散射矩阵幺正性限制

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目录

1 散射矩阵幺正性

由散射矩阵 $S = 1 + iT$ 的幺正性, 可得

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT) = 1 - iT^\dagger + iT + T^\dagger T \quad \Rightarrow \quad -i(T - T^\dagger) = T^\dagger T. \tag{1}$$

利用 $\langle\beta|\, T\, |\alpha\rangle = (2\pi)^4 \delta^{(4)}(p_\alpha - p_\beta)\mathcal{M}(\alpha \to \beta)$ 和 $\langle\beta|\, T^\dagger\, |\alpha\rangle = \langle\alpha|\, T |\beta\rangle^*$, 有

$$\begin{aligned}\langle\beta| - i(T - T^\dagger)\, |\alpha\rangle \;&=\; -i\left(\langle\beta|\, T\, |\alpha\rangle - \langle\beta|\, T^\dagger\, |\alpha\rangle\right) = -i\left(\langle\beta|\, T\, |\alpha\rangle - \langle\alpha|\, T|\beta\rangle^*\right) \\ &=\; (2\pi)^4\delta^{(4)}(p_\alpha - p_\beta)(-i)[\mathcal{M}(\alpha \to \beta) - \mathcal{M}^*(\beta \to \alpha)]. \end{aligned} \tag{2}$$

一组中间态的完备集 $\{|\gamma\rangle\}$ 满足 $\sum_\gamma \int d\Pi_\gamma \,|\gamma\rangle\langle\gamma| = 1$, 其中 $d\Pi_\gamma \equiv \prod_i \frac{d^3 p_{\gamma_i}}{(2\pi)^3 2E_{\gamma_i}}$. 从而,

$$
\begin{aligned}
\langle\beta| T^\dagger T |\alpha\rangle &= \sum_\gamma \int d\Pi_\gamma \langle\beta| T^\dagger |\gamma\rangle \langle\gamma| T |\alpha\rangle = \sum_\gamma \int d\Pi_\gamma \langle\gamma| T|\beta\rangle^* \langle\gamma| T |\alpha\rangle \\
&= \sum_\gamma \int d\Pi_\gamma (2\pi)^4 \delta^{(4)}(p_\beta - p_\gamma) \mathcal{M}^*(\beta\to\gamma)(2\pi)^4\delta^{(4)}(p_\alpha - p_\gamma)\mathcal{M}(\alpha\to\gamma) \\
&= (2\pi)^4\delta^{(4)}(p_\alpha - p_\beta) \sum_\gamma \int d\Pi_\gamma \mathcal{M}^*(\beta\to\gamma)\mathcal{M}(\alpha\to\gamma)(2\pi)^4\delta^{(4)}(p_\alpha - p_\gamma).
\end{aligned} \tag{3}
$$

于是

$$
-i[\mathcal{M}(\alpha\to\beta) - \mathcal{M}^*(\beta\to\alpha)] = \sum_\gamma \int d\Pi_\gamma \mathcal{M}^*(\beta\to\gamma)\mathcal{M}(\alpha\to\gamma)(2\pi)^4\delta^{(4)}(p_\alpha - p_\gamma), \tag{4}
$$

与 Peskin & Schroeder [1] 中的 (7.49) 式一致.

2 弹性散射

设质心系中粒子 1 与粒子 2 的弹性散射振幅为 $\mathcal{M}_{\text{el}}(s, \cos\theta)$, 其中 θ 为散射角. 考虑粒子 1 与粒子 2 的如下弹性散射过程:

$$
\begin{aligned}
\alpha(p_1, p_2) \to \beta(q_1, q_2), &\quad \mathcal{M}_{\text{el}}(s, \cos\theta_{\alpha\beta}), \quad \cos\theta_{\alpha\beta} = \frac{\mathbf{p}_1\cdot\mathbf{q}_1}{|\mathbf{p}_1||\mathbf{q}_1|}; \\
\alpha(p_1, p_2) \to \gamma_{\text{el}}(k_1, k_2), &\quad \mathcal{M}_{\text{el}}(s, \cos\theta_{\alpha\gamma}), \quad \cos\theta_{\alpha\gamma} = \frac{\mathbf{p}_1\cdot\mathbf{k}_1}{|\mathbf{p}_1||\mathbf{k}_1|}; \\
\beta(q_1, q_2) \to \gamma_{\text{el}}(k_1, k_2), &\quad \mathcal{M}_{\text{el}}(s, \cos\theta_{\beta\gamma}), \quad \cos\theta_{\beta\gamma} = \frac{\mathbf{q}_1\cdot\mathbf{k}_1}{|\mathbf{q}_1||\mathbf{k}_1|}.
\end{aligned} \tag{5}
$$

弹性末态 γ_{el} 的相空间积分

$$
\begin{aligned}
&\int d\Pi_{\gamma_{\text{el}}} (2\pi)^4 \delta^{(4)}(p_\alpha - p_\gamma) \\
&= \int \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2) = \frac{1}{4(2\pi)^2}\int \frac{d^3k_1}{E_1E_2}\delta(\sqrt{s} - E_1 - E_2) \\
&= \frac{1}{16\pi^2}\int \frac{|\mathbf{k}_1|^2 d|\mathbf{k}_1| d\Omega_k}{E_1E_2}\delta(\sqrt{s} - \sqrt{|\mathbf{k}_1|^2 + m_1^2} - \sqrt{|\mathbf{k}_1|^2 + m_1^2}) \\
&= \frac{1}{16\pi^2}\int \frac{|\mathbf{k}_1|^2 d\Omega_k}{E_1E_2}\left(\frac{|\mathbf{k}_1|}{E_1} + \frac{|\mathbf{k}_1|}{E_2}\right)^{-1} = \frac{1}{16\pi^2}\frac{|\mathbf{k}_1|}{E_1+E_2}\int d\Omega_k \\
&= \frac{\beta(s, m_1)}{32\pi^2}\int d\Omega_k,
\end{aligned} \tag{6}
$$

其中 $\beta(s, m_1) \equiv \sqrt{1 - 4m_1^2/s}$ 是粒子 1 的速度.

注意到 $\mathcal{M}(\alpha\to\beta) = \mathcal{M}(\beta\to\alpha) = \mathcal{M}_{\text{el}}(s, \cos\theta_{\alpha\beta})$, 方程 (4) 可化为

$$
\begin{aligned}
&-i[\mathcal{M}(\alpha\to\beta) - \mathcal{M}^*(\beta\to\alpha)] = 2\,\text{Im}\,\mathcal{M}_{\text{el}}(s, \cos\theta_{\alpha\beta}) \\
&= \sum_\gamma \int d\Pi_\gamma \mathcal{M}^*(\beta\to\gamma)\mathcal{M}(\alpha\to\gamma)(2\pi)^4\delta^{(4)}(p_\alpha - p_\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&= \int d\Pi_{\gamma_{\rm el}} \mathcal{M}^*(\beta \to \gamma_{\rm el}) \mathcal{M}(\alpha \to \gamma_{\rm el}) (2\pi)^4 \delta^{(4)}(p_\alpha - p_\gamma) \\
&\qquad + \sum_{\gamma \in \{\gamma_{\rm inel}\}} \int d\Pi_\gamma \mathcal{M}^*(\beta \to \gamma) \mathcal{M}(\alpha \to \gamma) (2\pi)^4 \delta^{(4)}(p_\alpha - p_\gamma) \\
&\geqslant \int d\Pi_{\gamma_{\rm el}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \mathcal{M}^*_{\rm el}(s, \cos\theta_{\beta\gamma}) \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\gamma}) \\
&= \frac{\beta(s, m_1)}{32\pi^2} \int d\Omega_k \mathcal{M}^*_{\rm el}(s, \cos\theta_{\beta\gamma}) \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\gamma}), \qquad (7)
\end{aligned}
$$

亦即

$$
\operatorname{Im} \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\beta}) \geqslant \frac{\beta(s, m_1)}{64\pi^2} \int d\Omega_k \mathcal{M}^*_{\rm el}(s, \cos\theta_{\beta\gamma}) \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\gamma}), \qquad (8)
$$

与 Bjorken & Drell [2] 中的 (18.148) 式一致.

对于在区间 $(-1, 1)$ 上具有连续一阶导数和逐段连续二阶导数的函数 $f(z)$, 可用 Legendre 多项式 $P_j(z)$ 将其展开为 $f(z) = \sum_{j=0}^{\infty} \tilde{a}_j P_j(z)$, 其中系数 $\tilde{a}_j = \frac{1}{2}(2j+1)\int_{-1}^{1} dz P_j(z) f(z)$. Legendre 多项式满足正交关系

$$
\int_{-1}^{1} dz P_j(z) P_k(z) = \int_0^\pi d\theta \sin\theta P_j(\cos\theta) P_k(\cos\theta) = \frac{2}{2j+1}\delta_{jk}. \qquad (9)
$$

用 Legendre 多项式 $P_j(\cos\theta)$ 将弹性散射振幅 $\mathcal{M}_{\rm el}(s, \cos\theta)$ 展开为

$$
\mathcal{M}_{\rm el}(s, \cos\theta) = \sum_{j=0}^{\infty} \tilde{a}_j(s) P_j(\cos\theta) = 32\pi \sum_{j=0}^{\infty} \frac{2j+1}{2} a_j(s) P_j(\cos\theta), \qquad (10)
$$

其中

$$
a_j(s) = \frac{1}{32\pi} \int_0^\pi d\theta \sin\theta P_j(\cos\theta) \mathcal{M}_{\rm el}(s, \cos\theta) = \frac{1}{32\pi} \frac{2}{2j+1} \tilde{a}_j(s). \qquad (11)
$$

由不等式 (8) 可得

$$
\begin{aligned}
\operatorname{Im} a_j(s) &= \frac{1}{32\pi} \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} P_j(\cos\theta_{\alpha\beta}) \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\beta}) \\
&\geqslant \frac{1}{32\pi} \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} P_j(\cos\theta_{\alpha\beta}) \frac{\beta(s, m_{\alpha_1})}{64\pi^2} \int d\Omega \mathcal{M}^*_{\rm el}(s, \cos\theta_{\beta\gamma}) \mathcal{M}_{\rm el}(s, \cos\theta_{\alpha\gamma}) \\
&= \frac{1}{32\pi} \frac{\beta(s, m_1)}{64\pi^2} \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} P_j(\cos\theta_{\alpha\beta}) \\
&\qquad \times \int d\Omega\, 32\pi \sum_{k=0}^{\infty} \frac{2k+1}{2} a_k^*(s) P_k(\cos\theta_{\beta\gamma})\, 32\pi \sum_{l=0}^{\infty} \frac{2l+1}{2} a_l(s) P_l(\cos\theta_{\alpha\gamma}) \\
&= \frac{\beta(s, m_1)}{8\pi} \sum_{k,l=0}^{\infty} (2k+1)(2l+1) a_k^*(s) a_l(s) \\
&\qquad \times \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} \int d\Omega_k P_j(\cos\theta_{\alpha\beta}) P_k(\cos\theta_{\beta\gamma}) P_l(\cos\theta_{\alpha\gamma}). \qquad (12)
\end{aligned}
$$

如 Fig. 1 所示, 在以 $\mathbf{p}_1$ 方向为 z 轴方向的坐标系中, 分别将 $\mathbf{p}_1$, $\mathbf{q}_1$ 和 $\mathbf{k}_1$ 表示为

$$
\mathbf{p}_1 = |\mathbf{p}_1|(0, 0, 1), \quad \mathbf{q}_1 = |\mathbf{q}_1|(\sin\theta_{\alpha\beta}, 0, \cos\theta_{\alpha\beta}), \quad \mathbf{k}_1 = |\mathbf{k}_1|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad (13)
$$

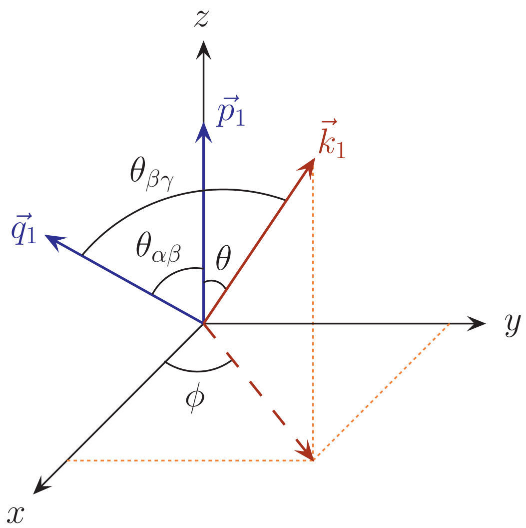


Figure 1: 球坐标系中各散射角的关系.

则

$$\cos\theta_{\alpha\gamma} = \cos\theta, \quad \cos\theta_{\beta\gamma} = \sin\theta_{\alpha\beta}\sin\theta\cos\phi + \cos\theta_{\alpha\beta}\cos\theta. \tag{14}$$

此时, Legendre 多项式满足如下加法公式 (参考 [3, 4])

$$P_k(\cos\theta_{\beta\gamma}) = P_k(\cos\theta_{\alpha\beta})P_k(\cos\theta) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_k^m(\cos\theta_{\alpha\beta})P_k^m(\cos\theta)\cos m\phi, \tag{15}$$

两边对 ϕ 积分, 利用 $\int_0^{2\pi} d\phi\cos m\phi = 0$, 可得

$$\int_0^{2\pi} d\phi P_k(\sin\theta_{\alpha\beta}\sin\theta\cos\phi + \cos\theta_{\alpha\beta}\cos\theta) = 2\pi P_k(\cos\theta_{\alpha\beta})P_k(\cos\theta), \tag{16}$$

由 Legendre 多项式的正交关系 (9), 有

$$\begin{aligned}
&\int_0^{\pi} d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}\int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\phi P_j(\cos\theta_{\alpha\beta})P_k(\sin\theta_{\alpha\beta}\sin\theta\cos\phi + \cos\theta_{\alpha\beta}\cos\theta)P_l(\cos\theta)\\
=\ & 2\pi\int_0^{\pi} d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}P_j(\cos\theta_{\alpha\beta})P_k(\cos\theta_{\alpha\beta})\int_0^{\pi} d\theta\sin\theta P_k(\cos\theta)P_l(\cos\theta)\\
=\ & \frac{8\pi}{(2k+1)(2l+1)}\delta_{jk}\delta_{kl}.
\end{aligned} \tag{17}$$

于是, 不等式 (12) 化为

$$\begin{aligned}
\mathrm{Im}\, a_j(s) \geqslant\ & \frac{\beta(s,m_1)}{8\pi}\sum_{k,l=0}^{\infty}(2k+1)(2l+1)a_k^*(s)a_l(s)\\
&\times\int_0^{\pi} d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}\int_0^{\pi} d\theta\sin\theta\int_0^{2\pi} d\phi P_j(\cos\theta_{\alpha\beta})P_k(\cos\theta_{\beta\gamma})P_l(\cos\theta_{\alpha\gamma})\\
=\ & \frac{\beta(s,m_1)}{8\pi}\sum_{k,l=0}^{\infty}(2k+1)(2l+1)a_k^*(s)a_l(s)\frac{8\pi}{(2k+1)(2l+1)}\delta_{jk}\delta_{kl}
\end{aligned}$$

$$= \beta(s, m_1)|a_j(s)|^2. \tag{18}$$

可见, 对于弹性散射, 散射矩阵幺正性要求振幅的每一分波满足

$$\mathrm{Im}\, a_j(s) \geqslant \beta(s, m_1)|a_j(s)|^2. \tag{19}$$

不等式 (19) 等价于

$$(\mathrm{Re}\, a_j)^2 + \left(\mathrm{Im}\, a_j - \frac{1}{2\beta}\right)^2 \leqslant \frac{1}{(2\beta)^2} \quad \text{或} \quad \left|a_j - \frac{i}{2\beta}\right| \leqslant \frac{1}{2\beta}. \tag{20}$$

这在 a_j 复平面上表现为以 $i(2\beta)^{-1}$ 为中心, 半径为 $(2\beta)^{-1}$ 的圆面, 如 Fig. 2 所示. 容易看出, 下列不等式也成立,

$$|\,\mathrm{Re}\, a_j(s)| \leqslant \frac{1}{2\beta(s, m_1)}. \tag{21}$$

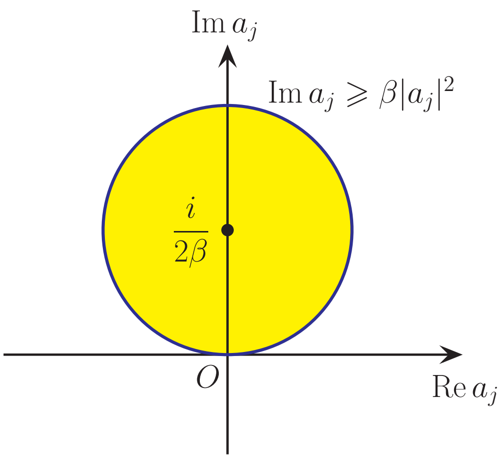


Figure 2: 不等式 (19) 在复平面上对应的区域.

对于无质量粒子散射的情形, $\beta(s, m_1) = 1$, 则有

$$\mathrm{Im}\, a_j(s) \geqslant |a_j(s)|^2 \quad \text{和} \quad |\,\mathrm{Re}\, a_j(s)| \leqslant \frac{1}{2}, \tag{22}$$

与文献 [5] 中 (2) 式和 (5) 式一致.

3 非弹性散射

3.1 两体非弹性散射

现在考虑两体非弹性散射过程 $1 + 2 \to 3 + 4$, 散射振幅为 $\mathcal{M}_{\mathrm{inel}}(s, \cos\theta)$:

$$\begin{aligned} &\alpha(p_1, p_2) \to \gamma_{34}(k_3, k_4), \quad \mathcal{M}_{\mathrm{inel}}(s, \cos\theta_{\alpha\gamma}), \quad \cos\theta_{\alpha\gamma} = \frac{\mathbf{p}_1 \cdot \mathbf{k}_3}{|\mathbf{p}_1||\mathbf{k}_3|}; \\ &\beta(q_1, q_2) \to \gamma_{34}(k_3, k_4), \quad \mathcal{M}_{\mathrm{inel}}(s, \cos\theta_{\beta\gamma}), \quad \cos\theta_{\beta\gamma} = \frac{\mathbf{q}_1 \cdot \mathbf{k}_3}{|\mathbf{q}_1||\mathbf{k}_3|}. \end{aligned} \tag{23}$$

仿照不等式 (8) 的推导过程, 可得

$$\begin{aligned}\operatorname{Im}\mathcal{M}_{\mathrm{el}}(s,\cos\theta_{\alpha\beta}) \geqslant\ & \frac{\beta(s,m_1)}{64\pi^2}\int d\Omega_k \mathcal{M}^*_{\mathrm{el}}(s,\cos\theta_{\beta\gamma})\mathcal{M}_{\mathrm{el}}(s,\cos\theta_{\alpha\gamma}) \\ & + \frac{\beta(s,m_3)}{64\pi^2}\int d\Omega_{k_3}\mathcal{M}^*_{\mathrm{inel}}(s,\cos\theta_{\beta\gamma})\mathcal{M}_{\mathrm{inel}}(s,\cos\theta_{\alpha\gamma}). \end{aligned} \tag{24}$$

而这一非弹性散射振幅的分波

$$a_j^{\mathrm{inel}}(s) = \frac{1}{32\pi}\int_0^\pi d\theta\sin\theta P_j(\cos\theta)\mathcal{M}_{\mathrm{inel}}(s,\cos\theta) \tag{25}$$

应满足

$$\operatorname{Im} a_j(s) \geqslant \beta(s,m_1)|a_j(s)|^2 + \beta(s,m_3)|a_j^{\mathrm{inel}}(s)|^2. \tag{26}$$

从而,

$$\begin{aligned}\beta(s,m_3)|a_j^{\mathrm{inel}}|^2 &\leqslant \operatorname{Im} a_j - \beta(s,m_1)|a_j|^2 \\ &= \frac{1}{4\beta(s,m_1)} - \beta(s,m_1)\left[(\operatorname{Re} a_j)^2 + \left(\operatorname{Im} a_j - \frac{1}{2\beta(s,m_1)}\right)^2\right] \\ &\leqslant \frac{1}{4\beta(s,m_1)}. \end{aligned} \tag{27}$$

可见, 任意两体非弹性散射过程的每一散射分波应满足不等式

$$|a_j^{\mathrm{inel}}(s)| \leqslant \frac{1}{2\sqrt{\beta(s,m_1)\beta(s,m_3)}}. \tag{28}$$

当入射粒子与出射粒子均无质量时, 不等式 (28) 化为

$$|a_j^{\mathrm{inel}}(s)| \leqslant \frac{1}{2}, \tag{29}$$

与文献 [5] 中 (28) 式一致.

若 1 与 2 互为反粒子, 3 与 4 互为反粒子, 且 $m_1 \leqslant m_3$, 则有 $\beta(s,m_1) \geqslant \beta(s,m_3)$, 从而

$$|a_j^{\mathrm{inel}}(s)| \leqslant \frac{1}{2\beta(s,m_1)}. \tag{30}$$

这一结果与文献 [6] 中 (3) 式以下段落里面给出的式子一致.

3.2 $2\to 3$ 非弹性散射

对于 $2\to 3$ 非弹性散射过程 $1+2\to 3+4+5$,

$$\alpha(p_1,p_2)\to\gamma_{345}(k_3,k_4,k_5),\quad \beta(q_1,q_2)\to\gamma_{345}(k_3,k_4,k_5),\quad \cos\theta_{\alpha\beta} = \frac{\mathbf{p}_1\cdot\mathbf{q}_1}{|\mathbf{p}_1||\mathbf{q}_1|}, \tag{31}$$

同样仿照不等式 (8) 的推导过程, 可得

$$\operatorname{Im}\mathcal{M}_{\mathrm{el}}(s,\cos\theta_{\alpha\beta}) \geqslant \frac{\beta(s,m_1)}{64\pi^2}\int d\Omega_k \mathcal{M}^*_{\mathrm{el}}(s,\cos\theta_{\beta\gamma})\mathcal{M}_{\mathrm{el}}(s,\cos\theta_{\alpha\gamma})$$

$$+\frac{1}{2}\int d\Pi_{345}\sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})(2\pi)^4\delta^{(4)}(p_\alpha-p_\gamma). \tag{32}$$

对弹性振幅进行分波, 有

$$\begin{aligned}
\operatorname{Im}a_j(s) \;\geqslant\; & \beta(s,m_1)|a_j(s)|^2 \\
& +\frac{1}{64\pi}\int_0^\pi d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}P_j(\cos\theta_{\alpha\beta}) \\
& \quad\times\int d\Pi_{345}\sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})(2\pi)^4\delta^{(4)}(p_\alpha-p_\gamma).
\end{aligned}\tag{33}$$

设

$$\begin{aligned}
b_j^{\text{inel}}(s) \;\equiv\; & \frac{1}{64\pi}\int_0^\pi d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}P_j(\cos\theta_{\alpha\beta}) \\
& \quad\times\int d\Pi_{345}\sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})(2\pi)^4\delta^{(4)}(p_\alpha-p_\gamma), \\
=\; & \frac{1}{64\pi}\int_0^\pi d\theta_{\alpha\beta}\sin\theta_{\alpha\beta}P_j(\cos\theta_{\alpha\beta})G(s,\theta_{\alpha\beta}),
\end{aligned}\tag{34}$$

其中

$$\begin{aligned}
G(s,\theta_{\alpha\beta}) \;\equiv\; & \int d\Pi_{345}\sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})(2\pi)^4\delta^{(4)}(p_\alpha-p_\gamma) \\
=\; & \int\frac{d^3k_3}{(2\pi)^3 2E_3}\frac{d^3k_4}{(2\pi)^3 2E_4}\frac{d^3k_5}{(2\pi)^3 2E_5}(2\pi)^4\delta^{(4)}(p_1+p_2-k_3-k_4-k_5) \\
& \times\sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345}),
\end{aligned}\tag{35}$$

则

$$\begin{aligned}
b_j^{\text{inel}}(s) \;\leqslant\; & \operatorname{Im}a_j-\beta(s,m_1)|a_j|^2 \\
=\; & \frac{1}{4\beta(s,m_1)}-\beta(s,m_1)\left[(\operatorname{Re}a_j)^2+\left(\operatorname{Im}a_j-\frac{1}{2\beta(s,m_1)}\right)^2\right] \\
\leqslant\; & \frac{1}{4\beta(s,m_1)}.
\end{aligned}\tag{36}$$

这就是散射矩阵幺正性对 $2\to 3$ 非弹性散射过程给出的限制.

4 复标量 WIMP 与光子的有效耦合

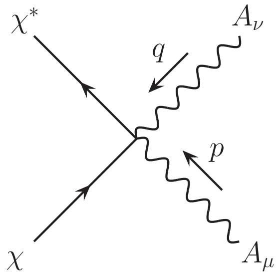
考虑复标量 WIMP (χ) 与光子具有如下形式的有效相互作用,

$$\mathcal{L}_{\text{int}}=\frac{1}{\Lambda^2}\chi^*\chi F_{\mu\nu}F^{\mu\nu}=\frac{2}{\Lambda^2}\chi^*\chi(\partial_\mu A_\nu\partial^\mu A^\nu-\partial_\mu A_\nu\partial^\nu A^\mu). \tag{37}$$

对于指向顶点的动量 k, 时空导数 ∂_μ 在动量空间中贡献一个 $-ik_\mu$ 因子, 因而,

$$\begin{aligned}
& i\frac{2}{\Lambda^2}\chi^*\chi(g^{\rho\sigma}g^{\mu\nu}\partial_\rho A_\mu\partial_\sigma A_\nu + g^{\rho\sigma}g^{\nu\mu}\partial_\rho A_\nu\partial_\sigma A_\mu - g^{\rho\nu}g^{\mu\sigma}\partial_\rho A_\mu\partial_\sigma A_\nu - g^{\rho\mu}g^{\nu\sigma}\partial_\rho A_\nu\partial_\sigma A_\mu) \\
\to\ & i\frac{2}{\Lambda^2}[g^{\rho\sigma}g^{\mu\nu}(-ip_\rho)(-iq_\sigma) + g^{\rho\sigma}g^{\nu\mu}(-iq_\rho)(-ip_\sigma) - g^{\rho\nu}g^{\mu\sigma}(-ip_\rho)(-iq_\sigma) - g^{\rho\mu}g^{\nu\sigma}(-iq_\rho)(-ip_\sigma)] \\
=\ & -i\frac{4}{\Lambda^2}[g^{\mu\nu}(p\cdot q) - q^\mu p^\nu]. \qquad (38)
\end{aligned}$$

于是, 相互作用顶点的 Feynman 规则如 Fig. 3 所示.



$$= -i\frac{4}{\Lambda^2}[g^{\mu\nu}(p\cdot q) - q^\mu p^\nu]$$

Figure 3: 复标量 WIMP 与光子有效相互作用顶点的 Feynman 规则.

4.1 湮灭过程

对于 WIMP 湮灭到双光子的过程 $\chi(p_1) + \chi^*(p_2) \to \gamma(k_1) + \gamma(k_2)$,

$$\begin{aligned}
i\mathcal{M}(\chi\chi^* \to 2\gamma) &= -i\frac{4}{\Lambda^2}[g^{\mu\nu}(-k_2)\cdot(-k_1) - k_1^\mu k_2^\nu]\varepsilon_\nu^*(k_1)\varepsilon_\mu^*(k_2) \\
&= i\frac{4}{\Lambda^2}[-(k_1\cdot k_2)\varepsilon_\nu^*(k_1)\varepsilon^{*\nu}(k_2) + k_{1\mu}\varepsilon^{*\mu}(k_2)k_2^\nu\varepsilon_\nu^*(k_1)], \qquad (39) \\
[i\mathcal{M}(\chi\chi^* \to 2\gamma)]^* &= -i\frac{4}{\Lambda^2}[-(k_1\cdot k_2)\varepsilon_\rho(k_1)\varepsilon^\rho(k_2) + k_{1\rho}\varepsilon^\rho(k_2)k_2^\sigma\varepsilon_\sigma(k_1)], \qquad (40)
\end{aligned}$$

$$\begin{aligned}
\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{16}{\Lambda^4}\sum_{\text{spins}}[-(k_1\cdot k_2)\varepsilon_\nu^*(k_1)\varepsilon^{*\nu}(k_2) + k_{1\mu}\varepsilon^{*\mu}(k_2)k_2^\nu\varepsilon_\nu^*(k_1)] \\
&\qquad \times[-(k_1\cdot k_2)\varepsilon_\rho(k_1)\varepsilon^\rho(k_2) + k_{1\rho}\varepsilon^\rho(k_2)k_2^\sigma\varepsilon_\sigma(k_1)] \\
&= \frac{16}{\Lambda^4}[(k_1\cdot k_2)^2 g_{\nu\rho}g^{\nu\rho} - (k_1\cdot k_2)k_{1\rho}k_2^\sigma g_{\nu\sigma}g^{\nu\rho} - (k_1\cdot k_2)k_{1\mu}k_2^\nu g^{\mu\rho}g_{\nu\rho} + k_{1\mu}k_2^\nu k_{1\rho}k_2^\sigma g^{\mu\rho}g_{\nu\sigma}] \\
&= \frac{32}{\Lambda^4}(k_1\cdot k_2)^2 = \frac{8}{\Lambda^4}s^2, \qquad (41)
\end{aligned}$$

由

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{p_1}2E_{p_2}|\mathbf{v}_1 - \mathbf{v}_2|}\frac{|\mathbf{k}_1|}{(2\pi)^2 4E_{\text{CM}}}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{32\pi^2 sv}\sum_{\text{spins}}|\mathcal{M}|^2, \qquad (42)$$

注意到末态光子的全同性并取低速极限, 可得

$$\sigma_{\text{ann}}v = \frac{1}{32\pi^2 s}\frac{1}{2}\int d\Omega\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{32\pi s}\int\sin\theta d\theta\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{16\pi s}\frac{8}{\Lambda^4}s^2 = \frac{s}{2\pi\Lambda^4} \simeq \frac{2m_\chi^2}{\pi\Lambda^4}, \qquad (43)$$

其中 $v \equiv |\mathbf{v}_1 - \mathbf{v}_2|$. 当

$$\langle \sigma_{\rm ann} v \rangle_{\chi\chi^* \to 2\gamma} \sim 1.27 \times 10^{-27}\ {\rm cm^3 s^{-1}} \simeq 10^{-10}\ {\rm GeV}^{-2}, \quad m_\chi \simeq 130\ {\rm GeV} \tag{44}$$

时 ($1\ {\rm cm}^3\ {\rm s}^{-1} = 8.5666 \times 10^{16}\ {\rm GeV}^{-2}$), 有

$$\Lambda = \left(\frac{2m_\chi^2}{\pi \langle \sigma_{\rm ann} v \rangle_{\chi\chi^* \to 2\gamma}} \right)^{1/4} \simeq 3220\ {\rm GeV}. \tag{45}$$

4.2 $2 \to 3$ 产生过程

对于标准模型费米子对湮灭导致的 $2 \to 3$ 产生过程 $f(p_1) + \bar{f}(p_2) \to \gamma(k_3) + \chi(k_4) + \chi^*(k_5)$,

$$\begin{aligned} i\mathcal{M}(f\bar{f} \to \gamma\chi\chi^*) &= iQ_f e \bar{v}(p_2)\gamma^\mu u(p_1) \frac{-ig_{\mu\nu}}{q^2} \left(-i\frac{4}{\Lambda^2} \right) [g^{\rho\nu}(-k_3 \cdot q) - q^\rho(-k_3^\nu)] \varepsilon_\rho^*(k_3) \\ &= i\frac{4}{\Lambda^2} Q_f e \frac{1}{q^2} \bar{v}(p_2)\gamma^\mu u(p_1)[(k_3 \cdot q)\varepsilon_\mu^*(k_3) - k_{3\mu} q^\nu \varepsilon_\nu^*(k_3)], \end{aligned} \tag{46}$$

$$[i\mathcal{M}(f\bar{f} \to \gamma\chi\chi^*)]^* = -i\frac{4}{\Lambda^2} Q_f e \frac{1}{q^2} \bar{u}(p_1)\gamma^\rho v(p_2)[(k_3 \cdot q)\varepsilon_\rho(k_3) - k_{3\rho} q^\sigma \varepsilon_\sigma(k_3)], \tag{47}$$

其中 $q = p_1 + p_2 = k_3 + k_4 + k_5$, 则

$$\begin{aligned} & \frac{1}{4} \sum_{\rm spins} |\mathcal{M}(f\bar{f} \to \gamma\chi\chi^*)|^2 \\ = & \frac{1}{4} \sum_{\rm spins} \frac{16}{\Lambda^4} Q_f^2 e^2 \frac{1}{q^4} \bar{v}(p_2)\gamma^\mu u(p_1)[(k_3 \cdot q)\varepsilon_\mu^*(k_3) - k_{3\mu} q^\nu \varepsilon_\nu^*(k_3)] \\ & \qquad \times \bar{u}(p_1)\gamma^\rho v(p_2)[(k_3 \cdot q)\varepsilon_\rho(k_3) - k_{3\rho} q^\sigma \varepsilon_\sigma(k_3)] \\ = & \frac{4}{\Lambda^4} Q_f^2 e^2 \sum_{\rm spins} \frac{1}{q^4} {\rm Tr}[v(p_2)\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\rho][(k_3 \cdot q)^2 \varepsilon_\mu^*(k_3)\varepsilon_\rho(k_3) \\ & \quad -(k_3 \cdot q) k_{3\rho} q^\sigma \varepsilon_\mu^*(k_3)\varepsilon_\sigma(k_3) - (k_3 \cdot q) k_{3\mu} q^\nu \varepsilon_\nu^*(k_3)\varepsilon_\rho(k_3) + k_{3\mu} q^\nu k_{3\rho} q^\sigma \varepsilon_\nu^*(k_3)\varepsilon_\sigma(k_3)] \\ = & \frac{4}{\Lambda^4} Q_f^2 e^2 \frac{1}{q^4} {\rm Tr}[(\not{p}_2 - m_f)\gamma^\mu(\not{p}_1 + m_f)\gamma^\rho][-(k_3 \cdot q)^2 g_{\mu\rho} + (k_3 \cdot q)(k_{3\mu} q_\rho + k_{3\rho} q_\mu) - q^2 k_{3\mu} k_{3\rho}] \\ = & \frac{16 Q_f^2 e^2}{\Lambda^4 q^4} \big\{ q^2 [k_3^2(p_1 \cdot p_2 + m_f^2) - 2(p_1 \cdot k_3)(p_2 \cdot k_3)] + 2m_f^2 (q \cdot k_3)^2 \\ & \qquad +2(q \cdot k_3)[(p_1 \cdot k_3)(p_2 \cdot q) + (p_2 \cdot k_3)(p_1 \cdot q)] \big\} \\ = & \frac{16 Q_f^2 e^2}{\Lambda^4 s^2} [(q \cdot k_3)^2 (s + 2m_f^2) - 2s(p_1 \cdot k_3)(p_2 \cdot k_3)]. \end{aligned} \tag{48}$$

利用

$$\int dk_{35}^0 \delta(s_{35} - k_{35}^2) = \int dk_{35}^0 \delta(s_{35} - (k_{35}^0)^2 + |\mathbf{k}_{35}|^2) = \left. \frac{1}{2k_{35}^0} \right|_{k_{35}^0 = \sqrt{s_{35} + |\mathbf{k}_{35}|^2}}, \tag{49}$$

可以将三体相空间积分化成两个二体相空间积分,

$$\begin{aligned} \int d\Phi^{(3)} &= \int \frac{d^3 k_3}{(2\pi)^3 2k_3^0} \frac{d^3 k_4}{(2\pi)^3 2k_4^0} \frac{d^3 k_5}{(2\pi)^3 2k_5^0} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_3 - k_4 - k_5) \\ &= \int ds_{35} \delta(s_{35} - k_{35}^2) d^4 k_{35} \delta^{(4)}(k_{35} - k_3 - k_5) \end{aligned}$$

$$
\begin{aligned}
&\times \frac{d^3k_3}{(2\pi)^3 2k_3^0}\frac{d^3k_4}{(2\pi)^3 2k_4^0}\frac{d^3k_5}{(2\pi)^3 2k_5^0}(2\pi)^4\delta^{(4)}(p_1+p_2-k_3-k_4-k_5)\\
= &\int ds_{35}\delta^{(4)}(k_{35}-k_3-k_5)\frac{d^3k_{35}}{2k_{35}^0}\\
&\times \frac{d^3k_3}{(2\pi)^3 2k_3^0}\frac{d^3k_4}{(2\pi)^3 2k_4^0}\frac{d^3k_5}{(2\pi)^3 2k_5^0}(2\pi)^4\delta^{(4)}(p_1+p_2-k_3-k_4-k_5)\\
= &\int \frac{ds_{35}}{2\pi}\times\frac{d^3k_4}{(2\pi)^3 2k_4^0}\frac{d^3k_{35}}{(2\pi)^3 2k_{35}^0}(2\pi)^4\delta^{(4)}(p_1+p_2-k_4-k_{35})\\
&\times \frac{d^3k_3}{(2\pi)^3 2k_3^0}\frac{d^3k_5}{(2\pi)^3 2k_5^0}(2\pi)^4\delta^{(4)}(k_{35}-k_3-k_5)\\
= &\int \frac{ds_{35}}{2\pi}d\Phi_1 d\Phi_2,
\end{aligned}
\tag{50}
$$

其中

$$
d\Phi_1 \equiv \frac{d^3k_4}{(2\pi)^3 2k_4^0}\frac{d^3k_{35}}{(2\pi)^3 2k_{35}^0}(2\pi)^4\delta^{(4)}(p_1+p_2-k_4-k_{35}),\tag{51}
$$

$$
d\Phi_2 \equiv \frac{d^3k_3}{(2\pi)^3 2k_3^0}\frac{d^3k_5}{(2\pi)^3 2k_5^0}(2\pi)^4\delta^{(4)}(k_{35}-k_3-k_5).\tag{52}
$$

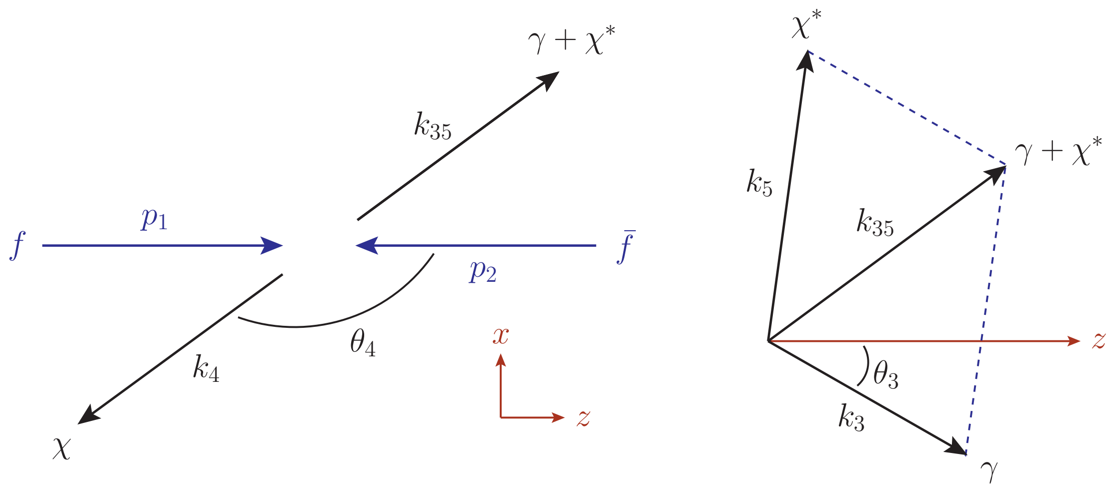


Figure 4: 各动量示意图.

如 Fig. 4 所示, 将 $\mathbf{k}_3$ 和 $\mathbf{k}_4$ 分别表达为

$$
\mathbf{k}_3 = |\mathbf{k}_3|(\sin\theta_3\cos\phi_3, \sin\theta_3\sin\phi_3, \cos\theta_3), \quad \mathbf{k}_4 = |\mathbf{k}_4|(\sin\theta_4, 0, \cos\theta_4) = -\mathbf{k}_{35}.\tag{53}
$$

依照两体运动学,

$$
k_{35}^0 = \frac{s+s_{35}-m_\chi^2}{2\sqrt{s}}, \quad |\mathbf{k}_4| = \frac{1}{2\sqrt{s}}\sqrt{\left[s-(\sqrt{s_{35}}+m_\chi)^2\right]\left[s-(\sqrt{s_{35}}-m_\chi)^2\right]}.\tag{54}
$$

在 γ 和 χ^* 的质心系中, 有

$$
\tilde{k}_{35}^\mu = (\sqrt{s_{35}}, 0, 0, 0), \quad \tilde{k}_3^0 = \frac{s_{35}-m_\chi^2}{2\sqrt{s_{35}}}.\tag{55}
$$

从而,

$$\frac{1}{2}(s_{35}-m_\chi^2)=\tilde{k}_{35}\cdot\tilde{k}_3=k_{35}\cdot k_3=k_{35}^0|\mathbf{k}_3|+|\mathbf{k}_3||\mathbf{k}_4|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3), \quad (56)$$

于是,

$$|\mathbf{k}_3|=\frac{s_{35}-m_\chi^2}{2}\frac{1}{k_{35}^0+|\mathbf{k}_4|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3)}. \quad (57)$$

另一方面,

$$\begin{aligned}
\mathbf{k}_5 &= -\mathbf{k}_3-\mathbf{k}_4=-(|\mathbf{k}_4|\sin\theta_4+|\mathbf{k}_3|\sin\theta_3\cos\phi_3,|\mathbf{k}_3|\sin\theta_3\sin\phi_3,|\mathbf{k}_4|\cos\theta_4+|\mathbf{k}_3|\cos\theta_3), \quad (58)\\
|\mathbf{k}_5|^2 &= (|\mathbf{k}_4|\sin\theta_4+|\mathbf{k}_3|\sin\theta_3\cos\phi_3)^2+|\mathbf{k}_3|^2\sin^2\theta_3\sin^2\phi_3+(|\mathbf{k}_4|\cos\theta_4+|\mathbf{k}_3|\cos\theta_3)^2\\
&= |\mathbf{k}_4|^2\sin^2\theta_4+|\mathbf{k}_3|^2\sin^2\theta_3\cos^2\phi_3+2|\mathbf{k}_4||\mathbf{k}_3|\sin\theta_4\sin\theta_3\cos\phi_3\\
&\quad +|\mathbf{k}_3|^2\sin^2\theta_3\sin^2\phi_3+|\mathbf{k}_4|^2\cos^2\theta_4+|\mathbf{k}_3|^2\cos^2\theta_3+2|\mathbf{k}_4||\mathbf{k}_3|\cos\theta_4\cos\theta_3\\
&= |\mathbf{k}_4|^2+|\mathbf{k}_3|^2+2|\mathbf{k}_4||\mathbf{k}_3|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3), \quad (59)
\end{aligned}$$

故

$$\frac{\partial\sqrt{|\mathbf{k}_5|^2+m_5^2}}{\partial|\mathbf{k}_3|}=\frac{1}{k_5^0}[|\mathbf{k}_3|+|\mathbf{k}_4|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3)]. \quad (60)$$

利用这些关系, 可得

$$\begin{aligned}
\int d\Phi_1 &= \int\frac{d^3k_4}{(2\pi)^3 2k_4^0}\frac{d^3k_{35}}{(2\pi)^3 2k_{35}^0}(2\pi)^4\delta^{(4)}(p_1+p_2-k_4-k_{35})\\
&= \int\frac{d^3k_4}{(2\pi)^2 2k_4^0 2k_{35}^0}\delta(p_1^0+p_2^0-k_4^0-k_{35}^0)\\
&= \int\frac{|\mathbf{k}_4|^2 d|\mathbf{k}_4|d\cos\theta_4}{8\pi k_4^0 k_{35}^0}\delta\left(p_1^0+p_2^0-\sqrt{|\mathbf{k}_4|^2+m_4^2}-\sqrt{|\mathbf{k}_4|^2+s_{35}}\right)\\
&= \int\frac{|\mathbf{k}_4|^2 d\cos\theta_4}{8\pi k_4^0 k_{35}^0}\left(\frac{|\mathbf{k}_4|}{k_4^0}+\frac{|\mathbf{k}_4|}{k_{35}^0}\right)^{-1}=\int\frac{|\mathbf{k}_4|d\cos\theta_4}{8\pi(k_4^0+k_{35}^0)}=\frac{|\mathbf{k}_4|}{8\pi\sqrt{s}}\int d\cos\theta_4, \quad (61)\\
\int d\Phi_2 &= \int\frac{d^3k_3}{(2\pi)^3 2k_3^0}\frac{d^3k_5}{(2\pi)^3 2k_5^0}(2\pi)^4\delta^{(4)}(k_{35}-k_3-k_5)\\
&= \int\frac{|\mathbf{k}_3|^2 d|\mathbf{k}_3|d\cos\theta_3 d\phi_3}{16\pi^2 k_3^0 k_5^0}\delta\left(k_{35}^0-\sqrt{|\mathbf{k}_3|^2+m_3^2}-\sqrt{|\mathbf{k}_5|^2+m_5^2}\right)\\
&= \int\frac{|\mathbf{k}_3|^2 d\cos\theta_3 d\phi_3}{16\pi^2 k_3^0 k_5^0}\left[\frac{|\mathbf{k}_3|}{k_3^0}+\frac{|\mathbf{k}_3|}{k_5^0}+\frac{|\mathbf{k}_4|}{k_5^0}(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3)\right]^{-1}\\
&= \int\frac{|\mathbf{k}_3|^2 d\cos\theta_3 d\phi_3}{16\pi^2 k_3^0 k_5^0}\frac{k_3^0 k_5^0}{|\mathbf{k}_3|k_5^0+|\mathbf{k}_3|k_3^0+|\mathbf{k}_4|k_3^0(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3)}\\
&= \frac{1}{16\pi^2}\int d\cos\theta_3 d\phi_3\frac{|\mathbf{k}_3|}{k_{35}^0+|\mathbf{k}_4|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3)}\\
&= \frac{1}{8\pi^2(s_{35}-m_\chi^2)}\int d\cos\theta_3 d\phi_3|\mathbf{k}_3|^2. \quad (62)
\end{aligned}$$

于是, $2\to 3$ 过程的截面

$$\sigma = \frac{1}{2p_1^0 2p_2^0|\mathbf{v}_1-\mathbf{v}_2|}\int d\Phi^{(3)}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{2s\beta_f}\int\frac{ds_{35}}{2\pi}d\Phi_1 d\Phi_2\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{1}{256\pi^4\beta_f}\int_{m_\chi^2/s}^{(\sqrt{s}-m_\chi)^2/s} d\frac{s_{35}}{s}\int_0^\pi \sin\theta_4 d\theta_4 \int_0^\pi \sin\theta_3 d\theta_3 \int_0^{2\pi} d\phi_3 \frac{|\mathbf{k}_4||\mathbf{k}_3|^2}{\sqrt{s}(s_{35}-m_\chi^2)}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2. \quad (63)$$

现在, 利用

$$q^\mu = (\sqrt{s},0,0,0), \quad p_1^\mu = \frac{\sqrt{s}}{2}(1,0,0,\beta_f), \quad p_2^\mu = \frac{\sqrt{s}}{2}(1,0,0,-\beta_f), \quad (64)$$

其中 $\beta_f \equiv \sqrt{1-4m_f^2/s}$, 有

$$q\cdot k_3 = \sqrt{s}|\mathbf{k}_3|, \quad p_1\cdot k_3 = \frac{\sqrt{s}}{2}|\mathbf{k}_3|(1-\beta_f\cos\theta_3), \quad p_2\cdot k_3 = \frac{\sqrt{s}}{2}|\mathbf{k}_3|(1+\beta_f\cos\theta_3), \quad (65)$$

从而, 可以将 (48) 式化为

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{16Q_f^2e^2}{\Lambda^4 s^2}[(q\cdot k_3)^2(s+2m_f^2) - 2s(p_1\cdot k_3)(p_2\cdot k_3)] \\ &= \frac{8Q_f^2e^2}{\Lambda^4 s}|\mathbf{k}_3|^2[s(1+\beta_f^2\cos^2\theta_3)+4m_f^2] \\ &= \frac{32\pi Q_f^2\alpha}{\Lambda^4}|\mathbf{k}_3|^2\left(1+\beta_f^2\cos^2\theta_3+\frac{4m_f^2}{s}\right). \quad (66)\end{aligned}$$

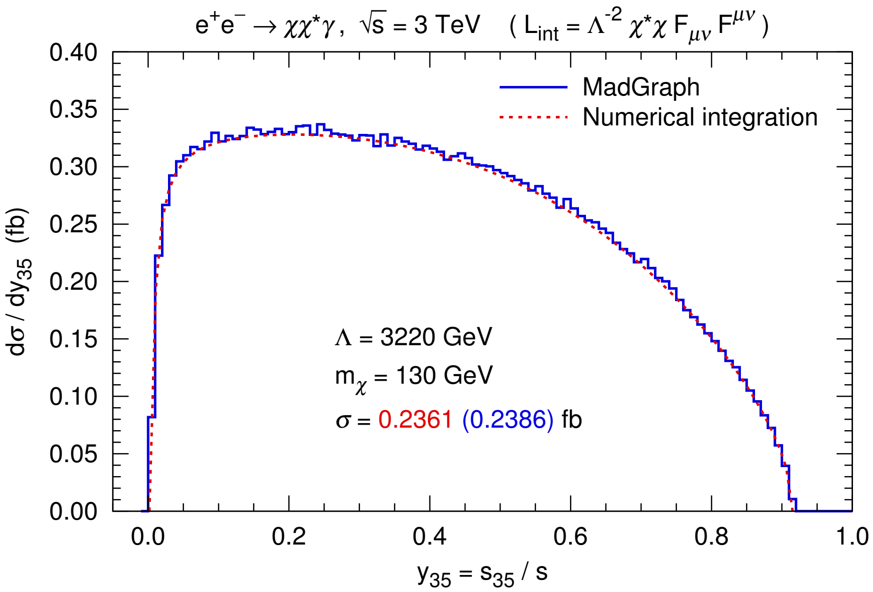


Figure 5: 关于 s_{35} 的微分截面分布.

对于 $e^+e^- \to \chi\chi^*\gamma$ 过程, 当 $\sqrt{s} = 3$ TeV, $m_\chi = 130$ GeV, $\Lambda = 3220$ GeV 时, `MadGraph5` 给出的结果是 $\sigma = 0.2386$ fb, 关于 s_{35} 的事例分布如 Fig. 5 所示. 将 (66) 式代入 (63) 式进行数值积分, 结果为 $\sigma = 0.2361$ fb, 关于 s_{35} 的微分截面分布亦示于 Fig. 5 中.

在进行数值积分时, 采用了 QED 单圈计算的结果来描述精细结构常数的能标跑动行为,

$$\alpha_{\text{eff}}(s) = \frac{\alpha}{1 - \dfrac{\alpha}{3\pi}\ln\left(\dfrac{s}{e^{5/3}m_e^2}\right)}, \tag{67}$$

其中 $\alpha = 1/137.0359991$. 由此, $\alpha_{\text{eff}}((3\ \text{TeV})^2) \simeq 1/134$. 另一方面, MadGraph5 采用 $\overline{\text{MS}}$ 方案的结果 $\alpha_{\overline{\text{MS}}}(m_Z^2) = 1/132.50698$ 进行计算. 这造成了两种方法在截面计算上的差异.

下面, 都采用 $\alpha_{\overline{\text{MS}}}(m_Z^2) = 1/132.50698$ (MadGraph5 默认值) 进行计算.

固定 $\Lambda = 3220$ GeV, 对于 $\sqrt{s} = 2$ TeV, 2.5 TeV 和 3 TeV, $e^+e^- \to \chi\chi^*\gamma$ 过程的产生截面随 m_χ 的变化如 Fig. 6 所示.

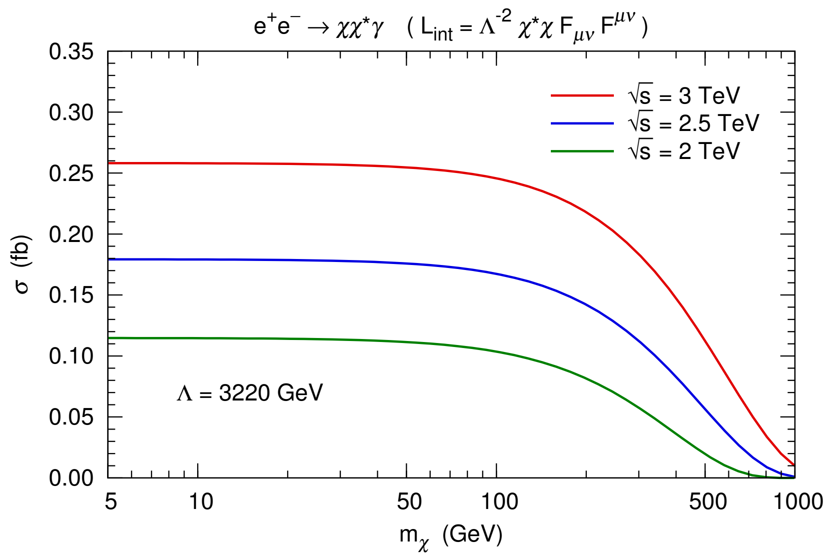


Figure 6: $e^+e^- \to \chi\chi^*\gamma$ 的产生截面随 m_χ 的变化.

4.3 产生过程的幺正性限制

为了得到幺正性限制, 必须计算如下过程的振幅,

$$\begin{aligned} \alpha \to \gamma_{345}: &\quad f(p_1) + \bar{f}(p_2) \to \gamma(k_3) + \chi(k_4) + \chi^*(k_5), \\ \beta \to \gamma_{345}: &\quad f(q_1) + \bar{f}(q_2) \to \gamma(k_3) + \chi(k_4) + \chi^*(k_5). \end{aligned} \tag{68}$$

它们的不变振幅分别为

$$\begin{aligned} i\mathcal{M}_{\text{inel}}(\alpha \to \gamma_{345}) &= i\frac{4Q_f e}{\Lambda^2 s}\bar{v}(p_2)\gamma^\mu u(p_1)[(k_3\cdot q)\varepsilon_\mu^*(k_3) - k_{3\mu}q^\nu\varepsilon_\nu^*(k_3)], \\ [i\mathcal{M}_{\text{inel}}(\beta \to \gamma_{345})]^* &= -i\frac{4Q_f e}{\Lambda^2 s}\bar{u}(q_1)\gamma^\rho v(q_2)[(k_3\cdot q)\varepsilon_\rho(k_3) - k_{3\rho}q^\sigma\varepsilon_\sigma(k_3)], \end{aligned} \tag{69}$$

其中 $q = p_1 + p_2 = q_1 + q_2$, 则

$$\begin{aligned} &\mathcal{M}_{\text{inel}}^*(\beta \to \gamma_{345})\mathcal{M}_{\text{inel}}(\alpha \to \gamma_{345}) \\ = &\frac{16Q_f^2 e^2}{\Lambda^4 s^2}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(q_1)\gamma^\rho v(q_2)[(k_3\cdot q)\varepsilon_\mu^*(k_3) - k_{3\mu}q^\nu\varepsilon_\nu^*(k_3)][(k_3\cdot q)\varepsilon_\rho^*(k_3) - k_{3\rho}q^\sigma\varepsilon_\sigma^*(k_3)] \end{aligned}$$

$$
\begin{aligned}
&= \frac{16Q_f^2e^2}{\Lambda^4s^2}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(q_1)\gamma^\rho v(q_2)\Big[(k_3\cdot q)^2\varepsilon^*_\mu(k_3)\varepsilon_\rho(k_3)-(k_3\cdot q)k_{3\rho}q^\sigma\varepsilon^*_\mu(k_3)\varepsilon_\sigma(k_3)\\
&\qquad -(k_3\cdot q)k_{3\mu}q^\nu\varepsilon^*_\nu(k_3)\varepsilon_\rho(k_3)+k_{3\mu}q^\nu k_{3\rho}q^\sigma\varepsilon^*_\nu(k_3)\varepsilon_\sigma(k_3)\Big].
\end{aligned} \tag{70}
$$

考虑极化的初态 $f_\lambda\bar{f}_{\lambda'}$ $(\lambda,\lambda'=\pm)$, 设 $F(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'})$ 为上式对末态粒子自旋态求和后的结果, 则

$$
\begin{aligned}
F(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'}) &\equiv \sum_{\text{spins of }\gamma_{345}}\mathcal{M}^*_{\text{inel}}(\beta(f_\lambda\bar{f}_{\lambda'})\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha(f_\lambda\bar{f}_{\lambda'})\to\gamma_{345})\\
&= \frac{16Q_f^2e^2}{\Lambda^4s^2}\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)\\
&\qquad\times[-(k_3\cdot q)^2g_{\mu\rho}+(k_3\cdot q)(q_\mu k_{3\rho}+q_\rho k_{3\mu})-sk_{3\mu}k_{3\rho}]\\
&= \frac{64\pi Q_f^2\alpha}{\Lambda^4s^2}\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)[-(k_3\cdot q)^2g_{\mu\rho}-sk_{3\mu}k_{3\rho}]\\
&= \frac{64\pi Q_f^2\alpha}{\Lambda^4s^2}\tilde{F}(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'}),
\end{aligned} \tag{71}
$$

其中

$$
\tilde{F}(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'})\equiv-\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)[(k_3\cdot q)^2g_{\mu\rho}+sk_{3\mu}k_{3\rho}]. \tag{72}
$$

在计算过程中用到了 Dirac 方程平面波解的性质

$$
\begin{aligned}
\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)q_\mu &= \bar{v}_{\lambda'}(p_2)(\not{p}_2+\not{p}_1)u_\lambda(p_1)=\bar{v}_{\lambda'}(p_2)(-m_f+m_f)u_\lambda(p_1)=0,\\
\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)q_\rho &= \bar{u}_\lambda(q_1)(\not{q}_1+\not{q}_2)v_{\lambda'}(q_2)=\bar{u}_\lambda(q_1)(-m_f+m_f)v_{\lambda'}(q_2)=0.
\end{aligned} \tag{73}
$$

用一组螺旋态基底 $\xi_\lambda(p)$ $(\lambda=\pm)$ 可以将 Dirac 方程的平面波解表示为

$$
u(p,\lambda)=\begin{pmatrix}\omega_{-\lambda}(p)\xi_\lambda(p)\\ \omega_\lambda(p)\xi_\lambda(p)\end{pmatrix},\quad v(p,\lambda)=\begin{pmatrix}-\lambda\omega_\lambda(p)\xi_{-\lambda}(p)\\ \lambda\omega_{-\lambda}(p)\xi_{-\lambda}(p)\end{pmatrix},\quad \omega_\lambda(p)=\sqrt{E+\lambda|\mathbf{p}|}. \tag{74}
$$

从而,

$$
\begin{aligned}
\bar{v}_+(p_2)\gamma^\mu u_-(p_1) &= -\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)+\frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1),\\
\bar{v}_-(p_2)\gamma^\mu u_+(p_1) &= \frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1)-\frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1),\\
\bar{v}_-(p_2)\gamma^\mu u_-(p_1) &= m_f\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)-m_f\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1),\\
\bar{v}_+(p_2)\gamma^\mu u_+(p_1) &= -m_f\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1)+m_f\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1),\\
\bar{u}_-(q_1)\gamma^\rho v_+(q_2) &= -\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(q_1)\bar{\sigma}^\rho\xi_-(q_2)+\frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(q_1)\sigma^\rho\xi_-(q_2),\\
\bar{u}_+(q_1)\gamma^\rho v_-(q_2) &= \frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(q_1)\bar{\sigma}^\rho\xi_+(q_2)-\frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(q_1)\sigma^\rho\xi_+(q_2),\\
\bar{u}_-(q_1)\gamma^\rho v_-(q_2) &= m_f\xi_-^\dagger(q_1)\bar{\sigma}^\rho\xi_+(q_2)-m_f\xi_-^\dagger(q_1)\sigma^\rho\xi_+(q_2),\\
\bar{u}_+(q_1)\gamma^\rho v_+(q_2) &= -m_f\xi_+^\dagger(q_1)\bar{\sigma}^\rho\xi_-(q_2)+m_f\xi_+^\dagger(q_1)\sigma^\rho\xi_-(q_2).
\end{aligned} \tag{75}
$$

初态粒子的动量和螺旋态可以表示成

$$
p_1=\frac{\sqrt{s}}{2}(1,0,0,\beta_f),\quad p_2=\frac{\sqrt{s}}{2}(1,0,0,-\beta_f), \tag{76}
$$

$$\xi_+(p_1)=\begin{pmatrix}1\\0\end{pmatrix},\quad \xi_-(p_1)=\begin{pmatrix}0\\1\end{pmatrix},\quad \xi_+(p_2)=\begin{pmatrix}0\\-1\end{pmatrix},\quad \xi_-(p_2)=\begin{pmatrix}1\\0\end{pmatrix},\tag{77}$$

$$q_1=\frac{\sqrt{s}}{2}(1,\beta_f s_{\theta_{\alpha\beta}},0,\beta_f c_{\theta_{\alpha\beta}}),\quad q_2=\frac{\sqrt{s}}{2}(1,-\beta_f s_{\theta_{\alpha\beta}},0,-\beta_f c_{\theta_{\alpha\beta}}),\tag{78}$$

$$\xi_+(q_1)=\begin{pmatrix}c_{\theta_{\alpha\beta}/2}\\s_{\theta_{\alpha\beta}/2}\end{pmatrix},\quad \xi_-(q_1)=\begin{pmatrix}-s_{\theta_{\alpha\beta}/2}\\c_{\theta_{\alpha\beta}/2}\end{pmatrix},\quad \xi_+(q_2)=\begin{pmatrix}s_{\theta_{\alpha\beta}/2}\\-c_{\theta_{\alpha\beta}/2}\end{pmatrix},\quad \xi_-(q_2)=\begin{pmatrix}c_{\theta_{\alpha\beta}/2}\\s_{\theta_{\alpha\beta}/2}\end{pmatrix}.\tag{79}$$

利用这些表达式, 计算各极化初态的 $\tilde{F}(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'})$.

$$\begin{aligned}
&\tilde{F}(\theta_{\alpha\beta},f_+,\bar{f}_-)\\
=\;&-\bar{v}_-(p_2)\gamma^\mu u_+(p_1)\bar{u}_+(q_1)\gamma^\rho v_-(q_2)[(k_3\cdot q)^2 g_{\mu\rho}+sk_{3\mu}k_{3\rho}]\\
=\;&-\bar{v}_-(p_2)\gamma^\mu u_+(p_1)\bar{u}_+(q_1)\gamma^\rho v_-(q_2)g_{\mu\nu}g_{\rho\sigma}[(k_3\cdot q)^2 g^{\nu\sigma}+sk_3^\nu k_3^\sigma]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2[(1+\cos^2\theta_3)(1+\cos\theta_{\alpha\beta})+e^{2i\phi_3}(1-\cos\theta_{\alpha\beta})\sin^2\theta_3+e^{i\phi_3}\sin 2\theta_3\sin\theta_{\alpha\beta}]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2[(1+\cos^2\theta_3)(1+\cos\theta_{\alpha\beta})+(\cos 2\phi_3+i\sin 2\phi_3)(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\\
&\qquad+(\cos\phi_3+i\sin\phi_3)\sin 2\theta_3\sin\theta_{\alpha\beta}]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2\{(1+\cos\theta_{\alpha\beta})(1+\cos^2\theta_3)+(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\cos 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3\\
&\qquad+i[(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\sin 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\sin\phi_3]\},
\end{aligned}\tag{80}$$

$$\begin{aligned}
&\tilde{F}(\theta_{\alpha\beta},f_-,\bar{f}_+)\\
=\;&-\bar{v}_+(p_2)\gamma^\mu u_-(p_1)\bar{u}_-(q_1)\gamma^\rho v_+(q_2)[(k_3\cdot q)^2 g_{\mu\rho}+sk_{3\mu}k_{3\rho}]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2[(1+\cos^2\theta_3)(1+\cos\theta_{\alpha\beta})+e^{-2i\phi_3}(1-\cos\theta_{\alpha\beta})\sin^2\theta_3+e^{-i\phi_3}\sin 2\theta_3\sin\theta_{\alpha\beta}]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2[(1+\cos^2\theta_3)(1+\cos\theta_{\alpha\beta})+(\cos 2\phi_3-i\sin 2\phi_3)(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\\
&\qquad+(\cos\phi_3-i\sin\phi_3)\sin 2\theta_3\sin\theta_{\alpha\beta}]\\
=\;&\frac{1}{2}s^2|\mathbf{k}_3|^2\{(1+\cos\theta_{\alpha\beta})(1+\cos^2\theta_3)+(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\cos 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3\\
&\qquad-i[(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\sin 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\sin\phi_3]\}\\
=\;&\tilde{F}^*(\theta_{\alpha\beta},f_+,\bar{f}_-),
\end{aligned}\tag{81}$$

$$\begin{aligned}
\tilde{F}(\theta_{\alpha\beta},f_-,\bar{f}_-)\;&=\;-\bar{v}_-(p_2)\gamma^\mu u_-(p_1)\bar{u}_-(q_1)\gamma^\rho v_-(q_2)[(k_3\cdot q)^2 g_{\mu\rho}+sk_{3\mu}k_{3\rho}]\\
&=\;2s|\mathbf{k}_3|^2 m_f^2(2\cos\theta_{\alpha\beta}\sin^2\theta_3-\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3),
\end{aligned}\tag{82}$$

$$\begin{aligned}
\tilde{F}(\theta_{\alpha\beta},f_+,\bar{f}_+)\;&=\;-\bar{v}_+(p_2)\gamma^\mu u_+(p_1)\bar{u}_+(q_1)\gamma^\rho v_+(q_2)[(k_3\cdot q)^2 g_{\mu\rho}+sk_{3\mu}k_{3\rho}]\\
&=\;2s|\mathbf{k}_3|^2 m_f^2(2\cos\theta_{\alpha\beta}\sin^2\theta_3-\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3)\\
&=\;\tilde{F}(\theta_{\alpha\beta},f_-,\bar{f}_-).
\end{aligned}\tag{83}$$

当 $\theta_{\alpha\beta}=0$ 时, 这些式子化为

$$\tilde{F}(\theta_{\alpha\beta}=0,f_+,\bar{f}_-)\;=\;\tilde{F}(\theta_{\alpha\beta}=0,f_-,\bar{f}_+)=s^2|\mathbf{k}_3|^2(1+\cos^2\theta_3),\tag{84}$$

$$\tilde{F}(\theta_{\alpha\beta}=0,f_-,\bar{f}_-)\;=\;\tilde{F}(\theta_{\alpha\beta}=0,f_+,\bar{f}_+)=4s|\mathbf{k}_3|^2 m_f^2\sin^2\theta_3.\tag{85}$$

$\tilde{F}(\theta_{\alpha\beta}=0,f_-,\bar{f}_-)$ 和 $\tilde{F}(\theta_{\alpha\beta}=0,f_+,\bar{f}_+)$ 正比于 m_f^2, 这是螺旋度压低的表现.

对于非极化初态, 有

$$
\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}}\mathcal{M}^*_{\text{inel}}(\beta\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})\\
=&\ \frac{1}{4}[F(\theta_{\alpha\beta},f_+,\bar f_-)+F(\theta_{\alpha\beta},f_-,\bar f_+)+F(\theta_{\alpha\beta},f_-,\bar f_-)+F(\theta_{\alpha\beta},f_+,\bar f_+)]\\
=&\ \frac{1}{4}\frac{64\pi Q_f^2\alpha}{\Lambda^4 s^2}[\tilde F(\theta_{\alpha\beta},f_+,\bar f_-)+\tilde F(\theta_{\alpha\beta},f_-,\bar f_+)+\tilde F(\theta_{\alpha\beta},f_-,\bar f_-)+\tilde F(\theta_{\alpha\beta},f_+,\bar f_+)]\\
=&\ \frac{1}{4}\frac{64\pi Q_f^2\alpha}{\Lambda^4 s^2}\{s^2|\mathbf{k}_3|^2[(1+\cos\theta_{\alpha\beta})(1+\cos^2\theta_3)+(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\cos 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3]\\
&\qquad\qquad +4s|\mathbf{k}_3|^2m_f^2(2\cos\theta_{\alpha\beta}\sin^2\theta_3-\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3)\}\\
=&\ \frac{16\pi Q_f^2\alpha}{\Lambda^4}\frac{|\mathbf{k}_3|^2}{s}\{s[(1+\cos\theta_{\alpha\beta})(1+\cos^2\theta_3)+(1-\cos\theta_{\alpha\beta})\sin^2\theta_3\cos 2\phi_3+\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3]\\
&\qquad\qquad +4m_f^2(2\cos\theta_{\alpha\beta}\sin^2\theta_3-\sin\theta_{\alpha\beta}\sin 2\theta_3\cos\phi_3)\}.
\end{aligned}
\tag{86}
$$

令 $\theta_{\alpha\beta}=0$, 就可以得到此散射过程的振幅模方,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345})|^2 &= \frac{32\pi Q_f^2\alpha}{\Lambda^4}\frac{|\mathbf{k}_3|^2}{s}[s(1+\cos^2\theta_3)+4m_f^2\sin^2\theta_3]\\
&= \frac{32\pi Q_f^2\alpha}{\Lambda^4}|\mathbf{k}_3|^2\left(1+\beta_f^2\cos^2\theta_3+\frac{4m_f^2}{s}\right).
\end{aligned}
\tag{87}
$$

这一结果与 (66) 式一致.

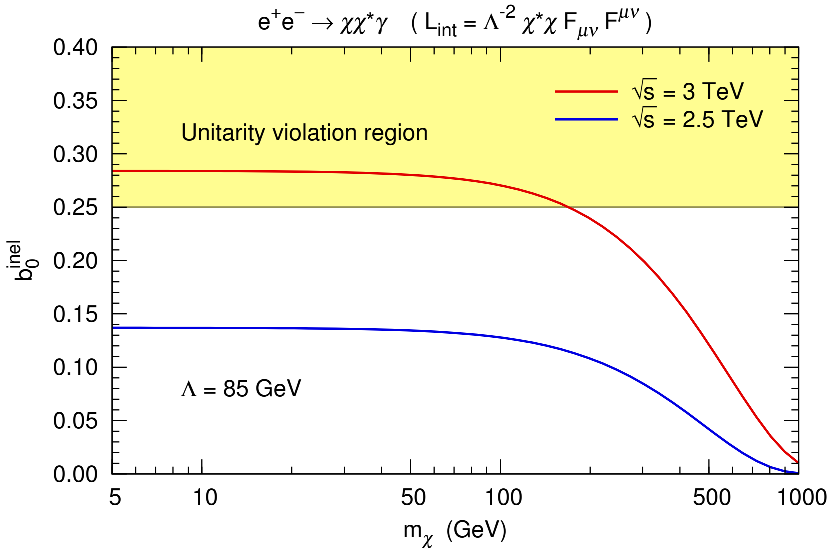


Figure 7: 固定 $\Lambda=85$ GeV 时, $b_0^{\text{inel}}(s,e_+^+,e_-^-)$ 随 m_χ 变化的情况. 幺正性在浅黄色区域中遭到破坏.

现在, 对于各极化初态相应过程, (35) 式化为

$$
G(s,\theta_{\alpha\beta},f_\lambda,\bar f_{\lambda'}) = \int\frac{d^3k_3}{(2\pi)^3 2E_3}\frac{d^3k_4}{(2\pi)^3 2E_4}\frac{d^3k_5}{(2\pi)^3 2E_5}(2\pi)^4\delta^{(4)}(p_1+p_2-k_3-k_4-k_5)
$$

$$\times \sum_{\text{spins of } \gamma_{345}} \mathcal{M}^*_{\text{inel}}(\beta(f_\lambda \bar{f}_{\lambda'}) \to \gamma_{345}) \mathcal{M}_{\text{inel}}(\alpha(f_\lambda \bar{f}_{\lambda'}) \to \gamma_{345})$$

$$= \int \frac{ds_{35}}{2\pi} d\Phi_1 d\Phi_2 F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'})$$

$$= \int \frac{ds_{35}}{2\pi} \frac{|\mathbf{k}_4|}{8\pi\sqrt{s}} \int d\cos\theta_4 \frac{1}{8\pi^2(s_{35} - m_\chi^2)} \int d\cos\theta_3 d\phi_3 |\mathbf{k}_3|^2 F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'})$$

$$= \frac{1}{128\pi^4} \int_{m_\chi^2/s}^{(\sqrt{s}-m_\chi)^2/s} d\frac{s_{35}}{s} \int_0^\pi \sin\theta_4 d\theta_4 \int_0^\pi \sin\theta_3 d\theta_3 \int_0^{2\pi} d\phi_3$$

$$\times \frac{|\mathbf{k}_4||\mathbf{k}_3|^2 s}{\sqrt{s}(s_{35} - m_\chi^2)} F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'}), \tag{88}$$

而 (34) 式变成

$$b_j^{\text{inel}}(s, f_\lambda, \bar{f}_{\lambda'}) = \frac{1}{64\pi} \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} P_j(\cos\theta_{\alpha\beta}) G(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'})$$

$$= \frac{1}{64\pi} \int_0^\pi d\theta_{\alpha\beta} \sin\theta_{\alpha\beta} P_j(\cos\theta_{\alpha\beta}) \frac{1}{128\pi^4} \int_{m_\chi^2/s}^{(\sqrt{s}-m_\chi)^2/s} d\frac{s_{35}}{s} \int_0^\pi \sin\theta_4 d\theta_4$$

$$\times \int_0^\pi \sin\theta_3 d\theta_3 \int_0^{2\pi} d\phi_3 \frac{|\mathbf{k}_4||\mathbf{k}_3|^2 s}{\sqrt{s}(s_{35} - m_\chi^2)} F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'}). \tag{89}$$

上式中对 ϕ_3 的积分范围可以替换成 $[-\pi, \pi]$ 而不会改变结果, 此时, 因为 $|\mathbf{k}_3|$ 是 ϕ 的偶函数, $F(\theta_{\alpha\beta}, f_+, \bar{f}_-)$ 和 $F(\theta_{\alpha\beta}, f_-, \bar{f}_+)$ 的虚部是 ϕ_3 的奇函数, 所以经过积分后, $b_j^{\text{inel}}(s, f_+, \bar{f}_-)$ 和 $b_j^{\text{inel}}(s, f_-, \bar{f}_+)$ 都是实数.

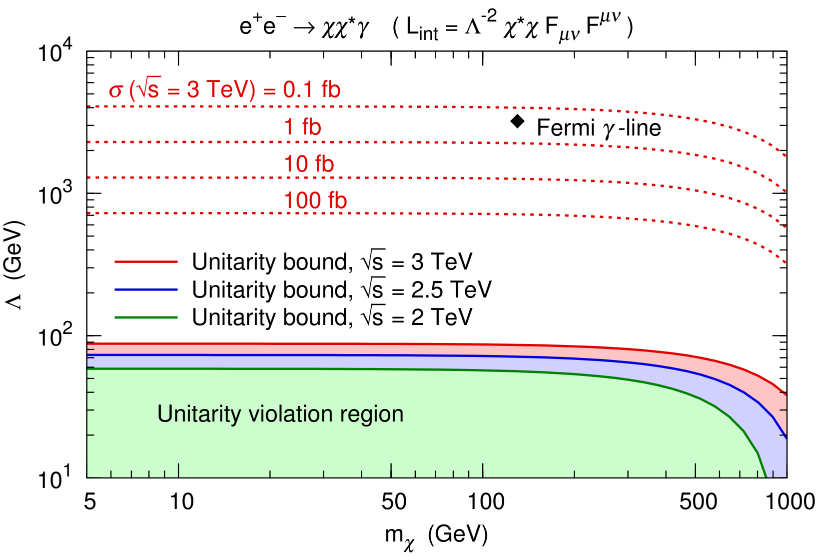


Figure 8: $b_0^{\text{inel}}(s, e_+^+, e_-^-) \leq 1/4$ 给出的幺正性限制. 红色虚等值线表示 $\sqrt{s} = 3$ TeV 处 $e^+e^- \to \chi\chi^*\gamma$ 过程的产生截面. 实心菱形对应解释 Fermi-LAT 观察到的银心 γ 线谱所需要的参数值.

另一方面, 散射截面可表达为

$$\sigma(f_\lambda \bar{f}_{\lambda'} \to \gamma\chi\chi^*) = \frac{1}{2s\beta_f} G(s, \theta_{\alpha\beta} = 0, f_\lambda, \bar{f}_{\lambda'}). \tag{90}$$

应用上式进行计算, 对于 $e^+e^- \to \chi\chi^*\gamma$ 过程, 当 $\sqrt{s} = 3$ TeV, $m_\chi = 130$ GeV, $\Lambda = 3220$ GeV 时, 用数值积分方法得出 $\sigma(f_+\bar{f}_- \to \gamma\chi\chi^*) = \sigma(f_-\bar{f}_+ \to \gamma\chi\chi^*) = 0.4771$ fb (`MadGraph5` 的结果为 0.4769 fb), $\sigma(f_-\bar{f}_- \to \gamma\chi\chi^*) = \sigma(f_+\bar{f}_+ \to \gamma\chi\chi^*) \sim 10^{-14}$ fb. 从而, 非极化截面

$$\sigma(f\bar{f} \to \gamma\chi\chi^*) = \frac{1}{4}\sum_{\lambda\lambda'} \sigma(f_\lambda \bar{f}_{\lambda'} \to \gamma\chi\chi^*) = 0.2386 \text{ fb}. \tag{91}$$

与前面的计算结果一致.

对于 $\sqrt{s} \sim$ TeV 的 e^+e^- 对撞, 可认为 $\beta(s, m_e) = 1$, 幺正性要求 $b_j^{\text{inel}}(s, e_\lambda^+, e_{\lambda'}^-) \leq 1/4$. 对于固定的 Λ, $b_0^{\text{inel}}(s, e_+^+, e_-^-)$ 或 $b_0^{\text{inel}}(s, e_-^+, e_+^-)$ 在各分波各螺旋态组合中的值最大, 如 Fig. 7 所示.

Fig. 8 在 m_χ-Λ 平面上画出由 $b_0^{\text{inel}}(s, e_+^+, e_-^-) \leq 1/4$ 给出的幺正性限制. 图中红色虚线表示 $\sqrt{s} = 3$ TeV 处 $e^+e^- \to \chi\chi^*\gamma$ 过程的产生截面. 此外, 还标出解释 Fermi-LAT 观察到的银心 γ 线谱所需要的参数值.

下面计算 $2 \to 2$ 产生过程 $\gamma(p_1) + \gamma(p_2) \to \chi(k_3) + \chi^*(k_4)$ 对应的幺正性限制. 极化振幅

$$\mathcal{M}(\gamma_{\lambda_1}\gamma_{\lambda_2} \to \chi\chi^*) = -\frac{4}{\Lambda^2}[g_{\mu\nu}(p_1 \cdot p_2) - p_{2\mu}p_{1\nu}]\varepsilon_{\lambda_1}^\mu(p_1)\varepsilon_{\lambda_2}^\nu(p_2). \tag{92}$$

初态光子的动量可表示成

$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1), \tag{93}$$

相应极化矢量为

$$\begin{aligned} \varepsilon(p_1, +) &= \frac{1}{\sqrt{2}}(0, -1, -i, 0), \quad \varepsilon(p_1, -) = \frac{1}{\sqrt{2}}(0, 1, -i, 0), \\ \varepsilon(p_2, +) &= \frac{1}{\sqrt{2}}(0, 1, -i, 0), \quad \varepsilon(p_2, -) = \frac{1}{\sqrt{2}}(0, -1, -i, 0). \end{aligned} \tag{94}$$

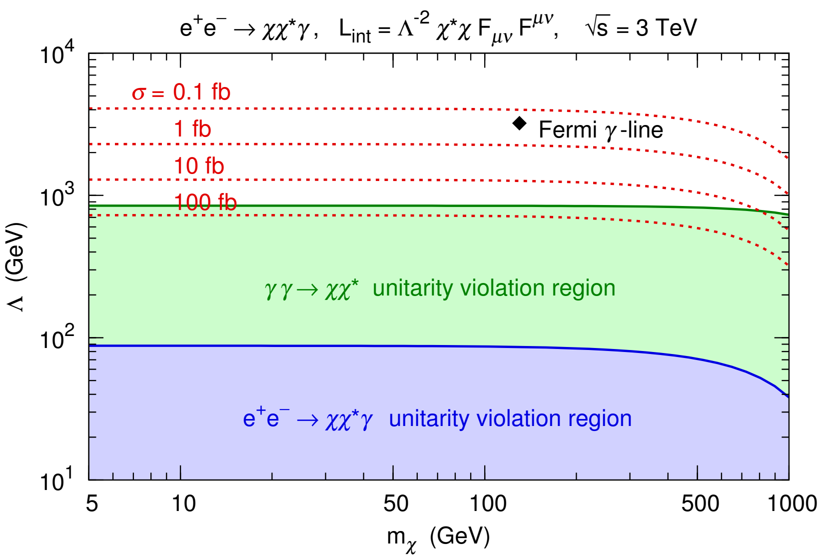


Figure 9: 当 $\sqrt{s} = 3$ TeV 时, $2 \to 2$ 过程和 $2 \to 3$ 过程分别对应的幺正性限制.

由此, 可求得

$$\begin{aligned}\mathcal{M}(\gamma_+\gamma_+ \to \chi\chi^*) &= \mathcal{M}(\gamma_-\gamma_- \to \chi\chi^*) = -\frac{2s}{\Lambda^2},\\ \mathcal{M}(\gamma_+\gamma_- \to \chi\chi^*) &= \mathcal{M}(\gamma_-\gamma_+ \to \chi\chi^*) = 0,\end{aligned} \tag{95}$$

则

$$\begin{aligned}a_0^{\text{inel}}(\gamma_-\gamma_- \to \chi\chi^*) = a_0^{\text{inel}}(\gamma_+\gamma_+ \to \chi\chi^*) &= \frac{1}{32\pi}\int_0^\pi d\theta \sin\theta P_0(\cos\theta)\mathcal{M}(\gamma_+\gamma_+ \to \chi\chi^*)\\ &= \frac{1}{32\pi}\int_0^\pi d\theta \sin\theta \left(-\frac{2s}{\Lambda^2}\right) = -\frac{s}{8\pi\Lambda^2},\end{aligned} \tag{96}$$

由 (28) 式, $|a_0^{\text{inel}}(\gamma_+\gamma_+ \to \chi\chi^*)| \leqslant \left(2\sqrt{\beta_\chi}\right)^{-1}$, 故

$$\Lambda \geqslant \sqrt{s}\left(\frac{\sqrt{\beta_\chi}}{4\pi}\right)^{1/2}. \tag{97}$$

当 $\sqrt{s} = 3$ TeV 时, $2 \to 2$ 过程和 $2 \to 3$ 过程分别对应的幺正性限制如 Fig. 9所示, 两者几乎相差一个量级.

5 Dirac WIMP 与光子的有效耦合

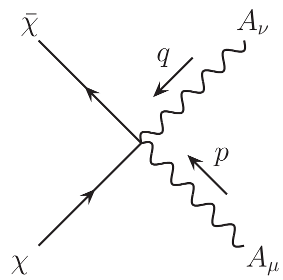
考虑 Dirac WIMP (χ) 与光子具有如下形式的有效相互作用,

$$\mathcal{L}_{\text{int}} = \frac{1}{\Lambda^3}\bar{\chi}i\gamma_5\chi F_{\mu\nu}F^{\mu\nu} = \frac{2}{\Lambda^3}\bar{\chi}i\gamma_5\chi(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu). \tag{98}$$

由于

$$\begin{aligned}& i\frac{2}{\Lambda^3}\bar{\chi}i\gamma_5\chi(g^{\rho\sigma}g^{\mu\nu}\partial_\rho A_\mu \partial_\sigma A_\nu + g^{\rho\sigma}g^{\nu\mu}\partial_\rho A_\nu \partial_\sigma A_\mu - g^{\rho\nu}g^{\mu\sigma}\partial_\rho A_\mu \partial_\sigma A_\nu - g^{\rho\mu}g^{\nu\sigma}\partial_\rho A_\nu \partial_\sigma A_\mu)\\ \to\ & -\frac{2}{\Lambda^3}\gamma_5[g^{\rho\sigma}g^{\mu\nu}(-ip_\rho)(-iq_\sigma) + g^{\rho\sigma}g^{\nu\mu}(-iq_\rho)(-ip_\sigma) - g^{\rho\nu}g^{\mu\sigma}(-ip_\rho)(-iq_\sigma) - g^{\rho\mu}g^{\nu\sigma}(-iq_\rho)(-ip_\sigma)]\\ =\ & \frac{4}{\Lambda^3}\gamma_5[g^{\mu\nu}(p\cdot q) - q^\mu p^\nu],\end{aligned} \tag{99}$$

相互作用顶点的 Feynman 规则如 Fig. 10 所示.



$$= \frac{4}{\Lambda^3}\gamma_5\left[g^{\mu\nu}(p\cdot q) - q^\mu p^\nu\right]$$

Figure 10: Dirac WIMP 与光子有效相互作用顶点的 Feynman 规则.

5.1 湮灭过程

对于 WIMP 湮灭到双光子的过程 $\chi(p_1)+\bar{\chi}(p_2)\to\gamma(k_1)+\gamma(k_2)$,

$$
\begin{aligned}
i\mathcal{M}(\chi\bar{\chi}\to 2\gamma) &= \frac{4}{\Lambda^3}\bar{v}(p_2)\gamma_5 u(p_1)[g^{\mu\nu}(-k_2)\cdot(-k_1)-k_1^\mu k_2^\nu]\varepsilon_\nu^*(k_1)\varepsilon_\mu^*(k_2) \\
&= \frac{4}{\Lambda^3}\bar{v}(p_2)\gamma_5 u(p_1)[(k_1\cdot k_2)\varepsilon_\nu^*(k_1)\varepsilon^{*\nu}(k_2)-k_{1\mu}\varepsilon^{*\mu}(k_2)k_2^\nu\varepsilon_\nu^*(k_1)], \qquad (100)\\
[i\mathcal{M}(\chi\bar{\chi}\to 2\gamma)]^* &= -\frac{4}{\Lambda^3}\bar{u}(p_1)\gamma_5 v(p_2)[(k_1\cdot k_2)\varepsilon_\rho(k_1)\varepsilon^{\rho}(k_2)-k_{1\rho}\varepsilon^{\rho}(k_2)k_2^\sigma\varepsilon_\sigma(k_1)], \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= -\frac{4}{\Lambda^6}\sum_{\text{spins}}\bar{v}(p_2)\gamma_5 u(p_1)\bar{u}(p_1)\gamma_5 v(p_2)[(k_1\cdot k_2)\varepsilon_\nu^*(k_1)\varepsilon^{*\nu}(k_2)-k_{1\mu}\varepsilon^{*\mu}(k_2)k_2^\nu\varepsilon_\nu^*(k_1)] \\
&\quad\times[(k_1\cdot k_2)\varepsilon_\rho(k_1)\varepsilon^{\rho}(k_2)-k_{1\rho}\varepsilon^{\rho}(k_2)k_2^\sigma\varepsilon_\sigma(k_1)] \\
&= -\frac{4}{\Lambda^6}\mathrm{Tr}[v(p_2)\bar{v}(p_2)\gamma_5 u(p_1)\bar{u}(p_1)\gamma_5][(k_1\cdot k_2)^2 g_{\nu\rho}g^{\nu\rho}-(k_1\cdot k_2)k_{1\rho}k_2^\sigma g_{\nu\sigma}g^{\nu\rho} \\
&\quad -(k_1\cdot k_2)k_{1\mu}k_2^\nu g^{\mu\rho}g_{\nu\rho}+k_{1\mu}k_2^\nu k_{1\rho}k_2^\sigma g^{\mu\rho}g_{\nu\sigma}] \\
&= -\frac{8}{\Lambda^6}\mathrm{Tr}[(\not{p}_2-m_\chi)\gamma_5(\not{p}_1+m_\chi)\gamma_5](k_1\cdot k_2)^2 \\
&= \frac{32}{\Lambda^6}(p_1\cdot p_2+m_\chi^2)(k_1\cdot k_2)^2=\frac{4}{\Lambda^6}s^3. \qquad (102)
\end{aligned}
$$

于是

$$
\sigma_{\mathrm{ann}}v=\frac{1}{32\pi^2 s}\frac{1}{2}\int d\Omega\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{32\pi s}\int\sin\theta d\theta\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{16\pi s}\frac{4}{\Lambda^6}s^3=\frac{s^2}{4\pi\Lambda^6}\simeq\frac{4m_\chi^4}{\pi\Lambda^6}. \qquad (103)
$$

当 $\langle\sigma_{\mathrm{ann}}v\rangle_{\chi\bar{\chi}\to2\gamma}\sim 1.27\times10^{-27}\ \mathrm{cm^3s^{-1}}\simeq10^{-10}\ \mathrm{GeV^{-2}}$, $m_\chi\simeq130$ GeV 时,

$$
\Lambda=\left(\frac{4m_\chi^4}{\pi\langle\sigma_{\mathrm{ann}}v\rangle_{\chi\bar{\chi}\to2\gamma}}\right)^{1/6}\simeq 1240\ \mathrm{GeV}. \qquad (104)
$$

5.2 $2\to3$ 产生过程

对于 $2\to3$ 产生过程 $f(p_1)+\bar{f}(p_2)\to\gamma(k_3)+\chi(k_4)+\bar{\chi}(k_5)$,

$$
\begin{aligned}
i\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi}) &= iQ_f e\bar{v}(p_2)\gamma^\mu u(p_1)\frac{-ig_{\mu\nu}}{q^2}\frac{4}{\Lambda^3}\bar{u}(k_4)\gamma_5 v(k_5)[g^{\rho\nu}(-k_3\cdot q)-q^\rho(-k_3^\nu)]\varepsilon_\rho^*(k_3) \\
&= -\frac{4}{\Lambda^3}Q_f e\frac{1}{q^2}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(k_4)\gamma_5 v(k_5)[(k_3\cdot q)\varepsilon_\mu^*(k_3)-k_{3\mu}q^\nu\varepsilon_\nu^*(k_3)], \qquad (105)\\
[i\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi})]^* &= \frac{4}{\Lambda^3}Q_f e\frac{1}{q^2}\bar{u}(p_1)\gamma^\rho v(p_2)\bar{v}(k_5)\gamma_5 u(k_4)[(k_3\cdot q)\varepsilon_\rho(k_3)-k_{3\rho}q^\sigma\varepsilon_\sigma(k_3)], \qquad (106)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi})|^2 \\
&= -\frac{1}{4}\sum_{\text{spins}}\frac{16}{\Lambda^6}Q_f^2 e^2\frac{1}{q^4}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\rho v(p_2)\bar{u}(k_4)\gamma_5 v(k_5)\bar{v}(k_5)\gamma_5 u(k_4) \\
&\quad\times[(k_3\cdot q)\varepsilon_\mu^*(k_3)-k_{3\mu}q^\nu\varepsilon_\nu^*(k_3)][(k_3\cdot q)\varepsilon_\rho(k_3)-k_{3\rho}q^\sigma\varepsilon_\sigma(k_3)]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{4}{\Lambda^6}Q_f^2e^2\sum_{\text{spins}}\frac{1}{q^4}\mathrm{Tr}[v(p_2)\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\rho]\mathrm{Tr}[u(k_4)\bar{u}(k_4)\gamma_5 v(k_5)\bar{v}(k_5)\gamma_5] \\
&\qquad \times[(k_3\cdot q)^2\varepsilon_\mu^*(k_3)\varepsilon_\rho(k_3)-(k_3\cdot q)k_{3\rho}q^\sigma\varepsilon_\mu^*(k_3)\varepsilon_\sigma(k_3) \\
&\qquad\qquad -(k_3\cdot q)k_{3\mu}q^\nu\varepsilon_\nu^*(k_3)\varepsilon_\rho(k_3)+k_{3\mu}q^\nu k_{3\rho}q^\sigma\varepsilon_\nu^*(k_3)\varepsilon_\sigma(k_3)] \\
&= -\frac{4}{\Lambda^6}Q_f^2e^2\frac{1}{q^4}\mathrm{Tr}[(\not{p}_2-m_f)\gamma^\mu(\not{p}_1+m_f)\gamma^\rho]\mathrm{Tr}[(\not{k}_4+m_\chi)\gamma_5(\not{k}_5-m_\chi)\gamma_5] \\
&\qquad \times[-(k_3\cdot q)^2 g_{\mu\rho}+(k_3\cdot q)(k_{3\mu}q_\rho+k_{3\rho}q_\mu)-q^2k_{3\mu}k_{3\rho}] \\
&= \frac{64Q_f^2e^2}{\Lambda^6 s^2}(k_4\cdot k_5+m_\chi^2)[2m_f^2(q\cdot k_3)^2+s(q\cdot k_3)(p_1\cdot k_3+p_2\cdot k_3)-2s(p_1\cdot k_3)(p_2\cdot k_3)] \\
&= \frac{32Q_f^2e^2}{\Lambda^6 s}|\mathbf{k}_3|^2(k_4\cdot k_5+m_\chi^2)[s(1+\beta_f^2\cos^2\theta_3)+4m_f^2] \\
&= \frac{128\pi Q_f^2\alpha}{\Lambda^6}|\mathbf{k}_3|^2(k_4\cdot k_5+m_\chi^2)\left(1+\beta_f^2\cos^2\theta_3+\frac{4m_f^2}{s}\right).
\end{aligned} \tag{107}
$$

这一结果与 (66) 式相比多了一个因子

$$
-\frac{1}{\Lambda^2}\mathrm{Tr}[(\not{k}_4+m_\chi)\gamma_5(\not{k}_5-m_\chi)\gamma_5]=\frac{4}{\Lambda^2}(k_4\cdot k_5+m_\chi^2). \tag{108}
$$

注意到 (53), (59) 式及

$$
\begin{aligned}
k_4^0 &= \frac{s+m_\chi^2-s_{35}}{2\sqrt{s}}, \quad k_5^0=\sqrt{|\mathbf{k}_5|^2+m_\chi^2}, \\
k_4\cdot k_5 &= k_4^0k_5^0+\mathbf{k}_4\cdot(\mathbf{k}_3+\mathbf{k}_4)=k_4^0k_5^0+|\mathbf{k}_4|^2+|\mathbf{k}_4||\mathbf{k}_3|(\sin\theta_4\sin\theta_3\cos\phi_3+\cos\theta_4\cos\theta_3),
\end{aligned} \tag{109}
$$

将这些结果代入 (63) 式进行数值积分, 当 $\sqrt{s}=3$ TeV, $m_\chi=130$ GeV, $\Lambda=1240$ GeV 时, 结果为 $\sigma=27.107$ fb. `MadGraph5` 给出的结果是 $\sigma=27.108$ fb. 微分截面分布如 Fig. 11所示.

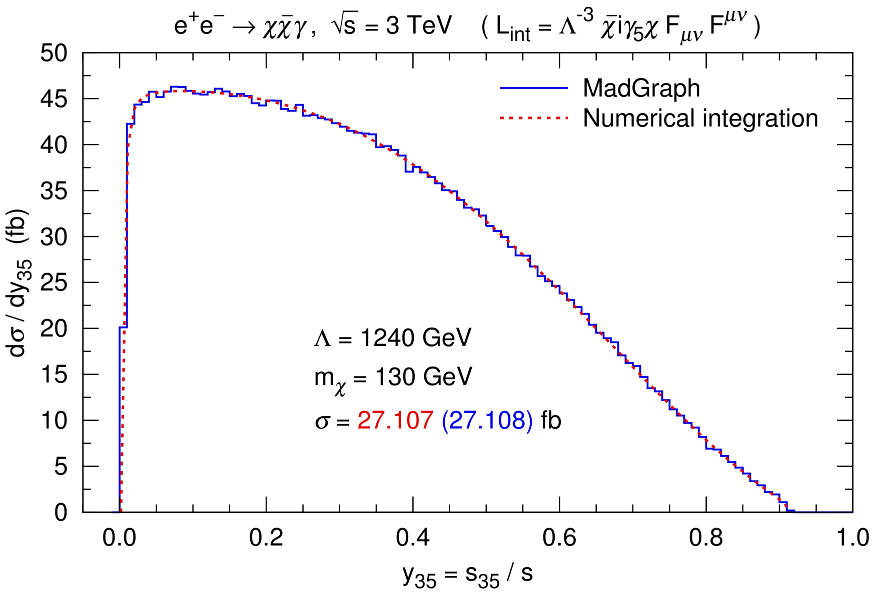


Figure 11: 关于 s_{35} 的微分截面分布.

5.3 产生过程的幺正性限制

为了得到幺正性限制, 必须计算如下过程的振幅,

$$
\begin{aligned}
\alpha \to \gamma_{345}: \quad & f(p_1) + \bar{f}(p_2) \to \gamma(k_3) + \chi(k_4) + \bar{\chi}(k_5), \\
\beta \to \gamma_{345}: \quad & f(q_1) + \bar{f}(q_2) \to \gamma(k_3) + \chi(k_4) + \bar{\chi}(k_5).
\end{aligned} \tag{110}
$$

它们的不变振幅分别为

$$
\begin{aligned}
i\mathcal{M}_{\text{inel}}(\alpha \to \gamma_{345}) &= -\frac{4Q_f e}{\Lambda^3 s}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(k_4)\gamma_5 v(k_5)[(k_3\cdot q)\varepsilon^*_\mu(k_3) - k_{3\mu}q^\nu\varepsilon^*_\nu(k_3)], \\
[i\mathcal{M}_{\text{inel}}(\beta \to \gamma_{345})]^* &= \frac{4Q_f e}{\Lambda^3 s}\bar{u}(q_1)\gamma^\rho v(q_2)\bar{v}(k_5)\gamma_5 u(k_4)[(k_3\cdot q)\varepsilon_\rho(k_3) - k_{3\rho}q^\sigma\varepsilon_\sigma(k_3)].
\end{aligned} \tag{111}
$$

考虑极化的初态 $f_\lambda \bar{f}_{\lambda'}$ $(\lambda, \lambda' = \pm)$, 则

$$
\begin{aligned}
F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'}) \equiv & \sum_{\text{spins of } \gamma_{345}} \mathcal{M}^*_{\text{inel}}(\beta(f_\lambda\bar{f}_{\lambda'}) \to \gamma_{345})\mathcal{M}_{\text{inel}}(\alpha(f_\lambda\bar{f}_{\lambda'}) \to \gamma_{345}) \\
= & -\frac{16Q_f^2 e^2}{\Lambda^6 s^2}\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)\sum_{\text{spins of } \gamma_{345}}\bar{u}(k_4)\gamma_5 v(k_5)\bar{v}(k_5)\gamma_5 u(k_4) \\
& \times[(k_3\cdot q)\varepsilon^*_\mu(k_3) - k_{3\mu}q^\nu\varepsilon^*_\nu(k_3)][(k_3\cdot q)\varepsilon_\rho(k_3) - k_{3\rho}q^\sigma\varepsilon_\sigma(k_3)] \\
= & -\frac{64\pi Q_f^2\alpha}{\Lambda^6 s^2}\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)\mathrm{Tr}[(\not{k}_4 + m_\chi)\gamma_5(\not{k}_5 - m_\chi)\gamma_5] \\
& \times[-(k_3\cdot q)^2 g_{\mu\rho} + (k_3\cdot q)(k_{3\mu}q_\rho + k_{3\rho}q_\mu) - s k_{3\mu}k_{3\rho}] \\
= & \frac{256\pi Q_f^2\alpha}{\Lambda^6 s^2}\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\rho v_{\lambda'}(q_2)(k_4\cdot k_5 + m_\chi^2)[-(k_3\cdot q)^2 g_{\mu\rho} - s k_{3\mu}k_{3\rho}] \\
= & \frac{256\pi Q_f^2\alpha}{\Lambda^6 s^2}(k_4\cdot k_5 + m_\chi^2)\tilde{F}(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'}),
\end{aligned} \tag{112}
$$

其中 $\tilde{F}(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'})$ 的定义与 (72) 式完全相同, 可以直接使用上一节的计算结果. 易见, $F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'})$ 的表达式同样比 (71) 式多一个因子 (108).

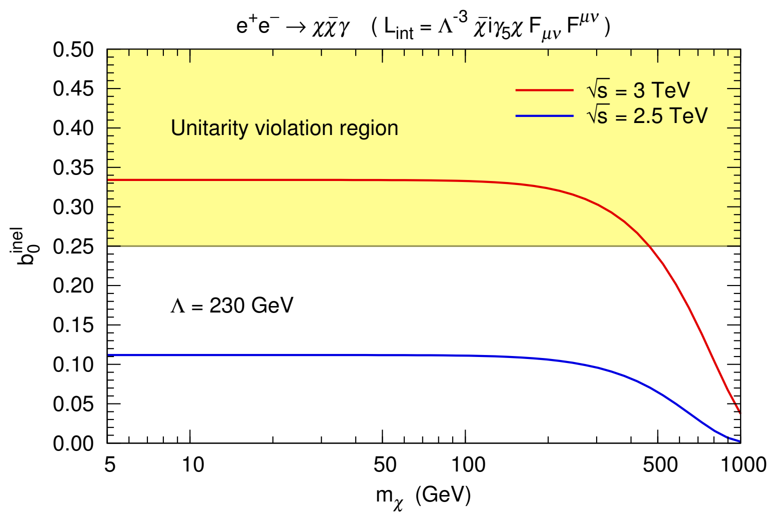


Figure 12: 固定 $\Lambda = 230$ GeV 时, $b_0^{\text{inel}}(s, e^+_+, e^-_-)$ 随 m_χ 变化的情况. 幺正性在浅黄色区域中遭到破坏.

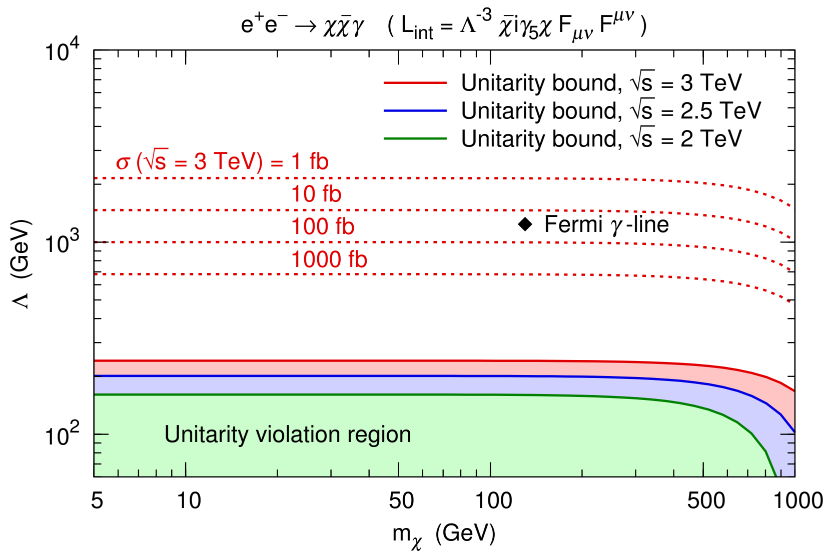


Figure 13: $b_0^{\rm inel}(s,e_+^+,e_-^-) \le 1/4$ 给出的幺正性限制. 红色虚等值线表示 $\sqrt{s}=3$ TeV 处 $e^+e^- \to \chi\bar{\chi}\gamma$ 过程的产生截面. 实心菱形对应解释 Fermi-LAT 观察到的银心 γ 线谱所需要的参数值.

对于 e^+e^- 对撞, 固定 Λ, $b_0^{\rm inel}(s,e_+^+,e_-^-)$ 或 $b_0^{\rm inel}(s,e_-^+,e_+^-)$ 在各分波各螺旋态组合中的值最大, 如 Fig. 12 所示. Fig. 13 在 m_χ-Λ 平面上画出由 $b_0^{\rm inel}(s,e_+^+,e_-^-) \le 1/4$ 给出的幺正性限制. 图中红色虚线表示 $\sqrt{s}=3$ TeV 处 $e^+e^- \to \chi\bar{\chi}\gamma$ 过程的产生截面. 此外, 还标出解释 Fermi-LAT 观察到的银心 γ 线谱所需要的参数值.

下面计算 $2\to 2$ 产生过程 $\gamma(p_1)+\gamma(p_2)\to\chi(k_3)+\bar{\chi}(k_4)$ 对应的幺正性限制. 极化振幅

$$\mathcal{M}(\gamma_{\lambda_1}\gamma_{\lambda_2}\to\chi_{\lambda_3}\bar{\chi}_{\lambda_4}) = -i\frac{4}{\Lambda^3}\bar{u}_{\lambda_3}(k_3)\gamma_5 v_{\lambda_4}(k_4)[g_{\mu\nu}(p_1\cdot p_2)-p_{2\mu}p_{1\nu}]\varepsilon^\mu_{\lambda_1}(p_1)\varepsilon^\nu_{\lambda_2}(p_2). \tag{113}$$

初态光子的动量和极化矢量如 (93) 式和 (94) 式所示. 末态 Dirac WIMP 的动量可表示成

$$k_3 = \frac{\sqrt{s}}{2}(1,\beta_\chi s_\theta,0,\beta_\chi c_\theta), \quad k_4 = \frac{\sqrt{s}}{2}(1,-\beta_\chi s_\theta,0,-\beta_\chi c_\theta), \tag{114}$$

相应螺旋态为

$$\xi_+(k_3) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix}, \quad \xi_-(k_3) = \begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2} \end{pmatrix}, \quad \xi_+(k_4) = \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix}, \quad \xi_-(k_4) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix}. \tag{115}$$

由此, 可求得

$$\begin{aligned} \mathcal{M}(\gamma_+\gamma_+\to\chi_+\bar{\chi}_+) &= \mathcal{M}(\gamma_+\gamma_+\to\chi_-\bar{\chi}_-) = -i\frac{2s^{3/2}}{\Lambda^3}, \\ \mathcal{M}(\gamma_-\gamma_-\to\chi_+\bar{\chi}_+) &= \mathcal{M}(\gamma_-\gamma_-\to\chi_-\bar{\chi}_-) = -i\frac{2s^{3/2}}{\Lambda^3}, \end{aligned} \tag{116}$$

其它过程均违反角动量守恒, 振幅为 0.

如果用 $\gamma_+\gamma_+\to\chi_+\bar{\chi}_+$ 过程计算幺正性限制,

$$a_0^{\rm inel}(\gamma_+\gamma_+\to\chi_+\bar{\chi}_+) = \frac{1}{32\pi}\int_0^\pi d\theta\sin\theta P_0(\cos\theta)\mathcal{M}(\gamma_+\gamma_+\to\chi_+\bar{\chi}_+)$$

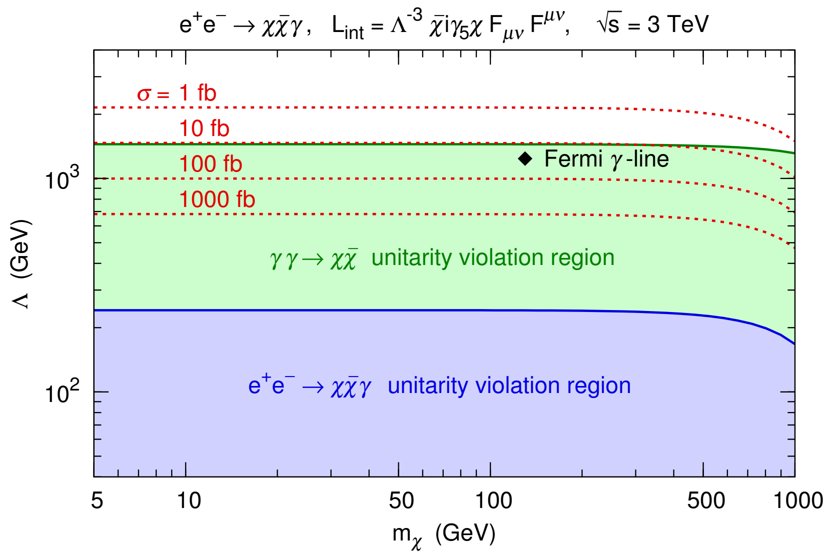


Figure 14: 当 $\sqrt{s} = 3$ TeV 时, $2 \to 2$ 过程和 $2 \to 3$ 过程分别对应的幺正性限制.

$$= \frac{1}{32\pi}\int_0^\pi d\theta \sin\theta \left(-i\frac{2s^{3/2}}{\Lambda^3}\right) = -i\frac{s^{3/2}}{8\pi\Lambda^3}, \tag{117}$$

由 $|a_0^{\text{inel}}(\gamma_+\gamma_+ \to \chi_+\bar{\chi}_+)| \leqslant \left(2\sqrt{\beta_\chi}\right)^{-1}$, 可得

$$\Lambda \geqslant \sqrt{s}\left(\frac{\sqrt{\beta_\chi}}{4\pi}\right)^{1/3}. \tag{118}$$

实际上, 可将 (27) 式推广为

$$\beta_\chi \sum_{\lambda\lambda'} |a_j^{\text{inel}}(\gamma_+\gamma_+ \to \chi_\lambda\bar{\chi}_{\lambda'})|^2 \leqslant \frac{1}{4}, \tag{119}$$

则由

$$\sum_{\lambda\lambda'} |a_0^{\text{inel}}(\gamma_+\gamma_+ \to \chi_\lambda\bar{\chi}_{\lambda'})|^2 = 2\left(\frac{s^{3/2}}{8\pi\Lambda^3}\right)^2 = \frac{s^3}{32\pi^2\Lambda^6}, \tag{120}$$

可得

$$\Lambda \geqslant \sqrt{s}\left(\frac{\sqrt{\beta_\chi}}{2\sqrt{2}\pi}\right)^{1/3}. \tag{121}$$

这一限制比 (118) 式要更强一些. 当 $\sqrt{s} = 3$ TeV 时, $2 \to 2$ 过程和 $2 \to 3$ 过程分别对应的幺正性限制如 Fig. 14所示, 两者几乎相差一个量级.

A 与另一种幺正性限制计算方法的比较

文献 [7] 中推导出了另一种计算 $2 \to n$ 过程幺正性限制的方法. 利用 Legendre 多项式的性质 $P_j(1) = 1$, $\forall j$, 可以验证, 文献 [7] 中 (22) 式的推导过程是严格成立的. 然而在文献 [7] 中, 由 (22) 式推导 (23) 式和 (24) 式的过程却是不严格的. 但我们知道, (23) 式和 (24) 式本身是严格成立的, 因为

它们就是本文档中的 (20) 式和 (21) 式. 另外, 在文献 [7] 中, 假设两体弹性散射的零阶分波主导并忽略其它分波贡献之后, 从 (22) 式推导出了可用于计算 $2 \to n$ 过程幺正性限制的 (26) 式. 对于我们要考虑的散射过程, 这一假设基本成立, 因此文献 [7] 中的 (26) 式也应该近似成立. Fig. 15 比较了用我们的方法计算出来的幺正性限制与用文献 [7] 中的 (26) 式计算出来的幺正性限制, 可以发现, 两者是一致的, 我们的限制略强一些.

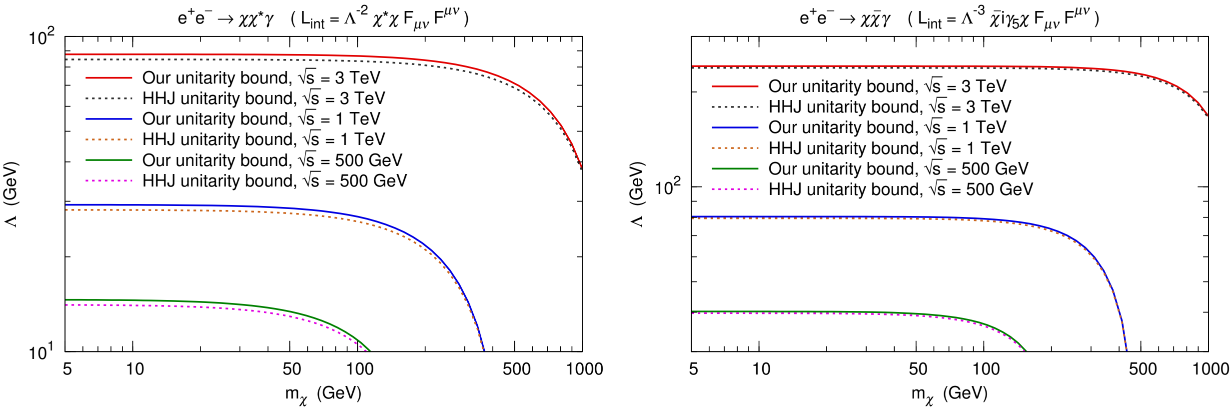


Figure 15: 用我们的方法计算出来的幺正性限制与用文献 [7] 中的 (26) 式计算出来的幺正性限制之间的比较.

B Dirac WIMP 与光子的另一种有效耦合

考虑 Dirac WIMP (χ) 与光子具有如下形式的有效相互作用,

$$\mathcal{L}_{\text{int}} = \frac{1}{\Lambda^3}\bar{\chi} i\gamma_5 \chi F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2\Lambda^3}\bar{\chi} i\gamma_5 \chi \varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho). \tag{122}$$

由

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) &= 2\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 4\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma \\ &= 4\varepsilon^{\rho\mu\sigma\nu}\partial_\rho A_\mu \partial_\sigma A_\nu,\end{aligned} \tag{123}$$

可得

$$\mathcal{L}_{\text{int}} = \frac{1}{\Lambda^3}\bar{\chi} i\gamma_5 \chi F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{2}{\Lambda^3}\bar{\chi} i\gamma_5 \chi \varepsilon^{\rho\mu\sigma\nu}\partial_\rho A_\mu \partial_\sigma A_\nu. \tag{124}$$

于是,

$$\begin{aligned}& i\frac{2}{\Lambda^3}\bar{\chi} i\gamma_5 \chi(\varepsilon^{\rho\mu\sigma\nu}\partial_\rho A_\mu \partial_\sigma A_\nu + \varepsilon^{\rho\nu\sigma\mu}\partial_\rho A_\nu \partial_\sigma A_\mu) \\ \to\ & -\frac{2}{\Lambda^3}\gamma_5[\varepsilon^{\rho\mu\sigma\nu}(-ip_\rho)(-iq_\sigma) + \varepsilon^{\rho\nu\sigma\mu}(-iq_\rho)(-ip_\sigma)] \\ =\ & \frac{2}{\Lambda^3}\gamma_5[\varepsilon^{\rho\mu\sigma\nu}p_\rho q_\sigma - \varepsilon^{\sigma\mu\rho\nu}q_\rho p_\sigma] = \frac{4}{\Lambda^3}\gamma_5 \varepsilon^{\rho\mu\sigma\nu}p_\rho q_\sigma \\ =\ & -\frac{4}{\Lambda^3}\gamma_5 \varepsilon^{\mu\nu\rho\sigma}p_\rho q_\sigma,\end{aligned} \tag{125}$$

相互作用顶点的 Feynman 规则如 Fig. 16 所示.

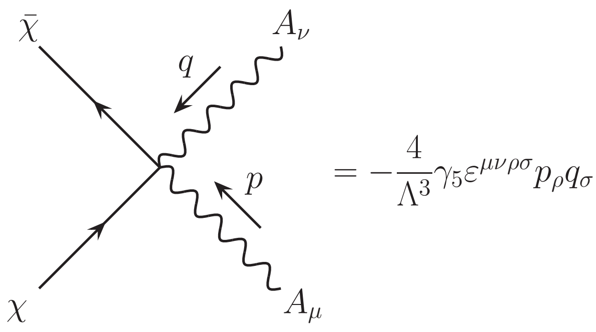


Figure 16: Dirac WIMP 与光子另一种有效相互作用顶点的 Feynman 规则.

对于 WIMP 湮灭到双光子的过程 $\chi(p_1)+\bar{\chi}(p_2)\to\gamma(k_1)+\gamma(k_2)$,

$$\begin{aligned} i\mathcal{M}(\chi\bar{\chi}\to 2\gamma) &= -\frac{4}{\Lambda^3}\bar{v}(p_2)\gamma_5 u(p_1)\varepsilon^{\mu\nu\rho\sigma}(-k_{1\rho})(-k_{2\sigma})\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2) \\ &= -\frac{4}{\Lambda^3}\bar{v}(p_2)\gamma_5 u(p_1)\varepsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2), \qquad (126) \\ [i\mathcal{M}(\chi\bar{\chi}\to 2\gamma)]^* &= \frac{4}{\Lambda^3}\bar{u}(p_1)\gamma_5 v(p_2)\varepsilon^{\alpha\beta\gamma\delta}k_{1\gamma}k_{2\delta}\varepsilon_\alpha(k_1)\varepsilon_\beta(k_2). \qquad (127) \end{aligned}$$

由

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\gamma\delta} = -2!2!\delta^{[\rho}_\gamma\delta^{\sigma]}_\delta = -2(\delta^\rho_\gamma\delta^\sigma_\delta-\delta^\sigma_\gamma\delta^\rho_\delta) = 2(\delta^\sigma_\gamma\delta^\rho_\delta-\delta^\rho_\gamma\delta^\sigma_\delta), \qquad (128)$$

可得

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\gamma\delta}k_{1\rho}k_{2\sigma}k_1^\gamma k_2^\delta = 2(\delta^\sigma_\gamma\delta^\rho_\delta k_{1\rho}k_{2\sigma}k_1^\gamma k_2^\delta-\delta^\rho_\gamma\delta^\sigma_\delta k_{1\rho}k_{2\sigma}k_1^\gamma k_2^\delta) = 2[(k_1\cdot k_2)^2-k_1^2k_2^2] = 2(k_1\cdot k_2)^2. \qquad (129)$$

于是,

$$\begin{aligned} \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= -\frac{4}{\Lambda^6}\sum_{\text{spins}}\bar{v}(p_2)\gamma_5 u(p_1)\bar{u}(p_1)\gamma_5 v(p_2)\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}k_{1\rho}k_{2\sigma}k_{1\gamma}k_{2\delta}\varepsilon^*_\mu(k_1)\varepsilon_\alpha(k_1)\varepsilon^*_\nu(k_2)\varepsilon_\beta(k_2) \\ &= -\frac{4}{\Lambda^6}\mathrm{Tr}[v(p_2)\bar{v}(p_2)\gamma_5 u(p_1)\bar{u}(p_1)\gamma_5]\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}k_{1\rho}k_{2\sigma}k_{1\gamma}k_{2\delta}g_{\mu\alpha}g_{\nu\beta} \\ &= -\frac{4}{\Lambda^6}\mathrm{Tr}[(\not{p}_2-m_\chi)\gamma_5(\not{p}_1+m_\chi)\gamma_5]\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\gamma\delta}k_{1\rho}k_{2\sigma}k_1^\gamma k_2^\delta \\ &= -\frac{8}{\Lambda^6}\mathrm{Tr}[(\not{p}_2-m_\chi)\gamma_5(\not{p}_1+m_\chi)\gamma_5](k_1\cdot k_2)^2 \\ &= \frac{8}{\Lambda^6}4(p_1\cdot p_2+m_\chi^2)(k_1\cdot k_2)^2 = \frac{4}{\Lambda^6}s^3. \qquad (130) \end{aligned}$$

从而,

$$\sigma_{\text{ann}}v = \frac{1}{32\pi^2 s}\frac{1}{2}\int d\Omega\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{32\pi s}\int\sin\theta d\theta\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{16\pi s}\frac{4}{\Lambda^6}s^3 = \frac{s^2}{4\pi\Lambda^6}\simeq\frac{4m_\chi^4}{\pi\Lambda^6}, \qquad (131)$$

与 (103) 式相同.

对于 $2\to 3$ 产生过程 $f(p_1)+\bar{f}(p_2)\to\gamma(k_3)+\chi(k_4)+\bar{\chi}(k_5)$,

$$i\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi}) = iQ_f e\bar{v}(p_2)\gamma^\mu u(p_1)\frac{-ig_{\mu\nu}}{q^2}\left(-\frac{4}{\Lambda^3}\right)\bar{u}(k_4)\gamma_5 v(k_5)\varepsilon^{\rho\nu\alpha\beta}(-k_{3\alpha})q_\beta\varepsilon^*_\rho(k_3) \qquad (132)$$

$$
\begin{aligned}
&= \frac{4}{\Lambda^3} Q_f e \frac{1}{q^2} \bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(k_4)\gamma_5 v(k_5)\varepsilon^{\nu\mu\rho\sigma} k_{3\rho} q_\sigma \varepsilon^*_\nu(k_3), \\
[i\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi})]^* &= -\frac{4}{\Lambda^3} Q_f e \frac{1}{q^2} \bar{u}(p_1)\gamma_\delta v(p_2)\bar{v}(k_5)\gamma_5 u(k_4)\varepsilon^{\gamma\delta\alpha\beta} k_{3\alpha} q_\beta \varepsilon_\gamma(k_3),
\end{aligned}
\tag{133}
$$

$$
\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}(f\bar{f}\to\gamma\chi\bar{\chi})|^2 \\
&= -\frac{1}{4}\sum_{\text{spins}} \frac{16}{\Lambda^6} Q_f^2 e^2 \frac{1}{q^4} \bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(k_4)\gamma_5 v(k_5)\bar{u}(p_1)\gamma_\delta v(p_2)\bar{v}(k_5)\gamma_5 u(k_4) \\
&\qquad \times \varepsilon^{\nu\mu\rho\sigma}\varepsilon^{\gamma\delta\alpha\beta} k_{3\rho} q_\sigma k_{3\alpha} q_\beta \varepsilon^*_\nu(k_3)\varepsilon_\gamma(k_3) \\
&= \frac{4}{\Lambda^6} Q_f^2 e^2 \frac{1}{q^4} \mathrm{Tr}[v(p_2)\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(p_1)\gamma_\delta]\mathrm{Tr}[u(k_4)\bar{u}(k_4)\gamma_5 v(k_5)\bar{v}(k_5)\gamma_5]\varepsilon^{\nu\mu\rho\sigma}\varepsilon^{\gamma\delta\alpha\beta} k_{3\rho} q_\sigma k_{3\alpha} q_\beta g_{\nu\gamma} \\
&= \frac{4}{\Lambda^6} Q_f^2 e^2 \frac{1}{q^4} \mathrm{Tr}[(\not{p}_2 - m_f)\gamma_\mu(\not{p}_1 + m_f)\gamma^\delta]\mathrm{Tr}[(\not{k}_4 + m_\chi)\gamma_5(\not{k}_5 - m_\chi)\gamma_5]\varepsilon^{\nu\mu\rho\sigma}\varepsilon^{\gamma\delta\alpha\beta} k_{3\rho} q_\sigma k_{3\alpha} q_\beta g_{\nu\gamma} \\
&= \frac{128 Q_f^2 e^2}{\Lambda^6 s^2}(k_4\cdot k_5 + m_\chi^2)[(q\cdot p_2)(p_1\cdot k_3)(q\cdot k_3) + (q\cdot p_1)(p_2\cdot k_3)(q\cdot k_3) \\
&\qquad\qquad + m_f^2(q\cdot k_3)^2 - s(p_1\cdot k_3)(p_2\cdot k_3)] \\
&= \frac{64 Q_f^2 e^2}{\Lambda^6 s^2}(k_4\cdot k_5 + m_\chi^2)[2m_f^2(q\cdot k_3)^2 + s(q\cdot k_3)(p_1\cdot k_3 + p_2\cdot k_3) - 2s(p_1\cdot k_3)(p_2\cdot k_3)] \\
&= \frac{32 Q_f^2 e^2}{\Lambda^6 s}|\mathbf{k}_3|^2(k_4\cdot k_5 + m_\chi^2)[s(1+\beta_f^2\cos^2\theta_3) + 4m_f^2] \\
&= \frac{128\pi Q_f^2 \alpha}{\Lambda^6}|\mathbf{k}_3|^2(k_4\cdot k_5 + m_\chi^2)\left(1+\beta_f^2\cos^2\theta_3 + \frac{4m_f^2}{s}\right),
\end{aligned}
\tag{134}
$$

结果与 (107) 式相同.

对于

$$
\begin{aligned}
\alpha\to\gamma_{345}: &\quad f(p_1) + \bar{f}(p_2) \to \gamma(k_3) + \chi(k_4) + \chi^*(k_5), \\
\beta\to\gamma_{345}: &\quad f(q_1) + \bar{f}(q_2) \to \gamma(k_3) + \chi(k_4) + \chi^*(k_5),
\end{aligned}
\tag{135}
$$

不变振幅分别为

$$
\begin{aligned}
i\mathcal{M}_{\text{inel}}(\alpha\to\gamma_{345}) &= \frac{4Q_f e}{\Lambda^3 s}\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(k_4)\gamma_5 v(k_5)\varepsilon^{\nu\mu\rho\sigma} k_{3\rho} q_\sigma \varepsilon^*_\nu(k_3), \\
[i\mathcal{M}_{\text{inel}}(\beta\to\gamma_{345})]^* &= -\frac{4Q_f e}{\Lambda^3 s}\bar{u}(q_1)\gamma_\delta v(q_2)\bar{v}(k_5)\gamma_5 u(k_4)\varepsilon^{\gamma\delta\alpha\beta} k_{3\alpha} q_\beta \varepsilon_\gamma(k_3).
\end{aligned}
\tag{136}
$$

于是,

$$
\begin{aligned}
&F(\theta_{\alpha\beta}, f_\lambda, \bar{f}_{\lambda'}) \\
&\equiv \sum_{\text{spins of }\gamma_{345}} \mathcal{M}^*_{\text{inel}}(\beta(f_\lambda\bar{f}_{\lambda'})\to\gamma_{345})\mathcal{M}_{\text{inel}}(\alpha(f_\lambda\bar{f}_{\lambda'})\to\gamma_{345}) \\
&= -\frac{16Q_f^2 e^2}{\Lambda^6 s^2}\bar{v}_{\lambda'}(p_2)\gamma_\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma_\delta v_{\lambda'}(q_2) \\
&\qquad \times \sum_{\text{spins of }\gamma_{345}} \bar{u}(k_4)\gamma_5 v(k_5)\bar{v}(k_5)\gamma_5 u(k_4)\varepsilon^{\nu\mu\rho\sigma} k_{3\rho} q_\sigma \varepsilon^*_\nu(k_3)\varepsilon^{\gamma\delta\alpha\beta} k_{3\alpha} q_\beta \varepsilon_\gamma(k_3)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{64\pi Q_f^2\alpha}{\Lambda^6 s^2}\mathrm{Tr}[(\not{k}_4+m_\chi)\gamma_5(\not{k}_5-m_\chi)\gamma_5]\bar{v}_{\lambda'}(p_2)\gamma_\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma_\delta v_{\lambda'}(q_2)\varepsilon^{\nu\mu\rho\sigma}\varepsilon^{\gamma\delta\alpha\beta}k_{3\rho}q_\sigma k_{3\alpha}q_\beta g_{\nu\gamma}\\
&= \frac{64\pi Q_f^2\alpha}{\Lambda^6 s^2}[-4(k_4\cdot k_5+m_\chi^2)]\bar{v}_{\lambda'}(p_2)\gamma_\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\delta v_{\lambda'}(q_2)\varepsilon^{\nu\mu\rho\sigma}\varepsilon_{\nu\delta\alpha\beta}k_{3\rho}q_\sigma k_3^\alpha q^\beta\\
&= -\frac{256\pi Q_f^2\alpha}{\Lambda^6 s^2}(k_4\cdot k_5+m_\chi^2)\bar{v}_{\lambda'}(p_2)\gamma_\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\delta v_{\lambda'}(q_2)\\
&\qquad\qquad\times[\delta_\delta^\mu(q\cdot k_3)^2+sk_3^\mu k_{3\delta}-(q\cdot k_3)(q^\mu k_{3\delta}+q_\delta k_3^\mu)]\\
&= -\frac{256\pi Q_f^2\alpha}{\Lambda^6 s^2}(k_4\cdot k_5+m_\chi^2)\bar{v}_{\lambda'}(p_2)\gamma^\mu u_\lambda(p_1)\bar{u}_\lambda(q_1)\gamma^\delta v_{\lambda'}(q_2)[g_{\mu\delta}(q\cdot k_3)^2+sk_{3\mu}k_{3\delta}]\\
&= \frac{256\pi Q_f^2\alpha}{\Lambda^6 s^2}(k_4\cdot k_5+m_\chi^2)\tilde{F}(\theta_{\alpha\beta},f_\lambda,\bar{f}_{\lambda'}),
\end{aligned}
\tag{137}
$$

这一结果与 (112) 式相同. 在上述计算过程中用到了

$$
\begin{aligned}
\varepsilon^{\nu\mu\rho\sigma}\varepsilon_{\nu\delta\alpha\beta}k_{3\rho}q_\sigma k_3^\alpha q^\beta &= -3!\delta_\delta^{[\mu}\delta_\alpha^\rho\delta_\beta^{\sigma]}k_{3\rho}q_\sigma k_3^\alpha q^\beta\\
&= -(\delta_\delta^\mu\delta_\alpha^\rho\delta_\beta^\sigma+\delta_\delta^\sigma\delta_\alpha^\mu\delta_\beta^\rho+\delta_\delta^\rho\delta_\alpha^\sigma\delta_\beta^\mu-\delta_\delta^\mu\delta_\alpha^\sigma\delta_\beta^\rho-\delta_\delta^\sigma\delta_\alpha^\rho\delta_\beta^\mu-\delta_\delta^\rho\delta_\alpha^\mu\delta_\beta^\sigma)k_{3\rho}q_\sigma k_3^\alpha q^\beta\\
&= -(q\cdot k_3)q_\delta k_3^\mu-(q\cdot k_3)k_{3\delta}q^\mu+\delta_\delta^\mu(q\cdot k_3)^2+sk_{3\delta}k_3^\mu\\
&= \delta_\delta^\mu(q\cdot k_3)^2+sk_3^\mu k_{3\delta}-(q\cdot k_3)(q^\mu k_{3\delta}+q_\delta k_3^\mu).
\end{aligned}
\tag{138}
$$

参考文献

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory," Reading, USA: Addison-Wesley (1995) 842 p.

[2] J. D. Bjorken and S. D. Drell, "Relativistic quantum fields," ISBN-0070054940.

[3] 王竹溪, 郭敦仁, "特殊函数概论", 北京大学出版社.

[4] G. B. Arfken and H. J. Weber, "Mathematical Methods for Physicists," Academic Press, 6th edition (2005).

[5] W. J. Marciano, G. Valencia and S. Willenbrock, "Renormalization Group Improved Unitarity Bounds On The Higgs Boson And Top Quark Masses," Phys. Rev. D **40**, 1725 (1989).

[6] I. M. Shoemaker and L. Vecchi, "Unitarity and Monojet Bounds on Models for DAMA, CoGeNT, and CRESST-II," Phys. Rev. D **86**, 015023 (2012) [arXiv:1112.5457 [hep-ph]].

[7] D. A. Dicus and H. -J. He, "Scales of fermion mass generation and electroweak symmetry breaking," Phys. Rev. D **71**, 093009 (2005) [hep-ph/0409131].