrix} -2i\varepsilon^{ijk} \sigma^k & \\ & -2i\varepsilon^{ijk} \sigma^k \end{pmatrix} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}, \end{aligned} \quad (4.92)$$

从 Pauli 矩阵的厄米性可知, S^{ij} 是厄米矩阵:

$$(S^{ij})^\dagger = S^{ij}. \quad (4.93)$$

由 (1.98) 式可得

$$S^i = \frac{1}{2} \varepsilon^{ijk} S^{jk} = \frac{1}{4} \varepsilon^{ijk} \varepsilon^{jkl} \begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{4} 2\delta^{il} \begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}. \quad (4.94)$$

于是, 自旋角动量矩阵的平方为

$$\mathbf{S}^2 = S^i S^i = \frac{1}{4} \begin{pmatrix} \sigma^i \sigma^i & \\ & \sigma^i \sigma^i \end{pmatrix} = \frac{3}{4} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \frac{1}{2} \left(\frac{1}{2} + 1 \right) = s(s+1). \quad (4.95)$$

上式最后两步省略了 4×4 单位矩阵。可见, Dirac 旋量场 $\psi(x)$ 的自旋量子数是

$$s = \frac{1}{2}. \quad (4.96)$$

经过量子化程序之后, $\psi(x)$ 应当描述自旋为 $1/2$ 的粒子。

4.3 Dirac 方程

为了写下 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量, 我们需要结合两个旋量场来得到 Lorentz 标量。在 Weyl 表象中, $S^{\mu\nu}$ 的 $0i$ 分量为

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{i}{2}\gamma^0\gamma^i = \frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix}. \quad (4.97)$$

由 Pauli 矩阵的厄米性可得

$$(S^{0i})^\dagger = -\frac{i}{2}\begin{pmatrix} -(\sigma^i)^\dagger & \\ & (\sigma^i)^\dagger \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} = -S^{0i}. \quad (4.98)$$

可见, S^{0i} 不是厄米矩阵。于是, 当 $\omega_{0i} \neq 0$ 时,

$$D^\dagger(\Lambda) = \left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \right]^\dagger = \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right] \neq \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = D^{-1}(\Lambda), \quad (4.99)$$

即 $D(\Lambda)$ 不是么正矩阵。因此, 一般地, $\psi^\dagger(x)\psi(x)$ 不是 Lorentz 标量:

$$\psi^\dagger(x')\psi(x') = \psi^\dagger(x)D^\dagger(\Lambda)D(\Lambda)\psi(x) \neq \psi^\dagger(x)\psi(x). \quad (4.100)$$

根据约定 (4.4), 可得

$$(\gamma^0)^\dagger\gamma^0 = \gamma^0\gamma^0, \quad (\gamma^i)^\dagger\gamma^0 = -\gamma^i\gamma^0 = \gamma^0\gamma^i. \quad (4.101)$$

这两条式子可以合起来写成

$$(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu. \quad (4.102)$$

从而, 有

$$(S^{\mu\nu})^\dagger\gamma^0 = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger\gamma^0 = -\frac{i}{4}[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0 = -\frac{i}{4}\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu) = \gamma^0S^{\mu\nu}. \quad (4.103)$$

于是, 可得

$$\begin{aligned} D^\dagger(\Lambda)\gamma^0 &= \exp\left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\gamma^0 = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]^n \gamma^0 = \gamma^0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)^n \\ &= \gamma^0 \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \gamma^0 D^{-1}(\Lambda). \end{aligned} \quad (4.104)$$

根据上式, 定义

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0, \quad (4.105)$$

则它的 Lorentz 变换形式为

$$\bar{\psi}'(x') = \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)D^\dagger(\Lambda)\gamma^0 = \psi^\dagger(x)\gamma^0 D^{-1}(\Lambda) = \bar{\psi}(x)D^{-1}(\Lambda). \quad (4.106)$$

这样一来, $\bar{\psi}(x)\psi(x)$ 就是一个 Lorentz 标量:

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)D(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x). \quad (4.107)$$

$\bar{\psi}(x)\psi(x)$ 这种形式的量属于旋量双线性型 (spinor bilinear), 我们可以使用 $\bar{\psi}(x)$ 构造一些 Lorentz 协变的其它旋量双线性型。 $\bar{\psi}(x)i\gamma^5\psi(x)$ 是一个 Lorentz 标量:

$$\bar{\psi}'(x')i\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)i\gamma^5D(\Lambda)\psi(x) = \bar{\psi}(x)i\gamma^5\psi(x). \quad (4.108)$$

$\bar{\psi}(x)\gamma^\mu\psi(x)$ 和 $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ 都是 Lorentz 矢量:

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\psi(x), \quad (4.109)$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu\gamma^5D(\Lambda)\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x). \quad (4.110)$$

$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$ 是一个 2 阶反对称 Lorentz 张量:

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)D^{-1}(\Lambda)\sigma^{\mu\nu}D(\Lambda)\psi(x) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}(x)\sigma^{\rho\sigma}\psi(x) \quad (4.111)$$

如果将 $\psi(x)$ 看作旋量空间中的列矢量, 则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 都是行矢量, 因而这些旋量双线性型都只是旋量空间中的 1×1 矩阵, 也就是数。由 γ^0 和 γ^5 的厄米性及 (4.102) 式可知, 这些旋量双线性型都是厄米的, 或者说, 都是实数:

$$(\bar{\psi}\psi)^\dagger = (\psi^\dagger\gamma^0\psi)^\dagger = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi, \quad (4.112)$$

$$(\bar{\psi}i\gamma^5\psi)^\dagger = -i\psi^\dagger\gamma^5\gamma^0\psi = i\psi^\dagger\gamma^0\gamma^5\psi = \bar{\psi}i\gamma^5\psi, \quad (4.113)$$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = \psi^\dagger(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \quad (4.114)$$

$$(\bar{\psi}\gamma^\mu\gamma^5\psi)^\dagger = \psi^\dagger\gamma^5(\gamma^\mu)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad (4.115)$$

$$(\bar{\psi}\sigma^{\mu\nu}\psi)^\dagger = -\frac{i}{2}\psi^\dagger[(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - (\gamma^\mu)^\dagger(\gamma^\nu)^\dagger]\gamma^0\psi = -\frac{i}{2}\psi^\dagger\gamma^0(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)\psi = \bar{\psi}\sigma^{\mu\nu}\psi. \quad (4.116)$$

此外, 包含时空导数的旋量双线性型 $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ 是 Lorentz 标量:

$$\begin{aligned} \bar{\psi}'(x')\gamma^\mu\partial'_\mu\psi'(x) &= \bar{\psi}(x)D^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) = \bar{\psi}(x)\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu\psi(x) \\ &= \bar{\psi}(x)\delta^\nu{}_\rho\gamma^\rho\partial_\nu\psi(x) = \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x). \end{aligned} \quad (4.117)$$

利用 $\bar{\psi}\gamma^\mu\partial_\mu\psi$ 和 $\bar{\psi}\psi$ 可以写下自由 Dirac 旋量场 $\psi(x)$ 的 Lorentz 不变拉氏量

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \quad (4.118)$$

于是, 有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}. \quad (4.119)$$

Euler-Lagrange 方程 (1.116) 给出

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = i(\partial_\mu\bar{\psi})\gamma^\mu + m\bar{\psi}. \quad (4.120)$$

对上式取厄米共轭, 得到

$$0 = -i(\gamma^\mu)^\dagger \partial_\mu (\psi^\dagger \gamma^0)^\dagger + m(\psi^\dagger \gamma^0)^\dagger = -i(\gamma^\mu)^\dagger \gamma^0 \partial_\mu \psi + m\gamma^0 \psi = -\gamma^0 (i\gamma^\mu \partial_\mu - m)\psi, \quad (4.121)$$

即

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (4.122)$$

上式就是 **Dirac 方程**, 标明旋量指标的形式为

$$[i(\gamma^\mu)_{ab} \partial_\mu - m\delta_{ab}]\psi_b(x) = 0. \quad (4.123)$$

可以验证, Dirac 方程具有 Lorentz 协变性:

$$\begin{aligned} (i\gamma^\mu \partial'_\mu - m)\psi'(x') &= [i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]D(\Lambda)\psi(x) = D(\Lambda)[iD^{-1}(\Lambda)\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\psi(x) \\ &= D(\Lambda)[i\Lambda^\mu{}_\rho \gamma^\rho (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\psi(x) = D(\Lambda)(i\delta^\nu{}_\rho \gamma^\rho \partial_\nu - m)\psi(x) \\ &= D(\Lambda)(i\gamma^\nu \partial_\nu - m)\psi(x) = 0. \end{aligned} \quad (4.124)$$

对 Dirac 方程 (4.122) 左边乘以 $(-i\gamma^\mu \partial_\mu - m)$, 利用反对易关系 (4.1), 可得

$$\begin{aligned} 0 &= (-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi = (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = \left[\frac{1}{2} \gamma^\mu \gamma^\nu (\partial_\mu \partial_\nu + \partial_\nu \partial_\mu) + m^2 \right] \psi \\ &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi = (g^{\mu\nu} \partial_\mu \partial_\nu + m^2)\psi = (\partial^2 + m^2)\psi. \end{aligned} \quad (4.125)$$

也就是说, 自由的 Dirac 旋量场 $\psi(x)$ 满足 *Klein-Gordon* 方程

$$(\partial^2 + m^2)\psi(x) = 0. \quad (4.126)$$

由 (4.92) 和 (4.97) 式可以看出, 旋量表示的生成元在 Weyl 表象中都是分块对角的, 因而它可以分解为两个 2 维表示的直和。相应地, 可以把具有 4 个分量的 Dirac 旋量场 ψ 分解为两个二分量旋量 φ_L 和 φ_R :

$$\psi = \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix}. \quad (4.127)$$

这样的二分量旋量称为 **Weyl 旋量**, 其中, φ_L 称为左手 (left-handed) Weyl 旋量, φ_R 称为右手 (right-handed) Weyl 旋量。

用 2×2 单位矩阵和 Pauli 矩阵定义

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma}), \quad (4.128)$$

那么, Weyl 表象中的 Dirac 矩阵 (4.88) 可以简洁地表示成

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}. \quad (4.129)$$

从而, Dirac 方程 (4.122) 化为

$$0 = (i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} = \begin{pmatrix} i\sigma^\mu \partial_\mu \varphi_R - m\varphi_L \\ i\bar{\sigma}^\mu \partial_\mu \varphi_L - m\varphi_R \end{pmatrix}, \quad (4.130)$$

即

$$\begin{cases} i\bar{\sigma}^\mu \partial_\mu \varphi_L - m\varphi_R = 0, \\ i\sigma^\mu \partial_\mu \varphi_R - m\varphi_L = 0. \end{cases} \quad (4.131)$$

这是一组相互耦合的方程。如果 $m = 0$, 方程组中的两个方程就变得相互独立了:

$$i\bar{\sigma}^\mu \partial_\mu \varphi_L = 0, \quad i\sigma^\mu \partial_\mu \varphi_R = 0. \quad (4.132)$$

这两个独立的方程称为 **Weyl 方程**。可见, 非零质量 m 的存在将左手和右手 Weyl 旋量耦合起来。

4.4 Dirac 旋量场的平面波展开

4.4.1 平面波解的一般形式

本小节讨论与表象选取无关。

对于确定的动量 \mathbf{p} , 我们假设 Dirac 方程具有如下形式的平面波解:

$$\psi_a(x; \mathbf{k}) = w_a(k^0, \mathbf{k})e^{-ik \cdot x}. \quad (4.133)$$

其中, 系数 $w_a(k^0, \mathbf{k})$ 是四分量旋量, 带着一个旋量指标 a 。隐去旋量指标, 将这个平面波解代入到 Dirac 方程 (4.122) 中, 可得

$$0 = (i\gamma^\mu \partial_\mu - m)\psi(x; \mathbf{k}) = (\gamma^\mu k_\mu - m)w(k^0, \mathbf{k})e^{-ik \cdot x} = (k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)w(k^0, \mathbf{k})e^{-ik \cdot x}. \quad (4.134)$$

因此, 有

$$(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)w(k^0, \mathbf{k}) = 0. \quad (4.135)$$

对上式左乘 γ^0 , 可得

$$[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]w(k^0, \mathbf{k}) = 0. \quad (4.136)$$

通过移项, 上式化为

$$[\gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) + m\gamma^0]w(k^0, \mathbf{k}) = k^0w(k^0, \mathbf{k}). \quad (4.137)$$

这是一个本征值方程, 它具有非平庸解的条件是特征多项式 $\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0]$ 为零, 即

$$\det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = 0. \quad (4.138)$$

这个方程的根给出 k^0 的本征值, 相应的非平庸解是本征矢量。

方程 (4.138) 可化为

$$0 = \det[k^0 - \gamma^0(\mathbf{k} \cdot \boldsymbol{\gamma}) - m\gamma^0] = \det[\gamma^0(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det(\gamma^0) \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m). \quad (4.139)$$

由 (4.3) 式可得 $[\det(\gamma^0)]^2 = \det(\gamma^0\gamma^0) = \det(\mathbf{1}) = 1$, 故 $\det(\gamma^0) \neq 0$ 。因而方程 (4.138) 等价于

$$\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = 0. \quad (4.140)$$

利用 (4.48) 式, 上式左边可化为

$$\begin{aligned} \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) &= \det[(\gamma^5)^2(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det[\gamma^5(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)\gamma^5] \\ &= \det[(\gamma^5)^2(-k^0\gamma^0 + \mathbf{k} \cdot \boldsymbol{\gamma} - m)] = \det[-(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m]. \end{aligned} \quad (4.141)$$

这里第二步用到行列式性质

$$\det(AB) = \det(BA), \quad (4.142)$$

第三步用到 γ^5 与 γ^μ 反对易的性质 (4.50)。由反对易关系 (4.1) 有

$$(k_\mu\gamma^\mu)^2 = k_\mu k_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} k_\mu k_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = k_\mu k_\nu g^{\mu\nu} \mathbf{1} = k^2 \mathbf{1} = [(k^0)^2 - |\mathbf{k}|^2] \mathbf{1}. \quad (4.143)$$

从而, 可得

$$\begin{aligned} [\det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m)]^2 &= \det[(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \det[-(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma}) - m] \\ &= \det(k_\mu\gamma^\mu - m) \det(-k_\mu\gamma^\mu - m) = \det[(k_\mu\gamma^\mu - m)(-k_\mu\gamma^\mu - m)] \\ &= \det[-(k_\mu\gamma^\mu)^2 + m^2] = \det\{[-(k^0)^2 + |\mathbf{k}|^2 + m^2] \mathbf{1}\} \\ &= [-(k^0)^2 + |\mathbf{k}|^2 + m^2]^4 = [E_{\mathbf{k}}^2 - (k^0)^2]^4, \end{aligned} \quad (4.144)$$

其中 $E_{\mathbf{k}} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$ 。于是, 方程 (4.140) 化为

$$0 = \det(k^0\gamma^0 - \mathbf{k} \cdot \boldsymbol{\gamma} - m) = [E_{\mathbf{k}}^2 - (k^0)^2]^2 = (E_{\mathbf{k}} + k^0)^2 (E_{\mathbf{k}} - k^0)^2. \quad (4.145)$$

这个方程有 2 个根 $k^0 = \pm E_{\mathbf{k}}$; 这 2 个根都是 2 重根, 各自对应于 2 个独立的本征矢量, 共有 4 个线性无关的本征矢量。

(1) $k^0 = E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.146)$$

因而平面波解中有 2 个正能解, 形式为

$$w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2. \quad (4.147)$$

(2) $k^0 = -E_{\mathbf{k}}$ 对应于 2 个本征矢量

$$w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.148)$$

因而平面波解中有 2 个负能解, 形式为

$$w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t + \mathbf{k} \cdot \mathbf{x})], \quad \sigma = 1, 2. \quad (4.149)$$

可以将这 4 个本征矢量的正交归一关系取为

$$\begin{aligned} w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma') &= 2E_{\mathbf{k}} \delta_{\sigma\sigma'}, & w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma') &= 2E_{\mathbf{k}} \delta_{\sigma\sigma'}, \\ w^{(+)\dagger}(E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma') &= w^{(-)\dagger}(-E_{\mathbf{k}}, \mathbf{k}; \sigma) w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma') = 0. \end{aligned} \quad (4.150)$$

按如下定义引入四分量旋量 $u(\mathbf{k}; \sigma)$ 和 $v(\mathbf{k}; \sigma)$:

$$u(\mathbf{k}; \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad v(-\mathbf{k}; \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, \mathbf{k}; \sigma), \quad \sigma = 1, 2. \quad (4.151)$$

第二个定义式等价于

$$v(\mathbf{k}; \sigma) = w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}; \sigma). \quad (4.152)$$

于是, Dirac 方程的正能解和负能解可以分别写作

$$\psi^{(+)}(x; \mathbf{k}; \sigma) \equiv w^{(+)}(E_{\mathbf{k}}, \mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = u(\mathbf{k}; \sigma) \exp[-i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})], \quad (4.153)$$

$$\psi^{(-)}(x; \mathbf{k}; \sigma) \equiv w^{(-)}(-E_{\mathbf{k}}, -\mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})] = v(\mathbf{k}; \sigma) \exp[i(E_{\mathbf{k}}t - \mathbf{k} \cdot \mathbf{x})]. \quad (4.154)$$

替换一下动量记号, 可得

$$\psi^{(+)}(x; \mathbf{p}; \sigma) = u(\mathbf{p}; \sigma) e^{-ip \cdot x}, \quad \psi^{(-)}(x; \mathbf{p}; \sigma) = v(\mathbf{p}; \sigma) e^{ip \cdot x}, \quad p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.155)$$

从而, Dirac 旋量场算符 $\psi(\mathbf{x}, t)$ 的平面波展开式可写作

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=1}^2 [\psi^{(+)}(x; \mathbf{p}; \sigma) a_{\mathbf{p};\sigma} + \psi^{(-)}(x; \mathbf{p}; \sigma) b_{\mathbf{p};\sigma}^{\dagger}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=1}^2 [u(\mathbf{p}; \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x} + v(\mathbf{p}; \sigma) b_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}]. \end{aligned} \quad (4.156)$$

其中, $a_{\mathbf{p};\sigma}$ 是湮灭算符, $b_{\mathbf{p};\sigma}^{\dagger}$ 是产生算符。一般地, $a_{\mathbf{p};\sigma} \neq b_{\mathbf{p};\sigma}$

旋量系数 $u(\mathbf{p}; \sigma)$ 和 $v(\mathbf{p}; \sigma)$ 的正交归一关系为

$$u^{\dagger}(\mathbf{p}; \sigma) u(\mathbf{p}; \sigma') = w^{(+)\dagger}(E_{\mathbf{p}}, \mathbf{p}; \sigma) w^{(+)}(E_{\mathbf{p}}, \mathbf{p}; \sigma') = 2E_{\mathbf{p}} \delta_{\sigma\sigma'}, \quad (4.157)$$

$$v^{\dagger}(\mathbf{p}; \sigma) v(\mathbf{p}; \sigma') = w^{(-)\dagger}(-E_{\mathbf{p}}, -\mathbf{p}; \sigma) w^{(-)}(-E_{\mathbf{p}}, -\mathbf{p}; \sigma') = 2E_{\mathbf{p}} \delta_{\sigma\sigma'}, \quad (4.158)$$

$$u^{\dagger}(\mathbf{p}; \sigma) v(-\mathbf{p}; \sigma') = w^{(+)\dagger}(E_{\mathbf{p}}, \mathbf{p}; \sigma) w^{(-)}(-E_{\mathbf{p}}, \mathbf{p}; \sigma') = 0. \quad (4.159)$$

4.4.2 Weyl 表象中的平面波解

本小节在 Weyl 表象中讨论 Dirac 方程的平面波解。

Dirac 旋量场描述自旋为 $1/2$ 的粒子, 因而粒子的自旋在动量方向上的投影有 2 种取值, $+1/2$ 和 $-1/2$, 归一化后对应于 2 种螺旋度 $\lambda = \pm$ 。类似于矢量场的情况, Dirac 旋量场所描述的粒子的状态可以用螺旋度本征值 λ 来表征。因此, 无论是平面波解的正能解还是负能解, 都能够以 2 种螺旋度本征态作为 2 个独立的本征矢量。

按照这个思路, 可以把正能解的 2 个本征矢量记作

$$\psi^{(+)}(x; \mathbf{p}, \lambda) = u(\mathbf{p}, \lambda)e^{-ip \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.160)$$

根据 Dirac 方程 (4.122), 有

$$0 = (i\gamma^\mu \partial_\mu - m)\psi^{(+)}(x; \mathbf{p}, \lambda) = (p_\mu \gamma^\mu - m)u(\mathbf{p}, \lambda)e^{-ip \cdot x}, \quad (4.161)$$

即

$$(\not{p} - m)u(\mathbf{p}, \lambda) = 0, \quad (4.162)$$

其中, \not{p} 的定义为

$$\not{p} \equiv p_\mu \gamma^\mu. \quad (4.163)$$

这种斜线记号称为 **Dirac 斜线** (slash), 是 R. Feynman 引进的。

将四分量旋量 $u(\mathbf{p}, \lambda)$ 分解为两个二分量旋量 $f_\lambda(\mathbf{p})$ 和 $g_\lambda(\mathbf{p})$,

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.164)$$

那么, 根据 Weyl 表象中的 Dirac 矩阵表达式 (4.129), 方程 (4.162) 化为

$$0 = (\not{p} - m)u(\mathbf{p}, \lambda) = \begin{pmatrix} -m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & -m \end{pmatrix} \begin{pmatrix} f_\lambda(\mathbf{p}) \\ g_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu f_\lambda(\mathbf{p}) - m g_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.165)$$

即

$$(p \cdot \sigma)g_\lambda(\mathbf{p}) - m f_\lambda(\mathbf{p}) = 0, \quad (4.166)$$

$$(p \cdot \bar{\sigma})f_\lambda(\mathbf{p}) - m g_\lambda(\mathbf{p}) = 0. \quad (4.167)$$

将 (4.129) 式代入反对易关系 (4.1), 可得

$$2g^{\mu\nu} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}, \quad (4.168)$$

故

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2g^{\mu\nu}, \quad (4.169)$$

$$\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu}. \quad (4.170)$$

因而, 有

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = \frac{1}{2} p_\mu p_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2} p_\mu p_\nu 2g^{\mu\nu} = p^2. \quad (4.171)$$

由方程 (4.167) 可得

$$g_\lambda(\mathbf{p}) = \frac{p \cdot \bar{\sigma}}{m} f_\lambda(\mathbf{p}). \quad (4.172)$$

将上式代入到由方程 (4.166) 得出的关系中, 有

$$f_\lambda(\mathbf{p}) = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{m} g_\lambda(\mathbf{p}) = \frac{1}{m^2} (\mathbf{p} \cdot \boldsymbol{\sigma})(\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}) f_\lambda(\mathbf{p}) = \frac{p^2}{m^2} f_\lambda(\mathbf{p}) = f_\lambda(\mathbf{p}). \quad (4.173)$$

可见, 关系式 (4.172) 是自洽的。这样的话, 只要选取合适的 $f_\lambda(\mathbf{p})$, 然后由 (4.164) 和 (4.172) 式得到

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}}{m} f_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.174)$$

就可以满足方程 (4.162)。

在 Weyl 表象中, 根据 (4.94) 式, 自旋角动量矩阵 \mathbf{S} 在动量 \mathbf{p} 方向上的投影为

$$\hat{\mathbf{p}} \cdot \mathbf{S} = \frac{1}{2} \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}. \quad (4.175)$$

归一化后, 得到螺旋度矩阵

$$2\hat{\mathbf{p}} \cdot \mathbf{S} = \begin{pmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & \\ & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}. \quad (4.176)$$

上式的两个分块相同, 因此, 左手和右手 Weyl 旋量对应的螺旋度矩阵是相同的, 都是

$$\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}|} = \frac{1}{|\mathbf{p}|} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}. \quad (4.177)$$

引入作为螺旋度本征态的二分量旋量 $\xi_\lambda(\mathbf{p})$, 满足

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda \xi_\lambda(\mathbf{p}), \quad \lambda = \pm. \quad (4.178)$$

我们要求 $\xi_\lambda(\mathbf{p})$ 具有正交归一关系

$$\xi_\lambda^\dagger(\mathbf{p}) \xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'} \quad (4.179)$$

和完备性关系

$$\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) = \mathbf{1}. \quad (4.180)$$

此外, 由 $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ 可得

$$(\mathbf{p} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) = \lambda |\mathbf{p}| \xi_\lambda(\mathbf{p}) \quad (4.181)$$

我们将 $\xi_\lambda(\mathbf{p})$ 称为螺旋态。在实际应用中, 可以把螺旋态 $\xi_\lambda(\mathbf{p})$ 取为如下形式:

$$\xi_+(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} -p^1 + ip^2 \\ |\mathbf{p}| + p^3 \end{pmatrix}. \quad (4.182)$$

可以验证, 它们确实是 $\lambda = \pm$ 的本征态:

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_+(\mathbf{p}) = \frac{1}{|\mathbf{p}| \sqrt{2|\mathbf{p}|(|\mathbf{p}| + p^3)}} \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix} \begin{pmatrix} |\mathbf{p}| + p^3 \\ p^1 + ip^2 \end{pmatrix}$$

$$\begin{aligned}
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} p^3(|\mathbf{p}|+p^3) + (p^1-ip^2)(p^1+ip^2) \\ (p^1+ip^2)(|\mathbf{p}|+p^3) - p^3(p^1+ip^2) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} p^3|\mathbf{p}| + |\mathbf{p}|^2 \\ (p^1+ip^2)|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} p^3 + |\mathbf{p}| \\ p^1 + ip^2 \end{pmatrix} \\
&= +\xi_+(\mathbf{p}), \tag{4.183}
\end{aligned}$$

$$\begin{aligned}
(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_-(\mathbf{p}) &= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix} \begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} -p^3(p^1-ip^2) + (p^1-ip^2)(|\mathbf{p}|+p^3) \\ (p^1+ip^2)(-p^1+ip^2) - p^3(|\mathbf{p}|+p^3) \end{pmatrix} \\
&= \frac{1}{|\mathbf{p}|\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} (p^1-ip^2)|\mathbf{p}| \\ -|\mathbf{p}|^2 - p^3|\mathbf{p}| \end{pmatrix} = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}} \begin{pmatrix} p^1-ip^2 \\ -|\mathbf{p}| - p^3 \end{pmatrix} \\
&= -\xi_-(\mathbf{p}). \tag{4.184}
\end{aligned}$$

而且, 满足正交归一关系:

$$\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_+(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} \begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2 \\ p^1+ip^2 \end{pmatrix} \begin{pmatrix} |\mathbf{p}|+p^3 \\ p^1+ip^2 \end{pmatrix} \\
&= \frac{(|\mathbf{p}|+p^3)^2 + |p^1+ip^2|^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2 + 2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1, \tag{4.185}
\end{aligned}$$

$$\begin{aligned}
\xi_-^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} \begin{pmatrix} -p^1-ip^2 & |\mathbf{p}|+p^3 \\ |\mathbf{p}|+p^3 \end{pmatrix} \begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{|-p^1+ip^2|^2 + (|\mathbf{p}|+p^3)^2}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = \frac{2|\mathbf{p}|^2 + 2p^3|\mathbf{p}|}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 1, \tag{4.186}
\end{aligned}$$

$$\begin{aligned}
\xi_+^\dagger(\mathbf{p})\xi_-(\mathbf{p}) &= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} \begin{pmatrix} |\mathbf{p}|+p^3 & p^1-ip^2 \\ p^1+ip^2 \end{pmatrix} \begin{pmatrix} -p^1+ip^2 \\ |\mathbf{p}|+p^3 \end{pmatrix} \\
&= \frac{-(|\mathbf{p}|+p^3)(p^1-ip^2) + (|\mathbf{p}|+p^3)(p^1-ip^2)}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} = 0. \tag{4.187}
\end{aligned}$$

也满足完备性关系:

$$\begin{aligned}
\sum_{\lambda=\pm} \xi_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) &= \xi_+(\mathbf{p})\xi_+^\dagger(\mathbf{p}) + \xi_-(\mathbf{p})\xi_-^\dagger(\mathbf{p}) \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} \begin{pmatrix} (|\mathbf{p}|+p^3)^2 + |-p^1+ip^2|^2 & (|\mathbf{p}|+p^3)(p^1-ip^2) + (|\mathbf{p}|+p^3)(-p^1+ip^2) \\ (|\mathbf{p}|+p^3)(p^1+ip^2) + (|\mathbf{p}|+p^3)(-p^1-ip^2) & |p^1+ip^2|^2 + (|\mathbf{p}|+p^3)^2 \end{pmatrix} \\
&= \frac{1}{2|\mathbf{p}|(|\mathbf{p}|+p^3)} \begin{pmatrix} 2|\mathbf{p}|^2 + 2p^3|\mathbf{p}| & 0 \\ 0 & 2|\mathbf{p}|^2 + 2p^3|\mathbf{p}| \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}. \tag{4.188}
\end{aligned}$$

当 $p^3 = -|\mathbf{p}|$ 时, (4.182) 式失去良好的定义, 此时我们可以将螺旋态取成

$$\xi_+(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_-(\mathbf{p}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{4.189}$$

现在, 将 $f_\lambda(\mathbf{p})$ 取为

$$f_\lambda(\mathbf{p}) = C_\lambda \xi_\lambda(\mathbf{p}), \quad (4.190)$$

其中 C_λ 是常数。从而, 利用 (4.181) 式, (4.174) 式可化为

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} f_\lambda(\mathbf{p}) \\ \frac{\mathbf{p} \cdot \bar{\boldsymbol{\sigma}}}{m} f_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix} = C_\lambda \begin{pmatrix} \xi_\lambda(\mathbf{p}) \\ \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} \xi_\lambda(\mathbf{p}) \end{pmatrix}. \quad (4.191)$$

再取

$$C_\lambda = \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \quad (4.192)$$

则由

$$\sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} = \sqrt{E_{\mathbf{p}}^2 - \lambda^2|\mathbf{p}|^2} = \sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2} = m, \quad (4.193)$$

有

$$\begin{aligned} C_\lambda \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} &= \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \frac{E_{\mathbf{p}} + \lambda|\mathbf{p}|}{m} = \frac{\sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}}{m} \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} \\ &= \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}. \end{aligned} \quad (4.194)$$

于是, 得到 $u(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$u(\mathbf{p}, \lambda) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \xi_\lambda(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \xi_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.195)$$

其中, $\omega_\lambda(\mathbf{p})$ 定义为

$$\omega_\lambda(\mathbf{p}) \equiv \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}, \quad (4.196)$$

它是关于 \mathbf{p} 的偶函数:

$$\omega_\lambda(-\mathbf{p}) = \omega_\lambda(\mathbf{p}). \quad (4.197)$$

这样的话, 根据 (4.176) 式, $u(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 λ :

$$(2\hat{\mathbf{p}} \cdot \mathbf{S})u(\mathbf{p}, \lambda) = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} = \lambda u(\mathbf{p}, \lambda). \quad (4.198)$$

另一方面, 可以把负能解的 2 个本征矢量记作

$$\psi^{(-)}(x; \mathbf{p}, \lambda) = v(\mathbf{p}, \lambda) e^{ip \cdot x}, \quad \lambda = \pm, \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (4.199)$$

根据 Dirac 方程 (4.122), 有

$$0 = (i\gamma^\mu \partial_\mu - m)\psi^{(-)}(x; \mathbf{p}, \lambda) = (-p_\mu \gamma^\mu - m)v(\mathbf{p}, \lambda) e^{ip \cdot x}, \quad (4.200)$$

即

$$(\not{p} + m)v(\mathbf{p}, \lambda) = 0. \quad (4.201)$$

同样, 将四分量旋量 $v(\mathbf{p}, \lambda)$ 分解为两个二分量旋量 $\tilde{f}_\lambda(\mathbf{p})$ 和 $\tilde{g}_\lambda(\mathbf{p})$,

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.202)$$

则有

$$0 = (\not{p} + m)v(\mathbf{p}, \lambda) = \begin{pmatrix} m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & m \end{pmatrix} \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} p_\mu \sigma^\mu \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) \\ p_\mu \bar{\sigma}^\mu \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) \end{pmatrix}, \quad (4.203)$$

即

$$(p \cdot \sigma) \tilde{g}_\lambda(\mathbf{p}) + m \tilde{f}_\lambda(\mathbf{p}) = 0, \quad (4.204)$$

$$(p \cdot \bar{\sigma}) \tilde{f}_\lambda(\mathbf{p}) + m \tilde{g}_\lambda(\mathbf{p}) = 0. \quad (4.205)$$

由方程 (4.205) 可得

$$\tilde{g}_\lambda(\mathbf{p}) = -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}). \quad (4.206)$$

将上式代入到由方程 (4.204) 得出的关系中, 根据 (4.171) 式, 有

$$\tilde{f}_\lambda(\mathbf{p}) = -\frac{p \cdot \sigma}{m} \tilde{g}_\lambda(\mathbf{p}) = \frac{1}{m^2} (p \cdot \sigma) (p \cdot \bar{\sigma}) \tilde{f}_\lambda(\mathbf{p}) = \frac{p^2}{m^2} \tilde{f}_\lambda(\mathbf{p}) = \tilde{f}_\lambda(\mathbf{p}). \quad (4.207)$$

可见, 关系式 (4.206) 是自洽的。

现在, 将 $\tilde{f}_\lambda(\mathbf{p})$ 取为

$$\tilde{f}_\lambda(\mathbf{p}) = \tilde{C}_\lambda \xi_{-\lambda}(\mathbf{p}), \quad (4.208)$$

其中 \tilde{C}_λ 是常数。在这里, 我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$, 而非 $\xi_\lambda(\mathbf{p})$ 。这种取法的原因将在 4.5.4 小节中说明, 现在姑且接受这种选择。从而, 有

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} \tilde{f}_\lambda(\mathbf{p}) \\ -\frac{p \cdot \bar{\sigma}}{m} \tilde{f}_\lambda(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \tilde{C}_\lambda \begin{pmatrix} \xi_{-\lambda}(\mathbf{p}) \\ -\frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}. \quad (4.209)$$

再取

$$\tilde{C}_\lambda = -\lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|}, \quad (4.210)$$

则由

$$\begin{aligned} -\tilde{C}_\lambda \frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} &= \lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \frac{E_{\mathbf{p}} - \lambda|\mathbf{p}|}{m} = \lambda \frac{\sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}}{m} \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} \\ &= \lambda \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|}, \end{aligned} \quad (4.211)$$

可得 $v(\mathbf{p}, \lambda)$ 的螺旋态表达式

$$v(\mathbf{p}, \lambda) = \begin{pmatrix} -\lambda \sqrt{E_{\mathbf{p}} + \lambda|\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \\ \lambda \sqrt{E_{\mathbf{p}} - \lambda|\mathbf{p}|} \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix}. \quad (4.212)$$

这样一来, $v(\mathbf{p}, \lambda)$ 是螺旋度本征态, 本征值为 $-\lambda$:

$$(2\hat{\mathbf{p}} \cdot \mathbf{S})v(\mathbf{p}, \lambda) = \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) (\hat{\mathbf{p}} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = -\lambda v(\mathbf{p}, \lambda). \quad (4.213)$$

根据 $\xi_{\lambda}(\mathbf{p})$ 的正交归一关系 (4.179), 可以验证, $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 满足 (4.157) 和 (4.158) 式表示的正交归一关系:

$$\begin{aligned} u^{\dagger}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) & \omega_{\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = [\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})\delta_{\lambda\lambda'} + \omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &= [\omega_{-\lambda}^2(\mathbf{p}) + \omega_{\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_{\mathbf{p}} - \lambda|\mathbf{p}|) + (E_{\mathbf{p}} + \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \end{aligned} \quad (4.214)$$

$$\begin{aligned} v^{\dagger}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} -\lambda \omega_{\lambda}(\mathbf{p})\xi_{-\lambda}^{\dagger}(\mathbf{p}) & \lambda \omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda' \omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda' \omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda\lambda'[\omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_{-\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) = \lambda\lambda'[\omega_{\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &= \lambda^2[\omega_{\lambda}^2(\mathbf{p}) + \omega_{-\lambda}^2(\mathbf{p})]\delta_{\lambda\lambda'} = [(E_{\mathbf{p}} + \lambda|\mathbf{p}|) + (E_{\mathbf{p}} - \lambda|\mathbf{p}|)]\delta_{\lambda\lambda'} = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \end{aligned} \quad (4.215)$$

依照螺旋态的本征值方程 (4.178), 可得

$$(-\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = -\lambda \xi_{-\lambda}(-\mathbf{p}), \quad (4.216)$$

从而, 有

$$(\hat{\mathbf{p}} \cdot \boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p}) = \lambda \xi_{-\lambda}(-\mathbf{p}). \quad (4.217)$$

可见, $\xi_{-\lambda}(-\mathbf{p})$ 与 $\xi_{\lambda}(\mathbf{p})$ 服从相同的本征值方程, 这意味着 $\xi_{-\lambda}(-\mathbf{p}) \propto \xi_{\lambda}(\mathbf{p})$, 故

$$\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \propto \xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}. \quad (4.218)$$

于是, (4.159) 式表示的正交关系也成立:

$$\begin{aligned} u^{\dagger}(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) & \omega_{\lambda}(\mathbf{p})\xi_{\lambda}^{\dagger}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda' \omega_{\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\ \lambda' \omega_{-\lambda'}(-\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \end{pmatrix} \\ &= \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(-\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda'}(-\mathbf{p})]\xi_{\lambda}^{\dagger}(\mathbf{p})\xi_{-\lambda'}(-\mathbf{p}) \\ &\propto \lambda'[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} \\ &\propto \lambda[-\omega_{-\lambda}(\mathbf{p})\omega_{\lambda}(\mathbf{p}) + \omega_{\lambda}(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{\lambda\lambda'} = 0. \end{aligned} \quad (4.219)$$

整理一下, 旋量系数 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 满足如下正交归一关系:

$$u^{\dagger}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = v^{\dagger}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = 2E_{\mathbf{p}}\delta_{\lambda\lambda'}, \quad u^{\dagger}(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda') = v^{\dagger}(-\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0. \quad (4.220)$$

此外, 由 (4.193) 式有

$$\omega_{\lambda}(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) = \sqrt{(E_{\mathbf{p}} + \lambda|\mathbf{p}|)(E_{\mathbf{p}} - \lambda|\mathbf{p}|)} = m. \quad (4.221)$$

从而, 利用

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda) &= u^\dagger(\mathbf{p}, \lambda)\gamma^0 = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix},\end{aligned}\quad (4.222)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda) &= v^\dagger(\mathbf{p}, \lambda)\gamma^0 = \begin{pmatrix} -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix},\end{aligned}\quad (4.223)$$

可得

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= [\omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = 2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} = 2m\delta_{\lambda\lambda'},\end{aligned}\quad (4.224)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda'\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda'\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= -\lambda\lambda'[\omega_{-\lambda}(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_\lambda(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda\lambda'} = -2\lambda^2\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})\delta_{\lambda\lambda'} \\ &= -2m\delta_{\lambda\lambda'},\end{aligned}\quad (4.225)$$

$$\begin{aligned}\bar{u}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') &= \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p})\xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} -\lambda'\omega_{\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ \lambda'\omega_{-\lambda'}(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\xi_\lambda^\dagger(\mathbf{p})\xi_{-\lambda'}(\mathbf{p}) \\ &= \lambda'[-\omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p})]\delta_{\lambda, -\lambda'} \\ &= -\lambda[-\omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p}) + \omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p})]\delta_{\lambda, -\lambda'} = 0,\end{aligned}\quad (4.226)$$

$$\begin{aligned}\bar{v}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') &= \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_{-\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ \omega_{\lambda'}(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \end{pmatrix} \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\xi_{-\lambda}^\dagger(\mathbf{p})\xi_{\lambda'}(\mathbf{p}) \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_{-\lambda'}(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{\lambda'}(\mathbf{p})]\delta_{-\lambda, \lambda'} \\ &= \lambda[\omega_{-\lambda}(\mathbf{p})\omega_\lambda(\mathbf{p}) - \omega_\lambda(\mathbf{p})\omega_{-\lambda}(\mathbf{p})]\delta_{-\lambda, \lambda'} = 0.\end{aligned}\quad (4.227)$$

整理一下, 有

$$\bar{u}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 2m\delta_{\lambda\lambda'}, \quad \bar{v}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = -2m\delta_{\lambda\lambda'}, \quad \bar{u}(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda') = \bar{v}(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda') = 0. \quad (4.228)$$

另一方面, 利用等式

$$(p \cdot \bar{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} + \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_\lambda^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \quad (4.229)$$

$$(p \cdot \sigma)\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma})\xi_\lambda(\mathbf{p}) = (E_{\mathbf{p}} - \lambda|\mathbf{p}|)\xi_\lambda(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p})\xi_\lambda(\mathbf{p}), \quad (4.230)$$

以及 (4.221) 式和 $\xi_\lambda(\mathbf{p})$ 的完备性关系 (4.180), 可得

$$\begin{aligned}
 \sum_{\lambda=\pm} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \omega_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
 &= \sum_{\lambda=\pm} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p}) \omega_\lambda(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_{-\lambda}^2(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \\ \omega_\lambda^2(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & \omega_\lambda(\mathbf{p}) \omega_{-\lambda}(\mathbf{p}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \end{pmatrix} \\
 &= \sum_{\lambda=\pm} \begin{pmatrix} m \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & (p \cdot \sigma) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) \\ (p \cdot \bar{\sigma}) \xi_\lambda(\mathbf{p}) \xi_\lambda^\dagger(\mathbf{p}) & m \xi_\lambda^\dagger(\mathbf{p}) \xi_\lambda(\mathbf{p}) \end{pmatrix} \\
 &= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = p_\mu \gamma^\mu + m. \tag{4.231}
 \end{aligned}$$

通过等式

$$(p \cdot \bar{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} + \mathbf{p} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} - \lambda |\mathbf{p}|) \xi_{-\lambda}(\mathbf{p}) = \omega_{-\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}), \tag{4.232}$$

$$(p \cdot \sigma) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} - \mathbf{p} \cdot \boldsymbol{\sigma}) \xi_{-\lambda}(\mathbf{p}) = (E_{\mathbf{p}} + \lambda |\mathbf{p}|) \xi_{-\lambda}(\mathbf{p}) = \omega_\lambda^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}), \tag{4.233}$$

则可以得到

$$\begin{aligned}
 \sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) &= \sum_{\lambda=\pm} \begin{pmatrix} -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \\ \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} \lambda \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda \omega_\lambda(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
 &= \sum_{\lambda=\pm} \begin{pmatrix} -\lambda^2 \omega_\lambda(\mathbf{p}) \omega_{-\lambda}(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & \lambda^2 \omega_\lambda^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \\ \lambda^2 \omega_{-\lambda}^2(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -\lambda^2 \omega_{-\lambda}(\mathbf{p}) \omega_\lambda(\mathbf{p}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
 &= \sum_{\lambda=\pm} \begin{pmatrix} -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & (p \cdot \sigma) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \\ (p \cdot \bar{\sigma}) \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) & -m \xi_{-\lambda}(\mathbf{p}) \xi_{-\lambda}^\dagger(\mathbf{p}) \end{pmatrix} \\
 &= \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = p_\mu \gamma^\mu - m. \tag{4.234}
 \end{aligned}$$

整理一下, 有如下螺旋度求和关系, 或者说, 自旋求和关系:

$$\sum_{\lambda=\pm} u(\mathbf{p}, \lambda) \bar{u}(\mathbf{p}, \lambda) = \not{p} + m, \quad \sum_{\lambda=\pm} v(\mathbf{p}, \lambda) \bar{v}(\mathbf{p}, \lambda) = \not{p} - m. \tag{4.235}$$

用 $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 可以把 Dirac 旋量场算符 $\psi(\mathbf{x}, t)$ 的平面波展开式写作

$$\begin{aligned}
 \psi(\mathbf{x}, t) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\psi^{(+)}(x; \mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} + \psi^{(-)}(x; \mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger \right] \\
 &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} \right]. \tag{4.236}
 \end{aligned}$$

从而, 有

$$\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right], \tag{4.237}$$

$$\bar{\psi}(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\bar{u}(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + \bar{v}(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]. \tag{4.238}$$

4.4.3 哈密顿量和产生湮灭算符

根据 (4.119) 式, $\psi(x)$ 对应的共轭动量密度是

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \quad (4.239)$$

它的平面波展开式为

$$\pi(\mathbf{x}, t) = i\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right]. \quad (4.240)$$

自由运动的旋量场 $\psi(x)$ 满足 Dirac 方程 (4.122), 相应地, 拉氏量 (4.118) 化为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (4.241)$$

于是, 根据 (1.119) 式, 自由 Dirac 旋量场的哈密顿量密度为

$$\mathcal{H} = \pi \partial_0 \psi - \mathcal{L} = \pi \partial_0 \psi = i\psi^\dagger \partial_0 \psi. \quad (4.242)$$

从而, 哈密顿量为

$$\begin{aligned} H &= \int d^3 x \mathcal{H} = \int d^3 x \psi^\dagger i \partial_0 \psi \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p}, \lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p}, \lambda} e^{-ip \cdot x} \right] \\ &\quad \times \left[q_0 u(\mathbf{q}, \lambda') a_{\mathbf{q}, \lambda'} e^{-iq \cdot x} - q_0 v(\mathbf{q}, \lambda') b_{\mathbf{q}, \lambda'}^\dagger e^{iq \cdot x} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} q_0 \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(p-q) \cdot x} \right. \\ &\quad - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger e^{-i(p-q) \cdot x} - u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(p+q) \cdot x} \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(p+q) \cdot x} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p d^3 q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} E_{\mathbf{q}} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{q}, \lambda'} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right. \right. \\ &\quad \left. \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{q}, \lambda'}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right] \right. \\ &\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[-u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{\mathbf{q}, \lambda'}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \right. \\ &\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda'} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \right\} \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} E_{\mathbf{p}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right. \\ &\quad \left. - u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p}, \lambda}^\dagger b_{-\mathbf{p}, \lambda'}^\dagger e^{2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p}, \lambda} a_{-\mathbf{p}, \lambda'} e^{-2iE_{\mathbf{p}}t} \right] \\ &= \sum_{\lambda \lambda'} \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} E_{\mathbf{p}} \left(2E_{\mathbf{p}} \delta_{\lambda \lambda'} a_{\mathbf{p}, \lambda}^\dagger a_{\mathbf{p}, \lambda'} - 2E_{\mathbf{p}} \delta_{\lambda \lambda'} b_{\mathbf{p}, \lambda} b_{\mathbf{p}, \lambda'}^\dagger \right) \end{aligned}$$

$$= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right). \quad (4.243)$$

倒数第二步用到正交归一关系 (4.220)。

另一方面, 利用正交归一关系 (4.220), 可得

$$\begin{aligned} & \int d^3x e^{ip \cdot x} u^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t) \\ &= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \right] \\ &= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right. \\ & \quad \left. + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda'} + u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}} \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda'}) = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p},\lambda}. \end{aligned} \quad (4.244)$$

从而, 湮灭算符 $a_{\mathbf{p},\lambda}$ 和产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 可以表示为

$$a_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} u^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad a_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} \psi^\dagger(\mathbf{x}, t) u(\mathbf{p}, \lambda). \quad (4.245)$$

同理, 可以推出

$$\begin{aligned} & \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t) \\ &= \int \frac{d^3x d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} \right] \\ &= \int \frac{d^3q}{\sqrt{2E_{\mathbf{q}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}+\mathbf{q}) \right. \\ & \quad \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p}-\mathbf{q}) \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} \left[v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p},\lambda'}^\dagger \right] \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda'=\pm} (2E_{\mathbf{p}} \delta_{\lambda\lambda'} b_{\mathbf{p},\lambda'}^\dagger) = \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p},\lambda}^\dagger. \end{aligned} \quad (4.246)$$

于是, 产生算符 $b_{\mathbf{p},\lambda}^\dagger$ 和湮灭算符 $b_{\mathbf{p},\lambda}$ 可以表示为

$$b_{\mathbf{p},\lambda}^\dagger = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ip \cdot x} v^\dagger(\mathbf{p}, \lambda) \psi(\mathbf{x}, t), \quad b_{\mathbf{p},\lambda} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{ip \cdot x} \psi^\dagger(\mathbf{x}, t) v(\mathbf{p}, \lambda). \quad (4.247)$$

4.5 Dirac 旋量场的正则量子化

4.5.1 用等时对易关系量子化 Dirac 旋量场的困难

在标量场和矢量场的正则量子化程序中，我们先假设场算符与其共轭动量密度算符满足等时对易关系 (2.57)，然后推导出产生湮灭算符的对易关系，再通过计算给出正定的哈密顿量（对于无质量矢量场，需要用弱 Lorenz 规范条件来得到正定的哈密顿量期待值），从而说明在量子场论中使用正则量子化方法是合理的。在本小节中，我们将尝试用类似的等时对易关系对 Dirac 旋量场进行量子化，不过，我们会发现这种方法并不能给出正定的哈密顿量，因而是不可行的。

假设 Dirac 旋量场算符 $\psi(x)$ 与其共轭动量密度算符 $\pi(x)$ 满足等时对易关系

$$[\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \quad (4.248)$$

这里已经将旋量指标明显地写出来。根据 (4.239) 式，这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时对易关系

$$[\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = 0. \quad (4.249)$$

接下来，我们计算产生湮灭算符的对易关系。由 (4.245) 式和正交归一关系 (4.220)，可得

$$\begin{aligned} [a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] u_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} u_a^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') \\ &= \frac{1}{2E_{\mathbf{p}}} u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.250)$$

另外，有

$$[a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} + \mathbf{q} \cdot \mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) u_b^\dagger(\mathbf{q}, \lambda') [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = 0. \quad (4.251)$$

由 (4.247) 式和正交归一关系 (4.220)，可得

$$\begin{aligned} [b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} v_b^\dagger(\mathbf{q}, \lambda') [\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] v_a(\mathbf{p}, \lambda) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} v_b^\dagger(\mathbf{q}, \lambda') v_a(\mathbf{p}, \lambda) (-\delta_{ba}) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= -\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} v^\dagger(\mathbf{q}, \lambda') v(\mathbf{p}, \lambda) \\ &= -\frac{1}{2E_{\mathbf{p}}} v^\dagger(\mathbf{p}, \lambda') v(\mathbf{p}, \lambda) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.252)$$

注意, 这个结果非同寻常地多了一个负号。此外, 还有

$$[b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] v_a(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') = 0, \quad (4.253)$$

$$[a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) v_b^\dagger(\mathbf{q}, \lambda') [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] = 0, \quad (4.254)$$

以及

$$\begin{aligned} [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] v_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} u_a^\dagger(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') \delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') \\ &= \frac{1}{2E_{\mathbf{p}}} e^{2iE_{\mathbf{p}}t} u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) = 0. \end{aligned} \quad (4.255)$$

上式最后一步用到正交归一关系 (4.220)。

整理起来, 通过等时对易关系 (4.248) 得到的产生湮灭算符对易关系为

$$\begin{aligned} [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{q},\lambda'}^\dagger] = 0, \\ [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] &= -(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad [b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0, \\ [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger] &= [b_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] = [a_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}] = [a_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{q},\lambda'}^\dagger] = 0. \end{aligned} \quad (4.256)$$

利用这样的对易关系, 可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) \\ &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \end{aligned} \quad (4.257)$$

上式最后一行第二项是零点能。在第一项中由 $a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为正, 但由 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 描述的粒子对总能量的贡献为负。从而, 粒子数密度 $b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}$ 越大, 场的总能量越少, 这显然是非物理的。因此, 用正则对易关系 (4.248) 对 Dirac 旋量场进行量子化是不可行的。

4.5.2 用等时反对易关系量子化 Dirac 旋量场

从 (4.257) 式的计算过程可以看出, 如果在交换 $b_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}^\dagger$ 位置的同时可以改变圆括号中第二项的符号, 就可以得到正定的哈密顿量。这意味着我们需要的不是 $b_{\mathbf{p},\lambda}$ 与 $b_{\mathbf{p},\lambda}^\dagger$ 的对易关系, 而是反对易关系。为了得到合适的 $b_{\mathbf{p},\lambda}$ 与 $b_{\mathbf{p},\lambda}^\dagger$ 的反对易关系, 则需要舍弃等时对易关系 (4.248), 代之以等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} = i\delta_{ab} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)\} = 0. \quad (4.258)$$

采用反对易关系进行量子化的方法称为 **Jordan-Wigner 量子化**。根据 (4.239) 式, 这些关系等价于 $\psi(x)$ 与 $\psi^\dagger(x)$ 的等时反对易关系

$$\{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = \{\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} = 0. \quad (4.259)$$

接下来, 我们计算产生湮灭算符的反对易关系。计算过程与上一小节类似, 只是我们要将 (4.250) 至 (4.255) 式中表示对易的方括号改成表示反对易的花括号。因此, 可得

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) \{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} u_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) u_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \end{aligned} \quad (4.260)$$

和

$$\{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) u_b^\dagger(\mathbf{q}, \lambda') \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = 0. \quad (4.261)$$

另外, 有

$$\begin{aligned} \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') \{\psi_a^\dagger(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} v_a(\mathbf{p}, \lambda) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} v_b^\dagger(\mathbf{q}, \lambda') v_a(\mathbf{p}, \lambda) \delta_{ba}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} e^{-i(\mathbf{p} - \mathbf{q}) \cdot \mathbf{x}} v^\dagger(\mathbf{q}, \lambda') v(\mathbf{p}, \lambda) \\ &= \frac{1}{2E_{\mathbf{p}}} v^\dagger(\mathbf{p}, \lambda') v(\mathbf{p}, \lambda) (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (4.262)$$

与 (4.252) 式不同, 上式的结果具有正常的符号。此外, 还有

$$\{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} \{\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} v_a(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') = 0, \quad (4.263)$$

$$\{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} = \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x - q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) v_b^\dagger(\mathbf{q}, \lambda') \{\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)\} = 0, \quad (4.264)$$

以及

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) \{\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)\} v_b(\mathbf{q}, \lambda') \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y e^{i(p \cdot x + q \cdot y)} u_a^\dagger(\mathbf{p}, \lambda) v_b(\mathbf{q}, \lambda') \delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}) = 0. \end{aligned} \quad (4.265)$$

整理起来, 通过等时反对易关系 (4.258) 得到的产生湮灭算符反对易关系为

$$\begin{aligned} \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad \{a_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}\} = \{a_{\mathbf{p}, \lambda}^\dagger, a_{\mathbf{q}, \lambda'}^\dagger\} = 0, \\ \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad \{b_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \{b_{\mathbf{p}, \lambda}^\dagger, b_{\mathbf{q}, \lambda'}^\dagger\} = 0, \\ \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}^\dagger\} &= \{b_{\mathbf{p}, \lambda}, a_{\mathbf{q}, \lambda'}^\dagger\} = \{a_{\mathbf{p}, \lambda}, b_{\mathbf{q}, \lambda'}\} = \{a_{\mathbf{p}, \lambda}^\dagger, b_{\mathbf{q}, \lambda'}^\dagger\} = 0. \end{aligned} \quad (4.266)$$

$a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 各自描述一种粒子。利用这样的反对易关系, 可以把哈密顿量 (4.243) 化为

$$\begin{aligned} H &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right) \\ &= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) - (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{d^3p}{(2\pi)^3} 2E_{\mathbf{p}}. \end{aligned} \quad (4.267)$$

上式最后一行第二项是零点能。第一项是所有动量模式所有螺旋度所有粒子贡献的能量之和, 它是正定的。可见, 用等时反对易关系对 Dirac 旋量场进行正则量子化是合适的。

利用 (4.8) 式和反对易关系 (4.266), 可得哈密顿量 H 与产生湮灭算符的对易子为

$$\begin{aligned} [H, a_{\mathbf{p},\lambda}^\dagger] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} \right] a_{\mathbf{p},\lambda}^\dagger \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left(a_{\mathbf{q},\lambda'}^\dagger \{a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} - \{a_{\mathbf{q},\lambda'}^\dagger, a_{\mathbf{p},\lambda}^\dagger\} a_{\mathbf{q},\lambda'} \right. \\ &\quad \left. + b_{\mathbf{q},\lambda'}^\dagger \{b_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} - \{b_{\mathbf{q},\lambda'}^\dagger, a_{\mathbf{p},\lambda}^\dagger\} b_{\mathbf{q},\lambda'} \right) \\ &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q},\lambda'}^\dagger \{a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger\} \\ &= \sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'}^\dagger \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger, \end{aligned} \quad (4.268)$$

$$\begin{aligned} [H, a_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} \right] a_{\mathbf{p},\lambda} \\ &= - \sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p},\lambda}, \end{aligned} \quad (4.269)$$

$$\begin{aligned} [H, b_{\mathbf{p},\lambda}^\dagger] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} \right] b_{\mathbf{p},\lambda}^\dagger \\ &= \sum_{\lambda'} \int d^3q E_{\mathbf{q}} b_{\mathbf{q},\lambda'}^\dagger \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger, \end{aligned} \quad (4.270)$$

$$\begin{aligned} [H, b_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} \left[a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'} + b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{q},\lambda'} \right] b_{\mathbf{p},\lambda} \\ &= - \sum_{\lambda'} \int d^3q E_{\mathbf{q}} b_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} b_{\mathbf{p},\lambda}. \end{aligned} \quad (4.271)$$

设 $|E\rangle$ 是 H 的本征态, 本征值为 E , 则

$$H |E\rangle = E |E\rangle. \quad (4.272)$$

从而, 可得

$$\begin{aligned} H a_{\mathbf{p},\lambda}^\dagger |E\rangle &= (a_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) a_{\mathbf{p},\lambda}^\dagger |E\rangle, \\ H a_{\mathbf{p},\lambda} |E\rangle &= (a_{\mathbf{p},\lambda} H - E_{\mathbf{p}} a_{\mathbf{p},\lambda}) |E\rangle = (E - E_{\mathbf{p}}) a_{\mathbf{p},\lambda} |E\rangle, \end{aligned}$$

$$\begin{aligned}
Hb_{\mathbf{p},\lambda}^\dagger |E\rangle &= (b_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger) |E\rangle = (E + E_{\mathbf{p}}) b_{\mathbf{p},\lambda}^\dagger |E\rangle, \\
Hb_{\mathbf{p},\lambda} |E\rangle &= (b_{\mathbf{p},\lambda} H - E_{\mathbf{p}} b_{\mathbf{p},\lambda}) |E\rangle = (E - E_{\mathbf{p}}) b_{\mathbf{p},\lambda} |E\rangle.
\end{aligned} \tag{4.273}$$

可见, 当 $a_{\mathbf{p},\lambda}^\dagger |E\rangle$ 和 $b_{\mathbf{p},\lambda}^\dagger |E\rangle$ 不为零时, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 和 $b_{\mathbf{p},\lambda}^\dagger$ 的作用都是使能量本征值增加 $E_{\mathbf{p}}$; 当 $a_{\mathbf{p},\lambda} |E\rangle$ 和 $b_{\mathbf{p},\lambda} |E\rangle$ 不为零时, 湮灭算符 $a_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的作用都是使能量本征值减少 $E_{\mathbf{p}}$ 。

根据 (1.158) 式, Dirac 旋量场的总动量为

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \pi \nabla \psi = \int d^3x \psi^\dagger (-i \nabla) \psi \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip \cdot x} \right] \\
&\quad \times \left[\mathbf{q} u(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'} e^{-iq \cdot x} - \mathbf{q} v(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \mathbf{q} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(p-q) \cdot x} \right. \\
&\quad - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger e^{-i(p-q) \cdot x} - u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{q},\lambda'}^\dagger e^{i(p+q) \cdot x} \\
&\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q) \cdot x} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \mathbf{q} \left\{ \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{q},\lambda'} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right. \right. \\
&\quad \left. - v^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right] \\
&\quad \left. + \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left[- u^\dagger(\mathbf{p}, \lambda) v(\mathbf{q}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{q},\lambda'}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right. \right. \\
&\quad \left. \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{q}, \lambda') b_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \right] \right\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[\mathbf{p} u^\dagger(\mathbf{p}, \lambda) u(\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} - \mathbf{p} v^\dagger(\mathbf{p}, \lambda) v(\mathbf{p}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda'}^\dagger \right. \\
&\quad \left. + \mathbf{p} u^\dagger(\mathbf{p}, \lambda) v(-\mathbf{p}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} - \mathbf{p} v^\dagger(\mathbf{p}, \lambda) u(-\mathbf{p}, \lambda') b_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \right] \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \mathbf{p} \left(2E_{\mathbf{p}} \delta_{\lambda\lambda'} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} - 2E_{\mathbf{p}} \delta_{\lambda\lambda'} b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda'}^\dagger \right) \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - b_{\mathbf{p},\lambda} b_{\mathbf{p},\lambda}^\dagger \right) \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) - 2\delta^{(3)}(\mathbf{0}) \int d^3p \mathbf{p} \\
&= \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right). \tag{4.274}
\end{aligned}$$

倒数第四步用到正交归一关系 (4.220), 倒数第二步用到反对易关系 (4.266)。总动量是所有动量模式所有螺旋度所有粒子贡献的动量之和。

4.5.3 U(1) 整体对称性

类似于复标量场，Dirac 旋量场也具有 U(1) 整体对称性。对 Dirac 旋量场 $\psi(x)$ 作 U(1) 整体变换

$$\psi'(x) = e^{iq\theta}\psi(x), \quad (4.275)$$

则 $\psi^\dagger(x)$ 和 $\bar{\psi}(x)$ 的相应变换为

$$[\psi^\dagger(x)]' = [\psi'(x)]^\dagger = \psi^\dagger(x)e^{-iq\theta}, \quad [\bar{\psi}(x)]' = \bar{\psi}'(x) = [\psi'(x)]^\dagger \gamma^0 = \bar{\psi}(x)e^{-iq\theta}. \quad (4.276)$$

在此变换下，拉氏量 (4.118) 不变：

$$\begin{aligned} \mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = \bar{\psi}(x)e^{-iq\theta}(i\gamma^\mu\partial_\mu - m)e^{iq\theta}\psi(x) \\ &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) = \mathcal{L}(x). \end{aligned} \quad (4.277)$$

容易验证，4.3 节中列举的旋量双线性型都在这种 U(1) 整体变换下不变。因此，用这些旋量双线性型构造的拉氏量都具有 U(1) 整体对称性。

U(1) 整体变换的无穷小形式为

$$\psi'(x) = \psi(x) + iq\theta\psi(x). \quad (4.278)$$

由于 $\delta x^\mu = 0$ ，根据 (1.136) 式可得

$$\bar{\delta}\psi = \delta\psi = iq\theta\psi. \quad (4.279)$$

按照 (1.141) 式，相应的 Noether 守恒流为

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar{\delta}\psi = i\bar{\psi}\gamma^\mu(iq\theta\psi) = -q\theta\bar{\psi}\gamma^\mu\psi. \quad (4.280)$$

扔掉无穷小参数 $-\theta$ ，定义

$$J^\mu \equiv q\bar{\psi}\gamma^\mu\psi, \quad (4.281)$$

则 Noether 定理给出

$$\partial_\mu J^\mu = 0. \quad (4.282)$$

相应的守恒荷为

$$\begin{aligned} Q &= \int d^3x J^0 = q \int d^3x \bar{\psi}\gamma^0\psi = q \int d^3x \psi^\dagger\psi \\ &= q \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[u^\dagger(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip\cdot x} + v^\dagger(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip\cdot x} \right] \\ &\quad \times \left[u(\mathbf{k}, \lambda') a_{\mathbf{k},\lambda'} e^{-ik\cdot x} + v(\mathbf{k}, \lambda') b_{\mathbf{k},\lambda'}^\dagger e^{ik\cdot x} \right] \\ &= q \sum_{\lambda\lambda'} \int \frac{d^3x d^3p d^3k}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left[u^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{k},\lambda'} e^{i(p-k)\cdot x} \right. \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') b_{\mathbf{p},\lambda} b_{\mathbf{k},\lambda'}^\dagger e^{-i(p-k)\cdot x} \right. \\ &\quad \left. + u^\dagger(\mathbf{p}, \lambda) v(\mathbf{k}, \lambda') a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{k},\lambda'} e^{i(p-k)\cdot x} \right. \\ &\quad \left. + v^\dagger(\mathbf{p}, \lambda) u(\mathbf{k}, \lambda') b_{\mathbf{p},\lambda} a_{\mathbf{k},\lambda'}^\dagger e^{-i(p-k)\cdot x} \right] \end{aligned}$$

$$\begin{aligned}
& +v^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{k},\lambda'}^\dagger e^{-i(p-k)\cdot x} + u^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{k},\lambda'}^\dagger e^{i(p+k)\cdot x} \\
& +v^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(p+k)\cdot x} \Big] \\
= & q \sum_{\lambda\lambda'} \int \frac{d^3p d^3k}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{k}}}} \left\{ \delta^{(3)}(\mathbf{p}-\mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{k},\lambda'}e^{i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right. \right. \\
& \left. \left. +v^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{k},\lambda'}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{k}})t} \right] \right. \\
& \left. +\delta^{(3)}(\mathbf{p}+\mathbf{k}) \left[u^\dagger(\mathbf{p}, \lambda)v(\mathbf{k}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{\mathbf{k},\lambda'}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right. \right. \\
& \left. \left. +v^\dagger(\mathbf{p}, \lambda)u(\mathbf{k}, \lambda')b_{\mathbf{p},\lambda}a_{\mathbf{k},\lambda'}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} \right] \right\} \\
= & q \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[u^\dagger(\mathbf{p}, \lambda)u(\mathbf{p}, \lambda')a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} + v^\dagger(\mathbf{p}, \lambda)v(\mathbf{p}, \lambda')b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda'}^\dagger \right. \\
& \left. +u^\dagger(\mathbf{p}, \lambda)v(-\mathbf{p}, \lambda')a_{\mathbf{p},\lambda}^\dagger b_{-\mathbf{p},\lambda'}^\dagger e^{2iE_{\mathbf{p}}t} + v^\dagger(\mathbf{p}, \lambda)u(-\mathbf{p}, \lambda')b_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} \right] \\
= & q \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left(2E_{\mathbf{p}}\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'} + 2E_{\mathbf{p}}\delta_{\lambda\lambda'}b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda'}^\dagger \right) \\
= & q \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \left(a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} + b_{\mathbf{p},\lambda}b_{\mathbf{p},\lambda}^\dagger \right) \\
= & \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3} \left(q a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda} - q b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda} \right) + 2\delta^{(3)}(\mathbf{0}) \int d^3p q. \tag{4.283}
\end{aligned}$$

上式第二项是零点荷。从第一项的形式可以看出，由 $a_{\mathbf{p},\lambda}^\dagger, a_{\mathbf{p},\lambda}$ 描述的粒子是正粒子，具有的荷为 q ；由 $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 描述的粒子是反粒子，具有的荷为 $-q$ 。除去零点荷，总荷是所有动量模式所有螺旋度所有正反粒子贡献的荷之和。

4.5.4 粒子态

对于自由的 Dirac 旋量场，真空态定义为被任意 $a_{\mathbf{p},\lambda}$ 和任意 $b_{\mathbf{p},\lambda}$ 湮灭的态，

$$a_{\mathbf{p},\lambda}|0\rangle = b_{\mathbf{p},\lambda}|0\rangle = 0, \tag{4.284}$$

满足

$$\langle 0|0\rangle = 1, \quad H|0\rangle = E_{\text{vac}}|0\rangle, \quad E_{\text{vac}} = -2\delta^{(3)}(\mathbf{0}) \int d^3p E_{\mathbf{p}}. \tag{4.285}$$

动量为 \mathbf{p} 、螺旋度为 λ 的单个正粒子态和单个反粒子态分别定义为

$$|\mathbf{p}, \lambda, +\rangle \equiv \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad |\mathbf{p}, \lambda, -\rangle \equiv \sqrt{2E_{\mathbf{p}}} b_{\mathbf{p},\lambda}^\dagger |0\rangle. \tag{4.286}$$

根据 (4.268) 和 (4.270) 式，有

$$\begin{aligned}
H|\mathbf{p}, \lambda, +\rangle &= \sqrt{2E_{\mathbf{p}}} H a_{\mathbf{p},\lambda}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (a_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger) |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) a_{\mathbf{p},\lambda}^\dagger |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}, \lambda, +\rangle, \\
H|\mathbf{p}, \lambda, -\rangle &= \sqrt{2E_{\mathbf{p}}} H b_{\mathbf{p},\lambda}^\dagger |0\rangle = \sqrt{2E_{\mathbf{p}}} (b_{\mathbf{p},\lambda}^\dagger H + E_{\mathbf{p}} b_{\mathbf{p},\lambda}^\dagger) |0\rangle
\end{aligned} \tag{4.287}$$

$$= \sqrt{2E_{\mathbf{p}}} (E_{\text{vac}} + E_{\mathbf{p}}) b_{\mathbf{p},\lambda}^{\dagger} |0\rangle = (E_{\text{vac}} + E_{\mathbf{p}}) |\mathbf{p}, \lambda, -\rangle. \quad (4.288)$$

可见, $|\mathbf{p}, \lambda, +\rangle$ 和 $|\mathbf{p}, \lambda, -\rangle$ 都比真空态多了一份能量 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ 。

将 $\psi(x)$ 的平面波解 (4.236) 代入 (4.81) 式左边, 得

$$[\psi(x), \mathbf{J}] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left\{ u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, \mathbf{J}] e^{-ip \cdot x} + v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, \mathbf{J}] e^{ip \cdot x} \right\}, \quad (4.289)$$

代入右边, 得

$$\begin{aligned} (\mathbf{L} + \mathbf{S})\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} (-i\mathbf{x} \times \nabla + \mathbf{S}) \left[u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[(\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right]. \end{aligned} \quad (4.290)$$

可见, 对于动量模式 \mathbf{p} 和螺旋度 λ , 有

$$u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, \mathbf{J}] = (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \quad v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, \mathbf{J}] = (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.291)$$

根据 (4.198) 和 (4.213) 式, $u(\mathbf{p}, \lambda)$ 和 $v(\mathbf{p}, \lambda)$ 分别是本征值为 λ 和 $-\lambda$ 的螺旋度本征态, 因而

$$u(\mathbf{p}, \lambda) [a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = 2\hat{\mathbf{p}} \cdot (\mathbf{x} \times \mathbf{p} + \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} = (2\hat{\mathbf{p}} \cdot \mathbf{S}) u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} = \lambda u(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}, \quad (4.292)$$

$$v(\mathbf{p}, \lambda) [b_{\mathbf{p},\lambda}^{\dagger}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = 2\hat{\mathbf{p}} \cdot (-\mathbf{x} \times \mathbf{p} + \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} = (2\hat{\mathbf{p}} \cdot \mathbf{S}) v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger} = -\lambda v(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.293)$$

故

$$[a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = \lambda a_{\mathbf{p},\lambda}, \quad [b_{\mathbf{p},\lambda}^{\dagger}, 2\hat{\mathbf{p}} \cdot \mathbf{J}] = -\lambda b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.294)$$

由于 \mathbf{J} 是厄米算符, 对第一式取厄米共轭可得

$$\lambda a_{\mathbf{p},\lambda}^{\dagger} = [a_{\mathbf{p},\lambda}, 2\hat{\mathbf{p}} \cdot \mathbf{J}]^{\dagger} = (2\hat{\mathbf{p}} \cdot \mathbf{J}) a_{\mathbf{p},\lambda}^{\dagger} - a_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) = [2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p},\lambda}^{\dagger}]. \quad (4.295)$$

于是, 有

$$[2\hat{\mathbf{p}} \cdot \mathbf{J}, a_{\mathbf{p},\lambda}^{\dagger}] = \lambda a_{\mathbf{p},\lambda}^{\dagger}, \quad [2\hat{\mathbf{p}} \cdot \mathbf{J}, b_{\mathbf{p},\lambda}^{\dagger}] = \lambda b_{\mathbf{p},\lambda}^{\dagger}. \quad (4.296)$$

\mathbf{J} 是总角动量算符, 真空态 $|0\rangle$ 不具有角动量, 所以满足

$$\mathbf{J} |0\rangle = \mathbf{0}. \quad (4.297)$$

由此, 可得

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) a_{\mathbf{p},\lambda}^{\dagger} |0\rangle = [a_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda a_{\mathbf{p},\lambda}^{\dagger}] |0\rangle = \lambda a_{\mathbf{p},\lambda}^{\dagger} |0\rangle, \quad (4.298)$$

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) b_{\mathbf{p},\lambda}^{\dagger} |0\rangle = [b_{\mathbf{p},\lambda}^{\dagger} (2\hat{\mathbf{p}} \cdot \mathbf{J}) + \lambda b_{\mathbf{p},\lambda}^{\dagger}] |0\rangle = \lambda b_{\mathbf{p},\lambda}^{\dagger} |0\rangle. \quad (4.299)$$

在没有轨道角动量的情况下, $2\hat{\mathbf{p}} \cdot \mathbf{J}$ 是螺旋度算符。因此, 上面两式说明 $|\mathbf{p}, \lambda, +\rangle$ 和 $|\mathbf{p}, \lambda, -\rangle$ 都是螺旋度本征态, 本征值为 λ :

$$(2\hat{\mathbf{p}} \cdot \mathbf{J}) |\mathbf{p}, \lambda, \pm\rangle = \lambda |\mathbf{p}, \lambda, \pm\rangle. \quad (4.300)$$

这正是我们所期望的。

以上讨论表明, 产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 的作用是产生一个动量为 \mathbf{p} 、螺旋度为 λ 的正粒子, 另一个产生算符 $b_{\mathbf{p},\lambda}^\dagger$ 的作用是产生一个动量为 \mathbf{p} 、螺旋度为 λ 的反粒子。正粒子和反粒子具有相同的质量 m 。

在 (4.208) 式中, 我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_{-\lambda}(\mathbf{p})$, 使得 $v(\mathbf{p}, \lambda)$ 的螺旋度本征值为 $-\lambda$, 从而得到 $b_{\mathbf{p},\lambda}^\dagger |0\rangle$ 的螺旋度本征值为 λ 的结果。如果我们选择让 $\tilde{f}_\lambda(\mathbf{p})$ 正比于 $\xi_\lambda(\mathbf{p})$, 依照上述推导, $b_{\mathbf{p},\lambda}^\dagger |0\rangle$ 的螺旋度本征值就会变成 $-\lambda$; 也就是说, $b_{\mathbf{p},\lambda}^\dagger, b_{\mathbf{p},\lambda}$ 将描述螺旋度为 $-\lambda$ 的反粒子。这不符合我们的记号, 因此, 我们将 $\tilde{f}_\lambda(\mathbf{p})$ 取为 (4.208) 式的形式。

由反对易关系 (4.266), 可得

$$\begin{aligned} a_{\mathbf{p},\lambda} |\mathbf{q}, \lambda', +\rangle &= \sqrt{2E_{\mathbf{q}}} a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{p},\lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle, \end{aligned} \quad (4.301)$$

$$\begin{aligned} b_{\mathbf{p},\lambda} |\mathbf{q}, \lambda', -\rangle &= \sqrt{2E_{\mathbf{q}}} b_{\mathbf{p},\lambda} b_{\mathbf{q},\lambda'}^\dagger |0\rangle = \sqrt{2E_{\mathbf{q}}} [(2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{p},\lambda}] |0\rangle \\ &= \sqrt{2E_{\mathbf{q}}} (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle. \end{aligned} \quad (4.302)$$

可以看出, 湮灭算符 $a_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 \mathbf{p} 、螺旋度为 λ 的正粒子, 湮灭算符 $b_{\mathbf{p},\lambda}$ 的作用是减少一个动量为 \mathbf{p} 、螺旋度为 λ 的反粒子。

将包含 2 个正粒子和 2 个反粒子的态记为

$$|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle \equiv \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{p}_3} E_{\mathbf{p}_4}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle. \quad (4.303)$$

根据反对易关系 (4.266), 有

$$\begin{aligned} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle &= -a_{\mathbf{p}_2, \lambda_2}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle = -a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle \\ &= -a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger |0\rangle = -b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger |0\rangle \\ &= -a_{\mathbf{p}_1, \lambda_1}^\dagger b_{\mathbf{p}_4, \lambda_4}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle = -b_{\mathbf{p}_4, \lambda_4}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger b_{\mathbf{p}_3, \lambda_3}^\dagger a_{\mathbf{p}_1, \lambda_1}^\dagger |0\rangle. \end{aligned} \quad (4.304)$$

从而, 可得

$$\begin{aligned} |\mathbf{p}_2, \lambda_2, +; \mathbf{p}_1, \lambda_1, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_4, \lambda_4, -; \mathbf{p}_3, \lambda_3, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_1, \lambda_1, +; \mathbf{p}_4, \lambda_4, -\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_1, \lambda_1, +; \mathbf{p}_4, \lambda_4, -; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_2, \lambda_2, +\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle, \\ |\mathbf{p}_4, \lambda_4, -; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_1, \lambda_1, +\rangle &= -|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +; \mathbf{p}_3, \lambda_3, -; \mathbf{p}_4, \lambda_4, -\rangle. \end{aligned} \quad (4.305)$$

也就是说, 交换任意两个粒子, 得到的态相差一个负号, 故多粒子态对于全同粒子交换是反对称的。这说明旋量场描述的粒子是费米子 (fermion), 服从 Fermi-Dirac 统计。得到这个结论的

关键在于两个产生算符相互反对易。对于两个相同的产生算符 $a_{\mathbf{p},\lambda}^\dagger$ 或 $b_{\mathbf{p},\lambda}^\dagger$ ，反对易关系导致

$$a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle = -a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle = -b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle, \quad (4.306)$$

故

$$a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}^\dagger |0\rangle = 0, \quad b_{\mathbf{p},\lambda}^\dagger b_{\mathbf{p},\lambda}^\dagger |0\rangle = 0. \quad (4.307)$$

这说明在没有其它自由度的情况下，不存在动量和螺旋度都相同的两个正费米子或两个反费米子组成的态，这就是 **Pauli 不相容原理**。

在第 2 章和第 3 章中，我们分别讨论了自旋为 0 的标量场和自旋为 1 的矢量场，合适的处理方式是通过反对易关系对它们进行量子化，因而它们都描述玻色子。另一方面，在本章中，我们需要采用反对易关系才能对自旋为 1/2 的旋量场进行合适的量子化，因而旋量场描述的粒子是费米子。实际上，这样的状况是普遍的，存在自旋—统计定理：整数自旋的物理场必须用反对易关系进行量子化，对应的粒子是玻色子；半整数自旋的物理场必须用反对易关系进行量子化，对应的粒子是费米子。

将两个正费米子组成的双粒子态记为

$$|\mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, +\rangle \equiv \sqrt{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger |0\rangle, \quad (4.308)$$

则双粒子态的内积关系是

$$\begin{aligned} & \langle \mathbf{q}_1, \lambda'_1, +; \mathbf{q}_2, \lambda'_2, + | \mathbf{p}_1, \lambda_1, +; \mathbf{p}_2, \lambda_2, + \rangle \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{q}_1, \lambda'_1} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\ & \quad \left. - \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_1, \lambda_1}^\dagger a_{\mathbf{q}_1, \lambda'_1} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right] \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^3 \delta_{\lambda_1 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_2, \lambda_2}^\dagger | 0 \rangle \right. \\ & \quad \left. - (2\pi)^3 \delta_{\lambda_2 \lambda'_1} \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \langle 0 | a_{\mathbf{q}_2, \lambda'_2} a_{\mathbf{p}_1, \lambda_1}^\dagger | 0 \rangle \right] \\ &= \sqrt{16E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{q}_1} E_{\mathbf{q}_2}} \left[(2\pi)^6 \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) \right. \\ & \quad \left. - (2\pi)^6 \delta_{\lambda_2 \lambda'_1} \delta_{\lambda_1 \lambda'_2} \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \right] \\ &= 4E_{\mathbf{p}_1} E_{\mathbf{p}_2} (2\pi)^6 \left[\delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) \right. \\ & \quad \left. - \delta_{\lambda_1 \lambda'_2} \delta_{\lambda_2 \lambda'_1} \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2) \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \right]. \quad (4.309) \end{aligned}$$

上式最后两行方括号中第二项前面有一个负号，由产生湮灭算符的反对易关系引起。这是双费米子态内积关系与双玻色子态内积关系 (2.127) 在形式上的不同之处。

第 5 章 量子场的相互作用

第 2、3、4 章分别讨论了标量场、矢量场、旋量场的正则量子化。不过，这些讨论只涉及自由量子场的拉氏量，没有考虑到量子场的相互作用。像 (2.60)、(3.84) 和 (4.118) 式这样的自由场拉氏量包含着动能项和质量项，它们都是双线性的，即每一项均包含 2 个场算符。如果我们更进一步，考虑拉氏量包含多于 2 个场算符的项，则这些项将描述场的相互作用 (interaction)。在局域场论中，拉氏量 $\mathcal{L}(x)$ 中的相互作用项只能包含同一个时空点处的几个场，例如 $[\phi(x)]^3$ ；不能包含处于不同时空点上的场，例如 $[\phi(x)]^2\phi(y)$ 。这样可以保持理论的因果性 (causality)。

相互作用项可以只包含同一种场，从而描述场的自相互作用 (self-interaction)。例如，对于实标量场 $\phi(x)$ ，可以构造如下拉氏量：

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \quad (5.1)$$

前两项与 (2.60) 式相同，第三项描述四个实标量场的自相互作用，其中， λ 是一个耦合常数 (coupling constant)，它的大小决定耦合的强度。 \mathcal{L}_{ϕ^4} 描述的理论称为实标量场的 ϕ^4 理论。

在自然单位制中，时空坐标 x^μ 的量纲是能量量纲的倒数，即 $[x^\mu] = [E]^{-1}$ ，故时空导数的量纲是 $[\partial_\mu] = [E]$ ，时空体积元的量纲则是 $[d^4x] = [E]^{-4}$ 。由于作用量 $S = \int d^4x \mathcal{L}$ 没有量纲，拉氏量的量纲是

$$[\mathcal{L}] = [E]^4. \quad (5.2)$$

于是，从拉氏量 (5.1) 的第一项可以看出，标量场的量纲是

$$[\phi] = [E]. \quad (5.3)$$

从而， $[\phi^4] = [E]^4$ ，故 $[\lambda] = 1$ ，即耦合常数 λ 是无量纲的。

相互作用项也可以涉及不同类型的场。例如，用实标量场 $\phi(x)$ 和 Dirac 旋量场 $\psi(x)$ 可以构造拉氏量

$$\mathcal{L}_{\text{Yukawa}} = \mathcal{L}_S + \mathcal{L}_D + \mathcal{L}_Y, \quad (5.4)$$

其中，

$$\mathcal{L}_S = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m_\phi^2\phi^2 \quad (5.5)$$

包含 ϕ 的动能项和质量项，

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_\psi\bar{\psi}\psi \quad (5.6)$$

包含 ψ 的动能项和质量项, 而相互作用项

$$\mathcal{L}_Y = -\kappa \phi \bar{\psi} \psi \quad (5.7)$$

描述标量场 ϕ 与旋量场 ψ 之间的 **Yukawa** 相互作用, 这里 κ 是耦合常数。由拉氏量 (5.6) 的第一项可以看出, 旋量场的量纲是 $[E]^{3/2}$, 故

$$[\psi] = [\bar{\psi}] = [E]^{3/2}. \quad (5.8)$$

因此, $[\phi \bar{\psi} \psi] = [E]^4$, 于是 Yukawa 耦合常数 κ 没有量纲。这类相互作用最先由汤川秀树 (Hideki Yukawa) 于 1935 年提出, 当时引入 π 介子 (对应于 ϕ) 来传递核子 (对应于 ψ) 之间的强相互作用。

存在相互作用时, 场的经典运动方程是非线性的。例如, 由 Euler-Lagrange 方程 (1.116) 可得, ϕ^4 理论的场方程为

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3. \quad (5.9)$$

如果像 Yukawa 相互作用那样, 相互作用项包含不同类型的场, 则会得到多个相互耦合的场方程。这样的场方程在经典场论中不容易求解, 在量子场论中就更加困难了。所幸的是, 当耦合常数 (如 λ 、 κ) 比较小时, 在微扰论 (perturbation theory) 中利用微扰级数展开可以得到比较可靠的近似解。本章主要介绍用微扰论处理量子场相互作用的思路。

如果拉氏量中的相互作用项 \mathcal{L}_{int} 不包含场 $\phi_a(x)$ 的时空导数 $\partial_\mu \phi_a$, 则 $\partial \mathcal{L}_{\text{int}} / \partial \dot{\phi}_a = 0$ 。上面两个例子都属于这种情况。按照定义式 (1.117), 此时场的共轭动量密度 $\pi_a(x)$ 不会受到 $\mathcal{L}_{\text{int}}(\phi_a)$ 的影响, 因而与没有相互作用时的量相同。这样的话, 等时对易关系 (2.57) 或等时反对易关系 (4.258) 不会受到影响, 我们可以继续采用这些关系。将哈密顿量密度 \mathcal{H} 分解成自由部分 $\mathcal{H}_{\text{free}}$ (与没有相互作用时的哈密顿量密度相同) 和相互作用部分 \mathcal{H}_{int} ,

$$\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}, \quad (5.10)$$

则根据定义式 (1.119) 有

$$\mathcal{H}_{\text{int}}(\phi_a) = -\mathcal{L}_{\text{int}}(\phi_a). \quad (5.11)$$

从而, 哈密顿量中描述相互作用的项是

$$H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}}(\phi_a) = - \int d^3x \mathcal{L}_{\text{int}}(\phi_a). \quad (5.12)$$

如果 \mathcal{L}_{int} 包含场的时空导数 $\partial_\mu \phi_a$, 则共轭动量密度 $\pi_a(x)$ 将与没有相互作用的情况不同。此时, 正则量子化方法并不方便, 更容易的处理方法是采用路径积分量子化。因此, 本章讨论仅局限于 \mathcal{L}_{int} 不包含 $\partial_\mu \phi_a$ 的情况, 其余情况留待路径积分量子化方法处理。

5.1 相互作用绘景

在 2.2 节中已经介绍过, 当系统的哈密顿量 H 不含时间 (这对于封闭系统是成立的) 时, 可以建立 Heisenberg 绘景。Heisenberg 绘景中的不含时态矢 $|\Psi\rangle^H$ 和含时算符 $O^H(t)$ (场算符或描述物理量的算符) 与 Schrödinger 绘景中的含时态矢 $|\Psi(t)\rangle^S$ 和不含时算符 O^S 之间的关系为

$$|\Psi\rangle^H = e^{iHt}|\Psi(t)\rangle^S, \quad O^H(t) = e^{iHt}O^S e^{-iHt}. \quad (5.13)$$

由 $[H, H] = 0$, 有

$$e^{iHt} H e^{-iHt} = H e^{iHt} e^{-iHt} = H. \quad (5.14)$$

可见, 哈密顿量 H 在这两种绘景中是相同的:

$$H^H = H^S = H. \quad (5.15)$$

此外, 可以得到

$$\begin{aligned} i\partial_0 O^H(t) &= (i\partial_0 e^{iHt})O^S e^{-iHt} + e^{iHt}O^S(i\partial_0 e^{-iHt}) = -H e^{iHt}O^S e^{-iHt} + e^{iHt}O^S e^{-iHt}H \\ &= [e^{iHt}O^S e^{-iHt}, H], \end{aligned} \quad (5.16)$$

即 Heisenberg 绘景中的含时算符 $O^H(t)$ 满足 **Heisenberg 运动方程**

$$i\frac{\partial}{\partial t}O^H(t) = [O^H(t), H]. \quad (5.17)$$

由于 Heisenberg 绘景能够明确地处理场算符的时间依赖性, 前面章节中自由场的正则量子化程序都是在这个绘景中进行的。为便于讨论, 接下来以实标量场为例进行表述。自由实标量场 $\phi(x)$ 的哈密顿量可以用产生湮灭算符表达成 (2.95) 式的形式:

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (5.18)$$

这里我们省略了零点能, 因为零点能是一个 c 数, 只决定总能量的零点, 不会影响下面的讨论。湮灭算符 $a_{\mathbf{p}}$ 和产生算符 $a_{\mathbf{p}}^\dagger$ 不依赖于时间 t , 它们实际上是 Schrödinger 绘景中的算符。由 (2.98) 式, 可得

$$\begin{aligned} [a_{\mathbf{p}}, (-iHt)^{(1)}] &= [a_{\mathbf{p}}, -iHt] = -it[a_{\mathbf{p}}, H] = -iE_{\mathbf{p}}t a_{\mathbf{p}}, \\ [a_{\mathbf{p}}, (-iHt)^{(2)}] &= [[a_{\mathbf{p}}, -iH^{(1)}t], -iHt] = -iE_{\mathbf{p}}t[a_{\mathbf{p}}, H] = (-iE_{\mathbf{p}}t)^2 a_{\mathbf{p}}, \\ &\dots \\ [a_{\mathbf{p}}, (-iHt)^{(n)}] &= (-iE_{\mathbf{p}}t)^n a_{\mathbf{p}}. \end{aligned} \quad (5.19)$$

从而, 由 (4.22) 式可以推出 Heisenberg 绘景中的湮灭算符为

$$a_{\mathbf{p}}^H(t) = e^{iHt} a_{\mathbf{p}} e^{-iHt} = \sum_{n=0}^{\infty} \frac{1}{n!} [a_{\mathbf{p}}, (-iHt)^{(n)}] = \sum_{n=0}^{\infty} \frac{1}{n!} (-iE_{\mathbf{p}}t)^n a_{\mathbf{p}} = e^{-iE_{\mathbf{p}}t} a_{\mathbf{p}}, \quad (5.20)$$

而相应的产生算符 $a_{\mathbf{p}}^{\text{H}\dagger}(t)$ 满足

$$e^{iHt} a_{\mathbf{p}}^{\dagger} e^{-iHt} = a_{\mathbf{p}}^{\text{H}\dagger}(t) = e^{iE_{\mathbf{p}}t} a_{\mathbf{p}}^{\dagger}. \quad (5.21)$$

根据这两条关系, 可以把自由实标量场的平面波展开式 (2.75) 表示成

$$\phi^{\text{H}}(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip \cdot x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}}^{\text{H}}(t) e^{i\mathbf{p} \cdot \mathbf{x}} + a_{\mathbf{p}}^{\text{H}\dagger}(t) e^{-i\mathbf{p} \cdot \mathbf{x}}]. \quad (5.22)$$

在最右边的表达式中, 场算符的时间依赖性只包含在 Heisenberg 绘景中的产生湮灭算符里面。反过来, 在 Schrödinger 绘景中, 自由实标量场的平面波展开式为

$$\begin{aligned} \phi^{\text{S}}(\mathbf{x}) &= e^{-iHt} \phi^{\text{H}}(\mathbf{x}, t) e^{iHt} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} [e^{-iHt} a_{\mathbf{p}}^{\text{H}}(t) e^{iHt} e^{i\mathbf{p} \cdot \mathbf{x}} + e^{-iHt} a_{\mathbf{p}}^{\text{H}\dagger}(t) e^{iHt} e^{-i\mathbf{p} \cdot \mathbf{x}}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p} \cdot \mathbf{x}}). \end{aligned} \quad (5.23)$$

可见, 场算符在 Schrödinger 绘景中确实不依赖于时间。同样, 将共轭动量密度的展开式 (2.77) 变换到 Schrödinger 绘景中, 则共轭动量密度也不依赖于时间:

$$\begin{aligned} \pi^{\text{S}}(\mathbf{x}) &= e^{-iHt} \pi^{\text{H}}(\mathbf{x}, t) e^{iHt} = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} e^{-iHt} [a_{\mathbf{p}}^{\text{H}}(t) e^{i\mathbf{p} \cdot \mathbf{x}} - a_{\mathbf{p}}^{\text{H}\dagger}(t) e^{-i\mathbf{p} \cdot \mathbf{x}}] e^{iHt} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} - a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p} \cdot \mathbf{x}}). \end{aligned} \quad (5.24)$$

我们在 2.2 节中提到, 正则对易关系的形式与绘景无关。这一点很容易验证, 比如, 实标量场的等时对易关系 (2.78) 在 Schrödinger 绘景中化为

$$[\phi^{\text{S}}(\mathbf{x}), \pi^{\text{S}}(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi^{\text{S}}(\mathbf{x}), \phi^{\text{S}}(\mathbf{y})] = [\pi^{\text{S}}(\mathbf{x}), \pi^{\text{S}}(\mathbf{y})] = 0. \quad (5.25)$$

如果从这些正则对易关系和展开式 (5.23)、(5.24) 出发, 可以推出产生湮灭算符的对易关系, 结果必定与在 Heisenberg 绘景中导出的 (2.92) 式相同。于是, 可以进一步导出哈密顿量的表达式 (5.18)。这说明在 Schrödinger 绘景中进行计算也会得到自洽结果。

现在, 考虑存在相互作用的情况, 假设系统的哈密顿量 $H = H^{\text{S}} = H^{\text{H}}$ 在 Schrödinger 绘景中分解为两个部分

$$H = H_0^{\text{S}} + H_1^{\text{S}}, \quad (5.26)$$

其中, 主要部分 H_0^{S} 是自由 (没有相互作用) 的哈密顿量, 微扰部分 H_1^{S} 描述相互作用, 只给出较小的影响。此时, 可以建立相互作用绘景 (interaction picture), 它也称为 Dirac 绘景。建立方式是把主要部分 H_0^{S} 的影响塞进态矢里面, 将态矢定义为

$$|\Psi(t)\rangle^{\text{I}} = e^{iH_0^{\text{S}}t} |\Psi(t)\rangle^{\text{S}}, \quad (5.27)$$

算符定义为

$$O^{\text{I}}(t) = e^{iH_0^{\text{S}}t} O^{\text{S}} e^{-iH_0^{\text{S}}t}. \quad (5.28)$$

这样一来，相互作用绘景中哈密顿量的自由部分与 Schrödinger 绘景相同，

$$H_0^I = e^{iH_0^S t} H_0^S e^{-iH_0^S t} = H_0^S; \quad (5.29)$$

但总哈密顿量不同，

$$H^I = e^{iH_0^S t} H e^{-iH_0^S t}; \quad (5.30)$$

微扰部分则满足

$$H_1^I = e^{iH_0^S t} H_1^S e^{-iH_0^S t} = e^{iH_0^S t} (H - H_0^S) e^{-iH_0^S t} = H^I - H_0^S = H^I - H_0^I. \quad (5.31)$$

此外，由 (5.13) 式有

$$|\Psi(t)\rangle^S = e^{-iHt} |\Psi\rangle^H, \quad O^S = e^{-iHt} O^H(t) e^{iHt}, \quad (5.32)$$

故相互作用绘景与 Heisenberg 绘景之间的关系为

$$|\Psi(t)\rangle^I = e^{iH_0^S t} e^{-iHt} |\Psi\rangle^H, \quad O^I(t) = e^{iH_0^S t} e^{-iHt} O^H(t) e^{iHt} e^{-iH_0^S t}. \quad (5.33)$$

于是，等时对易关系的形式不变，如

$$\begin{aligned} [\phi^I(\mathbf{x}, t), \pi^I(\mathbf{y}, t)] &= [e^{iH_0^S t} e^{-iHt} \phi^H(\mathbf{x}, t) e^{iHt} e^{-iH_0^S t}, e^{iH_0^S t} e^{-iHt} \pi^H(\mathbf{y}, t) e^{iHt} e^{-iH_0^S t}] \\ &= e^{iH_0^S t} e^{-iHt} [\phi^H(\mathbf{x}, t), \pi^H(\mathbf{y}, t)] e^{iHt} e^{-iH_0^S t} = e^{iH_0^S t} e^{-iHt} i\delta^{(3)}(\mathbf{x} - \mathbf{y}) e^{iHt} e^{-iH_0^S t} \\ &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (5.34)$$

当 $t = 0$ 时，三种绘景是一致的，

$$|\Psi(0)\rangle^I = |\Psi(0)\rangle^S = |\Psi\rangle^H, \quad O^I(0) = O^S = O^H(0). \quad (5.35)$$

在任意 t 时刻，均有

$${}^I\langle\Psi(t)| O^I(t) |\Psi(t)\rangle^I = {}^S\langle\Psi(t)| O^S |\Psi(t)\rangle^S = {}^H\langle\Psi| O^H(t) |\Psi\rangle^H, \quad (5.36)$$

因而三种绘景描述相同的物理。如果没有相互作用， $H = H_0^S$ ，则相互作用绘景与 Heisenberg 绘景相同。

在 Schrödinger 绘景中，Schrödinger 方程是

$$i\frac{\partial}{\partial t} |\Psi(t)\rangle^S = H |\Psi(t)\rangle^S. \quad (5.37)$$

由此可得

$$\begin{aligned} i\partial_0 |\Psi(t)\rangle^I &= \left(i\partial_0 e^{iH_0^S t} \right) |\Psi(t)\rangle^S + e^{iH_0^S t} i\partial_0 |\Psi(t)\rangle^S = \left(-H_0^S e^{iH_0^S t} + e^{iH_0^S t} H \right) |\Psi(t)\rangle^S \\ &= \left(-H_0^S + e^{iH_0^S t} H e^{-iH_0^S t} \right) e^{iH_0^S t} |\Psi(t)\rangle^S = (-H_0^I + H^I) e^{iH_0^S t} |\Psi(t)\rangle^S, \end{aligned} \quad (5.38)$$

即

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle^I = H_1^I |\Psi(t)\rangle^I. \quad (5.39)$$

这是态矢 $|\Psi(t)\rangle^I$ 的演化方程。可见，在相互作用绘景中，态矢的演化只由相互作用哈密顿量 H_1^I 决定。另一方面，有

$$\begin{aligned} i\partial_0 O^I(t) &= (i\partial_0 e^{iH_0^S t}) O^S e^{-iH_0^S t} + e^{iH_0^S t} O^S (i\partial_0 e^{-iH_0^S t}) \\ &= -H_0^S e^{iH_0^S t} O^S e^{-iH_0^S t} + e^{iH_0^S t} O^S e^{-iH_0^S t} H_0^S = [e^{iH_0^S t} O^S e^{-iH_0^S t}, H_0^S], \end{aligned} \quad (5.40)$$

即

$$i \frac{\partial}{\partial t} O^I(t) = [O^I(t), H_0^S]. \quad (5.41)$$

这个方程表明相互作用绘景中算符的演化只由自由哈密顿量 $H_0^S = H_0^I$ 决定。

综上，在相互作用绘景中，态矢的演化规律与 Schrödinger 绘景中的运动方程 (5.37) 相同，但必须将那里的总哈密顿量 H 换成相互作用哈密顿量 H_1^I ，这部分演化属于动力学 (dynamics) 演化；算符的演化规律与 Heisenberg 绘景中的运动方程 (5.17) 相同，但必须将那里的总哈密顿量 H 换成自由哈密顿量 H_0^I ，这部分演化属于运动学 (kinematics) 演化。在 Heisenberg 绘景中，对未加微扰的系统求出各个算符之间的关系之后，加入微扰一般会让这些关系发生改变。幸运的是，加入微扰之后各个算符在相互作用绘景中的关系仍然与加入微扰之前它们在 Heisenberg 绘景中的关系相同，可以直接套用原来的公式。这就是相互作用绘景的好处。

因此，在相互作用绘景中，具有相互作用的场算符的平面波展开式将与没有相互作用的场算符在 Heisenberg 绘景中的展开式相同。于是，在存在相互作用的情况下，我们仍然可以沿用第 2、3、4 章中导出的许多自由场关系式，比如产生湮灭算符的对易或反对易关系。

5.1.1 例：实标量场

下面以实标量场为例讨论相互作用绘景。假设 $t = 0$ 时，实标量场 $\phi(x)$ 的平面波展开式与自由场展开式 (5.23) 和 (5.24) 一样，

$$\phi^I(\mathbf{x}, 0) = \phi^H(\mathbf{x}, 0) = \phi^S(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}), \quad (5.42)$$

$$\pi^I(\mathbf{x}, 0) = \pi^H(\mathbf{x}, 0) = \pi^S(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}), \quad (5.43)$$

其中，产生湮灭算符 $a_{\mathbf{p}}^\dagger$ 和 $a_{\mathbf{p}}$ 满足对易关系 (2.92)。哈密顿量的自由部分 H_0^S 具有 (5.18) 式的形式：

$$H_0^S = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (5.44)$$

类似于 (5.19) 式，我们有

$$[a_{\mathbf{p}}, (-iH_0^S t)^{(n)}] = (-iE_{\mathbf{p}} t)^n a_{\mathbf{p}}. \quad (5.45)$$

从而由 (4.22) 式可得

$$a_{\mathbf{p}}^{\text{I}}(t) = e^{iH_0^{\text{S}}t} a_{\mathbf{p}} e^{-iH_0^{\text{S}}t} = e^{-iE_{\mathbf{p}}t} a_{\mathbf{p}}, \quad a_{\mathbf{p}}^{\text{I}\dagger}(t) = e^{iH_0^{\text{S}}t} a_{\mathbf{p}}^{\dagger} e^{-iH_0^{\text{S}}t} = e^{iE_{\mathbf{p}}t} a_{\mathbf{p}}^{\dagger}. \quad (5.46)$$

于是, 相互作用绘景中任意 t 时刻的场算符展开式为

$$\begin{aligned} \phi^{\text{I}}(\mathbf{x}, t) &= e^{iH_0^{\text{S}}t} \phi^{\text{S}}(\mathbf{x}) e^{-iH_0^{\text{S}}t} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} [a_{\mathbf{p}}^{\text{I}}(t) e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\text{I}\dagger}(t) e^{-i\mathbf{p}\cdot\mathbf{x}}] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{iE_{\mathbf{p}}t} e^{-i\mathbf{p}\cdot\mathbf{x}}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}), \end{aligned} \quad (5.47)$$

共轭动量密度的展开式为

$$\pi^{\text{I}}(\mathbf{x}, t) = e^{iH_0^{\text{S}}t} \pi^{\text{S}}(\mathbf{x}) e^{-iH_0^{\text{S}}t} = \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}). \quad (5.48)$$

正如所期望的, 这两个式子与自由实标量场在 Heisenberg 绘景中的展开式 (2.75) 和 (2.77) 一致。

因此, 根据产生湮灭算符的对易关系 (2.92), 可以证明 $\phi^{\text{I}}(x)$ 和 $\pi^{\text{I}}(x)$ 满足与 (2.78) 形式相同的等时对易关系

$$[\phi^{\text{I}}(\mathbf{x}, t), \pi^{\text{I}}(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi^{\text{I}}(\mathbf{x}, t), \phi^{\text{I}}(\mathbf{y}, t)] = [\pi^{\text{I}}(\mathbf{x}, t), \pi^{\text{I}}(\mathbf{y}, t)] = 0. \quad (5.49)$$

可以验证, 场算符展开式符合演化方程 (5.41): 类似于 (2.97) 式和 (2.98) 式, 可以推出

$$[a_{\mathbf{p}}, H_0^{\text{S}}] = E_{\mathbf{p}} a_{\mathbf{p}}, \quad [a_{\mathbf{p}}^{\dagger}, H_0^{\text{S}}] = -E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}, \quad (5.50)$$

从而, 有

$$\begin{aligned} i\frac{\partial}{\partial t} \phi^{\text{I}}(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (E_{\mathbf{p}} a_{\mathbf{p}} e^{-ip\cdot x} - E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} e^{ip\cdot x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} ([a_{\mathbf{p}}, H_0^{\text{S}}] e^{-ip\cdot x} + [a_{\mathbf{p}}^{\dagger}, H_0^{\text{S}}] e^{ip\cdot x}) = [\phi^{\text{I}}(\mathbf{x}, t), H_0^{\text{S}}]. \end{aligned} \quad (5.51)$$

5.1.2 例: 有质量矢量场

不难将上述讨论推广到复标量场、无质量矢量场和 Dirac 旋量场。但是, 推广到有质量矢量场 $A^{\mu}(x)$ 却会得到不同寻常的结果, 原因在于 $A^0(x)$ 不是一个独立的场分量, 不具备相应的共轭动量密度和正则对易关系, 因而在绘景变换中具有特殊的性质。

假设参与相互作用的有质量矢量场具有拉氏量

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad (5.52)$$

其中, 自由场的拉氏量为

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \quad (5.53)$$

相互作用项为

$$\mathcal{L}_1 = J_\mu A^\mu. \quad (5.54)$$

此处, $J_\mu(x)$ 是由其它的场组成的流, 如 $g\bar{\psi}(x)\gamma_\mu\psi(x)$ 。根据 (1.116) 式及

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu + J^\nu, \quad (5.55)$$

可得有质量矢量场的 Euler-Lagrange 方程为

$$\partial_\mu F^{\mu\nu} + m^2 A^{\nu} = -J^{\nu}. \quad (5.56)$$

这里我们将 Heisenberg 绘景的标记明确写出来。由于 $J_\mu(x)$ 不包含 A^μ 的时间导数, 正则动量密度与自由情况形式相同:

$$\pi_i^H = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^{H,i})} = -F_{0i}^H, \quad \pi^{H,i} = F^{H,i0} = -\partial^0 A^{H,i} + \partial^i A^{H,0}. \quad (5.57)$$

写成空间矢量的形式, 得

$$\boldsymbol{\pi}^H = -\dot{\mathbf{A}}^H - \nabla A^{H,0}, \quad \dot{\mathbf{A}}^H = -\boldsymbol{\pi}^H - \nabla A^{H,0}. \quad (5.58)$$

当 $\nu = 0$ 时, 运动方程变成

$$\partial_i F^{H,i0} + m^2 A^{H,0} = -J^{H,0}, \quad (5.59)$$

故

$$A^{H,0} = -\frac{1}{m^2}(\partial_i F^{H,i0} + J^{H,0}) = -\frac{1}{m^2}(\nabla \cdot \boldsymbol{\pi}^H + J^{H,0}). \quad (5.60)$$

与自由情况 (3.179) 不同, 此处 $A^{H,0}$ 还依赖于 $J^{H,0}$ 。

现在, 哈密顿量密度是

$$\begin{aligned} \mathcal{H}^H &= \pi_i^H \partial_0 A^{H,i} - \mathcal{L} = -\boldsymbol{\pi}^H \cdot \dot{\mathbf{A}}^H - \mathcal{L} \\ &= -\boldsymbol{\pi}^H \cdot \dot{\mathbf{A}}^H - \frac{1}{2}(\boldsymbol{\pi}^H)^2 + \frac{1}{2}(\nabla \times \mathbf{A}^H)^2 - \frac{1}{2}m^2[(A^{H,0})^2 - (\mathbf{A}^H)^2] - J^{H,0}A^{H,0} + \mathbf{J}^H \cdot \mathbf{A}^H. \end{aligned} \quad (5.61)$$

我们需要知道它比自由哈密顿量密度 (3.185) 多了什么。(5.61) 式第一项可化为

$$\begin{aligned} -\boldsymbol{\pi}^H \cdot \dot{\mathbf{A}}^H &= \boldsymbol{\pi}^H \cdot (\boldsymbol{\pi}^H + \nabla A^{H,0}) = (\boldsymbol{\pi}^H)^2 + \nabla \cdot (A^{H,0} \boldsymbol{\pi}^H) - A^{H,0} \nabla \cdot \boldsymbol{\pi}^H \\ &= (\boldsymbol{\pi}^H)^2 + \nabla \cdot (A^{H,0} \boldsymbol{\pi}^H) + \frac{1}{m^2}(\nabla \cdot \boldsymbol{\pi}^H)^2 + \frac{1}{m^2}J^{H,0} \nabla \cdot \boldsymbol{\pi}^H. \end{aligned} \quad (5.62)$$

最后一行第二项是全散度, 不会影响哈密顿量。(5.61) 式第四项中包括

$$-\frac{1}{2}m^2(A^{H,0})^2 = -\frac{1}{2}m^2 \frac{1}{m^4}(\nabla \cdot \boldsymbol{\pi}^H + J^{H,0})^2$$

$$= -\frac{1}{2m^2}(\nabla \cdot \boldsymbol{\pi}^H)^2 - \frac{1}{2m^2}(J^{H,0})^2 - \frac{1}{m^2}J^{H,0}\nabla \cdot \boldsymbol{\pi}^H, \quad (5.63)$$

而第五项为

$$-J^{H,0}(A^H)^0 = \frac{1}{m^2}J^{H,0}(\nabla \cdot \boldsymbol{\pi}^H + J^{H,0}) = \frac{1}{m^2}J^{H,0}\nabla \cdot \boldsymbol{\pi}^H + \frac{1}{m^2}(J^{H,0})^2. \quad (5.64)$$

这里包含 J^μ 的项都是自由场不具备的, 应该归为相互作用项。于是, 我们可以将哈密顿量分解为

$$H^H = \int d^3x \mathcal{H}^H = H_0^H + H_1^H, \quad (5.65)$$

其中,

$$H_0^H = \frac{1}{2} \int d^3x \left[(\boldsymbol{\pi}^H)^2 + \frac{1}{m^2}(\nabla \cdot \boldsymbol{\pi}^H)^2 + (\nabla \times \mathbf{A}^H)^2 + m^2(\mathbf{A}^H)^2 \right] \quad (5.66)$$

与自由哈密顿量密度 (3.186) 形式相同, 而

$$H_1^H = \int d^3x \left[\mathbf{J}^H \cdot \mathbf{A}^H + \frac{1}{m^2}J^{H,0}\nabla \cdot \boldsymbol{\pi}^H + \frac{1}{2m^2}(J^{H,0})^2 \right] \quad (5.67)$$

描述相互作用。

根据等时对易关系 (3.96), 有

$$\begin{aligned} [A^{H,i}(x), (\boldsymbol{\pi}^H(y))^2] &= [A^{H,i}(x), \pi_j^H(y)]\pi_j^H(y) + \pi_j^H(y)[A^{H,i}(x), \pi_j^H(y)] \\ &= 2i\delta_j^i\delta^{(3)}(\mathbf{x} - \mathbf{y})\pi_j^H(y) = -2i\delta^{(3)}(\mathbf{x} - \mathbf{y})\pi^{H,i}(y), \end{aligned} \quad (5.68)$$

写成空间矢量的形式是

$$[\mathbf{A}^H(x), (\boldsymbol{\pi}^H(y))^2] = -2i\delta^{(3)}(\mathbf{x} - \mathbf{y})\boldsymbol{\pi}^H(y). \quad (5.69)$$

另一方面, 用 ∇_y 表示对空间矢量 \mathbf{y} 的梯度算符, 可得

$$[A^{H,i}(x), \nabla_y \cdot \boldsymbol{\pi}^H(y)] = -\frac{\partial}{\partial y^j}[A^{H,i}(x), \pi_j^H(y)] = -i\delta_j^i\frac{\partial}{\partial y^j}\delta^{(3)}(\mathbf{x} - \mathbf{y}) = -i\frac{\partial}{\partial y^i}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (5.70)$$

即

$$[\mathbf{A}^H(x), \nabla_y \cdot \boldsymbol{\pi}^H(y)] = -i\nabla_y\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (5.71)$$

从而, 我们能够导出

$$\begin{aligned} [\mathbf{A}^H(x), H_0^H] &= \frac{1}{2} \int d^3y \left\{ [\mathbf{A}^H(x), (\boldsymbol{\pi}^H(y))^2] + \frac{1}{m^2}[\mathbf{A}^H(x), (\nabla_y \cdot \boldsymbol{\pi}^H(y))^2] \right\} \\ &= \int d^3y \left\{ -i\delta^{(3)}(\mathbf{x} - \mathbf{y})\boldsymbol{\pi}^H(y) - \frac{i}{m^2}(\nabla_y \cdot \boldsymbol{\pi}^H(y))\nabla_y\delta^{(3)}(\mathbf{x} - \mathbf{y}) \right\} \\ &= -i\boldsymbol{\pi}^H(x) + \frac{i}{m^2} \int d^3y \{ \delta^{(3)}(\mathbf{x} - \mathbf{y})\nabla_y(\nabla_y \cdot \boldsymbol{\pi}^H(y)) \} \\ &= -i\boldsymbol{\pi}^H(x) + \frac{i}{m^2}\nabla_x(\nabla_x \cdot \boldsymbol{\pi}^H(x)). \end{aligned} \quad (5.72)$$

接下来, 我们转换到相互作用绘景,

$$\mathbf{A}^I = e^{iH_0^S t} e^{-iHt} \mathbf{A}^H e^{iHt} e^{-iH_0^S t}, \quad \boldsymbol{\pi}^I = e^{iH_0^S t} e^{-iHt} \boldsymbol{\pi}^H e^{iHt} e^{-iH_0^S t}, \quad (5.73)$$

则有

$$\begin{aligned} H_0^S &= H_0^I = e^{iH_0^S t} e^{-iHt} H_0^H e^{iHt} e^{-iH_0^S t} \\ &= \frac{1}{2} \int d^3x \left[(\boldsymbol{\pi}^I)^2 + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi}^I)^2 + (\nabla \times \mathbf{A}^I)^2 + m^2 (\mathbf{A}^I)^2 \right]. \end{aligned} \quad (5.74)$$

将演化方程 (5.41) 应用到 \mathbf{A}^I 上, 利用 (5.72) 式, 可得

$$\begin{aligned} i\dot{\mathbf{A}}^I &= [\mathbf{A}^I, H_0^S] = e^{iH_0^S t} e^{-iHt} [\mathbf{A}^H, H_0^H] e^{iHt} e^{-iH_0^S t} \\ &= e^{iH_0^S t} e^{-iHt} \left[-i\boldsymbol{\pi}^H + \frac{i}{m^2} \nabla(\nabla \cdot \boldsymbol{\pi}^H) \right] e^{iHt} e^{-iH_0^S t} = -i\boldsymbol{\pi}^I + \frac{i}{m^2} \nabla(\nabla \cdot \boldsymbol{\pi}^I), \end{aligned} \quad (5.75)$$

即

$$\boldsymbol{\pi}^I = -\dot{\mathbf{A}}^I + \frac{1}{m^2} \nabla(\nabla \cdot \boldsymbol{\pi}^I). \quad (5.76)$$

与 (3.177) 式和 (3.179) 式比较, 可以看出, 这个等式与自由场情况形式相同。

现在, 假设 $t = 0$ 时 $A^\mu(x)$ 和 $\pi_i(x)$ 的平面波展开式与 $t = 0$ 时的自由场展开式 (3.146) 和 (3.151) 相同,

$$\begin{aligned} A^{I,\mu}(\mathbf{x}, 0) &= A^{H,\mu}(\mathbf{x}, 0) = A^{S,\mu}(\mathbf{x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{i\mathbf{p}\cdot\mathbf{x}} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right], \end{aligned} \quad (5.77)$$

$$\begin{aligned} \pi_i^I(\mathbf{x}, 0) &= \pi_i^H(\mathbf{x}, 0) = \pi_i^S(\mathbf{x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{i\mathbf{p}\cdot\mathbf{x}} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right], \end{aligned} \quad (5.78)$$

其中, 产生湮灭算符 $a_{\mathbf{p},\lambda}^\dagger$ 和 $a_{\mathbf{p},\lambda}$ 满足对易关系 (3.175)。哈密顿量的自由部分 H_0^S 具有 (3.205) 式的形式 (略去零点能):

$$H_0^S = \sum_{\lambda=\pm,0} \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}. \quad (5.79)$$

从而, 有

$$\begin{aligned} [H_0^S, a_{\mathbf{p},\lambda}^\dagger] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}^\dagger] = \sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'}^\dagger \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p},\lambda}^\dagger, \\ [H_0^S, a_{\mathbf{p},\lambda}] &= \sum_{\lambda'} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{q},\lambda'}, a_{\mathbf{p},\lambda}] = - \sum_{\lambda'} \int d^3q E_{\mathbf{q}} a_{\mathbf{q},\lambda'} \delta_{\lambda'\lambda} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p},\lambda}. \end{aligned} \quad (5.81)$$

于是, 我们能够得到与 (5.19) 形式相同的式子

$$[a_{\mathbf{p},\lambda}, (-iH_0^S t)^{(n)}] = (-iE_{\mathbf{p}} t)^{(n)} a_{\mathbf{p},\lambda}, \quad (5.82)$$

再根据 (4.22) 式, 可以导出

$$a_{\mathbf{p},\lambda}^{\text{I}}(t) = e^{iH_0^{\text{S}}t} a_{\mathbf{p},\lambda} e^{-iH_0^{\text{S}}t} = e^{-iE_{\mathbf{p}}t} a_{\mathbf{p},\lambda}, \quad a_{\mathbf{p},\lambda}^{\text{I}\dagger}(t) = e^{iH_0^{\text{S}}t} a_{\mathbf{p},\lambda}^{\dagger} e^{-iH_0^{\text{S}}t} = e^{iE_{\mathbf{p}}t} a_{\mathbf{p},\lambda}^{\dagger}. \quad (5.83)$$

更进一步, 推出

$$A^{\text{I},\mu}(\mathbf{x}, t) = e^{iH_0^{\text{S}}t} A^{\text{S},\mu}(\mathbf{x}) e^{-iH_0^{\text{S}}t} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\varepsilon^{\mu}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} + \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right], \quad (5.84)$$

$$\pi_i^{\text{I}}(\mathbf{x}, t) = e^{iH_0^{\text{S}}t} \pi_i^{\text{S}}(\mathbf{x}) e^{-iH_0^{\text{S}}t} = \int \frac{d^3p}{(2\pi)^3} \frac{ip_0}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \left[\tilde{\varepsilon}_i(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^{\dagger} e^{ip \cdot x} \right]. \quad (5.85)$$

也就是说, 对于任意 t 时刻, $A^{\text{I},\mu}(x)$ 和 $\pi_i^{\text{I}}(x)$ 的展开式与 Heisenberg 绘景中的自由场展开式 (3.146) 和 (3.151) 一致。这是我们期望的结果。

因此, $\pi_i^{\text{I}}(x)$ 和 $A^{\text{I},\mu}(x)$ 的关系也与自由场情况 (3.95) 式一样:

$$\pi_i^{\text{I}} = -\partial_0 A_i^{\text{I}} + \partial_i A_0^{\text{I}}, \quad (5.86)$$

即

$$\boldsymbol{\pi}^{\text{I}} = -\dot{\mathbf{A}}^{\text{I}} - \nabla A^{\text{I},0}. \quad (5.87)$$

与 (5.76) 式比较, 就得到

$$A^{\text{I},0} = -\frac{1}{m^2} \nabla \cdot \boldsymbol{\pi}^{\text{I}}. \quad (5.88)$$

这个式子不同于 Heisenberg 绘景中的关系式 (5.60), 反而与自由场中的关系式 (3.179) 一致。实际上, 由于 $A^{\text{H},0}$ 不是独立的场分量, 我们在 Heisenberg 绘景中可以利用场的 Euler-Lagrange 方程导出关系式 (5.60) 来确定它, 但我们无法保证这个关系式在相互作用绘景中成立, 因而不能通过相似变换定义 $A^{\text{H},0}$ 在相互作用绘景中对应的量。

根据 (5.88) 式, 相互作用哈密顿量 (5.67) 在相互作用绘景中将变成

$$\begin{aligned} H_1^{\text{I}} &= e^{iH_0^{\text{S}}t} e^{-iHt} H_1^{\text{H}} e^{iHt} e^{-iH_0^{\text{S}}t} = \int d^3x \left[\mathbf{J}^{\text{I}} \cdot \mathbf{A}^{\text{I}} + \frac{1}{m^2} J^{\text{I},0} \nabla \cdot \boldsymbol{\pi}^{\text{I}} + \frac{1}{2m^2} (J^{\text{I},0})^2 \right] \\ &= \int d^3x \left[\mathbf{J}^{\text{I}} \cdot \mathbf{A}^{\text{I}} - J^{\text{I},0} A^{\text{I},0} + \frac{1}{2m^2} (J^{\text{I},0})^2 \right] = \int d^3x \left[-J_{\mu}^{\text{I}} A^{\text{I},\mu} + \frac{1}{2m^2} (J^{\text{I},0})^2 \right] \\ &= \int d^3x \left[-\mathcal{L}_1^{\text{I}} + \frac{1}{2m^2} (J^{\text{I},0})^2 \right]. \end{aligned} \quad (5.89)$$

最后一行方括号中第一项 $-\mathcal{L}_1^{\text{I}} = -J_{\mu}^{\text{I}} A^{\text{I},\mu}$ 是我们期望得到的, 它是 Lorentz 不变的。但第二项异乎寻常, 不具有 Lorentz 不变性, 我们将它记为

$$\mathcal{H}_{J^0} = \frac{1}{2m^2} (J^{\text{I},0})^2. \quad (5.90)$$

在这里, \mathcal{H}_{J^0} 看起来会破坏理论的 Lorentz 协变性, 不过, 在后续微扰论分析中, 我们将看到它的贡献恰好抵消了矢量场传播子中的非协变项。最终, 理论仍然是 Lorentz 协变的。

5.2 时间演化算符和 S 矩阵

如前所述, 在相互作用绘景中, 态矢 $|\Psi(t)\rangle^I$ 承载着动力学演化, 它的演化方程 (5.39) 是微扰论处理量子场相互作用的一个出发点。引入时间演化算符 (time-evolution operator) $U(t, t_0)$, 用于联系 t_0 和 t 两个时刻的态矢:

$$|\Psi(t)\rangle^I = U(t, t_0)|\Psi(t_0)\rangle^I. \quad (5.91)$$

由 (5.33) 式, 有

$$|\Psi(t)\rangle^I = e^{iH_0^S t} e^{-iHt} |\Psi\rangle^H = e^{iH_0^S t} e^{-iH(t-t_0)} e^{-iH_0^S t_0} |\Psi(t_0)\rangle^I. \quad (5.92)$$

因此, 时间演化算符可以表示为

$$U(t, t_0) = e^{iH_0^S t} e^{-iH(t-t_0)} e^{-iH_0^S t_0}. \quad (5.93)$$

容易看出, 时间演化算符满足

$$U(t_0, t_0) = 1. \quad (5.94)$$

两个时间演化算符相继作用得出的乘法规则为

$$\begin{aligned} U(t_2, t_1)U(t_1, t_0) &= e^{iH_0^S t_2} e^{-iH(t_2-t_1)} e^{-iH_0^S t_1} e^{iH_0^S t_1} e^{-iH(t_1-t_0)} e^{-iH_0^S t_0} = e^{iH_0^S t_2} e^{-iH(t_2-t_0)} e^{-iH_0^S t_0} \\ &= U(t_2, t_0). \end{aligned} \quad (5.95)$$

上式取 $t_2 = t_0$, 即得

$$U(t_0, t_1)U(t_1, t_0) = U(t_0, t_0) = 1, \quad (5.96)$$

故时间演化算符的逆算符满足

$$U^{-1}(t, t_0) = U(t_0, t). \quad (5.97)$$

再由 H 和 H_0^S 的厄米性, 可得

$$U^\dagger(t, t_0) = e^{iH_0^S t_0} e^{iH(t-t_0)} e^{-iH_0^S t} = e^{iH_0^S t_0} e^{-iH(t_0-t)} e^{-iH_0^S t} = U(t_0, t) = U^{-1}(t_0, t), \quad (5.98)$$

也就是说, 时间演化算符是么正算符。取 $t_0 = 0$, 有

$$U(t, 0) = e^{iH_0^S t} e^{-iHt}, \quad U^{-1}(t, 0) = e^{iHt} e^{-iH_0^S t}, \quad (5.99)$$

因而根据 (5.33) 式和 (5.35) 式可得

$$|\Psi(t)\rangle^I = U(t, 0)|\Psi\rangle^H, \quad O^I(t) = U(t, 0)O^H(t)U^{-1}(t, 0). \quad (5.100)$$

可见, $U(t, 0)$ 就是联系 Heisenberg 绘景和相互作用绘景的么正变换算符。

从态矢的演化方程 (5.39) 可以得出

$$i \frac{\partial}{\partial t} U(t, t_0) |\Psi(t_0)\rangle^I = i \frac{\partial}{\partial t} |\Psi(t)\rangle^I = H_1^I(t) |\Psi(t)\rangle^I = H_1^I(t) U(t, t_0) |\Psi(t_0)\rangle^I, \quad (5.101)$$

即

$$i \frac{\partial}{\partial t} U(t, t_0) = H_1^I(t) U(t, t_0). \quad (5.102)$$

这是时间演化算符需要满足的微分方程，结合边值条件 (5.94)，可以将方程的解表达为

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_1^I(t_1) U(t_1, t_0). \quad (5.103)$$

上式左右两边均包含时间演化算符，可以进行重复迭代，从而得到级数

$$\begin{aligned} U(t, t_0) = & 1 + (-i) \int_{t_0}^t dt_1 H_1^I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_1^I(t_1) H_1^I(t_2) \\ & + \cdots + \left[(-i)^n \int_{t_0}^t dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_1^I(t_1) \cdots H_1^I(t_n) \right] + \cdots \end{aligned} \quad (5.104)$$

这个级数用起来不够方便，需要进一步化简。

从现在开始，我们将省略表示相互作用绘景的上标 I ，因为本章余下内容均在相互作用绘景中讨论。

在级数 (5.104) 中，作为积分上限的时刻是降序排列的，即 $t \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq \cdots \geq t_0$ 。由于积分上限相互依赖，这样的多重积分是很难处理的。为了将级数中每个积分的上限都扩展到 t 时刻，需要引入时序乘积 (time-ordered product) 的概念。时序乘积使若干个含时算符的乘积强行按照它们相应的时刻降序排列。以 n 个 $H_1(t)$ 算符为例，用 T 表示这种时序操作，有

$$T[H_1(t_1)H_1(t_2)\cdots H_1(t_n)] = H_1(t_{i_1})H_1(t_{i_2})\cdots H_1(t_{i_n}), \quad t_{i_1} \geq t_{i_2} \geq \cdots \geq t_{i_n}. \quad (5.105)$$

这里 $t_{i_1}, t_{i_2}, \cdots, t_{i_n}$ 是由 t_1, t_2, \cdots, t_n 降序排列得到的：

$$t_{i_1} \geq t_{i_2} \geq \cdots \geq t_{i_n}. \quad (5.106)$$

又如，两个标量场算符 $\phi(x)$ 和 $\phi(y)$ 的时序乘积可以用阶跃函数表示为

$$T[\phi(x)\phi(y)] = \phi(x)\phi(y)\theta(x^0 - y^0) + \phi(y)\phi(x)\theta(y^0 - x^0). \quad (5.107)$$

对于费米子算符，需要顾及到它们的反对易性质，因此，如果时序操作使费米子算符之间交换了奇数次，则应该额外加上一个负号。比如，两个旋量场算符 $\psi_a(x)$ 和 $\bar{\psi}_b(y)$ 的时序乘积是

$$T[\psi_a(x)\bar{\psi}_b(y)] = \psi_a(x)\bar{\psi}_b(y)\theta(x^0 - y^0) - \bar{\psi}_b(y)\psi_a(x)\theta(y^0 - x^0). \quad (5.108)$$

现在考虑级数 (5.104) 的第三项，它包含一个关于 t_1 和 t_2 的二重积分，积分区域如图 5.1(a) 所示，先对 t_2 积分，再对 t_1 积分。这个二重积分可以重新表达为

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_1(t_1) H_1(t_2) = \int_{t_0}^t dt_2 \int_{t_2}^t dt_1 H_1(t_1) H_1(t_2) = \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 H_1(t_2) H_1(t_1). \quad (5.109)$$

在上式第一步中, 我们改成先对 t_1 积分, 再对 t_2 积分, 积分区域不变, 如图 5.1(b) 所示。第二步, 我们交换了积分变量 t_1 和 t_2 , 对应的积分区域如图 5.1(c) 所示。由此, 可得

$$\begin{aligned} 2! \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_1(t_1) H_1(t_2) &= \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_1(t_1) H_1(t_2) + \int_{t_0}^t dt_1 \int_{t_1}^t dt_2 H_1(t_2) H_1(t_1) \\ &= \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \mathcal{T}[H_1(t_1) H_1(t_2)]. \end{aligned} \quad (5.110)$$

这里利用时序乘积将 t_1 和 t_2 的积分范围都扩展到整个 $[t_0, t]$ 区间, 因为图 5.1(a) 中的积分区域与图 5.1(c) 中的积分区域恰好拼成一个正方形。在上式第一步第一项中, t_1 是 t_2 的积分上限, 显然有 $t_1 \geq t_2$, 因而 $H_1(t_1)H_1(t_2)$ 是正确的时序乘积; 在第二项中, t_1 是 t_2 的积分下限, 故 $t_2 \geq t_1$, 此时 $H_1(t_2)H_1(t_1)$ 才是正确的时序乘积; 两项相加, 就得到第二步的结果。

将上述讨论推广到级数 (5.104) 中的第 n 项, 可得

$$n! \int_{t_0}^t dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_1(t_1) \cdots H_1(t_n) = \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_n \mathcal{T}[H_1(t_1) \cdots H_1(t_n)]. \quad (5.111)$$

上式出现 $n!$ 是因为此时对 n 个时间积分变量有 $n!$ 种排列方式。于是, 级数 (5.104) 可以用时序乘积表达为

$$\begin{aligned} U(t, t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_n \mathcal{T}[H_1(t_1) \cdots H_1(t_n)] \\ &\equiv \mathcal{T} \exp \left[-i \int_{t_0}^t dt' H_1(t') \right]. \end{aligned} \quad (5.112)$$

由于这个级数具有指数函数的级数展开形式, 这里进一步用指数记号来表示。

像 (5.12) 式一样, 在局域场论中 $H_1(t)$ 是相应哈密顿量密度 $\mathcal{H}_1(x)$ 的空间积分

$$H_1(t) = \int d^3x \mathcal{H}_1(x). \quad (5.113)$$

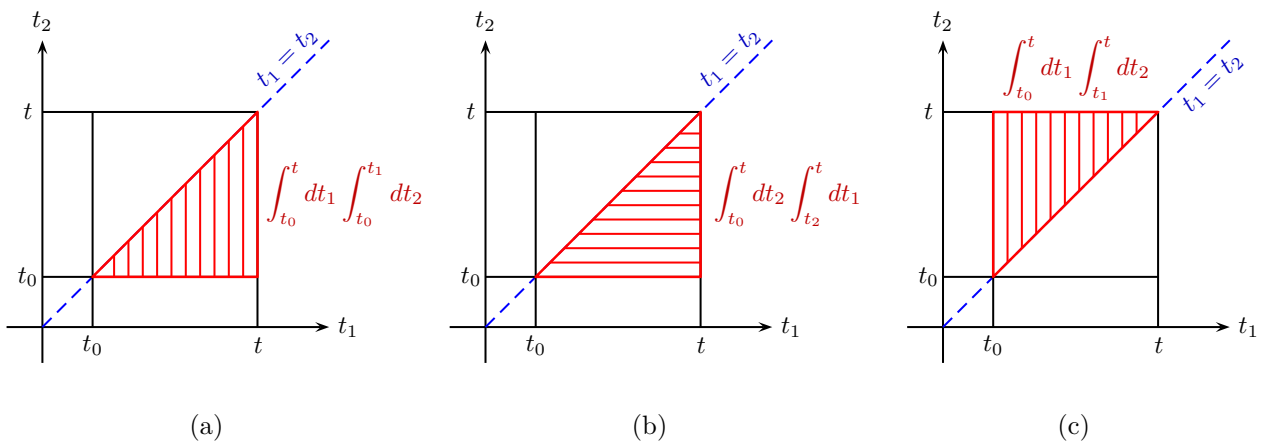


图 5.1: $t_1 - t_2$ 平面上的积分区域。

因此, 时间演化算符满足

$$U(t, t_0) = \mathsf{T} \exp \left[-i \int_{t_0}^t dt' \int d^3x' \mathcal{H}_1(x) \right]. \quad (5.114)$$

S 矩阵, 或者称为散射矩阵 (scattering matrix), 是量子散射理论的核心概念, 它描述系统从初态跃迁到末态的概率振幅。在相互作用绘景中, S 矩阵可以用时间演化算符来构造。

假设系统的初态 $|i\rangle$ 和末态 $|f\rangle$ 均处于自由状态, 而相互作用只发生在很短的时间间隔里, 则初始时刻处于遥远过去, 而终末时刻处于遥远未来。若将 t 时刻处描述系统的态矢记为 $|\Psi(t)\rangle$, 它从遥远过去 ($t \rightarrow -\infty$) 的初态 $|i\rangle$ 演化而来, 因而可以用时间演化算符表达为

$$|\Psi(t)\rangle = \lim_{t_0 \rightarrow -\infty} U(t, t_0) |i\rangle. \quad (5.115)$$

此过程相应的 S 矩阵元 S_{fi} 定义为态矢 $|\Psi(t)\rangle$ 演化到遥远未来 ($t \rightarrow +\infty$) 处与末态 $|f\rangle$ 的内积, 即

$$S_{fi} = \lim_{t \rightarrow +\infty} \langle f | \Psi(t) \rangle = \lim_{t \rightarrow +\infty} \lim_{t_0 \rightarrow -\infty} \langle f | U(t, t_0) | i \rangle. \quad (5.116)$$

引入 S 算符, 它在初态与末态之间的期待值就是 S 矩阵元 S_{fi} :

$$S_{fi} = \langle f | S | i \rangle. \quad (5.117)$$

那么, 我们可以得出

$$S = U(+\infty, -\infty). \quad (5.118)$$

从而, S 算符可以表达为相互作用哈密顿量的积分级数

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_n \mathsf{T}[H_1(t_1) \cdots H_1(t_n)] \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathsf{T}[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]. \end{aligned} \quad (5.119)$$

由时间演化算符的么正性可知, S 算符也是么正的,

$$S^\dagger S = 1. \quad (5.120)$$

5.3 散射截面和衰变宽度

在没有相互作用的理论中, S 算符就是单位算符 1, 因而 S 矩阵为 $S_{fi} = \langle f | i \rangle$ 。对于存在相互作用的理论, 上一节的计算告诉我们, S 算符可以展开为级数 (5.119)。这个级数的 $n = 0$ 项也是单位算符, 因此我们可以将 S 算符分解为

$$S = 1 + iT, \quad (5.121)$$

其中 iT 包含所有 $n \geq 1$ 的项。从而, S 矩阵可以分解为

$$S_{fi} = \langle f|i \rangle + \langle f|iT|i \rangle. \quad (5.122)$$

右边第一项意味着, 即使理论中存在相互作用, 初态也有一定概率自由地演化, 也就是说, 初态中的粒子仍然有一定概率不发生任何相互作用。由此可见, S 矩阵中真正描述相互作用的项是 $\langle f|iT|i \rangle$ 。由于能动量守恒定律, 初态中所有粒子的四维动量之和 p_i^μ 必定等于末态中所有粒子的四维动量之和 p_f^μ 。因此, $\langle f|iT|i \rangle$ 具有如下形式:

$$\langle f|iT|i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f) i\mathcal{M}_{fi}. \quad (5.123)$$

上式右边的四维 δ 函数体现了能动量守恒定律, 而 \mathcal{M}_{fi} 是 Lorentz 不变的, 称为不变矩阵元 (invariant matrix element), 或者不变散射振幅 (invariant scattering amplitude), 它是初态和末态动量的函数。

5.3.1 跃迁概率

在发生相互作用时, $i \rightarrow f$ 的跃迁概率可以表示成

$$P_{fi} = \frac{|\langle f|iT|i \rangle|^2}{\langle i|i \rangle \langle f|f \rangle}, \quad (5.124)$$

其中, $\langle i|i \rangle$ 和 $\langle f|f \rangle$ 分别是初态 $|i \rangle$ 和末态 $|f \rangle$ 的归一化因子。上式右边的分子为

$$|\langle f|iT|i \rangle|^2 = [(2\pi)^4 \delta^{(4)}(p_i - p_f)]^2 |\mathcal{M}_{fi}|^2 = (2\pi)^4 \delta^{(4)}(0) \cdot (2\pi)^4 \delta^{(4)}(p_i - p_f) |\mathcal{M}_{fi}|^2. \quad (5.125)$$

根据 Fourier 变换公式

$$\int d^4x e^{\pm ip \cdot x} = (2\pi)^4 \delta^{(4)}(p), \quad (5.126)$$

有

$$(2\pi)^4 \delta^{(4)}(0) = \int d^4x = \tilde{V} \tilde{T}. \quad (5.127)$$

其中, \tilde{V} 是空间积分区域的体积, \tilde{T} 是时间积分范围的长度, 对于全空间全时间积分, 它们趋于无穷大。于是, (5.125) 式可以写作

$$|\langle f|iT|i \rangle|^2 = \tilde{V} \tilde{T} (2\pi)^4 \delta^{(4)}(p_i - p_f) |\mathcal{M}_{fi}|^2. \quad (5.128)$$

现在, 讨论 2 体初态到 n 体末态的跃迁过程, 即初态包含 2 个粒子 \mathcal{A} 和 \mathcal{B} , 它们通过相互作用发生散射, 从而产生包含 n 个粒子的末态。设初态中两个粒子的动量分别为 $\mathbf{p}_\mathcal{A}$ 和 $\mathbf{p}_\mathcal{B}$, 则 $|i \rangle$ 可以用相应的产生算符表达为

$$|i \rangle = \sqrt{2E_\mathcal{A} 2E_\mathcal{B}} a_{\mathbf{p}_\mathcal{A}}^\dagger a_{\mathbf{p}_\mathcal{B}}^\dagger |0 \rangle, \quad E_{\mathcal{A},\mathcal{B}} = p_{\mathcal{A},\mathcal{B}}^0 = \sqrt{|\mathbf{p}_{\mathcal{A},\mathcal{B}}|^2 + m_{\mathcal{A},\mathcal{B}}^2}. \quad (5.129)$$

此处，我们省略了产生算符的螺旋度指标（或者说，自旋指标）。 $|0\rangle$ 是理论的真空态，理论中任意湮灭算符作用到它身上都将得到零。类似地，末态 $|f\rangle$ 可以写成

$$|f\rangle = \left(\prod_{j=1}^n \sqrt{2E_j} a_{\mathbf{p}_j}^\dagger \right) |0\rangle, \quad E_j = p_j^0 = \sqrt{|\mathbf{p}_j|^2 + m_j^2}. \quad (5.130)$$

其中， \mathbf{p}_j ($j = 1, \dots, n$) 是 n 个末态粒子的动量。此时，初态和末态的四维总动量分别是

$$p_i^\mu = p_{\mathcal{A}}^\mu + p_{\mathcal{B}}^\mu, \quad p_f^\mu = \sum_{j=1}^n p_j^\mu. \quad (5.131)$$

我们可以把初态 $|i\rangle$ 改写为直积，

$$|i\rangle = \sqrt{2E_{\mathcal{A}}} a_{\mathbf{p}_{\mathcal{A}}}^\dagger |0\rangle_{\mathcal{A}} \otimes \sqrt{2E_{\mathcal{B}}} a_{\mathbf{p}_{\mathcal{B}}}^\dagger |0\rangle_{\mathcal{B}} = |\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}} \otimes |\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}}. \quad (5.132)$$

这里 $|0\rangle_{\mathcal{A}}$ 和 $|0\rangle_{\mathcal{B}}$ 分别是描述 \mathcal{A} 和 \mathcal{B} 的两个量子场所对应的真空态。如同 (2.116) 式，单粒子态 $|\mathbf{p}_{\mathcal{A}}\rangle_{\mathcal{A}}$ 和 $|\mathbf{p}_{\mathcal{B}}\rangle_{\mathcal{B}}$ 的自我内积分别是

$$\langle \mathbf{p}_{\mathcal{A}} | \mathbf{p}_{\mathcal{A}} \rangle_{\mathcal{A}} = 2E_{\mathcal{A}} (2\pi)^3 \delta^{(3)}(\mathbf{0}) = 2E_{\mathcal{A}} \tilde{V}, \quad \langle \mathbf{p}_{\mathcal{B}} | \mathbf{p}_{\mathcal{B}} \rangle_{\mathcal{B}} = 2E_{\mathcal{B}} (2\pi)^3 \delta^{(3)}(\mathbf{0}) = 2E_{\mathcal{B}} \tilde{V}. \quad (5.133)$$

于是，我们得到

$$\langle i | i \rangle = \langle \mathbf{p}_{\mathcal{A}} | \mathbf{p}_{\mathcal{A}} \rangle_{\mathcal{A}} \langle \mathbf{p}_{\mathcal{B}} | \mathbf{p}_{\mathcal{B}} \rangle_{\mathcal{B}} = 4E_{\mathcal{A}} E_{\mathcal{B}} \tilde{V}^2. \quad (5.134)$$

同理可得

$$\langle f | f \rangle = \prod_{j=1}^n (2E_j \tilde{V}). \quad (5.135)$$

从而，跃迁概率化为

$$P_{fi} = \frac{|\langle f | iT | i \rangle|^2}{\langle i | i \rangle \langle f | f \rangle} = \frac{\tilde{V} \tilde{T} (2\pi)^4 \delta^{(4)}(p_i - p_f) |\mathcal{M}_{fi}|^2}{4E_{\mathcal{A}} E_{\mathcal{B}} \tilde{V}^2 \prod_{j=1}^n (2E_j \tilde{V})} = \frac{\tilde{T} (2\pi)^4 \delta^{(4)}(p_i - p_f) |\mathcal{M}_{fi}|^2}{4E_{\mathcal{A}} E_{\mathcal{B}} \tilde{V} \prod_{j=1}^n (2E_j \tilde{V})}. \quad (5.136)$$

对于一组特定的动量 $\{p_j\}$ ，单位时间内的跃迁概率为

$$R_{\{p_j\}} = \frac{P_{fi}}{\tilde{T}} = \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}} \tilde{V} \prod_{j=1}^n (2E_j \tilde{V})} (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.137)$$

此处四维 δ 函数可以分解为

$$\delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right) = \delta^{(3)}\left(\mathbf{p}_{\mathcal{A}} + \mathbf{p}_{\mathcal{B}} - \sum_{j=1}^n \mathbf{p}_j\right) \delta\left(E_{\mathcal{A}} + E_{\mathcal{B}} - \sum_{j=1}^n E_j\right). \quad (5.138)$$

在这样一个 $2 \rightarrow n$ 散射过程中，末态中 n 个粒子的动量可以取任意满足运动学要求的值，而运动学条件

$$p_{\mathcal{A}}^\mu + p_{\mathcal{B}}^\mu - \sum_{j=1}^n p_j^\mu = 0 \quad (5.139)$$

已经体现在 (5.137) 式的四维 δ 函数中。为了计算总的跃迁率，我们需要将 $\{p_j\}$ 的所有可能取值包含起来，也就是说，需要对末态的动量相空间积分。

接下来，我们讨论如何包含末态粒子所有可能的动量取值。考察一维情况，先假定粒子局限在 $x \in [-L/2, L/2]$ 范围内运动，最后让 $L \rightarrow \infty$ 。为了确保动量算符 $p_x = -i\partial/\partial x$ 在区间 $[-L/2, L/2]$ 上是厄米算符，必须要求描述粒子的波函数 $\varphi(x)$ 满足周期性边界条件

$$\varphi\left(-\frac{L}{2}\right) = \varphi\left(\frac{L}{2}\right). \quad (5.140)$$

作为动量本征态的波函数是平面波解 $\varphi_p(x) \propto \exp(ipx)$ ，结合周期性边界条件，有

$$\exp\left(-\frac{i}{2}pL\right) = \exp\left(\frac{i}{2}pL\right), \quad (5.141)$$

故

$$\exp(ipL) = 1, \quad \sin(pL) = 0, \quad \cos(pL) = 1. \quad (5.142)$$

上式成立意味着

$$pL = 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots \quad (5.143)$$

因此，动量本征值是

$$p_k = \frac{2\pi}{L}k, \quad k \in \mathbb{Z}. \quad (5.144)$$

当 $L \rightarrow \infty$ 时，相邻动量本征值之差变成动量的微分：

$$\Delta p_k = p_{k+1} - p_k = \frac{2\pi}{L} \rightarrow dp. \quad (5.145)$$

从而可得

$$\sum_{k=-\infty}^{+\infty} \Delta p_k = \frac{2\pi}{L} \sum_{k=-\infty}^{+\infty} \rightarrow \int_{-\infty}^{+\infty} dp, \quad (5.146)$$

即

$$\sum_{k=-\infty}^{+\infty} \rightarrow \frac{L}{2\pi} \int_{-\infty}^{+\infty} dp. \quad (5.147)$$

推广到三维情况，先假定粒子局限在体积为 $\tilde{V} = L^3$ 的立方体中运动，周期性边界条件相当于将立方体表面上任意一点视作与位于相对的面上的对应点等同。满足此条件的动量本征值为

$$\mathbf{p} = \frac{2\pi}{L}(k_1, k_2, k_3), \quad k_1, k_2, k_3 \in \mathbb{Z}. \quad (5.148)$$

当 $L \rightarrow \infty$ 时，我们得到

$$\sum_{k_1 k_2 k_3} \rightarrow \frac{L^3}{(2\pi)^3} \int d^3p = \frac{\tilde{V}}{(2\pi)^3} \int d^3p. \quad (5.149)$$

上式最左边代表对所有动量取值求和，当动量可取连续数值时，这种求和就化作最右边的动量相空间积分。将 n 个末态粒子的所有动量取值都考虑进来，要对 (5.137) 式积分，从而得到单位时间内 $2 \rightarrow n$ 散射过程的跃迁概率为

$$\begin{aligned} R &= \left(\prod_{j=1}^n \frac{\tilde{V}}{(2\pi)^3} \int d^3 p_j \right) R_{\{p_j\}} \\ &= \frac{1}{4E_A E_B \tilde{V}} \left(\prod_{j=1}^n \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_{j=1}^n p_j) |\mathcal{M}_{fi}|^2. \end{aligned} \quad (5.150)$$

根据 2.3.4 小节的讨论，我们知道体积元 (2.120) 是 Lorentz 不变的，因而上式中相空间体积元

$$\frac{d^3 p_j}{(2\pi)^3 2E_j} \quad (5.151)$$

也是 Lorentz 不变的。

5.3.2 散射截面

现在，我们讨论束流打靶实验，靶 (target) 由 \mathcal{A} 粒子组成，束流 (beam) 由 \mathcal{B} 粒子组成。设束流中每个 \mathcal{B} 粒子的运动速度相同，记为 \mathbf{v}_B ，按照狭义相对论，有 $\mathbf{v}_B \equiv \mathbf{p}_B/E_B$ 。记束流的横截面积为 A ，则 t 时间内束流的一个横截面经过的体积为 $V = A|\mathbf{v}_B|t$ 。再设束流中 \mathcal{B} 粒子的数密度为 n_B ，从而，体积 V 中的粒子数为 $N_B = n_B V = n_B A|\mathbf{v}_B|t$ 。在单位时间内穿过单位面积的 \mathcal{B} 粒子数称为流密度，记作 j_B ，可以通过下式计算，

$$j_B = \frac{N_B}{At} = \frac{n_B A|\mathbf{v}_B|t}{At} = n_B |\mathbf{v}_B|. \quad (5.152)$$

考虑流密度为 j_B 的束流打到由 N_A 个 \mathcal{A} 粒子组成的靶上，则 t 时间内散射发生的次数可以表示为

$$N = N_A j_B \sigma t. \quad (5.153)$$

这里引入了物理量 σ ，由量纲分析知道它具有面积量纲，称为散射截面 (scattering cross section)，或简称为截面 (cross section)。散射截面表征散射过程的强度，由 \mathcal{A} 粒子与 \mathcal{B} 粒子的相互作用性质决定，常用单位是靶 (barn)，记作 b，

$$1 \text{ b} = 10^{-28} \text{ m}^2 = 2.568 \times 10^3 \text{ GeV}^{-2}. \quad (5.154)$$

于是，单位时间单位体积内散射发生的次数为

$$\mathcal{R} = \frac{N}{Vt} = \frac{N_A j_B \sigma}{V} = \frac{N_A n_B |\mathbf{v}_B| \sigma}{V} = n_A n_B \sigma |\mathbf{v}_B|, \quad (5.155)$$

其中 $n_A = N_A/V$ 相当于 \mathcal{A} 粒子在体积 V 中的密度。

如果只考虑一个 \mathcal{B} 粒子打到一个 \mathcal{A} 粒子上, 那么, 可以看作在体积 \tilde{V} 中仅有这两个粒子, 因而 $n_{\mathcal{A}} = n_{\mathcal{B}} = 1/\tilde{V}$, 此时 \mathcal{R} 可以用单位时间内的跃迁概率 R 表示为 $\mathcal{R} = R/\tilde{V}$ 。于是, 根据 (5.150) 式, 我们得到

$$\begin{aligned}\sigma &= \frac{\mathcal{R}}{n_{\mathcal{A}}n_{\mathcal{B}}|\mathbf{v}_{\mathcal{B}}|} = \frac{R}{\tilde{V}|\mathbf{v}_{\mathcal{B}}|} = \frac{R\tilde{V}}{|\mathbf{v}_{\mathcal{B}}|} \\ &= \frac{1}{4E_{\mathcal{A}}E_{\mathcal{B}}|\mathbf{v}_{\mathcal{B}}|} \left(\prod_{j=1}^n \int \frac{d^3p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2.\end{aligned}\quad (5.156)$$

上式对 \mathcal{A} 粒子静止的参考系成立。我们想把它推广到任意惯性系, 从而可以处理 \mathcal{A} 粒子和 \mathcal{B} 粒子处于任意运动状态的情况。为此, 把散射截面 σ 定义为 Lorentz 不变量会比较方便。(5.156) 式最后一行中, 除了第一个因子 $(4E_{\mathcal{A}}E_{\mathcal{B}}|\mathbf{v}_{\mathcal{B}}|)^{-1}$ 之外, 其余部分是 Lorentz 不变的。在 \mathcal{A} 粒子静止的参考系中, $|\mathbf{v}_{\mathcal{B}}|$ 就是 \mathcal{B} 粒子与 \mathcal{A} 粒子之间的相对速度。相对速度可以定义为

$$v_{\text{rel}} \equiv |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|, \quad (5.157)$$

其中 $\mathbf{v}_{\mathcal{A}} \equiv \mathbf{p}_{\mathcal{A}}/E_{\mathcal{A}}$ 是 \mathcal{A} 粒子的运行速度。不过, $E_{\mathcal{A}}E_{\mathcal{B}}v_{\text{rel}}$ 并不是 Lorentz 不变量。我们要做的是将相对速度替换成另一个物理量 Møller 速度, 定义是

$$v_{\text{Møl}} \equiv \frac{1}{E_{\mathcal{A}}E_{\mathcal{B}}} \sqrt{(p_{\mathcal{A}} \cdot p_{\mathcal{B}})^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2}. \quad (5.158)$$

容易看出, $E_{\mathcal{A}}E_{\mathcal{B}}v_{\text{Møl}}$ 是 Lorentz 不变量。现在, 我们将散射截面定义为

$$\sigma = \frac{1}{4E_{\mathcal{A}}E_{\mathcal{B}}v_{\text{Møl}}} \left(\prod_{j=1}^n \int \frac{d^3p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.159)$$

它是 Lorentz 不变的, 而且 $\mathcal{R} = n_{\mathcal{A}}n_{\mathcal{B}}\sigma v_{\text{Møl}}$ 也是 Lorentz 不变的。当 \mathcal{A} 粒子静止时, $E_{\mathcal{A}} = m_{\mathcal{A}}$, $\mathbf{p}_{\mathcal{A}} = \mathbf{0}$, 故

$$v_{\text{Møl}} = \frac{1}{m_{\mathcal{A}}E_{\mathcal{B}}} \sqrt{m_{\mathcal{A}}^2 E_{\mathcal{B}}^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2} = \frac{\sqrt{E_{\mathcal{B}}^2 - m_{\mathcal{B}}^2}}{E_{\mathcal{B}}} = \frac{|\mathbf{p}_{\mathcal{B}}|}{E_{\mathcal{B}}} = |\mathbf{v}_{\mathcal{B}}|, \quad (5.160)$$

此时截面定义式 (5.159) 可以回复到 (5.156) 式。

在 (5.159) 式右边, 不变振幅模方 $|\mathcal{M}_{fi}|^2$ 是动力学因素, 而其它部分都属于运动学因素。在运动学因素中, 对末态动量的积分具有如下形式:

$$\int d\Pi_n = \left(\prod_{j=1}^n \int \frac{d^3p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right). \quad (5.161)$$

这个积分称为 n 体不变相空间。利用这个记号, 可以把 (5.159) 式写得简洁一些:

$$\sigma = \frac{1}{4E_{\mathcal{A}}E_{\mathcal{B}}v_{\text{Møl}}} \int d\Pi_n |\mathcal{M}_{fi}|^2. \quad (5.162)$$

如果 (5.159) 式右边不作积分, 则对应于微分散射截面

$$d\sigma = \frac{1}{4E_{\mathcal{A}}E_{\mathcal{B}}v_{\text{Møl}}} \left(\prod_{j=1}^n \frac{d^3p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.163)$$

下面进一步考察 Møller 速度 $v_{\text{Møl}}$ 的性质。设 \mathcal{A} 粒子与 \mathcal{B} 粒子运动方向之间的夹角为 α ，则有

$$\mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}} = |\mathbf{v}_{\mathcal{A}}| |\mathbf{v}_{\mathcal{B}}| \cos \alpha, \quad |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}| = |\mathbf{v}_{\mathcal{A}}| |\mathbf{v}_{\mathcal{B}}| \sin \alpha, \quad (5.164)$$

故

$$(\mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}})^2 = |\mathbf{v}_{\mathcal{A}}|^2 |\mathbf{v}_{\mathcal{B}}|^2 \cos^2 \alpha = |\mathbf{v}_{\mathcal{A}}|^2 |\mathbf{v}_{\mathcal{B}}|^2 (1 - \sin^2 \alpha) = |\mathbf{v}_{\mathcal{A}}|^2 |\mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2. \quad (5.165)$$

从而，可以推出

$$\begin{aligned} (1 - \mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}})^2 &= 1 - 2 \mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}} + (\mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}})^2 = 1 - 2 \mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}} + |\mathbf{v}_{\mathcal{A}}|^2 |\mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2 \\ &= 1 + |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}}|^2 - |\mathbf{v}_{\mathcal{B}}|^2 + |\mathbf{v}_{\mathcal{A}}|^2 |\mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2 \\ &= |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2 + (1 - |\mathbf{v}_{\mathcal{A}}|^2)(1 - |\mathbf{v}_{\mathcal{B}}|^2). \end{aligned} \quad (5.166)$$

将 \mathcal{A} 和 \mathcal{B} 的四维动量分解为时间分量和空间分量，得

$$p_{\mathcal{A}}^{\mu} = (E_{\mathcal{A}}, \mathbf{p}_{\mathcal{A}}) = E_{\mathcal{A}}(1, \mathbf{v}_{\mathcal{A}}), \quad p_{\mathcal{B}}^{\mu} = (E_{\mathcal{B}}, \mathbf{p}_{\mathcal{B}}) = E_{\mathcal{B}}(1, \mathbf{v}_{\mathcal{B}}). \quad (5.167)$$

这两个四维动量的内积为

$$p_{\mathcal{A}} \cdot p_{\mathcal{B}} = E_{\mathcal{A}} E_{\mathcal{B}} - \mathbf{p}_{\mathcal{A}} \cdot \mathbf{p}_{\mathcal{B}} = E_{\mathcal{A}} E_{\mathcal{B}} (1 - \mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}}). \quad (5.168)$$

于是，可以导出

$$\begin{aligned} (p_{\mathcal{A}} \cdot p_{\mathcal{B}})^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 &= E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (1 - \mathbf{v}_{\mathcal{A}} \cdot \mathbf{v}_{\mathcal{B}})^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 \\ &= E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2) + E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (1 - |\mathbf{v}_{\mathcal{A}}|^2)(1 - |\mathbf{v}_{\mathcal{B}}|^2) - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 \\ &= E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2) + (E_{\mathcal{A}}^2 - |\mathbf{p}_{\mathcal{A}}|^2)(E_{\mathcal{B}}^2 - |\mathbf{p}_{\mathcal{B}}|^2) - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 \\ &= E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2) + m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2 \\ &= E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2). \end{aligned} \quad (5.169)$$

这样的话，由 Møller 速度的定义 (5.158) 可得

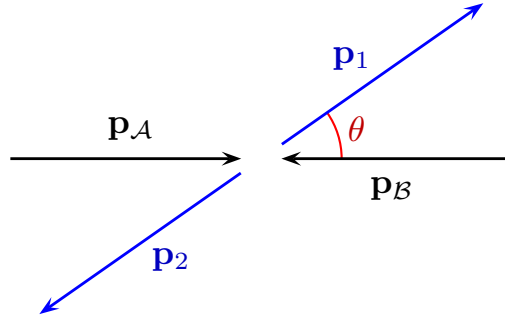
$$\begin{aligned} v_{\text{Møl}} &= \frac{1}{E_{\mathcal{A}} E_{\mathcal{B}}} \sqrt{(p_{\mathcal{A}} \cdot p_{\mathcal{B}})^2 - m_{\mathcal{A}}^2 m_{\mathcal{B}}^2} = \frac{1}{E_{\mathcal{A}} E_{\mathcal{B}}} \sqrt{E_{\mathcal{A}}^2 E_{\mathcal{B}}^2 (|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2)} \\ &= \sqrt{|\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|^2 - |\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}}|^2}. \end{aligned} \quad (5.170)$$

如果 \mathcal{A} 粒子与 \mathcal{B} 粒子的运动方向相同或相反，则 $\mathbf{v}_{\mathcal{A}} \times \mathbf{v}_{\mathcal{B}} = \mathbf{0}$ ，因而

$$v_{\text{Møl}} = |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}| = v_{\text{rel}}, \quad (5.171)$$

即 Møller 速度与相对速度相同。这种情况在对撞机 (collider) 实验中经常遇到，因为在束流迎头对撞时，两束束流中的粒子具有相反的运动方向。此时，散射截面 (5.159) 化为

$$\sigma = \frac{1}{4E_{\mathcal{A}} E_{\mathcal{B}} |\mathbf{v}_{\mathcal{A}} - \mathbf{v}_{\mathcal{B}}|} \left(\prod_{j=1}^n \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum_{j=1}^n p_j) |\mathcal{M}_{fi}|^2. \quad (5.172)$$

图 5.2: 质心系中 $2 \rightarrow 2$ 散射过程的动量示意图。

在非相对论极限下, $v_{\text{rel}} = |\mathbf{v}_A - \mathbf{v}_B|$ 确实是 A 与 B 的相对速度, 但是, 对于相对论极限下的束流对撞, $|\mathbf{v}_A| = |\mathbf{v}_B| = 1$ 且 $\mathbf{v}_B = -\mathbf{v}_A$, 故 $v_{\text{rel}} = |\mathbf{v}_A - \mathbf{v}_B| = 2$, 它是真空光速的 2 倍, 显然不是真正意义的相对速度。

接下来讨论 $2 \rightarrow 2$ 散射, 即 $n = 2$ 的情况, 此时末态包含 2 个粒子。在系统的质量中心参考系 (简称质心系, center-of-mass system) 中, 总动量为零, 即

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}, \quad (5.173)$$

因而

$$|\mathbf{p}_A| = |\mathbf{p}_B|, \quad |\mathbf{p}_1| = |\mathbf{p}_2|. \quad (5.174)$$

可见, 初态中 \mathbf{p}_A 与 \mathbf{p}_B 大小相等, 方向相反, 故 $v_{\text{Mol}} = v_{\text{rel}}$; 末态中 \mathbf{p}_1 与 \mathbf{p}_2 也是大小相等, 方向相反。这些动量在质心系中的关系如图 5.2 所示, 其中, 散射角 θ 是 \mathbf{p}_1 与 \mathbf{p}_A 之间的夹角。质心系中系统的总能量称为质心能 (center-of-mass energy) E_{CM} , 满足

$$E_{\text{CM}} = E_A + E_B = E_1 + E_2. \quad (5.175)$$

由

$$(p_A + p_B)^2 = (E_A + E_B)^2 - (\mathbf{p}_A + \mathbf{p}_B)^2 = (E_A + E_B)^2 = E_{\text{CM}}^2 \quad (5.176)$$

可知, 质心能 E_{CM} 是 Lorentz 不变量。

根据 (5.172) 和 (5.161) 式, 质心系中 $2 \rightarrow 2$ 散射截面可以写成

$$\sigma = \frac{1}{4E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \int d\Pi_2 |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2. \quad (5.177)$$

其中, 不变散射振幅 \mathcal{M} 的动量依赖性已经明显表示出来。计算 2 体不变相空间中的积分, 可得

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2) \\ &= \int \frac{d^3 p_1}{(2\pi)^2 4E_1 E_2} \delta(E_{\text{CM}} - E_1 - E_2) \\ &= \int d\Omega d|\mathbf{p}_1| \frac{|\mathbf{p}_1|^2}{16\pi^2 E_1 E_2} \delta\left(E_{\text{CM}} - \sqrt{|\mathbf{p}_1|^2 + m_1^2} - \sqrt{|\mathbf{p}_1|^2 + m_2^2}\right). \end{aligned} \quad (5.178)$$

第二步结合三维 δ 函数 $\delta^{(3)}(\mathbf{p}_A + \mathbf{p}_B - \mathbf{p}_1 - \mathbf{p}_2)$ 作出 \mathbf{p}_2 的三维积分。这样积分看起来没有效果，但实际上是要求 \mathbf{p}_2 满足动量守恒条件 $\mathbf{p}_A + \mathbf{p}_B - \mathbf{p}_1 - \mathbf{p}_2 = \mathbf{0}$ ，因此后续计算中出现的 \mathbf{p}_2 应该满足这个条件，在质心系中则体现为 $\mathbf{p}_2 = -\mathbf{p}_1$ ，故 $E_2 = \sqrt{|\mathbf{p}_2|^2 + m_2^2} = \sqrt{|\mathbf{p}_1|^2 + m_2^2}$ 。第三步利用球坐标将 \mathbf{p}_1 动量空间的体积元分解为 $d^3p_1 = |\mathbf{p}_1|^2 d|\mathbf{p}_1| d\Omega$ ，而立体角的微分可以用散射角 θ 表示为

$$d\Omega = \sin\theta d\theta d\phi, \quad (5.179)$$

其中方位角 ϕ 在垂直于 \mathbf{p}_A 方向的平面上定义。现在， δ 函数的宗量是关于 $|\mathbf{p}_1|$ 的函数，利用 (2.117) 式，可得作出 $|\mathbf{p}_1|$ 的积分，得到

$$\begin{aligned} & \int d|\mathbf{p}_1| \delta(E_{\text{CM}} - \sqrt{|\mathbf{p}_1|^2 + m_1^2} - \sqrt{|\mathbf{p}_1|^2 + m_2^2}) \\ &= \left| \frac{d}{d|\mathbf{p}_1|} (E_{\text{CM}} - \sqrt{|\mathbf{p}_1|^2 + m_1^2} - \sqrt{|\mathbf{p}_1|^2 + m_2^2}) \right|^{-1} = \left(\frac{2|\mathbf{p}_1|}{2\sqrt{|\mathbf{p}_1|^2 + m_1^2}} + \frac{2|\mathbf{p}_1|}{2\sqrt{|\mathbf{p}_1|^2 + m_2^2}} \right)^{-1} \\ &= \left[|\mathbf{p}_1| \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \right]^{-1} = \frac{E_1 E_2}{|\mathbf{p}_1|(E_1 + E_2)} = \frac{E_1 E_2}{|\mathbf{p}_1| E_{\text{CM}}}. \end{aligned} \quad (5.180)$$

于是，(5.178) 式化为

$$\int d\Pi_2 = \int d\Omega \frac{|\mathbf{p}_1|^2}{16\pi^2 E_1 E_2} \frac{E_1 E_2}{|\mathbf{p}_1| E_{\text{CM}}} = \int d\Omega \frac{|\mathbf{p}_1|}{16\pi^2 E_{\text{CM}}}. \quad (5.181)$$

将上式代入散射截面表达式 (5.177)，得

$$\sigma = \frac{1}{4E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \int d\Omega \frac{|\mathbf{p}_1|}{16\pi^2 E_{\text{CM}}} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2. \quad (5.182)$$

于是，质心系中关于立体角的微分散射截面是

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{64\pi^2} \frac{1}{E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \frac{|\mathbf{p}_1|}{E_{\text{CM}}} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2. \quad (5.183)$$

利用末态粒子在质心系中的动量关系 $|\mathbf{p}_1| = |\mathbf{p}_2|$ ，可得

$$E_{\text{CM}} = E_1 + E_2 = E_1 + \sqrt{|\mathbf{p}_1|^2 + m_2^2} = E_1 + \sqrt{E_1^2 - m_1^2 + m_2^2}, \quad (5.184)$$

故

$$E_1^2 - m_1^2 + m_2^2 = (E_{\text{CM}} - E_1)^2 = E_{\text{CM}}^2 - 2E_{\text{CM}}E_1 + E_1^2, \quad (5.185)$$

即

$$2E_{\text{CM}}E_1 = E_{\text{CM}}^2 + m_1^2 - m_2^2, \quad (5.186)$$

从而， E_1 可以表示为

$$E_1 = \frac{1}{2E_{\text{CM}}} (E_{\text{CM}}^2 + m_1^2 - m_2^2). \quad (5.187)$$

同理， E_2 可以表示为

$$E_2 = \frac{1}{2E_{\text{CM}}} (E_{\text{CM}}^2 + m_2^2 - m_1^2). \quad (5.188)$$

根据动量与能量的关系, 有

$$\begin{aligned}
 |\mathbf{p}_1|^2 &= E_1^2 - m_1^2 = \frac{1}{4E_{\text{CM}}^2} (E_{\text{CM}}^2 + m_1^2 - m_2^2)^2 - m_1^2 \\
 &= \frac{1}{4E_{\text{CM}}^2} [E_{\text{CM}}^4 + m_1^4 + m_2^4 + 2E_{\text{CM}}^2 m_1^2 - 2E_{\text{CM}}^2 m_2^2 - 2m_1^2 m_2^2 - 4E_{\text{CM}}^2 m_1^2] \\
 &= \frac{1}{4E_{\text{CM}}^2} (E_{\text{CM}}^4 + m_1^4 + m_2^4 - 2E_{\text{CM}}^2 m_1^2 - 2E_{\text{CM}}^2 m_2^2 - 4m_1^2 m_2^2) \\
 &= \frac{1}{4E_{\text{CM}}^2} \lambda(E_{\text{CM}}^2, m_1^2, m_2^2).
 \end{aligned} \tag{5.189}$$

其中, λ 函数定义为

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz, \tag{5.190}$$

它关于 x, y, z 对称。可见, 末态粒子的动量满足

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{1}{2E_{\text{CM}}} \lambda^{1/2}(E_{\text{CM}}^2, m_1^2, m_2^2) = \frac{E_{\text{CM}}}{2} \lambda^{1/2} \left(1, \frac{m_1^2}{E_{\text{CM}}^2}, \frac{m_2^2}{E_{\text{CM}}^2} \right). \tag{5.191}$$

于是, (5.183) 式可以改写成

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{128\pi^2 E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \lambda^{1/2} \left(1, \frac{m_1^2}{E_{\text{CM}}^2}, \frac{m_2^2}{E_{\text{CM}}^2} \right) |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2. \tag{5.192}$$

下面讨论几种特殊情况。

(1) 如果散射过程关于对撞轴 (\mathbf{p}_A 对应的直线) 对称, 则不变振幅 \mathcal{M} 与 ϕ 无关, 是 θ 的函数, 从而,

$$\int d\Omega |\mathcal{M}(\theta)|^2 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta |\mathcal{M}(\theta)|^2 = 2\pi \int_0^\pi d\theta \sin \theta |\mathcal{M}(\theta)|^2. \tag{5.193}$$

此时散射截面为

$$\sigma = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{64\pi E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \lambda^{1/2} \left(1, \frac{m_1^2}{E_{\text{CM}}^2}, \frac{m_2^2}{E_{\text{CM}}^2} \right) \int_0^\pi d\theta \sin \theta |\mathcal{M}(\theta)|^2. \tag{5.194}$$

(2) 如果末态 2 个粒子质量相同, $m_1 = m_2 = m$, 则由

$$\lambda(x, y, y) = x^2 + 2y^2 - 4xy - 2y^2 = x(x - 4y) \tag{5.195}$$

可得

$$\lambda^{1/2} \left(1, \frac{m_1^2}{E_{\text{CM}}^2}, \frac{m_2^2}{E_{\text{CM}}^2} \right) = \lambda^{1/2} \left(1, \frac{m^2}{E_{\text{CM}}^2}, \frac{m^2}{E_{\text{CM}}^2} \right) = \sqrt{1 - \frac{4m^2}{E_{\text{CM}}^2}}. \tag{5.196}$$

(3) 如果初末态 4 个粒子的质量相同, 即 $m_A = m_B = m_1 = m_2$, 则有

$$E_A = E_B = \frac{E_{\text{CM}}}{2} = E_1 = E_2, \quad |\mathbf{p}_A| = |\mathbf{p}_B| = |\mathbf{p}_1| = |\mathbf{p}_2|. \tag{5.197}$$

从而, 可得

$$|\mathbf{v}_A - \mathbf{v}_B| = \left| \frac{\mathbf{p}_A}{E_A} - \frac{\mathbf{p}_B}{E_B} \right| = \frac{2|\mathbf{p}_A|}{E_A} = \frac{4|\mathbf{p}_1|}{E_{\text{CM}}}. \quad (5.198)$$

于是, (5.183) 式化为

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} &= \frac{1}{64\pi^2} \frac{4}{E_{\text{CM}}^2} \frac{E_{\text{CM}}}{4|\mathbf{p}_1|} \frac{|\mathbf{p}_1|}{E_{\text{CM}}} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2 \\ &= \frac{1}{64\pi^2 E_{\text{CM}}^2} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2. \end{aligned} \quad (5.199)$$

5.3.3 衰变宽度

即使没有与其它粒子散射, 一个粒子也不一定是稳定的。不稳定粒子 \mathcal{A} 自身可以通过相互作用衰变 (decay) 成其它粒子。在 \mathcal{A} 粒子的静止参考系中, 它在衰变之前存活的时间 t 服从指数分布, 概率密度为

$$P(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) = \Gamma \exp(-\Gamma t). \quad (5.200)$$

其中, τ 是常数, 称为粒子的寿命 (lifetime), 由 t 的期待值

$$\langle t \rangle = \frac{1}{\tau} \int_0^\infty t e^{-t/\tau} dt = - \int_0^\infty t d e^{-t/\tau} = -t e^{-t/\tau} \Big|_0^\infty + \int_0^\infty e^{-t/\tau} dt = -\tau e^{-t/\tau} \Big|_0^\infty = \tau \quad (5.201)$$

可知, 寿命是粒子存活的平均时间。因此,

$$\Gamma \equiv \frac{1}{\tau} \quad (5.202)$$

是 \mathcal{A} 粒子在静止系中发生衰变的平均速率, 它在自然单位制中具有质量的量纲, 称为衰变宽度 (decay width), 简称宽度。

\mathcal{A} 粒子可能有多种衰变过程。在一次衰变中, 某个衰变过程 $i \rightarrow f$ 发生的概率称为此过程的分支比 (branching ratio), 记作 $\text{BR}(f)$ 。衰变过程 $i \rightarrow f$ 的分宽度 (partial decay width) 定义为

$$\Gamma_f = \Gamma \cdot \text{BR}(f), \quad (5.203)$$

它是 \mathcal{A} 粒子静止系中衰变过程 $i \rightarrow f$ 发生的平均速率。所有衰变过程的分支比之和应该是归一的, 故

$$\sum_f \text{BR}(f) = \frac{1}{\Gamma} \sum_f \Gamma_f = 1, \quad \Gamma = \sum_f \Gamma_f. \quad (5.204)$$

我们可以通过跃迁概率计算衰变过程 $i \rightarrow f$ 的分宽度。现在, 初态 $|i\rangle$ 只包含 1 个粒子 \mathcal{A} , 末态 $|f\rangle$ 则包含 $n \geq 2$ 个粒子。因此, $|i\rangle$ 的自我内积为

$$\langle i|i \rangle = 2E_A \tilde{V}, \quad (5.205)$$

跃迁概率是

$$P_{fi} = \frac{|\langle f|iT|i \rangle|^2}{\langle i|i \rangle \langle f|f \rangle} = \frac{\tilde{V} \tilde{T} (2\pi)^4 \delta^{(4)}(p_A - p_f) |\mathcal{M}_{fi}|^2}{2E_A \tilde{V} \prod_{j=1}^n (2E_j V)} = \frac{\tilde{T} (2\pi)^4 \delta^{(4)}(p_A - p_f) |\mathcal{M}_{fi}|^2}{2E_A \prod_{j=1}^n (2E_j \tilde{V})}. \quad (5.206)$$

对于一组特定的末态动量 $\{p_j\}$, 单位时间内的跃迁概率为

$$R_{\{p_j\}} = \frac{P_{fi}}{\tilde{T}} = \frac{1}{2E_A \prod_{j=1}^n (2E_j \tilde{V})} (2\pi)^4 \delta^{(4)}\left(p_A - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.207)$$

将末态动量的所有取值考虑进来, 可得单位时间内衰变过程 $i \rightarrow f$ 的发生概率为

$$R_f = \left(\prod_{j=1}^n \frac{\tilde{V}}{(2\pi)^3} \int d^3 p_j \right) R_{\{p_j\}} = \frac{1}{2E_A} \left(\prod_{j=1}^n \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_A - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.208)$$

在 \mathcal{A} 粒子静止系中, $E_A = m_A$, 而 R_f 的值就是分宽度 Γ_f , 故

$$\Gamma_f = \frac{1}{2m_A} \left(\prod_{j=1}^n \int \frac{d^3 p_j}{(2\pi)^3 2E_j} \right) (2\pi)^4 \delta^{(4)}\left(p_A - \sum_{j=1}^n p_j\right) |\mathcal{M}_{fi}|^2. \quad (5.209)$$

若 \mathcal{A} 粒子是标量粒子, 自旋为 0, 则 \mathcal{A} 粒子静止系没有特殊的方向, 于是, 任一末态粒子在动量方向上呈球对称分布。若 \mathcal{A} 粒子具有非零自旋, 则自旋方向是 \mathcal{A} 粒子静止系的特殊方向, 于是, 末态粒子在动量方向上呈轴对称分布, 以 \mathcal{A} 粒子自旋方向为轴; 在实际情况下, 初态中 \mathcal{A} 粒子自旋的取向往往是不确定的, 而且它取不同方向具有相同的概率, 那么, 我们可以对 \mathcal{A} 粒子的自旋方向取平均, 从而, 末态粒子在动量方向上也呈球对称分布。

下面分别讨论二体衰变和三体衰变。

(1) 对于二体衰变, $n = 2$, 末态两个粒子的质心系就是 \mathcal{A} 粒子的静止系, 故 $E_{\text{CM}} = m_A$ 。于是, (5.187) 和 (5.188) 式化为

$$E_1 = \frac{1}{2m_A} (E_{\text{CM}}^2 + m_1^2 - m_2^2), \quad E_2 = \frac{1}{2m_A} (E_{\text{CM}}^2 + m_2^2 - m_1^2). \quad (5.210)$$

而 (5.191) 式化为

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{m_A}{2} \lambda^{1/2} \left(1, \frac{m_1^2}{m_A^2}, \frac{m_2^2}{m_A^2} \right). \quad (5.211)$$

2 体不变相空间 (5.181) 变成

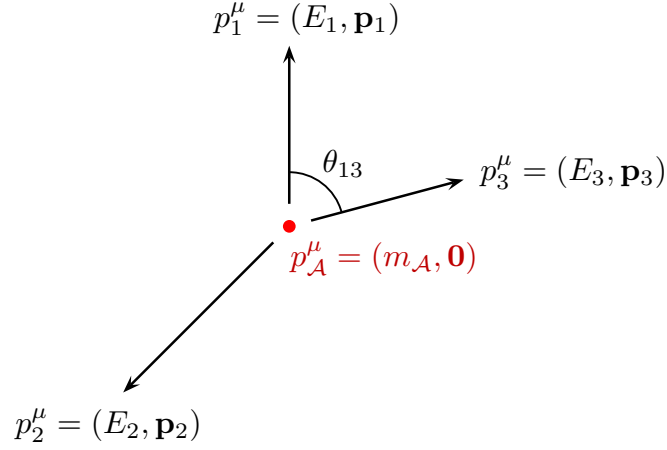
$$\int d\Pi_2 = \int d\Omega \frac{|\mathbf{p}_1|}{16\pi^2 m_A}. \quad (5.212)$$

此处, $d\Omega = \sin\theta d\theta d\phi$ 中的 θ 和 ϕ 分别是 \mathbf{p}_1 在某个球坐标系中的极角 (polar angle) 和方位角 (azimuthal angle)。于是, 衰变过程 $i \rightarrow f$ 的分宽度可以表达为

$$\begin{aligned} \Gamma_f &= \frac{1}{2m_A} \int d\Pi_2 |\mathcal{M}(p_A \rightarrow p_1, p_2)|^2 = \frac{|\mathbf{p}_1|}{32\pi^2 m_A^2} \int d\Omega |\mathcal{M}(p_A \rightarrow p_1, p_2)|^2 \\ &= \frac{1}{64\pi^2 m_A} \lambda^{1/2} \left(1, \frac{m_1^2}{m_A^2}, \frac{m_2^2}{m_A^2} \right) \int d\Omega |\mathcal{M}(p_A \rightarrow p_1, p_2)|^2. \end{aligned} \quad (5.213)$$

如果 \mathcal{A} 粒子的自旋为 0, 或者对它的自旋方向取平均, 按照前述讨论, 末态粒子在动量方向上呈球对称分布。此时, 不变振幅模方 $|\mathcal{M}|^2$ 与 θ 、 ϕ 无关, 对立体角积分只给出一个 4π 因子, 故分宽度为

$$\Gamma_f = \frac{|\mathbf{p}_1|}{8\pi m_A^2} |\mathcal{M}|^2 = \frac{|\mathcal{M}|^2}{16\pi m_A} \lambda^{1/2} \left(1, \frac{m_1^2}{m_A^2}, \frac{m_2^2}{m_A^2} \right). \quad (5.214)$$

图 5.3: \mathcal{A} 粒子静止系中三体衰变过程的动量示意图。

(2) 对于三体衰变, $n = 3$, 衰变过程 $i \rightarrow f$ 的分宽度可以表示成

$$\Gamma_f = \frac{1}{2m_{\mathcal{A}}} \int d\Pi_3 |\mathcal{M}(p_{\mathcal{A}} \rightarrow p_1, p_2, p_3)|^2, \quad (5.215)$$

其中, 3 体不变相空间为

$$\int d\Pi_3 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \frac{d^3 p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} - p_1 - p_2 - p_3). \quad (5.216)$$

这里, 我们只在 \mathcal{A} 粒子的静止系中讨论它没有自旋或者对它的自旋方向取平均的情况, 如前所述, 此时末态粒子在动量方向上呈球对称分布, 不变振幅模方 $|\mathcal{M}|^2$ 与末态粒子的运动方向无关。根据动量守恒定律, $\mathbf{0} = \mathbf{p}_{\mathcal{A}} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$, 即末态 3 个粒子的三维动量之和为零, 因而这 3 个三维动量矢量处在同一个平面内, 如图 5.3 所示。对于确定的 \mathbf{p}_1 和 \mathbf{p}_3 , 第 2 个粒子的三维动量 $\mathbf{p}_2 = -\mathbf{p}_1 - \mathbf{p}_3$ 由动量守恒定律决定。对 \mathbf{p}_2 积分, 可消去代表动量守恒定律的 $\delta^{(3)}(\mathbf{p}_{\mathcal{A}} - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)$, 得到

$$\begin{aligned} \int d\Pi_3 &= \frac{1}{8(2\pi)^5} \int \frac{d^3 p_1 d^3 p_3}{E_1 E_2 E_3} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3) \\ &= \frac{1}{8(2\pi)^5} \int d\Omega_1 d|\mathbf{p}_1| d\Omega_3 d|\mathbf{p}_3| \frac{|\mathbf{p}_1|^2 |\mathbf{p}_3|^2}{E_1 E_2 E_3} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3). \end{aligned} \quad (5.217)$$

其中, Ω_1 和 Ω_3 分别是 \mathbf{p}_1 和 \mathbf{p}_3 相对应的立方角。

对粒子 1 的质壳条件 $|\mathbf{p}_1|^2 + m_1^2 = E_1^2$ 两边求微分, 得 $2|\mathbf{p}_1|d|\mathbf{p}_1| = 2E_1 dE_1$, 对粒子 3 也可以得到类似的式子, 故

$$|\mathbf{p}_1|d|\mathbf{p}_1| = E_1 dE_1, \quad |\mathbf{p}_3|d|\mathbf{p}_3| = E_3 dE_3. \quad (5.218)$$

从而, 有

$$\int d\Pi_3 = \frac{1}{8(2\pi)^5} \int d\Omega_1 d\Omega_3 dE_1 dE_3 \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3)$$

$$= \frac{1}{4(2\pi)^4} \int d\Omega_3 dE_1 dE_3 \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3). \quad (5.219)$$

第二步对 Ω_1 作了积分, 由于粒子 1 在动量方向上呈球对称分布, 此积分只给出一个 4π 因子。

将 \mathbf{p}_1 与 \mathbf{p}_3 方向之间的夹角记为 θ_{13} , 则粒子 3 的立体角微分可以表示为

$$d\Omega_3 = \sin \theta_{13} d\theta_{13} d\phi_3 = d\cos \theta_{13} d\phi_3, \quad (5.220)$$

其中 ϕ_3 是粒子 3 的方位角。这样的话, 对 Ω_3 积分不是平庸的, 这是因为 E_2 依赖于 $\cos \theta_{13}$,

$$E_2 = \sqrt{m_2^2 + |\mathbf{p}_2|^2} = \sqrt{m_2^2 + |\mathbf{p}_1 + \mathbf{p}_3|^2} = \sqrt{m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos \theta_{13}}, \quad (5.221)$$

导致 $\delta(m_{\mathcal{A}} - E_1 - E_2 - E_3)$ 也依赖于 $\cos \theta_{13}$ 。由

$$\frac{\partial E_2}{\partial \cos \theta_{13}} = \frac{2|\mathbf{p}_1||\mathbf{p}_3|}{2\sqrt{m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_3|^2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos \theta_{13}}} = \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \quad (5.222)$$

有

$$\left| \frac{\partial(m_{\mathcal{A}} - E_1 - E_2 - E_3)}{\partial \cos \theta_{13}} \right| = \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2}. \quad (5.223)$$

再利用 (2.117) 式, 作出关于 Ω_3 的积分, 得

$$\begin{aligned} \int d\Pi_3 &= \frac{1}{4(2\pi)^4} \int_0^{2\pi} d\phi_3 \int_{-1}^1 d\cos \theta_{13} \int dE_1 dE_3 \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \delta(m_{\mathcal{A}} - E_1 - E_2 - E_3) \\ &= \frac{1}{4(2\pi)^4} \int_0^{2\pi} d\phi_3 \int dE_1 dE_3 \frac{|\mathbf{p}_1||\mathbf{p}_3|}{E_2} \left| \frac{\partial(m_{\mathcal{A}} - E_1 - E_2 - E_3)}{\partial \cos \theta_{13}} \right|^{-1} \\ &= \frac{1}{4(2\pi)^3} \int dE_1 dE_3. \end{aligned} \quad (5.224)$$

从而, 分宽度 (5.215) 化为

$$\Gamma_f = \frac{1}{(2\pi)^3} \frac{1}{8m_{\mathcal{A}}} \int_{E_1^{\min}}^{E_1^{\max}} dE_1 \int_{E_3^{\min}}^{E_3^{\max}} dE_3 |\mathcal{M}(E_1, E_2)|^2. \quad (5.225)$$

注意, 使用上式计算时需要把不变振幅 \mathcal{M} 表达为 E_1 和 E_2 的函数, 而且要仔细考虑 E_1 和 E_2 的积分上下限。

在实践中, 把 E_1 和 E_2 当作积分变量并不方便, 我们可以将它们替换成更加便利的变量。引入两个 Lorentz 不变量

$$s_{12} \equiv (p_1 + p_2)^2 = (p_{\mathcal{A}} - p_3)^2 = m_{\mathcal{A}}^2 + m_3^2 - 2m_{\mathcal{A}}E_3, \quad (5.226)$$

$$s_{23} \equiv (p_2 + p_3)^2 = (p_{\mathcal{A}} - p_1)^2 = m_{\mathcal{A}}^2 + m_1^2 - 2m_{\mathcal{A}}E_1, \quad (5.227)$$

它们在不同参考系中分别具有相同的值。我们可以把粒子 1 和 2 组成的系统看成一个等效粒子, 四维动量为 $p_{12}^\mu = p_1^\mu + p_2^\mu$ 。由于 $p_{12}^2 = (p_1 + p_2)^2 = s_{12}$, $\sqrt{s_{12}}$ 相当于这个等效粒子的质量, 称

为粒子 1 和 2 的不变质量 (**invariant mass**), 它也是粒子 1 和 2 的质心能。类似地, $\sqrt{s_{23}}$ 是粒子 2 和 3 的不变质量。 s_{12} 和 s_{23} 的微分分别正比于 E_1 和 E_2 的微分,

$$ds_{12} = -2m_A dE_3, \quad ds_{23} = -2m_A dE_1. \quad (5.228)$$

于是, 分宽度的积分式 (5.225) 可以改写为

$$\Gamma_f = \frac{1}{(2\pi)^3} \frac{1}{32m_A^3} \int_{s_{12}^{\min}}^{s_{12}^{\max}} ds_{12} \int_{s_{23}^{\min}}^{s_{23}^{\max}} ds_{23} |\mathcal{M}(s_{12}, s_{23})|^2. \quad (5.229)$$

使用上式计算时, 需要把不变振幅 \mathcal{M} 表达为 s_{12} 和 s_{23} 的函数。接下来, 我们讨论 s_{12} 和 s_{23} 的积分上下限。注意, 对 s_{23} 的积分位于内层, 积分上下限会依赖于 s_{12} 。

在粒子 1 和 2 的质心系中, $\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 = \mathbf{0}$, 质心能 $\tilde{E}_{\text{CM}} = \sqrt{s_{12}}$ 。这里我们用波浪线标记此参考系中的物理量。根据 (5.188) 式, 粒子 2 的能量为

$$\tilde{E}_2 = \frac{1}{2\sqrt{s_{12}}} (s_{12} - m_1^2 + m_2^2). \quad (5.230)$$

动量守恒定律给出 $\tilde{\mathbf{p}}_3 = \tilde{\mathbf{p}}_A - \tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2 = \tilde{\mathbf{p}}_A$, 由 s_{12} 的 Lorentz 不变性有

$$\begin{aligned} s_{12} &= (p_1 + p_2)^2 = (\tilde{p}_1 + \tilde{p}_2)^2 = (\tilde{p}_A - \tilde{p}_3)^2 = p_A^2 + p_3^2 - 2p_A \cdot p_3 \\ &= m_A^2 + m_3^2 - 2\tilde{E}_A \tilde{E}_3 + 2\tilde{\mathbf{p}}_A \cdot \tilde{\mathbf{p}}_3 = m_A^2 + m_3^2 - 2\sqrt{|\tilde{\mathbf{p}}_3|^2 + m_A^2} \tilde{E}_3 + 2|\tilde{\mathbf{p}}_3|^2 \\ &= m_A^2 + m_3^2 - 2\sqrt{\tilde{E}_3^2 - m_3^2 + m_A^2} \tilde{E}_3 + 2\tilde{E}_3^2 - 2m_3^2 \\ &= m_A^2 - 2\sqrt{\tilde{E}_3^2 - m_3^2 + m_A^2} \tilde{E}_3 + 2\tilde{E}_3^2 - m_3^2. \end{aligned} \quad (5.231)$$

整理, 得 $2\sqrt{\tilde{E}_3^2 - m_3^2 + m_A^2} \tilde{E}_3 = m_A^2 - s_{12} + 2\tilde{E}_3^2 - m_3^2$, 两边平方, 得

$$\begin{aligned} 4(\tilde{E}_3^2 - m_3^2 + m_A^2)\tilde{E}_3^2 &= (m_A^2 - s_{12} + 2\tilde{E}_3^2 - m_3^2)^2 \\ &= (m_A^2 - s_{12} - m_3^2)^2 + 4\tilde{E}_3^4 + 4(m_A^2 - s_{12} - m_3^2)\tilde{E}_3^2. \end{aligned} \quad (5.232)$$

再整理, 得 $4s_{12}\tilde{E}_3^2 = (m_A^2 - s_{12} - m_3^2)^2$, 故粒子 3 的能量为

$$\tilde{E}_3 = \frac{1}{2\sqrt{s_{12}}} (m_A^2 - s_{12} - m_3^2). \quad (5.233)$$

(5.230) 式和 (5.233) 式右边是 Lorentz 不变的, 而且, 对于确定的 s_{12} , \tilde{E}_2 和 \tilde{E}_3 是确定的。

另一方面, 由 s_{23} 的 Lorentz 不变性有

$$s_{23} = (p_2 + p_3)^2 = (\tilde{p}_2 + \tilde{p}_3)^2 = (\tilde{E}_2 + \tilde{E}_3)^2 - |\tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3|^2, \quad (5.234)$$

这里,

$$|\tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3|^2 = |\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|^2 + 2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3|\cos\tilde{\theta}_{23}, \quad (5.235)$$

其中 $\tilde{\theta}_{23}$ 是 $\tilde{\mathbf{p}}_2$ 与 $\tilde{\mathbf{p}}_3$ 方向之间的夹角。当 $\cos \tilde{\theta}_{23} = 1$ 时, $|\tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3|^2 = |\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|^2 + 2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3| = (|\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|)^2$, 而 s_{23} 取得最小值

$$s_{23}^{\min} = (\tilde{E}_2 + \tilde{E}_3)^2 - (|\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|)^2 = (\tilde{E}_2 + \tilde{E}_3)^2 - \left(\sqrt{\tilde{E}_2^2 - m_2^2} + \sqrt{\tilde{E}_3^2 - m_3^2} \right)^2. \quad (5.236)$$

当 $\cos \tilde{\theta}_{23} = -1$ 时, $|\tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3|^2 = |\tilde{\mathbf{p}}_2|^2 + |\tilde{\mathbf{p}}_3|^2 - 2|\tilde{\mathbf{p}}_2||\tilde{\mathbf{p}}_3| = (|\tilde{\mathbf{p}}_2|^2 - |\tilde{\mathbf{p}}_3|)^2$, 而 s_{23} 取得最大值

$$s_{23}^{\max} = (\tilde{E}_2 + \tilde{E}_3)^2 - (|\tilde{\mathbf{p}}_2|^2 - |\tilde{\mathbf{p}}_3|)^2 = (\tilde{E}_2 + \tilde{E}_3)^2 - \left(\sqrt{\tilde{E}_2^2 - m_2^2} - \sqrt{\tilde{E}_3^2 - m_3^2} \right)^2. \quad (5.237)$$

对于确定的 s_{12} , (5.236) 式和 (5.237) 式分别给出 s_{23} 的积分下限和上限。注意, 它们是 Lorentz 不变的。

在粒子 1 和 2 的质心系中,

$$s_{12} = (\tilde{p}_1 + \tilde{p}_2)^2 = (\tilde{E}_1 + \tilde{E}_2)^2 - |\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2|^2 = (\tilde{E}_1 + \tilde{E}_2)^2. \quad (5.238)$$

可见, 当 $\tilde{E}_1 = m_1$ 且 $\tilde{E}_2 = m_2$ 时, s_{12} 取得最小值

$$s_{12}^{\min} = (m_1 + m_2)^2. \quad (5.239)$$

在 \mathcal{A} 粒子的静止系中, 根据 (5.226) 式, 当 $E_3 = m_3$ 时, s_{12} 取得最大值

$$s_{12}^{\max} = m_{\mathcal{A}}^2 + m_3^2 - 2m_{\mathcal{A}}m_3 = (m_{\mathcal{A}} - m_3)^2. \quad (5.240)$$

注意 s_{12} 的积分下限 (5.239) 和积分上限 (5.240) 也是 Lorentz 不变的。

5.4 Wick 定理

5.4.1 正规乘积和 Wick 定理

在 5.2 节中, 借助时序乘积, 我们把 S 算符写成了一个紧凑的级数形式 (5.119)。不过, 如何适当地处理级数每一项中的时序乘积 $T[\mathcal{H}_1(x_1) \cdots \mathcal{H}_1(x_n)]$ 呢? 在量子场论中, 相互作用哈密顿量密度 $\mathcal{H}_1(x)$ 是由若干个场算符构成的, 因而我们需要处理的是多个场算符的时序乘积。这看来不是一个简单的问题, 幸好接下来将要介绍的 Wick 定理为我们提供了一个简便的方法。

在相互作用绘景中, 实标量场 $\phi(x)$ 的平面波展开式 (5.47) 可以分解成两个部分:

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \quad (5.241)$$

其中正能解部分为

$$\phi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ip \cdot x}, \quad (5.242)$$

负能解部分为

$$\phi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}^{\dagger} e^{ip \cdot x}. \quad (5.243)$$

根据 (5.84) 式, 我们同样可以把有质量矢量场 $A^\mu(x)$ 分为正能解和负能解两部分:

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x), \quad (5.244)$$

其中,

$$A^{\mu(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x}, \quad (5.245)$$

$$A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^{\mu*}(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}. \quad (5.246)$$

前面提到, Dirac 旋量场 $\psi_a(x)$ 在相互作用绘景中的平面波展开式也具有 Heisenberg 绘景中自由场展开式 (4.236) 的形式, 即

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x), \quad (5.247)$$

其中,

$$\psi_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} u_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x}, \quad (5.248)$$

$$\psi_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} v_a(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}. \quad (5.249)$$

可以看到, 正能解部分只包含湮灭算符, 而负能解部分只包含产生算符。

引入正规乘积 (normal product) 的概念, 以 N 为记号, 它的作用是将乘积中的所有湮灭算符移动到所有产生算符的右边, 形成正规次序 (normal order); 考虑到费米子算符的反对易性, 移动过程中若涉及奇数次费米子算符之间的交换, 则应额外增加一个负号。例如, 对于标量场的产生湮灭算符, 有

$$N(a_{\mathbf{p}} a_{\mathbf{q}}^\dagger a_{\mathbf{k}} a_{\mathbf{l}}^\dagger) = a_{\mathbf{q}}^\dagger a_{\mathbf{l}}^\dagger a_{\mathbf{p}} a_{\mathbf{k}} = a_{\mathbf{l}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{p}} a_{\mathbf{k}} = a_{\mathbf{l}}^\dagger a_{\mathbf{k}} a_{\mathbf{p}} a_{\mathbf{q}}; \quad (5.250)$$

对于旋量场的产生湮灭算符, 则有

$$N(b_{\mathbf{p},\lambda_1} a_{\mathbf{q},\lambda_2}^\dagger a_{\mathbf{k},\lambda_3} b_{\mathbf{l},\lambda_4}^\dagger) = -a_{\mathbf{q},\lambda_2}^\dagger b_{\mathbf{l},\lambda_4}^\dagger b_{\mathbf{p},\lambda_1} a_{\mathbf{k},\lambda_3} = b_{\mathbf{l},\lambda_4}^\dagger a_{\mathbf{q},\lambda_2}^\dagger b_{\mathbf{p},\lambda_1} a_{\mathbf{k},\lambda_3} = -b_{\mathbf{l},\lambda_4}^\dagger a_{\mathbf{k},\lambda_3} a_{\mathbf{q},\lambda_2}^\dagger b_{\mathbf{p},\lambda_1}. \quad (5.251)$$

于是, 可以得到两个标量场的正规乘积为

$$N[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x), \quad (5.252)$$

最后一项中 $\phi^{(+)}(x)$ 被正规操作移动到 $\phi^{(-)}(y)$ 的右边。而两个旋量场的正规乘积为

$$N[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y) + \psi_a^{(-)}(x)\psi_b^{(+)}(y) + \psi_a^{(+)}(x)\psi_b^{(+)}(y) - \psi_b^{(-)}(y)\psi_a^{(+)}(x), \quad (5.253)$$

最后一项中 $\psi_a^{(+)}(x)$ 被正规操作移动到 $\psi_b^{(-)}(y)$ 的右边, 并出现一个负号。湮灭算符对真空态 $|0\rangle$ 的作用为零, 如 $a_{\mathbf{p}}|0\rangle = 0$, $\langle 0|a_{\mathbf{p}}^\dagger = 0$, 因此, 对一组产生湮灭算符的任意乘积取正规次序之后, 真空期待值为零:

$$\langle 0|N(\text{产生湮灭算符的乘积})|0\rangle = 0. \quad (5.254)$$

用统一的记号 $\Phi_a(x)$ 代表一般的场算符，它可以是标量场 $\phi(x)$ 或 $\phi^\dagger(x)$ ，也可以是矢量场 $A^\mu(x)$ 的一个分量，还可以是旋量场 $\psi_a(x)$ 、 $\psi_a^\dagger(x)$ 或 $\bar{\psi}_a(x)$ 的一个分量。比如， $\Phi_a(x)\Phi_b(x)\Phi_c(x)$ 可以表示 $\phi(x)\phi(x)\phi(x)$ ，也可以表示 $A_\mu(x)\bar{\psi}_a(x)\psi_b(x)$ 。后者不是 Lorentz 不变的，但利用 Dirac 矩阵可以线性地组合出 Lorentz 不变量 $A_\mu(x)\bar{\psi}_a(x)(\gamma^\mu)_{ab}\psi_b(x) = A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$ 。将 $\Phi_a(x)$ 分解为正能解部分 $\Phi_a^{(+)}(x)$ 和负能解部分 $\Phi_a^{(-)}(x)$ ，

$$\Phi_a(x) = \Phi_a^{(+)}(x) + \Phi_a^{(-)}(x), \quad (5.255)$$

则可得

$$\Phi_a(x)\Phi_b(y) = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y). \quad (5.256)$$

由于正能解部分和负能解部分分别只包含湮灭算符和产生算符，我们有

$$\Phi_a^{(+)}(x)|0\rangle = 0, \quad \langle 0|\Phi_a^{(-)}(x) = 0, \quad (5.257)$$

从而，可以推出

$$\langle 0|\Phi_a(x)\Phi_b(y)|0\rangle = \langle 0|\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)|0\rangle. \quad (5.258)$$

现在， $\Phi_a(x)$ 与 $\Phi_b(y)$ 的正规乘积可以表达为

$$N[\Phi_a(x)\Phi_b(y)] = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x), \quad (5.259)$$

其中，因子 $\epsilon_{ab} = \pm 1$ 来自费米子算符的反对易性。若 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符，则 $\epsilon_{ab} = -1$ ；其余情况 $\epsilon_{ab} = +1$ 。利用 ϵ_{ab} ，我们可以交换 (5.259) 式右边第一项和第三项各自的两个场算符，得到

$$\begin{aligned} N[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x)], \end{aligned} \quad (5.260)$$

即

$$N[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}N[\Phi_b(y)\Phi_a(x)]. \quad (5.261)$$

也就是说，两个场算符的位置交换后，正规乘积只相差一个由费米子算符的反对易性导致的符号。另一方面， $\Phi_a(x)\Phi_b(y)$ 的时序乘积可以写作

$$\begin{aligned} T[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0 - x^0) \\ &= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0 - y^0) + \Phi_b(y)\Phi_a(x)\theta(y^0 - x^0)], \end{aligned} \quad (5.262)$$

因此，两个场算符的位置交换后，时序乘积也只相差一个由费米子算符的反对易性导致的符号：

$$T[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}T[\Phi_b(y)\Phi_a(x)]. \quad (5.263)$$

当 $x^0 \geq y^0$ 时, $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积为

$$\begin{aligned} T[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y) \\ &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y). \end{aligned} \quad (5.264)$$

最后一项可以改写成

$$\begin{aligned} \Phi_a^{(+)}(x)\Phi_b^{(-)}(y) &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_a^{(+)}(x), \Phi_b^{(-)}(y) - \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\ &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}. \end{aligned} \quad (5.265)$$

这里 $[\cdot, \cdot]_- = [\cdot, \cdot]$ 代表对易子, $[\cdot, \cdot]_+ = \{\cdot, \cdot\}$ 代表反对易子。 \mp 号仅当 $\Phi_a(x)$ 和 $\Phi_b(y)$ 都是费米子算符时取负号, 其余情况取正号。于是, 由 (5.259) 式可以得到

$$T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}. \quad (5.266)$$

注意, $[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}$ 必定是一个 c 数, 因为 $\Phi_a^{(+)}(x)$ 中的湮灭算符与 $\Phi_b^{(-)}(y)$ 中的产生算符的对易子或反对易子并不是算符, 而是 c 数。从而, 根据 (5.257) 式和 (5.258) 式可得

$$\begin{aligned} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} &= \langle 0 | [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} | 0 \rangle = \langle 0 | \Phi_a^{(+)}(x)\Phi_b^{(-)}(y) | 0 \rangle = \langle 0 | \Phi_a(x)\Phi_b(y) | 0 \rangle \\ &= \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle. \end{aligned} \quad (5.267)$$

当 $x^0 \leq y^0$ 时, $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积变成

$$\begin{aligned} T[\Phi_a(x)\Phi_b(y)] &= \epsilon_{ab}\Phi_b(y)\Phi_a(x) \\ &= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(-)}(x)] \\ &= \epsilon_{ab}\{\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) \\ &\quad + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}\} \\ &= \epsilon_{ab}N[\Phi_b(y)\Phi_a(x)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} \\ &= N[\Phi_a(x)\Phi_b(y)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}. \end{aligned} \quad (5.268)$$

最后一步用到 (5.261) 式。根据 (5.263) 式, 有

$$\begin{aligned} \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} &= \epsilon_{ab} \langle 0 | [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp} | 0 \rangle = \epsilon_{ab} \langle 0 | \Phi_b^{(+)}(y)\Phi_a^{(-)}(x) | 0 \rangle \\ &= \epsilon_{ab} \langle 0 | \Phi_b(y)\Phi_a(x) | 0 \rangle = \epsilon_{ab} \langle 0 | T[\Phi_b(y)\Phi_a(x)] | 0 \rangle = \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle. \end{aligned} \quad (5.269)$$

综合这两种情况, 我们发现 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积可以统一地表达为

$$T[\Phi_a(x)\Phi_b(y)] = N[\Phi_a(x)\Phi_b(y)] + \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle. \quad (5.270)$$

引入场算符的缩并 (contraction) 概念, 将两个场算符 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

$$\overline{\Phi_a(x)\Phi_b(y)} \equiv \langle 0 | T[\Phi_a(x)\Phi_b(y)] | 0 \rangle = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}, & x^0 \geq y^0, \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}, & x^0 < y^0. \end{cases} \quad (5.271)$$

上式仅当 $\Phi_a^{(+)}(x)$ 中的湮灭算符与 $\Phi_b^{(-)}(y)$ 中的产生算符属于同一套产生湮灭算符时非零, 因而不同类型的场算符的缩并为零。两个场算符的缩并是一个 c 数, 不会受到正规操作 N 的影响。在正规乘积中出现缩并记号时, 参与缩并的一对场算符可以不相邻。为了使它们相邻, 需要适当地交换场算符, 交换时应计入费米子算符的反对易性引起的符号差异, 我们约定这样得到的式子与原先的式子相等。例如,

$$N(\overbrace{\Phi_a \Phi_b \Phi_c \Phi_d \Phi_e \Phi_f}) = \epsilon_{cd} \epsilon_{ef} N(\overbrace{\Phi_a \Phi_b \Phi_d \Phi_c \Phi_f \Phi_e}) = \epsilon_{cd} \epsilon_{ef} \overbrace{\Phi_b \Phi_d \Phi_c \Phi_f} N(\Phi_a \Phi_e) \quad (5.272)$$

于是, (5.270) 式可以改记为

$$T[\Phi_a(x) \Phi_b(y)] = N[\Phi_a(x) \Phi_b(y) + \overbrace{\Phi_a(x) \Phi_b(y)}]. \quad (5.273)$$

上式表明, 两个场算符的时序乘积等于它们的正规乘积加上它们的缩并。

这个结论可以推广成 **Wick 定理**: 一组场算符的时序乘积可以分解为它们的正规乘积与所有可能缩并的正规乘积之和, 也就是说,

$$T[\Phi_{a_1}(x_1) \Phi_{a_2}(x_2) \cdots \Phi_{a_n}(x_n)] = N[\Phi_{a_1}(x_1) \Phi_{a_2}(x_2) \cdots \Phi_{a_n}(x_n) + (\Phi_{a_1} \Phi_{a_2} \cdots \Phi_{a_n} \text{ 的所有可能缩并})]. \quad (5.274)$$

例如, 对于四个场算符的情况, 有

$$\begin{aligned} T(\Phi_a \Phi_b \Phi_c \Phi_d) = & N(\Phi_a \Phi_b \Phi_c \Phi_d + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} \\ & + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} \\ & + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d}). \end{aligned} \quad (5.275)$$

根据正规乘积的性质 (5.254), 上式的真空期待值为

$$\begin{aligned} \langle 0 | T(\Phi_a \Phi_b \Phi_c \Phi_d) | 0 \rangle &= \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} \\ &= \overbrace{\Phi_a \Phi_b \Phi_c \Phi_d} + \epsilon_{bc} \overbrace{\Phi_a \Phi_c \Phi_b \Phi_d} + \epsilon_{cd} \epsilon_{bd} \overbrace{\Phi_a \Phi_d \Phi_b \Phi_c} \\ &= \langle 0 | T(\Phi_a \Phi_b) | 0 \rangle \langle 0 | T(\Phi_c \Phi_d) | 0 \rangle + \epsilon_{bc} \langle 0 | T(\Phi_a \Phi_c) | 0 \rangle \langle 0 | T(\Phi_b \Phi_d) | 0 \rangle \\ &\quad + \epsilon_{cd} \epsilon_{bd} \langle 0 | T(\Phi_a \Phi_d) | 0 \rangle \langle 0 | T(\Phi_b \Phi_c) | 0 \rangle. \end{aligned} \quad (5.276)$$

5.4.2 Wick 定理的证明

为了证明 Wick 定理, 我们需要先证明如下引理。

引理 如果场算符 $\Phi_b(x_b)$ 的时间坐标比 n 个场算符 $\Phi_{a_1}(x_1), \cdots, \Phi_{a_n}(x_n)$ 的时间坐标都小, 即 $x_b^0 \leq x_1^0, \cdots, x_n^0$, 那么, 以下等式成立:

$$\begin{aligned} N[\Phi_{a_1}(x_1) \cdots \Phi_{a_n}(x_n)] \Phi_b(x_b) = & N[\Phi_{a_1}(x_1) \cdots \Phi_{a_n}(x_n) \Phi_b(x_b) + \overbrace{\Phi_{a_1}(x_1) \cdots \Phi_{a_n}(x_n) \Phi_b(x_b)} \\ & + \Phi_{a_1}(x_1) \overbrace{\Phi_{a_2}(x_2) \cdots \Phi_{a_n}(x_n) \Phi_b(x_b)} + \cdots + \Phi_{a_1}(x_1) \cdots \overbrace{\Phi_{a_n}(x_n) \Phi_b(x_b)}]. \end{aligned} \quad (5.277)$$

如果 $\Phi_{a_1}, \dots, \Phi_{a_n}$ 中有些算符已经先彼此缩并了, 也存在与 (5.277) 形式相同的等式, 如

$$\begin{aligned} N(\overbrace{\Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n}}) \Phi_b &= N(\overbrace{\Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n} \Phi_b} \\ &+ \overbrace{\Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n} \Phi_b} + \overbrace{\Phi_{a_1} \Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n} \Phi_b} \\ &+ \overbrace{\Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n} \Phi_b} + \dots + \overbrace{\Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \Phi_{a_4} \Phi_{a_5} \dots \Phi_{a_n} \Phi_b}). \end{aligned} \quad (5.278)$$

证明 我们分四步来证明。

(1) 将 Φ_b 分解为正能解部分和负能解部分, $\Phi_b = \Phi_b^{(+)} + \Phi_b^{(-)}$, 则可以证明正能解部分 $\Phi_b^{(+)}$ 满足

$$\begin{aligned} N(\Phi_{a_1} \dots \Phi_{a_n}) \Phi_b^{(+)} &= N(\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(+)} + \overbrace{\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(+)}} + \overbrace{\Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_n} \Phi_b^{(+)}} \\ &+ \dots + \overbrace{\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(+)}}). \end{aligned} \quad (5.279)$$

由于 $x_b^0 \leq x_1^0, \dots, x_n^0$, $\Phi_{a_i}(x_i)$ ($i = 1, \dots, n$) 与 $\Phi_b^{(+)}$ 的缩并为零:

$$\overbrace{\Phi_{a_i}(x_i) \Phi_b^{(+)}(x_b)} = \langle 0 | T[\Phi_{a_i}(x_i) \Phi_b^{(+)}(x_b)] | 0 \rangle = \langle 0 | \Phi_{a_i}(x_i) \Phi_b^{(+)}(x_b) | 0 \rangle = 0. \quad (5.280)$$

因此, (5.279) 式右边除第一项外的其它项均为零。另一方面, (5.279) 式左边和右边第一项已经按正规次序排列了, 故 (5.279) 式成立。现在, 只需要证明负能解部分 $\Phi_b^{(-)}$ 满足

$$\begin{aligned} N(\Phi_{a_1} \dots \Phi_{a_n}) \Phi_b^{(-)} &= N(\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(-)} + \overbrace{\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(-)}} + \overbrace{\Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_n} \Phi_b^{(-)}} \\ &+ \dots + \overbrace{\Phi_{a_1} \dots \Phi_{a_n} \Phi_b^{(-)}}). \end{aligned} \quad (5.281)$$

将 $\Phi_{a_1}, \dots, \Phi_{a_n}$ 都分解为正能解部分和负能解部分, 则 $N(\Phi_{a_1} \dots \Phi_{a_n})$ 将包含 2^n 项, 每一项是 j 个负能解部分 ($j = 0, \dots, n$) 与 $n - j$ 个正能解部分之积

$$\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)}, \quad (5.282)$$

负能解部分都处于正能解部分的左边。

(2) 可以证明, 通项 (5.282) 中右边正能解部分之积 $\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)}$ 满足

$$\begin{aligned} N(\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)}) \Phi_b^{(-)} &= N(\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overbrace{\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \overbrace{\Phi_{a_{j+1}}^{(+)} \Phi_{a_{j+2}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ &+ \dots + \overbrace{\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.283)$$

下面用数学归纳法证明 (5.283) 式。

对于 $N(\Phi_{a_n}^{(+)} \Phi_b^{(-)})$, 存在与 (5.283) 形式相同的等式, 这是因为由 (5.273) 式可以得到

$$N(\Phi_{a_n}^{(+)} \Phi_b^{(-)}) = \Phi_{a_n}^{(+)} \Phi_b^{(-)} = T(\Phi_{a_n}^{(+)} \Phi_b^{(-)}) = N(\Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overbrace{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \quad (5.284)$$

这样的话, 需要证明的是可以从上式递推地导出 (5.283) 式。

假设 $N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)}) \Phi_b^{(-)}$ ($j + 2 \leq k \leq n$) 满足与 (5.283) 形式相同的等式

$$N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)}) \Phi_b^{(-)} = N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overbrace{\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \overbrace{\Phi_{a_k}^{(+)} \Phi_{a_{k+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}})$$

$$+ \dots + \Phi_{a_k}^{(+)} \dots \overline{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}, \quad (5.285)$$

那么, 可以得到

$$\begin{aligned} N(\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) &= \Phi_{a_{k-1}}^{(+)} N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) \\ &= \Phi_{a_{k-1}}^{(+)} N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) + N(\Phi_{a_{k-1}}^{(+)} \overline{\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \overline{\Phi_{a_{k+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ &\quad + \dots + \Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \overline{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.286)$$

接着, 我们进一步整理上式第二步的第一项,

$$\begin{aligned} &\Phi_{a_{k-1}}^{(+)} N(\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) \\ &= \Phi_{a_{k-1}}^{(+)} \epsilon_1 N(\Phi_b^{(-)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)}) = \epsilon_1 \Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \\ &= \epsilon_1 T(\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)}) \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} = \epsilon_1 N(\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)} + \overline{\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)}}) \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \\ &= \epsilon_1 N(\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)}) \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} + \epsilon_1 \overline{\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)}} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \\ &= \epsilon_1 \epsilon_{a_{k-1}b} N(\Phi_b^{(-)} \Phi_{a_{k-1}}^{(+)}) \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} + \epsilon_1 N(\overline{\Phi_{a_{k-1}}^{(+)} \Phi_b^{(-)}} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)}) \\ &= \epsilon_1 \epsilon_{a_{k-1}b} N(\Phi_b^{(-)} \Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)}) + N(\overline{\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}}) \\ &= N(\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) + N(\overline{\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.287)$$

第一步重复利用 (5.261) 式, 将 $\Phi_b^{(-)}$ 从正规乘积中的最左边移动到最右边, 因而出现因子

$$\epsilon_1 = \epsilon_{a_n b} \epsilon_{a_{n-1} b} \dots \epsilon_{a_{k+1} b} \epsilon_{a_k b}. \quad (5.288)$$

第三步利用到 $x_b^0 \leq x_{k-1}^0$ 的条件。第四步使用了 (5.273) 式。第六至八步再多次利用 (5.261) 式。将 (5.287) 式代入 (5.286) 式, 立即得到

$$\begin{aligned} N(\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) &= N(\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overline{\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ &\quad + \Phi_{a_{k-1}}^{(+)} \overline{\Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \dots + \Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \overline{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.289)$$

因此, $N(\Phi_{a_{k-1}}^{(+)} \Phi_{a_k}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)})$ 也满足与 (5.283) 形式相同的等式。结合 (5.284) 式, 可知 (5.283) 式成立。

(3) 根据 (5.283) 式, 通项 (5.282) 满足

$$\begin{aligned} &N(\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) = \Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} N(\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) \\ &= \Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} N(\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overline{\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ &\quad + \Phi_{a_{j+1}}^{(+)} \overline{\Phi_{a_{j+2}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \dots + \Phi_{a_1}^{(-)} \dots \overline{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}) \\ &= N(\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \overline{\Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ &\quad + \Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \overline{\Phi_{a_{j+2}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \dots + \Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \overline{\Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.290)$$

由

$$\overline{\Phi_{a_i}^{(-)}(x_i)}\Phi_b^{(-)}(x_b) = \langle 0 | T[\Phi_{a_i}^{(-)}(x_i)\Phi_b^{(-)}(x_b)] | 0 \rangle = 0, \quad (5.291)$$

可得

$$\begin{aligned} N(\overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \overline{\Phi_{a_1}^{(-)} \Phi_{a_2}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ + \dots + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}}) = 0. \end{aligned} \quad (5.292)$$

因此，将上式左边添加到 (5.290) 式右边，等式仍然成立：

$$\begin{aligned} N(\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}) \\ = N(\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)} + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ + \overline{\Phi_{a_1}^{(-)} \Phi_{a_2}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \dots + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \Phi_{a_{j+2}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}} \\ + \dots + \overline{\Phi_{a_1}^{(-)} \dots \Phi_{a_j}^{(-)} \Phi_{a_{j+1}}^{(+)} \dots \Phi_{a_n}^{(+)} \Phi_b^{(-)}}). \end{aligned} \quad (5.293)$$

也就是说， $N(\Phi_{a_1} \dots \Phi_{a_n})$ 分解后每一项都满足与 (5.281) 形式相同的等式，故 (5.281) 式成立。结合第 (1) 步结论，(5.277) 式成立。

(4) 如果 $\Phi_{a_1}, \dots, \Phi_{a_n}$ 中有些算符已经先彼此缩并了，可以按照第 (1)、(2)、(3) 步的方法进行类似的证明。因此，像 (5.278) 这样的等式也成立。引理证毕。

现在，我们可以利用这个引理来证明 Wick 定理。

证明 用数学归纳法证明。

当 $n = 2$ 时，(5.274) 式变成

$$T[\Phi_{a_1}(x)\Phi_{a_2}(y)] = N[\Phi_{a_1}(x)\Phi_{a_2}(y) + \overline{\Phi_{a_1}(x)\Phi_{a_2}(y)}]. \quad (5.294)$$

这是成立的，因为它的形式与 (5.273) 式相同。

假设当 $n = k$ 时，(5.274) 式成立，即

$$T[\Phi_{a_1}(x_1) \dots \Phi_{a_k}(x_k)] = N[\Phi_{a_1}(x_1) \dots \Phi_{a_k}(x_k) + (\Phi_{a_1} \dots \Phi_{a_k} \text{ 的所有可能缩并})]. \quad (5.295)$$

如果 $x_{k+1}^0 \leq x_1^0, \dots, x_k^0$ ，我们就可以得到

$$\begin{aligned} T[\Phi_{a_1}(x_1) \dots \Phi_{a_k}(x_k)\Phi_{a_{k+1}}(x_{k+1})] &= T[\Phi_{a_1}(x_1) \dots \Phi_{a_k}(x_k)]\Phi_{a_{k+1}}(x_{k+1}) \\ &= N(\Phi_{a_1} \dots \Phi_{a_k})\Phi_{a_{k+1}} + N(\Phi_{a_1} \dots \Phi_{a_k} \text{ 的所有可能缩并})\Phi_{a_{k+1}}. \end{aligned} \quad (5.296)$$

根据上述引理中的 (5.277) 式，(5.296) 式第二行第一项为

$$\begin{aligned} N(\Phi_{a_1} \dots \Phi_{a_k})\Phi_{a_{k+1}} &= N(\Phi_{a_1} \dots \Phi_{a_k} \Phi_{a_{k+1}} + \overline{\Phi_{a_1} \dots \Phi_{a_k} \Phi_{a_{k+1}}} + \overline{\Phi_{a_1} \Phi_{a_2} \dots \Phi_{a_k} \Phi_{a_{k+1}}} \\ &\quad + \dots + \overline{\Phi_{a_1} \dots \Phi_{a_k} \Phi_{a_{k+1}}}), \end{aligned} \quad (5.297)$$

上式右边的缩并项穷尽了只有一次缩并时与 $\Phi_{a_{k+1}}$ 有关的缩并。另一方面, 上述引理中有些算符已经先彼此缩并的情况可以应用到 (5.296) 式第二行的其它项上, 得到的项都包含缩并, 在这些项里面, 只包含一次缩并的项中的缩并必定与 $\Phi_{a_{k+1}}$ 无关, 余下的项则穷尽了 $\Phi_{a_1} \cdots \Phi_{a_{k+1}}$ 的包含一次以上缩并的所有情况。因此, (5.296) 式已经包含了 $\Phi_{a_1} \cdots \Phi_{a_{k+1}}$ 的所有可能缩并, 故

$$T[\Phi_{a_1}(x_1) \cdots \Phi_{a_{k+1}}(x_{k+1})] = N[\Phi_{a_1}(x_1) \cdots \Phi_{a_{k+1}}(x_{k+1}) + (\Phi_{a_1} \cdots \Phi_{a_{k+1}} \text{ 的所有可能缩并})]. \quad (5.298)$$

因此, 对于 $x_{k+1}^0 \leq x_1^0, \dots, x_k^0$ 的情形, 当 $n = k + 1$ 时 (5.274) 式也成立。结合 (5.294) 式, 我们就证明了 (5.274) 式对 $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$ 成立。

当 $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$ 这个条件不成立时, 我们可以交换 $\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)$ 中各个算符的位置, 得到符合时序的乘积

$$\Phi'_{a_1}(x'_1)\Phi'_{a_2}(x'_2)\cdots\Phi'_{a_n}(x'_n),$$

其中时间坐标已经按降序排列, $x_1'^0 \geq x_2'^0 \geq \cdots \geq x_n'^0$ 。从而, 等式

$$T[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n)] = N[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n) + (\Phi'_{a_1}\cdots\Phi'_{a_n} \text{ 的所有可能缩并})] \quad (5.299)$$

成立。(5.263) 式和 (5.261) 式表明, 时序乘积与正规乘积关于算符交换的性质是相同的。因此, 如果我们分别在时序乘积和正规乘积中通过交换算符将 $\Phi'_{a_1}(x'_1)\Phi'_{a_2}(x'_2)\cdots\Phi'_{a_n}(x'_n)$ 调回到原来的形式 $\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)$, 将出现一个共同的因子 $\epsilon_2 = \pm 1$, 它由费米子算符的反对易性所致。也就是说, 我们得到了

$$T[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n)] = \epsilon_2 T[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n)], \quad (5.300)$$

和

$$\begin{aligned} & N[\Phi'_{a_1}(x'_1)\cdots\Phi'_{a_n}(x'_n) + (\Phi'_{a_1}\cdots\Phi'_{a_n} \text{ 的所有可能缩并})] \\ &= \epsilon_2 N[\Phi_{a_1}(x_1)\cdots\Phi_{a_n}(x_n) + (\Phi_{a_1}\cdots\Phi_{a_n} \text{ 的所有可能缩并})]. \end{aligned} \quad (5.301)$$

将以上两式分别代入到 (5.299) 式的左右两边, 消去 ϵ_2 , 我们就证明了 (5.274) 式对 $x_1^0, x_2^0, \dots, x_n^0$ 的任意次序成立。证毕。

5.5 Feynman 传播子

在应用 Wick 定理时, 两个场算符的缩并是一种基本要素。在上一节中我们已经指出, 仅当参与缩并的场算符中含有同一套产生湮灭算符时, 缩并的结果才不为零。这些非零缩并就是 **Feynman 传播子** (propagator), 在本节中, 我们将导出它们的显式结果。

5.5.1 实标量场的 Feynman 传播子

实标量场 $\phi(x)$ 的 Feynman 传播子 $D_F(x-y)$ 定义为

$$D_F(x-y) \equiv \overline{\phi(x)\phi(y)} = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle. \quad (5.302)$$

根据展开式 (5.242) 和 (5.243), 当 $x^0 > y^0$ 时, 有

$$\begin{aligned} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \langle 0 | \phi(x)\phi(y) | 0 \rangle = \langle 0 | \phi^{(+)}(x)\phi^{(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_p} \sqrt{2E_q}} \langle 0 | a_p e^{-ip \cdot x} a_q^\dagger e^{iq \cdot y} | 0 \rangle = \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_p} \sqrt{2E_q}} \langle 0 | ([a_p, a_q^\dagger] + a_q^\dagger a_p) | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_p} \sqrt{2E_q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_p} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} \frac{e^{-iE_p(x^0-y^0)}}{2E_p}. \end{aligned} \quad (5.303)$$

第四步用到产生湮灭算符的对易关系 (2.92)。借助复变函数的知识, 可以将上式最后一行中的因子 $e^{-iE_p(x^0-y^0)}/(2E_p)$ 化为一维积分的结果。

将 p^0 视作复变量, 在 p^0 的复平面上考虑函数

$$\frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p)(p^0 + E_p)} \quad (5.304)$$

的曲线积分。这个函数具有两个一阶极点, $p^0 = \pm E_p$, 均位于实轴上。图 5.4(a) 中画出了 p^0 复平面上的几条积分路径。路径 Γ_F 在两个极点处分别通过一个半径无穷小的半圆绕过极点, 当

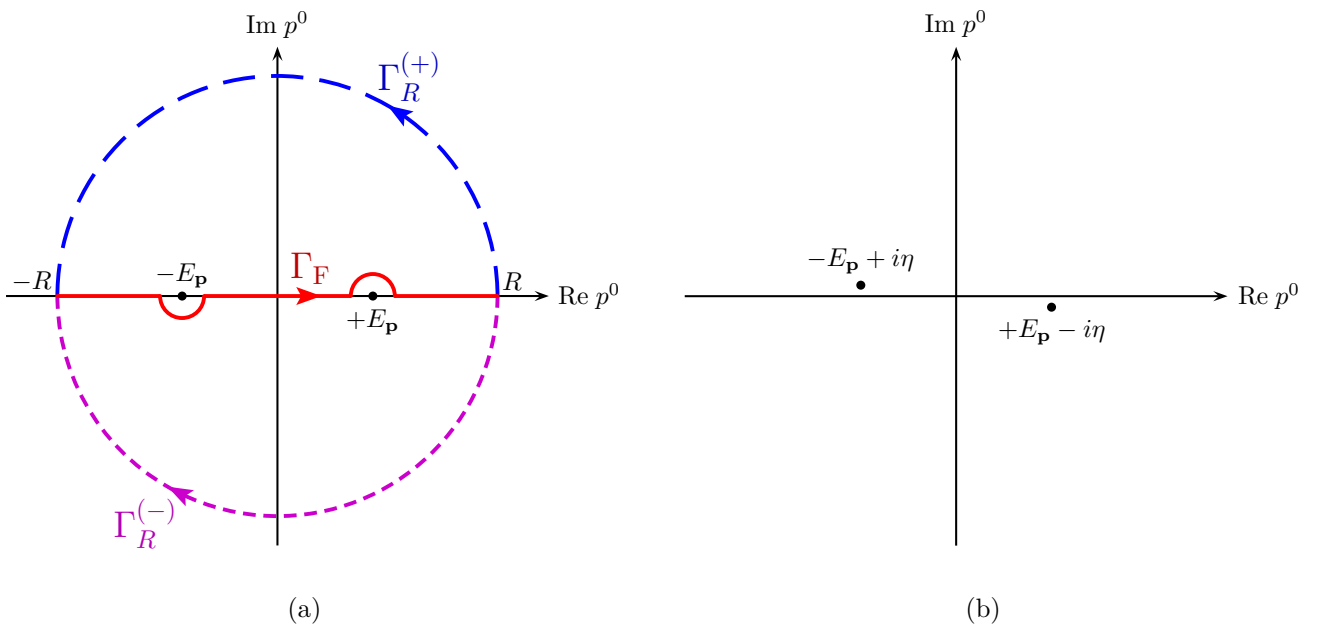


图 5.4: Feynman 传播子的极点和积分路径。

$R \rightarrow \infty$ 时, Γ_F 将从 $p^0 = -\infty$ 一直延伸到 $p^0 = +\infty$ 。将 Γ_F 与下半平面上的半圆弧 $\Gamma_R^{(-)}$ 组成一条围线 $C_F^{(-)} = \Gamma_F + \Gamma_R^{(-)}$, 方向为顺时针方向, 即反方向。由于 $x^0 - y^0 > 0$, 根据复变函数的 Jordan 引理, 可得

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^{(-)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} = 0. \quad (5.305)$$

从而, 当 $R \rightarrow \infty$ 时, 由留数定理可以计算相应的积分主值,

$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} &= \int_{C_F^{(-)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} \\ &= -2\pi i \operatorname{Res} \left[\frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})}, +E_{\mathbf{p}} \right] = -2\pi i \frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}. \end{aligned} \quad (5.306)$$

利用

$$(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}}) = (p^0)^2 - E_{\mathbf{p}}^2 = (p^0)^2 - |\mathbf{p}|^2 - m^2 = p^2 - m^2, \quad (5.307)$$

我们进一步得到

$$\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = -\frac{1}{2\pi i} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} = \int_{\Gamma_F} \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{p^2 - m^2}. \quad (5.308)$$

如图 5.4(b) 所示, 如果我们将左边极点向正虚轴方向移动一个无穷小量 $\eta > 0$, 右边极点向负虚轴方向同样移动无穷小量 η , 则沿正实轴积分将等价于原来沿 Γ_F 积分。此时, 极点位置为 $p^0 = \pm(E_{\mathbf{p}} - i\eta)$, 积分项中的分母应改为

$$[p^0 - (E_{\mathbf{p}} - i\eta)][p^0 + (E_{\mathbf{p}} - i\eta)] = (p^0)^2 - (E_{\mathbf{p}} - i\eta)^2 = (p^0)^2 - E_{\mathbf{p}}^2 + 2i\eta E_{\mathbf{p}} + \eta^2 \simeq p^2 - m^2 + i\epsilon. \quad (5.309)$$

最后一步忽略了二阶小量, 而 $\epsilon = 2\eta E_{\mathbf{p}} > 0$ 也是一个无穷小量。于是, 我们可以得到

$$\frac{e^{-iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{[p^0 - (E_{\mathbf{p}} - i\eta)][p^0 + (E_{\mathbf{p}} - i\eta)]} = \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon}. \quad (5.310)$$

将上式代入到 (5.303) 式, 立即推出

$$\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{ie^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}. \quad (5.311)$$

当 $x^0 < y^0$ 时, 时序操作将改变 $\phi(x)$ 和 $\phi(y)$ 的次序, 有

$$\begin{aligned} \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle &= \langle 0 | \phi(y)\phi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip \cdot (y-x)} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_{\mathbf{p}}} \\ &= \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}. \end{aligned} \quad (5.312)$$

最后一步将积分变量 \mathbf{p} 替换成 $-\mathbf{p}$ 。将 Γ_F 与上半平面上的半圆弧 $\Gamma_R^{(+)}$ 组成一条围线 $C_F^{(+)} = \Gamma_F + \Gamma_R^{(+)}$, 方向为逆时针方向, 即正方向。由于 $x^0 - y^0 < 0$, 根据 Jordan 引理, 可得

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^{(+)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} = 0. \quad (5.313)$$

从而, 当 $R \rightarrow \infty$ 时, 可以推出

$$\begin{aligned} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} &= \int_{C_F^{(+)}} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} \\ &= 2\pi i \operatorname{Res} \left[\frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})}, -E_{\mathbf{p}} \right] = -2\pi i \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}. \end{aligned} \quad (5.314)$$

故

$$\frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = -\frac{1}{2\pi i} \int_{\Gamma_F} dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_{\mathbf{p}})(p^0 + E_{\mathbf{p}})} = \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon}, \quad (5.315)$$

代入到 (5.312) 式, 即得

$$\langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{ie^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} e^{-ip^0(x^0-y^0)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}. \quad (5.316)$$

(5.311) 式和 (5.316) 式是一样的。因此, 无论 x^0 和 y^0 孰大孰小, Feynman 传播子都可以表达为

$$D_F(x-y) = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}. \quad (5.317)$$

它是 Lorentz 不变的, 而且是一个偶函数:

$$D_F(y-x) = D_F(x-y). \quad (5.318)$$

5.5.2 复标量场的 Feynman 传播子

在相互作用绘景中, 复标量场 $\phi(x)$ 的平面波展开式仍然具有 (2.144) 的形式。将 $\phi(x)$ 和 $\phi^\dagger(x)$ 分解为正能解和负能解两部分, 得

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \quad \phi^\dagger(x) = \phi^{\dagger(+)}(x) + \phi^{\dagger(-)}(x), \quad (5.319)$$

其中,

$$\phi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} e^{-ip\cdot x}, \quad \phi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} b_{\mathbf{p}}^\dagger e^{ip\cdot x}, \quad (5.320)$$

$$\phi^{\dagger(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} b_{\mathbf{p}} e^{-ip\cdot x}, \quad \phi^{\dagger(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}^\dagger e^{ip\cdot x}. \quad (5.321)$$

容易看出,

$$\overline{\phi(x)\phi(y)} = \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle = 0, \quad \overline{\phi^\dagger(x)\phi^\dagger(y)} = \langle 0 | T[\phi^\dagger(x)\phi^\dagger(y)] | 0 \rangle = 0. \quad (5.322)$$

复标量场的 Feynman 传播子定义为

$$D_F(x-y) \equiv \overline{\phi(x)\phi^\dagger(y)} = \langle 0 | T[\phi(x)\phi^\dagger(y)] | 0 \rangle. \quad (5.323)$$

类似于上一小节的计算，利用产生湮灭算符的对易关系 (2.164)，可以得到

$$\begin{aligned} \langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle &= \langle 0 | \phi^{(+)}(x) \phi^{\dagger(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \langle 0 | a_{\mathbf{p}} e^{-ip \cdot x} a_{\mathbf{q}}^\dagger e^{iq \cdot y} | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_{\mathbf{p}}}, \end{aligned} \quad (5.324)$$

以及

$$\begin{aligned} \langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle &= \langle 0 | \phi^{\dagger(+)}(y) \phi^{(-)}(x) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \langle 0 | b_{\mathbf{p}} e^{-ip \cdot y} b_{\mathbf{q}}^\dagger e^{iq \cdot x} | 0 \rangle = \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \langle 0 | ([b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] + b_{\mathbf{q}}^\dagger b_{\mathbf{p}}) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} e^{-i(p \cdot y - q \cdot x)} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_{\mathbf{p}}}. \end{aligned} \quad (5.325)$$

归纳上一小节的计算过程，可得

$$\theta(x^0 - y^0) \frac{e^{-iE_{\mathbf{p}}(x^0 - y^0)}}{2E_{\mathbf{p}}} + \theta(y^0 - x^0) \frac{e^{iE_{\mathbf{p}}(x^0 - y^0)}}{2E_{\mathbf{p}}} = \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon}, \quad (5.326)$$

其中 $\epsilon > 0$ 是一个无穷小量。从而，有

$$\begin{aligned} &\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \left[\theta(x^0 - y^0) \frac{e^{-iE_{\mathbf{p}}(x^0 - y^0)}}{2E_{\mathbf{p}}} + \theta(y^0 - x^0) \frac{e^{iE_{\mathbf{p}}(x^0 - y^0)}}{2E_{\mathbf{p}}} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0 - y^0)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}, \end{aligned} \quad (5.327)$$

于是，复标量场的 Feynman 传播子能够表达为

$$\begin{aligned} D_F(x - y) &= \langle 0 | T[\phi(x) \phi^\dagger(y)] | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \end{aligned} \quad (5.328)$$

可以看出，复标量场与实标量场具有相同形式的 Feynman 传播子。此外，由 (5.263) 式有

$$\overline{\phi^\dagger(x) \phi(y)} = \langle 0 | T[\phi^\dagger(x) \phi(y)] | 0 \rangle = \langle 0 | T[\phi(y) \phi^\dagger(x)] | 0 \rangle = D_F(y - x) = D_F(x - y). \quad (5.329)$$

也就是说， $\overline{\phi^\dagger(x) \phi(y)}$ 与 $\overline{\phi(x) \phi^\dagger(y)}$ 相等。

5.5.3 有质量矢量场的 Feynman 传播子

有质量矢量场 $A^\mu(x)$ 的 Feynman 传播子 $\Delta_F(x-y)$ 定义为

$$\Delta_F^{\mu\nu}(x-y) \equiv \overline{A^\mu(x)A^\nu(y)} = \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle. \quad (5.330)$$

根据展开式 (5.245) 和 (5.246), 及产生湮灭算符的对易关系 (3.175), 可得

$$\begin{aligned} \langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle &= \langle 0 | A^{\mu(+)}(x)A^{\nu(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0 | \varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} \varepsilon^{\nu*}(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{q}, \lambda') \langle 0 | ([a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger] + a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{p},\lambda}) | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{q}, \lambda') (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3p e^{-ip \cdot (x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} \varepsilon^\mu(\mathbf{p}, \lambda) \varepsilon^{\nu*}(\mathbf{p}, \lambda) = \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{e^{-ip \cdot (x-y)}}{2E_{\mathbf{p}}}, \end{aligned} \quad (5.331)$$

以及

$$\langle 0 | A^\nu(y)A^\mu(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \left(-g^{\nu\mu} + \frac{p^\nu p^\mu}{m^2} \right) \frac{e^{-ip \cdot (y-x)}}{2E_{\mathbf{p}}} = \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{e^{ip \cdot (x-y)}}{2E_{\mathbf{p}}}. \quad (5.332)$$

从而, 有

$$\begin{aligned} \Delta_F^{\mu\nu}(x-y) &= \langle 0 | T[A^\mu(x)A^\nu(y)] | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | A^\mu(x)A^\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A^\nu(y)A^\mu(x) | 0 \rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}]. \end{aligned} \quad (5.333)$$

最后一行圆括号中的项 $p^\mu p^\nu / m^2$ 与 p^0 有关, 因此直接应用 (5.326) 式不能得到适当的结果。

为了得到简洁的表达式, 我们需要将 $p^\mu p^\nu / m^2$ 转换为时空导数。记 $\partial_x^\mu \equiv \partial / \partial x_\mu$, 利用阶跃函数与 δ 函数的关系

$$\theta'(x) = \delta(x), \quad (5.334)$$

可以推出

$$\begin{aligned} &\partial_x^\mu \partial_x^\nu [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= \partial_x^\mu [-ip^\nu \theta(x^0 - y^0) e^{-ip \cdot (x-y)} + g^{\nu 0} \delta(x^0 - y^0) e^{-ip \cdot (x-y)} + ip^\nu \theta(y^0 - x^0) e^{ip \cdot (x-y)} \\ &\quad - g^{\nu 0} \delta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= -p^\mu p^\nu \theta(x^0 - y^0) e^{-ip \cdot (x-y)} - ig^{\mu 0} p^\nu \delta(x^0 - y^0) e^{-ip \cdot (x-y)} - ip^\mu g^{\nu 0} \delta(x^0 - y^0) e^{-ip \cdot (x-y)} \\ &\quad + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(x^0 - y^0) e^{-ip \cdot (x-y)} - p^\mu p^\nu \theta(y^0 - x^0) e^{ip \cdot (x-y)} - ig^{\mu 0} p^\nu \delta(y^0 - x^0) e^{ip \cdot (x-y)} \\ &\quad + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(y^0 - x^0) e^{ip \cdot (x-y)} \end{aligned}$$

$$\begin{aligned}
& -ip^\mu g^{\nu 0} \delta(y^0 - x^0) e^{ip \cdot (x-y)} + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(y^0 - x^0) e^{ip \cdot (x-y)} \\
= & -p^\mu p^\nu [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
& -i(g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}] \\
& + g^{\mu 0} g^{\nu 0} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}],
\end{aligned} \tag{5.335}$$

故

$$\begin{aligned}
& \frac{p^\mu p^\nu}{m^2} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
= & -\frac{\partial_x^\mu \partial_x^\nu}{m^2} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\
& -\frac{i}{m^2} (g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}] \\
& + \frac{g^{\mu 0} g^{\nu 0}}{m^2} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}].
\end{aligned} \tag{5.336}$$

因此, $\Delta_F^{\mu\nu}(x-y)$ 可以分解成三个部分,

$$\Delta_F^{\mu\nu}(x-y) = f_1^{\mu\nu}(x,y) + f_2^{\mu\nu}(x,y) + f_3^{\mu\nu}(x,y), \tag{5.337}$$

分别为

$$f_1^{\mu\nu}(x,y) \equiv -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}], \tag{5.338}$$

$$f_2^{\mu\nu}(x,y) \equiv -\frac{i}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (g^{\mu 0} p^\nu + g^{\nu 0} p^\mu) \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} + e^{ip \cdot (x-y)}], \tag{5.339}$$

$$f_3^{\mu\nu}(x,y) \equiv \frac{g^{\mu 0} g^{\nu 0}}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \partial_x^0 \delta(x^0 - y^0) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}]. \tag{5.340}$$

根据 (5.327) 式, $f_1^{\mu\nu}(x,y)$ 化为

$$f_1^{\mu\nu}(x,y) = -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right) \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon} = \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}. \tag{5.341}$$

$\delta(x^0 - y^0)$ 只在 $x^0 - y^0 = 0$ 处非零, 此时有 $e^{-iE_{\mathbf{p}}(x^0 - y^0)} = e^{iE_{\mathbf{p}}(x^0 - y^0)} = 1$, 故

$$f_2^{i0}(x,y) = f_2^{0i}(x,y) = -\frac{i}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{p^i}{2E_{\mathbf{p}}} \delta(x^0 - y^0) [e^{-i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})}] = 0. \tag{5.342}$$

上式中积分项是关于 \mathbf{p} 的奇函数, 因而对整个三维动量空间积分为零。此外, 利用 Fourier 变换公式

$$\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p} \cdot \mathbf{x}} = \delta^{(3)}(\mathbf{x}), \tag{5.343}$$

可以导出

$$f_2^{00}(x,y) = -\frac{i}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{2p^0}{2E_{\mathbf{p}}} \delta(x^0 - y^0) [e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})}]$$

$$= -\frac{2i}{m^2}\delta(x^0 - y^0)\delta^{(3)}(\mathbf{x} - \mathbf{y}) = -\frac{2i}{m^2}\delta^{(4)}(x - y). \quad (5.344)$$

归纳起来, 得到

$$f_2^{\mu\nu}(x, y) = -\frac{2i}{m^2}g^{\mu 0}g^{\nu 0}\delta^{(4)}(x - y). \quad (5.345)$$

另一方面, 根据 δ 函数的导数的定义, 有

$$\int dx f(x)\delta'(x - a) = -f'(a) = -\int dx f'(x)\delta(x - a), \quad (5.346)$$

因而对 (5.340) 式中的积分项可作替换

$$\partial_x^0\delta(x^0 - y^0)[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}] \rightarrow -\delta(x^0 - y^0)\partial_x^0[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}], \quad (5.347)$$

则

$$\begin{aligned} f_3^{\mu\nu}(x, y) &= -\frac{g^{\mu 0}g^{\nu 0}}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \delta(x^0 - y^0) \partial_x^0 [e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}] \\ &= -\frac{g^{\mu 0}g^{\nu 0}}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \delta(x^0 - y^0) [-ip^0 e^{-ip\cdot(x-y)} - ip^0 e^{ip\cdot(x-y)}] \\ &= \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \int \frac{d^3p}{(2\pi)^3} \delta(x^0 - y^0) [e^{ip\cdot(\mathbf{x}-\mathbf{y})} + e^{-ip\cdot(\mathbf{x}-\mathbf{y})}] = \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x - y). \end{aligned} \quad (5.348)$$

综合起来, 有质量矢量场 Feynman 传播子的表达式为

$$\Delta_F^{\mu\nu}(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} - \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x - y). \quad (5.349)$$

第一项是 Lorentz 协变的, 但第二项是非协变的。幸好, 这个非协变项在微扰论中的贡献被相互作用哈密顿量密度中的非协变项 (5.90) 精确抵消, 从而理论是 Lorentz 协变的。因此, 在实际计算中可以只保留协变项:

$$\Delta_F^{\mu\nu}(x - y) \rightarrow \int \frac{d^4p}{(2\pi)^4} \frac{-i(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}. \quad (5.350)$$

5.5.4 无质量矢量场的 Feynman 传播子

无质量矢量场的 Feynman 传播子依赖于规范的选择, 这里我们取 Feynman 规范 ($\xi = 1$)。在相互作用绘景中, 无质量矢量场 $A^\mu(x)$ 的平面波展开式仍然具有 (3.250) 的形式, 把它分解为正能解和负能解两部分, 得

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x), \quad (5.351)$$

其中,

$$A^{\mu(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^{\mu}(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} e^{-ip\cdot x}, \quad (5.352)$$

$$A^{\mu(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^3 e^{\mu}(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma}^{\dagger} e^{ip \cdot x}. \quad (5.353)$$

相应的 Feynman 传播子定义为

$$\Delta_F^{\mu\nu}(x-y) \equiv \overline{A^{\mu}(x)A^{\nu}(y)} = \langle 0 | T[A^{\mu}(x)A^{\nu}(y)] | 0 \rangle. \quad (5.354)$$

根据产生湮灭算符的对易关系 (3.261) 和极化矢量的完备性关系 (3.103), 可以得到

$$\begin{aligned} \langle 0 | A^{\mu}(x)A^{\nu}(y) | 0 \rangle &= \langle 0 | A^{\mu(+)}(x)A^{\nu(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} \langle 0 | e^{\mu}(\mathbf{p}, \sigma) a_{\mathbf{p};\sigma} e^{-ip \cdot x} e^{\nu}(\mathbf{q}, \sigma') a_{\mathbf{q};\sigma'}^{\dagger} e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^{\mu}(\mathbf{p}, \sigma) e^{\nu}(\mathbf{q}, \sigma') \langle 0 | ([a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}^{\dagger}] + a_{\mathbf{q};\sigma'}^{\dagger} a_{\mathbf{p};\sigma}) | 0 \rangle \\ &= - \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^{\mu}(\mathbf{p}, \sigma) e^{\nu}(\mathbf{q}, \sigma') (2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= - \int \frac{d^3p e^{-ip \cdot (x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\sigma} g_{\sigma\sigma} e^{\mu}(\mathbf{p}, \lambda) e^{\nu}(\mathbf{p}, \lambda) = -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_{\mathbf{p}}}, \end{aligned} \quad (5.355)$$

以及

$$\langle 0 | A^{\nu}(y)A^{\mu}(x) | 0 \rangle = -g^{\nu\mu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot (y-x)}}{2E_{\mathbf{p}}} = -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip \cdot (x-y)}}{2E_{\mathbf{p}}}. \quad (5.356)$$

当质量 $m = 0$ 时, (5.327) 式化为

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 + i\epsilon}. \quad (5.357)$$

于是, Feynman 规范下无质量矢量场的 Feynman 传播子可以表达为

$$\begin{aligned} \Delta_F^{\mu\nu}(x-y) &= \langle 0 | T[A^{\mu}(x)A^{\nu}(y)] | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | A^{\mu}(x)A^{\nu}(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | A^{\nu}(y)A^{\mu}(x) | 0 \rangle \\ &= -g^{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)}] \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} e^{-ip \cdot (x-y)}. \end{aligned} \quad (5.358)$$

5.5.5 Dirac 旋量场的 Feynman 传播子

Dirac 旋量场 $\psi_a(x)$ 的 Feynman 传播子 $S_{F,ab}(x-y)$ 定义为

$$S_{F,ab}(x-y) \equiv \overline{\psi_a(x)\psi_b(y)} = \langle 0 | T[\psi_a(x)\bar{\psi}_b(y)] | 0 \rangle. \quad (5.359)$$

在相互作用绘景中, $\bar{\psi}_a(x)$ 的平面波展开式仍然具有 (4.238) 的形式, 将它分解为正能解和负能解两个部分, 有

$$\bar{\psi}_a(x) = \bar{\psi}_a^{(+)}(x) + \bar{\psi}_a^{(-)}(x), \quad (5.360)$$

其中,

$$\bar{\psi}_a^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \bar{u}_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda}^\dagger e^{ip \cdot x}, \quad (5.361)$$

$$\bar{\psi}_a^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \bar{v}_a(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip \cdot x}. \quad (5.362)$$

再利用 $\psi_a^{(\pm)}(x)$ 的展开式 (5.248) 和 (5.249)、产生湮灭算符的反对易关系 (4.266)、自旋求和关系 (4.235), 可得

$$\begin{aligned} \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle &= \langle 0 | \psi_a^{(+)}(x) \bar{\psi}_b^{(-)}(y) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0 | u_a(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} e^{-ip \cdot x} \bar{u}_b(\mathbf{q}, \lambda') a_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot y} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{q}, \lambda') \langle 0 | (\{a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}^\dagger\} - a_{\mathbf{q},\lambda'}^\dagger a_{\mathbf{p},\lambda}) | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot x - q \cdot y)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{q}, \lambda') (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3p e^{-ip \cdot (x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} u_a(\mathbf{p}, \lambda) \bar{u}_b(\mathbf{p}, \lambda) = \int \frac{d^3p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} \frac{e^{-ip \cdot (x-y)}}{2E_{\mathbf{p}}} \\ &= \int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} e^{-ip \cdot (x-y)} \delta(p^2 - m^2) \theta(p^0), \end{aligned} \quad (5.363)$$

最后一步逆向利用 (2.119) 式的推导过程将 d^3p 积分化为 d^4p 积分。类似地, 还可以导出

$$\begin{aligned} \langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle &= \langle 0 | \bar{\psi}_b^{(+)}(y) \psi_a^{(-)}(x) | 0 \rangle \\ &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0 | \bar{v}_b(\mathbf{p}, \lambda) b_{\mathbf{p},\lambda} e^{-ip \cdot y} v_a(\mathbf{q}, \lambda') b_{\mathbf{q},\lambda'}^\dagger e^{iq \cdot x} | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{p}, \lambda) \langle 0 | (\{b_{\mathbf{p},\lambda}, b_{\mathbf{q},\lambda'}^\dagger\} - b_{\mathbf{q},\lambda'}^\dagger b_{\mathbf{p},\lambda}) | 0 \rangle \\ &= \int \frac{d^3p d^3q e^{-i(p \cdot y - q \cdot x)}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{\lambda\lambda'} v_a(\mathbf{q}, \lambda') \bar{v}_b(\mathbf{p}, \lambda) (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3p e^{-ip \cdot (y-x)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} v_a(\mathbf{p}, \lambda) \bar{v}_b(\mathbf{p}, \lambda) = \int \frac{d^3p}{(2\pi)^3} (\gamma^\mu p_\mu - m)_{ab} \frac{e^{ip \cdot (x-y)}}{2E_{\mathbf{p}}} \\ &= \int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu - m)_{ab} e^{ip \cdot (x-y)} \delta(p^2 - m^2) \theta(p^0) \\ &= - \int \frac{d^4p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} e^{-ip \cdot (x-y)} \delta(p^2 - m^2) \theta(-p^0). \end{aligned} \quad (5.364)$$

最后一步作了变量替换 $p^\mu \rightarrow -p^\mu$ 。于是, Feynman 传播子为

$$S_{F,ab}(x-y) = \langle 0 | T[\psi_a(x) \bar{\psi}_b(y)] | 0 \rangle$$

$$\begin{aligned}
&= \theta(x^0 - y^0) \langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle \\
&= \int \frac{d^4 p}{(2\pi)^3} (\gamma_\mu p^\mu + m)_{ab} e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2). \quad (5.365)
\end{aligned}$$

现在要想办法将 (5.365) 式转化为简洁的表达式。由

$$\begin{aligned}
&\partial_x^\mu \{ e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \} \\
&= -ip^\mu e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \\
&\quad + g^{\mu 0} e^{-ip \cdot (x-y)} [\delta(x^0 - y^0) \theta(p^0) - \delta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2), \quad (5.366)
\end{aligned}$$

可得

$$\begin{aligned}
&p^\mu e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \\
&= i \partial_x^\mu \{ e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \} \\
&\quad - i g^{\mu 0} e^{-ip \cdot (x-y)} [\theta(p^0) - \theta(-p^0)] \delta(x^0 - y^0) \delta(p^2 - m^2). \quad (5.367)
\end{aligned}$$

将上式代入 (5.365) 式, 得到

$$\begin{aligned}
&S_{F,ab}(x-y) \\
&= \int \frac{d^4 p}{(2\pi)^3} [(i\gamma_\mu \partial_x^\mu + m)_{ab} \{ e^{-ip \cdot (x-y)} [\theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \} \\
&\quad - i(\gamma_\mu)_{ab} g^{\mu 0} e^{-ip \cdot (x-y)} [\theta(p^0) - \theta(-p^0)] \delta(x^0 - y^0) \delta(p^2 - m^2)] \\
&= (i\gamma_\mu \partial_x^\mu + m)_{ab} \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-y)} \theta(x^0 - y^0) \theta(p^0) + \theta(y^0 - x^0) \theta(-p^0)] \delta(p^2 - m^2) \\
&\quad - i(\gamma^0)_{ab} \int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-y)} [\theta(p^0) - \theta(-p^0)] \delta(x^0 - y^0) \delta(p^2 - m^2). \quad (5.368)
\end{aligned}$$

先计算 (5.368) 式最后一行。利用 δ 函数的性质 (2.49), 有

$$e^{-ip^0(x^0-y^0)} \delta(x^0 - y^0) = e^{-ip^0(x^0-x^0)} \delta(x^0 - y^0) = \delta(x^0 - y^0), \quad (5.369)$$

由此可得

$$\begin{aligned}
&\int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-y)} \theta(p^0) \delta(x^0 - y^0) \delta(p^2 - m^2) \\
&= \int \frac{d^4 p}{(2\pi)^3} e^{-ip^0(x^0-y^0)} e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} \theta(p^0) \delta(x^0 - y^0) \delta(p^2 - m^2) \\
&= \int \frac{d^4 p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x}-\mathbf{y})} \theta(p^0) \delta(x^0 - y^0) \delta(p^2 - m^2), \quad (5.370)
\end{aligned}$$

以及

$$\int \frac{d^4 p}{(2\pi)^3} e^{-ip \cdot (x-y)} \theta(-p^0) \delta(x^0 - y^0) \delta(p^2 - m^2)$$

$$\begin{aligned}
&= \int \frac{d^4 p}{(2\pi)^3} e^{-ip^0(x^0-y^0)} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(-p^0) \delta(x^0-y^0) \delta(p^2-m^2) \\
&= \int \frac{d^4 p}{(2\pi)^3} e^{ip^0(x^0-y^0)} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(p^0) \delta(x^0-y^0) \delta(p^2-m^2) \\
&= \int \frac{d^4 p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \theta(p^0) \delta(x^0-y^0) \delta(p^2-m^2).
\end{aligned} \tag{5.371}$$

第二步作了变量替换 $p^0 \rightarrow -p^0$ 。结合以上两式，有

$$\int \frac{d^4 p}{(2\pi)^3} e^{-ip\cdot(x-y)} [\theta(p^0) - \theta(-p^0)] \delta(x^0-y^0) \delta(p^2-m^2) = 0. \tag{5.372}$$

故 (5.368) 式最后一行为零。另一方面，(5.368) 式倒数第二行中积分可化为

$$\begin{aligned}
&\int \frac{d^4 p}{(2\pi)^3} e^{-ip\cdot(x-y)} [\theta(x^0-y^0)\theta(p^0) + \theta(y^0-x^0)\theta(-p^0)] \delta(p^2-m^2) \\
&= \int \frac{d^4 p}{(2\pi)^3} [e^{-ip\cdot(x-y)} \theta(x^0-y^0) + e^{ip\cdot(x-y)} \theta(y^0-x^0)] \theta(p^0) \delta(p^2-m^2) \\
&= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [\theta(x^0-y^0) e^{-ip\cdot(x-y)} + \theta(y^0-x^0) e^{ip\cdot(x-y)}] = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}.
\end{aligned} \tag{5.373}$$

第一步作了变量替换 $p^\mu \rightarrow -p^\mu$ ，第二步利用 (2.119) 式的推导过程将 $d^4 p$ 积分化为 $d^3 p$ 积分，第三步用到 (5.327) 式。将上式代入 (5.368) 式，则 Dirac 旋量场的 Feynman 传播子可以表达为

$$S_{F,ab}(x-y) = (i\gamma_\mu \partial_x^\mu + m)_{ab} \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}. \tag{5.374}$$

写成旋量空间矩阵的形式是

$$S_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}. \tag{5.375}$$

根据 Dirac 矩阵的反对易关系 (4.1)，有

$$\not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = p_\mu p_\nu g^{\mu\nu} = p^2, \tag{5.376}$$

从而，可得

$$(\not{p}+m)(\not{p}-m) = \not{p}\not{p} - m^2 = p^2 - m^2, \tag{5.377}$$

故

$$(\not{p}+m)(\not{p}-m+i\epsilon) = p^2 - m^2 + i\epsilon(\not{p}+m). \tag{5.378}$$

$i\epsilon(\not{p}+m)$ 是一个无穷小量，因而上式右边与 (5.375) 式右边分式中的分母等价。于是，(5.375) 式也可以表示成

$$S_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p}+m)}{(\not{p}+m)(\not{p}-m+i\epsilon)} e^{-ip\cdot(x-y)} = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{\not{p}-m+i\epsilon} e^{-ip\cdot(x-y)}. \tag{5.379}$$

上式最右边在表达方式上更为简洁，但在矩阵的意义上不好理解，应将它转化回到 (5.375) 式来理解。

附录 A 英汉对照

Annihilation operator: 湮灭算符	External line: 外线
Antichronous: 反时向	Fermion: 费米子
Anti-particle: 反粒子	Feynman diagram: Feynman 图
Axial vector: 轴矢量	Feynman propagator: Feynman 传播子
Azimuthal angle: 方位角	Feynman rule: Feynman 规则
Beam: 束流	Field strength tensor: 场强张量
Boost: 增速	Gauge-fixing term: 规范固定项
Boson: 玻色子	Gauge invariant: 规范不变量
Branching ratio: 分支比	Gauge symmetry: 规范对称性
Canonical quantization: 正则量子化	Gauge transformation: 规范变换
Causality: 因果性	Generalized coordinate: 广义坐标
Center-of-mass energy: 质心能	Generator: 生成元
Center-of-mass system: 质心系	Global: 整体
Chiral representation: 手征表象	Hamiltonian: 哈密顿量
Collider: 对撞机	Helicity: 螺旋度
Conjugate momentum density: 共轭动量密度	Hermitian conjugate: 厄米共轭
Conserved charge: 守恒荷	Hermitian operator: 厄米算符
Conserved current: 守恒流	Homomorphic: 同态
Contraction: 缩并	Improper: 非固有
Contravariant vector: 逆变矢量	Interaction: 相互作用
Coupling constant: 耦合常数	Interaction picture: 相互作用绘景
Covariant vector: 协变矢量	Internal line: 内线
Creation operator: 产生算符	Invariant mass: 不变质量
Cross section: 截面	Invariant matrix element: 不变矩阵元
Decay: 衰变	Invariant scattering amplitude: 不变散射振幅
Decay width: 衰变宽度	Kinematics: 运动学
Dirac slash: Dirac 斜线	Lagrangian: 拉格朗日量
Dynamics: 动力学	Left-handed: 左手
Electron: 电子	Lifetime: 寿命
Energy-momentum tensor: 能动张量	Local: 局域
Expectation value: 期待值	Lowering operator: 降算符

Metric: 度规	Vector: 矢量
Mode: 模式	Vertex: 顶点
Normal order: 正规次序	External line: 外线
Normal product: 正规乘积	Zero-point energy: 零点能
Orthochronous: 保时向	
Parity: 宇称	
Partial decay width: 分宽度	
Perturbation theory: 微扰论	
Phonon: 声子	
Picture: 绘景	
Plane-wave solution: 平面波解	
Polar angle: 极角	
Polarization vector: 极化矢量	
Positron: 正电子	
Proper: 固有	
Pseudoscalar: 赝标量	
Raising operator: 升算符	
Right-handed: 右手	
Real orthogonal matrix: 实正交矩阵	
Scalar: 标量	
Scattering cross section: 散射截面	
Scattering matrix: 散射矩阵	
Self-conjugate: 自共轭	
Self-interaction: 自相互作用	
Simple harmonic oscillator: 简谐振子	
Space inversion: 空间反射	
Spinor: 旋量	
Spinor bilinear: 旋量双线性型	
Spinor representation: 旋量表示	
Step function: 阶跃函数	
Target: 靶	
Tensor: 张量	
Time-evolution operator: 时间演化算符	
Time-ordered product: 时序乘积	
Time reversal: 时间反演	
Unitary: 么正	
Vacuum: 真空	