离散对称性 P, T, C

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目录

度规及 Dirac 矩阵约定与文献 [1] 相同. 标量场与旋量场部分主要参考文献 [2].
度规张量

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1}$$

Pauli 矩阵

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{2}$$

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}). \tag{3}$$

手征表示下的 Dirac 矩阵

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}. \tag{4}$$

固有 Lorentz 群变换对时空坐标的作用为

$$x^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{5}$$

$\Lambda^\mu{}_\nu$ 满足

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}, \quad \Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}, \quad (\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho. \tag{6}$$

1 标量场

固有 Lorentz 群变换 $\Lambda^\mu{}_\nu$ 相应幺正算符 $U(\Lambda)$ 对标量场 $\phi(x)$ 的作用为

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{7}$$

然而, 宇称变换

$$\mathcal{P}^{\mu}{}_{\nu} = (\mathcal{P}^{-1})^{\mu}{}_{\nu} = \mathcal{P}_{\nu}{}^{\mu} = \mathrm{diag}(+1, -1, -1, -1) \tag{8}$$

和时间反演变换

$$\mathcal{T}^{\mu}{}_{\nu} = (\mathcal{T}^{-1})^{\mu}{}_{\nu} = \mathcal{T}_{\nu}{}^{\mu} = \mathrm{diag}(-1, +1, +1, +1) \tag{9}$$

不属于固有 Lorentz 群. 由于宇称变换和时间反演变换均为自身的逆变换, 作用两次不会改变任意可观测量. 标量场 $\phi(x)$ 在原理上是一个可观测量. 记 $P \equiv U(\mathcal{P})$, $T \equiv U(\mathcal{T})$, 则对标量场有

$$P^{-2}\phi(x)P^2 = \phi(x), \quad T^{-2}\phi(x)T^2 = \phi(x). \tag{10}$$

从而, 标量场的 P 变换可以有两种形式:

$$\text{对于标量场}, \quad P^{-1}\phi(x)P = +\phi(\mathcal{P}x); \tag{11}$$

$$\text{对于赝标量场}, \; P^{-1}\phi(x)P = -\phi(\mathcal{P}x). \tag{12}$$

标量场的 T 变换也有两种形式:

$$T^{-1}\phi(x)T \;=\; +\phi(\mathcal{T}x); \tag{13}$$

$$T^{-1}\phi(x)T \;=\; -\phi(\mathcal{T}x). \tag{14}$$

若拉氏量满足 $P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x)$ 和 $T^{-1}\mathcal{L}(x)T = +\mathcal{L}(\mathcal{T}x)$, 则对 d^4x 积分而得的作用量 S 对 P 变换和 T 变换也是不变的, 此时宇称和时间反演守恒.

在固有 Lorentz 变换下, 能量动量矢量 $P^{\mu} = (H, P^i)$ 的变换为

$$U(\Lambda)^{-1}P^{\mu}U(\Lambda) = \Lambda^{\mu}{}_{\nu}P^{\nu}. \tag{15}$$

如果哈密顿量 H 在 P 变换和 T 变换下不变, 即 $P^{-1}HP = +H$ 和 $T^{-1}HT = +H$, 则应要求

$$P^{-1}P^{\mu}P = \mathcal{P}^{\mu}{}_{\nu}P^{\nu}, \quad T^{-1}P^{\mu}T = -\mathcal{T}^{\mu}{}_{\nu}P^{\nu}. \tag{16}$$

值得注意的是, 时间反演算符 T 是反幺正的 (注 1), 即

$$T^{-1}iT = -i. \tag{17}$$

注 1: 时空平移算符 $\mathsf{T}(a) = \exp(-iP\cdot a)$ 对标量场的作用为

$$\mathsf{T}(a)^{-1}\phi(x)\mathsf{T}(a) = \phi(x-a), \tag{18}$$

由其构造方式易知它是 Lorentz 标量:

$$U(\Lambda)^{-1}\mathsf{T}(a)U(\Lambda) = \mathsf{T}(\Lambda^{-1}a). \tag{19}$$

将上式展开至第 1 阶得 $U(\Lambda)^{-1}(1 - ia_{\mu}P^{\mu})U(\Lambda) = 1 - i(\Lambda^{-1})_{\nu}{}^{\mu}a_{\mu}P^{\nu} = 1 - i\Lambda^{\mu}{}_{\nu}a_{\mu}P^{\nu}$, 对于 T 变换, 此式化为

$$T^{-1}(1 - ia_{\mu}P^{\mu})T = 1 - i\mathcal{T}^{\mu}{}_{\nu}a_{\mu}P^{\nu}. \tag{20}$$

可见, 若要求 $T^{-1}P^{\mu}T = -\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$, 则 T 变换必须满足 $T^{-1}iT = -i$.

考虑复标量场 $\phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$ 和拉氏量

$$\mathcal{L} \;=\; \partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - m^2\phi^{\dagger}\phi - \frac{1}{4}\lambda(\phi^{\dagger}\phi)^2$$

$$= \frac{1}{2}\partial^\mu\varphi_1\partial_\mu\varphi_1 + \frac{1}{2}\partial^\mu\varphi_2\partial_\mu\varphi_2 - \frac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - \frac{1}{16}\lambda(\varphi_1^2+\varphi_2^2)^2. \tag{21}$$

则 $\mathcal{L}$ 除了具有 $U(1)$ 对称性 (对两个实标量场而言是 $SO(2)$ 对称性)

$$\phi(x) \to e^{-i\alpha}\phi(x), \quad \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} \to \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} \tag{22}$$

之外, 还具有离散对称性

$$\phi(x) \to \phi^\dagger(x), \quad \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix} \to \begin{pmatrix}+1 & \\ & -1\end{pmatrix}\begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\end{pmatrix}. \tag{23}$$

此离散对称性被称为 电荷共轭 对称性, 它总是伴随 $U(1)$ 连续对称性而来.

可以将标量场的电荷共轭算符 C 定义为

$$C^{-1}\phi(x)C = \phi^\dagger(x), \tag{24}$$

亦即

$$C^{-1}\varphi_1(x)C = +\varphi_1(x), \quad C^{-1}\varphi_2(x)C = -\varphi_2(x), \tag{25}$$

从而 $C^{-1}\mathcal{L}(x)C = \mathcal{L}(x)$ 对应着电荷共轭对称性.

算符 $\phi^\dagger\phi$ 的变换性质为

$$P^{-1}\phi^\dagger(x)\phi(x)P = +\phi^\dagger(\mathcal{P}x)\phi(\mathcal{P}x), \tag{26}$$
$$T^{-1}\phi^\dagger(x)\phi(x)T = +\phi^\dagger(\mathcal{T}x)\phi(\mathcal{T}x), \tag{27}$$
$$C^{-1}\phi^\dagger(x)\phi(x)C = +\phi^\dagger(x)\phi(x). \tag{28}$$

由于

$$P^{-1}\partial^\mu P = \mathcal{P}^\mu{}_\nu\partial^\nu, \quad T^{-1}\partial^\mu T = \mathcal{T}^\mu{}_\nu\partial^\nu, \quad C^{-1}\partial^\mu C = \partial^\mu, \tag{29}$$

算符 $\phi^\dagger i\overleftrightarrow{\partial^\mu}\phi$ 的变换性质为

$$P^{-1}\phi^\dagger(x)i\overleftrightarrow{\partial^\mu}\phi(x)P = +\mathcal{P}^\mu{}_\nu\phi^\dagger(\mathcal{P}x)i\overleftrightarrow{\partial^\nu}\phi(\mathcal{P}x), \tag{30}$$
$$T^{-1}\phi^\dagger(x)i\overleftrightarrow{\partial^\mu}\phi(x)T = -\mathcal{T}^\mu{}_\nu\phi^\dagger(\mathcal{T}x)i\overleftrightarrow{\partial^\nu}\phi(\mathcal{T}x), \tag{31}$$
$$C^{-1}\phi^\dagger(x)i\overleftrightarrow{\partial^\mu}\phi(x)C = -\phi^\dagger(x)i\overleftrightarrow{\partial^\mu}\phi(x). \tag{32}$$

动能项 $\partial^\mu\phi^\dagger\partial_\mu\phi$ 的变换性质为

$$P^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)P = \mathcal{P}^\mu{}_\sigma\mathcal{P}_\mu{}^\rho\partial^\sigma\phi^\dagger(\mathcal{P}x)\partial_\rho\phi(\mathcal{P}x) = +\partial^\mu\phi^\dagger(\mathcal{P}x)\partial_\mu\phi(\mathcal{P}x), \tag{33}$$
$$T^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)T = \mathcal{T}^\mu{}_\sigma\mathcal{T}_\mu{}^\rho\partial^\sigma\phi^\dagger(\mathcal{T}x)\partial_\rho\phi(\mathcal{T}x) = +\partial^\mu\phi^\dagger(\mathcal{T}x)\partial_\mu\phi(\mathcal{T}x), \tag{34}$$
$$C^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)C = \partial^\mu\phi(x)\partial_\mu\phi^\dagger(x) = +\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x). \tag{35}$$

2 旋量场

固有 Lorentz 群变换 $\Lambda^\mu{}_\nu$ 相应幺正算符 $U(\Lambda)$ 对 (Dirac 或 Majorana) 旋量场 $\psi(x)$ 的作用为

$$U(\Lambda)^{-1}\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x). \tag{36}$$

无穷小变换 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ 对应于 $D(\Lambda) = 1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$, 其中 $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

现在, 对于宇称变换, 有

$$P^{-1}\psi(x)P = D(\mathcal{P})\psi(\mathcal{P}x), \tag{37}$$

作用两次得 $P^{-2}\psi(x)P^2 = D(\mathcal{P})^2\psi(x)$. 由于需要偶数个旋量场才能构造可观测量, 要求 $D(\mathcal{P})^2 = \pm 1$.

由于

$$P^{-1}\mathbf{x}P = -\mathbf{x}, \quad P^{-1}\mathbf{P}P = -\mathbf{P}, \quad P^{-1}\mathbf{J}P = +\mathbf{J}, \tag{38}$$

宇称变换反转 3 动量的方向, 但会保持自旋方向不变, 故对产生算符的变换形式为

$$P^{-1}b_s^\dagger(\mathbf{p})P = \eta b_s^\dagger(-\mathbf{p}), \quad P^{-1}d_s^\dagger(\mathbf{p})P = \eta d_s^\dagger(-\mathbf{p}). \tag{39}$$

依照上述讨论, $\eta^2 = \pm 1$. 这里对算符 b 和 d 用了同一个 η, 这样宇称变换就与 Majorana 条件 $b_s(\mathbf{p}) = d_s(\mathbf{p})$ 一致了.

将旋量场 $\psi(x)$ 展开为

$$\psi(x) = \sum_s \int \widetilde{dp}\big[b_s(\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}\big], \quad s = \pm, \quad \widetilde{dp} \equiv \frac{d^3\mathbf{p}}{(2\pi^3)\sqrt{2E_\mathbf{p}}}, \tag{40}$$

则

$$\begin{aligned} P^{-1}\psi(x)P &= \sum_s \int \widetilde{dp}\big[\eta^* b_s(-\mathbf{p})u_s(\mathbf{p})e^{-ip\cdot x} + \eta d_s^\dagger(-\mathbf{p})v_s(\mathbf{p})e^{ip\cdot x}\big] \\ &= \sum_s \int \widetilde{dp}\big[\eta^* b_s(\mathbf{p})u_s(-\mathbf{p})e^{-ip\cdot \mathcal{P}x} + \eta d_s^\dagger(\mathbf{p})v_s(-\mathbf{p})e^{ip\cdot \mathcal{P}x}\big]. \end{aligned} \tag{41}$$

由平面波解

$$u_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar\sigma}\xi_s \end{pmatrix}, \quad v_s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_s \\ -\sqrt{p\cdot\bar\sigma}\eta_s \end{pmatrix}, \tag{42}$$

有

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p}), \quad v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p}), \quad \beta \equiv \gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{43}$$

如果我们取 $\eta = -i$, 就可以得到与 $P^{-1}\psi(x)P = D(\mathcal{P})\psi(\mathcal{P}x)$ 对应的表达式

$$P^{-1}\psi(x)P = \sum_s \int \widetilde{dp}\big[ib_s(\mathbf{p})\beta u_s(\mathbf{p})e^{-ip\cdot\mathcal{P}x} - id_s^\dagger(\mathbf{p})(-\beta)v_s(\mathbf{p})e^{ip\cdot\mathcal{P}x}\big] = i\beta\psi(\mathcal{P}x). \tag{44}$$

此时

$$D(\mathcal{P}) = i\beta, \tag{45}$$

即

$$P^{-1}\psi(x)P = i\beta\psi(\mathcal{P}x). \tag{46}$$

η 为纯虚数具有物理意义. 在质心系中考虑一对正反费米子的态 $|\Phi\rangle = \int \widetilde{dp}\Phi(\mathbf{p})b_s^\dagger(\mathbf{p})d_{s'}^\dagger(-\mathbf{p})|0\rangle$. 假设动量空间波函数 $\Phi(\mathbf{p})$ 具有确定的轨道宇称 $\Phi(-\mathbf{p}) = (-)^l\Phi(\mathbf{p})$, 而真空是宇称变换不变的, 则

$$\begin{aligned} P^{-1}|\Phi\rangle &= \int \widetilde{dp}\Phi(\mathbf{p})P^{-1}b_s^\dagger(\mathbf{p})PP^{-1}d_{s'}^\dagger(-\mathbf{p})PP^{-1}|0\rangle = \eta^2\int\widetilde{dp}\Phi(\mathbf{p})b_s^\dagger(-\mathbf{p})d_{s'}^\dagger(\mathbf{p})|0\rangle \\ &= (-i)^2\int\widetilde{dp}\Phi(-\mathbf{p})b_s^\dagger(\mathbf{p})d_{s'}^\dagger(-\mathbf{p})|0\rangle = -(-)^l|\Phi\rangle. \end{aligned} \tag{47}$$

可见, 一对正反费米子的 内禀宇称 为 -1, 这一结论对 Majorana 费米子也成立.

用 Weyl 场表示 Dirac 场, 有

$$\psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \xi^a & \chi^\dagger_{\dot{a}} \end{pmatrix}. \tag{48}$$

对 (37) 式应用 $D(\mathcal{P}) = i\beta$, 得

$$P^{-1}\chi_a(x)P = i\xi^{\dagger\dot{a}}(\mathcal{P}x), \quad P^{-1}\xi^{\dagger\dot{a}}(x)P = i\chi_a(\mathcal{P}x). \tag{49}$$

可见, 宇称变换使左手场与右手场互换. 对上式取厄米共轭, 并利用

$$\varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab}, \quad \varepsilon^{\dot{a}\dot{b}}(\chi_b)^\dagger = \varepsilon^{\dot{a}\dot{b}}\chi^\dagger_{\dot{b}} = \chi^{\dagger\dot{a}}, \quad \varepsilon^{\dot{a}\dot{b}}(\xi^{\dagger\dot{b}})^\dagger = -\varepsilon_{ab}\xi^b = -\xi_a, \tag{50}$$

有

$$P^{-1}\chi^{\dagger\dot{a}}(x)P = i\xi_a(\mathcal{P}x), \quad P^{-1}\xi_a(x)P = i\chi^{\dagger\dot{a}}(\mathcal{P}x), \tag{51}$$

这与 Majorana 条件 $\chi_a(x) = \xi_a(x)$ 是一致的.

下面讨论旋量场双线性型 $\bar{\psi}A\psi$ 的变换性质, 其中 A 是 Dirac 矩阵的组合. 记 $\bar{A} \equiv \beta A^\dagger \beta$, 若 $\bar{A} = A$, 则

$$(\bar{\psi}A\psi)^\dagger = \psi^\dagger A^\dagger \beta\psi = \psi^\dagger\beta\beta A^\dagger\beta\psi = \bar{\psi}\bar{A}\psi = \bar{\psi}A\psi, \tag{52}$$

此时 $\bar{\psi}A\psi$ 是厄米的. 由

$$P^{-1}\bar{\psi}(x)P = P^{-1}\psi^\dagger(x)\beta P = [P^{-1}\beta\psi(x)P]^\dagger = [i\beta\beta\psi(\mathcal{P}x)]^\dagger = -i\bar{\psi}(\mathcal{P}x)\beta \tag{53}$$

可知

$$P^{-1}\bar{\psi}(x)A\psi(x)P = \bar{\psi}(\mathcal{P}x)\beta A\beta\psi(\mathcal{P}x). \tag{54}$$

因此, $\bar{\psi}A\psi$ 的 P 变换性质由 $\beta A\beta$ 的形式所决定.

利用

$$\begin{gathered} \beta 1\beta = +1, \quad \beta i\gamma_5\beta = -i\gamma_5, \quad \beta\gamma^0\beta = +\gamma^0, \quad \beta\gamma^i\beta = -\gamma^i, \\ \beta\gamma^0\gamma_5\beta = -\gamma^0\gamma_5, \quad \beta\gamma^i\gamma_5\beta = +\gamma^i\gamma_5, \quad \beta\gamma^0\gamma^0\beta = \gamma^0\gamma^0, \quad \beta\gamma^0\gamma^i\beta = -\gamma^0\gamma^i, \quad \beta\gamma^i\gamma^j\beta = \gamma^i\gamma^j, \end{gathered} \tag{55}$$

有

$$\beta\gamma^\mu\beta = \mathcal{P}^\mu{}_\nu\gamma^\nu, \quad \beta\gamma^\mu\gamma_5\beta = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma_5, \quad \beta\gamma^\mu\gamma^\nu\beta = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\gamma^\rho\gamma^\sigma, \tag{56}$$

而对于 $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$,

$$\beta\sigma^{\mu\nu}\beta = +\frac{i}{2}(\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\gamma^\rho\gamma^\sigma - \mathcal{P}^\nu{}_\rho\mathcal{P}^\mu{}_\sigma\gamma^\rho\gamma^\sigma) = +\frac{i}{2}(\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\gamma^\rho\gamma^\sigma - \mathcal{P}^\nu{}_\sigma\mathcal{P}^\mu{}_\rho\gamma^\sigma\gamma^\rho) = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\sigma^{\rho\sigma}, \tag{57}$$

于是,

$$\begin{aligned} P^{-1}\bar{\psi}(x)\psi(x)P &= +\bar{\psi}(\mathcal{P}x)\psi(\mathcal{P}x), && (58) \\ P^{-1}\bar{\psi}(x)i\gamma_5\psi(x)P &= -\bar{\psi}(\mathcal{P}x)i\gamma_5\psi(\mathcal{P}x), && (59) \\ P^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)P &= +\mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\psi(\mathcal{P}x), && (60) \\ P^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)P &= -\mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\gamma_5\psi(\mathcal{P}x), && (61) \\ P^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)P &= +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\bar{\psi}(\mathcal{P}x)\sigma^{\rho\sigma}\psi(\mathcal{P}x). && (62) \end{aligned}$$

由此可知, $\bar{\psi}\psi$ 是标量, $\bar{\psi}i\gamma_5\psi$ 是赝标量, $\bar{\psi}\gamma^\mu\psi$ 是 (极) 矢量, $\bar{\psi}\gamma^\mu\gamma_5\psi$ 是轴矢量.

左手流 $\bar{\psi}_L\gamma^\mu\psi_L \equiv \frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma_5)\psi$ 和右手流 $\bar{\psi}_R\gamma^\mu\psi_R \equiv \frac{1}{2}\bar{\psi}\gamma^\mu(1+\gamma_5)\psi$ 的变换关系为

$$
\begin{aligned}
P^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)P &= \mathcal{P}^\mu{}_\nu\bar{\psi}_R(\mathcal{P}x)\gamma^\nu\psi_R(\mathcal{P}x), & (63)\\
P^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)P &= \mathcal{P}^\mu{}_\nu\bar{\psi}_L(\mathcal{P}x)\gamma^\nu\psi_L(\mathcal{P}x). & (64)
\end{aligned}
$$

带有电荷 Q 的 Dirac 场与电磁场相互作用时, 运动方程为 (参考文献 [3])

$$
[\gamma^\mu(i\partial_\mu + QeA_\mu) - m]\psi = 0, \tag{65}
$$

取厄米共轭并右乘 γ^0, 得

$$
0 = \psi^\dagger[\gamma^{\mu\dagger}(-i\partial_\mu + QeA_\mu) - m]\gamma^0 = \bar{\psi}[-\gamma^\mu(i\partial_\mu - QeA_\mu) - m], \tag{66}
$$

转置, 得

$$
[-(\gamma^\mu)^{\mathrm{T}}(i\partial_\mu - QeA_\mu) - m]\bar{\psi}^{\mathrm{T}} = 0. \tag{67}
$$

设 $\mathcal{C} \equiv i\gamma^0\gamma^2$, 它有如下性质:

$$
\mathcal{C}^{\mathrm{T}} = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C}, \quad \mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}, \quad \mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5. \tag{68}
$$

由此, 可得

$$
0 = [\mathcal{C}^{-1}\gamma^\mu\mathcal{C}(i\partial_\mu - QeA_\mu) - m]\bar{\psi}^{\mathrm{T}} = \mathcal{C}^{-1}[\gamma^\mu(i\partial_\mu - QeA_\mu) - m]\mathcal{C}\bar{\psi}^{\mathrm{T}}. \tag{69}
$$

令

$$
\psi_c \equiv \mathcal{C}\bar{\psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix}, \tag{70}
$$

则

$$
[\gamma^\mu(i\partial_\mu - QeA_\mu) - m]\psi_c = 0. \tag{71}
$$

与 (65) 式比较, 可知 ψ_c 是与 ψ 带相反电荷的场. 因此, Dirac 场的电荷共轭变换为

$$
C^{-1}\psi(x)C = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x), \tag{72}
$$

而 $\mathcal{C}$ 是旋量空间的电荷共轭变换矩阵. 从而,

$$
\begin{aligned}
C^{-1}\bar{\psi}C &= C^{-1}\psi^\dagger(x)\beta C = [C^{-1}\beta\psi C]^\dagger = [\beta\mathcal{C}\bar{\psi}^{\mathrm{T}}]^\dagger = \{[\bar{\psi}\mathcal{C}^{\mathrm{T}}\beta^{\mathrm{T}}]^\dagger\}^{\mathrm{T}} = \{[\psi^\dagger\beta\mathcal{C}^{\mathrm{T}}\beta^{\mathrm{T}}]^\dagger\}^{\mathrm{T}}\\
&= [\beta^*\mathcal{C}^*\beta\psi]^{\mathrm{T}} = \psi^{\mathrm{T}}\beta^{\mathrm{T}}\mathcal{C}^\dagger\beta = \psi^{\mathrm{T}}\beta^{\mathrm{T}}\mathcal{C}^{-1}\beta = -\psi^{\mathrm{T}}\mathcal{C}^{-1}\beta\beta = \psi^{\mathrm{T}}\mathcal{C}. & (73)
\end{aligned}
$$

于是, 注意到转置时交换两个费米子场应多出一个额外的负号, 有

$$
C^{-1}\bar{\psi}A\psi C = \psi^{\mathrm{T}}\mathcal{C}A\mathcal{C}\bar{\psi}^{\mathrm{T}} = -\bar{\psi}\mathcal{C}^{\mathrm{T}}A^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi, \tag{74}
$$

亦即

$$
C^{-1}\bar{\psi}(x)A\psi(x)C = \bar{\psi}(x)\mathcal{C}^{-1}A^{\mathrm{T}}\mathcal{C}\psi(x). \tag{75}
$$

利用

$$
\begin{aligned}
\mathcal{C}^{-1}1^{\mathrm{T}}\mathcal{C} &= +1, \quad \mathcal{C}^{-1}(i\gamma_5)^{\mathrm{T}}\mathcal{C} = +i\gamma_5, \quad \mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu,\\
\mathcal{C}^{-1}(\gamma^\mu\gamma_5)^{\mathrm{T}}\mathcal{C} &= +\gamma^\mu\gamma_5, \quad \mathcal{C}^{-1}(\gamma^\mu\gamma^\nu)^{\mathrm{T}}\mathcal{C} = \gamma^\nu\gamma^\mu, \quad \mathcal{C}^{-1}(\sigma^{\mu\nu})^{\mathrm{T}}\mathcal{C} = -\sigma^{\mu\nu}, & (76)
\end{aligned}
$$

有

$$
\begin{aligned}
C^{-1}\bar{\psi}(x)\psi(x)C &= +\bar{\psi}(x)\psi(x), &(77)\\
C^{-1}\bar{\psi}(x)i\gamma_5\psi(x)C &= +\bar{\psi}(x)i\gamma_5\psi(x), &(78)\\
C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C &= -\bar{\psi}(x)\gamma^\mu\psi(x), &(79)\\
C^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)C &= +\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x), &(80)\\
C^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)C &= -\bar{\psi}(x)\sigma^{\mu\nu}\psi(x), &(81)\\
C^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)C &= -\bar{\psi}_R(x)\gamma^\mu\psi_R(x), &(82)\\
C^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)C &= -\bar{\psi}_L(x)\gamma^\mu\psi_L(x). &(83)
\end{aligned}
$$

对于 Majorana 场, $\psi = \mathcal{C}\bar{\psi}^{\mathrm{T}}$, 则

$$
\bar{\psi} = (\mathcal{C}\bar{\psi}^{\mathrm{T}})^\dagger\beta = [(\bar{\psi}\mathcal{C}^{\mathrm{T}})^\dagger]^{\mathrm{T}}\beta = [(\psi^\dagger\beta\mathcal{C}^{\mathrm{T}})^\dagger]^{\mathrm{T}}\beta = (\mathcal{C}\beta\psi)^{\mathrm{T}}\beta = \psi^{\mathrm{T}}\beta^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\beta = \psi^{\mathrm{T}}\mathcal{C}, \tag{84}
$$

于是,

$$
C^{-1}\psi(x)C = \psi(x), \quad C^{-1}\bar{\psi}(x)C = \bar{\psi}(x), \tag{85}
$$

从而,

$$
C^{-1}\bar{\psi}(x)A\psi(x)C = \bar{\psi}(x)A\psi(x). \tag{86}
$$

由此可见, 对于 Majorana 场, 必有

$$
\bar{\psi}(x)\gamma^\mu\psi(x) = 0, \quad \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) = 0. \tag{87}
$$

对于时间反演变换, 有

$$
T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x), \tag{88}
$$

同理 $D(\mathcal{T})^2 = \pm 1$. 注意到 (16) 式, 有

$$
T^{-1}\mathbf{x}T = +\mathbf{x}, \quad T^{-1}\mathbf{P}T = -\mathbf{P}, \quad T^{-1}\mathbf{J}T = -\mathbf{J}, \tag{89}
$$

因而时间反演变换同时反转 3 动量方向和自旋方向, 故对产生算符的变换形式为

$$
T^{-1}b_s^\dagger(\mathbf{p})T = \zeta_s b_{-s}^\dagger(-\mathbf{p}), \quad T^{-1}d_s^\dagger(\mathbf{p})T = \zeta_s d_{-s}^\dagger(-\mathbf{p}). \tag{90}
$$

注意到 T 变换的反幺正性 $T^{-1}iT = -i$, 可得

$$
\begin{aligned}
T^{-1}\psi(x)T &= \sum_s \int \widetilde{dp}[\zeta_s^* b_{-s}(-\mathbf{p})u_s^*(\mathbf{p})e^{ip\cdot x} + \zeta_s d_{-s}^\dagger(-\mathbf{p})v_s^*(\mathbf{p})e^{-ip\cdot x}]\\
&= \sum_s \int \widetilde{dp}[\zeta_{-s}^* b_s(\mathbf{p})u_{-s}^*(-\mathbf{p})e^{-ip\cdot\mathcal{T}x} + \zeta_{-s} d_s^\dagger(\mathbf{p})v_{-s}^*(-\mathbf{p})e^{ip\cdot\mathcal{T}x}]. \qquad (91)
\end{aligned}
$$

由于 (注 2)

$$
u_{-s}^*(-\mathbf{p}) = -s\mathcal{C}\gamma_5 u_s(\mathbf{p}), \quad v_{-s}^*(-\mathbf{p}) = -s\mathcal{C}\gamma_5 v_s(\mathbf{p}), \tag{92}
$$

如果我们取 $\zeta_s = s$, 就可以得到与 $T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$ 对应的表达式

$$
T^{-1}\psi(x)T = \sum_s \int \widetilde{dp}[(-s)^2 b_s(\mathbf{p})\mathcal{C}\gamma_5 u_s(\mathbf{p})e^{-ip\cdot\mathcal{T}x} + (-s)^2 d_s^\dagger(\mathbf{p})\mathcal{C}\gamma_5 v_s(\mathbf{p})e^{ip\cdot\mathcal{T}x}] = \mathcal{C}\gamma_5\psi(\mathcal{T}x). \tag{93}
$$

此时

$$D(\mathcal{T}) = \mathcal{C}\gamma_5, \tag{94}$$

即

$$T^{-1}\psi(x)T = \mathcal{C}\gamma_5\psi(\mathcal{T}x). \tag{95}$$

注 2: 对于动量方向矢量 $\widehat{\mathbf{p}} = (s_\theta c_\phi, s_\theta s_\phi, c_\theta)$, 其中 $s_\theta \equiv \sin\theta$, $c_\theta \equiv \cos\theta$, 螺旋度算符可用矩阵表示为

$$\widehat{\mathbf{p}} \cdot \sigma = \begin{pmatrix} c_\theta & e^{-i\phi}s_\theta \\ e^{i\phi}s_\theta & -c_\theta \end{pmatrix}. \tag{96}$$

螺旋态基底取为

$$\xi_+ = \begin{pmatrix} c_{\theta/2} \\ e^{i\phi}s_{\theta/2} \end{pmatrix}, \quad \xi_- = \begin{pmatrix} -e^{-i\phi}s_{\theta/2} \\ c_{\theta/2} \end{pmatrix}, \tag{97}$$

就可以满足

$$(\widehat{\mathbf{p}} \cdot \sigma)\xi_s = s\xi_s, \quad s = \pm. \tag{98}$$

定义 $\tilde{\xi}_s \equiv -is\sigma^2\xi_s^*$, 则有 $\tilde{\xi}_+ = \xi_-$, $\tilde{\xi}_- = \xi_+$, 于是

$$(\widehat{\mathbf{p}} \cdot \sigma)\tilde{\xi}_s = -s\tilde{\xi}_s. \tag{99}$$

因此, 基底 $\tilde{\xi}_s$ 是对应于螺旋度 $-s$.

利用 $\sigma^i\sigma^2 = -\sigma^2(\sigma^i)^*$ 和

$$\mathcal{C} = i\gamma^0\gamma^2 = \begin{pmatrix} -i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}, \quad \mathcal{C}\gamma_5 = \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}, \tag{100}$$

便可以得到

$$\begin{aligned} u_{-s}^*(-\mathbf{p}) &= \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\tilde{\xi}_s \\ \sqrt{p\cdot\sigma}\tilde{\xi}_s \end{pmatrix}^* = \begin{pmatrix} -is\sigma^2\sqrt{p\cdot\sigma^*}\xi_s^* \\ -is\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\xi_s^* \end{pmatrix}^* = \begin{pmatrix} -is\sigma^2\sqrt{p\cdot\sigma}\xi_s \\ -is\sigma^2\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \\ &= -s\begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 u_s(\mathbf{p}), \end{aligned} \tag{101}$$

$$\begin{aligned} v_{-s}^*(-\mathbf{p}) &= \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\tilde{\xi}_s \\ -\sqrt{p\cdot\sigma}\tilde{\xi}_s \end{pmatrix}^* = \begin{pmatrix} -is\sigma^2\sqrt{p\cdot\sigma^*}\xi_s^* \\ +is\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\xi_s^* \end{pmatrix}^* = \begin{pmatrix} -is\sigma^2\sqrt{p\cdot\sigma}\xi_s \\ +is\sigma^2\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \\ &= -s\begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} = -s\mathcal{C}\gamma_5 v_s(\mathbf{p}). \end{aligned} \tag{102}$$

用 Weyl 场表示 Dirac 场, 留意到 $\mathcal{C}\gamma_5 = \mathrm{diag}(i\sigma^2, i\sigma^2) = \mathrm{diag}(\varepsilon^{ab}, -\varepsilon_{\dot{a}\dot{b}})$, 有

$$T^{-1}\chi_a(x)T = +\chi^a(\mathcal{T}x), \quad T^{-1}\xi^{\dagger\dot{a}}(x)T = -\xi^\dagger_{\dot{a}}(\mathcal{T}x). \tag{103}$$

可见, T 变换不改变 Weyl 场的手征. 对上式取厄米共轭, 利用 $\varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}}$, 和 $\varepsilon^{ab} = -\varepsilon_{ab}$, 可得

$$T^{-1}\chi^{\dagger\dot{a}}(x)T = -\chi^\dagger_{\dot{a}}(\mathcal{T}x), \quad T^{-1}\xi_a(x)T = +\xi^a(\mathcal{T}x), \tag{104}$$

这与 Majorana 条件 $\chi_a(x) = \xi_a(x)$ 一致.

由 $T^{-1}\psi(x)T = \mathcal{C}\gamma_5\psi(\mathcal{T}x)$ 可得

$$T^{-1}\bar{\psi}(x)T = T^{-1}\psi^\dagger(x)\beta T = [\mathcal{C}\gamma_5\psi(\mathcal{T}x)]^\dagger\beta = \psi^\dagger(\mathcal{T}x)\gamma_5\mathcal{C}^\dagger\beta = \psi^\dagger(\mathcal{T}x)\beta\gamma_5\mathcal{C}^\dagger = \bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}, \tag{105}$$

因而, 利用 $T^{-1}AT=A^*$, 有

$$T^{-1}\bar{\psi}(x)A\psi(x)T=\bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\psi(\mathcal{T}x). \tag{106}$$

注意到

$$\begin{aligned}
\gamma_5\mathcal{C}^{-1}1^*\mathcal{C}\gamma_5 &= +1, \quad \gamma_5\mathcal{C}^{-1}(i\gamma_5)^*\mathcal{C}\gamma_5=-i\gamma_5,\\
\gamma_5\mathcal{C}^{-1}(\gamma^0)^*\mathcal{C}\gamma_5 &= +\gamma^0, \quad \gamma_5\mathcal{C}^{-1}(\gamma^i)^*\mathcal{C}\gamma_5=-\gamma^i,\\
\gamma_5\mathcal{C}^{-1}(\gamma^0\gamma_5)^*\mathcal{C}\gamma_5 &= +\gamma^0\gamma_5, \quad \gamma_5\mathcal{C}^{-1}(\gamma^i\gamma_5)^*\mathcal{C}\gamma_5=-\gamma^i\gamma_5,
\end{aligned}$$

$$\gamma_5\mathcal{C}^{-1}(\gamma^0\gamma^0)^*\mathcal{C}\gamma_5=+\gamma^0\gamma^0, \quad \gamma_5\mathcal{C}^{-1}(\gamma^0\gamma^i)^*\mathcal{C}\gamma_5=-\gamma^0\gamma^i, \quad \gamma_5\mathcal{C}^{-1}(\gamma^i\gamma^j)^*\mathcal{C}\gamma_5=+\gamma^i\gamma^j, \tag{107}$$

可得

$$\begin{aligned}
\gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*\mathcal{C}\gamma_5=-\mathcal{T}^\mu{}_\nu\gamma^\nu, &\quad \gamma_5\mathcal{C}^{-1}(\gamma^\mu\gamma_5)^*\mathcal{C}\gamma_5=-\mathcal{T}^\mu{}_\nu\gamma^\nu\gamma_5,\\
\gamma_5\mathcal{C}^{-1}(\sigma^{\mu\nu})^*\mathcal{C}\gamma_5=-\frac{i}{2}\gamma_5\mathcal{C}^{-1}[\gamma^\mu,\gamma^\nu]^*\mathcal{C}\gamma_5 &= -\frac{i}{2}\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[\gamma^\rho,\gamma^\sigma]=-\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\sigma^{\rho\sigma}.
\end{aligned} \tag{108}$$

于是

$$T^{-1}\bar{\psi}(x)\psi(x)T = +\bar{\psi}(\mathcal{T}x)\psi(\mathcal{T}x), \tag{109}$$

$$T^{-1}\bar{\psi}(x)i\gamma_5\psi(x)T = -\bar{\psi}(\mathcal{T}x)i\gamma_5\psi(\mathcal{T}x), \tag{110}$$

$$T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\psi(\mathcal{T}x), \tag{111}$$

$$T^{-1}\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\gamma_5\psi(\mathcal{T}x), \tag{112}$$

$$T^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\bar{\psi}(\mathcal{T}x)\sigma^{\rho\sigma}\psi(\mathcal{T}x), \tag{113}$$

$$T^{-1}\bar{\psi}_L(x)\gamma^\mu\psi_L(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}_L(\mathcal{T}x)\gamma^\nu\psi_L(\mathcal{T}x), \tag{114}$$

$$T^{-1}\bar{\psi}_R(x)\gamma^\mu\psi_R(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}_R(\mathcal{T}x)\gamma^\nu\psi_R(\mathcal{T}x). \tag{115}$$

动能项 $\bar{\psi}i\gamma^\mu\partial_\mu\psi$ 的 P,T 变换性质为

$$\begin{aligned}
P^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)P &= \mathcal{P}_\mu{}^\rho\bar{\psi}(\mathcal{P}x)i\beta\gamma^\mu\beta\partial_\rho\psi(\mathcal{P}x)=\mathcal{P}_\mu{}^\rho\mathcal{P}^\mu{}_\sigma\bar{\psi}(\mathcal{P}x)i\gamma^\sigma\partial_\rho\psi(\mathcal{P}x)\\
&= +\bar{\psi}(\mathcal{P}x)i\gamma^\mu\partial_\mu\psi(\mathcal{P}x),
\end{aligned} \tag{116}$$

$$\begin{aligned}
T^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)T &= -i\mathcal{T}_\mu{}^\rho\bar{\psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*\mathcal{C}\gamma_5\partial_\rho\psi(\mathcal{T}x)=i\mathcal{T}_\mu{}^\rho\mathcal{T}^\mu{}_\sigma\bar{\psi}(\mathcal{T}x)\gamma^\sigma\partial_\rho\psi(\mathcal{T}x)\\
&= +\bar{\psi}(\mathcal{T}x)i\gamma^\mu\partial_\mu\psi(\mathcal{T}x),
\end{aligned} \tag{117}$$

而在 C 变换下,

$$C^{-1}\bar{\psi}i\gamma^\mu\partial_\mu\psi C=\psi^{\mathrm{T}}\mathcal{C}i\gamma^\mu\mathcal{C}\partial_\mu\bar{\psi}^{\mathrm{T}}=-i\partial_\mu\bar{\psi}\mathcal{C}^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi=-i\partial_\mu\bar{\psi}\gamma^\mu\psi=-i\partial_\mu(\bar{\psi}\gamma^\mu\psi)+\bar{\psi}i\gamma^\mu\partial_\mu\psi, \tag{118}$$

丢掉全散度项, 化为

$$C^{-1}\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x)C=+\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x). \tag{119}$$

可见动能项 $\bar{\psi}i\gamma^\mu\partial_\mu\psi$ 在 P, T, C 变换下分别保持不变.

对于带有向旋量场求导操作的 2 阶张量 (twist-2) 算符

$$\begin{aligned}
\bar{\psi}\gamma^\mu\partial^\nu\psi+\mathrm{h.c.}=\bar{\psi}\gamma^\mu\partial^\nu\psi+(\partial^\nu\bar{\psi})\gamma^\mu\psi, &\quad i\bar{\psi}\gamma^\mu\partial^\nu\psi+\mathrm{h.c.}=i\bar{\psi}\gamma^\mu\partial^\nu\psi-i(\partial^\nu\bar{\psi})\gamma^\mu\psi,\\
\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi+\mathrm{h.c.}=\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi+(\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi, &\quad i\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi+\mathrm{h.c.}=i\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi-i(\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi,
\end{aligned}$$

$$\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.} = \bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + (\partial^\nu\bar{\psi}_{L/R})\gamma^\mu\psi_{L/R},$$
$$i\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.} = i\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} - i(\partial^\nu\bar{\psi}_{L/R})\gamma^\mu\psi_{L/R},$$

P 和 T 的变换性质为

$$P^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \tag{120}$$
$$P^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]P = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \tag{121}$$
$$P^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]P = +\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}_{R/L}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi_{R/L}(\mathcal{P}x) + \text{h.c.}], \tag{122}$$
$$T^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[\bar{\psi}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \tag{123}$$
$$T^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[\bar{\psi}(\mathcal{T}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \tag{124}$$
$$T^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]T = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[\bar{\psi}_{L/R}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{T}x) + \text{h.c.}], \tag{125}$$

由于 T 变换的反幺正性,

$$T^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]T = +\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[i\bar{\psi}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \tag{126}$$
$$T^{-1}[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]T = +\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[i\bar{\psi}(\mathcal{T}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{T}x) + \text{h.c.}], \tag{127}$$
$$T^{-1}[i\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]T = +\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma[i\bar{\psi}_{L/R}(\mathcal{T}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{T}x) + \text{h.c.}]. \tag{128}$$

另一方面, C 变换会联系互为厄米共轭的两项,

$$C^{-1}\bar{\psi}\gamma^\mu\partial^\nu\psi C = \psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\mathcal{C}\partial^\nu\bar{\psi}^{\mathrm{T}} = -\partial^\nu\bar{\psi}\mathcal{C}^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi = -(\partial^\nu\bar{\psi})\gamma^\mu\psi, \tag{129}$$
$$C^{-1}(\partial^\nu\bar{\psi})\gamma^\mu\psi C = (\partial^\nu\psi^{\mathrm{T}})\mathcal{C}\gamma^\mu\mathcal{C}\bar{\psi}^T = -\bar{\psi}\mathcal{C}^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\partial^\nu\psi = -\bar{\psi}\gamma^\mu\partial^\nu\psi, \tag{130}$$
$$C^{-1}\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi C = \psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\gamma_5\mathcal{C}\partial^\nu\bar{\psi}^{\mathrm{T}} = -\partial^\nu\bar{\psi}\mathcal{C}^{\mathrm{T}}(\gamma^\mu\gamma_5)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi = (\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi, \tag{131}$$
$$C^{-1}(\partial^\nu\bar{\psi})\gamma^\mu\gamma_5\psi C = (\partial^\nu\psi^{\mathrm{T}})\mathcal{C}\gamma^\mu\gamma_5\mathcal{C}\bar{\psi}^T = -\bar{\psi}\mathcal{C}^{\mathrm{T}}(\gamma^\mu\gamma_5)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\partial^\nu\psi = \bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi. \tag{132}$$

故

$$C^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]C = -[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}], \tag{133}$$
$$C^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]C = +[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}], \tag{134}$$
$$C^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]C = -[\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}], \tag{135}$$
$$C^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]C = +[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}], \tag{136}$$
$$C^{-1}[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]C = -[i\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}], \tag{137}$$
$$C^{-1}[i\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]C = +[i\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}]. \tag{138}$$

于是, CP 变换性质为

$$\begin{aligned}(CP)^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]CP &= -P^{-1}[\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \end{aligned}\tag{139}$$
$$\begin{aligned}(CP)^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]CP &= +P^{-1}[\bar{\psi}(x)\gamma^\mu\gamma_5\partial^\nu\psi(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}(\mathcal{P}x)\gamma^\rho\gamma_5\partial^\sigma\psi(\mathcal{P}x) + \text{h.c.}], \end{aligned}\tag{140}$$
$$\begin{aligned}(CP)^{-1}[\bar{\psi}_{L/R}(x)\gamma^\mu\partial^\nu\psi_{L/R}(x) + \text{h.c.}]CP &= -P^{-1}[\bar{\psi}_{R/L}(x)\gamma^\mu\partial^\nu\psi_{R/L}(x) + \text{h.c.}]P \\ &= -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma[\bar{\psi}_{L/R}(\mathcal{P}x)\gamma^\rho\partial^\sigma\psi_{L/R}(\mathcal{P}x) + \text{h.c.}], \end{aligned}\tag{141}$$
$$(CP)^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]CP = +P^{-1}[i\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x) + \text{h.c.}]P$$

$$
\begin{aligned}
&= +\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}[i\bar{\psi}(\mathcal{P}x)\gamma^{\rho}\partial^{\sigma}\psi(\mathcal{P}x)+\text{h.c.}], \qquad (142)\\
(CP)^{-1}[i\bar{\psi}(x)\gamma^{\mu}\gamma_5\partial^{\nu}\psi(x)+\text{h.c.}]CP &= -P^{-1}[i\bar{\psi}(x)\gamma^{\mu}\gamma_5\partial^{\nu}\psi(x)+\text{h.c.}]P\\
&= +\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}[i\bar{\psi}(\mathcal{P}x)\gamma^{\rho}\gamma_5\partial^{\sigma}\psi(\mathcal{P}x)+\text{h.c.}], \qquad (143)\\
(CP)^{-1}[i\bar{\psi}_{L/R}(x)\gamma^{\mu}\partial^{\nu}\psi_{L/R}(x)+\text{h.c.}]CP &= +P^{-1}[i\bar{\psi}_{R/L}(x)\gamma^{\mu}\partial^{\nu}\psi_{R/L}(x)+\text{h.c.}]P\\
&= +\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}[i\bar{\psi}_{L/R}(\mathcal{P}x)\gamma^{\rho}\partial^{\sigma}\psi_{L/R}(\mathcal{P}x)+\text{h.c.}]. \qquad (144)
\end{aligned}
$$

值得注意的是, 有无虚数 i 对应的两组算符的 CP 变换性质相反.

3 电磁场

电磁场 A^{μ} 的 P, T, C 变换性质可以通过分析有源 Maxwell 方程得到 [4]. 在经典电动力学中, Maxwell 方程在 P, T 和 C 变换下分别保持不变. 根据对应原理, 量子电动力学中在 Lorenz 规范下的电磁场运动方程

$$\partial^2 A^{\mu} = ej^{\mu} \qquad (145)$$

也应在这些变换下保持不变, 其中 $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$. 从而, 由

$$P^{-1}j^{\mu}(x)P = \mathcal{P}^{\mu}{}_{\nu}j^{\nu}(\mathcal{P}x), \quad T^{-1}j^{\mu}(x)T = -\mathcal{T}^{\mu}{}_{\nu}j^{\nu}(\mathcal{T}x), \quad C^{-1}j^{\mu}(x)C = -j^{\mu}(x), \qquad (146)$$

和

$$P^{-1}\partial^{\mu}P = \mathcal{P}^{\mu}{}_{\nu}\partial^{\nu}, \quad T^{-1}\partial^{\mu}T = \mathcal{T}^{\mu}{}_{\nu}\partial^{\nu}, \quad C^{-1}\partial^{\mu}C = \partial^{\mu}, \qquad (147)$$

可以推出

$$P^{-1}A^{\mu}(x)P = \mathcal{P}^{\mu}{}_{\nu}A^{\nu}(\mathcal{P}x), \quad T^{-1}A^{\mu}(x)T = -\mathcal{T}^{\mu}{}_{\nu}A^{\nu}(\mathcal{T}x), \quad C^{-1}A^{\mu}(x)C = -A^{\mu}(x). \qquad (148)$$

场强张量 $F^{\mu\nu} \equiv \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ 的变换性质如下,

$$
\begin{aligned}
P^{-1}F^{\mu\nu}(x)P &= \mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\partial^{\alpha}A^{\beta}(\mathcal{P}x) - \mathcal{P}^{\nu}{}_{\beta}\mathcal{P}^{\mu}{}_{\alpha}\partial^{\beta}A^{\alpha}(\mathcal{P}x) = \mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}F^{\alpha\beta}(\mathcal{P}x), \qquad (149)\\
T^{-1}F^{\mu\nu}(x)T &= -\mathcal{T}^{\mu}{}_{\alpha}\mathcal{T}^{\nu}{}_{\beta}\partial^{\alpha}A^{\beta}(\mathcal{T}x) + \mathcal{T}^{\nu}{}_{\beta}\mathcal{T}^{\mu}{}_{\alpha}\partial^{\beta}A^{\alpha}(\mathcal{T}x) = -\mathcal{T}^{\mu}{}_{\alpha}\mathcal{T}^{\nu}{}_{\beta}F^{\alpha\beta}(\mathcal{T}x), \qquad (150)\\
C^{-1}F^{\mu\nu}(x)C &= -\partial^{\mu}A^{\nu}(x) + \partial^{\nu}A^{\mu}(x) = -F^{\mu\nu}(x). \qquad (151)
\end{aligned}
$$

若 λ_1 和 λ_2 分别取 $\{0,1,2,3\}$ 中的两个数字, 而 λ_3 和 λ_4 分别取剩余的另外两个数字, 则有 $\mathcal{P}_{\lambda_3}{}^{\lambda_3}\mathcal{P}_{\lambda_4}{}^{\lambda_4} = -\mathcal{P}^{\lambda_1}{}_{\lambda_1}\mathcal{P}^{\lambda_2}{}_{\lambda_2}$, 从而

$$\varepsilon^{\lambda_1\lambda_2\alpha\beta}\mathcal{P}_{\alpha}{}^{\lambda_3}\mathcal{P}_{\beta}{}^{\lambda_4} = \varepsilon^{\lambda_1\lambda_2\lambda_3\lambda_4}\mathcal{P}_{\lambda_3}{}^{\lambda_3}\mathcal{P}_{\lambda_4}{}^{\lambda_4} = -\mathcal{P}^{\lambda_1}{}_{\lambda_1}\mathcal{P}^{\lambda_2}{}_{\lambda_2}\varepsilon^{\lambda_1\lambda_2\lambda_3\lambda_4} = -\mathcal{P}^{\lambda_1}{}_{\alpha}\mathcal{P}^{\lambda_2}{}_{\beta}\varepsilon^{\alpha\beta\lambda_3\lambda_4}, \qquad (152)$$

于是, 下式成立:

$$\varepsilon^{\mu\nu\alpha\beta}\mathcal{P}_{\alpha}{}^{\rho}\mathcal{P}_{\beta}{}^{\sigma} = -\mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}. \qquad (153)$$

同理, 可以推出

$$\varepsilon^{\mu\nu\alpha\beta}\mathcal{T}_{\alpha}{}^{\rho}\mathcal{T}_{\beta}{}^{\sigma} = -\mathcal{T}^{\mu}{}_{\alpha}\mathcal{T}^{\nu}{}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}. \qquad (154)$$

是故, 对偶场强张量 $\tilde{F}^{\mu\nu} \equiv \dfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ 的变换性质如下,

$$
\begin{aligned}
P^{-1}\tilde{F}^{\mu\nu}(x)P &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P^{-1}F_{\alpha\beta}(x)P = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{P}_{\alpha}{}^{\rho}\mathcal{P}_{\beta}{}^{\sigma}F_{\rho\sigma}(\mathcal{P}x)\\
&= -\frac{1}{2}\mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}(\mathcal{P}x) = -\mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\tilde{F}^{\alpha\beta}(\mathcal{P}x), \qquad (155)
\end{aligned}
$$

$$
\begin{aligned}
T^{-1}\tilde{F}^{\mu\nu}(x)T &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}T^{-1}F_{\alpha\beta}(x)T = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{T}_{\alpha}{}^{\rho}\mathcal{T}_{\beta}{}^{\sigma}F_{\rho\sigma}(\mathcal{T}x) \\
&= +\frac{1}{2}\mathcal{T}^{\mu}{}_{\alpha}\mathcal{T}^{\nu}{}_{\beta}\varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}(\mathcal{T}x) = +\mathcal{T}^{\mu}{}_{\alpha}\mathcal{T}^{\nu}{}_{\beta}\tilde{F}^{\alpha\beta}(\mathcal{T}x), \qquad (156) \\
C^{-1}\tilde{F}^{\mu\nu}(x)C &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}C^{-1}F_{\rho\sigma}(x)C = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}(x) = -\tilde{F}^{\mu\nu}(x). \qquad (157)
\end{aligned}
$$

由此, $F_{\mu\nu}(x)F^{\mu\nu}(x)$ 和 $F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)$ 的变换性质分别为:

$$
\begin{aligned}
P^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)P &= +\mathcal{P}_{\mu}{}^{\alpha}\mathcal{P}_{\nu}{}^{\beta}\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}F_{\alpha\beta}(\mathcal{P}x)F^{\gamma\delta}(\mathcal{P}x) = +F_{\mu\nu}(\mathcal{P}x)F^{\mu\nu}(\mathcal{P}x), \qquad (158) \\
T^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)T &= +\mathcal{T}_{\mu}{}^{\alpha}\mathcal{T}_{\nu}{}^{\beta}\mathcal{T}^{\mu}{}_{\gamma}\mathcal{T}^{\nu}{}_{\delta}F_{\alpha\beta}(\mathcal{T}x)F^{\gamma\delta}(\mathcal{T}x) = +F_{\mu\nu}(\mathcal{T}x)F^{\mu\nu}(\mathcal{T}x), \qquad (159) \\
C^{-1}F_{\mu\nu}(x)F^{\mu\nu}(x)C &= +F_{\mu\nu}(x)F^{\mu\nu}(x); \qquad (160) \\
P^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)P &= -\mathcal{P}_{\mu}{}^{\alpha}\mathcal{P}_{\nu}{}^{\beta}\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}F_{\alpha\beta}(\mathcal{P}x)\tilde{F}^{\gamma\delta}(\mathcal{P}x) = -F_{\mu\nu}(\mathcal{P}x)\tilde{F}^{\mu\nu}(\mathcal{P}x), \qquad (161) \\
T^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)T &= -\mathcal{T}_{\mu}{}^{\alpha}\mathcal{T}_{\nu}{}^{\beta}\mathcal{T}^{\mu}{}_{\gamma}\mathcal{T}^{\nu}{}_{\delta}F_{\alpha\beta}(\mathcal{T}x)\tilde{F}^{\gamma\delta}(\mathcal{T}x) = -F_{\mu\nu}(\mathcal{T}x)\tilde{F}^{\mu\nu}(\mathcal{T}x), \qquad (162) \\
C^{-1}F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)C &= +F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x). \qquad (163)
\end{aligned}
$$

4 电弱规范场

电弱理论由 $SU(2)_L \times U(1)_Y$ 规范群描述, 它同时违反 P 和 C 对称性. 然而, 在大多数弱作用过程里, CP 联合对称性依然得以保持. CP 破坏只出现在少数稀有过程里, 被认为是由 CKM 矩阵里的 CP 相角引起的.

电弱理论的协变导数可表达为

$$
D_\mu = \partial_\mu - ig_1 B_\mu Y - ig_2 W_\mu^a T^a = \partial_\mu - ig_1 B_\mu Y - ig_2 W_\mu, \qquad (164)
$$

其中 $W_\mu \equiv W_\mu^a T^a$. 对于左手费米子二重态 ψ_L(规范本征态) 而言, $T^a = \dfrac{\sigma^a}{2}$, 而

$$
\bar{\psi}_L i\gamma^\mu D_\mu \psi_L = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + g_2 \bar{\psi}_L \gamma^\mu W_\mu^a T^a \psi_L. \qquad (165)
$$

CP 守恒体现在相互作用项 $g_2\bar{\psi}_L\gamma^\mu W_\mu^a T^a\psi_L$ 上, 应有

$$
(CP)^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)CP = \bar{\psi}_L(\mathcal{P}x)\gamma^\mu W_\mu(\mathcal{P}x)\psi_L(\mathcal{P}x). \qquad (166)
$$

利用

$$
\begin{aligned}
P^{-1}\psi_L(x)P &= \frac{1}{2}(1-\gamma_5)i\beta\psi(\mathcal{P}x) = i\beta\frac{1}{2}(1+\gamma_5)\psi(\mathcal{P}x) = i\beta\psi_R(\mathcal{P}x), \qquad (167) \\
P^{-1}\bar{\psi}_L(x)P &= P^{-1}\psi^\dagger(x)\frac{1}{2}(1-\gamma_5)\beta P = \left[P^{-1}\beta\frac{1}{2}(1-\gamma_5)\psi(x)P\right]^\dagger \\
&= \left[\beta\frac{1}{2}(1-\gamma_5)i\beta\psi(\mathcal{P}x)\right]^\dagger = -i\left[\frac{1}{2}(1+\gamma_5)\psi(\mathcal{P}x)\right]^\dagger = -i\bar{\psi}_R(\mathcal{P}x)\beta, \qquad (168) \\
(CP)^{-1}\psi_L(x)CP &= i\beta\frac{1}{2}(1+\gamma_5)C^{-1}\psi(\mathcal{P}x)C = i\beta\frac{1}{2}(1+\gamma_5)\mathcal{C}\bar{\psi}^{\mathrm{T}}(\mathcal{P}x), \qquad (169) \\
(CP)^{-1}\bar{\psi}_L(x)CP &= -iC^{-1}\left[\frac{1}{2}(1+\gamma_5)\psi(\mathcal{P}x)\right]^\dagger C = -i\left[C^{-1}\frac{1}{2}(1+\gamma_5)\psi(\mathcal{P}x)C\right]^\dagger \\
&= -i\left[\frac{1}{2}(1+\gamma_5)\mathcal{C}\bar{\psi}^{\mathrm{T}}(\mathcal{P}x)\right]^\dagger = -i\left\{\left[\psi^\dagger(\mathcal{P}x)\beta\mathcal{C}^{\mathrm{T}}\frac{1}{2}(1+\gamma_5^{\mathrm{T}})\right]^\dagger\right\}^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
&= -i\left[\frac{1}{2}(1+\gamma_5^{\mathrm{T}})\mathcal{C}^*\beta\psi(\mathcal{P}x)\right]^{\mathrm{T}} = -i\psi^{\mathrm{T}}(\mathcal{P}x)\beta^{\mathrm{T}}\mathcal{C}^{\dagger}\frac{1}{2}(1+\gamma_5) \\
&= -i\psi^{\mathrm{T}}(\mathcal{P}x)\beta^{\mathrm{T}}\mathcal{C}^{-1}\frac{1}{2}(1+\gamma_5) = i\psi^{\mathrm{T}}(\mathcal{P}x)\mathcal{C}^{-1}\beta\frac{1}{2}(1+\gamma_5) \\
&= -i\psi^{\mathrm{T}}(\mathcal{P}x)\mathcal{C}\beta\frac{1}{2}(1+\gamma_5) = -i\psi^{\mathrm{T}}(\mathcal{P}x)\mathcal{C}\frac{1}{2}(1-\gamma_5)\beta,
\end{aligned} \tag{170}
$$

可得

$$
\begin{aligned}
&(CP)^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)CP \\
={}& (CP)^{-1}\bar{\psi}_L(x)\gamma^\mu CP(CP)^{-1}W_\mu(x)CP(CP)^{-1}\psi_L(x)CP \\
={}& -i\psi^{\mathrm{T}}(\mathcal{P}x)\mathcal{C}\frac{1}{2}(1-\gamma_5)\beta\gamma^\mu(CP)^{-1}W_\mu(x)CPi\beta\frac{1}{2}(1+\gamma_5)\mathcal{C}\bar{\psi}^{\mathrm{T}}(\mathcal{P}x) \\
={}& \psi^{\mathrm{T}}(\mathcal{P}x)\mathcal{C}\frac{1}{2}(1-\gamma_5)\beta\gamma^\mu(CP)^{-1}W_\mu(x)CP\beta\frac{1}{2}(1+\gamma_5)\mathcal{C}\bar{\psi}^{\mathrm{T}}(\mathcal{P}x) \\
={}& -\bar{\psi}(\mathcal{P}x)\mathcal{C}^{\mathrm{T}}\frac{1}{2}(1+\gamma_5^{\mathrm{T}})\beta^{\mathrm{T}}[(CP)^{-1}W_\mu(x)CP]^{\mathrm{T}}\gamma^{\mu\mathrm{T}}\beta^{\mathrm{T}}\frac{1}{2}(1-\gamma_5^{\mathrm{T}})\mathcal{C}^{\mathrm{T}}\psi(\mathcal{P}x) \\
={}& \bar{\psi}(\mathcal{P}x)\mathcal{C}^{-1}\frac{1}{2}(1+\gamma_5^{\mathrm{T}})\beta^{\mathrm{T}}[(CP)^{-1}W_\mu(x)CP]^{\mathrm{T}}\gamma^{\mu\mathrm{T}}\beta^{\mathrm{T}}\frac{1}{2}(1-\gamma_5^{\mathrm{T}})\mathcal{C}\psi(\mathcal{P}x) \\
={}& -\bar{\psi}(\mathcal{P}x)\frac{1}{2}(1+\gamma_5)\beta\gamma^\mu\beta[(CP)^{-1}W_\mu(x)CP]^{\mathrm{T}}\frac{1}{2}(1-\gamma_5)\psi(\mathcal{P}x) \\
={}& -\bar{\psi}_L(\mathcal{P}x)\mathcal{P}^\mu{}_\nu\gamma^\nu[(CP)^{-1}W_\mu(x)CP]^{\mathrm{T}}\psi_L(\mathcal{P}x).
\end{aligned} \tag{171}
$$

可见,

$$
-\mathcal{P}^\nu{}_\mu\gamma^\mu[(CP)^{-1}W_\nu(x)CP]^{\mathrm{T}} = \gamma^\mu W_\mu(\mathcal{P}x), \tag{172}
$$

即 $-\mathcal{P}^\nu{}_\mu[(CP)^{-1}W_\nu(x)CP]^{\mathrm{T}} = W_\mu(\mathcal{P}x)$. 因此, 规范场 W^μ 的 CP 变换性质为

$$
(CP)^{-1}W^\mu(x)CP = -\mathcal{P}^\mu{}_\nu W^{\nu\mathrm{T}}(\mathcal{P}x), \quad (CP)^{-1}W^{a\mu}(x)T^aCP = -\mathcal{P}^\mu{}_\nu W^{a\nu}(\mathcal{P}x)(T^a)^*. \tag{173}
$$

令

$$
T^\pm \equiv T^1 \pm iT^2 = \frac{1}{2}(\sigma^1 \pm i\sigma^2) = \sigma^\pm, \quad W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \tag{174}
$$

则有

$$
W_\mu^1T^1 + W_\mu^2T^2 = \frac{1}{\sqrt{2}}(W_\mu^+T^+ + W_\mu^-T^-). \tag{175}
$$

由于

$$
(T^\pm)^* = \frac{1}{2}(\sigma^1 \pm i\sigma^2)^* = \frac{1}{2}(\sigma^1 \pm i\sigma^2) = T^\pm, \quad (T^3)^* = T^3, \tag{176}
$$

$W^\pm$ 和 W^3 的 CP 变换性质为

$$
(CP)^{-1}W^{\pm\mu}(x)CP = -\mathcal{P}^\mu{}_\nu W^{\pm\nu}(\mathcal{P}x), \quad (CP)^{-1}W^{3\mu}(x)CP = -\mathcal{P}^\mu{}_\nu W^{3\nu}(\mathcal{P}x). \tag{177}
$$

同理, 可以得到

$$
(CP)^{-1}B^\mu(x)CP = -\mathcal{P}^\mu{}_\nu B^\nu(\mathcal{P}x). \tag{178}
$$

于是, $Z^\mu \equiv W^{3\mu}\cos\theta_W - B^\mu\sin\theta_W$ 和 $A^\mu \equiv W^{3\mu}\sin\theta_W + B^\mu\cos\theta_W$ 的 CP 变换性质为

$$
(CP)^{-1}Z^\mu(x)CP = -\mathcal{P}^\mu{}_\nu Z^\nu(\mathcal{P}x), \quad (CP)^{-1}A^\mu(x)CP = -\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x). \tag{179}
$$

另一方面, T 对称性要求

$$T^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)T=\bar{\psi}_L(\mathcal{T}x)\gamma^\mu W_\mu(\mathcal{T}x)\psi_L(\mathcal{T}x). \tag{180}$$

利用

$$T^{-1}\psi_L(x)T = \frac{1}{2}(1-\gamma_5)\mathcal{C}\gamma_5\psi(\mathcal{T}x)=\mathcal{C}\gamma_5\frac{1}{2}(1-\gamma_5)\psi(\mathcal{T}x)=\mathcal{C}\gamma_5\psi_L(x), \tag{181}$$

$$\begin{aligned}T^{-1}\bar{\psi}_L(x)T &= T^{-1}\psi^\dagger(x)\frac{1}{2}(1-\gamma_5)\beta T=[\mathcal{C}\gamma_5\psi(\mathcal{T}x)]^\dagger\frac{1}{2}(1-\gamma_5)\beta\\ &= \psi^\dagger(\mathcal{T}x)\gamma_5\mathcal{C}^\dagger\frac{1}{2}(1-\gamma_5)\beta=\psi^\dagger(\mathcal{T}x)\frac{1}{2}(1-\gamma_5)\beta\gamma_5\mathcal{C}^{-1}=\bar{\psi}_L(\mathcal{T}x)\gamma_5\mathcal{C}^{-1},\end{aligned} \tag{182}$$

可得

$$\begin{aligned}T^{-1}\bar{\psi}_L(x)\gamma^\mu W_\mu(x)\psi_L(x)T &= T^{-1}\bar{\psi}_L(x)TT^{-1}\gamma^\mu TT^{-1}W_\mu(x)TT^{-1}\psi_L(x)T\\ &= \bar{\psi}_L(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}(\gamma^\mu)^*T^{-1}W_\mu(x)T\mathcal{C}\gamma_5\psi_L(\mathcal{T}x)\\ &= -\bar{\psi}_L(\mathcal{T}x)\mathcal{T}^\mu{}_\nu\gamma^\nu T^{-1}W_\mu(x)T\psi_L(\mathcal{T}x),\end{aligned} \tag{183}$$

可见,

$$-\mathcal{T}^\nu{}_\mu\gamma^\mu T^{-1}W_\nu(x)T=\gamma^\mu W_\mu(\mathcal{T}x). \tag{184}$$

注意到时间反演算符 T 的反幺正性, 规范场 W^μ 的 T 变换性质为

$$T^{-1}W^\mu(x)T=-\mathcal{T}^\mu{}_\nu W^\nu(\mathcal{T}x),\quad T^{-1}W^{a\mu}(x)T^aT=T^{-1}W^{a\mu}(x)T(T^a)^*=-\mathcal{T}^\mu{}_\nu W^{a\nu}(\mathcal{T}x)T^a, \tag{185}$$

$W^\pm$ 和 W^3 的 T 变换性质为

$$T^{-1}W^{\pm\mu}(x)T=-\mathcal{T}^\mu{}_\nu W^{\pm\nu}(\mathcal{T}x),\quad T^{-1}W^{3\mu}(x)T=-\mathcal{T}^\mu{}_\nu W^{3\nu}(\mathcal{T}x). \tag{186}$$

同理, 可以得到

$$T^{-1}B^\mu(x)T=-\mathcal{T}^\mu{}_\nu B^\nu(\mathcal{T}x). \tag{187}$$

于是, Z^μ 和 A^μ 的 T 变换性质为

$$T^{-1}Z^\mu(x)T=-\mathcal{T}^\mu{}_\nu Z^\nu(\mathcal{T}x),\quad T^{-1}A^\mu(x)T=-\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x). \tag{188}$$

$W^{\pm\mu}$, Z^μ 和 B^μ 的 CP 和 T 变换性质均与电磁场 A^μ 的变换性质相同, 因此 $W^{\pm\mu\nu}\equiv\partial^\mu W^{\pm\nu}-\partial^\nu W^{\pm\mu}$, $Z^{\mu\nu}\equiv\partial^\mu Z^\nu-\partial^\nu Z^\mu$ 和 $B^{\mu\nu}\equiv\partial^\mu B^\nu-\partial^\nu B^\mu$ 均与 $F^{\mu\nu}$ 具有相同的 CP 和 T 变换性质, 即

$$(CP)^{-1}W^{\pm\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma W^{\pm\rho\sigma}(\mathcal{P}x),\quad T^{-1}W^{\pm\mu\nu}(x)T=-\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma W^{\pm\rho\sigma}(\mathcal{T}x), \tag{189}$$

$$(CP)^{-1}Z^{\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma Z^{\rho\sigma}(\mathcal{P}x),\quad T^{-1}Z^{\mu\nu}(x)T=-\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma Z^{\rho\sigma}(\mathcal{T}x), \tag{190}$$

$$(CP)^{-1}B^{\mu\nu}(x)CP = -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma B^{\rho\sigma}(\mathcal{P}x),\quad T^{-1}B^{\mu\nu}(x)T=-\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma B^{\rho\sigma}(\mathcal{T}x). \tag{191}$$

$\tilde{W}^{\pm\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}W^\pm_{\rho\sigma}$, $\tilde{Z}^{\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}Z_{\rho\sigma}$ 和 $\tilde{B}^{\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}B_{\rho\sigma}$ 的 CP 和 T 变换性质则与 $\tilde{F}^{\mu\nu}$ 相同:

$$(CP)^{-1}\tilde{W}^{\pm\mu\nu}(x)CP = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\tilde{W}^{\pm\alpha\beta}(\mathcal{P}x),\quad T^{-1}\tilde{W}^{\pm\mu\nu}(x)T=\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\tilde{W}^{\pm\alpha\beta}(\mathcal{T}x), \tag{192}$$

$$(CP)^{-1}\tilde{Z}^{\mu\nu}(x)CP = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\tilde{Z}^{\alpha\beta}(\mathcal{P}x),\quad T^{-1}\tilde{Z}^{\mu\nu}(x)T=\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\tilde{Z}^{\alpha\beta}(\mathcal{T}x), \tag{193}$$

$$(CP)^{-1}\tilde{B}^{\mu\nu}(x)CP = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\tilde{B}^{\alpha\beta}(\mathcal{P}x),\quad T^{-1}\tilde{B}^{\mu\nu}(x)T=\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\tilde{B}^{\alpha\beta}(\mathcal{T}x). \tag{194}$$

李群的生成元 t^a 满足

$$[t^a, t^b] = if^{abc}t^c, \quad [t^{a*}, t^{b*}] = -if^{abc}t^{c*}, \quad \mathrm{tr}(t^a t^b) = \mathrm{tr}(t^{a*}t^{b*}) = \frac{1}{2}\delta^{ab}. \tag{195}$$

对于 $SU(2)_L$ 规范场, 场强张量

$$W^{\mu\nu} = W^{a\mu\nu}T^a = \partial^\mu W^{a\nu}T^a - \partial^\nu W^{a\mu}T^a - ig_2[W^{a\mu}T^a, W^{b\nu}T^b], \tag{196}$$

其中

$$W^{a\mu\nu} = \partial^\mu W^{a\nu} - \partial^\nu W^{a\mu} + g_2\varepsilon^{abc}W^{b\mu}W^{c\nu}. \tag{197}$$

场强张量 $W^{\mu\nu}$ 的变换性质为

$$\begin{aligned}
& (CP)^{-1}W^{a\mu\nu}(x)T^aCP \\
=\ & (CP)^{-1}(\partial^\mu W^{a\nu}(x)T^a - \partial^\nu W^{a\mu}(x)T^a - ig_2[W^{a\mu}(x)T^a, W^{b\nu}(x)T^b])CP \\
=\ & -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\partial^\rho W^{a\sigma}(\mathcal{P}x)T^{a*} + \mathcal{P}^\nu{}_\rho\mathcal{P}^\mu{}_\sigma\partial^\rho W^{a\sigma}(\mathcal{P}x)T^{a*} - ig_2[\mathcal{P}^\mu{}_\rho W^{a\rho}(\mathcal{P}x)T^{a*}, \mathcal{P}^\nu{}_\sigma W^{b\sigma}(\mathcal{P}x)T^{b*}] \\
=\ & -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\{\partial^\rho W^{a\sigma}(\mathcal{P}x)T^{a*} - \partial^\sigma W^{a\rho}(\mathcal{P}x)T^{a*} + ig_2[W^{a\rho}(\mathcal{P}x)T^{a*}, W^{b\sigma}(\mathcal{P}x)T^{b*}]\} \\
=\ & -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\{\partial^\rho W^{a\sigma}(\mathcal{P}x)T^{a*} - \partial^\sigma W^{a\rho}(\mathcal{P}x)T^{a*} + g_2\varepsilon^{abc}W^{a\rho}(\mathcal{P}x)W^{b\sigma}(\mathcal{P}x)T^{c*}\} \\
=\ & -\mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma W^{a\rho\sigma}(\mathcal{P}x)T^{a*},
\end{aligned} \tag{198}$$

$$\begin{aligned}
& T^{-1}W^{a\mu\nu}(x)T(T^a)^* = T^{-1}W^{a\mu\nu}(x)T^aT \\
=\ & T^{-1}(\partial^\mu W^{a\nu}(x)T^a - \partial^\nu W^{a\mu}(x)T^a - ig_2[W^{a\mu}(x)T^a, W^{b\nu}(x)T^b])T \\
=\ & -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\partial^\rho W^{a\sigma}(\mathcal{T}x)T^a + \mathcal{T}^\nu{}_\rho\mathcal{T}^\mu{}_\sigma\partial^\rho W^{a\sigma}(\mathcal{T}x)T^a + ig_2[\mathcal{T}^\mu{}_\rho W^{a\rho}(\mathcal{T}x)T^a, \mathcal{T}^\nu{}_\sigma W^{b\sigma}(\mathcal{T}x)T^b] \\
=\ & -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\{\partial^\rho W^{a\sigma}(\mathcal{T}x)T^a - \partial^\sigma W^{a\rho}(\mathcal{T}x)T^a - ig_2[W^{a\rho}(\mathcal{T}x)T^a, W^{b\sigma}(\mathcal{T}x)T^b]\} \\
=\ & -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\{\partial^\rho W^{a\sigma}(\mathcal{T}x)T^a - \partial^\sigma W^{a\rho}(\mathcal{T}x)T^a + g_2\varepsilon^{abc}W^{a\rho}(\mathcal{T}x)W^{b\sigma}(\mathcal{T}x)T^c\} \\
=\ & -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma W^{a\rho\sigma}(\mathcal{T}x)T^a.
\end{aligned} \tag{199}$$

注意到 (153) 和 (154) 式, 对偶场强张量 $\tilde{W}^{\mu\nu} \equiv \dfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}W_{\alpha\beta}$ 的变换性质为

$$\begin{aligned}
(CP)^{-1}\tilde{W}^{a\mu\nu}(x)T^aCP &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(CP)^{-1}W^a_{\alpha\beta}(x)T^aCP = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{P}_\alpha{}^\rho\mathcal{P}_\beta{}^\sigma W^a_{\rho\sigma}(\mathcal{P}x)T^{a*} \\
&= \frac{1}{2}\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\varepsilon^{\alpha\beta\rho\sigma}W^a_{\rho\sigma}(\mathcal{P}x)T^{a*} = \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\tilde{W}^{a\alpha\beta}(\mathcal{P}x)T^{a*},
\end{aligned} \tag{200}$$

$$\begin{aligned}
T^{-1}\tilde{W}^{a\mu\nu}(x)T(T^a)^* &= T^{-1}\tilde{W}^{a\mu\nu}(x)T^aT \\
&= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}T^{-1}W^a_{\alpha\beta}(x)T^aT = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{T}_\alpha{}^\rho\mathcal{T}_\beta{}^\sigma W^a_{\rho\sigma}(\mathcal{T}x)T^a \\
&= \frac{1}{2}\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\varepsilon^{\alpha\beta\rho\sigma}W^a_{\rho\sigma}(\mathcal{T}x)T^a = \mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\tilde{W}^{a\alpha\beta}(\mathcal{T}x)T^a.
\end{aligned} \tag{201}$$

于是, $W^{a\mu\nu}W^a_{\mu\nu}$ 和 $W^{a\mu\nu}\tilde{W}^a_{\mu\nu}$ 的变换性质为

$$\begin{aligned}
& (CP)^{-1}W^{a\mu\nu}(x)W^a_{\mu\nu}(x)CP \\
=\ & (CP)^{-1}2\mathrm{tr}[W^{a\mu\nu}(x)T^aW^b_{\mu\nu}(x)T^b]CP = 2\mathrm{tr}[(CP)^{-1}W^{a\mu\nu}(x)T^aCP(CP)^{-1}W^b_{\mu\nu}(x)T^bCP] \\
=\ & 2\mathrm{tr}[\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta W^{a\alpha\beta}(\mathcal{P}x)T^{a*}\mathcal{P}_\mu{}^\gamma\mathcal{P}_\nu{}^\delta W^b_{\gamma\delta}(\mathcal{P}x)T^{b*}] = 2\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\mathcal{P}_\mu{}^\gamma\mathcal{P}_\nu{}^\delta W^{a\alpha\beta}(\mathcal{P}x)W^b_{\gamma\delta}(\mathcal{P}x)\mathrm{tr}(T^{a*}T^{b*}) \\
=\ & \mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\mathcal{P}_\mu{}^\gamma\mathcal{P}_\nu{}^\delta W^{a\alpha\beta}(\mathcal{P}x)W^a_{\gamma\delta}(\mathcal{P}x) = +W^{a\mu\nu}(\mathcal{P}x)W^a_{\mu\nu}(\mathcal{P}x),
\end{aligned} \tag{202}$$

$$T^{-1}W^{a\mu\nu}(x)W^a_{\mu\nu}(x)T$$

$$
\begin{aligned}
&= T^{-1}2\mathrm{tr}[W^{a\mu\nu}(x)T^aW^b_{\mu\nu}(x)T^b]T = 2\mathrm{tr}[T^{-1}W^{a\mu\nu}(x)T^aTT^{-1}W^b_{\mu\nu}(x)T^bT]\\
&= 2\mathrm{tr}[\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta W^{a\rho\sigma}(\mathcal{T}x)T^a\mathcal{T}_\mu{}^\gamma\mathcal{T}_\nu{}^\delta W^b_{\gamma\delta}(\mathcal{T}x)T^b] = 2\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\mathcal{T}_\mu{}^\gamma\mathcal{T}_\nu{}^\delta W^{a\rho\sigma}(\mathcal{T}x)W^b_{\gamma\delta}(\mathcal{T}x)\mathrm{tr}(T^aT^b)\\
&= \mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\mathcal{T}_\mu{}^\gamma\mathcal{T}_\nu{}^\delta W^{a\rho\sigma}(\mathcal{T}x)W^a_{\gamma\delta}(\mathcal{T}x) = +W^{a\mu\nu}(\mathcal{T}x)W^a_{\mu\nu}(\mathcal{T}x), \qquad (203)
\end{aligned}
$$

$$
\begin{aligned}
(CP)^{-1}W^{a\mu\nu}(x)\tilde{W}^a_{\mu\nu}(x)CP &= 2\mathrm{tr}[(CP)^{-1}W^{a\mu\nu}(x)T^aCP(CP)^{-1}\tilde{W}^b_{\mu\nu}(x)T^bCP]\\
&= -2\mathcal{P}^\mu{}_\alpha\mathcal{P}^\nu{}_\beta\mathcal{P}_\mu{}^\gamma\mathcal{P}_\nu{}^\delta W^{a\alpha\beta}(\mathcal{P}x)\tilde{W}^b_{\gamma\delta}(\mathcal{P}x)\mathrm{tr}(T^{a*}T^{b*})\\
&= -W^{a\mu\nu}(\mathcal{P}x)\tilde{W}^a_{\mu\nu}(\mathcal{P}x), \qquad (204)\\
T^{-1}W^{a\mu\nu}(x)\tilde{W}^a_{\mu\nu}(x)T &= 2\mathrm{tr}[T^{-1}W^{a\mu\nu}(x)T^aTT^{-1}\tilde{W}^b_{\mu\nu}(x)T^bT]\\
&= -2\mathcal{T}^\mu{}_\alpha\mathcal{T}^\nu{}_\beta\mathcal{T}_\mu{}^\gamma\mathcal{T}_\nu{}^\delta W^{a\rho\sigma}(\mathcal{T}x)\tilde{W}^a_{\gamma\delta}(\mathcal{T}x)\mathrm{tr}(T^aT^b)\\
&= -W^{a\mu\nu}(\mathcal{T}x)\tilde{W}^a_{\mu\nu}(\mathcal{T}x). \qquad (205)
\end{aligned}
$$

下面讨论算符 $\varepsilon^{abc}W^a_{\mu\rho}W^b_\nu W^{c\rho}F^{\mu\nu}$ [5]. 仅当 $SU(2)$ 指标 a, b, c 彼此不同时, 这个算符才不为零, 故必有一个 $SU(2)$ 指标等于 2. 在成生元 T^1, T^2, T^3 中, 只有 T^2 的复共轭不等于自身而表达成 $(T^2)^* = -T^2$. 参考 (173), (185), (198) 和 (199) 式可知, $\varepsilon^{abc}W^a_{\mu\rho}W^b_\nu W^{c\rho}$ 的 CP 和 T 变换都会有一个额外的负号, 来自生成元 T^2 的非自复共轭性质. 于是,

$$
\begin{aligned}
&\quad (CP)^{-1}\varepsilon^{abc}W^a_{\mu\rho}(x)W^b_\nu(x)W^{c\rho}(x)F^{\mu\nu}(x)CP\\
&= -\varepsilon^{abc}(-\mathcal{P}_\mu{}^\alpha\mathcal{P}_\rho{}^\beta)W^a_{\alpha\beta}(\mathcal{P}x)(-\mathcal{P}_\nu{}^\gamma)W^b_\gamma(\mathcal{P}x)(-\mathcal{P}^\rho{}_\delta)W^{c\delta}(\mathcal{P}x)(-\mathcal{P}^\mu{}_\lambda\mathcal{P}^\nu{}_\tau)F^{\lambda\tau}(\mathcal{P}x)\\
&= -\varepsilon^{abc}\mathcal{P}_\mu{}^\alpha\mathcal{P}_\rho{}^\beta\mathcal{P}_\nu{}^\gamma\mathcal{P}^\rho{}_\delta\mathcal{P}^\mu{}_\lambda\mathcal{P}^\nu{}_\tau W^a_{\alpha\beta}(\mathcal{P}x)W^b_\gamma(\mathcal{P}x)W^{c\delta}(\mathcal{P}x)F^{\lambda\tau}(\mathcal{P}x)\\
&= -\varepsilon^{abc}W^a_{\mu\rho}(\mathcal{P}x)W^b_\nu(\mathcal{P}x)W^{c\rho}(\mathcal{P}x)F^{\mu\nu}(\mathcal{P}x), \qquad (206)\\
&\quad T^{-1}\varepsilon^{abc}W^a_{\mu\rho}(x)W^b_\nu(x)W^{c\rho}(x)F^{\mu\nu}(x)T\\
&= -\varepsilon^{abc}(-\mathcal{T}_\mu{}^\alpha\mathcal{T}_\rho{}^\beta)W^a_{\alpha\beta}(\mathcal{T}x)(-\mathcal{T}_\nu{}^\gamma)W^b_\gamma(\mathcal{T}x)(-\mathcal{T}^\rho{}_\delta)W^{c\delta}(\mathcal{T}x)(-\mathcal{T}^\mu{}_\lambda\mathcal{T}^\nu{}_\tau)F^{\lambda\tau}(\mathcal{T}x)\\
&= -\varepsilon^{abc}\mathcal{T}_\mu{}^\alpha\mathcal{T}_\rho{}^\beta\mathcal{T}_\nu{}^\gamma\mathcal{T}^\rho{}_\delta\mathcal{T}^\mu{}_\lambda\mathcal{T}^\nu{}_\tau W^a_{\alpha\beta}(\mathcal{T}x)W^b_\gamma(\mathcal{T}x)W^{c\delta}(\mathcal{T}x)F^{\lambda\tau}(\mathcal{T}x)\\
&= -\varepsilon^{abc}W^a_{\mu\rho}(\mathcal{T}x)W^b_\nu(\mathcal{T}x)W^{c\rho}(\mathcal{T}x)F^{\mu\nu}(\mathcal{T}x). \qquad (207)
\end{aligned}
$$

5 总结

一些算符的 P, T, C 变换性质如 Tab. 1 所示.

可以看到, 带有 n 个矢量指标的旋量场双线性型在 CPT 变换下的奇偶性与数字 n 的奇偶性相同. 由于 ∂_μ 在 CPT 变换下是奇的, 这一结论也可以推广到含有时空导数的情况. 对于标量场和矢量场, 总可以选择它们在 C, P, T 变换下的相位因子, 使得它们满足: 由场和时空导数组成的厄米算符在 CPT 变换下的奇偶性与未收缩矢量指标个数的奇偶性相同. 于是, 由任意场和时空导数组成的厄米 Lorentz 标量在 CPT 变换下是偶的. 由于拉氏量必须由这样的标量构成, 在 CPT 变换下有 $\mathcal{L}(x)\to\mathcal{L}(-x)$, 从而作用量 $S=\int d^4x\mathcal{L}(x)$ 是 CPT 不变量. 这就是 <u>CPT 定理</u>.

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Table 1: 一些算符的 P, T, C 变换性质. 对于 $\mu = 0$, $[-]^\mu \equiv 1$; 对于 $\mu = 1, 2, 3$, $[-]^\mu \equiv -1$.

Operator	P	T	C	CP	CPT
i	$+$	$-$	$+$	$+$	$-$
∂^μ	$[-]^\mu$	$-[-]^\mu$	$+$	$[-]^\mu$	$-$
$\phi^\dagger\phi$	$+$	$+$	$+$	$+$	$+$
$\phi^\dagger i\overleftrightarrow{\partial^\mu}\phi$	$[-]^\mu$	$[-]^\mu$	$-$	$-[-]^\mu$	$-$
$\partial^\mu\phi^\dagger\partial_\mu\phi$	$+$	$+$	$+$	$+$	$+$
$\bar{\psi}\psi$	$+$	$+$	$+$	$+$	$+$
$\bar{\psi}i\gamma_5\psi$	$-$	$-$	$+$	$-$	$+$
$\bar{\psi}\gamma^\mu\psi$	$[-]^\mu$	$[-]^\mu$	$-$	$-[-]^\mu$	$-$
$\bar{\psi}\gamma^\mu\gamma_5\psi$	$-[-]^\mu$	$[-]^\mu$	$+$	$-[-]^\mu$	$-$
$\bar{\psi}\sigma^{\mu\nu}\psi$	$[-]^\mu[-]^\nu$	$-[-]^\mu[-]^\nu$	$-$	$-[-]^\mu[-]^\nu$	$+$
$\bar{\psi}_L\gamma^\mu\psi_L$	$[-]^\mu\bar{\psi}_R\gamma^\mu\psi_R$	$[-]^\mu$	$-\bar{\psi}_R\gamma^\mu\psi_R$	$-[-]^\mu$	$-$
$\bar{\psi}_R\gamma^\mu\psi_R$	$[-]^\mu\bar{\psi}_L\gamma^\mu\psi_L$	$[-]^\mu$	$-\bar{\psi}_L\gamma^\mu\psi_L$	$-[-]^\mu$	$-$
$\bar{\psi}i\gamma^\mu\partial_\mu\psi$	$+$	$+$	$+$	$+$	$+$
$\bar{\psi}\gamma^\mu\partial^\nu\psi + \text{h.c.}$	$[-]^\mu[-]^\nu$	$-[-]^\mu[-]^\nu$	$-$	$-[-]^\mu[-]^\nu$	$+$
$\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi + \text{h.c.}$	$-[-]^\mu[-]^\nu$	$-[-]^\mu[-]^\nu$	$+$	$-[-]^\mu[-]^\nu$	$+$
$\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.}$	$[-]^\mu[-]^\nu(L \leftrightarrow R)$	$-[-]^\mu[-]^\nu$	$-(L \leftrightarrow R)$	$-[-]^\mu[-]^\nu$	$+$
$i\bar{\psi}\gamma^\mu\partial^\nu\psi + \text{h.c.}$	$[-]^\mu[-]^\nu$	$[-]^\mu[-]^\nu$	$+$	$[-]^\mu[-]^\nu$	$+$
$i\bar{\psi}\gamma^\mu\gamma_5\partial^\nu\psi + \text{h.c.}$	$-[-]^\mu[-]^\nu$	$[-]^\mu[-]^\nu$	$-$	$[-]^\mu[-]^\nu$	$+$
$i\bar{\psi}_{L/R}\gamma^\mu\partial^\nu\psi_{L/R} + \text{h.c.}$	$[-]^\mu[-]^\nu(L \leftrightarrow R)$	$[-]^\mu[-]^\nu$	$+(L \leftrightarrow R)$	$[-]^\mu[-]^\nu$	$+$
A^μ	$[-]^\mu$	$[-]^\mu$	$-$	$-[-]^\mu$	$-$
$F^{\mu\nu}$	$[-]^\mu[-]^\nu$	$-[-]^\mu[-]^\nu$	$-$	$-[-]^\mu[-]^\nu$	$+$
$\tilde{F}^{\mu\nu}$	$-[-]^\mu[-]^\nu$	$[-]^\mu[-]^\nu$	$-$	$[-]^\mu[-]^\nu$	$+$
$F^{\mu\nu}F_{\mu\nu}$	$+$	$+$	$+$	$+$	$+$
$F^{\mu\nu}\tilde{F}_{\mu\nu}$	$-$	$-$	$+$	$-$	$+$
$W^{\pm\mu}$, Z^μ		$[-]^\mu$		$-[-]^\mu$	$-$
$W^{\pm\mu\nu}$, $Z^{\mu\nu}$		$-[-]^\mu[-]^\nu$		$-[-]^\mu[-]^\nu$	$+$
$\tilde{W}^{\pm\mu\nu}$, $\tilde{Z}^{\mu\nu}$		$[-]^\mu[-]^\nu$		$[-]^\mu[-]^\nu$	$+$
$W^{a\mu\nu}W^a_{\mu\nu}$, $B^{\mu\nu}B_{\mu\nu}$		$+$		$+$	$+$
$W^{a\mu\nu}\tilde{W}^a_{\mu\nu}$, $B^{\mu\nu}\tilde{B}_{\mu\nu}$		$-$		$-$	$+$
$\varepsilon^{abc}W^a_{\mu\rho}W^b_\nu W^{c\rho}F^{\mu\nu}$		$-$		$-$	$+$

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