Electroweak oblique parameters

Ref: Peskin & Takeuchi, PRD **46**, 381 (1992)

Vacuum polarization amplitude of gauge bosons I and J:

$$i\Pi_{IJ}^{\mu\nu}(p) = ig^{\mu\nu}\Pi_{IJ}(p^2) + (p^\mu p^\nu \text{ terms})$$

$$\Pi_{IJ}(p^2) \equiv \Pi_{IJ}(0) + p^2\Pi'_{IJ}(p^2) = \Pi_{IJ}(0) + p^2\Pi'_{IJ}(0) + \mathcal{O}(p^4), \quad \Pi'_{IJ}(0) = \left.\frac{\partial\Pi_{IJ}(p^2)}{\partial p^2}\right|_{p^2=0}$$

$$\Pi_{AA}(p^2) = e^2\Pi_{QQ}(p^2), \quad \Pi_{ZA}(p^2) = \frac{e^2}{s_\mathrm{W}c_\mathrm{W}}[\Pi_{3Q}(p^2) - s_\mathrm{W}^2\Pi_{QQ}(p^2)]$$

$$\Pi_{ZZ}(p^2) = \frac{e^2}{s_\mathrm{W}^2c_\mathrm{W}^2}[\Pi_{33}(p^2) - 2s_\mathrm{W}^2\Pi_{3Q}(p^2) + s_\mathrm{W}^4\Pi_{QQ}(p^2)], \quad \Pi_{WW}(p^2) = \frac{e^2}{s_\mathrm{W}^2}\Pi_{11}(p^2)$$

QED Ward identity $\Rightarrow$ $\Pi_{3Q}(0) = \Pi_{QQ}(0) = 0, \quad \Pi_{AA}(0) = \Pi_{ZA}(0) = 0$

$$S \equiv \frac{4e^2}{\alpha}[\Pi'_{33}(0) - \Pi'_{3Q}(0)] = \frac{4s_\mathrm{W}^2c_\mathrm{W}^2}{\alpha}\left[\Pi'_{ZZ}(0) - \frac{c_\mathrm{W}^2 - s_\mathrm{W}^2}{s_\mathrm{W}c_\mathrm{W}}\Pi'_{ZA}(0) - \Pi'_{AA}(0)\right]$$

$$T \equiv \frac{e^2}{\alpha s_\mathrm{W}^2c_\mathrm{W}^2m_Z^2}[\Pi_{11}(0) - \Pi_{33}(0)] = \frac{1}{\alpha}\left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2}\right]$$

$$U \equiv \frac{4e^2}{\alpha}[\Pi'_{11}(0) - \Pi'_{33}(0)] = \frac{4s_\mathrm{W}^2}{\alpha}\left[\Pi'_{WW}(0) - c_\mathrm{W}^2\Pi'_{ZZ}(0) - 2s_\mathrm{W}c_\mathrm{W}\Pi'_{ZA}(0) - s_\mathrm{W}^2\Pi'_{AA}(0)\right]$$

$$c_\mathrm{W} \equiv \cos\theta_\mathrm{W} = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad s_\mathrm{W} \equiv \sin\theta_\mathrm{W} = \frac{g}{\sqrt{g^2 + g'^2}}, \quad e = \frac{gg'}{\sqrt{g^2 + g'^2}} = gs_\mathrm{W} = g'c_\mathrm{W}, \quad m_W = m_Zc_\mathrm{W}$$

Fan, Reece & Wang, 1411.1054:

S is contributed by $\mathrm{SU(2)_L}$ multiplets that are split by the electroweak symmetry breaking

T and U are contributed by $\mathrm{SU(2)_L}$ multiplets that are split by the electroweak symmetry breaking with custodial symmetry [firstly called in Sikivie et al., NPB **173**, 189 (1980)] violating effects

<u>EFT viewpoint</u> [Zhenyu Han, 0807.0490]

$$\frac{1}{\Lambda^2}H^\dagger W_{\mu\nu}^a\sigma^a HB^{\mu\nu} \to S, \quad \frac{1}{\Lambda^2}H^\dagger D_\mu H(D^\mu H)^\dagger H \to T, \quad \frac{1}{\Lambda^4}H^\dagger W_{\mu\nu}^a\sigma^a HH^\dagger W^{b\mu\nu}\sigma^b H \to U$$

U corresponds to higher dimensional operators and is typically much smaller than S and T

$\underline{H^\dagger W^a_{\mu\nu}\sigma^a HB^{\mu\nu}\ \text{operator}}$

$$H \to \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \quad W^3_{\mu\nu} = s_\mathrm{W}A_{\mu\nu} + c_\mathrm{W}Z_{\mu\nu} - gW^1_\mu W^2_\nu + gW^2_\mu W^1_\nu, \quad B_{\mu\nu} = c_\mathrm{W}A_{\mu\nu} - s_\mathrm{W}Z_{\mu\nu}$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$H^\dagger W^a_{\mu\nu}\sigma^a H \supset H^\dagger W^3_{\mu\nu}\sigma^3 H \to \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}W^3_{\mu\nu} = -\frac{v^2}{2}W^3_{\mu\nu}$$

$$\mathcal{O}_{WB} = a_{WB}H^\dagger W^a_{\mu\nu}\sigma^a HB^{\mu\nu} \to -\frac{a_{WB}v^2}{2}W^3_{\mu\nu}B^{\mu\nu} \supset -\frac{a_{WB}v^2}{2}(s_\mathrm{W}A_{\mu\nu} + c_\mathrm{W}Z_{\mu\nu})(c_\mathrm{W}A^{\mu\nu} - s_\mathrm{W}Z^{\mu\nu})$$

$$= -\frac{a_{WB}v^2}{2}[s_\mathrm{W}c_\mathrm{W}(A_{\mu\nu}A^{\mu\nu} - Z_{\mu\nu}Z^{\mu\nu}) + (c^2_\mathrm{W} - s^2_\mathrm{W})Z_{\mu\nu}A^{\mu\nu}]$$

[Note: Feynman rule for momenta pointing into the vertex: $\partial_\mu \to -ip_\mu$]

Feynman rule for $Z_\mu(p) - \times - Z_\nu(p)$ from $Z_{\mu\nu}Z^{\mu\nu}$

$$Z_{\mu\nu}Z^{\mu\nu} = 2(\partial_\mu Z_\nu \partial^\mu Z^\nu - \partial_\mu Z_\nu \partial^\nu Z^\mu)$$

$$\to 2[g^{\rho\sigma}g^{\mu\nu}\partial_\rho Z_\mu(p)\partial_\sigma Z_\nu(p) + g^{\rho\sigma}g^{\nu\mu}\partial_\rho Z_\nu(p)\partial_\sigma Z_\mu(p)$$

$$- g^{\rho\nu}g^{\mu\sigma}\partial_\rho Z_\mu(p)\partial_\sigma Z_\nu(p) - g^{\rho\mu}g^{\nu\sigma}\partial_\rho Z_\nu(p)\partial_\sigma Z_\mu(p)]$$

$$\to 2i[(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})(-ip_\rho)(ip_\sigma) + (g^{\rho\sigma}g^{\nu\mu} - g^{\rho\mu}g^{\nu\sigma})(ip_\rho)(-ip_\sigma)] = 4i(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

Contribution to $\Pi^{\mu\nu}_{ZZ}$ from $\mathcal{O}_{WB}$:

$$i\Pi^{\mu\nu}_{ZZ}(p^2) = \frac{s_\mathrm{W}c_\mathrm{W}a_{WB}v^2}{2}4i(g^{\mu\nu}p^2 - p^\mu p^\nu) = 2is_\mathrm{W}c_\mathrm{W}a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{ZZ}(p^2) = 2s_\mathrm{W}c_\mathrm{W}a_{WB}v^2p^2 \quad \Rightarrow \quad \Pi'_{ZZ}(0) = 2s_\mathrm{W}c_\mathrm{W}a_{WB}v^2$$

Feynman rule for $A_\mu(p) - \times - A_\nu(p)$ from $A_{\mu\nu}A^{\mu\nu}$: $\quad A_{\mu\nu}A^{\mu\nu} \to 4i(g^{\mu\nu}p^2 - p^\mu p^\nu)$

Contribution to $\Pi^{\mu\nu}_{AA}$ from $\mathcal{O}_{WB}$:

$$i\Pi^{\mu\nu}_{AA}(p^2) = -\frac{s_\mathrm{W}c_\mathrm{W}a_{WB}v^2}{2}4i(g^{\mu\nu}p^2 - p^\mu p^\nu) = -2is_\mathrm{W}c_\mathrm{W}a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{AA}(p^2) = -2s_\mathrm{W}c_\mathrm{W}a_{WB}v^2p^2 \quad \Rightarrow \quad \Pi'_{AA}(0) = -2s_\mathrm{W}c_\mathrm{W}a_{WB}v^2$$

Feynman rule for $Z_\mu(p) - \times - A_\nu(p)$ from $Z_{\mu\nu}A^{\mu\nu}$:

$$Z_{\mu\nu}A^{\mu\nu} = 2(\partial_\mu Z_\nu\partial^\mu A^\nu - \partial_\mu Z_\nu \partial^\nu A^\mu) \to 2[g^{\rho\sigma}g^{\mu\nu}\partial_\rho Z_\mu(p)\partial_\sigma A_\nu(p) - g^{\rho\nu}g^{\mu\sigma}\partial_\rho Z_\mu(p)\partial_\sigma A_\nu(p)]$$

$$\to 2i(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})(-ip_\rho)(ip_\sigma) = 2i(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

Contribution to $\Pi^{\mu\nu}_{ZA}$ from $\mathcal{O}_{WB}$:

$$i\Pi^{\mu\nu}_{ZA}(p^2) = -\frac{a_{WB}v^2}{2}(c^2_\mathrm{W} - s^2_\mathrm{W})2i(g^{\mu\nu}p^2 - p^\mu p^\nu) = -i(c^2_\mathrm{W} - s^2_\mathrm{W})a_{WB}v^2(g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\Pi_{ZA}(p^2) = -(c^2_\mathrm{W} - s^2_\mathrm{W})a_{WB}v^2p^2 \quad \Rightarrow \quad \Pi'_{ZA}(0) = -(c^2_\mathrm{W} - s^2_\mathrm{W})a_{WB}v^2$$

$$S = \frac{4s^2_\mathrm{W}c^2_\mathrm{W}}{\alpha}\left[\Pi'_{ZZ}(0) - \frac{c^2_\mathrm{W} - s^2_\mathrm{W}}{s_\mathrm{W}c_\mathrm{W}}\Pi'_{ZA}(0) - \Pi'_{AA}(0)\right] = \frac{4s^2_\mathrm{W}c^2_\mathrm{W}}{\alpha}a_{WB}v^2\left[2s_\mathrm{W}c_\mathrm{W} + \frac{(c^2_\mathrm{W} - s^2_\mathrm{W})^2}{s_\mathrm{W}c_\mathrm{W}} + 2s_\mathrm{W}c_\mathrm{W}\right] = \frac{4s_\mathrm{W}c_\mathrm{W}}{\alpha}a_{WB}v^2$$

$$T = \frac{1}{\alpha}\left[\frac{\Pi_{WW}(0)}{m^2_W} - \frac{\Pi_{ZZ}(0)}{m^2_Z}\right] = -\frac{\Pi_{ZZ}(0)}{\alpha m^2_Z} = 0$$

$$U = \frac{4s^2_\mathrm{W}}{\alpha}\left[\Pi'_{WW}(0) - c^2_\mathrm{W}\Pi'_{ZZ}(0) - 2s_\mathrm{W}c_\mathrm{W}\Pi'_{ZA}(0) - s^2_\mathrm{W}\Pi'_{AA}(0)\right] = \frac{4s^2_\mathrm{W}}{\alpha}a_{WB}v^2\left[-2s_\mathrm{W}c^3_\mathrm{W} + 2s_\mathrm{W}c_\mathrm{W}(c^2_\mathrm{W} - s^2_\mathrm{W}) + 2s^3_\mathrm{W}c_\mathrm{W}\right] = 0$$

$\underline{H^\dagger D_\mu H(D^\mu H)^\dagger H}$ operator

$$D_\mu H = (\partial_\mu - ig' B_\mu Y_H - igW_\mu^a t_{\mathrm{D}}^a)H, \quad Y_H = \frac{1}{2}, \quad t_{\mathrm{D}}^a = \frac{\sigma^a}{2}$$

$$D_\mu H \supset -\frac{i}{2}\begin{pmatrix} g'B_\mu + gW_\mu^3 & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & g'B_\mu - gW_\mu^3 \end{pmatrix} H \to -\frac{iv}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}gW_\mu^+ \\ g'B_\mu - gW_\mu^3 \end{pmatrix}, \quad H^\dagger D_\mu H \to -\frac{iv}{4}(g'B_\mu - gW_\mu^3)$$

$$g'B_\mu - gW_\mu^3 = -\sqrt{g^2 + g'^2}\, Z_\mu, \quad m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}$$

$$\mathcal{O}_h = a_h H^\dagger D_\mu H(D^\mu H)^\dagger H \to \frac{a_h v^2}{16}(g'B_\mu - gW_\mu^3)^2 = \frac{a_h v^2}{16}(g^2 + g'^2) Z_\mu Z^\mu = \frac{a_h v^2}{4} m_Z^2 Z_\mu Z^\mu$$

Contribution to $\Pi_{ZZ}^{\mu\nu}$ from $\mathcal{O}_{WB}$: $\quad i\Pi_{ZZ}^{\mu\nu}(p^2) = i\dfrac{a_h v^2}{2} m_Z^2 g^{\mu\nu}$

$$\Pi_{ZZ}(p^2) = \frac{a_h v^2}{2} m_Z^2$$

$$S = \frac{4s_{\mathrm{W}}^2 c_{\mathrm{W}}^2}{\alpha}\Pi'_{ZZ}(0) = 0, \quad T = -\frac{1}{\alpha}\frac{1}{m_Z^2}\frac{a_h v^2}{2} m_Z^2 = -\frac{1}{2\alpha} a_h v^2, \quad U = -\frac{4s_{\mathrm{W}}^2}{\alpha} c_{\mathrm{W}}^2 \Pi'_{ZZ}(0) = 0$$

$\underline{H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu}\sigma^b H}$ operator

$$H^\dagger W_{\mu\nu}^a \sigma^a H \to -\frac{v^2}{2} W_{\mu\nu}^3$$

$$\mathcal{O}_{WW} = a_{WW} H^\dagger W_{\mu\nu}^a \sigma^a H H^\dagger W^{b\mu\nu}\sigma^b H \to \frac{a_{WW} v^4}{4} W_{\mu\nu}^3 W^{3,\mu\nu}$$

$$= \frac{a_{WW} v^4}{4}(s_{\mathrm{W}} A_{\mu\nu} + c_{\mathrm{W}} Z_{\mu\nu})(s_{\mathrm{W}} A^{\mu\nu} + c_{\mathrm{W}} Z^{\mu\nu}) = \frac{a_{WW} v^4}{4}(s_{\mathrm{W}}^2 A_{\mu\nu} A^{\mu\nu} + 2s_{\mathrm{W}} c_{\mathrm{W}} Z_{\mu\nu} A^{\mu\nu} + c_{\mathrm{W}}^2 Z_{\mu\nu} Z^{\mu\nu})$$

$$\Pi_{ZZ}(p^2) = c_{\mathrm{W}}^2 a_{WW} v^4 p^2, \quad \Pi_{ZA}(p^2) = s_{\mathrm{W}} c_{\mathrm{W}} a_{WW} v^4 p^2, \quad \Pi_{AA}(p^2) = s_{\mathrm{W}}^2 a_{WW} v^4 p^2$$

$$\Pi'_{ZZ}(0) = c_{\mathrm{W}}^2 a_{WW} v^4, \quad \Pi'_{ZA}(0) = s_{\mathrm{W}} c_{\mathrm{W}} a_{WW} v^4, \quad \Pi'_{AA}(0) = s_{\mathrm{W}}^2 a_{WW} v^4$$

$$S = \frac{4s_{\mathrm{W}}^2 c_{\mathrm{W}}^2}{\alpha} a_{WW} v^4 \left[c_{\mathrm{W}}^2 - \frac{c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2}{s_{\mathrm{W}} c_{\mathrm{W}}} s_{\mathrm{W}} c_{\mathrm{W}} - s_{\mathrm{W}}^2 \right] = 0, \quad T = -\frac{\Pi_{ZZ}(0)}{\alpha m_Z^2} = 0$$

$$U = \frac{4s_{\mathrm{W}}^2}{\alpha} a_{WW} v^4 (0 - c_{\mathrm{W}}^4 - 2s_{\mathrm{W}}^2 c_{\mathrm{W}}^2 - s_{\mathrm{W}}^4) = -\frac{4s_{\mathrm{W}}^2}{\alpha} a_{WW} v^4$$

Normal and χ^2 distributions

n-dim normal distribution $f(\mathbf{x}) = \dfrac{1}{(2\pi)^{n/2} |\mathbf{V}|^{1/2}} e^{-Q/2}$

Quadratic form $Q = (\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \mathbf{V}^{-1} (\mathbf{x}-\boldsymbol{\mu}) = (x_i - \mu_i) V_{ij}^{-1} (x_j - \mu_j)$

$\mathbf{V}$ is the positive definite symmetric covariance matrix

$$V_{ij} = \operatorname{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \rho_{1n}\sigma_1\sigma_n & \cdots & & \sigma_n^2 \end{pmatrix}$$

Correlation coefficients $\rho_{ij} = \dfrac{\operatorname{cov}(X_i, X_j)}{\sqrt{V(X_i)V(X_j)}}$

Probability inside the n-dim ellipsoid $Q = Q_0$: $\quad P(Q < Q_0) = F_{\chi^2(n)}(Q_0)$

$F_{\chi^2(n)}(x)$ is the cumulative χ^2 distribution function with n d.o.f.

$F_{\chi^2(1)}(1) = 68.3\%, \quad F_{\chi^2(1)}(4) = 95.4\%, \quad F_{\chi^2(1)}(9) = 99.7\%$

$F_{\chi^2(2)}(2.295749) = 68.3\%, \quad F_{\chi^2(2)}(6.180074) = 95.4\%, \quad F_{\chi^2(2)}(11.82916) = 99.7\%$

$F_{\chi^2(3)}(3.526741) = 68.3\%, \quad F_{\chi^2(3)}(8.024882) = 95.4\%, \quad F_{\chi^2(3)}(14.15641) = 99.7\%$

2-dim normal distribution

$$V_{ij} = \operatorname{cov}(X_i, X_j) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}, \quad V_{ij}^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-Q/2}$$

$$Q = \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2}\right]$$

Ellipse parametric equation with parameter t:

$$\begin{cases} x_1 = \mu_1 + a\cos\phi\cos t - b\sin\phi\sin t \\ x_2 = \mu_2 + a\sin\phi\cos t + b\cos\phi\sin t \end{cases}$$

$$\phi = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}\right), \quad a = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2\cos^2\phi - 2\rho\sigma_1\sigma_2\sin\phi\cos\phi + \sigma_1^2\sin^2\phi}}$$

$$b = \frac{\sigma_1\sigma_2\sqrt{Q(1-\rho^2)}}{\sqrt{\sigma_2^2\sin^2\phi + 2\rho\sigma_1\sigma_2\sin\phi\cos\phi + \sigma_1^2\cos^2\phi}}$$

Precision directly extracted from previous works

Gfitter, 1407.3792

Current constraints on S and T with $U=0$ fixed [below Eq. (4)]:

$$S = 0.06 \pm 0.09, \quad T = 0.10 \pm 0.07, \quad \rho_{ST} = +0.91$$

Current precision on S and T [Fig. 7]:

$$\sigma_S = 0.093, \quad \sigma_T = 0.082, \quad \rho_{ST} = +0.93$$

Fan, Reece & Wang, 1411.1054

Current precision [Fig. 1]:

$$\sigma_S = 0.086, \quad \sigma_T = 0.074, \quad \rho_{ST} = +0.91$$

CEPC optimistic baseline precision [Fig. 3]:

$$\sigma_S = 0.024, \quad \sigma_T = 0.019, \quad \rho_{ST} = +0.84$$

CEPC precision with improvements of m_t, Γ_Z and $\sin^2\theta_{\mathrm{eff}}^{\ell}$ [Fig. 3]:

$$\sigma_S = 0.011, \quad \sigma_T = 0.0072, \quad \rho_{ST} = +0.78$$

CEPC-SPPC Preliminary Conceptual Design Report

CEPC baseline precision [Fig. 4.2]:

$$\sigma_S = 0.016, \quad \sigma_T = 0.015, \quad \rho_{ST} = +0.82$$

CEPC precision with improvements of m_t, m_Z and Γ_Z [Fig. 4.2]:

$$\sigma_S = 0.011, \quad \sigma_T = 0.0073, \quad \rho_{ST} = +0.78$$

Dependence of EW precision observables on S,T,U

Ref: Ciuchini et al., 1306.4644; Fan, Reece & Wang, 1411.1054

Tree level:

$$m_W = 80.385\ \text{GeV},\quad m_Z = 91.1876\ \text{GeV},\quad c_\text{W}^2 = \frac{m_W^2}{m_Z^2} = 0.77737,\quad s_\text{W}^2 = 1 - c_\text{W}^2 = 0.22263$$

Loop level:

$$m_W = m_W^\text{SM} + \Delta m_W,\quad \Gamma_W = \Gamma_W^\text{SM} + \Delta\Gamma_W,\quad \sin^2\theta_\text{eff}^\ell = (\sin^2\theta_\text{eff}^\ell)^\text{SM} + \Delta\sin^2\theta_\text{eff}^\ell,\quad \Gamma_Z = \Gamma_Z^\text{SM} + \Delta\Gamma_Z$$

$$F_1(S,T,U) \equiv S - 2c_\text{W}^2 T - \frac{c_\text{W}^2 - s_\text{W}^2}{2s_\text{W}^2}U = S - 1.55T - 1.24U$$

$$F_2(S,T) \equiv S - 4s_\text{W}^2 c_\text{W}^2 T = S - 0.693T$$

$$F_3(S,T) \equiv -10(3 - 8s_\text{W}^2)S + (63 - 126s_\text{W}^2 - 40s_\text{W}^4)T = -12.2S + 32.9T = -12.2(S - 2.71T)$$

$$\Delta m_W = -\frac{\alpha m_W^\text{SM}}{4(c_\text{W}^2 - s_\text{W}^2)}\left(S - 2c_\text{W}^2 T - \frac{c_\text{W}^2 - s_\text{W}^2}{2s_\text{W}^2}U\right) = -\frac{\alpha m_W^\text{SM}}{4(c_\text{W}^2 - s_\text{W}^2)}F_1(S,T,U)$$

$$\Delta\Gamma_W = -\frac{3\alpha\Gamma_W^\text{SM}}{4(c_\text{W}^2 - s_\text{W}^2)}\left(S - 2c_\text{W}^2 T - \frac{c_\text{W}^2 - s_\text{W}^2}{2s_\text{W}^2}U\right) = -\frac{3\alpha\Gamma_W^\text{SM}}{4(c_\text{W}^2 - s_\text{W}^2)}F_1(S,T,U)$$

$$\mathcal{L} \supset \sum_f \frac{g}{2c_\text{W}}\bar{f}\gamma^\mu(g_\text{V}^f - g_\text{A}^f\gamma_5)fZ_\mu,\quad g_\text{V}^f \equiv T_f^3 - 2Q_f s_\text{W}^2,\quad g_\text{A}^f \equiv T_f^3$$

$$\left[\begin{array}{l} Q_{\ell_i} = -1,\quad Q_{\nu_i} = 0,\quad Q_{d_i} = -\dfrac{1}{3},\quad Q_{u_i} = \dfrac{2}{3} \\ g_\text{V}^{\ell_i} = -\dfrac{1}{2} + 2s_\text{W}^2,\quad g_\text{A}^{\ell_i} = -\dfrac{1}{2};\quad g_\text{V}^{\nu_i} = \dfrac{1}{2},\quad g_\text{A}^{\nu_i} = \dfrac{1}{2} \\ g_\text{V}^{d_i} = -\dfrac{1}{2} + \dfrac{2}{3}s_\text{W}^2,\quad g_\text{A}^{d_i} = -\dfrac{1}{2};\quad g_\text{V}^{u_i} = \dfrac{1}{2} - \dfrac{4}{3}s_\text{W}^2,\quad g_\text{A}^{u_i} = \dfrac{1}{2} \end{array}\right]$$

$$\delta g_\text{V}^f = \frac{g_\text{V}^f\alpha}{2}T + \frac{(g_\text{V}^f - g_\text{A}^f)\alpha}{4s_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}(S - 4s_\text{W}^2 c_\text{W}^2 T),\quad \delta g_\text{A}^f = \frac{g_\text{A}^f\alpha}{2}T$$

$$\Delta\sin^2\theta_\text{eff}^\ell = -\frac{g_\text{A}^e\delta g_\text{V}^e - g_\text{V}^e\delta g_\text{A}^e}{4(g_\text{A}^e)^2} = -\frac{1}{4(g_\text{A}^e)^2}\left[g_\text{A}^e\left(\frac{g_\text{V}^e\alpha}{2}T + \frac{(g_\text{V}^e - g_\text{A}^e)\alpha}{4s_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}(S - 4s_\text{W}^2 c_\text{W}^2 T)\right) - g_\text{V}^e\frac{g_\text{A}^e\alpha}{2}T\right]$$

$$= -\frac{(g_\text{V}^e - g_\text{A}^e)\alpha}{16g_\text{A}^e s_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}(S - 4s_\text{W}^2 c_\text{W}^2 T) = \frac{\alpha}{4(c_\text{W}^2 - s_\text{W}^2)}F_2(S,T)$$

$$G_f \equiv (g_\text{V}^f)^2 + (g_\text{A}^f)^2$$

$$\delta G_f \equiv 2(g_\text{V}^f\delta g_\text{V}^f + g_\text{A}^f\delta g_\text{A}^f) = \frac{g_\text{V}^f(g_\text{V}^f - g_\text{A}^f)\alpha}{2s_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}(S - 4s_\text{W}^2 c_\text{W}^2 T) + [(g_\text{V}^f)^2 + (g_\text{A}^f)^2]\alpha T$$

$$\left[\begin{array}{l} g_\text{V}^{\ell_i}(g_\text{V}^{\ell_i} - g_\text{A}^{\ell_i}) = -s_\text{W}^2 + 4s_\text{W}^4,\quad (g_\text{V}^{\ell_i})^2 + (g_\text{A}^{\ell_i})^2 = \dfrac{1}{2} - 2s_\text{W}^2 + 4s_\text{W}^4 \\ g_\text{V}^{\nu_i}(g_\text{V}^{\nu_i} - g_\text{A}^{\nu_i}) = 0,\quad (g_\text{V}^{\nu_i})^2 + (g_\text{A}^{\nu_i})^2 = \dfrac{1}{2} \\ g_\text{V}^{d_i}(g_\text{V}^{d_i} - g_\text{A}^{d_i}) = -\dfrac{1}{3}s_\text{W}^2 + \dfrac{4}{9}s_\text{W}^4,\quad (g_\text{V}^{d_i})^2 + (g_\text{A}^{d_i})^2 = \dfrac{1}{2} - \dfrac{2}{3}s_\text{W}^2 + \dfrac{4}{9}s_\text{W}^4 \\ g_\text{V}^{u_i}(g_\text{V}^{u_i} - g_\text{A}^{u_i}) = -\dfrac{2}{3}s_\text{W}^2 + \dfrac{16}{9}s_\text{W}^4,\quad (g_\text{V}^{u_i})^2 + (g_\text{A}^{u_i})^2 = \dfrac{1}{2} - \dfrac{4}{3}s_\text{W}^2 + \dfrac{16}{9}s_\text{W}^4 \end{array}\right]$$

$$f = e, \nu_e, \mu, \nu_\mu, \tau, \mu_\tau, d, u, s, c, b$$

$$\sum_f N_\text{c}^f g_\text{V}^f(g_\text{V}^f - g_\text{A}^f) = 3(-s_\text{W}^2 + 4s_\text{W}^4) + 3\cdot 3\left(-\frac{1}{3}s_\text{W}^2 + \frac{4}{9}s_\text{W}^4\right) + 2\cdot 3\left(-\frac{2}{3}s_\text{W}^2 + \frac{16}{9}s_\text{W}^4\right) = -\frac{10}{3}s_\text{W}^2(3 - 8s_\text{W}^2)$$

$$\sum_f N_\text{c}^f[(g_\text{V}^f)^2 + (g_\text{A}^f)^2] = \left[3\left(\frac{1}{2} + \frac{1}{2} - 2s_\text{W}^2 + 4s_\text{W}^4\right) + 3\cdot 3\left(\frac{1}{2} - \frac{2}{3}s_\text{W}^2 + \frac{4}{9}s_\text{W}^4\right) + 2\cdot 3\left(\frac{1}{2} - \frac{4}{3}s_\text{W}^2 + \frac{16}{9}s_\text{W}^4\right)\right] = \frac{21}{2} - 20s_\text{W}^2 + \frac{80}{3}s_\text{W}^4$$

$$\sum_f N_\text{c}^f\delta G_f = \frac{\alpha}{2s_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}(S - 4c_\text{W}^2 s_\text{W}^2 T)\sum_f N_\text{c}^f g_\text{V}^f(g_\text{V}^f - g_\text{A}^f) + \alpha T\sum_f N_\text{c}^f[(g_\text{V}^f)^2 + (g_\text{A}^f)^2]$$

$$= -\frac{5\alpha}{3(c_\text{W}^2 - s_\text{W}^2)}(S - 4c_\text{W}^2 s_\text{W}^2 T)(3 - 8s_\text{W}^2) + \alpha T\left(\frac{21}{2} - 20s_\text{W}^2 + \frac{80}{3}s_\text{W}^4\right)$$

$$= -\frac{5(3 - 8s_\text{W}^2)}{3(c_\text{W}^2 - s_\text{W}^2)}\alpha S + \frac{20\alpha T}{3(c_\text{W}^2 - s_\text{W}^2)}\left[c_\text{W}^2 s_\text{W}^2(3 - 8s_\text{W}^2) + \frac{3(c_\text{W}^2 - s_\text{W}^2)}{20}\left(\frac{21}{2} - 20s_\text{W}^2 + \frac{80}{3}s_\text{W}^4\right)\right]$$

$$= \frac{\alpha}{6(c_\text{W}^2 - s_\text{W}^2)}[-10(3 - 8s_\text{W}^2)S + (63 - 126s_\text{W}^2 - 40s_\text{W}^4)T]$$

$$\Delta\Gamma_Z = \frac{\alpha m_Z}{12s_\text{W}^2 c_\text{W}^2}\sum_f N_\text{c}^f\delta G_f = \frac{\alpha^2 m_Z}{72s_\text{W}^2 c_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}[-10(3 - 8s_\text{W}^2)S + (63 - 126s_\text{W}^2 - 40s_\text{W}^4)T] = \frac{\alpha^2 m_Z}{72s_\text{W}^2 c_\text{W}^2(c_\text{W}^2 - s_\text{W}^2)}F_3(S,T)$$

Parametrization of the SM correction to m_W

Ref: Awramik et al., hep-ph/0311148

Loop level: $m_W^2\left(1-\frac{m_W^2}{m_Z^2}\right)=\frac{\pi\alpha}{\sqrt{2}G_{\rm F}}(1+\Delta r) \quad \Rightarrow \quad m_W^2=m_Z^2\left[\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\pi\alpha}{\sqrt{2}G_{\rm F}m_Z^2}(1+\Delta r)}\right]$

Parametrization formula:

$$m_W^{\rm SM}=m_W^0-c_1 dH-c_2 dH^2+c_3 dH^4+c_4(dh-1)-c_5 d\alpha+c_6 dt-c_7 dt^2-c_8 dH dt+c_9 dh dt-c_{10} d\alpha_{\rm s}+c_{11} dZ$$

$$dH=\ln\frac{m_h}{100\ \text{GeV}},\quad dh=\left(\frac{m_h}{100\ \text{GeV}}\right)^2,\quad dt=\left(\frac{m_t}{174.3\ \text{GeV}}\right)^2-1$$

$$dZ=\frac{m_Z}{91.1875\ \text{GeV}}-1,\quad d\alpha=\frac{\Delta\alpha}{0.05907}-1,\quad d\alpha_{\rm s}=\frac{\alpha_{\rm s}(m_Z)}{0.119}-1$$

$\Delta\alpha\equiv\Delta\alpha_{\rm lep}+\Delta\alpha_{\rm had}^{(5)},\quad \Delta\alpha_{\rm lep}=0.0314977$

Coefficient values can be found in Eq.(9) for $m_h>100$ GeV

Parametrization of the SM correction to $\sin^2\theta_{\rm eff}^{\ell}$

Ref: Awramik et al., hep-ph/0608099

Loop level: $\sin^2\theta_{\rm eff}^{\ell}=\left(1-\frac{m_W^2}{m_Z^2}\right)(1+\Delta\kappa)$

Parametrization formula:

$$(\sin^2\theta_{\rm eff}^{\ell})^{\rm SM}=s_0+d_1 L_H+d_2 L_H^2+d_3 L_H^4+d_4(\Delta_H^2-1)+d_5\Delta_\alpha+d_6\Delta_t+d_7\Delta_t^2+d_8\Delta_t(\Delta_H-1)+d_9\Delta_{\alpha_{\rm s}}+d_{10}\Delta_Z$$

$$L_H=\ln\frac{m_h}{100\ \text{GeV}},\quad \Delta_H=\frac{m_h}{100\ \text{GeV}},\quad \Delta_\alpha=\frac{\Delta\alpha}{0.05907}-1$$

$$\Delta_t=\left(\frac{m_t}{178\ \text{GeV}}\right)^2-1,\quad \Delta_{\alpha_{\rm s}}=\frac{\alpha_{\rm s}(m_Z)}{0.117}-1,\quad \Delta_Z=\frac{m_Z}{91.1876\ \text{GeV}}-1$$

Coefficient values can be found in the 2nd column of Table 5

Parametrization of the SM correction to Γ_Z

Ref: Freitas, 1401.2447

Parametrization formula:

$$\Gamma_Z^{\rm SM}=\Gamma_Z^0+c_1 L_H+c_2\Delta_t+c_3\Delta_{\alpha_{\rm s}}+c_4\Delta_{\alpha_{\rm s}}^2+c_5\Delta_{\alpha_{\rm s}}\Delta_t+c_6\Delta_\alpha+c_7\Delta_Z$$

$$L_H=\ln\frac{m_h}{125.7\ \text{GeV}},\quad \Delta_t=\left(\frac{m_t}{173.2\ \text{GeV}}\right)^2-1$$

$$\Delta_{\alpha_{\rm s}}=\frac{\alpha_{\rm s}(m_Z)}{0.1184}-1,\quad \Delta_\alpha=\frac{\Delta\alpha}{0.059}-1,\quad \Delta_Z=\frac{m_Z}{91.1876\ \text{GeV}}-1$$

$\Delta\alpha\equiv\Delta\alpha_{\rm lep}+\Delta\alpha_{\rm had}^{(5)},\quad \Delta\alpha_{\rm lep}=0.0314976$

Coefficient values can be found in the 9th row of Table 5

Least squares fitting

Ref: Fan, Reece & Wang, 1411.1054

Likelihood for n normal variables $x_i \sim N(\mu_i,\sigma_i)$

$$L=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left[-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2\right]$$

$$\ln L=\sum_{i=1}^{n}\left[-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2-\ln(\sqrt{2\pi}\sigma_i)\right]$$

$$\chi^2\equiv\sum_{i=1}^{n}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2 \quad\Rightarrow\quad \chi^2=-2\ln L+\text{const.}$$

Assume a flat theoretical uncertainty δ_j for a observable O_j with theoretical parameters $\{\alpha_k\}$:

$$P(O_j\,|\,\{\alpha_k\})=\begin{cases}\dfrac{1}{2\delta_j}, & \left|O_j-O_j^{\text{pred}}(\{\alpha_k\})\right|\le\delta_j\\ 0, & \left|O_j-O_j^{\text{pred}}(\{\alpha_k\})\right|>\delta_j\end{cases}$$

$O_j=$ true value, $O_j^{\text{pred}}=$ theoretically predicted value

Assume a normal experimental uncertainty σ_j for the measured value O_j^{meas}:

$$P(O_j^{\text{meas}}\,|\,O_j)=\frac{1}{\sqrt{2\pi}\sigma_j}\exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}}-O_j}{\sigma_j}\right)^2\right]$$

Extract the probability for O_j^{meas} in terms of $\{\alpha_k\}$ as a convolution, integrating out the unknown true value O_j:

$$P(O_j^{\text{meas}}\,|\,\{\alpha_k\})=\int dO_j P(O_j^{\text{meas}}\,|\,O_j)P(O_j\,|\,\{\alpha_k\})=q(O_j^{\text{meas}};O_j^{\text{pred}}(\{\alpha_k\}),\sigma_j,\delta_j) \quad\Rightarrow\quad \chi^2_{\text{mod}}\supset-2\ln q(O_j^{\text{meas}};O_j^{\text{pred}}(\{\alpha_k\}),\sigma_j,\delta_j)$$

$$q(x;\mu,\sigma,\delta)\equiv\frac{1}{4\delta}\left[\text{erf}\left(\frac{x-\mu+\delta}{\sqrt{2}\sigma}\right)-\text{erf}\left(\frac{x-\mu-\delta}{\sqrt{2}\sigma}\right)\right]$$

Cumulative distribution function for $x\sim N(\mu,\sigma)$: $\quad \text{CDF}(x;\mu,\sigma)=\Phi\left(\dfrac{x-\mu}{\sigma}\right)=\dfrac{1}{2}+\dfrac{1}{2}\text{erf}\left(\dfrac{x-\mu}{\sqrt{2}\sigma}\right),\quad \text{erf}(-x)=-\text{erf}(x)$

$$\int dO_j P(O_j^{\text{meas}}\,|\,O_j)P(O_j\,|\,\{\alpha_k\})=\int dO_j\frac{1}{\sqrt{2\pi}\sigma_j}\exp\left[-\frac{1}{2}\left(\frac{O_j^{\text{meas}}-O_j}{\sigma_j}\right)^2\right]\frac{\theta\left(\delta_j-\left|O_j-O_j^{\text{pred}}\right|\right)}{2\delta_j}$$

$$=\frac{1}{2\delta_j}\int_{O_j^{\text{pred}}-\delta_j}^{O_j^{\text{pred}}+\delta_j}dO_j\frac{1}{\sqrt{2\pi}\sigma_j}\exp\left[-\frac{1}{2}\left(\frac{O_j-O_j^{\text{meas}}}{\sigma_j}\right)^2\right]=\frac{1}{2\delta_j}\left[\Phi\left(\frac{O_j^{\text{pred}}+\delta_j-O_j^{\text{meas}}}{\sigma_j}\right)-\Phi\left(\frac{O_j^{\text{pred}}-\delta_j-O_j^{\text{meas}}}{\sigma_j}\right)\right]$$

$$=\frac{1}{4\delta_j}\left[\text{erf}\left(\frac{O_j^{\text{pred}}+\delta_j-O_j^{\text{meas}}}{\sqrt{2}\sigma_j}\right)-\text{erf}\left(\frac{O_j^{\text{pred}}-\delta_j-O_j^{\text{meas}}}{\sqrt{2}\sigma_j}\right)\right]=\frac{1}{4\delta_j}\left[\text{erf}\left(\frac{O_j^{\text{meas}}-O_j^{\text{pred}}+\delta_j}{\sqrt{2}\sigma_j}\right)-\text{erf}\left(\frac{O_j^{\text{meas}}-O_j^{\text{pred}}-\delta_j}{\sqrt{2}\sigma_j}\right)\right]$$

Define modified χ^2 for n observables O_i without theoretical uncertainties
and m observables O_j with flat theoretical uncertainties:

$$\chi^2_{\text{mod}}=\sum_{i=1}^{n}\left(\frac{O_i^{\text{meas}}-O_i^{\text{pred}}}{\sigma_i}\right)^2+\sum_{j=1}^{m}\left\{-2\ln\left[\text{erf}\left(\frac{O_j^{\text{meas}}-O_j^{\text{pred}}+\delta_j}{\sqrt{2}\sigma_j}\right)-\text{erf}\left(\frac{O_j^{\text{meas}}-O_j^{\text{pred}}-\delta_j}{\sqrt{2}\sigma_j}\right)\right]\right\}$$

Current data

$\alpha_s(m_Z^2) = 0.1185 \pm 0.0006$ [PDG 2014]

$\Delta\alpha_{\rm had}^{(5)}(m_Z^2) = 0.02765 \pm 0.00008$ [1209.4802]

$m_Z = 91.1875 \pm 0.0021$ GeV [hep-ex/0509008]

$m_t = 173.34 \pm 0.76_{\rm exp}$ [1403.4427] $\pm 0.5_{\rm the}$ [1412.4435] GeV

$m_h = 125.09 \pm 0.24$ GeV [1503.07589]

$m_W = 80.385 \pm 0.015_{\rm exp}$ [PDG 2014] $\pm 0.004_{\rm the}$ GeV [hep-ph/0311148]

$\sin^2\theta_{\rm eff}^{\ell} = 0.23153 \pm 0.00016$ [hep-ex/0509008]

$\Gamma_Z = 2.4952 \pm 0.0023$ GeV [hep-ex/0509008]

$\sigma_S = 0.10, \quad \sigma_T = 0.12, \quad \sigma_U = 0.094, \quad \rho_{ST} = +0.89, \quad \rho_{SU} = -0.55, \quad \rho_{TU} = -0.80$

$U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.085, \quad \sigma_T = 0.072, \quad \rho_{ST} = +0.90$

$S = 0$ fixed $\Rightarrow$ $\sigma_T = 0.054, \quad \sigma_U = 0.078, \quad \rho_{TU} = -0.81$

$T = U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.037$

$S = U = 0$ fixed $\Rightarrow$ $\sigma_T = 0.032$

CEPC baseline precision

$\alpha_s(m_Z^2)$: $\pm 1.0 \times 10^{-4}$ [1404.0319]

$\Delta\alpha_{\rm had}^{(5)}(m_Z^2)$: $\pm 4.7 \times 10^{-5}$ [1407.3792]

m_Z: $\pm 5 \times 10^{-4}$ GeV [CEPC-SPPC pre-CDR]

m_t: $\pm 0.2_{\rm exp}$ [CMS-PAS-FTR-13-017] $\pm 0.5_{\rm the}$ [1412.4435] GeV

m_h: $\pm 5.9 \times 10^{-3}$ GeV [CEPC-SPPC pre-CDR]

m_W: ($\pm 3_{\rm exp}$ [CEPC-SPPC pre-CDR] $\pm 1_{\rm the}$ [1307.3962]) $\times 10^{-3}$ GeV

$\sin^2\theta_{\rm eff}^{\ell}$: ($\pm 2.3_{\rm exp}$ [CEPC-SPPC pre-CDR] $\pm 1.5_{\rm the}$ [1307.3962]) $\times 10^{-5}$

Γ_Z: ($\pm 5_{\rm exp}$ [CEPC-SPPC pre-CDR] $\pm 0.8_{\rm the}$ [1411.1054, Mishima's talk]) $\times 10^{-4}$ GeV

$\sigma_S = 0.021, \quad \sigma_T = 0.026, \quad \sigma_U = 0.020, \quad \rho_{ST} = +0.90, \quad \rho_{SU} = -0.68, \quad \rho_{TU} = -0.84$

$U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.015, \quad \sigma_T = 0.014, \quad \rho_{ST} = +0.83$

$S = 0$ fixed $\Rightarrow$ $\sigma_T = 0.011, \quad \sigma_U = 0.015, \quad \rho_{TU} = -0.72$

$T = U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.0085$

$S = U = 0$ fixed $\Rightarrow$ $\sigma_T = 0.0079$

Potential improvements for CEPC

Reduced systematic uncertainty in the CEPC measurement

$\rightarrow$ m_Z: $\pm 1 \times 10^{-4}$ GeV [CEPC-SPPC pre-CDR]

Γ_Z: ($\pm 1_{\rm exp}$ [CEPC-SPPC pre-CDR] $\pm 0.8_{\rm the}$ [1411.1054, Mishima's talk]) $\times 10^{-4}$ GeV

ILC top threshold scan $\rightarrow$ m_t: $\pm 0.03_{\rm exp} \pm 0.1_{\rm the}$ GeV [1306.6352]

m_Z, Γ_Z & m_t improvements:

$\sigma_S = 0.011, \quad \sigma_T = 0.0071, \quad \sigma_U = 0.010, \quad \rho_{ST} = +0.74, \quad \rho_{SU} = +0.15, \quad \rho_{TU} = -0.21$

$U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.011, \quad \sigma_T = 0.0069, \quad \rho_{ST} = +0.80$

$S = 0$ fixed $\Rightarrow$ $\sigma_T = 0.0048, \quad \sigma_U = 0.010, \quad \rho_{TU} = -0.48$

$T = U = 0$ fixed $\Rightarrow$ $\sigma_S = 0.0068$

$S = U = 0$ fixed $\Rightarrow$ $\sigma_T = 0.0042$

DM models

Convention:

$v^i \in \mathbf{2}, \quad v_i \in \bar{\mathbf{2}}, \quad \varepsilon^{12} = +1, \quad \varepsilon_{12} = -1, \quad \varepsilon^{ij} = -\varepsilon^{ji}, \quad \varepsilon_{ij} = -\varepsilon_{ji}$

SM Higgs field $\quad H = H^i = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} G^+(x) \\ \frac{1}{\sqrt{2}}[v + h(x) + iG^0(x)] \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right)$ under $(\mathrm{SU}(2)_\mathrm{L}, \mathrm{U}(1)_\mathrm{Y})$

$$H_i = \varepsilon_{ij} H^j = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} -H^0 \\ H^+ \end{pmatrix} \in \left(\bar{\mathbf{2}}, \frac{1}{2}\right)$$

$$H_i^\dagger = (H^i)^* = \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} \in \left(\bar{\mathbf{2}}, -\frac{1}{2}\right)$$

$$\tilde{H} = H^{\dagger i} = \varepsilon^{ij} H_j^\dagger = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} H^- \\ H^{0*} \end{pmatrix} = \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right)$$

$$\left[\text{Note:} \quad H^\dagger H = H_i^\dagger H^i = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = H^+ H^- + H^0 H^{0*} = -H^{\dagger i} H_i \right]$$

Singlet-Doublet Fermionic Dark Matter (SDFDM)

Ref: D'Eramo, 0705.4493; Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604

Left-handed Weyl fermions:

$$S \in (\mathbf{1}, 0), \quad D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right)$$

$$\mathcal{L}_\mathrm{S} = iS^\dagger \bar{\sigma}^\mu \partial_\mu S - \frac{1}{2}(m_S SS + \text{h.c.}), \quad \mathcal{L}_\mathrm{D} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.})$$

$$\mathcal{L}_\mathrm{HSD} = y_1 H_i S D_1^i - y_2 H_i^\dagger S D_2^i + \text{h.c.}$$

$$-m_D \varepsilon_{ij} D_1^i D_2^j = m_D D_1^0 D_2^0 - m_D D_1^- D_2^+$$

$$y_1 H_i S D_1^i = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} S \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = -y_1 H^0 S D_1^0 + y_1 H^+ S D_1^- \to -\frac{1}{\sqrt{2}} y_1 (v+h) S D_1^0$$

$$-y_2 H_i^\dagger S D_2^i = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} S \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 H^- S D_2^+ - y_2 H^{0*} S D_2^0 \to -\frac{1}{\sqrt{2}} y_2 (v+h) S D_2^0$$

$$\mathcal{L}_\mathrm{mass} = -\frac{1}{2} \begin{pmatrix} S & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_\mathrm{N} \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} - m_D D_1^- D_2^+ + \text{h.c.} = -\frac{1}{2} \sum_{i=1}^{3} m_{\chi_i^0} \chi_i^0 \chi_i^0 - m_{\chi^\pm} \chi^- \chi^+ + \text{h.c.}$$

$$\mathcal{M}_\mathrm{N} = \begin{pmatrix} m_S & \frac{1}{\sqrt{2}} y_1 v & \frac{1}{\sqrt{2}} y_2 v \\ \frac{1}{\sqrt{2}} y_1 v & 0 & -m_D \\ \frac{1}{\sqrt{2}} y_2 v & -m_D & 0 \end{pmatrix}, \quad m_{\chi^\pm} = m_D, \quad \chi^+ = D_2^+, \quad \chi^- = D_1^-$$

$$\mathcal{N}^\mathrm{T} \mathcal{M}_\mathrm{N} \mathcal{N} = \mathrm{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \begin{pmatrix} S \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}$$

$y = y_1 = y_2 \quad \Rightarrow \quad$ <u>Custodial $\mathrm{SU(2)_R}$ global symmetry</u>

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an } \mathrm{SU(2)_R} \text{ indice}$$

$$H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = -\frac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = -\frac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j$$

$$\mathcal{L}_{\mathrm{HSD}} = y(H_i S D_1^i - H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S(\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_{\mathrm{D}} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A - \frac{1}{2}[m_D \varepsilon_{AB}\varepsilon_{ij}(\mathcal{D}^A)^i(\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_S \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(-D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_{\chi^\pm} = m_D \\ m_{\chi_2^0} = \frac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} + m_D - m_S\right] \\ m_{\chi_3^0} = \frac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} - m_D + m_S\right] \end{cases}$$

$$m_D > m_S \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = \frac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} - m_D + m_S\right] \\ m_{\chi_2^0} = m_{\chi^\pm} = m_D \\ m_{\chi_3^0} = \frac{1}{2}\left[\sqrt{(m_D+m_S)^2 + 4y^2v^2} + m_D - m_S\right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_S)}$$

$$\begin{bmatrix} y = y_1 = -y_2 \quad \Rightarrow \quad \text{Another custodial symmetry limit} \\ (\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} -H_i^\dagger \\ H_i \end{pmatrix} \\ H^\dagger H = H_i^\dagger H^i = \frac{1}{2}(\varepsilon^{ij} H_i^\dagger H_j - \varepsilon^{ij} H_i H_j^\dagger) = \frac{1}{2}[\varepsilon_{12}\varepsilon^{ij}(\mathcal{H}^1)_i(\mathcal{H}^2)_j + \varepsilon_{21}\varepsilon^{ij}(\mathcal{H}^2)_i(\mathcal{H}^1)_j] = \frac{1}{2}\varepsilon_{AB}\varepsilon^{ij}(\mathcal{H}^A)_i(\mathcal{H}^B)_j \\ \mathcal{L}_{\mathrm{HSD}} = y(H_i S D_1^i + H_i^\dagger S D_2^i) + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i S(\mathcal{D}^B)^j + \text{h.c.} \end{bmatrix}$$

<u>Gauge interactions</u>

$$D_\mu D_i = (\partial_\mu - ig' B_\mu Y_{D_i} - igW_\mu^a t_{\mathrm{D}}^a) D_i$$

$$Y_{D_1} = -\frac{1}{2}, \quad Y_{D_2} = \frac{1}{2}, \quad t_{\mathrm{D}}^1 = \frac{\sigma^1}{2} = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad t_{\mathrm{D}}^2 = \frac{\sigma^2}{2} = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad t_{\mathrm{D}}^3 = \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$g' B_\mu Y_{D_1} + gW_\mu^a t_{\mathrm{D}}^a = \frac{1}{2}\begin{pmatrix} -g'B_\mu + gW_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -g'B_\mu - gW_\mu^a \end{pmatrix} = \begin{pmatrix} \frac{g}{2c_{\mathrm{W}}} Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -eA_\mu + \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu \end{pmatrix}$$

$$g' B_\mu Y_{D_2} + gW_\mu^a t_{\mathrm{D}}^a = \frac{1}{2}\begin{pmatrix} g'B_\mu + gW_\mu^a & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & g'B_\mu - gW_\mu^a \end{pmatrix} = \begin{pmatrix} eA_\mu - \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{g}{2c_{\mathrm{W}}} Z_\mu \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_{\mathrm{D}} \supset{}& iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 \\ ={}& \frac{g}{2c_{\mathrm{W}}} Z_\mu (D_1^0)^\dagger \bar{\sigma}^\mu D_1^0 + \left[-eA_\mu + \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu\right](D_1^-)^\dagger \bar{\sigma}^\mu D_1^- + \frac{g}{\sqrt{2}}[W_\mu^+ (D_1^0)^\dagger \bar{\sigma}^\mu D_1^- + W_\mu^- (D_1^-)^\dagger \bar{\sigma}^\mu D_1^0] \\ &+ \left[eA_\mu - \frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2 - c_{\mathrm{W}}^2) Z_\mu\right](D_2^+)^\dagger \bar{\sigma}^\mu D_2^+ - \frac{g}{2c_{\mathrm{W}}} Z_\mu (D_2^0)^\dagger \bar{\sigma}^\mu D_2^0 + \frac{g}{\sqrt{2}}[W_\mu^+ (D_2^+)^\dagger \bar{\sigma}^\mu D_2^0 + W_\mu^- (D_2^0)^\dagger \bar{\sigma}^\mu D_2^+] \end{aligned}$$

$$X_i^0=\begin{pmatrix}(\chi_{iL}^0)_\alpha\\(\chi_{iR}^0)^{\dagger\dot\alpha}\end{pmatrix}=\begin{pmatrix}\chi_{iL}^0\\(\chi_{iR}^0)^\dagger\end{pmatrix},\quad \chi_L^0=\chi_R^0=\mathcal{N}^\dagger\psi_L^0=\mathcal{N}^\dagger\psi_R^0=\begin{pmatrix}\chi_1^0 & \chi_2^0 & \chi_3^0\end{pmatrix}^T,\quad \bar{X}_i^0=\begin{pmatrix}\chi_{iR}^0 & (\chi_{iL}^0)^\dagger\end{pmatrix}$$

$$\Psi_i^0=\begin{pmatrix}\psi_{iL}^0\\(\psi_{iR}^0)^\dagger\end{pmatrix},\quad \psi_L^0=\psi_R^0=\mathcal{N}\chi_L^0=\mathcal{N}\chi_R^0=\begin{pmatrix}S & D_1^0 & D_2^0\end{pmatrix}^T,\quad \bar\Psi_i^0=\begin{pmatrix}\psi_{iR}^0 & (\psi_{iL}^0)^\dagger\end{pmatrix}$$

$$X^+=\begin{pmatrix}\chi^+\\(\chi^-)^\dagger\end{pmatrix},\quad \chi^+=D_2^+,\quad \chi^-=D_1^-$$

$$\Psi_{iL}^0=\begin{pmatrix}\psi_{iL}^0\\0\end{pmatrix}=\begin{pmatrix}(\mathcal{N}\chi_L^0)_i\\0\end{pmatrix}=\mathcal{N}_{ij}X_{jL}^0,\quad \Psi_{iR}^0=\begin{pmatrix}0\\(\psi_{iR}^0)^\dagger\end{pmatrix}=\begin{pmatrix}0\\(\mathcal{N}\chi_R^0)_i^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*X_{jR}^0$$

$$\bar\Psi_{iL}^0=\begin{pmatrix}0 & (\psi_{iL}^0)^\dagger\end{pmatrix}=\mathcal{N}_{ij}^*\bar{X}_{jL}^0,\quad \bar\Psi_{iR}^0=\begin{pmatrix}\psi_{iR}^0 & 0\end{pmatrix}=\mathcal{N}_{ij}\bar{X}_{jR}^0$$

$$\bar\Psi_{iL}^0\gamma^\mu\Psi_{iL}^0=\begin{pmatrix}0 & (\psi_{iL}^0)^\dagger\end{pmatrix}\begin{pmatrix} & \sigma^\mu\\ \bar\sigma^\mu & \end{pmatrix}\begin{pmatrix}\psi_{iL}^0\\0\end{pmatrix}=(\psi_{iL}^0)^\dagger\bar\sigma^\mu\psi_{iL}^0$$

$$\bar\Psi_{iR}^0\gamma^\mu\Psi_{iR}^0=\begin{pmatrix}\psi_{iR}^0 & 0\end{pmatrix}\begin{pmatrix} & \sigma^\mu\\ \bar\sigma^\mu & \end{pmatrix}\begin{pmatrix}0\\(\psi_{iR}^0)^\dagger\end{pmatrix}=\psi_{iR}^0\sigma^\mu(\psi_{iR}^0)^\dagger=-(\psi_{iR}^0)^\dagger\bar\sigma^\mu\psi_{iR}^0=-(\psi_{iL}^0)^\dagger\bar\sigma^\mu\psi_{iL}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0}=\frac{1}{2}a_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar\Psi_{iL}^0\gamma^\mu\Psi_{iL}^0+\frac{1}{2}b_{Z\Psi_i^0\Psi_i^0}Z_\mu\bar\Psi_{iR}^0\gamma^\mu\Psi_{iR}^0$$

$$=\frac{1}{2}a_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}^*\mathcal{N}_{kj}Z_\mu\bar{X}_{iL}^0\gamma^\mu X_{jL}^0+\frac{1}{2}b_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}\mathcal{N}_{kj}^*Z_\mu\bar{X}_{iR}^0\gamma^\mu X_{jR}^0=\frac{1}{2}(a_{ZX_i^0X_j^0}Z_\mu\bar{X}_{iL}^0\gamma^\mu X_{jL}^0+b_{ZX_i^0X_j^0}Z_\mu\bar{X}_{iR}^0\gamma^\mu X_{jR}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0}=b_{Z\Psi_1^0\Psi_1^0}=0,\quad a_{Z\Psi_2^0\Psi_2^0}=-b_{Z\Psi_2^0\Psi_2^0}=\frac{g}{2c_W},\quad a_{Z\Psi_3^0\Psi_3^0}=-b_{Z\Psi_3^0\Psi_3^0}=-\frac{g}{2c_W}$$

$$a_{ZX_i^0X_j^0}=a_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}^*\mathcal{N}_{kj},\quad b_{ZX_i^0X_j^0}=b_{Z\Psi_k^0\Psi_k^0}\mathcal{N}_{ki}\mathcal{N}_{kj}^*$$

$$\left(\begin{array}{l}
\text{Note: } a_{ZX_j^0X_i^0}=a^*_{ZX_i^0X_j^0},\quad b_{ZX_j^0X_i^0}=b^*_{ZX_i^0X_j^0},\quad a_{ZX_i^0X_j^0}=-b_{ZX_j^0X_i^0}\\
C^{-1}=-C,\quad X_i^0=C(\bar{X}_i^0)^T,\quad \bar{X}_i^0=(X_i^0)^TC\\
C^{-1}(\gamma^\mu P_L)^TC=\frac{1}{2}C^{-1}(\gamma^\mu)^TC-\frac{1}{2}C^{-1}(\gamma^\mu\gamma_5)^TC=-\frac{1}{2}\gamma^\mu-\frac{1}{2}\gamma^\mu\gamma_5=-\gamma^\mu P_R\\
a_{ZX_1^0X_2^0}\bar{X}_{1L}^0\gamma^\mu X_{2L}^0=a_{ZX_1^0X_2^0}(\bar{X}_1^0\gamma^\mu P_L X_2^0)^T=-a_{ZX_1^0X_2^0}(X_2^0)^TCC^{-1}(\gamma^\mu P_L)^TC^{-1}C(\bar{X}_1^0)^T=-a_{ZX_1^0X_2^0}\bar{X}_2^0C^{-1}(\gamma^\mu P_L)^TC^{-1}X_1^0\\
=a_{ZX_1^0X_2^0}\bar{X}_2^0C^{-1}(\gamma^\mu P_L)^TCX_1^0=-a_{ZX_1^0X_2^0}\bar{X}_2^0\gamma^\mu P_R X_1^0=b_{ZX_2^0X_1^0}\bar{X}_{2R}^0\gamma^\mu X_{1R}^0\\
\frac{1}{2}a_{ZX_1^0X_2^0}Z_\mu\bar{X}_{1L}^0\gamma^\mu X_{2L}^0 \text{ and } \frac{1}{2}b_{ZX_2^0X_1^0}Z_\mu\bar{X}_{2R}^0\gamma^\mu X_{1R}^0 \text{ give an identical vertex!}
\end{array}\right)$$

$$\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}X^{+}_{\mathrm{L}}=\begin{pmatrix}0 & (\chi^{+})^{\dagger}\end{pmatrix}\begin{pmatrix} & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & \end{pmatrix}\begin{pmatrix}\chi^{+}\\ 0\end{pmatrix}=(D_2^{+})^{\dagger}\bar{\sigma}^{\mu}D_2^{+}$$

$$\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}X^{+}_{\mathrm{R}}=\begin{pmatrix}\chi^{-} & 0\end{pmatrix}\begin{pmatrix} & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & \end{pmatrix}\begin{pmatrix}0\\ (\chi^{-})^{\dagger}\end{pmatrix}=\chi^{-}\sigma^{\mu}(\chi^{-})^{\dagger}=-(\chi^{-})^{\dagger}\bar{\sigma}^{\mu}\chi^{-}=-(D_1^{-})^{\dagger}\bar{\sigma}^{\mu}D_1^{-}$$

$$\mathcal{L}_{AX^+X^+}=a_{AX^+X^+}A_{\mu}\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}X^{+}_{\mathrm{L}}+b_{AX^+X^+}A_{\mu}\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}X^{+}_{\mathrm{R}}$$

$$a_{AX^+X^+}=b_{AX^+X^+}=e$$

$$\mathcal{L}_{ZX^+X^+}=a_{ZX^+X^+}Z_{\mu}\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}X^{+}_{\mathrm{L}}+b_{ZX^+X^+}Z_{\mu}\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}X^{+}_{\mathrm{R}}$$

$$a_{ZX^+X^+}=b_{ZX^+X^+}=-\frac{g}{2c_{\mathrm{W}}}(s_{\mathrm{W}}^2-c_{\mathrm{W}}^2)$$

$$\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}\Psi^{0}_{i\mathrm{L}}=\begin{pmatrix}0 & (\chi^{+})^{\dagger}\end{pmatrix}\begin{pmatrix} & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & \end{pmatrix}\begin{pmatrix}\psi^{0}_{i\mathrm{L}}\\ 0\end{pmatrix}=(\chi^{+})^{\dagger}\bar{\sigma}^{\mu}\psi^{0}_{i\mathrm{L}}$$

$$\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}\Psi^{0}_{i\mathrm{R}}=\begin{pmatrix}\chi^{-} & 0\end{pmatrix}\begin{pmatrix} & \sigma^{\mu}\\ \bar{\sigma}^{\mu} & \end{pmatrix}\begin{pmatrix}0\\ (\psi^{0}_{i\mathrm{R}})^{\dagger}\end{pmatrix}=\chi^{-}\sigma^{\mu}(\psi^{0}_{i\mathrm{R}})^{\dagger}=-(\psi^{0}_{i\mathrm{R}})^{\dagger}\bar{\sigma}^{\mu}\chi^{-}$$

$$\mathcal{L}_{WX^+\Psi_i^0}=a_{WX^+\Psi_i^0}(W^{+}_{\mu}\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}\Psi^{0}_{i\mathrm{L}}+\mathrm{h.c.})+b_{WX^+\Psi_i^0}(W^{+}_{\mu}\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}\Psi^{0}_{i\mathrm{R}}+\mathrm{h.c.})$$

$$=a_{WX^+\Psi_i^0}[\mathcal{N}_{ij}W^{+}_{\mu}\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}X^{0}_{j\mathrm{L}}+\mathrm{h.c.}]+b_{WX^+\Psi_i^0}[\mathcal{N}^{*}_{ij}W^{+}_{\mu}\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}X^{0}_{j\mathrm{R}}+\mathrm{h.c.}]$$

$$=a_{WX^+X_i^0}W^{+}_{\mu}\bar{X}^{+}_{\mathrm{L}}\gamma^{\mu}X^{0}_{i\mathrm{L}}+a^{*}_{WX^+X_i^0}W^{-}_{\mu}\bar{X}^{0}_{i\mathrm{L}}\gamma^{\mu}X^{+}_{\mathrm{L}}+b_{WX^+X_i^0}W^{+}_{\mu}\bar{X}^{+}_{\mathrm{R}}\gamma^{\mu}X^{0}_{i\mathrm{R}}+b^{*}_{WX^+X_i^0}W^{-}_{\mu}\bar{X}^{0}_{i\mathrm{R}}\gamma^{\mu}X^{+}_{\mathrm{R}}$$

$$b_{WX^+\Psi_2^0}=-\frac{g}{\sqrt{2}},\quad a_{WX^+\Psi_3^0}=\frac{g}{\sqrt{2}},\quad \text{others}=0$$

$$a_{WX^+X_i^0}=a_{WX^+\Psi_i^0}\mathcal{N}_{ij},\quad b_{WX^+X_i^0}=b_{WX^+\Psi_i^0}\mathcal{N}^{*}_{ij}$$

$$\bar{\Psi}^{0}_{i\mathrm{R}}\Psi^{0}_{j\mathrm{L}}=\begin{pmatrix}\psi^{0}_{i\mathrm{R}} & 0\end{pmatrix}\begin{pmatrix}\psi^{0}_{j\mathrm{L}}\\ 0\end{pmatrix}=\psi^{0}_{i\mathrm{R}}\psi^{0}_{j\mathrm{L}},\quad \bar{\Psi}^{0}_{i\mathrm{L}}\Psi^{0}_{j\mathrm{R}}=\begin{pmatrix}0 & (\psi^{0}_{i\mathrm{L}})^{\dagger}\end{pmatrix}\begin{pmatrix}0\\ (\psi^{0}_{j\mathrm{R}})^{\dagger}\end{pmatrix}=(\psi^{0}_{i\mathrm{L}})^{\dagger}(\psi^{0}_{j\mathrm{R}})^{\dagger}$$

$$\mathcal{L}_{h\Psi_i^0\Psi_j^0}=\frac{1}{2}a_{h\Psi_i^0\Psi_j^0}h\bar{\Psi}^{0}_{i\mathrm{R}}\Psi^{0}_{j\mathrm{L}}+\frac{1}{2}b_{h\Psi_i^0\Psi_j^0}h\bar{\Psi}^{0}_{i\mathrm{L}}\Psi^{0}_{j\mathrm{R}}$$

$$=\frac{1}{2}a_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}\mathcal{N}_{lj}h\bar{X}^{0}_{i\mathrm{R}}X^{0}_{j\mathrm{L}}+\frac{1}{2}b_{h\Psi_k^0\Psi_l^0}\mathcal{N}^{*}_{ki}\mathcal{N}^{*}_{lj}h\bar{X}^{0}_{i\mathrm{L}}X^{0}_{j\mathrm{R}}=\frac{1}{2}(a_{hX_i^0X_j^0}h\bar{X}^{0}_{i\mathrm{R}}X^{0}_{j\mathrm{L}}+b_{hX_i^0X_j^0}h\bar{X}^{0}_{i\mathrm{L}}X^{0}_{j\mathrm{R}})$$

$$a_{h\Psi_1^0\Psi_2^0}=b_{h\Psi_1^0\Psi_2^0}=-\frac{y_1}{\sqrt{2}}=a_{h\Psi_2^0\Psi_1^0}=b_{h\Psi_2^0\Psi_1^0},\quad a_{h\Psi_1^0\Psi_3^0}=b_{h\Psi_1^0\Psi_3^0}=-\frac{y_2}{\sqrt{2}}=a_{h\Psi_3^0\Psi_1^0}=b_{h\Psi_3^0\Psi_1^0},\quad \text{others}=0$$

$$a_{hX_i^0X_j^0}=a_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{ki}\mathcal{N}_{lj},\quad b_{hX_i^0X_j^0}=b_{h\Psi_k^0\Psi_l^0}\mathcal{N}^{*}_{ki}\mathcal{N}^{*}_{lj}$$

$$\left(\begin{array}{l}
\text{Note: } a_{hX_j^0X_i^0}=a_{h\Psi_k^0\Psi_l^0}\mathcal{N}_{kj}\mathcal{N}_{li}=a_{h\Psi_l^0\Psi_k^0}\mathcal{N}_{kj}\mathcal{N}_{li}=a_{hX_i^0X_j^0},\quad b_{hX_j^0X_i^0}=b_{hX_i^0X_j^0},\quad a_{hX_i^0X_j^0}=b^{*}_{hX_i^0X_j^0}\\
C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C=\frac{1}{2}C^{-1}(1-\gamma_5)^{\mathrm{T}}C=\frac{1}{2}(1-\gamma_5)=P_{\mathrm{L}}\\
a_{hX_1^0X_2^0}\bar{X}^{0}_{1\mathrm{R}}X^{0}_{2\mathrm{L}}=a_{hX_1^0X_2^0}(\bar{X}^{0}_{1}P_{\mathrm{L}}X^{0}_{2})^{\mathrm{T}}=-a_{hX_1^0X_2^0}(X^{0}_{2})^{\mathrm{T}}(P_{\mathrm{L}})^{\mathrm{T}}(\bar{X}^{0}_{1})^{\mathrm{T}}=-a_{hX_1^0X_2^0}(X^{0}_{2})^{\mathrm{T}}CC^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C^{-1}C(\bar{X}^{0}_{1})^{\mathrm{T}}=-a_{hX_1^0X_2^0}\bar{X}^{0}_{2}C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}C^{-1}X^{0}_{1}\\
=a_{hX_1^0X_2^0}\bar{X}^{0}_{2}C^{-1}(P_{\mathrm{L}})^{\mathrm{T}}CX^{0}_{1}=a_{hX_1^0X_2^0}\bar{X}^{0}_{2}P_{\mathrm{L}}X^{0}_{1}=a_{hX_2^0X_1^0}\bar{X}^{0}_{2\mathrm{R}}X^{0}_{1\mathrm{L}}\\
\frac{1}{2}a_{hX_1^0X_2^0}h\bar{X}^{0}_{1\mathrm{R}}X^{0}_{2\mathrm{L}}\left(\frac{1}{2}b_{hX_1^0X_2^0}h\bar{X}^{0}_{1\mathrm{L}}X^{0}_{2\mathrm{R}}\right)\text{ and }\frac{1}{2}a_{hX_2^0X_1^0}h\bar{X}^{0}_{2\mathrm{R}}X^{0}_{1\mathrm{L}}\left(\frac{1}{2}b_{hX_2^0X_1^0}h\bar{X}^{0}_{2\mathrm{L}}X^{0}_{1\mathrm{R}}\right)\text{ give an identical vertex!}
\end{array}\right)$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$a_{h\Psi_i^0\Psi_j^0} = b_{h\Psi_i^0\Psi_j^0}$$

$$\mathcal{L}_{hX_i^0X_j^0} = \frac{1}{2}(a_{hX_i^0X_j^0} h\bar{X}_{i\mathrm{R}}^0 X_{j\mathrm{L}}^0 + b_{hX_i^0X_j^0} h\bar{X}_{i\mathrm{L}}^0 X_{j\mathrm{R}}^0) = \frac{1}{2} h\bar{X}_i^0 (a_{hX_i^0X_j^0} P_\mathrm{L} + b_{hX_i^0X_j^0} P_\mathrm{R}) X_j^0$$

$$= \frac{1}{4}(a_{hX_i^0X_j^0} + b_{hX_i^0X_j^0}) h\bar{X}_i^0 X_j^0 + \frac{1}{4}(b_{hX_i^0X_j^0} - a_{hX_i^0X_j^0}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}\mathcal{N}_{lj} + \mathcal{N}_{ki}^*\mathcal{N}_{lj}^*) h\bar{X}_i^0 X_j^0 + \frac{1}{4} a_{h\Psi_k^0\Psi_l^0} (\mathcal{N}_{ki}^*\mathcal{N}_{lj}^* - \mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 \gamma_5 X_j^0$$

$$= \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 X_j^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Im}(\mathcal{N}_{ki}\mathcal{N}_{lj}) h\bar{X}_i^0 i\gamma_5 X_j^0$$

$$\operatorname{Im}(\mathcal{N}_{ki}\mathcal{N}_{li}) = 0$$

$$\mathcal{L}_{hX_1^0X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 - \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Im}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 i\gamma_5 X_1^0$$

$$= [a_{h\Psi_1^0\Psi_2^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= -\frac{1}{\sqrt{2}}[y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0X_1^0} = \frac{1}{2}(a_{hX_1^0X_1^0} + b_{hX_1^0X_1^0}) = -\sqrt{2}[y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

Effective operators: $\mathcal{L}_{S,q} = \sum_q G_{S,q} \bar{X}_1^0 X_1^0 \bar{q}q, \quad \mathcal{L}_{S,N} = \sum_{N=p,n} G_{S,N} \bar{X}_1^0 X_1^0 \bar{N}N$

$$G_{S,N} = m_N \left(\sum_{q=u,d,s} \frac{G_{S,q}}{m_q} f_q^N + \sum_{q=c,b,t} \frac{G_{S,q}}{m_q} f_Q^N \right), \quad f_Q^N = \frac{2}{27}\left(1 - \sum_{q=u,d,s} f_q^N\right)$$

hep-ph/0001005:

$$f_u^p = 0.020 \pm 0.004, \quad f_d^p = 0.026 \pm 0.005, \quad f_u^n = 0.014 \pm 0.003, \quad f_d^n = 0.036 \pm 0.008, \quad f_s^p = f_s^n = 0.118 \pm 0.062$$

$$\Rightarrow \quad f_Q^p = 0.0619, \quad f_Q^n = 0.0616$$

$$G_{S,q} = -\frac{g_{hX_1^0X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0X_1^0} m_N}{2vm_h^2} \left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N \right) \quad \Rightarrow \quad G_{S,n} \simeq G_{S,p}$$

$$\sigma_{\chi N}^{\mathrm{SI}} = \frac{4}{\pi} \mu_{\chi N}^2 G_{S,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$a_{Z\Psi_k^0\Psi_k^0} = -b_{Z\Psi_k^0\Psi_k^0}$$

$$\mathcal{L}_{ZX_i^0X_j^0} = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{L}}^0 \gamma^\mu X_{j\mathrm{L}}^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{R}}^0 \gamma^\mu X_{j\mathrm{R}}^0) = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_\mathrm{L} X_j^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_i^0 \gamma^\mu P_\mathrm{R} X_j^0)$$

$$= \frac{1}{4}(a_{ZX_i^0X_j^0} + b_{ZX_i^0X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{ZX_i^0X_j^0} - a_{ZX_i^0X_j^0}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4}(a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj} + b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 + \frac{1}{4}(b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^* - a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$= \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki}^* \mathcal{N}_{kj} - \mathcal{N}_{ki} \mathcal{N}_{kj}^*) Z_\mu \bar{X}_i^0 \gamma^\mu X_j^0 - \frac{1}{4} a_{Z\Psi_k^0\Psi_k^0} (\mathcal{N}_{ki} \mathcal{N}_{kj}^* + \mathcal{N}_{ki}^* \mathcal{N}_{kj}) Z_\mu \bar{X}_i^0 \gamma^\mu \gamma_5 X_j^0$$

$$\mathcal{L}_{ZX_1^0X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0X_1^0} = \frac{1}{2}(b_{ZX_1^0X_1^0} - a_{ZX_1^0X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

Effective operators: $\mathcal{L}_{A,q} = \sum_q G_{A,q} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{q} \gamma^\mu \gamma_5 q, \quad \mathcal{L}_{A,N} = \sum_{N=p,n} G_{A,N} \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \bar{N} \gamma^\mu \gamma_5 N$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q} \Delta_q^N$$

hep-ex/0609039:

$$\Delta_u^p = \Delta_d^n = 0.842 \pm 0.012, \quad \Delta_d^p = \Delta_u^n = -0.427 \pm 0.013, \quad \Delta_s^p = \Delta_s^n = -0.085 \pm 0.018$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0X_1^0}}{4 c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$\sigma_{\chi N}^{\mathrm{SD}} = \frac{12}{\pi} \mu_{\chi N}^2 G_{A,N}^2, \quad \mu_{\chi N} \equiv \frac{m_\chi m_N}{m_\chi + m_N}$$

Doublet-Triplet Fermionic Dark Matter (DTFDM)

Ref: Dedes & Karamitros, 1403.7744

Left-handed Weyl fermions:

$$D_1 = \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} \in \left(\mathbf{2}, -\frac{1}{2}\right), \quad D_2 = \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} \in \left(\mathbf{2}, \frac{1}{2}\right), \quad T = \begin{pmatrix} T^+ \\ T^0 \\ T^- \end{pmatrix} \in (\mathbf{3}, 0)$$

$$\mathcal{L}_{\mathrm{D}} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 + (m_D \varepsilon_{ij} D_1^i D_2^j + \mathrm{h.c.}), \quad \mathcal{L}_{\mathrm{T}} = iT^\dagger \bar{\sigma}^\mu D_\mu T - \frac{1}{2}(m_T T^a T^a + \mathrm{h.c.})$$

$$\mathcal{L}_{\mathrm{HDT}} = y_1 H_i T^a (\sigma^a)^i_{\ j} D_1^j - y_2 H_i^\dagger T^a (\sigma^a)^i_{\ j} D_2^j + \mathrm{h.c.}, \quad i,j = 1,2, \quad a = 1,2,3$$

$$m_D \varepsilon_{ij} D_1^i D_2^j = -m_D D_1^0 D_2^0 + m_D D_1^- D_2^+$$

$$T^\pm = -\frac{1}{\sqrt{2}}(T^1 \mp iT^2), \quad T^0 = T^3, \quad T^1 = -\frac{1}{\sqrt{2}}(T^+ + T^-), \quad T^2 = -\frac{i}{\sqrt{2}}(T^+ - T^-)$$

$$-\frac{1}{2} m_T T^a T^a = -\frac{1}{2} m_T \left[\frac{1}{2}(T^+ + T^-)^2 - \frac{1}{2}(T^+ - T^-)^2 + T^0 T^0 \right] = -m_T T^- T^+ - \frac{1}{2} m_T T^0 T^0$$

$$T^a \sigma^a = \begin{pmatrix} T^3 & T^1 - iT^2 \\ T^1 + iT^2 & -T^3 \end{pmatrix} = \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix}$$

$$y_1 H_i T^a (\sigma^a)^i_{\ j} D_1^j = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_1^0 \\ D_1^- \end{pmatrix} = y_1 \begin{pmatrix} -H^0 & H^+ \end{pmatrix} \begin{pmatrix} T^0 D_1^0 - \sqrt{2} T^+ D_1^- \\ -\sqrt{2} T^- D_1^0 - T^0 D_1^- \end{pmatrix}$$

$$= y_1 (-H^0 T^0 D_1^0 + \sqrt{2} H^0 T^+ D_1^- - \sqrt{2} H^+ T^- D_1^0 - H^+ T^0 D_1^-) \to -\frac{1}{\sqrt{2}} y_1 (v+h) T^0 D_1^0 + y_1 (v+h) T^+ D_1^-$$

$$-y_2 H_i^\dagger T^a (\sigma^a)^i_{\ j} D_2^j = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 & -\sqrt{2}T^+ \\ -\sqrt{2}T^- & -T^0 \end{pmatrix} \begin{pmatrix} D_2^+ \\ D_2^0 \end{pmatrix} = -y_2 \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} T^0 D_2^+ - \sqrt{2} T^+ D_2^0 \\ -\sqrt{2} T^- D_2^+ - T^0 D_2^0 \end{pmatrix}$$

$$= -y_2 (H^- T^0 D_2^+ - \sqrt{2} H^- T^+ D_2^0 - \sqrt{2} H^{0*} T^- D_2^+ - H^{0*} T^0 D_2^0) \to y_2 (v+h) T^- D_2^+ + \frac{1}{\sqrt{2}} y_2 (v+h) T^0 D_2^0$$

$$\left[\begin{aligned}
&\text{Note: } T^i_{\ j} = u^i v_j - \frac{1}{2}\delta^i_j u^k v_k = \frac{1}{\sqrt{2}} T^a (\sigma^a)^i_{\ j} \\
&T^+ = -T^1_{\ 2}, \quad T^- = -T^2_{\ 1}, \quad T^0 = \sqrt{2} T^1_{\ 1} = -\sqrt{2} T^2_{\ 2} \quad \Rightarrow \quad -\frac{1}{2} m_T T^j_{\ i} T^i_{\ j} = -m_T T^- T^+ - \frac{1}{2} m_T T^0 T^0 \\
&\sqrt{2} y_1 H_i T^i_{\ j} D_1^j = \sqrt{2} y_1 (H_1 T^1_{\ 1} D_1^1 + H_1 T^1_{\ 2} D_1^2 + H_2 T^2_{\ 1} D_1^1 + H_2 T^2_{\ 2} D_1^2) \\
&\quad = \sqrt{2} y_1 \left(-\frac{1}{\sqrt{2}} H^0 T^0 D_1^0 + H^0 T^+ D_1^- - H^+ T^- D_1^0 - \frac{1}{\sqrt{2}} H^+ T^0 D_1^- \right) \\
&\quad = y_1 (-H^0 T^0 D_1^0 + \sqrt{2} H^0 T^+ D_1^- - \sqrt{2} H^+ T^- D_1^0 - H^+ T^0 D_1^-) \\
&-\sqrt{2} y_2 H_i^\dagger T^i_{\ j} D_2^j = -\sqrt{2} y_2 (H_1^\dagger T^1_{\ 1} D_2^1 + H_1^\dagger T^1_{\ 2} D_2^2 + H_2^\dagger T^2_{\ 1} D_2^1 + H_2^\dagger T^2_{\ 2} D_2^2) \\
&\quad = -\sqrt{2} y_2 \left(\frac{1}{\sqrt{2}} H^- T^0 D_2^+ - H^- T^+ D_2^0 - H^{0*} T^- D_2^+ - \frac{1}{\sqrt{2}} H^{0*} T^0 D_2^0 \right) \\
&\quad = -y_2 (H^- T^0 D_2^+ - \sqrt{2} H^- T^+ D_2^0 - \sqrt{2} H^{0*} T^- D_2^+ - H^{0*} T^0 D_2^0)
\end{aligned}\right]$$

$$\mathcal{L}_{\mathrm{mass}} = -\frac{1}{2} \begin{pmatrix} T^0 & D_1^0 & D_2^0 \end{pmatrix} \mathcal{M}_{\mathrm{N}} \begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} - \begin{pmatrix} T^- & D_1^- \end{pmatrix} \mathcal{M}_{\mathrm{C}} \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} + \mathrm{h.c.} = -\frac{1}{2} \sum_{i=1}^{3} m_{\chi_i^0} \chi_i^0 \chi_i^0 - \sum_{i=1}^{2} m_{\chi_i^\pm} \chi_i^- \chi_i^+ + \mathrm{h.c.}$$

$$\mathcal{M}_{\mathrm{N}} = \begin{pmatrix} m_T & \frac{1}{\sqrt{2}} y_1 v & -\frac{1}{\sqrt{2}} y_2 v \\ \frac{1}{\sqrt{2}} y_1 v & 0 & m_D \\ -\frac{1}{\sqrt{2}} y_2 v & m_D & 0 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{C}} = \begin{pmatrix} m_T & -y_2 v \\ -y_1 v & -m_D \end{pmatrix}$$

$$\mathcal{N}^{\mathrm{T}} \mathcal{M}_{\mathrm{N}} \mathcal{N} = \mathrm{diag}(m_{\chi_1^0}, m_{\chi_2^0}, m_{\chi_3^0}), \quad \mathcal{C}_{\mathrm{R}}^{\mathrm{T}} \mathcal{M}_{\mathrm{C}} \mathcal{C}_{\mathrm{L}} = \mathrm{diag}(m_{\chi_1^\pm}, m_{\chi_2^\pm}), \quad \mathcal{N}^{-1} = \mathcal{N}^\dagger, \quad \mathcal{C}_{\mathrm{L}}^{-1} = \mathcal{C}_{\mathrm{L}}^\dagger, \quad \mathcal{C}_{\mathrm{R}}^{-1} = \mathcal{C}_{\mathrm{R}}^\dagger$$

$$\mathcal{C}_{\mathrm{L}}^\dagger \mathcal{M}_{\mathrm{C}}^\dagger \mathcal{M}_{\mathrm{C}} \mathcal{C}_{\mathrm{L}} = (\mathcal{C}_{\mathrm{L}}^\dagger \mathcal{M}_{\mathrm{C}}^\dagger \mathcal{C}_{\mathrm{R}}^*)(\mathcal{C}_{\mathrm{R}}^{\mathrm{T}} \mathcal{M}_{\mathrm{C}} \mathcal{C}_{\mathrm{L}}) = \mathrm{diag}(m^2_{\chi_1^\pm}, m^2_{\chi_2^\pm}), \quad \mathcal{C}_{\mathrm{R}}^{\mathrm{T}} \mathcal{M}_{\mathrm{C}} \mathcal{M}_{\mathrm{C}}^\dagger \mathcal{C}_{\mathrm{R}}^* = (\mathcal{C}_{\mathrm{R}}^{\mathrm{T}} \mathcal{M}_{\mathrm{C}} \mathcal{C}_{\mathrm{L}})(\mathcal{C}_{\mathrm{L}}^\dagger \mathcal{M}_{\mathrm{C}}^\dagger \mathcal{C}_{\mathrm{R}}^*) = \mathrm{diag}(m^2_{\chi_1^\pm}, m^2_{\chi_2^\pm})$$

$$\begin{pmatrix} T^0 \\ D_1^0 \\ D_2^0 \end{pmatrix} = \mathcal{N} \begin{pmatrix} \chi_1^0 \\ \chi_2^0 \\ \chi_3^0 \end{pmatrix}, \quad \begin{pmatrix} T^+ \\ D_2^+ \end{pmatrix} = \mathcal{C}_{\mathrm{L}} \begin{pmatrix} \chi_1^+ \\ \chi_2^+ \end{pmatrix}, \quad \begin{pmatrix} T^- \\ D_1^- \end{pmatrix} = \mathcal{C}_{\mathrm{R}} \begin{pmatrix} \chi_1^- \\ \chi_2^- \end{pmatrix}$$

$y = y_1 = y_2 \quad \Rightarrow \quad$ <u>Custodial $\mathrm{SU}(2)_\mathrm{R}$ global symmetry</u>

$$(\mathcal{D}^A)^i = \begin{pmatrix} D_1^i \\ D_2^i \end{pmatrix}, \quad (\mathcal{H}^A)_i = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad A \text{ is an } \mathrm{SU}(2)_\mathrm{R} \text{ indice}$$

$$\mathcal{L}_{\mathrm{HDT}} = y[H_i T^a (\sigma^a)^i_j D_1^j - H_i^\dagger T^a (\sigma^a)^i_j D_2^j] + \text{h.c.} = y\varepsilon_{AB}(\mathcal{H}^A)_i T^a (\sigma^a)^i_j (\mathcal{D}^B)^j + \text{h.c.}$$

$$\mathcal{L}_{\mathrm{D}} = iD_1^\dagger \bar{\sigma}^\mu D_\mu D_1 + iD_2^\dagger \bar{\sigma}^\mu D_\mu D_2 - (m_D \varepsilon_{ij} D_1^i D_2^j + \text{h.c.}) = iD_A^\dagger \bar{\sigma}^\mu D_\mu \mathcal{D}^A + \frac{1}{2}[m_D \varepsilon_{AB}\varepsilon_{ij}(\mathcal{D}^A)^i(\mathcal{D}^B)^j + \text{h.c.}]$$

$$m_D < m_T \quad \Rightarrow \quad \chi_1^0 = \frac{1}{\sqrt{2}}(D_1^0 + D_2^0) \quad \text{and} \quad \begin{cases} m_{\chi_1^0} = m_D \\ m_{\chi_2^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2v^2} + m_D - m_T\right] \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2v^2} - m_D + m_T\right] \end{cases}$$

$$m_D > m_T \quad \Rightarrow \quad \begin{cases} m_{\chi_1^0} = m_{\chi_1^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2v^2} - m_D + m_T\right] \\ m_{\chi_2^0} = m_D \\ m_{\chi_3^0} = m_{\chi_2^\pm} = \frac{1}{2}\left[\sqrt{(m_D + m_T)^2 + 4y^2v^2} + m_D - m_T\right] \end{cases} \quad \text{for} \quad |yv| < \sqrt{2m_D(m_D - m_T)}$$

<u>Gauge interactions</u>

$$D_\mu T = (\partial_\mu - igW_\mu^a t_\mathrm{T}^a)T$$

$$t_\mathrm{T}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & -1 \\ & -1 & \end{pmatrix}, \quad t_\mathrm{T}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & i \\ & -i & \end{pmatrix}, \quad t_\mathrm{T}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$gW_\mu^a t_\mathrm{T}^a = \begin{pmatrix} gW_\mu^3 & g(W_\mu^1 - iW_\mu^2)/\sqrt{2} & 0 \\ g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & 0 & -g(W_\mu^1 - iW_\mu^2)/\sqrt{2} \\ 0 & -g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & -gW_\mu^3 \end{pmatrix}$$

$$= \begin{pmatrix} eA_\mu + gc_\mathrm{W} Z_\mu & gW_\mu^+ & 0 \\ gW_\mu^- & 0 & -gW_\mu^+ \\ 0 & -gW_\mu^- & -eA_\mu - gc_\mathrm{W} Z_\mu \end{pmatrix}$$

$$\begin{aligned} \mathcal{L}_\mathrm{T} \supset\ & T^\dagger \bar{\sigma}^\mu g W_\mu^a t_\mathrm{T}^a T \\ =\ & g[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\ & + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\ & - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-] \\ =\ & (eA_\mu + gc_\mathrm{W} Z_\mu)(T^+)^\dagger \bar{\sigma}^\mu T^+ + gW_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\ & + gW_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - gW_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\ & - gW_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - (eA_\mu + gc_\mathrm{W} Z_\mu)(T^-)^\dagger \bar{\sigma}^\mu T^- \end{aligned}$$

$$X_i^0 = \begin{pmatrix} \chi_{i\mathrm{L}}^0 \\ (\chi_{i\mathrm{R}}^0)^\dagger \end{pmatrix}, \quad \chi_\mathrm{L}^0 = \chi_\mathrm{R}^0 = \mathcal{N}^\dagger \psi_\mathrm{L}^0 = \mathcal{N}^\dagger \psi_\mathrm{R}^0 = \begin{pmatrix} \chi_1^0 & \chi_2^0 & \chi_3^0 \end{pmatrix}^\mathrm{T}, \quad \bar{X}_i^0 = \begin{pmatrix} \chi_{i\mathrm{R}}^0 & (\chi_{i\mathrm{L}}^0)^\dagger \end{pmatrix}$$

$$X_i^+ = \begin{pmatrix} \chi_{i\mathrm{L}}^+ \\ (\chi_{i\mathrm{R}}^-)^\dagger \end{pmatrix}, \quad \chi_\mathrm{L}^+ = \mathcal{C}_\mathrm{L}^\dagger \psi_\mathrm{L}^+ = \begin{pmatrix} \chi_1^+ & \chi_2^+ \end{pmatrix}^\mathrm{T}, \quad \chi_\mathrm{R}^- = \mathcal{C}_\mathrm{R}^\dagger \psi_\mathrm{R}^- = \begin{pmatrix} \chi_1^- & \chi_2^- \end{pmatrix}^\mathrm{T}, \quad \bar{X}_i^+ = \begin{pmatrix} \chi_{i\mathrm{R}}^- & (\chi_{i\mathrm{L}}^+)^\dagger \end{pmatrix}$$

$$\Psi_i^0 = \begin{pmatrix} \psi_{i\mathrm{L}}^0 \\ (\psi_{i\mathrm{R}}^0)^\dagger \end{pmatrix}, \quad \psi_\mathrm{L}^0 = \psi_\mathrm{R}^0 = \mathcal{N}\chi_\mathrm{L}^0 = \mathcal{N}\chi_\mathrm{R}^0 = \begin{pmatrix} T^0 & D_1^0 & D_2^0 \end{pmatrix}^\mathrm{T}, \quad \bar{\Psi}_i^0 = \begin{pmatrix} \psi_{i\mathrm{R}}^0 & (\psi_{i\mathrm{L}}^0)^\dagger \end{pmatrix}$$

$$\Psi_i^+ = \begin{pmatrix} \psi_{i\mathrm{L}}^+ \\ (\psi_{i\mathrm{R}}^-)^\dagger \end{pmatrix}, \quad \psi_\mathrm{L}^+ = \mathcal{C}_\mathrm{L}\chi_\mathrm{L}^+ = \begin{pmatrix} T^+ & D_2^+ \end{pmatrix}^\mathrm{T}, \quad \psi_\mathrm{R}^- = \mathcal{C}_\mathrm{R}\chi_\mathrm{R}^- = \begin{pmatrix} T^- & D_1^- \end{pmatrix}^\mathrm{T}, \quad \bar{\Psi}_i^+ = \begin{pmatrix} \psi_{i\mathrm{R}}^- & (\psi_{i\mathrm{L}}^+)^\dagger \end{pmatrix}$$

$$\Psi_{i\mathrm{L}}^0 = \begin{pmatrix} \psi_{i\mathrm{L}}^0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{N}\chi_\mathrm{L}^0)_i \\ 0 \end{pmatrix} = \mathcal{N}_{ij} X_{j\mathrm{L}}^0, \quad \Psi_{i\mathrm{R}}^0 = \begin{pmatrix} 0 \\ (\psi_{i\mathrm{R}}^0)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{N}\chi_\mathrm{R}^0)_i^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* X_{j\mathrm{R}}^0$$

$$\bar{\Psi}_{i\mathrm{L}}^0 = \begin{pmatrix} 0 & (\psi_{i\mathrm{L}}^0)^\dagger \end{pmatrix} = \mathcal{N}_{ij}^* \bar{X}_{j\mathrm{L}}^0, \quad \bar{\Psi}_{i\mathrm{R}}^0 = \begin{pmatrix} \psi_{i\mathrm{R}}^0 & 0 \end{pmatrix} = \mathcal{N}_{ij} \bar{X}_{j\mathrm{R}}^0$$

$$\Psi_{i\mathrm{L}}^+ = \begin{pmatrix} \psi_{i\mathrm{L}}^+ \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{C}_\mathrm{L}\chi_\mathrm{L}^+)_i \\ 0 \end{pmatrix} = (\mathcal{C}_\mathrm{L})_{ij} X_{j\mathrm{L}}^+, \quad \Psi_{i\mathrm{R}}^+ = \begin{pmatrix} 0 \\ (\psi_{i\mathrm{R}}^-)^\dagger \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathcal{C}_\mathrm{R}\chi_\mathrm{R}^-)_i^\dagger \end{pmatrix} = (\mathcal{C}_\mathrm{R})_{ij}^* X_{j\mathrm{R}}^+$$

$$\bar{\Psi}_{i\mathrm{L}}^+ = \begin{pmatrix} 0 & (\psi_{i\mathrm{L}}^+)^\dagger \end{pmatrix} = (\mathcal{C}_\mathrm{L})_{ij}^* \bar{X}_{j\mathrm{L}}^+, \quad \bar{\Psi}_{i\mathrm{R}}^+ = \begin{pmatrix} \psi_{i\mathrm{R}}^- & 0 \end{pmatrix} = (\mathcal{C}_\mathrm{R})_{ij} \bar{X}_{j\mathrm{R}}^+$$

$$\bar{\Psi}_{i\mathrm{L}}^0 \gamma^\mu \Psi_{i\mathrm{L}}^0 = (\psi_{i\mathrm{L}}^0)^\dagger \bar{\sigma}^\mu \psi_{i\mathrm{L}}^0, \quad \bar{\Psi}_{i\mathrm{R}}^0 \gamma^\mu \Psi_{i\mathrm{R}}^0 = \psi_{i\mathrm{R}}^0 \sigma^\mu (\psi_{i\mathrm{R}}^0)^\dagger = -(\psi_{i\mathrm{R}}^0)^\dagger \bar{\sigma}^\mu \psi_{i\mathrm{R}}^0 = -(\psi_{i\mathrm{L}}^0)^\dagger \bar{\sigma}^\mu \psi_{i\mathrm{L}}^0$$

$$\mathcal{L}_{Z\Psi_i^0\Psi_i^0} = \frac{1}{2} a_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{i\mathrm{L}}^0 \gamma^\mu \Psi_{i\mathrm{L}}^0 + \frac{1}{2} b_{Z\Psi_i^0\Psi_i^0} Z_\mu \bar{\Psi}_{i\mathrm{R}}^0 \gamma^\mu \Psi_{i\mathrm{R}}^0 = \frac{1}{2}(a_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{L}}^0 \gamma^\mu X_{j\mathrm{L}}^0 + b_{ZX_i^0X_j^0} Z_\mu \bar{X}_{i\mathrm{R}}^0 \gamma^\mu X_{j\mathrm{R}}^0)$$

$$a_{Z\Psi_1^0\Psi_1^0} = b_{Z\Psi_1^0\Psi_1^0} = 0, \quad a_{Z\Psi_2^0\Psi_2^0} = -b_{Z\Psi_2^0\Psi_2^0} = \frac{g}{2c_W}, \quad a_{Z\Psi_3^0\Psi_3^0} = -b_{Z\Psi_3^0\Psi_3^0} = -\frac{g}{2c_W}$$

$$a_{ZX_i^0X_j^0} = a_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki}^* \mathcal{N}_{kj}, \quad b_{ZX_i^0X_j^0} = b_{Z\Psi_k^0\Psi_k^0} \mathcal{N}_{ki} \mathcal{N}_{kj}^*$$

$$\bar{\Psi}_{i\mathrm{L}}^+ \gamma^\mu \Psi_{i\mathrm{L}}^0 = (\psi_{i\mathrm{L}}^+)^\dagger \bar{\sigma}^\mu \psi_{i\mathrm{L}}^0, \quad \bar{\Psi}_{i\mathrm{R}}^+ \gamma^\mu \Psi_{i\mathrm{R}}^0 = \psi_{i\mathrm{R}}^- \sigma^\mu (\psi_{i\mathrm{R}}^0)^\dagger = -(\psi_{i\mathrm{R}}^0)^\dagger \bar{\sigma}^\mu \psi_{i\mathrm{R}}^-$$

$$\mathcal{L}_{W\Psi_i^+\Psi_j^0} = a_{W\Psi_i^+\Psi_j^0}(W_\mu^+ \bar{\Psi}_{i\mathrm{L}}^+ \gamma^\mu \Psi_{j\mathrm{L}}^0 + \mathrm{h.c.}) + b_{W\Psi_i^+\Psi_j^0}(W_\mu^+ \bar{\Psi}_{i\mathrm{R}}^+ \gamma^\mu \Psi_{j\mathrm{R}}^0 + \mathrm{h.c.})$$

$$= a_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_\mathrm{L})_{ki}^* \mathcal{N}_{lj} W_\mu^+ \bar{X}_{i\mathrm{L}}^+ \gamma^\mu X_{j\mathrm{L}}^0 + \mathrm{h.c.}] + b_{W\Psi_k^+\Psi_l^0}[(\mathcal{C}_\mathrm{R})_{ki} \mathcal{N}_{lj}^* W_\mu^+ \bar{X}_{i\mathrm{R}}^+ \gamma^\mu X_{j\mathrm{R}}^0 + \mathrm{h.c.}]$$

$$= a_{WX_i^+X_j^0} W_\mu^+ \bar{X}_{i\mathrm{L}}^+ \gamma^\mu X_{j\mathrm{L}}^0 + a_{WX_i^+X_j^0}^* W_\mu^- \bar{X}_{j\mathrm{L}}^0 \gamma^\mu X_{i\mathrm{L}}^+ + b_{WX_i^+X_j^0} W_\mu^+ \bar{X}_{i\mathrm{R}}^+ \gamma^\mu X_{j\mathrm{R}}^0 + b_{WX_i^+X_j^0}^* W_\mu^- \bar{X}_{j\mathrm{R}}^0 \gamma^\mu X_{i\mathrm{R}}^+$$

$$a_{W\Psi_1^+\Psi_1^0} = b_{W\Psi_1^+\Psi_1^0} = g, \quad b_{W\Psi_2^+\Psi_2^0} = -\frac{g}{\sqrt{2}}, \quad a_{W\Psi_2^+\Psi_3^0} = \frac{g}{\sqrt{2}}, \quad \text{others} = 0$$

$$a_{WX_i^+X_j^0} = a_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_L)_{ki}^* \mathcal{N}_{lj}, \quad b_{WX_i^+X_j^0} = b_{W\Psi_k^+\Psi_l^0} (\mathcal{C}_R)_{ki} \mathcal{N}_{lj}^*$$

$$\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} = (\psi^+_{i\mathrm{L}})^\dagger\bar{\sigma}^\mu\psi^+_{i\mathrm{L}}, \quad \bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}} = \psi^-_{i\mathrm{R}}\sigma^\mu(\psi^-_{i\mathrm{R}})^\dagger = -(\psi^-_{i\mathrm{R}})^\dagger\bar{\sigma}^\mu\psi^-_{i\mathrm{R}}$$

$$\mathcal{L}_{A\Psi^+_i\Psi^+_i} = a_{A\Psi^+_i\Psi^+_i}A_\mu\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} + b_{A\Psi^+_i\Psi^+_i}A_\mu\bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}}$$

$$= a_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}A_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}A_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$= a_{AX^+_iX^+_j}A_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{AX^+_iX^+_j}A_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$a_{A\Psi^+_1\Psi^+_1} = b_{A\Psi^+_1\Psi^+_1} = a_{A\Psi^+_2\Psi^+_2} = b_{A\Psi^+_2\Psi^+_2} = e$$

$$a_{AX^+_iX^+_j} = a_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj} = e\delta_{ij}, \quad b_{AX^+_iX^+_j} = b_{A\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj} = e\delta_{ij}$$

$$\mathcal{L}_{Z\Psi^+_i\Psi^+_i} = a_{Z\Psi^+_i\Psi^+_i}Z_\mu\bar{\Psi}^+_{i\mathrm{L}}\gamma^\mu\Psi^+_{i\mathrm{L}} + b_{Z\Psi^+_i\Psi^+_i}Z_\mu\bar{\Psi}^+_{i\mathrm{R}}\gamma^\mu\Psi^+_{i\mathrm{R}}$$

$$= a_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}Z_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}Z_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$= a_{ZX^+_iX^+_j}Z_\mu\bar{X}^+_{i\mathrm{L}}\gamma^\mu X^+_{j\mathrm{L}} + b_{ZX^+_iX^+_j}Z_\mu\bar{X}^+_{i\mathrm{R}}\gamma^\mu X^+_{j\mathrm{R}}$$

$$a_{Z\Psi^+_1\Psi^+_1} = b_{Z\Psi^+_1\Psi^+_1} = gc_\mathrm{W}, \quad a_{Z\Psi^+_2\Psi^+_2} = b_{Z\Psi^+_2\Psi^+_2} = -\frac{g}{2c_\mathrm{W}}(s^2_\mathrm{W} - c^2_\mathrm{W})$$

$$a_{ZX^+_iX^+_j} = a_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{L})_{kj}, \quad b_{ZX^+_iX^+_j} = b_{Z\Psi^+_k\Psi^+_k}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{R})^*_{kj}$$

$$\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} = \psi^0_{i\mathrm{R}}\psi^0_{j\mathrm{L}}, \quad \bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}} = (\psi^0_{i\mathrm{L}})^\dagger(\psi^0_{j\mathrm{R}})^\dagger$$

$$\mathcal{L}_{h\Psi^0_i\Psi^0_j} = \frac{1}{2}a_{h\Psi^0_i\Psi^0_j}h\bar{\Psi}^0_{i\mathrm{R}}\Psi^0_{j\mathrm{L}} + \frac{1}{2}b_{h\Psi^0_i\Psi^0_j}h\bar{\Psi}^0_{i\mathrm{L}}\Psi^0_{j\mathrm{R}}$$

$$= \frac{1}{2}a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj}h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + \frac{1}{2}b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj}h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}} = \frac{1}{2}(a_{hX^0_iX^0_j}h\bar{X}^0_{i\mathrm{R}}X^0_{j\mathrm{L}} + b_{hX^0_iX^0_j}h\bar{X}^0_{i\mathrm{L}}X^0_{j\mathrm{R}})$$

$$a_{h\Psi^0_1\Psi^0_2} = b_{h\Psi^0_1\Psi^0_2} = -\frac{y_1}{\sqrt{2}} = a_{h\Psi^0_2\Psi^0_1} = b_{h\Psi^0_2\Psi^0_1}, \quad a_{h\Psi^0_1\Psi^0_3} = b_{h\Psi^0_1\Psi^0_3} = \frac{y_2}{\sqrt{2}} = a_{h\Psi^0_3\Psi^0_1} = b_{h\Psi^0_3\Psi^0_1}, \quad \text{others} = 0$$

$$a_{hX^0_iX^0_j} = a_{h\Psi^0_k\Psi^0_l}\mathcal{N}_{ki}\mathcal{N}_{lj}, \quad b_{hX^0_iX^0_j} = b_{h\Psi^0_k\Psi^0_l}\mathcal{N}^*_{ki}\mathcal{N}^*_{lj}$$

$$\bar{\Psi}^+_{i\mathrm{R}}\Psi^+_{j\mathrm{L}} = \begin{pmatrix}\psi^-_{i\mathrm{R}} & 0\end{pmatrix}\begin{pmatrix}\psi^+_{j\mathrm{L}} \\ 0\end{pmatrix} = \psi^-_{i\mathrm{R}}\psi^+_{j\mathrm{L}}, \quad \bar{\Psi}^+_{i\mathrm{L}}\Psi^+_{j\mathrm{R}} = \begin{pmatrix}0 & (\psi^+_{i\mathrm{L}})^\dagger\end{pmatrix}\begin{pmatrix}0 \\ (\psi^-_{j\mathrm{R}})^\dagger\end{pmatrix} = (\psi^+_{i\mathrm{L}})^\dagger(\psi^-_{j\mathrm{R}})^\dagger$$

$$\mathcal{L}_{h\Psi^+_i\Psi^+_j} = a_{h\Psi^+_i\Psi^+_j}h\bar{\Psi}^+_{i\mathrm{R}}\Psi^+_{j\mathrm{L}} + b_{h\Psi^+_i\Psi^+_j}h\bar{\Psi}^+_{i\mathrm{L}}\Psi^+_{j\mathrm{R}}$$

$$= a_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{L})_{lj}h\bar{X}^+_{i\mathrm{R}}X^+_{j\mathrm{L}} + b_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{R})^*_{lj}h\bar{X}^+_{i\mathrm{L}}X^+_{j\mathrm{R}} = a_{hX^+_iX^+_j}h\bar{X}^+_{i\mathrm{R}}X^+_{j\mathrm{L}} + b_{hX^+_iX^+_j}h\bar{X}^+_{i\mathrm{L}}X^+_{j\mathrm{R}}$$

$$a_{h\Psi^+_1\Psi^+_2} = b_{h\Psi^+_2\Psi^+_1} = y_2, \quad a_{h\Psi^+_2\Psi^+_1} = b_{h\Psi^+_1\Psi^+_2} = y_1, \quad \text{others} = 0$$

$$a_{hX^+_iX^+_j} = a_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{R})_{ki}(\mathcal{C}_\mathrm{L})_{lj}, \quad b_{hX^+_iX^+_j} = b_{h\Psi^+_k\Psi^+_l}(\mathcal{C}_\mathrm{L})^*_{ki}(\mathcal{C}_\mathrm{R})^*_{lj}$$

Direct detection

Higgs-mediated spin-independent (SI) $\chi_1^0 N$ scattering

$$\mathcal{L}_{hX_1^0X_1^0} = \frac{1}{2} a_{h\Psi_k^0\Psi_l^0} \operatorname{Re}(\mathcal{N}_{k1}\mathcal{N}_{l1}) h\bar{X}_1^0 X_1^0 = [a_{h\Psi_1^0\Psi_2^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + a_{h\Psi_1^0\Psi_3^0} \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$= \frac{1}{\sqrt{2}}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})] h\bar{X}_1^0 X_1^0$$

$$\equiv \frac{1}{2} g_{hX_1^0X_1^0} h\bar{X}_1^0 X_1^0$$

$$g_{hX_1^0X_1^0} = \frac{1}{2}(a_{hX_1^0X_1^0} + b_{hX_1^0X_1^0}) = \sqrt{2}[-y_1 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{21}) + y_2 \operatorname{Re}(\mathcal{N}_{11}\mathcal{N}_{31})]$$

$$G_{S,q} = -\frac{g_{hX_1^0X_1^0} m_q}{2vm_h^2}, \quad G_{S,N} = -\frac{g_{hX_1^0X_1^0} m_N}{2vm_h^2}\left(\sum_{q=u,d,s} f_q^N + 3f_Q^N\right), \quad \sigma_{\chi N}^{\mathrm{SI}} = \frac{4}{\pi}\mu_{\chi N}^2 G_{S,N}^2$$

Z-mediated spin-dependent (SD) $\chi_1^0 N$ scattering

$$\mathcal{L}_{ZX_1^0X_1^0} = -\frac{1}{2} a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0 \equiv \frac{1}{2} g_{ZX_1^0X_1^0} Z_\mu \bar{X}_1^0 \gamma^\mu \gamma_5 X_1^0$$

$$g_{ZX_1^0X_1^0} = \frac{1}{2}(b_{ZX_1^0X_1^0} - a_{ZX_1^0X_1^0}) = -a_{Z\Psi_k^0\Psi_k^0} |\mathcal{N}_{k1}|^2 = \frac{g}{2c_W}(|\mathcal{N}_{31}|^2 - |\mathcal{N}_{21}|^2)$$

$$G_{A,q} = \frac{g g_A^q g_{ZX_1^0X_1^0}}{4c_W m_Z^2}, \quad g_A^u = \frac{1}{2}, \quad g_A^d = g_A^s = -\frac{1}{2}$$

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q}\Delta_q^N, \quad \sigma_{\chi N}^{\mathrm{SD}} = \frac{12}{\pi}\mu_{\chi N}^2 G_{A,N}^2$$

Singlet-Doublet Scalar Dark Matter (SDSDM)

Ref: Cohen, Kearney, Pierce & Tucker-Smith, 1109.2604; Cheung & Sanford, 1311.5896

CP-even real scalar singlet S and complex scalar doublet Φ:

$$S \in (\mathbf{1},0), \quad \Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} \in \left(\mathbf{2},\frac{1}{2}\right)$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)\partial^\mu S + (D_\mu \Phi)^\dagger D^\mu \Phi - V(S,\Phi)$$

$$V(S,\Phi) = \frac{1}{2}m_S^2 S^2 + m_D^2 |\Phi|^2 + (\kappa S \Phi^\dagger H + h.c.) + \frac{1}{2}\lambda_{Sh} S^2 |H|^2 + \lambda_1 |H|^2 |\Phi|^2$$
$$+ (\lambda_2 \Phi^\dagger H \Phi^\dagger H + h.c.) + \lambda_3 |\Phi^\dagger H|^2 + \text{(irrelevant terms)}$$

$$|H|^2 \to \frac{1}{2}(v+h)^2, \quad |\Phi|^2 = \phi^+\phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2, \quad \kappa S \Phi^\dagger H + h.c. \to \kappa(v+h)S\phi^0$$

$$\frac{1}{2}\lambda_{Sh} S^2 |H|^2 \to \frac{\lambda_{Sh}}{4} S^2 (v+h)^2, \quad \lambda_1 |H|^2 |\Phi|^2 = \frac{\lambda_1}{2}(v+h)^2 \left[\phi^+\phi^- + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2\right]$$

$$\lambda_2 \Phi^\dagger H \Phi^\dagger H + h.c. \to \frac{\lambda_2}{2}(v+h)^2[(\phi^0)^2 - (a^0)^2], \quad \lambda_3 |\Phi^\dagger H|^2 = \frac{\lambda_3}{4}(v+h)^2[(\phi^0)^2 + (a^0)^2]$$

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix} S & \phi^0 \end{pmatrix} M_0^2 \begin{pmatrix} S \\ \phi^0 \end{pmatrix} - \frac{1}{2}m_a^2 (a^0)^2 - m_C^2 |\phi^+|^2$$

$$M_0^2 = \begin{pmatrix} m_S^2 + \frac{1}{2}\lambda_{Sh} v^2 & \kappa v \\ \kappa v & m_D^2 + \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3)v^2 \end{pmatrix}, \quad m_a^2 = m_D^2 + \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3)v^2, \quad m_C^2 = m_D^2 + \frac{1}{2}\lambda_1 v^2$$

$$U^T M_0^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix} = U \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} c_\theta X_1 - s_\theta X_2 \\ s_\theta X_1 + c_\theta X_2 \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{(M_0^2)_{11} + (M_0^2)_{22} \mp \sqrt{[(M_0^2)_{11} - (M_0^2)_{22}]^2 + 4(M_0^2)_{12}^2}\right\}$$

$$\text{DM candidate} = \begin{cases} X_1, & m_1^2 < m_a^2, m_C^2 \\ a^0, & m_a^2 < m_1^2, m_C^2 \end{cases}$$

$$\mathcal{L}_{hXX,haa} = -\kappa h S \phi^0 - \frac{\lambda_{Sh} v}{2} h S^2 - \frac{\lambda_1}{2} v h[(\phi^0)^2 + (a^0)^2] - \lambda_2 v h[(\phi^0)^2 - (a^0)^2] - \frac{\lambda_3}{2} v h[(\phi^0)^2 + (a^0)^2]$$
$$= -\kappa h (c_\theta X_1 - s_\theta X_2)(s_\theta X_1 + c_\theta X_2) - \frac{\lambda_{Sh} v}{2} h (c_\theta X_1 - s_\theta X_2)^2$$
$$- \frac{1}{2} v h (\lambda_1 + 2\lambda_2 + \lambda_3)(s_\theta X_1 + c_\theta X_2)^2 - \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3) v h (a^0)^2$$
$$= \frac{1}{2}\lambda_{hX_1X_1} v h X_1^2 + \frac{1}{2}\lambda_{hX_2X_2} v h X_2^2 + \lambda_{hX_1X_2} v h X_1 X_2 + \frac{1}{2}\lambda_{haa} v h (a^0)^2$$

$$\lambda_{hX_1X_1} = -2\frac{\kappa}{v} s_\theta c_\theta - \lambda_{Sh} c_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3) s_\theta^2, \quad \lambda_{hX_2X_2} = 2\frac{\kappa}{v} s_\theta c_\theta - \lambda_{Sh} s_\theta^2 - (\lambda_1 + 2\lambda_2 + \lambda_3) c_\theta^2$$

$$\lambda_{hX_1X_2} = -\frac{\kappa}{v}(c_\theta^2 - s_\theta^2) - (-\lambda_{Sh} + \lambda_1 + 2\lambda_2 + \lambda_3) s_\theta c_\theta, \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

Direct detection

Effective operators: $\mathcal{L}_{S,q}=\frac{1}{2}\sum_q F_{S,q}\chi^2\bar{q}q,\quad \mathcal{L}_{S,N}=\frac{1}{2}\sum_{N=p,n}F_{S,N}\chi^2\bar{N}N,\quad \chi=X_1,a^0$

$$F_{S,N}=m_N\left(\sum_{q=u,d,s}\frac{F_{S,q}}{m_q}f_q^N+\sum_{q=c,b,t}\frac{F_{S,q}}{m_q}f_Q^N\right),\quad f_Q^N=\frac{2}{27}\left(1-\sum_{q=u,d,s}f_q^N\right)$$

$$\chi=X_1\quad\Rightarrow\quad F_{S,q}=\frac{1}{i}i\lambda_{hX_1X_1}v\frac{i}{-m_h^2}\left(-i\frac{m_q}{v}\right)=-\frac{\lambda_{hX_1X_1}m_q}{m_h^2},\quad F_{S,N}=-\frac{\lambda_{hX_1X_1}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N+3f_Q^N\right)$$

$$\chi=a^0\quad\Rightarrow\quad F_{S,q}=-\frac{\lambda_{haa}m_q}{m_h^2},\quad F_{S,N}=-\frac{\lambda_{haa}m_N}{m_h^2}\left(\sum_{q=u,d,s}f_q^N+3f_Q^N\right)$$

$$\sigma_{\chi N}^{\mathrm{SI}}=\frac{m_N^2F_{S,N}^2}{4\pi(m_\chi+m_N)^2}$$

Gauge interactions

$$D^\mu\Phi=\left(\partial_\mu-\frac{1}{2}ig'B_\mu-\frac{1}{2}igW_\mu^a\sigma^a\right)\Phi$$

$$\frac{1}{2}(g'B_\mu+gW_\mu^a\sigma^a)\Phi=\begin{pmatrix} eA_\mu-\frac{g}{2c_\mathrm{W}}(s_\mathrm{W}^2-c_\mathrm{W}^2)Z_\mu & \frac{g}{\sqrt{2}}W_\mu^+\\ \frac{g}{\sqrt{2}}W_\mu^- & -\frac{g}{2c_\mathrm{W}}Z_\mu\end{pmatrix}\Phi=\begin{pmatrix}\left[eA_\mu+\frac{g}{2c_\mathrm{W}}(c_\mathrm{W}^2-s_\mathrm{W}^2)Z_\mu\right]\phi^++\frac{g}{2}W_\mu^+(\phi^0+ia^0)\\ \frac{g}{\sqrt{2}}W_\mu^-\phi^+-\frac{g}{2\sqrt{2}c_\mathrm{W}}Z_\mu(\phi^0+ia^0)\end{pmatrix}$$

$$\varphi_1\overleftrightarrow{\partial^\mu}\varphi_2\equiv\varphi_1\partial^\mu\varphi_2-(\partial^\mu\varphi_1)\varphi_2=-\varphi_2\overleftrightarrow{\partial^\mu}\varphi_1,\quad(\varphi_1 i\overleftrightarrow{\partial^\mu}\varphi_2)^\dagger=\varphi_1 i\overleftrightarrow{\partial^\mu}\varphi_2$$

$$\mathcal{L}_{\text{gauge}}=-\frac{i}{2}(\partial^\mu\Phi^\dagger)(g'B_\mu+gW_\mu^a\sigma^a)\Phi+\frac{i}{2}[(g'B_\mu+gW_\mu^a\sigma^a)\Phi]^\dagger\partial^\mu\Phi+\left|\frac{1}{2}(g'B_\mu+gW_\mu^a\sigma^a)\Phi\right|^2$$

$$=-i\partial^\mu\phi^-\left\{\left[eA_\mu+\frac{g}{2c_\mathrm{W}}(c_\mathrm{W}^2-s_\mathrm{W}^2)Z_\mu\right]\phi^++\frac{g}{2}W_\mu^+(\phi^0+ia^0)\right\}-\frac{i}{\sqrt{2}}(\partial^\mu\phi^0-i\partial^\mu a^0)\left[\frac{g}{\sqrt{2}}W_\mu^-\phi^+-\frac{g}{2\sqrt{2}c_\mathrm{W}}Z_\mu(\phi^0+ia^0)\right]$$

$$+i\partial^\mu\phi^+\left\{\left[eA_\mu+\frac{g}{2c_\mathrm{W}}(c_\mathrm{W}^2-s_\mathrm{W}^2)Z_\mu\right]\phi^-+\frac{g}{2}W_\mu^-(\phi^0-ia^0)\right\}+\frac{i}{\sqrt{2}}(\partial^\mu\phi^0+i\partial^\mu a^0)\left[\frac{g}{\sqrt{2}}W_\mu^+\phi^--\frac{g}{2\sqrt{2}c_\mathrm{W}}Z_\mu(\phi^0-ia^0)\right]$$

$$+\left|\left[eA_\mu+\frac{g}{2c_\mathrm{W}}(c_\mathrm{W}^2-s_\mathrm{W}^2)Z_\mu\right]\phi^++\frac{g}{2}W_\mu^+(\phi^0+iA^0)\right|^2+\left|\frac{g}{\sqrt{2}}W_\mu^-\phi^+-\frac{g}{2\sqrt{2}c_\mathrm{W}}Z_\mu(\phi^0+ia^0)\right|^2$$

$$=\frac{g}{2}[W_\mu^+\phi^-i\overleftrightarrow{\partial^\mu}(\phi^0+ia^0)+h.c.]+eA_\mu\phi^-i\overleftrightarrow{\partial^\mu}\phi^++\frac{g}{2c_\mathrm{W}}Z_\mu[ia^0i\overleftrightarrow{\partial^\mu}\phi^0+(c_\mathrm{W}^2-s_\mathrm{W}^2)\phi^-i\overleftrightarrow{\partial^\mu}\phi^+]$$

$$+\frac{g^2}{4}W_\mu^+W^{-\mu}[2\phi^+\phi^-+(\phi^0)^2+(a^0)^2]+e^2A_\mu A^\mu\phi^+\phi^-+\frac{g^2}{4c_\mathrm{W}^2}Z_\mu Z^\mu\left[(c_\mathrm{W}^2-s_\mathrm{W}^2)^2\phi^+\phi^-+\frac{1}{2}(\phi^0)^2+\frac{1}{2}(a^0)^2\right]$$

$$+\left[\frac{eg}{2}W_\mu^+A^\mu\phi^-(\phi^0+ia^0)-\frac{g^2s_\mathrm{W}^2}{2c_\mathrm{W}}W_\mu^+Z^\mu\phi^-(\phi^0+ia^0)+h.c.\right]+\frac{eg}{c_\mathrm{W}}(c_\mathrm{W}^2-s_\mathrm{W}^2)A_\mu Z^\mu\phi^+\phi^-$$

Custodial symmetry: $T = U = 0$

1) $\lambda_3 = 2\lambda_2$

Bidoublet formalism: $\mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}$, $\mathcal{Z}_2 = \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix}$

$$\tilde{H}^\dagger \tilde{H} = \begin{pmatrix} H^0 & -H^+ \end{pmatrix} \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = H^0 H^{0*} + H^+ H^- = H^\dagger H$$

$$\mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = \mathrm{tr}\left(\begin{pmatrix} \tilde{H}^\dagger \\ H^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2|H|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2|\Phi|^2$$

$$\tilde{\Phi}^\dagger \tilde{H} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi^0 + ia^0) & -\phi^+ \end{pmatrix} \begin{pmatrix} H^{0*} \\ -H^- \end{pmatrix} = \frac{1}{\sqrt{2}}(\phi^0 + ia^0) H^{0*} + \phi^+ H^- = \begin{pmatrix} H^- & H^{0*} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\phi^0 + ia^0) \end{pmatrix} = H^\dagger \Phi$$

$$\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \left(\begin{pmatrix} \tilde{\Phi}^\dagger \\ \Phi^\dagger \end{pmatrix} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \right) = \tilde{\Phi}^\dagger \tilde{H} + \Phi^\dagger H = \Phi^\dagger H + H^\dagger \Phi$$

$$[\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2 = (\Phi^\dagger H + H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi + 2|\Phi^\dagger H|^2$$

$$V(S,\Phi) \supset \frac{1}{2} m_D^2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \kappa S \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_{Sh} S^2 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_1 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1)\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 [\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)]^2$$

$$\left[\begin{aligned}
&\text{Tensor formalism: } \mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \mathcal{F}_i^A = \begin{pmatrix} \Phi_i^\dagger \\ \Phi_i \end{pmatrix}, \quad A \text{ is an SU(2)}_\mathrm{R} \text{ indice} \\
&\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A \mathcal{H}_j^B = \varepsilon^{ij}(\varepsilon_{12}\mathcal{H}_i^1 \mathcal{H}_j^2 + \varepsilon_{21}\mathcal{H}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij}(-H_i^\dagger H_j + H_i H_j^\dagger) \\
&\quad = -H_i^\dagger H^i - H^j H_j^\dagger = -2|H|^2 \\
&\mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = -\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A \mathcal{H}_j^B, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = -\varepsilon_{AB}\varepsilon^{ij}\mathcal{F}_i^A \mathcal{F}_j^B \\
&\varepsilon_{AB}\varepsilon^{ij}\mathcal{F}_i^A \mathcal{H}_j^B = \varepsilon^{ij}(\varepsilon_{12}\mathcal{F}_i^1 \mathcal{H}_j^2 + \varepsilon_{21}\mathcal{F}_i^2 \mathcal{H}_j^1) = \varepsilon^{ij}(-\Phi_i^\dagger H_j + \Phi_i H_j^\dagger) \\
&\quad = -\Phi_i^\dagger H^i - \Phi^j H_j^\dagger = -\Phi^\dagger H - H^\dagger \Phi = -\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)
\end{aligned}\right]$$

2) $\lambda_3 = -2\lambda_2$ and $\kappa = 0$

$$\mathcal{Z}_1 = \begin{pmatrix} \tilde{H} & H \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} -\tilde{\Phi} & \Phi \end{pmatrix}$$

$$\mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) = 2|H|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) = 2|\Phi|^2, \quad \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1) = \Phi^\dagger H - H^\dagger \Phi$$

$$\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2 = (\Phi^\dagger H - H^\dagger \Phi)^2 = \Phi^\dagger H \Phi^\dagger H + H^\dagger \Phi H^\dagger \Phi - 2|\Phi^\dagger H|^2$$

$$V(S,\Phi) \supset \frac{1}{2} m_D^2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \frac{1}{4}\lambda_{Sh} S^2 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1) + \frac{1}{4}\lambda_1 \mathrm{tr}(\mathcal{Z}_1^\dagger \mathcal{Z}_1)\mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_2) + \lambda_2 \mathrm{tr}(\mathcal{Z}_2^\dagger \mathcal{Z}_1)^2$$

$\lambda_2 = \lambda_3 = 0$ and $\kappa = 0$: $S = T = U = 0$

$$\mathcal{L}_\mathrm{mass} = -\frac{1}{2}\left(m_S^2 + \frac{1}{2}\lambda_{Sh} v^2 \right) S^2 - \left(m_D^2 + \frac{1}{2}\lambda_1 v^2 \right)\left[\frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + |\phi^+|^2 \right]$$

$\Rightarrow$ No mixing, components of Φ have exactly degenerate masses

$\Rightarrow$ $S = T = U = 0$

$\kappa = 0$ and $m_S \to \infty$: inert Higgs doublet model [Deshpande & Ma, PRD **18**, 2574 (1978)]

$$s_\theta = 0, \quad c_\theta = 1$$

$$m_{\phi^0}^2 = m_D^2 + \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3) v^2, \quad m_a^2 = m_D^2 + \frac{1}{2}(\lambda_1 - 2\lambda_2 + \lambda_3) v^2, \quad m_\mathrm{C}^2 = m_D^2 + \frac{1}{2}\lambda_1 v^2$$

$$\lambda_3 < 2|\lambda_2|, \quad \text{DM candidate} = \begin{cases} \phi^0, & \lambda_2 < 0 \\ a^0, & \lambda_2 > 0 \end{cases}$$

$$\lambda_{h\phi^0\phi^0} = -(\lambda_1 + 2\lambda_2 + \lambda_3), \quad \lambda_{haa} = -(\lambda_1 - 2\lambda_2 + \lambda_3)$$

Singlet-Triplet Scalar Dark Matter (STSDM)

CP-even real scalar singlet S and complex scalar doublet Φ :

$$S\in(\mathbf{1},0),\quad \Delta=\begin{pmatrix}\Delta^+\\ \Delta^0\\ \Delta^-\end{pmatrix}\in(\mathbf{3},0),\quad \Delta^0=\frac{1}{\sqrt{2}}(\phi^0+ia^0)$$

$$\Delta^+=\Delta^1_2,\quad \Delta^-=\Delta^2_1,\quad \Delta^0=\sqrt{2}\Delta^2_2=-\sqrt{2}\Delta^1_1$$

$$\sum_i\Delta^i_i=0,\quad \Delta^i_j=\varepsilon^{ik}\varepsilon_{jl}\Delta^l_k,\quad (\Delta^\dagger)^j_i=(\Delta^i_j)^\dagger$$

$$(\Delta^+)^*=(\Delta^\dagger)^2_1,\quad (\Delta^-)^*=(\Delta^\dagger)^1_2,\quad (\Delta^0)^*=\sqrt{2}(\Delta^\dagger)^2_2=-\sqrt{2}(\Delta^\dagger)^1_1$$

$$\mathcal{L}=\frac{1}{2}(\partial_\mu S)\partial^\mu S+(D_\mu\Delta)^\dagger D^\mu\Delta-V(S,\Delta)$$

$$V(S,\Delta)=\frac{1}{2}m_S^2S^2+m_\Delta^2|\Delta|^2+\frac{1}{2}\lambda_{Sh}S^2|H|^2+\lambda_0|H|^2|\Delta|^2+\lambda_1H^\dagger_i\Delta^i_j(\Delta^\dagger)^j_kH^k+\lambda_2H^\dagger_i(\Delta^\dagger)^i_j\Delta^j_kH^k$$
$$+(-\lambda_3H^\dagger_i\Delta^i_j\Delta^j_kH^k-\lambda_3'|H|^2\,\Delta^i_j\Delta^j_i-\lambda_4SH^\dagger_i\Delta^i_jH^j+h.c.)+(\text{irrelevant terms})$$

$$|\Delta|^2=(\Delta^\dagger)^j_i\Delta^i_j=(\Delta^+)^*\Delta^++\Delta^-(\Delta^-)^*+(\Delta^0)^*\Delta^0,\quad (\Delta^0)^*\Delta^0=\frac{1}{2}[(\phi^0)^2+(a^0)^2],\quad (\Delta^0)^2+h.c.=(\phi^0)^2-(a^0)^2$$

$$H^1\to 0,\quad H^2\to\frac{v+h}{\sqrt{2}},\quad H^\dagger_1\to 0,\quad H^\dagger_2\to\frac{v+h}{\sqrt{2}}$$

$$\lambda_0|H|^2|\Delta|^2\to\frac{\lambda_0}{2}(v+h)^2[(\Delta^+)^*\Delta^++\Delta^-(\Delta^-)^*+(\Delta^0)^*\Delta^0]$$

$$\lambda_1H^\dagger_i\Delta^i_j(\Delta^\dagger)^j_kH^k\to\lambda_1H^\dagger_2\Delta^2_j(\Delta^\dagger)^j_2H^2=\frac{\lambda_1}{2}(v+h)^2\left[\Delta^-(\Delta^-)^*+\frac{1}{2}(\Delta^0)^*\Delta^0\right]$$

$$\lambda_2H^\dagger_i(\Delta^\dagger)^i_j\Delta^j_kH^k\to\lambda_2H^\dagger_2(\Delta^\dagger)^2_j\Delta^j_2H^2=\frac{\lambda_2}{2}(v+h)^2\left[(\Delta^+)^*\Delta^++\frac{1}{2}(\Delta^0)^*\Delta^0\right]$$

$$-\lambda_3H^\dagger_i\Delta^i_j\Delta^j_kH^k\to-\lambda_3H^\dagger_2\Delta^2_j\Delta^j_2H^2=-\frac{\lambda_3}{2}(v+h)^2\left[\Delta^-\Delta^++\frac{1}{2}(\Delta^0)^2\right]$$

$$-\lambda_3'|H|^2\,\Delta^i_j\Delta^j_i\to-\frac{\lambda_3'}{2}(v+h)^2[2\Delta^-\Delta^++(\Delta^0)^2]\propto\lambda_3H^\dagger_i\Delta^i_j\Delta^j_kH^k\quad[\lambda_3'\text{ can be absorbed into }\lambda_3\text{ in the unitary gauge}]$$

$$-\lambda_4SH^\dagger_i\Delta^i_jH^j\to-\lambda_4SH^\dagger_2\Delta^2_2H^2=-\frac{\lambda_4}{2\sqrt{2}}(v+h)^2S\Delta^0=-\frac{\lambda_4}{4}(v+h)^2S(\phi^0+ia^0)$$

$$\mathcal{L}_{\text{mass}}=-\frac{1}{2}m_a^2(a^0)^2-\frac{1}{2}\begin{pmatrix}S & \phi^0\end{pmatrix}M_0^2\begin{pmatrix}S\\ \phi^0\end{pmatrix}-\begin{pmatrix}(\Delta^+)^* & \Delta^-\end{pmatrix}M_{\mathrm{C}}^2\begin{pmatrix}\Delta^+\\ (\Delta^-)^*\end{pmatrix}$$

$$m_a^2=m_\Delta^2+\frac{1}{4}(2\lambda_0+\lambda_1+\lambda_2+2\lambda_3+4\lambda_3')v^2$$

$$M_0^2=\begin{pmatrix}m_S^2+\frac{1}{2}\lambda_{Sh}v^2 & -\frac{1}{2}\lambda_4v^2\\ -\frac{1}{2}\lambda_4v^2 & m_\Delta^2+\frac{1}{4}(2\lambda_0+\lambda_1+\lambda_2-2\lambda_3-4\lambda_3')v^2\end{pmatrix},\quad M_{\mathrm{C}}^2=\begin{pmatrix}m_\Delta^2+\frac{1}{2}(\lambda_0+\lambda_2)v^2 & -\frac{1}{2}(\lambda_3+2\lambda_3')v^2\\ -\frac{1}{2}(\lambda_3+2\lambda_3')v^2 & m_\Delta^2+\frac{1}{2}(\lambda_0+\lambda_1)v^2\end{pmatrix}$$

$$\lambda_\pm\equiv\lambda_1\pm\lambda_2$$

$$\lambda_1H^\dagger_i\Delta^i_j(\Delta^\dagger)^j_kH^k+\lambda_2H^\dagger_i(\Delta^\dagger)^i_j\Delta^j_kH^k\to\frac{\lambda_+}{4}(v+h)^2[(\Delta^+)^*\Delta^++\Delta^-(\Delta^-)^*+(\Delta^0)^*\Delta^0]+\frac{\lambda_-}{4}(v+h)^2[\Delta^-(\Delta^-)^*-(\Delta^+)^*\Delta^+]$$

$\Rightarrow$ λ_0 can be absorbed into λ_+ in the unitary gauge

$$m_a^2\to m_\Delta^2+\frac{1}{4}(\lambda_++2\lambda_3)v^2,\quad M_0^2\to\begin{pmatrix}m_S^2+\frac{1}{2}\lambda_{Sh}v^2 & -\frac{1}{2}\lambda_4v^2\\ -\frac{1}{2}\lambda_4v^2 & m_\Delta^2+\frac{1}{4}(\lambda_+-2\lambda_3)v^2\end{pmatrix},\quad M_{\mathrm{C}}^2\to\begin{pmatrix}m_\Delta^2+\frac{1}{4}(\lambda_+-\lambda_-)v^2 & -\frac{1}{2}\lambda_3v^2\\ -\frac{1}{2}\lambda_3v^2 & m_\Delta^2+\frac{1}{4}(\lambda_++\lambda_-)v^2\end{pmatrix}$$

$$U^{\mathrm{T}} M_{\mathrm{C}}^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} \Delta^+ \\ (\Delta^-)^* \end{pmatrix} = U \begin{pmatrix} \Delta_1^+ \\ \Delta_2^+ \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{(M_{\mathrm{C}}^2)_{11} + (M_{\mathrm{C}}^2)_{22} \mp \sqrt{[(M_{\mathrm{C}}^2)_{11} - (M_{\mathrm{C}}^2)_{22}]^2 + 4(M_{\mathrm{C}}^2)_{12}^2}\right\} = m_\Delta^2 + \frac{v^2}{4}\left(\lambda_+ \mp \sqrt{\lambda_-^2 + 4\lambda_3^2}\right)$$

$$V^{\mathrm{T}} M_0^2 V = \begin{pmatrix} \mu_1^2 & \\ & \mu_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix}, \quad \begin{pmatrix} S \\ \phi^0 \end{pmatrix} = V \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} c_\alpha X_1 - s_\alpha X_2 \\ s_\alpha X_1 + c_\alpha X_2 \end{pmatrix}$$

$$\mu_{1,2}^2 = \frac{1}{2}\left\{(M_0^2)_{11} + (M_0^2)_{22} \mp \sqrt{[(M_0^2)_{11} - (M_0^2)_{22}]^2 + 4(M_0^2)_{12}^2}\right\}$$

$$\text{DM candidate} = \begin{cases} X_1, & \mu_1^2 < m_a^2, m_1^2 \\ a^0, & m_a^2 < \mu_1^2, m_C^2 \end{cases}$$

$$\mathcal{L}_{hXX,haa} = -\frac{1}{2}\lambda_{Sh} vhS^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(\phi^0)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4 vhS\phi^0$$

$$= -\frac{1}{2}\lambda_{Sh} vh(c_\alpha X_1 - s_\alpha X_2)^2 - \frac{1}{4}(\lambda_+ - 2\lambda_3)vh(s_\alpha X_1 + c_\alpha X_2)^2 - \frac{1}{4}(\lambda_+ + 2\lambda_3)vh(a^0)^2 + \lambda_4 vh(c_\alpha X_1 - s_\alpha X_2)(s_\alpha X_1 + c_\alpha X_2)$$

$$= \frac{1}{2}\lambda_{hX_1X_1} vhX_1^2 + \frac{1}{2}\lambda_{hX_2X_2} vhX_2^2 + \lambda_{hX_1X_2} vhX_1X_2 + \frac{1}{2}\lambda_{haa} vh(a^0)^2$$

$$\lambda_{hX_1X_1} = -\left[\lambda_{Sh} c_\alpha^2 + \frac{1}{2}(\lambda_+ - 2\lambda_3)s_\alpha^2 - 2\lambda_4 s_\alpha c_\alpha\right], \quad \lambda_{hX_2X_2} = -\left[\lambda_{Sh} s_\alpha^2 + \frac{1}{2}(\lambda_+ - 2\lambda_3)c_\alpha^2 + 2\lambda_4 s_\alpha c_\alpha\right]$$

$$\lambda_{hX_1X_2} = \left[\lambda_{Sh} - \frac{1}{2}(\lambda_+ - 2\lambda_3)\right] s_\alpha c_\alpha + \lambda_4(c_\alpha^2 - s_\alpha^2), \quad \lambda_{haa} = -\frac{1}{2}(\lambda_+ + 2\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{\mathrm{SI}} = \frac{m_p^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}, \quad \begin{cases} \text{DM candidate } \chi = X_1 \;\Rightarrow\; F_{S,N} = -\dfrac{\lambda_{hX_1X_1} m_N}{m_h^2}\left(\displaystyle\sum_{q=u,d,s} f_q^N + 3f_Q^N\right) \\ \text{DM candidate } \chi = a^0 \;\Rightarrow\; F_{S,N} = -\dfrac{\lambda_{haa} m_N}{m_h^2}\left(\displaystyle\sum_{q=u,d,s} f_q^N + 3f_Q^N\right) \end{cases}$$

Gauge interactions

$$D^\mu \Delta = (\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a)\Delta$$

$$t_{\mathrm{T}}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} & -1 & \\ -1 & & 1 \\ & 1 & \end{pmatrix}, \quad t_{\mathrm{T}}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} & i & \\ -i & & -i \\ & i & \end{pmatrix}, \quad t_{\mathrm{T}}^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$gW_\mu^a t_{\mathrm{T}}^a \Delta = \begin{pmatrix} gW_\mu^3 & -g(W_\mu^1 - iW_\mu^2)/\sqrt{2} & 0 \\ -g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & 0 & g(W_\mu^1 - iW_\mu^2)/\sqrt{2} \\ 0 & g(W_\mu^1 + iW_\mu^2)/\sqrt{2} & -gW_\mu^3 \end{pmatrix}\begin{pmatrix} \Delta^+ \\ \Delta^0 \\ \Delta^- \end{pmatrix} = \begin{pmatrix} (eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^+ - gW_\mu^+ \Delta^0 \\ -gW_\mu^- \Delta^+ + gW_\mu^+ \Delta^- \\ gW_\mu^- \Delta^0 - (eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^- \end{pmatrix}$$

$$\mathcal{L}_{\text{gauge}} = -i(\partial^\mu \Delta^\dagger) gW_\mu^a t_{\mathrm{T}}^a \Delta + i(gW_\mu^a t_{\mathrm{T}}^a \Phi)^\dagger \partial^\mu \Delta + |gW_\mu^a t_{\mathrm{T}}^a \Delta|^2$$

$$= -i\partial^\mu(\Delta^+)^*[(eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^+ - gW_\mu^+\Delta^0] - i\partial^\mu(\Delta^0)^*(-gW_\mu^-\Delta^+ + gW_\mu^+\Delta^-) - i\partial^\mu(\Delta^-)^*[gW_\mu^-\Delta^0 - (eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^-]$$

$$+ i\partial^\mu\Delta^+[(eA_\mu + gc_{\mathrm{W}} Z_\mu)(\Delta^+)^* - gW_\mu^-(\Delta^0)^*] + i\partial^\mu\Delta^0[-gW_\mu^+(\Delta^+)^* + gW_\mu^-(\Delta^-)^*] + i\partial^\mu\Delta^-[gW_\mu^+(\Delta^0)^* - (eA_\mu + gc_{\mathrm{W}} Z_\mu)(\Delta^-)^*]$$

$$+ |(eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^+ - gW_\mu^+\Delta^0|^2 + |-gW_\mu^-\Delta^+ + gW_\mu^+\Delta^-|^2 + |gW_\mu^-\Delta^0 - (eA_\mu + gc_{\mathrm{W}} Z_\mu)\Delta^-|^2$$

$$= \frac{g}{\sqrt{2}}[W_\mu^+(\phi^0 + ia^0) i\overleftrightarrow{\partial^\mu}(\Delta^+)^* + W_\mu^+(\phi^0 - ia^0) i\overleftrightarrow{\partial^\mu}\Delta^- + h.c.] + (eA_\mu + gc_{\mathrm{W}} Z_\mu)[(\Delta^+)^* i\overleftrightarrow{\partial^\mu}\Delta^+ + \Delta^- i\overleftrightarrow{\partial^\mu}(\Delta^-)^*]$$

$$+ g^2 W_\mu^+ W^{-\mu}[|\Delta^+|^2 + |\Delta^-|^2 + (\phi^0)^2 + (a^0)^2] - g^2[W_\mu^+ W^{+\mu}(\Delta^+)^*\Delta^- + h.c.]$$

$$+ (e^2 A_\mu A^\mu + g^2 c_{\mathrm{W}}^2 Z_\mu Z^\mu + 2egc_{\mathrm{W}} A_\mu Z^\mu)(|\Delta^+|^2 + |\Delta^-|^2)$$

$$- \frac{g}{\sqrt{2}}[W_\mu^+(\Delta^+)^*(\phi^0 + ia^0) + W_\mu^+\Delta^-(\phi^0 - ia^0) + h.c.](eA^\mu + gc_{\mathrm{W}} Z^\mu)$$

Custodial symmetry limit $\lambda_- = \lambda_4 = 0$: $T = U = 0$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}$$

$$\varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B = \varepsilon_{12}\mathcal{H}_i^1\mathcal{H}_j^2 + \varepsilon_{21}\mathcal{H}_i^2\mathcal{H}_j^1 = -H_i^\dagger H_j + H_i H_j^\dagger$$

$$\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = \varepsilon^{ij}(-H_i^\dagger H_j + H_i H_j^\dagger) = -H_i^\dagger H^i - H^j H_j^\dagger = -2|H|^2$$

Custodial symmetric potential:

$$V_{\text{cust}} \supset \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B\left[-\frac{1}{4}\lambda_{Sh}S^2\varepsilon^{ij} - \frac{1}{2}\lambda_0|\Delta|^2\,\varepsilon^{ij} + \frac{1}{2}\lambda_3'\Delta_n^m\Delta_m^n\varepsilon^{ij} + (\lambda_a S\Delta_k^i\varepsilon^{kj} + h.c.) + \lambda_b\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl} + (\lambda_c\Delta_k^i\Delta_l^j\varepsilon^{kl} + h.c.)\right]$$

$$\left[\begin{array}{l}\text{For scalars } A^i \text{ and } B^i,\ A^iB_i = A^i\varepsilon_{ij}B^j = -A^i\varepsilon_{ji}B^j = -B^j\varepsilon_{ji}A^i = -B^iA_i = -A_iB^i \\ \quad\Rightarrow\quad \text{Raising an index and lowering the index contracted to it give a minus sign}\end{array}\right]$$

$$V_a = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_a S\Delta_k^i\varepsilon^{kj}] = \lambda_a S(-H_i^\dagger H_j + H_iH_j^\dagger)\Delta_k^i\varepsilon^{kj} = \lambda_a S(-H_i^\dagger\Delta_j^iH^j + H^{\dagger i}\Delta_i^jH_j) = \lambda_a S(-H_i^\dagger\Delta_j^iH^j + H_i^\dagger\Delta_j^iH^j) = 0$$

$$V_b = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_b\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl}] = \lambda_b(-H_i^\dagger H_j + H_iH_j^\dagger)\Delta_k^i(\Delta^\dagger)_l^j\varepsilon^{kl} = \lambda_b[-H_i^\dagger\Delta_k^i\varepsilon^{kl}(\Delta^\dagger)_l^jH_j - H_j^\dagger(\Delta^\dagger)_l^j\varepsilon^{lk}\Delta_k^iH_i]$$
$$= \lambda_b[H_i^\dagger\Delta_k^i(\Delta^\dagger)_j^kH^j + H_j^\dagger(\Delta^\dagger)_l^j\Delta_i^lH^i] = \lambda_b[H_i^\dagger\Delta_j^i(\Delta^\dagger)_k^jH^k + H_i^\dagger(\Delta^\dagger)_j^i\Delta_k^jH^k]$$

$$V_c = \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B[\lambda_c\Delta_k^i\Delta_l^j\varepsilon^{kl}] = \lambda_c(-H_i^\dagger H_j + H_iH_j^\dagger)\Delta_k^i\Delta_l^j\varepsilon^{kl} = \lambda_c(-H_i^\dagger\Delta_k^i\varepsilon^{kl}\Delta_l^jH_j - H_j^\dagger\Delta_l^j\varepsilon^{lk}\Delta_k^iH_i)$$
$$= \lambda_c(H_i^\dagger\Delta_k^i\Delta_j^kH^j + H_j^\dagger\Delta_l^j\Delta_i^lH^i) = 2\lambda_cH_i^\dagger\Delta_j^i\Delta_k^jH^k$$

$$\lambda_4 = 0,\quad \lambda_1 = \lambda_2 = \lambda_b(\Rightarrow \lambda_- = 0),\quad -\lambda_3 = 2\lambda_c \quad\Rightarrow\quad \text{Custodial symmetry}$$

$\lambda_- = \lambda_4 = \lambda_3 = 0$: components of Δ have exactly degenerate masses

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2}\left(m_S^2 + \frac{1}{2}\lambda_{Sh}v^2\right)S^2 - \left(m_\Delta^2 + \frac{1}{4}\lambda_+v^2\right)^2\left[\frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 + |\Delta^+|^2 + |\Delta^-|^2\right]$$

$\lambda_4 = 0$ and $m_S \to \infty$: complex triplet model

$$s_\alpha = 0,\quad c_\alpha = 1$$

$$m_{\phi^0}^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2\lambda_3)v^2,\quad m_a^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ + 2\lambda_3)v^2$$

$$m_1^2 = m_\Delta^2 + \frac{1}{4}\lambda_+v^2 - \frac{1}{4}\sqrt{\lambda_-^2 + 4\lambda_3^2}\,v^2 \le m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2|\lambda_3|)v^2 \le \min(m_{\phi^0}^2, m_a^2)$$

$$\lambda_- = 0 \quad\Rightarrow\quad m_1^2 = m_\Delta^2 + \frac{1}{4}(\lambda_+ - 2|\lambda_3|)v^2 = \min(m_{\phi^0}^2, m_a^2)$$

$\lambda_- \ne 0 \quad\Rightarrow\quad$ unstable DM candidate

Quadruplet Scalar Dark Matter (QSDM)

Complex scalar quadruplet X :

$$X = \begin{pmatrix} X^{++} \\ X^{+} \\ X^{0} \\ X^{-} \end{pmatrix} \in \left(\mathbf{4}, \frac{1}{2}\right), \quad X^0 = \frac{1}{\sqrt{2}}(\phi^0 + ia^0)$$

$$X^{++} = X_2^{11}, \quad X^{+} = \sqrt{3}X_1^{11} = -\sqrt{3}X_2^{12} = -\sqrt{3}X_2^{21}, \quad X^0 = \sqrt{3}X_2^{22} = -\sqrt{3}X_1^{12} = -\sqrt{3}X_1^{21}, \quad X^- = X_1^{22}$$

$$X_k^{ij} = X_k^{ji}, \quad \sum_k X_k^{ik} = \sum_k X_k^{kj} = 0, \quad (X^\dagger)_{ij}^k = (X_k^{ij})^\dagger$$

$$(X^{++})^* = (X^\dagger)_{11}^2, \quad (X^{+})^* = \sqrt{3}(X^\dagger)_{11}^1 = -\sqrt{3}(X^\dagger)_{12}^2 = -\sqrt{3}(X^\dagger)_{21}^2, \quad (X^0)^* = \sqrt{3}(X^\dagger)_{22}^2 = -\sqrt{3}(X^\dagger)_{12}^1 = -\sqrt{3}(X^\dagger)_{21}^1, \quad (X^-)^* = (X^\dagger)_{22}^1$$

$$\mathcal{L} = (D_\mu X)^\dagger D^\mu X - V(X)$$

$$V(X) = m_X^2 |X|^2 + \lambda_0 |H|^2 |X|^2 + \lambda_1 H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l - (\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} + h.c.) + (\text{irrelevant terms})$$

$$|X|^2 = (X^\dagger)_{ij}^k X_k^{ij} = (X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-, \quad (X^0)^* X^0 = \frac{1}{2}[(\phi^0)^2 + (a^0)^2], \quad (X^0)^2 + h.c. = (\phi^0)^2 - (a^0)^2$$

$$H^1 \to 0, \quad H^2 \to \frac{v+h}{\sqrt{2}}, \quad H_1^\dagger \to 0, \quad H_2^\dagger \to \frac{v+h}{\sqrt{2}}$$

$$\lambda_0 |H|^2 |X|^2 \to \frac{\lambda_0}{2}(v+h)^2[(X^{++})^* X^{++} + (X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^-]$$

$$\lambda_1 H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l \to \lambda_1 H_2^\dagger X_k^{2j}(X^\dagger)_{j2}^k H^2 = \frac{\lambda_1}{2}(v+h)^2[X_1^{21}(X^\dagger)_{12}^1 + X_1^{22}(X^\dagger)_{22}^1 + X_2^{21}(X^\dagger)_{12}^2 + X_2^{22}(X^\dagger)_{22}^2]$$

$$= \frac{\lambda_1}{2}(v+h)^2\left[\frac{1}{3}(X^+)^* X^+ + \frac{2}{3}(X^0)^* X^0 + (X^-)^* X^-\right]$$

$$\lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l \to \lambda_2 H_2^\dagger (X^\dagger)_{jk}^2 X_2^{jk} H^2 = \frac{\lambda_2}{2}(v+h)^2[(X^\dagger)_{11}^2 X_2^{11} + (X^\dagger)_{12}^2 X_2^{12} + (X^\dagger)_{21}^2 X_2^{21} + (X^\dagger)_{22}^2 X_2^{22}]$$

$$= \frac{\lambda_2}{2}(v+h)^2[(X^{++})^* X^{++} + \frac{2}{3}(X^+)^* X^+ + \frac{1}{3}(X^0)^* X^0]$$

$$-\lambda_3 H_i^\dagger H_j^\dagger X_l^{ik} X_k^{jl} \to -\lambda_3 H_2^\dagger H_2^\dagger X_l^{2k} X_k^{2l} = -\frac{\lambda_3}{2}(v+h)^2(X_1^{21}X_1^{21} + X_2^{21}X_1^{22} + X_1^{22}X_2^{21} + X_2^{22}X_2^{22}) = -\frac{\lambda_3}{2}(v+h)^2\left[\frac{2}{3}(X^0)^2 - \frac{2}{\sqrt{3}}X^+ X^-\right]$$

$$\mathcal{L}_{\text{mass}} = -m_{++}^2 |X^{++}|^2 - \begin{pmatrix}(X^+)^* & X^-\end{pmatrix} M_{\mathrm{C}}^2 \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} - \frac{1}{2}m_\phi^2(\phi^0)^2 - \frac{1}{2}m_a^2(a^0)^2$$

$$m_{++}^2 = m_X^2 + \frac{1}{2}(\lambda_0 + \lambda_2)v^2, \quad m_\phi^2 = m_X^2 + \left(\frac{1}{2}\lambda_0 + \frac{1}{3}\lambda_1 + \frac{1}{6}\lambda_2 - \frac{2}{3}\lambda_3\right)v^2, \quad m_a^2 = m_X^2 + \left(\frac{1}{2}\lambda_0 + \frac{1}{3}\lambda_1 + \frac{1}{6}\lambda_2 + \frac{2}{3}\lambda_3\right)v^2$$

$$M_{\mathrm{C}}^2 = \begin{pmatrix} m_X^2 + \left(\frac{\lambda_0}{2} + \frac{\lambda_1}{6} + \frac{\lambda_2}{3}\right)v^2 & \frac{\lambda_3}{\sqrt{3}}v^2 \\ \frac{\lambda_3}{\sqrt{3}}v^2 & m_X^2 + \frac{1}{2}(\lambda_0 + \lambda_1)v^2 \end{pmatrix}$$

$\lambda_\pm \equiv \lambda_1 \pm \lambda_2$

$$\lambda_1 H_i^\dagger X_k^{ij} (X^\dagger)_{jl}^k H^l + \lambda_2 H_i^\dagger (X^\dagger)_{jk}^i X_l^{jk} H^l \to \frac{\lambda_+}{4}(v+h)^2 \left[(X^+)^* X^+ + (X^0)^* X^0 + (X^-)^* X^- + (X^{++})^* X^{++} \right]$$
$$+ \frac{\lambda_-}{4}(v+h)^2 \left[-(X^{++})^* X^{++} - \frac{1}{3}(X^+)^* X^+ + \frac{1}{3}(X^0)^* X^0 + (X^-)^* X^- \right]$$

$\Rightarrow$ λ_0 can be absorbed into λ_+ in the unitary gauge

$$m_{++}^2 \to m_X^2 + \frac{1}{4}(\lambda_+ - \lambda_-)v^2, \quad m_\phi^2 \to m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- - 8\lambda_3)v^2, \quad m_a^2 \to m_X^2 + \frac{1}{12}(3\lambda_+ + \lambda_- + 8\lambda_3)v^2$$

$$M_C^2 \to \begin{pmatrix} m_X^2 + \frac{1}{12}(3\lambda_+ - \lambda_-)v^2 & \frac{\lambda_3}{\sqrt{3}}v^2 \\ \frac{\lambda_3}{\sqrt{3}}v^2 & m_X^2 + \frac{1}{4}(\lambda_+ + \lambda_-)v^2 \end{pmatrix}$$

$$U^\mathrm{T} M_C^2 U = \begin{pmatrix} m_1^2 & \\ & m_2^2 \end{pmatrix}, \quad U = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \begin{pmatrix} X^+ \\ (X^-)^* \end{pmatrix} = U \begin{pmatrix} X_1^+ \\ X_2^+ \end{pmatrix}$$

$$m_{1,2}^2 = \frac{1}{2}\left\{ (M_C^2)_{11} + (M_C^2)_{22} \mp \sqrt{[(M_C^2)_{11} - (M_C^2)_{22}]^2 + 4(M_C^2)_{12}^2} \right\} = m_X^2 + \frac{v^2}{12}\left(3\lambda_+ + \lambda_- \mp 2\sqrt{\lambda_-^2 + 12\lambda_3^2} \right)$$

$$\text{DM candidate} = \begin{cases} \phi^0, & \lambda_3 > 0 \text{ and } |\lambda_-| < 2\lambda_3 \\ a^0, & \lambda_3 < 0 \text{ and } |\lambda_-| < -2\lambda_3 \end{cases}$$

$$\mathcal{L}_{h\phi\phi,haa} = \frac{1}{2}\lambda_{h\phi\phi} vh(\phi^0)^2 + \frac{1}{2}\lambda_{haa} vh(a^0)^2, \quad \lambda_{h\phi\phi} = -\frac{1}{6}(3\lambda_+ + \lambda_- - 8\lambda_3), \quad \lambda_{haa} = -\frac{1}{6}(3\lambda_+ + \lambda_- + 8\lambda_3)$$

Direct detection

$$\sigma_{\chi N}^{\mathrm{SI}} = \frac{m_p^2 F_{S,N}^2}{4\pi(m_\chi + m_N)^2}, \quad \begin{cases} \text{DM candidate } \chi = \phi^0 \quad \Rightarrow \quad F_{S,N} = -\frac{\lambda_{h\phi\phi} m_N}{m_h^2}\left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N \right) \\ \text{DM candidate } \chi = a^0 \quad \Rightarrow \quad F_{S,N} = -\frac{\lambda_{haa} m_N}{m_h^2}\left(\sum_{q=u,d,s} f_q^N + 3 f_Q^N \right) \end{cases}$$

$\lambda_- = \lambda_3 = 0$: $S = T = U = 0$

$$\mathcal{L}_{\text{mass}} = -\left(m_X^2 + \frac{1}{4}\lambda_+ v^2 \right)\left[|X^{++}|^2 + |X^+|^2 + |X^-|^2 + \frac{1}{2}(\phi^0)^2 + \frac{1}{2}(a^0)^2 \right]$$

$\Rightarrow$ No mixing between X^+ and $(X^-)^*$, components of X have exactly degenerate masses

$\Rightarrow$ $S = T = U = 0$

Gauge interactions

$$D^\mu X = (\partial_\mu - \frac{1}{2} i g' B_\mu - i g W_\mu^a t_{\mathrm{Q}}^a) X$$

$$t_{\mathrm{Q}}^1 = \begin{pmatrix} & -\sqrt{3}/2 & & \\ -\sqrt{3}/2 & & -1 & \\ & -1 & & -\sqrt{3}/2 \\ & & -\sqrt{3}/2 & \end{pmatrix}, \quad t_{\mathrm{Q}}^2 = \begin{pmatrix} & \sqrt{3}i/2 & & \\ -\sqrt{3}i/2 & & i & \\ & -i & & \sqrt{3}i/2 \\ & & -\sqrt{3}i/2 & \end{pmatrix}, \quad t_{\mathrm{Q}}^3 = \begin{pmatrix} 3/2 & & & \\ & 1/2 & & \\ & & -1/2 & \\ & & & -3/2 \end{pmatrix}$$

$$\left(\frac{1}{2} g' B_\mu + g W_\mu^a t_{\mathrm{Q}}^a\right) X = \begin{pmatrix} 2eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu & -\frac{\sqrt{6}}{2} g W_\mu^+ & 0 & 0 \\ -\frac{\sqrt{6}}{2} g W_\mu^- & eA_\mu + \frac{g}{2c_{\mathrm{W}}}(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu & -\sqrt{2} g W_\mu^+ & 0 \\ 0 & -\sqrt{2} g W_\mu^- & -\frac{g}{2c_{\mathrm{W}}} Z_\mu & -\frac{\sqrt{6}}{2} g W_\mu^+ \\ 0 & 0 & -\frac{\sqrt{6}}{2} g W_\mu^- & -eA_\mu - \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2) Z_\mu \end{pmatrix} \begin{pmatrix} X^{++} \\ X^+ \\ X^0 \\ X^- \end{pmatrix}$$

$$= \begin{pmatrix} \left[2eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^{++} - \frac{\sqrt{6}}{2} g W_\mu^+ X^+ \\ -\frac{\sqrt{6}}{2} g W_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^+ - \sqrt{2} g W_\mu^+ X^0 \\ -\sqrt{2} g W_\mu^- X^+ - \frac{g}{2c_{\mathrm{W}}} Z_\mu X^0 - \frac{\sqrt{6}}{2} g W_\mu^+ X^- \\ -\frac{\sqrt{6}}{2} g W_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2) Z_\mu\right] X^- \end{pmatrix}$$

$$\mathcal{L}_{\mathrm{gauge}} = \left[-i(\partial^\mu \Delta^\dagger)\left(\frac{1}{2} g' B_\mu + g W_\mu^a t_{\mathrm{Q}}^a\right) X + h.c.\right] + \left|\left(\frac{1}{2} g' B_\mu + g W_\mu^a t_{\mathrm{Q}}^a\right) X\right|^2$$

$$\begin{aligned}
&= \Bigg\{ -i\partial^\mu (X^{++})^* \left(\left[2eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^{++} - \frac{\sqrt{6}}{2} g W_\mu^+ X^+ \right) - i\partial^\mu (X^+)^* \left(-\frac{\sqrt{6}}{2} g W_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^+ - \sqrt{2} g W_\mu^+ X^0 \right) \\
&\quad -i\partial^\mu (X^0)^* \left(-\sqrt{2} g W_\mu^- X^+ - \frac{g}{2c_{\mathrm{W}}} Z_\mu X^0 - \frac{\sqrt{6}}{2} g W_\mu^+ X^- \right) - i\partial^\mu (X^-)^* \left(-\frac{\sqrt{6}}{2} g W_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2) Z_\mu\right] X^- \right) + h.c. \Bigg\} \\
&\quad + \left| \left[2eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^{++} - \frac{\sqrt{6}}{2} g W_\mu^+ X^+ \right|^2 + \left| -\frac{\sqrt{6}}{2} g W_\mu^- X^{++} + \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2) Z_\mu\right] X^+ - \sqrt{2} g W_\mu^+ X^0 \right|^2 \\
&\quad + \left| -\sqrt{2} g W_\mu^- X^+ - \frac{g}{2c_{\mathrm{W}}} Z_\mu X^0 - \frac{\sqrt{6}}{2} g W_\mu^+ X^- \right|^2 + \left| -\frac{\sqrt{6}}{2} g W_\mu^- X^0 - \left[eA_\mu + \frac{g}{2c_{\mathrm{W}}}(3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2) Z_\mu\right] X^- \right|^2 \\
&= g\left[\frac{\sqrt{6}}{2} W_\mu^+ X^+ i\overleftrightarrow{\partial^\mu} (X^{++})^* + W_\mu^+ (\phi^0 + i a^0) i\overleftrightarrow{\partial^\mu} (X^+)^* + \frac{\sqrt{3}}{2} W_\mu^+ X^- i\overleftrightarrow{\partial^\mu} (\phi^0 - i a^0) + h.c. \right] \\
&\quad + eA_\mu [2(X^{++})^* i\overleftrightarrow{\partial^\mu} X^{++} + (X^+)^* i\overleftrightarrow{\partial^\mu} X^+ - (X^-)^* i\overleftrightarrow{\partial^\mu} X^-] \\
&\quad + \frac{g}{2c_{\mathrm{W}}} Z_\mu [(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)(X^{++})^* i\overleftrightarrow{\partial^\mu} X^{++} + (c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)(X^+)^* i\overleftrightarrow{\partial^\mu} X^+ + i a^0 i\overleftrightarrow{\partial^\mu} \phi^0 - (3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2)(X^-)^* i\overleftrightarrow{\partial^\mu} X^-] \\
&\quad + g^2 W_\mu^+ W^{-\mu} \left[\frac{3}{2} |X^{++}|^2 + \frac{7}{2} |X^+|^2 + \frac{7}{4} (\phi^0)^2 + \frac{7}{4} (a^0)^2 + \frac{3}{2} |X^-|^2 \right] + g^2 \left[\sqrt{3} W_\mu^+ W^{+\mu} (X^{++})^* X^0 + \sqrt{3} W_\mu^+ W^{+\mu} (X^+)^* X^- + h.c. \right] \\
&\quad + e^2 A_\mu A^\mu (4|X^{++}|^2 + |X^+|^2 + |X^-|^2) + \frac{eg}{c_{\mathrm{W}}} A_\mu Z^\mu [2(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)|X^{++}|^2 + (c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)|X^+|^2 + (3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2)|X^-|^2] \\
&\quad + \frac{g^2}{4c_{\mathrm{W}}^2} Z_\mu Z^\mu \left[(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2 |X^{++}|^2 + (c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)^2 |X^+|^2 + \frac{1}{2} (\phi^0)^2 + \frac{1}{2} (a^0)^2 + (3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2)^2 |X^-|^2 \right] \\
&\quad + \Bigg\{ -\frac{\sqrt{6}}{2} \left[2egA^\mu + \frac{g^2(3c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} Z^\mu \right] W_\mu^+ X^+ (X^{++})^* - \left[egA^\mu + \frac{g^2(c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} Z^\mu \right] W_\mu^+ \left(\frac{\sqrt{6}}{2} X^+ (X^{++})^* + (X^+)^* (\phi^0 + i a^0) \right) \\
&\qquad + \frac{\sqrt{3} g^2}{4c_{\mathrm{W}}} Z^\mu W_\mu^+ X^- (\phi^0 + i a^0) + \frac{g^2}{2c_{\mathrm{W}}} Z^\mu W_\mu^+ (X^+)^* (\phi^0 - i a^0) + \frac{\sqrt{3}}{2} \left[egA^\mu + \frac{g^2(3c_{\mathrm{W}}^2 + s_{\mathrm{W}}^2)}{2c_{\mathrm{W}}} Z^\mu \right] W_\mu^+ X^- (\phi^0 - i a^0) + h.c. \Bigg\}
\end{aligned}$$

Custodial symmetry: $T=U=0$

1) $\lambda_- = 2\lambda_3$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad \varepsilon_{AB}\mathcal{H}_i^A\mathcal{H}_j^B = -H_i^\dagger H_j + H_i H_j^\dagger, \quad \varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = -2|H|^2$$

$$(\mathcal{X}^A)_k^{ij} = \begin{pmatrix} (X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}, \quad (X^\dagger)_k^{ij} = \varepsilon^{il}(X^\dagger)_{lk}^j$$

$$\varepsilon_{AB}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = \varepsilon_{12}(\mathcal{X}^1)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{21}(\mathcal{X}^2)_l^{ik}(\mathcal{X}^1)_k^{jl} = -(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}$$

$$\varepsilon_{AB}\varepsilon_{ij}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = \varepsilon_{ij}[-(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}] = (X^\dagger)_{jl}^k X_k^{jl} + X_l^{ik}(X^\dagger)_{ik}^l = 2|X|^2$$

Custodial symmetric potential:

$$V_{\text{cust}} \supset \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} + \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$
$$+\lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} + \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$

[Note: raising an index and lowering the index contracted to it give a minus sign]

$$V_{0a} = \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} = -4\lambda_{0a}|H|^2|X|^2$$

$$V_{0b} = \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$
$$= \lambda_{0b}[\varepsilon_{12}\varepsilon_{12}\varepsilon^{ij}\mathcal{H}_i^1\mathcal{H}_j^1\varepsilon_{mn}(\mathcal{X}^2)_l^{mk}(\mathcal{X}^2)_k^{nl} + \varepsilon_{12}\varepsilon_{21}\varepsilon^{ij}\mathcal{H}_i^1\mathcal{H}_j^2\varepsilon_{mn}(\mathcal{X}^2)_l^{mk}(\mathcal{X}^1)_k^{nl}$$
$$+\varepsilon_{21}\varepsilon_{12}\varepsilon^{ij}\mathcal{H}_i^2\mathcal{H}_j^1\varepsilon_{mn}(\mathcal{X}^1)_l^{mk}(\mathcal{X}^2)_k^{nl} + \varepsilon_{21}\varepsilon_{21}\varepsilon^{ij}\mathcal{H}_i^2\mathcal{H}_j^2\varepsilon_{mn}(\mathcal{X}^1)_l^{mk}(\mathcal{X}^1)_k^{nl}]$$
$$= \lambda_{0b}[\varepsilon^{ij}H_i^\dagger H_j^\dagger\varepsilon_{mn}X_l^{mk}X_k^{nl} - \varepsilon^{ij}H_i^\dagger H_j\varepsilon_{mn}X_l^{mk}(X^\dagger)_k^{nl} - \varepsilon^{ij}H_iH_j^\dagger\varepsilon_{mn}(X^\dagger)_l^{mk}X_k^{nl} + \varepsilon^{ij}H_iH_j\varepsilon_{mn}(X^\dagger)_l^{mk}(X^\dagger)_k^{nl}]$$
$$= \lambda_{0b}[-H_i^\dagger H^i(X^\dagger)_{mk}^l X_l^{mk} - H_j^\dagger H^j\varepsilon_{nm}(X^\dagger)_{nl}^k X_k^{nl}] = -2\lambda_{0b}|H|^2|X|^2$$

$$V_a = \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} = \lambda_a(-H_i^\dagger H_j + H_iH_j^\dagger)[-(X^\dagger)_l^{ik}X_k^{jl} + X_l^{ik}(X^\dagger)_k^{jl}]$$
$$= \lambda_a[H_i^\dagger(X^\dagger)_l^{ik}X_k^{jl}H_j - H_i^\dagger X_l^{ik}(X^\dagger)_k^{jl}H_j - H_j^\dagger X_k^{jl}(X^\dagger)_l^{ik}H_i + H_j^\dagger(X^\dagger)_k^{jl}X_l^{ik}H_i]$$
$$= 2\lambda_a[H_i^\dagger(X^\dagger)_l^{ik}X_k^{jl}H_j - H_i^\dagger X_l^{ik}(X^\dagger)_k^{jl}H_j] = 2\lambda_a[H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_b = \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$
$$= \lambda_b[\varepsilon_{12}\varepsilon_{12}\mathcal{H}_i^1\mathcal{H}_j^1(\mathcal{X}^2)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{12}\varepsilon_{21}\mathcal{H}_i^1\mathcal{H}_j^2(\mathcal{X}^2)_l^{ik}(\mathcal{X}^1)_k^{jl} + \varepsilon_{21}\varepsilon_{12}\mathcal{H}_i^2\mathcal{H}_j^1(\mathcal{X}^1)_l^{ik}(\mathcal{X}^2)_k^{jl} + \varepsilon_{21}\varepsilon_{21}\mathcal{H}_i^2\mathcal{H}_j^2(\mathcal{X}^1)_l^{ik}(\mathcal{X}^1)_k^{jl}]$$
$$= \lambda_b[H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} - H_i^\dagger H_jX_l^{ik}(X^\dagger)_k^{jl} - H_iH_j^\dagger(X^\dagger)_l^{ik}X_k^{jl} + H_iH_j(X^\dagger)_l^{ik}(X^\dagger)_k^{jl}]$$
$$= \lambda_b[(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.) + 2H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset -(4\lambda_{0a} + 2\lambda_{0b})|H|^2|X|^2 + 2\lambda_a H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + 2(\lambda_a+\lambda_b)H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + \lambda_b(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.)$$

$$\text{Custodial symmetry} \quad \Rightarrow \quad \begin{cases} \lambda_0 = -(4\lambda_{0a} + 2\lambda_{0b}) \\ \lambda_1 = 2\lambda_a \\ \lambda_2 = 2(\lambda_a + \lambda_b) \\ -\lambda_3 = \lambda_b \end{cases} \quad \Rightarrow \quad \lambda_- = \lambda_1 - \lambda_2 = -2\lambda_b = 2\lambda_3$$

$\underline{2)\ \lambda_- = -2\lambda_3}$

$$\mathcal{H}_i^A = \begin{pmatrix} H_i^\dagger \\ H_i \end{pmatrix}, \quad (\mathcal{X}^A)_k^{ij} = \begin{pmatrix} -(X^\dagger)_k^{ij} \\ X_k^{ij} \end{pmatrix}$$

$$\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B = -2\,|H|^2, \quad \varepsilon_{AB}\varepsilon_{ij}(\mathcal{X}^A)_l^{ik}(\mathcal{X}^B)_k^{jl} = -2\,|X|^2$$

$$V_{\text{cust}} \supset \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} + \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl}$$
$$+\lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} + \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl}$$

$$V_{0a} = \lambda_{0a}\varepsilon_{AB}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^B\varepsilon_{CD}\varepsilon_{mn}(\mathcal{X}^C)_l^{mk}(\mathcal{X}^D)_k^{nl} = 4\lambda_{0a}\,|H|^2|X|^2$$

$$V_{0b} = \lambda_{0b}\varepsilon_{AB}\varepsilon_{CD}\varepsilon^{ij}\mathcal{H}_i^A\mathcal{H}_j^C\varepsilon_{mn}(\mathcal{X}^B)_l^{mk}(\mathcal{X}^D)_k^{nl} = 2\lambda_{0b}\,|H|^2|X|^2$$

$$V_a = \lambda_a\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^B(\mathcal{X}^C)_l^{ik}(\mathcal{X}^D)_k^{jl} = -2\lambda_a[H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l + H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_b = \lambda_b\varepsilon_{AB}\varepsilon_{CD}\mathcal{H}_i^A\mathcal{H}_j^C(\mathcal{X}^B)_l^{ik}(\mathcal{X}^D)_k^{jl} = \lambda_b[(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.) - 2H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l]$$

$$V_{\text{cust}} \supset (4\lambda_{0a}+2\lambda_{0b})\,|H|^2|X|^2 - 2\lambda_a H_i^\dagger(X^\dagger)_{jk}^i X_l^{jk}H^l - 2(\lambda_a+\lambda_b)H_i^\dagger X_k^{ij}(X^\dagger)_{jl}^k H^l + \lambda_b(H_i^\dagger H_j^\dagger X_l^{ik}X_k^{jl} + h.c.)$$

$$\text{Custodial symmetry} \quad \Rightarrow \quad \begin{cases} \lambda_0 = 4\lambda_{0a} + 2\lambda_{0b} \\ \lambda_1 = -2\lambda_a \\ \lambda_2 = -2(\lambda_a+\lambda_b) \\ -\lambda_3 = \lambda_b \end{cases} \quad \Rightarrow \quad \lambda_- = \lambda_1 - \lambda_2 = 2\lambda_b = -2\lambda_3$$

Passarino-Veltman scalar functions

D-dim one-loop integrals defined by Denner, 0709.1075:

$$T^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1}) = F\int d^Dq\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0D_1\cdots D_{N-1}},\quad F\equiv\frac{(2\pi\mu)^{4-D}}{i\pi^2}$$

$N=$ number of propagator factors in the denominator, $P=$ number of integration momenta in the numerator

$$D_0=q^2-m_0^2+i\varepsilon,\quad D_i=(q+p_i)^2-m_i^2+i\varepsilon,\quad i=1,\cdots,N-1,\quad \varepsilon=\frac{4-D}{2},\quad D=4-2\varepsilon$$

These integrals give rise to a UV-divergent term $\Delta=\frac{1}{\varepsilon}-\gamma_{\mathrm{E}}+\log 4\pi$

Subtracting Δ corresponds to the $\overline{\mathrm{MS}}$ scheme

$$(2\pi)^{4-D}=(2\pi)^{2\varepsilon}=[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)],\quad F=\frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+2\varepsilon\log 2\pi+\mathcal{O}(\varepsilon^2)]$$

Conventionally, $T^1\equiv A,\quad T^2\equiv B,\quad T^3\equiv C,\quad\cdots$

$$A(m_0^2)=A_0(m_0^2),\quad B(p_1^2,m_0^2,m_1^2)=B_0(p_1^2,m_0^2,m_1^2),\quad B_\mu(p_1^2,m_0^2,m_1^2)=p_{1\mu}B_1(p_1^2,m_0^2,m_1^2)$$

$$B_{\mu\nu}(p_1^2,m_0^2,m_1^2)=g_{\mu\nu}B_{00}(p_1^2,m_0^2,m_1^2)+p_{1\mu}p_{1\nu}B_{11}(p_1^2,m_0^2,m_1^2)$$

$$A_0(m^2)\sim m^2\Delta,\quad B_0(p^2,m_1^2,m_2^2)\sim\Delta,\quad B_1(p^2,m_1^2,m_2^2)\sim-\frac{1}{2}\Delta,\quad B_{00}(p^2,m_1^2,m_2^2)\sim-\frac{1}{12}(p^2-3m_1^2-3m_2^2)\Delta$$

D-dim one-loop integrals defined by LoopTools User's Guide:

$$T'^N_{\mu_1\cdots\mu_P}(p_1,\cdots,p_{N-1},m_0,\cdots,m_{N-1})=F'\int d^Dq\,\frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0D_1\cdots D_{N-1}},\quad F'\equiv\frac{\mu^{4-D}}{i\pi^{D/2}r_\Gamma},\quad r_\Gamma=\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\frac{1}{r_\Gamma}=1+\gamma_{\mathrm{E}}\varepsilon+\mathcal{O}(\varepsilon^2),\quad \frac{1}{\pi^{D/2}}=\frac{1}{\pi^{2-\varepsilon}}=\frac{1}{\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$F'=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)][1+\gamma_{\mathrm{E}}\varepsilon+\mathcal{O}(\varepsilon^2)]=\frac{\mu^{2\varepsilon}}{i\pi^2}[1+\gamma_{\mathrm{E}}\varepsilon+\varepsilon\log\pi+\mathcal{O}(\varepsilon^2)]$$

$$\frac{T'^N_{\mu_1\cdots\mu_P}}{T^N_{\mu_1\cdots\mu_P}}=\frac{F'}{F}=\frac{\mu^{2\varepsilon}}{i\pi^{2-\varepsilon}r_\Gamma}\frac{i\pi^2}{(2\pi\mu)^{2\varepsilon}}=\frac{1}{(2\pi)^{2\varepsilon}\pi^{-\varepsilon}r_\Gamma}=1+(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon^2)$$

$$\Delta'=\frac{F'}{F}\Delta=\Delta+\Delta(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}-\gamma_{\mathrm{E}}+\log 4\pi+\frac{1}{\varepsilon}(\gamma_{\mathrm{E}}-\log 4\pi)\varepsilon+\mathcal{O}(\varepsilon)=\frac{1}{\varepsilon}+\mathcal{O}(\varepsilon)$$

The UV-divergent term from $T'^N_{\mu_1\cdots\mu_P}$ is Δ', and subtracting Δ' corresponds to the $\overline{\mathrm{MS}}$ scheme

$$\frac{1}{K^{2-D/2}} = 1 - \frac{4-D}{2}\ln K + \mathcal{O}((4-D)^2), \quad \Gamma(2-D/2) = \frac{2}{4-D} - \gamma_{\mathrm{E}} + \mathcal{O}(4-D)$$

$$\frac{1}{(4\pi)^{D/2}} = \frac{1}{(4\pi)^2(4\pi)^{(D-4)/2}} = \frac{1}{16\pi^2}(4\pi)^{2-D/2} = \frac{1}{16\pi^2}\left[1 + \frac{4-D}{2}\ln 4\pi + \mathcal{O}((4-D)^2)\right]$$

$$\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}} = \frac{1}{16\pi^2}\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2-K)^n} = \frac{(-1)^n i}{(4\pi)^{D/2}}\frac{\Gamma(n-D/2)}{\Gamma(n)}\frac{1}{K^{n-D/2}}$$

$$\Gamma(2-D/2) = (1-D/2)\Gamma(1-D/2), \quad \Gamma(1) = 1, \quad \frac{1}{1-(4-D)/2} = 1 + \frac{4-D}{2} + \mathcal{O}((4-D)^2)$$

$$\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2-K} = \frac{-i}{(4\pi)^{D/2}}\frac{\Gamma(1-D/2)}{\Gamma(1)}\frac{1}{K^{1-D/2}} = \frac{-iK}{1-D/2}\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}}$$

$$= \frac{iK}{1-(4-D)/2}\frac{1}{16\pi^2}\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$= \frac{iK}{16\pi^2}\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln K + 1 + \mathcal{O}(4-D)\right]$$

$$\mu^{4-D} = 1 + \frac{4-D}{2}\ln\mu^2 + \mathcal{O}((4-D)^2)$$

$$A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{1}{q^2-m^2+i\varepsilon} = -16i\pi^2\mu^{4-D}\int\frac{d^D q}{(2\pi)^D}\frac{1}{q^2-(m^2-i\varepsilon)}$$

$$= (m^2-i\varepsilon)\mu^{4-D}\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln(m^2-i\varepsilon) + 1 + \mathcal{O}(4-D)\right] = m^2\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi + \ln\mu^2 - \ln m^2 + 1 + \mathcal{O}(4-D)\right]$$

$$= m^2\left(\Delta + 1 - \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}(4-D)$$

$$\Delta \equiv \frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi$$

$$x(q^2-m_1^2+i\varepsilon) + (1-x)[(q+p)^2-m_2^2+i\varepsilon] = q^2 + 2(1-x)q\cdot p + (1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon$$

$$= [q+(1-x)p]^2 + x(1-x)p^2 - xm_1^2 - (1-x)m_2^2 + i\varepsilon$$

$$= \ell^2 - K$$

$$\ell \equiv q + (1-x)p, \quad K = -x(1-x)p^2 + xm_1^2 + (1-x)m_2^2 - i\varepsilon = x^2p^2 - x(p^2-m_1^2+m_2^2) + m_2^2 - i\varepsilon$$

$$\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \int_0^1 dx\frac{1}{\{x[q^2-m_1^2+i\varepsilon]+(1-x)[(q+p)^2-m_2^2+i\varepsilon]\}^2} = \int_0^1 dx\frac{1}{(\ell^2-K)^2}$$

$$\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2} = i\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}K^{2-D/2}} = \frac{i}{16\pi^2}\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right]$$

$$B_0(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q\frac{1}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{16\pi^2\mu^{4-D}}{i}\int_0^1 dx\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-K)^2}$$

$$= \mu^{4-D}\int_0^1 dx\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi - \ln K + \mathcal{O}(4-D)\right] = \int_0^1 dx\left[\frac{2}{4-D} - \gamma_{\mathrm{E}} + \ln 4\pi + \ln\mu^2 - \ln K + \mathcal{O}(4-D)\right]$$

$$= \Delta - \int_0^1 dx\ln\frac{x^2p^2 - x(p^2-m_1^2+m_2^2) + m_2^2 - i\varepsilon}{\mu^2} + \mathcal{O}(4-D)$$

$$x^2p^2 - x(p^2-m_1^2+m_2^2) + m_2^2 - i\varepsilon = p^2(x-x_+)(x-x_-)$$

$$x_\pm \equiv \frac{1}{2p^2}\left[p^2 - m_1^2 + m_2^2 \pm\sqrt{(p^2-m_1^2+m_2^2)^2 - 4p^2(m_2^2-i\varepsilon)}\right]$$

$$\int_0^1 dx\ln\frac{x^2p^2 - x(p^2-m_1^2+m_2^2) + m_2^2 - i\varepsilon}{\mu^2} = \int_0^1 dx\ln\frac{p^2(x-x_+)(x-x_-)}{\mu^2} = \ln\frac{p^2}{\mu^2} + \int_0^1 dx[\ln(x-x_+) + \ln(x-x_-)]$$

$$= \ln\frac{p^2}{\mu^2} + [(x-x_+)\ln(x-x_+) - x + (x-x_-)\ln(x-x_-) - x]\Big|_0^1 = \ln\frac{p^2}{\mu^2} + f(x_+) + f(x_-)$$

$$f(z) \equiv (1-z)\ln(1-z) + z\ln(-z) - 1 = \ln(1-z) - z\ln(1-z^{-1}) - 1$$

$$B_0(p^2,m_1^2,m_2^2) = \Delta - \ln\frac{p^2}{\mu^2} - f(x_+) - f(x_-) + \mathcal{O}(4-D)$$

$$p_\mu B_1(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q_\mu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$p^2 B_1(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$2q\cdot p = [(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-p^2-m_1^2+m_2^2$$

$$\frac{2q\cdot p}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{1}{q^2-m_1^2+i\varepsilon}-\frac{1}{(q+p)^2-m_2^2+i\varepsilon}-\frac{p^2+m_1^2-m_2^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$2p^2 B_1(p^2,m_1^2,m_2^2) = A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)$$

$$B_1(p^2,m_1^2,m_2^2) = \frac{1}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(p^2+m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$g_{\mu\nu}B_{00}(p^2,m_1^2,m_2^2)+p_\mu p_\nu B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q_\mu q_\nu}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$DB_{00}(p^2,m_1^2,m_2^2)+p^2 B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$q^2 = (q^2-m_1^2+i\varepsilon)+m_1^2-i\varepsilon$$

$$\frac{q^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} = \frac{1}{(q+p)^2-m_2^2+i\varepsilon}+\frac{m_1^2-i\varepsilon}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$DB_{00}(p^2,m_1^2,m_2^2)+p^2 B_{11}(p^2,m_1^2,m_2^2) = A_0(m_2^2)+m_1^2 B_0(p^2,m_1^2,m_2^2)$$

$$4Dp^2 B_{00}(p^2,m_1^2,m_2^2)+4p^4 B_{11}(p^2,m_1^2,m_2^2) = 4p^2 A_0(m_2^2)+4p^2 m_1^2 B_0(p^2,m_1^2,m_2^2)$$

$$p^2 B_{00}(p^2,m_1^2,m_2^2)+p^4 B_{11}(p^2,m_1^2,m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^D q \frac{(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}$$

$$\begin{aligned}
4(q\cdot p)^2 &= \{[(q+p)^2-m_2^2+i\varepsilon]-[q^2-m_1^2+i\varepsilon]-[p^2+m_1^2-m_2^2]\}^2 \\
&= [(q+p)^2-m_2^2+i\varepsilon]\{[(q+p)^2-m_2^2+i\varepsilon]-2[q^2-m_1^2+i\varepsilon]-2[p^2+m_1^2-m_2^2]\} \\
&\quad +[q^2-m_1^2+i\varepsilon]\{[q^2-m_1^2+i\varepsilon]+2[p^2+m_1^2-m_2^2]\}+[p^2+m_1^2-m_2^2]^2 \\
&= [(q+p)^2-m_2^2+i\varepsilon](-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon)+(q^2-m_1^2+i\varepsilon)(q^2+2p^2+m_1^2-2m_2^2+i\varepsilon)+(p^2+m_1^2-m_2^2)^2
\end{aligned}$$

$$\begin{aligned}
\frac{4(q\cdot p)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} &= \frac{-q^2+2q\cdot p-p^2+m_2^2-i\varepsilon}{q^2-m_1^2+i\varepsilon}+\frac{q^2+2p^2+m_1^2-2m_2^2+i\varepsilon}{(q+p)^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} \\
&= -1+\frac{2q\cdot p-p^2-m_1^2+m_2^2}{q^2-m_1^2+i\varepsilon}+1+\frac{-2q\cdot p+p^2+m_1^2-m_2^2}{(q+p)^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]} \\
&= \frac{2q\cdot p-(p^2+m_1^2-m_2^2)}{q^2-m_1^2+i\varepsilon}+\frac{-2q'\cdot p+3p^2+m_1^2-m_2^2}{q'^2-m_2^2+i\varepsilon}+\frac{(p^2+m_1^2-m_2^2)^2}{[q^2-m_1^2+i\varepsilon][(q+p)^2-m_2^2+i\varepsilon]}
\end{aligned}$$

$$q' = q+p$$

$$4p^2 B_{00}(p^2,m_1^2,m_2^2)+4p^4 B_{11}(p^2,m_1^2,m_2^2) = -(p^2+m_1^2-m_2^2)A_0(m_1^2)+(3p^2+m_1^2-m_2^2)A_0(m_2^2)+(p^2+m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$

$$4(D-1)p^2 B_{00}(p^2,m_1^2,m_2^2) = 4p^2 A_0(m_2^2) + 4p^2 m_1^2 B_0(p^2,m_1^2,m_2^2)$$
$$+(p^2+m_1^2-m_2^2)A_0(m_1^2) - (3p^2+m_1^2-m_2^2)A_0(m_2^2) - (p^2+m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$
$$= p^2 A_0(m_1^2) + p^2 A_0(m_2^2) + 2p^2(m_1^2+m_2^2)B_0(p^2,m_1^2,m_2^2) - p^4 B_0(p^2,m_1^2,m_2^2)$$
$$+(m_1^2-m_2^2)A_0(m_1^2) - (m_1^2-m_2^2)A_0(m_2^2) - (m_1^2-m_2^2)^2 B_0(p^2,m_1^2,m_2^2)$$

$$[1-(4-D)/3]6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]$$

$$\frac{1}{1-(4-D)/3} = 1+\frac{1}{3}(4-D)+\mathcal{O}((4-D)^2)$$

$$6B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{1-(4-D)/3}\left\{\frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left.+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$
$$= \frac{1}{3}(m_1^2+m_2^2) + \frac{2}{3}\left(m_1^2+m_2^2-\frac{1}{2}p^2\right) + \frac{m_1^2-m_2^2}{3p^2}[m_1^2-m_2^2-(m_1^2-m_2^2)]$$
$$+\frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)$$
$$+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)] + \mathcal{O}(4-D)$$

$$B_{00}(p^2,m_1^2,m_2^2) = \frac{1}{6}\left\{m_1^2+m_2^2-\frac{1}{3}p^2+\frac{1}{2}[A_0(m_1^2)+A_0(m_2^2)] + \left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0(p^2,m_1^2,m_2^2)\right.$$
$$\left.+\frac{m_1^2-m_2^2}{2p^2}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]\right\}$$

$$J_1(p^2,m_1^2,m_2^2) \equiv A_0(m_1^2)+A_0(m_2^2)-(p^2-m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)-4B_{00}(p^2,m_1^2,m_2^2)$$

$$J_1'(p^2,m_1^2,m_2^2) = -B_0(p^2,m_1^2,m_2^2)-(p^2-m_1^2-m_2^2)B_0'(p^2,m_1^2,m_2^2)-4B_{00}'(p^2,m_1^2,m_2^2)$$

$$B_0'(p^2,m_1^2,m_2^2) = -\int_0^1 dx \frac{x^2-x}{x^2p^2-x(p^2-m_1^2+m_2^2)+m_2^2}$$

$$B_{00}'(p^2,m_1^2,m_2^2) = \frac{1}{6}\left\{-\frac{1}{3}-\frac{1}{2}B_0(p^2,m_1^2,m_2^2)+\left(m_1^2+m_2^2-\frac{1}{2}p^2\right)B_0'(p^2,m_1^2,m_2^2)\right.$$
$$\left.-\frac{m_1^2-m_2^2}{2p^4}[A_0(m_1^2)-A_0(m_2^2)-(m_1^2-m_2^2)B_0(p^2,m_1^2,m_2^2)]-\frac{(m_1^2-m_2^2)^2}{2p^2}B_0'(p^2,m_1^2,m_2^2)\right\}$$

$\underline{m = m_1 = m_2}$

$$-\ln\frac{x^2p^2-xp^2+m^2}{\mu^2} = -\ln\frac{m^2}{\mu^2}+\mathcal{O}(p^2)$$

$$B_0(p^2,m^2,m^2) = \Delta-\int_0^1 dx \ln\frac{x^2p^2-xp^2+m^2-i\varepsilon}{\mu^2} = \Delta-\ln\frac{m^2}{\mu^2}+\mathcal{O}(p^2)$$

$$-\frac{x^2-x}{x^2p^2-xp^2+m^2} = -\frac{x^2-x}{m^2}+\mathcal{O}(p^2)$$

$$B_0'(p^2,m^2,m^2) = -\int_0^1 dx \frac{x^2-x}{x^2p^2-xp^2+m^2} = \frac{1}{6m^2}+\mathcal{O}(p^2)$$

$$B_{00}(p^2,m^2,m^2) = \frac{1}{3}m^2-\frac{1}{18}p^2+\frac{1}{6}A_0(m^2)+\left(\frac{1}{3}m^2-\frac{1}{12}p^2\right)B_0(p^2,m^2,m^2)$$

$$B_{00}'(p^2,m^2,m^2) = -\frac{1}{18}-\frac{1}{12}B_0(p^2,m^2,m^2)+\left(\frac{1}{3}m^2-\frac{1}{12}p^2\right)B_0'(p^2,m^2,m^2)$$

$$J_1(p^2,m^2,m^2) = 2A_0(m^2)-(p^2-2m^2)B_0(p^2,m^2,m^2)-4B_{00}(p^2,m^2,m^2)$$

$$J_1'(p^2,m^2,m^2) = -B_0(p^2,m^2,m^2)-(p^2-2m^2)B_0'(p^2,m^2,m^2)+\frac{2}{9}+\frac{1}{3}B_0(p^2,m^2,m^2)-\left(\frac{4}{3}m^2-\frac{1}{3}p^2\right)B_0'(p^2,m^2,m^2)$$
$$=\frac{2}{9}-\frac{2}{3}B_0(p^2,m^2,m^2)+\frac{2}{3}(m^2-p^2)B_0'(p^2,m^2,m^2)$$
$$=\frac{2}{9}-\frac{2}{3}\left(\Delta-\ln\frac{m^2}{\mu^2}\right)+\frac{2}{3}m^2\frac{1}{6m^2}+\mathcal{O}(p^2) = -\frac{2}{3}\Delta+\frac{1}{3}+\frac{2}{3}\ln\frac{m^2}{\mu^2}+\mathcal{O}(p^2)$$

$\underline{m_1 = 0}$

$$-\ln\frac{x^2p^2-x(p^2+m_2^2)+m_2^2}{\mu^2} = -\ln\frac{(1-x)m_2^2}{\mu^2}+\frac{xp^2}{m_2^2}+\frac{x^2p^4}{2m_2^4}+\mathcal{O}(p^6)$$

$$B_0(p^2,0,m_2^2) = \Delta-\int_0^1 dx \ln\frac{x^2p^2-x(p^2+m_2^2)+m_2^2-i\varepsilon}{\mu^2} = \Delta+1-\ln\frac{m_2^2}{\mu^2}+\frac{p^2}{2m_2^2}+\frac{p^4}{6m_2^4}+\mathcal{O}(p^6)$$

$$-\frac{x^2-x}{x^2p^2-x(p^2+m_2^2)+m_2^2} = \frac{x}{m_2^2}+\frac{x^2p^2}{m_2^4}+\frac{x^3p^4}{m_2^6}+\mathcal{O}(p^6)$$

$$B_0'(p^2,0,m_2^2) = -\int_0^1 dx \frac{x^2-x}{x^2p^2-x(p^2+m_2^2)+m_2^2-i\varepsilon} = \frac{1}{2m_2^2}+\frac{p^2}{3m_2^4}+\frac{p^4}{4m_2^6}+\mathcal{O}(p^6)$$

$$-A_0(m_2^2)+m_2^2B_0(p^2,0,m_2^2) = \frac{p^2}{2}+\frac{p^4}{6m_2^2}+\mathcal{O}(p^6)$$

$$B_{00}'(p^2,0,m_2^2) = \frac{1}{6}\left[-\frac{1}{3}-\frac{1}{2}B_0(p^2,0,m_2^2)+\left(m_2^2-\frac{1}{2}p^2\right)B_0'(p^2,0,m_2^2)+\frac{m_2^2}{2p^4}[-A_0(m_2^2)+m_2^2B_0(p^2,0,m_2^2)]-\frac{m_2^4}{2p^2}B_0'(p^2,0,m_2^2)\right]$$
$$=\frac{1}{6}\left[-\frac{1}{3}-\frac{1}{2}\left(\Delta+1-\ln\frac{m_2^2}{\mu^2}\right)+m_2^2\frac{1}{2m_2^2}+\frac{m_2^2}{2p^4}\left(\frac{p^2}{2}+\frac{p^4}{6m_2^2}\right)-\frac{m_2^4}{2p^2}\left(\frac{1}{2m_2^2}+\frac{p^2}{3m_2^4}\right)\right]+\mathcal{O}(p^2)$$
$$=-\frac{1}{12}\Delta-\frac{5}{72}+\frac{1}{12}\ln\frac{m_2^2}{\mu^2}+\mathcal{O}(p^2)$$

$$J_1'(p^2,0,m_2^2) = -B_0(p^2,0,m_2^2)-(p^2-m_2^2)B_0'(p^2,0,m_2^2)-4B_{00}'(p^2,0,m_2^2)$$
$$=-\left(\Delta+1-\ln\frac{m_2^2}{\mu^2}\right)+m_2^2\frac{1}{2m_2^2}-4\left(-\frac{1}{12}\Delta-\frac{5}{72}+\frac{1}{12}\ln\frac{m_2^2}{\mu^2}\right)+\mathcal{O}(p^2)$$
$$=-\frac{2}{3}\Delta-\frac{2}{9}+\frac{2}{3}\ln\frac{m_2^2}{\mu^2}+\mathcal{O}(p^2)$$

Custodial symmetry

Ref: Jose Santiago's lecture note "the Physics of Electroweak Symmetry Breaking", 2009;
Montero & Pleitez, hep-ph/0607144; Branco, Ferreira, Lavoura, Rebelo, Sher & Silva, 1106.0034

$\mathrm{SU(2)_L \times SU(2)_R}$ bidoublet $\mathbf{H} = \begin{pmatrix} \tilde{H} & H \end{pmatrix} = \begin{pmatrix} H^{0*} & H^+ \\ -H^- & H^0 \end{pmatrix}$

$\tilde{H}^\dagger \tilde{H} = H^0 H^{0*} + H^+ H^- = H^\dagger H, \quad \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) = \tilde{H}^\dagger \tilde{H} + H^\dagger H = 2|H|^2$

$U_\mathrm{L} \in$ global $\mathrm{SU(2)_L}$, $U_\mathrm{R} \in$ global $\mathrm{SU(2)_R}$, $\mathbf{H} \to U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger \quad \Rightarrow \quad \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) \to \mathrm{tr}(U_\mathrm{R} \mathbf{H}^\dagger U_\mathrm{L}^\dagger U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger) = \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H})$

SM scalar potential $V = -\mu^2 |H|^2 + \lambda |H|^4 = -\dfrac{\mu^2}{2} \mathrm{tr}(\mathbf{H}^\dagger \mathbf{H}) + \dfrac{\lambda}{4} [\mathrm{tr}(\mathbf{H}^\dagger \mathbf{H})]^2$

$W_\mu \equiv W_\mu^a \dfrac{\sigma^a}{2}, \quad (\sigma^2)^2 = 1, \quad \sigma^a \sigma^2 = -\sigma^2 (\sigma^a)^*$

$$|(\partial_\mu - igW_\mu)\tilde{H}|^2 = [(\partial^\mu - igW^\mu) i\sigma^2 H^*]^\dagger (\partial_\mu - igW_\mu) i\sigma^2 H^* = H^\mathrm{T} \sigma^2 (\overleftarrow{\partial^\mu} + igW^\mu)(\partial_\mu - igW_\mu)\sigma^2 H^*$$
$$= H^\mathrm{T} (\overleftarrow{\partial^\mu} - igW^{\mu *})(\partial_\mu + igW_\mu^*) H^* = H^\dagger (\overleftarrow{\partial^\mu} + igW^\mu)(\partial_\mu - igW_\mu) H = |(\partial_\mu - igW_\mu) H|^2$$

$\mathbf{D}_\mu \mathbf{H} \equiv (\partial_\mu - igW_\mu)\mathbf{H} = \begin{pmatrix} (\partial_\mu - igW_\mu)\tilde{H} & (\partial_\mu - igW_\mu) H \end{pmatrix}$

$\mathrm{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}] = |(\partial_\mu - igW_\mu)\tilde{H}|^2 + |(\partial_\mu - igW_\mu) H|^2 = 2|(\partial_\mu - igW_\mu) H|^2$

$W_\mu \to U_\mathrm{L} W_\mu U_\mathrm{L}^\dagger$

$\mathbf{D}_\mu \mathbf{H} \to (\partial_\mu - igU_\mathrm{L} W_\mu U_\mathrm{L}^\dagger) U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger = U_\mathrm{L} \mathbf{D}_\mu \mathbf{H} U_\mathrm{R}^\dagger$

$\mathrm{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}] \to \mathrm{tr}(U_\mathrm{R} \mathbf{D}^\mu \mathbf{H}^\dagger U_\mathrm{L}^\dagger U_\mathrm{L} \mathbf{D}_\mu \mathbf{H} U_\mathrm{R}^\dagger) = \mathrm{tr}[(\mathbf{D}^\mu \mathbf{H})^\dagger \mathbf{D}_\mu \mathbf{H}]$

An $\mathrm{SU(2)_L \times SU(2)_R}$ global symmetry exists in the potential and the $\mathrm{SU(2)_L}$ gauge interaction of the Higgs field

$\langle H \rangle = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle \mathbf{H} \rangle = \dfrac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \to U_\mathrm{L} \langle \mathbf{H} \rangle U_\mathrm{R}^\dagger = \dfrac{1}{\sqrt{2}} U_\mathrm{L} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} U_\mathrm{R}^\dagger = \dfrac{v}{\sqrt{2}} U_\mathrm{L} U_\mathrm{R}^\dagger$

$U_\mathrm{L} = U_\mathrm{R} \quad \Rightarrow \quad U_\mathrm{L} \langle \mathbf{H} \rangle U_\mathrm{R}^\dagger = \dfrac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \langle \mathbf{H} \rangle$

$\Rightarrow$ the vacuum is invariant under the $\mathrm{SU(2)_{L+R}}$ global symmetry (equal left and right rotations)

$\Rightarrow$ $\mathrm{SU(2)_L \times SU(2)_R} \to$ $\mathrm{SU(2)_{L+R}}$ custodial symmetry

$\Rightarrow$ W_μ^a transform as a triplet under $\mathrm{SU(2)_{L+R}}$

$\Rightarrow$ identical contributions to the masses of W_μ^a from electroweak symmetry breaking

$\Rightarrow$ the custodial symmetry protects the parameter $\rho = \dfrac{m_W^2}{m_Z^2 c_\mathrm{W}^2}$, and leads to $T = U = 0$

Custodial symmetry violation in the SM

1) $\mathrm{U(1)_Y}$ gauge interaction

$\mathrm{U(1)_Y}$ gauge transformation $H \to e^{iY_H \theta(x)} H, \quad \tilde{H} \to e^{-iY_H \theta(x)} \tilde{H}, \quad Y_H = \dfrac{1}{2}$

$$\exp[-iY_H \theta(x) \sigma^3] = \sum_{k=0}^{\infty} \frac{1}{k!} [-iY_H \theta(x) \sigma^3]^k = \sum_{k=0}^{\infty} \frac{1}{(2k)!} [-iY_H \theta(x)]^{2k} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} [-iY_H \theta(x)]^{2k+1} \sigma^3$$
$$= \cos[Y_H \theta(x)] - i[Y_H \theta(x)]\sigma^3 = \begin{pmatrix} \cos[Y_H \theta(x)] - i\sin[Y_H \theta(x)] & \\ & \cos[Y_H \theta(x)] + i\sin[Y_H \theta(x)] \end{pmatrix} = \begin{pmatrix} e^{-iY_H \theta(x)} & \\ & e^{iY_H \theta(x)} \end{pmatrix}$$

$\mathbf{H} \to \begin{pmatrix} e^{-iY_H \theta(x)} \tilde{H} & e^{iY_H \theta(x)} H \end{pmatrix} = \mathbf{H} \exp[-iY_H \theta(x) \sigma^3], \quad \dfrac{\sigma^3}{2}$ is just one $\mathrm{SU(2)_R}$ generator

$Y_H \neq 0$ violates custodial symmetry

2) Difference between Yukawa couplings of up-type and down-type fermions

$Q_\mathrm{L} \equiv \begin{pmatrix} t_\mathrm{L} \\ b'_\mathrm{L} \end{pmatrix} \to U_\mathrm{L} Q_\mathrm{L}, \quad \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix} \to U_\mathrm{R} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix}, \quad \bar{Q}_\mathrm{L} \mathbf{H} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix} \to \bar{Q}_\mathrm{L} U_\mathrm{L}^\dagger U_\mathrm{L} \mathbf{H} U_\mathrm{R}^\dagger U_\mathrm{R} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix} = \bar{Q}_\mathrm{L} \mathbf{H} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix}$

$-y \bar{Q}_\mathrm{L} \mathbf{H} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix} = -y \bar{Q}_\mathrm{L} \begin{pmatrix} \tilde{H} & H \end{pmatrix} \begin{pmatrix} t_\mathrm{R} \\ b'_\mathrm{R} \end{pmatrix} = -y(\bar{Q}_\mathrm{L} \tilde{H} t_\mathrm{R} + \bar{Q}_\mathrm{L} H b'_\mathrm{R})$

Yukawa couplings in the SM: $-y_t \bar{Q}_\mathrm{L} \tilde{H} t_\mathrm{R} - y_b \bar{Q}_\mathrm{L} H b'_\mathrm{R} = -\bar{Q}_\mathrm{L} \mathbf{H} \begin{pmatrix} y_t t_\mathrm{R} \\ y_b b'_\mathrm{R} \end{pmatrix}$

$y_t \neq y_b$ violates custodial symmetry

$\mathrm{U(1)_Y}$ gauge symmetry

$$\mathbf{H} \to \mathbf{H} \exp\left(-i\theta(x)\frac{\sigma^3}{2}\right), \quad B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu \theta$$

$$D_\mu H = \left(\partial_\mu - igW_\mu - \frac{i}{2}g' B_\mu\right)H, \quad D_\mu \tilde{H} = \left(\partial_\mu - igW_\mu + \frac{i}{2}g' B_\mu\right)\tilde{H}$$

$$D_\mu \mathbf{H} = \partial_\mu \mathbf{H} - igW_\mu \mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu$$

$$D_\mu \mathbf{H} \to \left[\partial_\mu \mathbf{H} + \mathbf{H}\left(-i\partial_\mu\theta\frac{\sigma^3}{2}\right) - igW_\mu \mathbf{H} + ig'\mathbf{H}\frac{\sigma^3}{2}B_\mu + ig'\mathbf{H}\frac{1}{g'}\partial_\mu\theta\frac{\sigma^3}{2}\right]\exp\left(-i\theta(x)\frac{\sigma^3}{2}\right)$$

$$= D_\mu \mathbf{H} \exp\left(-i\theta(x)\frac{\sigma^3}{2}\right)$$

$H^\dagger W^a_{\mu\nu}\sigma^a H B^{\mu\nu}$ operator

$$W_{\mu\nu} = W^a_{\mu\nu}\frac{\sigma^a}{2} = (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon^{abc}W^b_\mu W^c_\nu)\frac{\sigma^a}{2}$$

$\mathrm{SU(2)_L \times SU(2)_R}$ global transformation:

$$\mathbf{H} \to U_\mathrm{L}\mathbf{H}U_\mathrm{R}^\dagger, \quad W_\mu \to U_\mathrm{L}W_\mu U_\mathrm{L}^\dagger, \quad W_{\mu\nu} \to U_\mathrm{L}W_{\mu\nu}U_\mathrm{L}^\dagger$$

$$\mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu} \to \mathrm{tr}\left(U_\mathrm{R}\mathbf{H}^\dagger U_\mathrm{L}^\dagger U_\mathrm{L} W_{\mu\nu} U_\mathrm{L}^\dagger U_\mathrm{L}\mathbf{H}U_\mathrm{R}^\dagger\right)B^{\mu\nu} = \mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu}$$

$$\tilde{H}^\dagger W_{\mu\nu}\tilde{H} = H^\mathrm{T}\sigma^2 W_{\mu\nu}\sigma^2 H^* = -H^\mathrm{T}W^*_{\mu\nu}H^* = -H^\dagger W_{\mu\nu}H$$

$$\mathrm{tr}(\mathbf{H}^\dagger W_{\mu\nu}\mathbf{H})B^{\mu\nu} = \mathrm{tr}\left[\begin{pmatrix}\tilde{H}^\dagger \\ H^\dagger\end{pmatrix}W_{\mu\nu}\begin{pmatrix}\tilde{H} & H\end{pmatrix}\right]B^{\mu\nu} = \mathrm{tr}\begin{pmatrix}\tilde{H}^\dagger W_{\mu\nu}\tilde{H} & \tilde{H}^\dagger W_{\mu\nu}H \\ H^\dagger W_{\mu\nu}\tilde{H} & H^\dagger W_{\mu\nu}H\end{pmatrix}B^{\mu\nu}$$

$$= (\tilde{H}^\dagger W_{\mu\nu}\tilde{H} + H^\dagger W_{\mu\nu}H)B^{\mu\nu} = 0$$

$\Rightarrow$ The operator $H^\dagger W^a_{\mu\nu}\sigma^a H B^{\mu\nu}$ does not respect the custodial symmetry

$\left\{\right.$ However, scalar potential terms or Yukawa terms for EW multiplets that respect the custodial symmetry, along with the $\mathrm{U(1)_Y}$ gauge interaction that violates the custodial symmetry, can contribute to this operator and hence to S $\left.\right\}$

Gauge symmetry: matrix notation vs tensor notation

Triplet in SU(2) as an example

Matrix notation

$U(x) = \exp[i\theta^a(x) t_{\mathrm{T}}^a], \quad U^\dagger = \exp(-i\theta^a t_{\mathrm{T}}^a)$

$D_\mu T \equiv (\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a)T$

$$T \to UT, \quad W_\mu^a t_{\mathrm{T}}^a \to UW_\mu^a t_{\mathrm{T}}^a U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger$$

$$D_\mu T \to \left[\partial_\mu - ig\left(UW_\mu^a t_{\mathrm{T}}^a U^\dagger + \frac{i}{g} U\partial_\mu U^\dagger\right)\right]UT$$

$$= UU^\dagger(\partial_\mu U)T + U\partial_\mu T - igUW_\mu^a t_{\mathrm{T}}^a T + U(\partial_\mu U^\dagger)UT$$

$$= U(\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a)T + U\partial_\mu(U^\dagger U)UT = UD_\mu T$$

$D_\mu T$ transforms as T

Infinitesimal transformation $U \simeq 1 + i\theta^a t_{\mathrm{T}}^a$

$$\delta T = i\theta^a t_{\mathrm{T}}^a T, \quad \delta(W_\mu^a t_{\mathrm{T}}^a) = i\theta^b [t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a] + \frac{1}{g}(\partial_\mu \theta^a) t_{\mathrm{T}}^a$$

$$\delta(D_\mu T) = (\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a)\delta T - ig\delta(W_\mu^a t_{\mathrm{T}}^a)T$$

$$= (\partial_\mu - igW_\mu^a t_{\mathrm{T}}^a) i\theta^b t_{\mathrm{T}}^b T - ig\left\{ i\theta^b [t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a] + \frac{1}{g}(\partial_\mu \theta^a) t_{\mathrm{T}}^a \right\} T$$

$$= i(\partial_\mu \theta^a) t_{\mathrm{T}}^a T + i\theta^a t_{\mathrm{T}}^a \partial_\mu T + g\theta^b W_\mu^a t_{\mathrm{T}}^a t_{\mathrm{T}}^b T + g\theta^b [t_{\mathrm{T}}^b, W_\mu^a t_{\mathrm{T}}^a]T - i(\partial_\mu \theta^a) t_{\mathrm{T}}^a T$$

$$= i\theta^a t_{\mathrm{T}}^a \partial_\mu T + g\theta^b t_{\mathrm{T}}^b W_\mu^a t_{\mathrm{T}}^a T = i\theta^a t_{\mathrm{T}}^a (\partial_\mu - igW_\mu^b t_{\mathrm{T}}^b)T = i\theta^a t_{\mathrm{T}}^a D_\mu T$$

$$D_\mu D_\nu T = \partial_\mu \partial_\nu T - ig(\partial_\mu W_\nu^a) t_{\mathrm{T}}^a T - igW_\nu^a t_{\mathrm{T}}^a \partial_\mu T - igW_\mu^a t_{\mathrm{T}}^a (\partial_\nu T - igW_\nu^b t_{\mathrm{T}}^b T)$$

$$[D_\mu, D_\nu]T = -ig(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) t_{\mathrm{T}}^a T - g^2 [W_\mu^a t_{\mathrm{T}}^a, W_\nu^b t_{\mathrm{T}}^b]T$$

$$W_{\mu\nu}^a t_{\mathrm{T}}^a T \equiv \frac{i}{g}[D_\mu, D_\nu]T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) t_{\mathrm{T}}^a T - ig[W_\mu^a t_{\mathrm{T}}^a, W_\nu^b t_{\mathrm{T}}^b]T = (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc} W_\mu^b W_\nu^c) t_{\mathrm{T}}^a T$$

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\varepsilon^{abc} W_\mu^b W_\nu^c$$

$$[D_\mu, D_\nu]T \to U[D_\mu, D_\nu]T = U[D_\mu, D_\nu]U^\dagger UT \quad \Rightarrow \quad W_{\mu\nu}^a t_{\mathrm{T}}^a \to UW_{\mu\nu}^a t_{\mathrm{T}}^a U^\dagger$$

$\mathrm{tr}(W_{\mu\nu}^a t_{\mathrm{T}}^a W^{b\mu\nu} t_{\mathrm{T}}^b)$ is a gauge invariant

Tensor notation

$$V^i_j(x)=\exp[i\theta^a(x)\tau^a]^i_j,\quad (V^\dagger)^i_j=\exp(-i\theta^a\tau^a)^i_j,\quad \tau^a\equiv\frac{\sigma^a}{2}$$

$$\underline{(D_\mu T)^i_j\equiv\partial_\mu T^i_j-ig(W_\mu)^i_k T^k_j+igT^i_k(W_\mu)^k_j}$$

[Ref: Ta-Pei Cheng & Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, Eq.(4.140)]

$$T^i_j\to V^i_k T^k_l (V^\dagger)^l_j,\quad (W_\mu)^i_j\to V^i_k(W_\mu)^k_l(V^\dagger)^l_j+\frac{i}{g}V^i_k\partial_\mu(V^\dagger)^k_j$$

Infinitesimal transformation $V^i_j\simeq\delta^i_j+i\theta^a(\tau^a)^i_j,\quad (V^\dagger)^i_j\simeq\delta^i_j-i\theta^a(\tau^a)^i_j$

$$\delta T^i_j=i\theta^a(\tau^a)^i_k T^k_j-i\theta^a T^i_l(\tau^a)^l_j=i\theta^a[(\tau^a)^i_k T^k_j-T^i_k(\tau^a)^k_j]$$

$$\delta(W_\mu)^i_j=i\theta^a[(\tau^a)^i_k(W_\mu)^k_j-(W_\mu)^i_k(\tau^a)^k_j]+\frac{1}{g}(\partial_\mu\theta^a)(\tau^a)^i_j$$

$$\begin{aligned}
\delta(D_\mu T)^i_j&=\delta(\partial_\mu T^i_j)-ig\delta(W_\mu)^i_l T^l_j+ig\delta(W_\mu)^l_j T^i_l-ig(W_\mu)^i_l\delta T^l_j+ig\delta T^i_l(W_\mu)^l_j\\
&=i(\partial_\mu\theta^a)[(\tau^a)^i_k T^k_j-T^i_k(\tau^a)^k_j]+i\theta^a[(\tau^a)^i_k\partial_\mu T^k_j-(\partial_\mu T^i_k)(\tau^a)^k_j]\\
&\quad+g\theta^a[(\tau^a)^i_k(W_\mu)^k_l-(W_\mu)^i_k(\tau^a)^k_l]T^l_j-i(\partial_\mu\theta^a)(\tau^a)^i_l T^l_j\\
&\quad-g\theta^a T^i_l[(\tau^a)^l_k(W_\mu)^k_j-(W_\mu)^l_k(\tau^a)^k_j]+i(\partial_\mu\theta^a)T^i_l(\tau^a)^l_j\\
&\quad+g(W_\mu)^i_l\theta^a[(\tau^a)^l_k T^k_j-T^l_k(\tau^a)^k_j]-g\theta^a[(\tau^a)^i_k T^k_l-T^i_k(\tau^a)^k_l](W_\mu)^l_j\\
&=i\theta^a(\tau^a)^i_k\partial_\mu T^k_j+g\theta^a(\tau^a)^i_k(W_\mu)^k_l T^l_j-g\theta^a(\tau^a)^i_k T^k_l(W_\mu)^l_j\\
&\quad-i\theta^a(\partial_\mu T^i_k)(\tau^a)^k_j-g\theta^a(W_\mu)^i_l T^l_k(\tau^a)^k_j+g\theta^a T^i_l(W_\mu)^l_k(\tau^a)^k_j\\
&=i\theta^a(\tau^a)^i_k[\partial_\mu T^k_j-ig(W_\mu)^k_l T^l_j+igT^k_l(W_\mu)^l_j]-i\theta^a[\partial_\mu T^i_k-ig(W_\mu)^i_l T^l_k+igT^i_l(W_\mu)^l_k](\tau^a)^k_j\\
&=i\theta^a[(\tau^a)^i_k(D_\mu T)^k_j-(D_\mu T)^i_k(\tau^a)^k_j]
\end{aligned}$$

$\underline{(D_\mu T)^i_j \text{ transforms as } T^i_j}$

$$\begin{aligned}
(D_\mu D_\nu T)^i_j&=\partial_\mu\partial_\nu T^i_j-ig\partial_\mu(W_\nu)^i_k T^k_j-ig(W_\nu)^i_k\partial_\mu T^k_j+ig(\partial_\mu T^i_k)(W_\nu)^k_j+igT^i_k\partial_\mu(W_\nu)^k_j\\
&\quad-ig(W_\mu)^i_l[\partial_\nu T^l_j-ig(W_\nu)^l_k T^k_j+igT^l_k(W_\nu)^k_j]+ig[\partial_\nu T^i_l-ig(W_\nu)^i_k T^k_l+igT^i_k(W_\nu)^k_l](W_\mu)^l_j
\end{aligned}$$

$$\begin{aligned}
([D_\mu,D_\nu]T)^i_j&=-ig[\partial_\mu(W_\nu)^i_k-\partial_\nu(W_\mu)^i_k]T^k_j+igT^i_k[\partial_\mu(W_\nu)^k_j-\partial_\nu(W_\mu)^k_j]\\
&\quad-g^2[(W_\mu)^i_l(W_\nu)^l_k-(W_\nu)^i_l(W_\mu)^l_k]T^k_j+g^2T^i_k[(W_\mu)^k_l(W_\nu)^l_j-(W_\nu)^k_l(W_\mu)^l_j]
\end{aligned}$$

$$\begin{aligned}
(W_{\mu\nu})^i_k T^k_j-T^i_k(W_{\mu\nu})^k_j&\equiv\frac{i}{g}([D_\mu,D_\nu]T)^i_j=\{\partial_\mu(W_\nu)^i_k-\partial_\nu(W_\mu)^i_k-ig[(W_\mu)^i_l(W_\nu)^l_k-(W_\nu)^i_l(W_\mu)^l_k]\}T^k_j\\
&\quad-T^i_k\{\partial_\mu(W_\nu)^k_j-\partial_\nu(W_\mu)^k_j-ig[(W_\mu)^k_l(W_\nu)^l_j-(W_\nu)^k_l(W_\mu)^l_j]\}
\end{aligned}$$

$$(W_{\mu\nu})^i_k=\partial_\mu(W_\nu)^i_k-\partial_\nu(W_\mu)^i_k-ig[(W_\mu)^i_l(W_\nu)^l_k-(W_\nu)^i_l(W_\mu)^l_k]$$

$$\begin{aligned}
(W_{\mu\nu})^i_k T^k_j-T^i_k(W_{\mu\nu})^k_j&\to V^i_m[(W_{\mu\nu})^m_k T^k_n-T^m_k(W_{\mu\nu})^k_n](V^\dagger)^n_j\\
&=V^i_m(W_{\mu\nu})^m_k(V^\dagger)^k_l V^l_p T^p_n(V^\dagger)^n_j-V^i_m T^m_k(V^\dagger)^k_l V^l_p(W_{\mu\nu})^p_n(V^\dagger)^n_j
\end{aligned}$$

$\Rightarrow\quad (W_{\mu\nu})^i_j$ transforms as T^i_j

$(W_{\mu\nu})^i_j(W^{\mu\nu})^j_i$ is a gauge invariant

Gauge interactions

For an SU(2) doublet D, $D_\mu D = (\partial_\mu - igW_\mu^a \tau^a)D$, or equivalently, $\underline{(D_\mu D)^i = \partial_\mu D^i - ig(W_\mu)^i_{\ j} D^j}$

$$\Rightarrow \quad (W_\mu)^i_{\ j} = W_\mu^a (\tau^a)^i_{\ j} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}$$

$$W_\mu^+ = \sqrt{2}(W_\mu)^1_{\ 2}, \quad W_\mu^- = \sqrt{2}(W_\mu)^2_{\ 1}, \quad W_\mu^3 = 2(W_\mu)^1_{\ 1} = -2(W_\mu)^2_{\ 2}$$

For an SU(2) triplet T, $\underline{(D_\mu T)^i_{\ j} = \partial_\mu T^i_{\ j} - ig(W_\mu)^i_{\ k} T^k_{\ j} + igT^i_{\ k}(W_\mu)^k_{\ j}}$

$$T^+ = -T^1_{\ 2}, \quad T^- = -T^2_{\ 1}, \quad T^0 = \sqrt{2}T^1_{\ 1} = -\sqrt{2}T^2_{\ 2}$$

$$(T^+)^\dagger = -(T^\dagger)^2_{\ 1}, \quad (T^-)^\dagger = -(T^\dagger)^1_{\ 2}, \quad (T^0)^\dagger = \sqrt{2}(T^\dagger)^1_{\ 1} = -\sqrt{2}(T^\dagger)^2_{\ 2}$$

$$\begin{aligned}
(T^\dagger)^i_{\ j}\bar{\sigma}^\mu (W_\mu)^j_{\ k} T^k_{\ i} &= (T^\dagger)^1_{\ 1}\bar{\sigma}^\mu (W_\mu)^1_{\ 1} T^1_{\ 1} + (T^\dagger)^1_{\ 1}\bar{\sigma}^\mu (W_\mu)^1_{\ 2} T^2_{\ 1} + (T^\dagger)^1_{\ 2}\bar{\sigma}^\mu (W_\mu)^2_{\ 1} T^1_{\ 1} + (T^\dagger)^1_{\ 2}\bar{\sigma}^\mu (W_\mu)^2_{\ 2} T^2_{\ 1} \\
&\quad + (T^\dagger)^2_{\ 1}\bar{\sigma}^\mu (W_\mu)^1_{\ 1} T^1_{\ 2} + (T^\dagger)^2_{\ 1}\bar{\sigma}^\mu (W_\mu)^1_{\ 2} T^2_{\ 2} + (T^\dagger)^2_{\ 2}\bar{\sigma}^\mu (W_\mu)^2_{\ 1} T^1_{\ 2} + (T^\dagger)^2_{\ 2}\bar{\sigma}^\mu (W_\mu)^2_{\ 2} T^2_{\ 2} \\
&= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^+ T^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^- T^0 - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu W_\mu^3 T^- \\
&\quad + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^3 T^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu W_\mu^+ T^0 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^- T^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu W_\mu^3 T^0 \\
&= \frac{1}{2}[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]
\end{aligned}$$

$$\begin{aligned}
(T^\dagger)^i_{\ j}\bar{\sigma}^\mu T^j_{\ k} (W_\mu)^k_{\ i} &= (T^\dagger)^1_{\ 1}\bar{\sigma}^\mu T^1_{\ 1} (W_\mu)^1_{\ 1} + (T^\dagger)^1_{\ 1}\bar{\sigma}^\mu T^1_{\ 2} (W_\mu)^2_{\ 1} + (T^\dagger)^1_{\ 2}\bar{\sigma}^\mu T^2_{\ 1} (W_\mu)^1_{\ 1} + (T^\dagger)^1_{\ 2}\bar{\sigma}^\mu T^2_{\ 2} (W_\mu)^2_{\ 1} \\
&\quad + (T^\dagger)^2_{\ 1}\bar{\sigma}^\mu T^1_{\ 1} (W_\mu)^1_{\ 2} + (T^\dagger)^2_{\ 1}\bar{\sigma}^\mu T^1_{\ 2} (W_\mu)^2_{\ 2} + (T^\dagger)^2_{\ 2}\bar{\sigma}^\mu T^2_{\ 1} (W_\mu)^1_{\ 2} + (T^\dagger)^2_{\ 2}\bar{\sigma}^\mu T^2_{\ 2} (W_\mu)^2_{\ 2} \\
&= \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3 + \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^+ W_\mu^- - \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^- W_\mu^3 + \frac{1}{2}(T^-)^\dagger \bar{\sigma}^\mu T^0 W_\mu^- \\
&\quad - \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^0 W_\mu^+ + \frac{1}{2}(T^+)^\dagger \bar{\sigma}^\mu T^+ W_\mu^3 - \frac{1}{2}(T^0)^\dagger \bar{\sigma}^\mu T^- W_\mu^+ - \frac{1}{4}(T^0)^\dagger \bar{\sigma}^\mu T^0 W_\mu^3 \\
&= \frac{1}{2}[-W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- + W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 + W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} &\supset g[(T^\dagger)^i_{\ j}\bar{\sigma}^\mu (W_\mu)^j_{\ k} T^k_{\ i} - (T^\dagger)^i_{\ j}\bar{\sigma}^\mu T^j_{\ k} (W_\mu)^k_{\ i}] \\
&= g[W_\mu^3 (T^+)^\dagger \bar{\sigma}^\mu T^+ + W_\mu^+ (T^+)^\dagger \bar{\sigma}^\mu T^0 \\
&\qquad + W_\mu^- (T^0)^\dagger \bar{\sigma}^\mu T^+ - W_\mu^+ (T^0)^\dagger \bar{\sigma}^\mu T^- \\
&\qquad - W_\mu^- (T^-)^\dagger \bar{\sigma}^\mu T^0 - W_\mu^3 (T^-)^\dagger \bar{\sigma}^\mu T^-]
\end{aligned}$$

For a general SU(2) multiplet $\psi^{i_1\cdots i_p}_{j_1\cdots j_q}$,

$$(D_\mu \psi)^{i_1\cdots i_p}_{j_1\cdots j_q} = \partial_\mu \psi^{i_1\cdots i_p}_{j_1\cdots j_q} - ig\left[\sum_{m=1}^{p} (W_\mu)^{i_m}_{\ k_m} \psi^{i_1\cdots k_m \cdots i_p}_{j_1\cdots j_q} - \sum_{n=1}^{q} \psi^{i_1\cdots i_p}_{j_1\cdots l_n \cdots j_q} (W_\mu)^{l_n}_{\ j_n}\right]$$

[Note: in differential geometry, the gauge field $(W_\mu)^i_{\ j}$ is a connection form. The gauge connection defines a principal bundle whose base space is the spacetime and structure group is the gauge group.]