Minimal DM with $Y = 0$

$$\tau^+ \left| e_k^{(n)} \right\rangle = \begin{cases} -\sqrt{(j-k)(j+k+1)} \left| e_{k+1}^{(n)} \right\rangle, & k \le 0; \\ \sqrt{(j-k)(j+k+1)} \left| e_{k+1}^{(n)} \right\rangle, & k < 0. \end{cases} \quad j = \frac{n-1}{2}, \quad k = -j, -j+1, \cdots, j. \quad t_{(n)}^- = (t_{(n)}^+)^{\mathrm{T}}$$

Odd n:

$$\tau^+ \left| e_0^{(n)} \right\rangle = -\frac{1}{2}\sqrt{n^2-1} \left| e_{+1}^{(n)} \right\rangle, \quad \tau^+ \left| e_{-1}^{(n)} \right\rangle = \frac{1}{2}\sqrt{n^2-1} \left| e_0^{(n)} \right\rangle, \quad \tau^- \left| e_{+1}^{(n)} \right\rangle = -\frac{1}{2}\sqrt{n^2-1} \left| e_0^{(n)} \right\rangle, \quad \tau^- \left| e_0^{(n)} \right\rangle = \frac{1}{2}\sqrt{n^2-1} \left| e_{-1}^{(n)} \right\rangle$$

$$\tau^+ \left| e_{+1}^{(n)} \right\rangle = -\frac{1}{2}\sqrt{n^2-9} \left| e_{+2}^{(n)} \right\rangle, \quad \tau^+ \left| e_{-2}^{(n)} \right\rangle = \frac{1}{2}\sqrt{n^2-9} \left| e_{-1}^{(n)} \right\rangle, \quad \tau^- \left| e_{+2}^{(n)} \right\rangle = -\frac{1}{2}\sqrt{n^2-9} \left| e_{+1}^{(n)} \right\rangle, \quad \tau^- \left| e_{-1}^{(n)} \right\rangle = \frac{1}{2}\sqrt{n^2-9} \left| e_{-2}^{(n)} \right\rangle$$

Dirac fermionic MDM

$$\bar{\mathcal{X}} i \not{D} \bar{\mathcal{X}} \supset g \bar{\mathcal{X}} \gamma^\mu W_\mu^a \tau^a \bar{\mathcal{X}} \supset \frac{g}{\sqrt{2}} \bar{\mathcal{X}} \gamma^\mu (W_\mu^+ \tau_{(n)}^+ + W_\mu^- \tau_{(n)}^-) \bar{\mathcal{X}}$$

$$\supset \frac{g}{\sqrt{2}} W_\mu^+ \left(-\frac{1}{2}\sqrt{n^2-1}\bar{\chi}^+\gamma^\mu\chi^0 + \frac{1}{2}\sqrt{n^2-1}\bar{\chi}^0\gamma^\mu\chi^- - \frac{1}{2}\sqrt{n^2-9}\bar{\chi}^{++}\gamma^\mu\chi^+ + \frac{1}{2}\sqrt{n^2-9}\bar{\chi}^-\gamma^\mu\chi^{--} \right)$$

$$+ \frac{g}{\sqrt{2}} W_\mu^- \left(-\frac{1}{2}\sqrt{n^2-1}\bar{\chi}^0\gamma^\mu\chi^+ + \frac{1}{2}\sqrt{n^2-1}\bar{\chi}^-\gamma^\mu\chi^0 - \frac{1}{2}\sqrt{n^2-9}\bar{\chi}^+\gamma^\mu\chi^{++} + \frac{1}{2}\sqrt{n^2-9}\bar{\chi}^{--}\gamma^\mu\chi^- \right)$$

$$= \frac{\sqrt{n^2-1}}{2\sqrt{2}} g[W_\mu^+(-\bar{\chi}^+\gamma^\mu\chi^0 + \bar{\chi}^0\gamma^\mu\chi^-) + W_\mu^-(-\bar{\chi}^0\gamma^\mu\chi^+ + \bar{\chi}^-\gamma^\mu\chi^0)]$$

$$+ \frac{\sqrt{n^2-9}}{2\sqrt{2}} g[W_\mu^+(-\bar{\chi}^{++}\gamma^\mu\chi^+ + \bar{\chi}^-\gamma^\mu\chi^{--}) + W_\mu^-(-\bar{\chi}^+\gamma^\mu\chi^{++} + \bar{\chi}^{--}\gamma^\mu\chi^-)]$$

Majorana fermionic MDM

$$\chi^- = (\chi^+)^c = C(\bar{\chi}^+)^{\mathrm{T}}, \quad \chi^0 = (\chi^0)^c = C(\bar{\chi}^0)^{\mathrm{T}}$$

$$\bar{\chi}^0\gamma^\mu\chi^- = \bar{\chi}^0\gamma^\mu C(\bar{\chi}^+)^{\mathrm{T}} = -\bar{\chi}^+ C^{\mathrm{T}} \gamma^{\mu\mathrm{T}} (\bar{\chi}^0)^{\mathrm{T}} = -\bar{\chi}^+ C^{\mathrm{T}} \gamma^{\mu\mathrm{T}} C^{\mathrm{T}} C (\bar{\chi}^0)^{\mathrm{T}} = -\bar{\chi}^+ \gamma^\mu C (\bar{\chi}^0)^{\mathrm{T}} = -\bar{\chi}^+\gamma^\mu\chi^0$$

$$\frac{1}{2}\bar{\mathcal{X}} i \not{D} \bar{\mathcal{X}} \supset \frac{1}{2} g \bar{\mathcal{X}} \gamma^\mu W_\mu^a \tau^a \bar{\mathcal{X}} \supset \frac{g}{2\sqrt{2}} \bar{\mathcal{X}} \gamma^\mu (W_\mu^+ \tau_{(n)}^+ + W_\mu^- \tau_{(n)}^-) \bar{\mathcal{X}}$$

$$\supset -\frac{\sqrt{n^2-1}}{2\sqrt{2}} g(W_\mu^+\bar{\chi}^+\gamma^\mu\chi^0 + W_\mu^-\bar{\chi}^0\gamma^\mu\chi^+) - \frac{\sqrt{n^2-9}}{2\sqrt{2}} g(W_\mu^+\bar{\chi}^{++}\gamma^\mu\chi^+ + W_\mu^-\bar{\chi}^+\gamma^\mu\chi^{++})$$

$$\left[\begin{array}{l} \text{2-component decomposition:} \quad \chi^0 = \begin{pmatrix} \psi_0 \\ \psi_0^\dagger \end{pmatrix}, \quad \chi^+ = \begin{pmatrix} \psi_+ \\ \psi_-^\dagger \end{pmatrix}, \quad \chi^{++} = \begin{pmatrix} \psi_{++} \\ \psi_{--}^\dagger \end{pmatrix} \\ \psi^\dagger \bar{\sigma}^\mu g W_\mu^a t_{(n)}^a \psi \supset \frac{g}{\sqrt{2}} \psi^\dagger \bar{\sigma}^\mu (W_\mu^+ \tau_{(n)}^+ + W_\mu^- \tau_{(n)}^-) \psi \\ \supset \frac{g}{\sqrt{2}} W_\mu^+ \left[-\frac{1}{2}\sqrt{n^2-1}\psi_+^\dagger\bar{\sigma}^\mu\psi_0 + \frac{1}{2}\sqrt{n^2-1}\psi_0^\dagger\bar{\sigma}^\mu\psi_- \right] + \frac{g}{\sqrt{2}} W_\mu^- \left[-\frac{1}{2}\sqrt{n^2-1}\psi_0^\dagger\bar{\sigma}^\mu\psi_+ + \frac{1}{2}\sqrt{n^2-1}\psi_-^\dagger\bar{\sigma}^\mu\psi_0 \right] \\ = \frac{\sqrt{n^2-1}}{2\sqrt{2}} g[W_\mu^+(-\psi_+^\dagger\bar{\sigma}^\mu\psi_0 + \psi_0^\dagger\bar{\sigma}^\mu\psi_-) + W_\mu^-(-\psi_0^\dagger\bar{\sigma}^\mu\psi_+ + \psi_-^\dagger\bar{\sigma}^\mu\psi_0)] \\ = -\frac{\sqrt{n^2-1}}{2\sqrt{2}} g W_\mu^+ (\bar{\chi}_L^+\gamma^\mu\chi_L^0 + \bar{\chi}_R^+\gamma^\mu\chi_R^0) + h.c. = -\frac{\sqrt{n^2-1}}{2\sqrt{2}} g(W_\mu^+\bar{\chi}^+\gamma^\mu\chi^0 + W_\mu\bar{\chi}^0\gamma^\mu\chi^+) \end{array} \right],$$

Fermionic MDM mass split with $Y = 0$

$$\mathrm{m}_Q - m_0 = \frac{\alpha_2 M}{4\pi} Q^2 \left[s_W^2 f\left(\frac{m_Z}{M}\right) + f\left(\frac{m_W}{M}\right) - f\left(\frac{m_Z}{M}\right) \right]$$

$$f(r) = \frac{r}{2}\left[2r^3 \ln r - 2r + \sqrt{r^2-4}\,(r^2+2)\ln A \right], \quad A = \frac{1}{2}\left(r^2 - 2 - r\sqrt{r^2-4} \right)$$

$$0 < r < 2 \quad \Rightarrow \quad \sqrt{r^2-4} = i\sqrt{4-r^2}$$

$$A = \frac{1}{2}\left(r^2 - 2 - ir\sqrt{4-r^2} \right) = |A|\,[\cos(\arg(A)) + i\sin(\arg(A))]$$

$$|A| = \frac{1}{2}\sqrt{(r^2-2)^2 + r^2(4-r^2)} = 1, \quad \arg(A) = \begin{cases} \tan^{-1}\dfrac{r\sqrt{4-r^2}}{2-r^2} - \pi, & 0 < r < \sqrt{2} \\ -\dfrac{\pi}{2}, & r = \sqrt{2} \\ \tan^{-1}\dfrac{r\sqrt{4-r^2}}{2-r^2}, & \sqrt{2} < r < 2 \end{cases}$$

$$\ln A = \ln|A| + i\arg(A) = i\arg(A)$$

$$f(r) = \frac{r}{2}\left[2r^3 \ln r - 2r - \sqrt{4-r^2}\,(r^2+2)\arg(A) \right]$$

$$\chi^+(p) \to \chi^0(k_1) + \ell^+(k_2) + \nu_\ell(k_3), \quad q = p - k_1 = k_2 + k_3, \quad \Delta m = m_{\chi^\pm} - m_{\chi^0}$$

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

$$\int d\Phi^{(3)} = \int \frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3k_2}{(2\pi)^3 2k_2^0}\frac{d^3k_3}{(2\pi)^3 2k_3^0}(2\pi)^4\delta^{(4)}(p - k_1 - k_2 - k_3)$$

$$= \int ds_{23}\delta(s_{23} - q^2)d^4q\delta^{(4)}(q - k_2 - k_3)\frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3k_2}{(2\pi)^3 2k_2^0}\frac{d^3k_3}{(2\pi)^3 2k_3^0}(2\pi)^4\delta^{(4)}(p - k_1 - k_2 - k_3)$$

$$= \int ds_{23}\delta^{(4)}(q - k_2 - k_3)\frac{d^3q}{2q^0}\frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3k_2}{(2\pi)^3 2k_2^0}\frac{d^3k_3}{(2\pi)^3 2k_3^0}(2\pi)^4\delta^{(4)}(p - k_1 - k_2 - k_3)$$

$$= \int \frac{ds_{23}}{2\pi}\frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3q}{(2\pi)^3 2q^0}(2\pi)^4\delta^{(4)}(p - k_1 - q)\frac{d^3k_2}{(2\pi)^3 2k_2^0}\frac{d^3k_3}{(2\pi)^3 2k_3^0}(2\pi)^4\delta^{(4)}(q - k_2 - k_3)$$

$$= \int \frac{ds_{23}}{2\pi}d\Phi_1^{(2)}d\Phi_2^{(2)}$$

$$d\Phi_1^{(2)} \equiv \frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3q}{(2\pi)^3 2q^0}(2\pi)^4\delta^{(4)}(p - k_1 - q), \quad d\Phi_2^{(2)} \equiv \frac{d^3k_2}{(2\pi)^3 2k_2^0}\frac{d^3k_3}{(2\pi)^3 2k_3^0}(2\pi)^4\delta^{(4)}(q - k_2 - k_3)$$

$$q^0 = \frac{m_{\chi^\pm}^2 + s_{23} - m_{\chi^0}^2}{2m_{\chi^\pm}}, \quad |\mathbf{k}_1| = |\mathbf{q}| = \frac{1}{2m_{\chi^\pm}}\sqrt{\left[m_{\chi^\pm}^2 - (\sqrt{s_{23}} + m_{\chi^0})^2\right]\left[m_{\chi^\pm}^2 - (\sqrt{s_{23}} - m_{\chi^0})^2\right]}$$

$$\tilde{q} = (\sqrt{s_{23}}, 0, 0, 0), \quad \tilde{k}_3^0 = \frac{s_{23} - m_\ell^2}{2\sqrt{s_{23}}}$$

$$\frac{1}{2}(s_{23} - m_\ell^2) = \tilde{q}\cdot\tilde{k}_3 = q\cdot k_3 = |\mathbf{k}_3|(q^0 - |\mathbf{q}|\cos\theta) \quad \Rightarrow \quad |\mathbf{k}_3| = \frac{s_{23} - m_\ell^2}{2(q^0 - |\mathbf{q}|\cos\theta)}$$

$$|\mathbf{k}_2| = |\mathbf{q} - \mathbf{k}_3| = \sqrt{|\mathbf{q}|^2 + |\mathbf{k}_3|^2 - 2|\mathbf{q}||\mathbf{k}_3|\cos\theta}$$

$$\int d\Phi_1^{(2)} = \int \frac{d^3k_1}{(2\pi)^2 2k_1^0 2q^0}\delta(p^0 - k_1^0 - q^0) = \int d|\mathbf{k}_1|\frac{|\mathbf{k}_1|^2}{4\pi k_1^0 q^0}\delta\left(p^0 - \sqrt{|\mathbf{k}_1|^2 + m_{\chi^0}^2} - \sqrt{|\mathbf{k}_1|^2 + s_{23}}\right)$$

$$= \frac{|\mathbf{k}_1|^2}{4\pi k_1^0 q^0}\left(\frac{|\mathbf{k}_1|}{k_1^0} + \frac{|\mathbf{k}_1|}{q^0}\right)^{-1} = \frac{|\mathbf{k}_1|}{4\pi m_{\chi^\pm}}$$

$$d\Phi_2^{(2)} = \int \frac{d^3k_3}{(2\pi)^2 2k_2^0 2k_3^0}\delta(q^0 - k_2^0 - k_3^0) = \int d|\mathbf{k}_3|d\cos\theta\frac{|\mathbf{k}_3|^2}{8\pi k_2^0 k_3^0}\delta\left(q^0 - \sqrt{|\mathbf{q}|^2 + |\mathbf{k}_3|^2 - 2|\mathbf{q}||\mathbf{k}_3|\cos\theta + m_\ell^2} - |\mathbf{k}_3|\right)$$

$$= \int d\cos\theta\frac{|\mathbf{k}_3|^2}{8\pi k_2^0 k_3^0}\left(\frac{|\mathbf{k}_3| - |\mathbf{q}|\cos\theta}{k_2^0} + 1\right)^{-1} = \int d\cos\theta\frac{|\mathbf{k}_3|}{8\pi(q^0 - |\mathbf{q}|\cos\theta)} = \frac{1}{4\pi(s_{23} - m_\ell^2)}\int d\cos\theta|\mathbf{k}_3|^2$$

$$\Gamma = \frac{1}{2m_{\chi^\pm}}\int d\Phi^{(3)}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{64\pi^3 m_{\chi^\pm}^2}\int_{m_\ell^2}^{\Delta m^2}ds_{23}\int_{-1}^{1}d\cos\theta\frac{|\mathbf{k}_1||\mathbf{k}_3|^2}{s_{23} - m_\ell^2}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$s_{23} = q^2 = (p-k_1)^2 \le \Delta m^2 \ll m_W^2, \quad G_F = \frac{g^2}{4\sqrt{2}m_W^2}$$

$$i\mathcal{M} = -i\frac{\sqrt{n^2-1}}{2\sqrt{2}} g\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\frac{-i(g_{\mu\nu}-q_\mu q_\nu / m_W^2)}{q^2-m_W^2} i\frac{1}{\sqrt{2}} g\bar{u}_{\nu_\ell}(k_3)\gamma^\nu P_L v_\ell(k_2)$$

$$= -i\frac{\sqrt{n^2-1}}{4} g^2 \frac{(g_{\mu\nu}-q_\mu q_\nu / m_W^2)}{q^2-m_W^2}\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\bar{u}_{\nu_\ell}(k_3)\gamma^\nu P_L v_\ell(k_2)$$

$$\simeq i\sqrt{2(n^2-1)}G_F\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\bar{u}_{\nu_\ell}(k_3)\gamma_\mu P_L v_\ell(k_2)$$

$$(i\mathcal{M})^* \simeq -i\sqrt{2(n^2-1)}G_F\bar{u}_{\chi^+}(p)\gamma^\nu u_{\chi^0}(k_1)\bar{v}_\ell(k_2)\gamma_\nu P_L u_{\nu_\ell}(k_3)$$

$$p = (m_{\chi^\pm},0,0,0), \quad p\cdot k_2 = m_{\chi^\pm}k_2^0, \quad p\cdot k_3 = m_{\chi^\pm}k_3^0$$

$$k_1\cdot k_2 = (p-q)\cdot k_2 = p\cdot k_2 - q\cdot k_2, \quad k_1\cdot k_3 = (p-q)\cdot k_3 = p\cdot k_3 - q\cdot k_3,$$

$$q\cdot k_2 = (k_2+k_3)\cdot k_2 = m_l^2 + k_2\cdot k_3, \quad q\cdot k_3 = (k_2+k_3)\cdot k_3 = k_2\cdot k_3$$

$$s_{23} = (k_2+k_3)^2 = m_\ell^2 + 2k_2\cdot k_3 \quad \Rightarrow \quad k_2\cdot k_3 = \frac{1}{2}(s_{23}-m_\ell^2)$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq \frac{1}{2}\sum_{\text{spins}} 2(n^2-1)G_F^2\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\bar{u}_{\nu_\ell}(k_3)\gamma_\mu P_L v_\ell(k_2)\bar{u}_{\chi^+}(p)\gamma^\nu u_{\chi^0}(k_1)\bar{v}_\ell(k_2)\gamma_\nu P_L u_{\nu_\ell}(k_3)$$

$$= (n^2-1)G_F^2\mathrm{Tr}[(\not{k}_1 + m_{\chi^0})\gamma^\mu(\not{p}+m_{\chi^\pm})\gamma^\nu]\mathrm{Tr}[\not{k}_3\gamma_\mu P_L(\not{k}_2 - m_\ell)\gamma_\nu P_L]$$

$$= 16(n^2-1)G_F^2[(k_1\cdot k_2)(p\cdot k_3) + (k_1\cdot k_3)(p\cdot k_2) - m_{\chi^0}m_{\chi^\pm}k_2\cdot k_3]$$

$$\Gamma = \frac{1}{64\pi^3 m_{\chi^\pm}^2}\int_{m_\ell^2}^{\Delta m^2} ds_{23}\int_{-1}^{1} d\cos\theta\frac{|\mathbf{k}_1||\mathbf{k}_3|^2}{s_{23}-m_\ell^2}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$\simeq \frac{n^2-1}{60\pi^3}G_F^2\Delta m^5\sqrt{1-\frac{m_l^2}{\Delta m^2}}P\left(\frac{m_l}{\Delta m}\right)\xrightarrow{m_l << \Delta m}\frac{n^2-1}{60\pi^3}G_F^2\Delta m^5$$

$$P(x) \equiv 1 - \frac{9}{2}x^2 - 4x^4 + \frac{15x^4}{2\sqrt{1-x^2}}\tanh^{-1}\sqrt{1-x^2}$$

$$\lim_{x\to 0}P(x) = 1, \quad \sqrt{1-\left(\frac{m_\mu}{167\text{ MeV}}\right)^2}P\left(\frac{m_\mu}{167\text{ MeV}}\right) \simeq 0.12$$

(Consistent with 0909.4549 & hep-ph/0512090)

$$\chi^+(p) \to \chi^0(k_1) + \pi^+(k_2)$$

$$\langle 0|\bar{Q}\gamma^\mu\gamma_5\tau^a Q\big|\pi^b(p)\rangle = -i\frac{1}{\sqrt{2}}p^\mu f_\pi \delta^{ab} e^{-ip\cdot x}, \quad Q = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \tau^a = \frac{\sigma^a}{2}, \quad f_\pi = 130.41\,\text{MeV}$$

[The definition of f_π here differs from Eq.(19.88) in Peskin's book by a factor of $\sqrt{2}$]

$$\tau^\pm = \tau^1 \pm i\tau^2, \quad \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \mp i\pi^2), \quad \tau^a\pi^a = \frac{1}{\sqrt{2}}(\tau^+\pi^+ + \tau^-\pi^-) + \tau^3\pi^3$$

$$\tau^+\pi^+ = \frac{1}{\sqrt{2}}(\tau^1\pi^1 - i\tau^1\pi^2 + i\tau^2\pi^1 + \tau^2\pi^2), \quad \tau^-\pi^- = \frac{1}{\sqrt{2}}(\tau^1\pi^1 + i\tau^1\pi^2 - i\tau^2\pi^1 + \tau^2\pi^2)$$

$$\tau^+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}, \quad \tau^- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}, \quad \bar{Q}\gamma^\mu\gamma_5\tau^+ Q = \bar{u}\gamma^\mu\gamma_5 d, \quad \bar{Q}\gamma^\mu\gamma_5\tau^- Q = \bar{d}\gamma^\mu\gamma_5 u$$

$$\langle 0|\bar{d}\gamma^\mu\gamma_5 u\big|\pi^-(p)\rangle = -ip^\mu f_\pi e^{-ip\cdot x}, \quad \langle\pi^+(p)\big|\bar{u}\gamma^\mu\gamma_5 d\,|0\rangle = ip^\mu f_\pi e^{ip\cdot x}$$

$$i\mathcal{M}_{\text{parton}} = -i\frac{\sqrt{n^2-1}}{2\sqrt{2}} g\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\frac{-i(g_{\mu\nu} - k_{2\mu}k_{2\nu}/m_W^2)}{k_2^2 - m_W^2} i\frac{1}{\sqrt{2}} gV_{ud}\bar{u}_u\gamma^\nu P_L v_d$$

$$\simeq i\sqrt{2}\sqrt{n^2-1}G_F V_{ud}\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\bar{u}_u\gamma_\mu P_L v_d$$

$$\bar{u}_u\gamma_\mu P_L v_d = \frac{1}{2}\bar{u}_u\gamma_\mu v_d - \frac{1}{2}\bar{u}_u\gamma_\mu\gamma_5 v_d$$

$$i\mathcal{M} \simeq i\sqrt{2}\sqrt{n^2-1}G_F V_{ud}\bar{u}_{\chi^0}(k_1)\gamma^\mu u_{\chi^+}(p)\left(-\frac{1}{2}ik_{2\mu}f_\pi\right) = \frac{\sqrt{n^2-1}}{\sqrt{2}}G_F V_{ud} f_\pi \bar{u}_{\chi^0}(k_1)\not{k}_2 u_{\chi^+}(p)$$

$$= \frac{\sqrt{n^2-1}}{\sqrt{2}}G_F V_{ud} f_\pi \bar{u}_{\chi^0}(k_1)(\not{p} - \not{k}_1)u_{\chi^+}(p) = \frac{\sqrt{n^2-1}}{\sqrt{2}}G_F V_{ud} f_\pi \Delta m \bar{u}_{\chi^0}(k_1)u_{\chi^+}(p)$$

$$(i\mathcal{M})^* \simeq \frac{\sqrt{n^2-1}}{\sqrt{2}}G_F V_{ud}^* f_\pi \Delta m \bar{u}_{\chi^+}(p)u_{\chi^0}(k_1)$$

$$k_1^0 = \frac{m_{\chi^\pm}^2 + m_{\chi^0}^2 - m_{\pi^\pm}^2}{2m_{\chi^\pm}}, \quad |\mathbf{k}_1| = \frac{1}{2m_{\chi^\pm}}\sqrt{\left[m_{\chi^\pm}^2 - (m_{\chi^0} + m_{\pi^\pm})^2\right]\left[m_{\chi^\pm}^2 - (m_{\chi^0} - m_{\pi^\pm})^2\right]}, \quad p\cdot k_1 = m_{\chi^\pm}k_1^0$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq \frac{1}{2}\sum_{\text{spins}}\frac{n^2-1}{2}G_F^2|V_{ud}|^2 f_\pi^2 \Delta m^2 \bar{u}_{\chi^0}(k_1)u_{\chi^+}(p)\bar{u}_{\chi^+}(p)u_{\chi^0}(k_1)$$

$$= \frac{1}{4}(n^2-1)G_F^2|V_{ud}|^2 f_\pi^2\Delta m^2 \text{Tr}[u_{\chi^0}(k_1)\bar{u}_{\chi^0}(k_1)u_{\chi^+}(p)\bar{u}_{\chi^+}(p)] = \frac{1}{4}(n^2-1)G_F^2|V_{ud}|^2 f_\pi^2\Delta m^2\text{Tr}[(\not{k}_1 + m_{\chi^0})(\not{p} + m_{\chi^\pm})]$$

$$= \frac{1}{4}(n^2-1)G_F^2|V_{ud}|^2 f_\pi^2\Delta m^2\, 4(p\cdot k_1 + m_{\chi^0}m_{\chi^+}) = (n^2-1)G_F^2|V_{ud}|^2 f_\pi^2\Delta m^2 m_{\chi^\pm}(k_1^0 + m_{\chi^0})$$

$$\Gamma = \frac{1}{8\pi}\frac{|\mathbf{k}_1|}{m_{\chi^\pm}^2}\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq \frac{n^2-1}{4\pi}G_F^2|V_{ud}|^2 f_\pi^2\Delta m^3\sqrt{1 - \frac{m_{\pi^\pm}^2}{\Delta m^2}}$$

$$\chi^{++}(p) \to \chi^{+}(k_1) + \ell^{+}(k_2) + \nu_\ell(k_3), \quad q = p - k_1 = k_2 + k_3, \quad \Delta m_1 = m_{\chi^{\pm\pm}} - m_{\chi^{\pm}}$$

$$i\mathcal{M} = -i\frac{\sqrt{n^2-9}}{2\sqrt{2}} g\bar{u}_{\chi^+}(k_1)\gamma^\mu u_{\chi^{++}}(p)\frac{-i(g_{\mu\nu} - q_\mu q_\nu / m_W^2)}{q^2 - m_W^2} i\frac{1}{\sqrt{2}} g\bar{u}_{\nu_\ell}(k_3)\gamma^\nu P_L v_\ell(k_2)$$

$$\simeq i\sqrt{2(n^2-9)} G_F \bar{u}_{\chi^+}(k_1)\gamma^\mu u_{\chi^{++}}(p)\bar{u}_{\nu_\ell}(k_3)\gamma_\mu P_L v_\ell(k_2)$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq 16(n^2-9)G_F^2[(k_1\cdot k_2)(p\cdot k_3) + (k_1\cdot k_3)(p\cdot k_2) - m_{\chi^\pm} m_{\chi^{\pm\pm}} k_2\cdot k_3]$$

$$\Gamma \simeq \frac{n^2-9}{60\pi^3} G_F^2 \Delta m_1^5 \sqrt{1 - \frac{m_l^2}{\Delta m_1^2}} P\left(\frac{m_l}{\Delta m_1}\right)$$

$$\chi^{++}(p) \to \chi^{+}(k_1) + \pi^{+}(k_2)$$

$$i\mathcal{M}_{\text{parton}} = -i\frac{\sqrt{n^2-9}}{2\sqrt{2}} g\bar{u}_{\chi^+}(k_1)\gamma^\mu u_{\chi^{++}}(p)\frac{-i(g_{\mu\nu} - k_{2\mu} k_{2\nu} / m_W^2)}{k_2^2 - m_W^2} i\frac{1}{\sqrt{2}} gV_{ud}\bar{u}_u\gamma^\nu P_L v_d$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq (n^2-9)G_F^2 |V_{ud}|^2 f_\pi^2 \Delta m_1^2 m_{\chi^{\pm\pm}}(k_1^0 + m_{\chi^\pm})$$

$$\Gamma \simeq \frac{n^2-9}{4\pi} G_F^2 |V_{ud}|^2 f_\pi^2 \Delta m_1^3 \sqrt{1 - \frac{m_{\pi^\pm}^2}{\Delta m_1^2}}$$

$$\chi^{++}(p) \to \chi^{+}(k_1) + K^{+}(k_2)$$

$$\left\langle K^+(p)\right|\bar{u}\gamma^\mu\gamma_5 s\left|0\right\rangle = ip^\mu f_K e^{ip\cdot x}, \quad f_K = 156.2\ \text{MeV}$$

$$i\mathcal{M}_{\text{parton}} = -i\frac{\sqrt{n^2-9}}{2\sqrt{2}} g\bar{u}_{\chi^+}(k_1)\gamma^\mu u_{\chi^{++}}(p)\frac{-i(g_{\mu\nu} - k_{2\mu} k_{2\nu} / m_W^2)}{k_2^2 - m_W^2} i\frac{1}{\sqrt{2}} gV_{us}\bar{u}_u\gamma^\nu P_L v_s$$

$$\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 \simeq (n^2-9)G_F^2 |V_{us}|^2 f_K^2 \Delta m_1^2 m_{\chi^{\pm\pm}}(k_1^0 + m_{\chi^\pm})$$

$$\Gamma \simeq \frac{n^2-9}{4\pi} G_F^2 |V_{us}|^2 f_K^2 \Delta m_1^3 \sqrt{1 - \frac{m_{K^\pm}^2}{\Delta m_1^2}}$$

Self-conjugated Fermionic Quintuplet

4-component spinor $\Sigma = \begin{pmatrix} \Sigma^{++} \\ \Sigma^{+} \\ \Sigma^{0} \\ \Sigma^{-} \\ \Sigma^{--} \end{pmatrix} \in (\mathbf{5}, 0)$

Self-conjugated: $\Sigma^0 = (\Sigma^0)^c = \mathcal{C}(\bar{\Sigma}^0)^T, \quad \Sigma^- = (\Sigma^-)^c = \mathcal{C}(\bar{\Sigma}^+)^T, \quad \Sigma^{--} = (\Sigma^{--})^c = \mathcal{C}(\bar{\Sigma}^{++})^T$

$$\mathcal{L}_\Sigma = \frac{i}{2}\bar{\Sigma}\gamma^\mu D_\mu \Sigma - \frac{1}{2}m_\Sigma \bar{\Sigma}\Sigma, \quad D_\mu = (\partial_\mu - igW_\mu^a t_{(5)}^a)\Sigma$$

$$\bar{\Sigma}^-\Sigma^- = (\Sigma^+)^T \mathcal{C}\mathcal{C}(\bar{\Sigma}^+)^T = -\bar{\Sigma}^+\mathcal{C}^T\mathcal{C}^T\Sigma^+ = \bar{\Sigma}^+\mathcal{C}^{-1}\mathcal{C}\Sigma^+ = \bar{\Sigma}^+\Sigma^+$$

$$\bar{\Sigma}\Sigma = \bar{\Sigma}^{++}\Sigma^{++} + \bar{\Sigma}^{+}\Sigma^{+} + \bar{\Sigma}^{0}\Sigma^{0} + \bar{\Sigma}^{-}\Sigma^{-} + \bar{\Sigma}^{--}\Sigma^{---}$$

$$\mathcal{L}_{\Sigma,\mathrm{mass}} = -\frac{1}{2}m_\Sigma\bar{\Sigma}\Sigma = -\frac{1}{2}m_\Sigma\bar{\Sigma}^0\Sigma^0 - m_\Sigma\bar{\Sigma}^+\Sigma^+ - m_\Sigma\bar{\Sigma}^{++}\Sigma^{++}$$

$$t_{(5)}^+ = \begin{pmatrix} 0 & -2 & & & \\ & 0 & -\sqrt{6} & & \\ & & 0 & \sqrt{6} & \\ & & & 0 & 2 \\ & & & & 0 \end{pmatrix}, \quad t_{(5)}^- = (t_{(5)}^+)^T, \quad t_{(5)}^1 = \frac{1}{2}(t_{(5)}^+ + t_{(5)}^-), \quad t_{(5)}^2 = -\frac{i}{2}(t_{(5)}^+ - t_{(5)}^-)$$

$$t_{(5)}^1 = \frac{1}{2}\begin{pmatrix} 0 & -2 & & & \\ -2 & 0 & -\sqrt{6} & & \\ & -\sqrt{6} & 0 & \sqrt{6} & \\ & & \sqrt{6} & 0 & 2 \\ & & & 2 & 0 \end{pmatrix}, \quad t_{(5)}^2 = \frac{i}{2}\begin{pmatrix} 0 & 2 & & & \\ -2 & 0 & \sqrt{6} & & \\ & -\sqrt{6} & 0 & -\sqrt{6} & \\ & & \sqrt{6} & 0 & -2 \\ & & & 2 & 0 \end{pmatrix}, \quad t_{(5)}^3 = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -2 \end{pmatrix}$$

$$gW_\mu^a t_{(5)}^a = \begin{pmatrix} 2gW_\mu^3 & -(W_\mu^1 - iW_\mu^2) & & & \\ -(W_\mu^1 + iW_\mu^2) & gW_\mu^3 & -\frac{\sqrt{6}}{2}(W_\mu^1 - iW_\mu^2) & & \\ & -\frac{\sqrt{6}}{2}(W_\mu^1 + iW_\mu^2) & 0 & \frac{\sqrt{6}}{2}(W_\mu^1 - iW_\mu^2) & \\ & & \frac{\sqrt{6}}{2}(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 & W_\mu^1 - iW_\mu^2 \\ & & & W_\mu^1 + iW_\mu^2 & -2gW_\mu^3 \end{pmatrix}$$

$$= \begin{pmatrix} 2eA_\mu + 2gc_W Z_\mu & -\sqrt{2}gW_\mu^+ & & & \\ -\sqrt{2}gW_\mu^- & eA_\mu + gc_W Z_\mu & -\sqrt{3}gW_\mu^+ & & \\ & -\sqrt{3}gW_\mu^- & 0 & \sqrt{3}gW_\mu^+ & \\ & & \sqrt{3}gW_\mu^- & -eA_\mu - gc_W Z_\mu & \sqrt{2}gW_\mu^+ \\ & & & \sqrt{2}gW_\mu^- & -2eA_\mu - 2gc_W Z_\mu \end{pmatrix}$$

$$\begin{aligned} \bar{\Sigma}\gamma^\mu gW_\mu^a t_{(5)}^a \Sigma = {} & 2(eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{++}\gamma^\mu\Sigma^{++} + (eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{+}\gamma^\mu\Sigma^{+} \\ & - (eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{-}\gamma^\mu\Sigma^{-} - 2(eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{--}\gamma^\mu\Sigma^{--} \\ & + g(-\sqrt{2}W_\mu^+\bar{\Sigma}^{++}\gamma^\mu\Sigma^{+} - \sqrt{3}W_\mu^+\bar{\Sigma}^{+}\gamma^\mu\Sigma^{0} \\ & + \sqrt{3}W_\mu^+\bar{\Sigma}^{0}\gamma^\mu\Sigma^{-} + \sqrt{2}W_\mu^+\bar{\Sigma}^{-}\gamma^\mu\Sigma^{--} + \mathrm{h.c.}) \end{aligned}$$

$$\bar{\Sigma}^0\gamma^\mu\Sigma^- = (\Sigma^0)^T\mathcal{C}\gamma^\mu\mathcal{C}(\bar{\Sigma}^+)^T = -\bar{\Sigma}^+\mathcal{C}^T(\gamma^\mu)^T\mathcal{C}^T\Sigma^0 = \bar{\Sigma}^+\mathcal{C}^{-1}(\gamma^\mu)^T\mathcal{C}\Sigma^0 = -\bar{\Sigma}^+\gamma^\mu\Sigma^0$$

$$\bar{\Sigma}^-\gamma^\mu\Sigma^{--} = -\bar{\Sigma}^{++}\gamma^\mu\Sigma^+, \quad \bar{\Sigma}^-\gamma^\mu\Sigma^- = -\bar{\Sigma}^+\gamma^\mu\Sigma^+, \quad \bar{\Sigma}^{--}\gamma^\mu\Sigma^{--} = -\bar{\Sigma}^{++}\gamma^\mu\Sigma^{++}$$

$$\mathcal{L}_{\Sigma,\mathrm{gauge}} = \frac{1}{2}\bar{\Sigma}\gamma^\mu gW_\mu^a t_{(5)}^a\Sigma$$

$$\begin{aligned} &= 2(eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{++}\gamma^\mu\Sigma^{++} + (eA_\mu + gc_W Z_\mu)\bar{\Sigma}^{+}\gamma^\mu\Sigma^{+} \\ &\quad -\sqrt{2}gW_\mu^+\bar{\Sigma}^{++}\gamma^\mu\Sigma^{+} - \sqrt{3}gW_\mu^+\bar{\Sigma}^{+}\gamma^\mu\Sigma^{0} - \sqrt{2}gW_\mu^-\bar{\Sigma}^{+}\gamma^\mu\Sigma^{++} - \sqrt{3}gW_\mu^-\bar{\Sigma}^{0}\gamma^\mu\Sigma^{+} \end{aligned}$$

2-component language

$$\text{Left-handed weyl spinor } \Sigma = \begin{pmatrix} \Sigma^{++} \\ \Sigma^{+} \\ \Sigma^{0} \\ \Sigma^{-} \\ \Sigma^{--} \end{pmatrix} \in (\mathbf{5}, 0)$$

$$\mathcal{L}_\Sigma = i\Sigma^\dagger \bar{\sigma}^\mu D_\mu \Sigma - \frac{1}{2}(m_\Sigma \Sigma\Sigma + \text{h.c.}), \quad D_\mu = (\partial_\mu - igW_\mu^a t_{(5)}^a)\Sigma$$

$$\mathcal{L}_{\Sigma,\text{gauge}} = \Sigma^\dagger \bar{\sigma}^\mu g W_\mu^a t_{(5)}^a \Sigma$$

$$\begin{aligned}
&= 2(eA_\mu + gc_\text{W} Z_\mu)(\Sigma^{++})^\dagger \bar{\sigma}^\mu \Sigma^{++} + (eA_\mu + gc_\text{W} Z_\mu)(\Sigma^{+})^\dagger \bar{\sigma}^\mu \Sigma^{+} \\
&\quad - (eA_\mu + gc_\text{W} Z_\mu)(\Sigma^{-})^\dagger \bar{\sigma}^\mu \Sigma^{-} - 2(eA_\mu + gc_\text{W} Z_\mu)(\Sigma^{--})^\dagger \bar{\sigma}^\mu \Sigma^{--} \\
&\quad + g[-\sqrt{2}W_\mu^+ (\Sigma^{++})^\dagger \bar{\sigma}^\mu \Sigma^{+} - \sqrt{3}W_\mu^+ (\Sigma^{+})^\dagger \bar{\sigma}^\mu \Sigma^{0} \\
&\qquad + \sqrt{3}W_\mu^+ (\Sigma^{0})^\dagger \bar{\sigma}^\mu \Sigma^{-} + \sqrt{2}W_\mu^+ (\Sigma^{-})^\dagger \bar{\sigma}^\mu \Sigma^{--} + \text{h.c.}]
\end{aligned}$$

$$\text{4-component spinors } \Sigma^0 = \begin{pmatrix} \Sigma^0 \\ (\Sigma^0)^\dagger \end{pmatrix}, \quad \Sigma^+ = \begin{pmatrix} \Sigma^+ \\ (\Sigma^-)^\dagger \end{pmatrix}, \quad \Sigma^{++} = \begin{pmatrix} \Sigma^{++} \\ (\Sigma^{--})^\dagger \end{pmatrix}$$

$$\begin{aligned}
\mathcal{L}_{\Sigma,\text{gauge}} &= 2(eA_\mu + gc_\text{W} Z_\mu)\bar{\Sigma}^{++}\gamma^\mu \Sigma^{++} + (eA_\mu + gc_\text{W} Z_\mu)\bar{\Sigma}^{+}\gamma^\mu \Sigma^{+} \\
&\quad - \sqrt{2}gW_\mu^+ \bar{\Sigma}^{++}\gamma^\mu \Sigma^{+} - \sqrt{3}gW_\mu^+ \bar{\Sigma}^{+}\gamma^\mu \Sigma^{0} - \sqrt{2}gW_\mu^- \bar{\Sigma}^{+}\gamma^\mu \Sigma^{++} - \sqrt{3}gW_\mu^- \bar{\Sigma}^{0}\gamma^\mu \Sigma^{+}
\end{aligned}$$