量子场论讲义

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http://yzhxxzxy.github.io/cn/teaching.html

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目　录

第 1 章　预备知识

1.1　量子场论的必要性

量子力学是描述微观世界的物理理论。然而，非相对论性量子力学的适用范围有限，不能正确地描述伴随着高速粒子产生和湮灭的相对论性系统。为了合理而自洽地描述这样的系统，需要用到量子场论，它结合了量子力学、相对性原理和场的概念。

在量子力学的基础课程中，量子化的对象通常是由**粒子**组成的动力学系统。如果对相对论性的粒子作类似的量子化，会遇到一些困难。考虑到相对论效应，可以用相对论性的波函数方程来描述单个粒子的运动。此类方程中第一个被提出的是 **Klein-Gordon 方程**：

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=(-\hbar^2c^2\nabla^2+m^2c^4)\psi(\mathbf{x},t). \tag{1.1}$$

它给出的自由粒子能量为

$$E=\pm\sqrt{|\mathbf{p}|^2c^2+m^2c^4}, \tag{1.2}$$

其中 $\mathbf{p}$ 为粒子的动量，m 为粒子的静止质量。可见，能量 E 可以为正，取值范围为 $mc^2\leq E<\infty$；也可以为负，取值范围为 $-\infty<E\leq mc^2$。一个粒子具有负无穷大的能量，在物理上是不可接受的。而且，即使粒子的初始能量为正，也可以通过跃迁到负能态而改变能量的符号。这就是**负能量困难**。另一方面，据此计算粒子在空间中的概率密度

$$\rho=\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t}-\frac{\partial\psi^*}{\partial t}\psi\right), \tag{1.3}$$

会发现 ρ 不总是正的，有可能在一些空间区域中为负。这是一个非物理的结果，称为**负概率困难**。

Klein-Gordon 方程出现负概率困难的根源在于方程中含有波函数对时间的二阶导数。为了克服这个问题，Dirac 方程被提出来，它只包含对时间的一阶导数，且具有 Lorentz 不变性。它描述的是自旋 1/2 的粒子，一开始是用来描述电子 (electron) 的。Dirac 方程能够保证概率密度正定和概率守恒。但是，负能量困难仍然存在。

为了解决负能量困难，P. A. M. Dirac 提出真空 (vacuum) 是所有 $E<0$ 的态都被填满而所有 $E>0$ 的态都为空的状态。这样一来，Pauli 不相容原理会阻止一个 $E>0$ 的电子跃迁到 $E<0$ 的态。如果负能海中缺失一个带有电荷 $-|e|$ 和能量 $-|E|$ 的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 $+|e|$ 和能量 $+|E|$ 的“反粒子”，称为**正电子** (positron)。正电子在 1932 年被 Carl Anderson 发现。

但是，Dirac 的空穴理论仍然面临一些困难，比如，为何没有观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场？另一方面，Dirac 方程一开始作为描述单个粒子波函数的方程提出来，但 Dirac 的解释却包含了无穷多个粒子。而且，像光子和 π 介子这些不满足 Pauli 不相容原理的粒子，空穴理论是不能成立的。此外，Dirac 方程只能描述自旋 1/2 的粒子，不能解决描述整数自旋粒子的困难。

用相对论性的波函数方程描述单个粒子会遇到这么多困难，是否意味着处理这些问题的基础本身就不正确呢？确实是这样的。量子力学的一条基本原理是：观测量由 Hilbert 空间中的厄米算符 (Hermitian operator) 描写。然而，**时间**显然是一个观测量，却没有用一个厄米算符来描写它。在 Schrödinger 绘景 (picture) 中，描述系统的量子态时可以让态依赖于一个时间参数 t，这是时间的概念进入量子力学的方式，但并没有假定这个参数是某个厄米算符的本征值。另一方面，粒子的空间位置 $\mathbf{x}$ 则是位置算符 $\hat{\mathbf{x}}$ 的本征值。可见，在量子力学中，对时间和空间的处理方式是完全不同的。而在狭义相对论中，Lorentz 对称性将两者混合起来。因此，在结合量子力学与狭义相对论的过程中出现困难，也是正常的。

那么，如何在量子力学中平等地处理时间和空间呢？一种途径是将时间提升为一个厄米算符，但这样做在实际操作中非常困难。另一种途径是将空间位置降格为一个参数，不再由厄米算符描写。这样，我们可以在每个空间点 $\mathbf{x}$ 处定义一个算符 $\hat{\phi}(\mathbf{x})$，所有这些算符的集合称为**量子场**。在 Heisenberg 绘景中，量子场算符也依赖于时间 t：

$$\hat{\phi}(\mathbf{x},t) = e^{i\hat{H}t/\hbar}\hat{\phi}(\mathbf{x})e^{-i\hat{H}t/\hbar}. \tag{1.4}$$

如此，量子化的对象变成是由依赖于时空坐标的**场**组成的动力学系统，这就是**量子场论**。这里的量子算符用 ˆ 符号标记，为了简化记号，后面将省略 ˆ 符号。

在量子场论中，前面提到的困难都可以得到解决。现在，Klein-Gordon 方程和 Dirac 方程这样的相对论性方程描述的是自由量子场的运动。真空是量子场的基态，包含粒子的态则是激发态，激发态可以包含任意多个粒子。量子场论平等地描述正粒子和反粒子，由正反粒子的产生算符和湮灭算符表达出来的哈密顿量是正定的，不再出现负能量困难。概率密度 ρ 的空间积分 $\int d^3x\,\rho$ 也可以用产生湮灭算符表达出来，虽然它不一定是正定的，但是它不再被解释为总概率，而是被解释为正粒子数与反粒子数之差，因而也不再出现负概率困难。

1.2 自然单位制

量子场论是结合量子力学和相对论的理论，因而时常出现约化 Planck 常量 $\hbar$ 和光速 c，这一点可以从上一节的几个公式中看出来。于是，为了简化表述，通常采用**自然单位制**，取

$$\hbar = c = 1. \tag{1.5}$$

从而，Klein-Gordon 方程 (1.1) 化为

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(\mathbf{x},t) = 0. \tag{1.6}$$

在自然单位制中，速度没有量纲 (dimension)；长度量纲与时间量纲相同，是能量量纲的倒数；能量、质量和动量具有相同的量纲。可以将能量单位电子伏特 (eV) 视作上述有量纲物理量的基本单位。利用转换关系

$$1 = \hbar = 6.582 \times 10^{-22}\ \mathrm{MeV \cdot s}, \quad 1 = \hbar c = 1.973 \times 10^{-11}\ \mathrm{MeV \cdot cm}, \tag{1.7}$$

可得

$$1\ \mathrm{s}^{-1} = 6.582 \times 10^{-22}\ \mathrm{MeV}, \quad 1\ \mathrm{cm}^{-1} = 1.973 \times 10^{-11}\ \mathrm{MeV}. \tag{1.8}$$

精细结构常数

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.036} \tag{1.9}$$

是没有量纲的，它的数值在任何单位制下都应该相同。因此，自然单位制不可能将 $\hbar$、c、ε_0 和 e 这四个常数同时归一化。在量子场论中，通常再取真空介电常数

$$\varepsilon_0 = 1, \tag{1.10}$$

同时可得真空磁导率 $\mu_0 = 1/(\varepsilon_0 c^2) = 1$，这样做其实是取了 Heaviside-Lorentz 单位制。从而，不同于 Gauss 单位制，Maxwell 方程组中不会出现无理数 4π：

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{1.11}$$

此处的单位制称为**有理化**的自然单位制。现在，精细结构常数可以简便地表达为 $\alpha = e^2/(4\pi)$，而单位电荷量 $e = \sqrt{4\pi\alpha} = 0.3028$ 是没有量纲的；4π 因子会出现在 Coulomb 定律中，点电荷 Q 的 Coulomb 势表达成

$$\Phi = \frac{Q}{4\pi r}. \tag{1.12}$$

1.3 Lorentz 变换和 Lorentz 群

描述高速运动的系统需要用到**狭义相对论**，它的基本原理如下。

(1) *光速不变原理*：在任意惯性参考系中，光速的大小不变。

(2) *狭义相对性原理*：在任意惯性参考系中，物理定律具有相同的形式。

两个惯性参考系的 Descartes 坐标由 Lorentz 变换联系起来。

设惯性坐标系 O' 沿着惯性坐标系 O 的 x 方向以速度 β 匀速运动，则 Lorentz 变换的形式是

$$t' = \gamma(t - \beta x), \quad x' = \gamma(x - \beta t), \quad y' = y, \quad z' = z, \tag{1.13}$$

其中 Lorentz 因子 $\gamma \equiv (1-\beta^2)^{-1/2}$. 这种 Lorentz 变换称为沿 x 方向的**增速** (boost)。在此变换下，有

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2. \tag{1.14}$$

可见，$t^2 - x^2 - y^2 - z^2$ 在 Lorentz 变换下不变，是一个 **Lorentz 不变量**。Lorentz 不变量在不同惯性系中具有相同的值，这是 Lorentz 变换对应的对称性，称为 **Lorentz 对称性**。

将时间坐标和空间坐标结合起来，可以构成 Minkowski 时空，坐标记为

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (x^0, \mathbf{x}), \quad 其中\ \mu = 0, 1, 2, 3. \tag{1.15}$$

上式中四种记法是等价的。x^μ 是一个逆变 (contravariant) 的 Lorentz 四维矢量 (vector)，“逆变”指它的指标 (index) μ 写在右上角。受到 (1.14) 式的启发，可以定义 Lorentz 不变的内积[1]

$$x^2 \equiv x \cdot x \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^0)^2 - |\mathbf{x}|^2. \tag{1.16}$$

引入对称的 **Minkowski 度规** (metric)

$$g_{\mu\nu} = g_{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.17}$$

可以把内积 (1.16) 简洁地写成

$$x^2 = g_{\mu\nu} x^\mu x^\nu. \tag{1.18}$$

这里采用了 **Einstein 求和约定**：*不写出求和符号，重复的指标即表示求和*。除非特别指出，后面都默认使用这个约定。在上式中，用同个字母表示的指标分别在上标和下标重复出现并求和，这称为**缩并** (contraction)，是 Lorentz 不变量的特点。

为了进一步简化记号，定义协变 (covariant) 的 Lorentz 四维矢量

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -x^1, -x^2, -x^3) = (x^0, -\mathbf{x}). \tag{1.19}$$

“协变”指的是指标 μ 写在右下角。于是，内积 x^2 的表达式 (1.18) 可以简化为

$$x^2 = x^\mu x_\mu. \tag{1.20}$$

(1.19) 式可以看作是用度规 $g_{\mu\nu}$ 通过缩并将逆变矢量 x^ν 的指标降下来，变成协变矢量 x_μ。从方阵的角度看，$g_{\mu\nu}$ 的逆为

$$g^{\mu\nu} = g^{\nu\mu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.21}$$

满足

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu, \tag{1.22}$$

[1]这里的记号有些不一致，第一个 x^2 是内积的记号，而第二个 x^2 是第 2 个空间坐标。

其中 **Kronecker 符号** $\delta^\mu{}_\nu$ 定义为

$$\delta^\mu{}_\nu = \delta_\mu{}^\nu = \delta^{\mu\nu} = \delta_{\mu\nu} = \begin{cases} 1, & \mu = \nu, \\ 0, & \mu \neq \nu. \end{cases} \tag{1.23}$$

对于 Minkowski 度规，$g_{\mu\nu}$ 的逆 $g^{\mu\nu}$ 与自己的矩阵形式相同，但更一般的度规有可能与它的逆不同. 将 (1.19) 式 $x_\mu = g_{\mu\nu}x^\nu$ 两边都乘以 $g^{\sigma\mu}$，对 μ 求和，得

$$g^{\sigma\mu}x_\mu = g^{\sigma\mu}g_{\mu\nu}x^\nu = \delta^\sigma{}_\nu x^\nu = x^\sigma, \tag{1.24}$$

这相当于用 $g^{\sigma\mu}$ 通过缩并将协变矢量 x_μ 的指标升起来，变成逆变矢量 x^σ。可见，逆变矢量与协变矢量是一一对应的，是对同一个 Lorentz 矢量的两种等价描述。

利用 Kronecker 符号的定义和 (1.22) 式，可得

$$g^{\mu\nu} = g^{\mu\rho}\delta^\nu{}_\rho = g^{\mu\rho}g^{\nu\sigma}g_{\sigma\rho} = g^{\mu\rho}g^{\nu\sigma}g_{\rho\sigma}, \tag{1.25}$$

$$g_{\mu\nu} = g_{\mu\rho}\delta^\rho{}_\nu = g_{\mu\rho}g^{\rho\sigma}g_{\sigma\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma}. \tag{1.26}$$

这两条式子表明，度规也可以用来对度规自身的指标进行升降。

利用四维矢量的记号，可以把 Lorentz 增速变换 (1.13) 改写为

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.27}$$

其中

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{1.28}$$

注意：在将 $\Lambda^\mu{}_\nu$ 视作矩阵时，偏左的指标 μ 表示行的编号，偏右的指标 ν 表示列的编号。$\Lambda^\mu{}_\nu$ 的特点是保持内积 $x^2 = x^\mu x_\mu$ 不变，从而使 $x^\mu x_\mu$ 在不同惯性系中具有相同的值。我们可以将 $\Lambda^\mu{}_\nu$ 推广为所有保持 $x^\mu x_\mu$ 不变的线性变换，称为（齐次）**Lorentz 变换**，使下式成立：

$$x'^2 = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = g_{\alpha\beta}x^\alpha x^\beta = x^2. \tag{1.29}$$

可见，Lorentz 变换 $\Lambda^\mu{}_\nu$ 必须满足**保度规条件**

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta}. \tag{1.30}$$

空间旋转变换能够保持 $|\mathbf{x}|^2$ 不变，根据 (1.16) 式，易见它也是一种 Lorentz 变换。例如，绕 z 轴旋转 θ 角的变换可以表示为

$$[R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}. \tag{1.31}$$

容易验证，它满足保度规条件 (1.30)。

将 (1.30) 式两边都乘以 $g^{\gamma\alpha}$ 并对 α 缩并，可得

$$\Lambda_\nu{}^\gamma \Lambda^\nu{}_\beta = g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g^{\gamma\alpha} g_{\alpha\beta} = \delta^\gamma{}_\beta, \tag{1.32}$$

其中

$$\Lambda_\nu{}^\gamma \equiv g^{\gamma\alpha} g_{\mu\nu} \Lambda^\mu{}_\alpha \tag{1.33}$$

可以看作是用度规对 $\Lambda^\mu{}_\alpha$ 的两个指标分别升降的结果。定义

$$(\Lambda^{-1})^\mu{}_\nu \equiv \Lambda_\nu{}^\mu, \tag{1.34}$$

则由 (1.32) 式可得

$$(\Lambda^{-1})^\mu{}_\rho \Lambda^\rho{}_\nu = \delta^\mu{}_\nu. \tag{1.35}$$

$\delta^\mu{}_\nu$ 也是一个 Lorentz 变换，它使得 $x'^\mu = \delta^\mu{}_\nu x^\nu = x^\mu$，即 x^μ 在这个变换下不变。可见，$\delta^\mu{}_\nu$ 是一个恒等变换。(1.35) 式表明，对时空坐标矢量先作 Λ 变换，再作 Λ^{-1} 变换，得到的矢量还是原来的矢量。也就是说，由 (1.34) 式定义的 Λ^{-1} 是 Λ 的逆变换，也是一个 Lorentz 变换。在这些记号下，协变矢量 x_μ 的 Lorentz 变换可以表达为

$$x'_\mu = g_{\mu\nu} x'^\nu = g_{\mu\nu} \Lambda^\nu{}_\rho x^\rho = g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma} x_\sigma = \Lambda_\mu{}^\sigma x_\sigma = x_\sigma (\Lambda^{-1})^\sigma{}_\mu. \tag{1.36}$$

Λ^{-1} 既然是一个 Lorentz 变换，必定满足保度规条件

$$g_{\mu\nu} (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta = g_{\alpha\beta}, \tag{1.37}$$

于是有

$$\begin{aligned} g^{\rho\sigma} &= g_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma} = g_{\mu\nu} (\Lambda^{-1})^\mu{}_\alpha (\Lambda^{-1})^\nu{}_\beta g^{\alpha\rho} g^{\beta\sigma} = g^{\gamma\delta} g_{\gamma\mu} g_{\delta\nu} \Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu g^{\alpha\rho} g^{\beta\sigma} \\ &= g^{\gamma\delta} (g^{\alpha\rho} g_{\gamma\mu} \Lambda_\alpha{}^\mu)(g^{\beta\sigma} g_{\delta\nu} \Lambda_\beta{}^\nu) = g^{\gamma\delta} \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta. \end{aligned} \tag{1.38}$$

这给出了保度规条件 (1.30) 的一个等价形式：

$$g^{\mu\nu} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = g^{\alpha\beta}. \tag{1.39}$$

将 $\Lambda^\mu{}_\nu$ 视作矩阵 Λ，则其转置矩阵 Λ^{T} 的分量满足 $(\Lambda^{\mathrm{T}})_\nu{}^\mu = \Lambda^\mu{}_\nu$，由保度规条件 (1.30) 可得

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = (\Lambda^{\mathrm{T}})_\alpha{}^\mu g_{\mu\nu} \Lambda^\nu{}_\beta, \tag{1.40}$$

写成矩阵等式是

$$\mathbf{g} = \Lambda^{\mathrm{T}} \mathbf{g}\, \Lambda. \tag{1.41}$$

取行列式得 $\det \mathbf{g} = \det \Lambda^{\mathrm{T}} \cdot \det \mathbf{g} \cdot \det \Lambda = \det \mathbf{g} \cdot (\det \Lambda)^2$，因此，

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \tag{1.42}$$

Lorentz 坐标变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ 的 Jacobi 行列式为

$$\mathcal{J} = \det\left[\frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)}\right] = \det\Lambda, \tag{1.43}$$

故体积元 d^4x 在 Lorentz 变换下的变化是

$$d^4x' = |\mathcal{J}|d^4x = |\det\Lambda|d^4x = d^4x. \tag{1.44}$$

可见，Minkowski 时空的体积元是 Lorentz 不变的。

$\det\Lambda$ 的值可以用来为 Lorentz 变换分类：$\det\Lambda = +1$ 的变换称为固有 (proper) Lorentz 变换，$\det\Lambda = -1$ 的则是非固有 (improper) Lorentz 变换。此外，由保度规条件 (1.30) 可得

$$1 = g_{00} = g_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2, \tag{1.45}$$

则 $(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1$，故有 $\Lambda^0{}_0 \geq +1$ 或 $\Lambda^0{}_0 \leq -1$。$\Lambda^0{}_0 \geq +1$ 的 Lorentz 变换称为保时向 (orthochronous) Lorentz 变换，$\Lambda^0{}_0 \leq -1$ 的称为反时向 (antichronous) Lorentz 变换。

在数学上，对称性由群论描述。对称变换的集合称为**群**，群元素具有乘法，满足下列四个条件。

(1) 两个群元素的乘积即是两次对称变换相继作用，乘法满足结合律。

(2) 群中任意两个元素的乘积仍属于此群（封闭性）。

(3) 群中必有一个恒元（对应于恒等变换），它与任一元素的乘积仍为此元素。

(4) 任一元素都可以在群中找到一个逆元（对应于逆变换），两者之积为恒元。

所有 Lorentz 变换组成的集合称为 **Lorentz 群**。

Lorentz 变换可以用一组连续变化的参数（如 β、θ 等）来描述，因而是一种连续变换，所以 Lorentz 群是一个连续群，参数的变化区域称为群空间。Lorentz 群的整个群空间不是连通的，它有四个连通分支，如图 1.1 所示，分别是固有保时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \geq +1$)、固有反时向分支 ($\det\Lambda = +1$ 且 $\Lambda^0{}_0 \leq -1$)、非固有保时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \geq +1$) 和非固有反时向分支 ($\det\Lambda = -1$ 且 $\Lambda^0{}_0 \leq -1$)，四个分支之间彼此不连通。恒元（即恒等变换）在固有保时向分支里，这个分支也称为**固有保时向 Lorentz 群**。

这里引入两个特殊的 Lorentz 变换。定义宇称 (parity) 变换为

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{1.46}$$

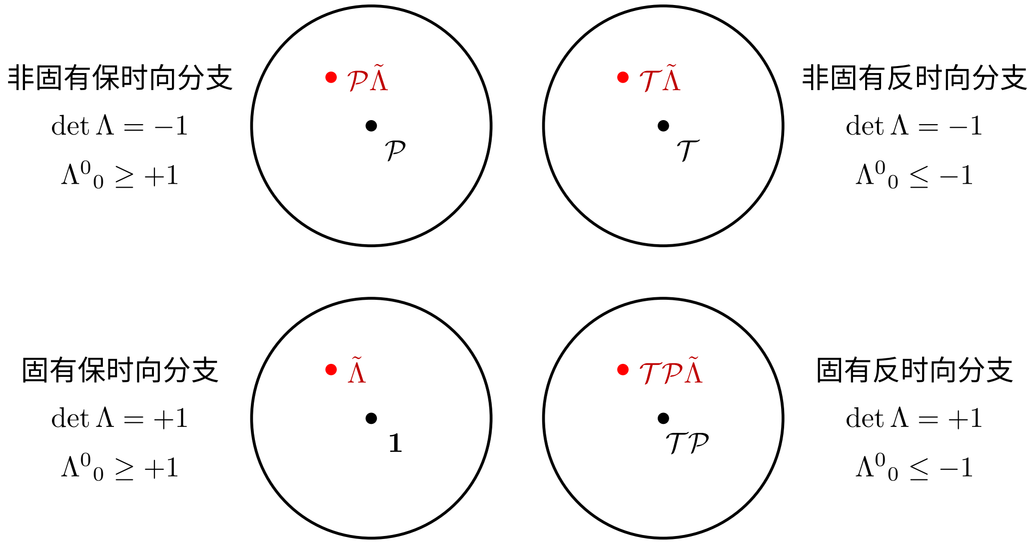


图 1.1: Lorentz 群的四个连通分支示意图。$\mathbf{1}$、$\mathcal{P}$ 和 $\mathcal{T}$ 分别代表恒等变换、宇称变换和时间反演变换，$\tilde{\Lambda}$ 是固有保时向分支中的任意元素。

它是非固有保时向的，亦称为空间反射 (space inversion) 变换。定义时间反演 (time reversal) 变换为

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \tag{1.47}$$

它是非固有反时向的。一个固有保时向 Lorentz 群中的元素，乘上宇称变换或（和）时间反演变换，就可以到达 Lorentz 群的其它分支。

1.4　Lorentz 矢量

如果一些 $m \times m$ 矩阵的乘法关系与某个群中元素的乘法关系完全相同，就可以用这些矩阵来表示这个群，这些矩阵构成了这个群的一个 m 维**线性表示**。利用群的线性表示，可以将对称变换视作矩阵，将变换作用的态视作列矩阵。

在上一节中，我们已经用矩阵的形式表示过 Lorentz 变换 $\Lambda^\mu{}_\nu$，可见，$\Lambda^\mu{}_\nu$ 自然而然地构成了 Lorentz 群的一个 4 维线性表示。这个表示被称为**矢量表示**，因为 Lorentz 矢量 x^ν 可以看作是变换 $\Lambda^\mu{}_\nu$ 所作用的态。一般地，一个 **Lorentz 矢量** A^μ 的定义是它在 Lorentz 变换下满足

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{1.48}$$

类似于 (1.36) 式，逆变矢量 A^μ 对应的协变矢量 $A_\mu = g_{\mu\nu}A^\nu$ 在 Lorentz 变换下满足

$$A_\mu = A_\nu(\Lambda^{-1})^\nu{}_\mu. \tag{1.49}$$

两个 Lorentz 矢量 $A^\mu = (A^0, \mathbf{A})$ 和 $B^\mu = (B^0, \mathbf{B})$ 的内积定义为

$$A \cdot B \equiv A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \tag{1.50}$$

由保度规条件 (1.30) 可知这个内积是 Lorentz 不变量：

$$A' \cdot B' = g_{\mu\nu} A'^\mu B'^\nu = g_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^\alpha B^\beta = g_{\alpha\beta} A^\alpha B^\beta = A \cdot B. \tag{1.51}$$

这样的 Lorentz 不变量是 **Lorentz 标量** (scalar)。由于度规 $g_{\mu\nu}$ 的对角元有正有负，Lorentz 矢量 A^μ 的自我内积不是正定的，可以分为三类。

(1) 若 $A^2 > 0$，则称 A^μ 为**类时**矢量。

(2) 若 $A^2 < 0$，则称 A^μ 为**类空**矢量。

(3) 若 $A^2 = 0$，则称 A^μ 为**类光**矢量。

由于 A^2 是 Lorentz 不变量，不能通过 Lorentz 变换改变 A^μ 的类型。

在狭义相对论中，质点的能量 E、动量 $\mathbf{p}$ 和（静止）质量 m 之间的关系为

$$E = \sqrt{|\mathbf{p}|^2 + m^2}. \tag{1.52}$$

可以用 E 和 $\mathbf{p}$ 组成一个 Lorentz 矢量

$$p^\mu = (E, \mathbf{p}), \tag{1.53}$$

称为**四维动量**，它的内积为

$$p^2 = p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - |\mathbf{p}|^2 = m^2. \tag{1.54}$$

这是合理的，因为质量 m 在狭义相对论中是一个 Lorentz 不变量。p^μ 在 $m > 0$ 时是类时矢量，在 $m = 0$ 时是类光矢量。

将对时空坐标的导数记为

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right) = g^{\mu\nu} \partial_\nu, \tag{1.55}$$

则有

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho{}^\nu = g^{\mu\nu}. \tag{1.56}$$

可见，这里关于时空导数指标位置的写法是合理的。对时空坐标作 Lorentz 变换 $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ 时，时空导数的 Lorentz 变换形式为

$$\partial'^\mu = \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \partial^\nu. \tag{1.57}$$

由上式、(1.56) 式和保度规条件 (1.39) 可得，

$$\partial'^\mu x'^\nu = \Lambda^\mu{}_\rho \partial^\rho (\Lambda^\nu{}_\sigma x^\sigma) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho x^\sigma = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} = g^{\mu\nu}, \tag{1.58}$$

说明 (1.56) 式在惯性坐标系 O' 中也成立。这显然是正确的，从而验证了时空导数 Lorentz 变换形式 (1.57) 的正确性。

(1.57) 式表明，时空导数的 Lorentz 变换形式与 Lorentz 矢量相同，因而我们可以将时空导数看作一个 Lorentz 矢量。定义 **d'Alembert 算符**

$$\partial^2 \equiv \partial^\mu \partial_\mu = \partial_0^2 - \nabla^2, \tag{1.59}$$

则由保度规条件 (1.30) 可得

$$\partial'^2 = g_{\mu\nu}\partial'^\mu \partial'^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \partial^\rho \partial^\sigma = g_{\rho\sigma}\partial^\rho \partial^\sigma = \partial^2. \tag{1.60}$$

可见，∂^2 算符是 Lorentz 不变的。用它可以把 Klein-Gordon 方程 (1.6) 改写成紧凑的形式

$$(\partial^2 + m^2)\psi(x) = 0, \tag{1.61}$$

其中 x 表示四维时空坐标。这样可以明显地看出 Klein-Gordon 方程的 Lorentz 不变性。

1.5　Lorentz 张量

Lorentz 张量 (tensor) 是 Lorentz 矢量的推广。一个 $p+q$ 阶的 (p,q) 型 **Lorentz 张量** $T^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ 应满足如下 Lorentz 变换规则：

$$T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_p}{}_{\rho_q} T^{\rho_1\cdots\rho_p}{}_{\sigma_1\cdots\sigma_q}(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\sigma_q}{}_{\nu_q}. \tag{1.62}$$

Lorentz 标量是 0 阶 Lorentz 张量，Lorentz 矢量是 1 阶 Lorentz 张量。Minkowski 度规 $g_{\mu\nu}$ 是一个 2 阶的 $(0,2)$ 型 Lorentz 张量，不过它在任何惯性系中不变，Lorentz 变换规则就是保度规条件 (1.37)。

利用 (1.35) 式和 Lorentz 张量的变换规则 (1.62)，可以验证，如下表达式都是 Lorentz 标量（亦即是 Lorentz 不变量）：

$$g_{\mu\nu}T^{\mu\nu}, \quad T^{\mu\nu}A_\mu B_\nu, \quad T^{\mu\nu}T_{\mu\nu}, \quad g_{\mu\sigma}T^{\mu\nu}{}_\rho T^{\sigma\rho}{}_\nu. \tag{1.63}$$

实际上，可以通过缩并若干个 Lorentz 张量的所有指标来构造 Lorentz 不变量。对 (p,q) 型 Lorentz 张量的一个逆变指标和一个协变指标进行缩并，可以得到一个 $(p-1,q-1)$ 型 Lorentz 张量。例如，由

$$T'^{\mu\nu}{}_\mu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma (\Lambda^{-1})^\gamma{}_\mu = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\gamma \delta^\gamma{}_\alpha = \Lambda^\nu{}_\beta T^{\alpha\beta}{}_\alpha \tag{1.64}$$

可知，$T^{\mu\nu}{}_\mu$ 是一个 Lorentz 矢量。

引入四维 **Levi-Civita 符号**

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的偶次置换}, \\ -1, & (\mu,\nu,\rho,\sigma) \text{ 是 } (0,1,2,3) \text{ 的奇次置换}, \\ 0, & \text{其它情况。} \end{cases} \tag{1.65}$$

这样定义出来的 $\varepsilon^{\mu\nu\rho\sigma}$ 是**全反对称**的，即关于任意两个指标反对称，如 $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma} = -\varepsilon^{\rho\nu\mu\sigma} = -\varepsilon^{\sigma\nu\rho\mu}$。它的协变形式为

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.66}$$

根据这些定义，$\varepsilon^{0123} = +1$，$\varepsilon_{0123} = -1$。从而，

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 4!\,\varepsilon^{0123}\varepsilon_{0123} = -4!. \tag{1.67}$$

利用 Levi-Civita 符号可以把 $\det\Lambda$ 按照行列式定义写成

$$\det\Lambda = \Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.68}$$

对于固有 Lorentz 变换，$\det\Lambda = +1$，有

$$\varepsilon^{0123} = \varepsilon^{0123}\det\Lambda = \Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.69}$$

利用 $\varepsilon^{\mu\nu\rho\sigma}$ 的全反对称性质，可得

$$\varepsilon^{1023} = -\varepsilon^{0123} = -\Lambda^0{}_\alpha\Lambda^1{}_\beta\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = -\Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\alpha\beta\gamma\delta} = \Lambda^1{}_\beta\Lambda^0{}_\alpha\Lambda^2{}_\gamma\Lambda^3{}_\delta\varepsilon^{\beta\alpha\gamma\delta}. \tag{1.70}$$

依此类推，可以证明

$$\varepsilon^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta\varepsilon^{\alpha\beta\gamma\delta}. \tag{1.71}$$

可见，在固有 Lorentz 变换下，$\varepsilon^{\mu\nu\rho\sigma}$ 可以看成是一个 4 阶 Lorentz 张量，不过它在任何惯性系中不变。

接下来讨论 Maxwell 方程组在 Lorentz 张量语言中的形式。在 Maxwell 方程组 (1.11) 中，ρ 是电荷密度，$\mathbf{J}$ 是电流密度，它们可以组成一个 Lorentz 矢量 $J^\mu = (\rho, \mathbf{J})$，从而，电流连续性方程

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{1.72}$$

可以写成 Lorentz 不变的形式

$$\partial_\mu J^\mu = 0. \tag{1.73}$$

此外，电场强度 $\mathbf{E}$ 和磁感应强度 $\mathbf{B}$ 可以用电势 Φ 和矢势 $\mathbf{A}$ 表达为

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}. \tag{1.74}$$

这样，方程

$$\nabla\cdot\mathbf{B} = 0 \tag{1.75}$$

是自动满足的。Φ 和 $\mathbf{A}$ 可以组成一个 Lorentz 矢量 $A^\mu = (\Phi, \mathbf{A})$，称为**四维矢势**，则 (1.74) 式的分量形式为

$$E^i = -\partial_i A^0 - \partial_0 A^i, \quad B^k = \varepsilon^{kij}\partial_i A^j, \quad i,j,k = 1,2,3. \tag{1.76}$$

这里的三维 Levi-Civita 符号可以用四维 Levi-Civita 符号定义为

$$\varepsilon^{ijk} \equiv \varepsilon^{0ijk}, \tag{1.77}$$

因而 $\varepsilon^{123} = +1$。

引入电磁场的**场强张量** (field strength tensor)

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{1.78}$$

它是一个 2 阶反对称 Lorentz 张量。由于两个时空导数可以交换次序，从上述定义可得

$$\begin{aligned}\partial^\rho F^{\mu\nu} &= \partial^\rho(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu\partial^\rho A^\nu - \partial^\mu\partial^\nu A^\rho + \partial^\nu\partial^\mu A^\rho - \partial^\nu\partial^\rho A^\mu \\ &= \partial^\mu F^{\rho\nu} + \partial^\nu F^{\mu\rho} = -\partial^\mu F^{\nu\rho} - \partial^\nu F^{\rho\mu},\end{aligned} \tag{1.79}$$

即

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \tag{1.80}$$

$F^{\mu\nu}$ 的 $0i$ 分量为

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \partial_0 A^i + \partial_i A^0 = -E^i, \tag{1.81}$$

可见，F^{0i} 对应于电场强度。由三维 Levi-Civita 符号的全反对称性有 $\varepsilon^{12k}\varepsilon^{12k} = \varepsilon^{123}\varepsilon^{123} = 1$ 和 $\varepsilon^{12k}\varepsilon^{21k} = \varepsilon^{123}\varepsilon^{213} = -1$，依此类推，可以归纳出如下求和关系：

$$\varepsilon^{ijk}\varepsilon^{kmn} = \varepsilon^{ijk}\varepsilon^{mnk} = \delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}, \tag{1.82}$$

利用这个关系，可得

$$\varepsilon^{ijk}B^k = \varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n = \delta^{im}\delta^{jn}\partial_m A^n - \delta^{in}\delta^{jm}\partial_m A^n = \partial_i A^j - \partial_j A^i, \tag{1.83}$$

从而，

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_i A^j + \partial_j A^i = -\varepsilon^{ijk}B^k, \tag{1.84}$$

故 $F^{\mu\nu}$ 的 ij 分量对应于磁感应强度。把 $F^{\mu\nu}$ 写成矩阵形式是

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \tag{1.85}$$

Gauss 定律对应的方程

$$\nabla \cdot \mathbf{E} = \rho \tag{1.86}$$

等价于

$$J^0 = \rho = \partial_i E^i = -\partial_i F^{0i} = \partial_i F^{i0} = \partial_i F^{i0} + \partial_0 F^{00} = \partial_\mu F^{\mu 0}, \tag{1.87}$$

而 Ampère 定律对应的方程

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{1.88}$$

等价于

$$J^i = \varepsilon^{ijk}\partial_j B^k - \partial_0 E^i = -\partial_j F^{ij} + \partial_0 F^{0i} = \partial_j F^{ji} + \partial_0 F^{0i} = \partial_\mu F^{\mu i}. \tag{1.89}$$

归纳起来，有

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.90}$$

这个方程完全是用 Lorentz 张量写出来的，它在不同惯性系中具有相同的形式，即具有 **Lorentz 协变性**，因而满足狭义相对性原理。

现在，Maxwell 方程组中还有一个方程没有讨论，它是 Maxwell–Faraday 方程

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.91}$$

将它写成分量的形式，得

$$\varepsilon^{kmn}\partial_m E^n = -\varepsilon^{kmn}\partial_m F^{0n} = \varepsilon^{kmn}\partial_m F^{n0} = -\partial_0 B^k, \tag{1.92}$$

从而

$$\partial_0 F^{ij} = -\varepsilon^{ijk}\partial_0 B^k = \varepsilon^{ijk}\varepsilon^{kmn}\partial_m F^{n0} = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\partial_m F^{n0} = \partial_i F^{j0} - \partial_j F^{i0}, \tag{1.93}$$

即

$$\partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = 0. \tag{1.94}$$

这个方程与 Maxwell–Faraday 方程等价，不过，它只是前面得到的方程 (1.80) 取特定分量的形式。

利用四维 Levi-Civita 符号，可以定义电磁场的对偶场强张量 (duel field strength tensor)

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{1.95}$$

它也是一个 2 阶反对称 Lorentz 张量。由 $\varepsilon^{1jk}\varepsilon^{1jk} = \varepsilon^{123}\varepsilon^{123} + \varepsilon^{132}\varepsilon^{132} = 2$ 和 $\varepsilon^{1jk}\varepsilon^{2jk} = \varepsilon^{123}\varepsilon^{223} + \varepsilon^{132}\varepsilon^{232} = 0$ 可以归纳出三维 Levi-Civita 符号的另一条求和关系

$$\varepsilon^{ijk}\varepsilon^{ljk} = 2\delta^{il}, \tag{1.96}$$

利用这个关系，可得

$$\begin{aligned}\tilde{F}^{0i} &= \frac{1}{2}\varepsilon^{0i\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{0ijk}F_{jk} = \frac{1}{2}\varepsilon^{0ijk}g_{j\mu}g_{k\nu}F^{\mu\nu} = \frac{1}{2}\varepsilon^{0ijk}g_{jm}g_{kn}F^{mn} = -\frac{1}{2}\varepsilon^{ijk}\delta^{jm}\delta^{kn}\varepsilon^{mnl}B^l \\ &= -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{jkl}B^l = -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}B^l = -\frac{1}{2}2\delta^{il}B^l = -B^i,\end{aligned} \tag{1.97}$$

故 $\tilde{F}^{0i}$ 对应于磁感应强度。另一方面，

$$\tilde{F}^{ij} = \frac{1}{2}\varepsilon^{ij\rho\sigma}F_{\rho\sigma} = \frac{1}{2}(\varepsilon^{ij0k}F_{0k} + \varepsilon^{ijk0}F_{k0}) = \varepsilon^{0ijk}F_{0k} = \varepsilon^{0ijk}g_{0\mu}g_{k\nu}F^{\mu\nu}$$

$$= \varepsilon^{ijk} g_{00} g_{kl} F^{0l} = -\varepsilon^{ijk} \delta^{kl} F^{0l} = -\varepsilon^{ijk} F^{0k} = \varepsilon^{ijk} E^k, \tag{1.98}$$

说明 $\tilde{F}^{ij}$ 对应于电场强度。$\tilde{F}^{\mu\nu}$ 的矩阵形式是

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \tag{1.99}$$

由 $\tilde{F}^{ij}$ 的定义，有

$$\begin{aligned} \partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{2}\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} = -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\sigma\mu\rho}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\rho\sigma\mu}\partial_\mu F_{\rho\sigma}) \\ &= -\frac{1}{6}(\varepsilon^{\nu\mu\rho\sigma}\partial_\mu F_{\rho\sigma} + \varepsilon^{\nu\mu\rho\sigma}\partial_\rho F_{\sigma\mu} + \varepsilon^{\nu\mu\rho\sigma}\partial_\sigma F_{\mu\rho}) = -\frac{1}{6}\varepsilon^{\nu\mu\rho\sigma}(\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho}), \end{aligned} \tag{1.100}$$

因此，方程 (1.80) 等价于

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{1.101}$$

从这些讨论可以看到，用 Lorentz 张量语言表达 Maxwell 方程组是十分简单的，而且方程的 Lorentz 协变性非常明确。

1.6 作用量原理

1.6.1 经典力学中的作用量原理

在经典力学中，质点力学系统可以用**拉格朗日量**（Lagrangian）描述。对于具有 n 个自由度的系统，可以定义 n 个相互独立的广义坐标 (generalized coordinate) q_i，它们的时间导数是广义速度 (generalized velocity) $\dot{q}_i = dq_i/dt$。拉格朗日量是广义坐标和广义速度的函数 $L(q_i, \dot{q}_i)$。拉格朗日量的时间积分

$$S = \int_{t_1}^{t_2} dt\, L[q_i(t), \dot{q}_i(t)] \tag{1.102}$$

称为**作用量**。

作用量原理指出，作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹。假设时间 t 的变分为零，则有

$$\delta\dot{q}_i = \delta\frac{dq_i}{dt} = \frac{d}{dt}\delta q_i, \tag{1.103}$$

即时间导数的变分等于变分的时间导数。从而可得

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt\, \delta L[q_i(t), \dot{q}_i(t)] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right) \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right] \end{aligned}$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta q_i \bigg|_{t_1}^{t_2}, \tag{1.104}$$

其中第四步用了分部积分。再假设初始和结束时刻处广义坐标的变分为零，即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$，则上式最后一行第二项为零，而 $\delta S = 0$ 等价于

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \cdots, n. \tag{1.105}$$

这是 **Euler-Lagrange 方程**，它给出质点系统的经典运动方程。

引入广义动量 (generalized momentum)

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \cdots, n. \tag{1.106}$$

反解上式表示的 n 个方程，则可以用 q_i 和 p_i 将 $\dot{q}_i$ 表达出来，然后用 Legendre 变换定义**哈密顿量** (Hamiltonian)

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L, \tag{1.107}$$

它是 q_i 和 p_i 的函数。可以用 H 取替 L 来表示作用量，变分为

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt\, \delta(p_i \dot{q}_i - H) = \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i \right) \\
&= \int_{t_1}^{t_2} dt \left(\dot{q}_i \delta p_i + p_i \frac{d}{dt}\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i \right) \\
&= \int_{t_1}^{t_2} dt \left[\dot{q}_i \delta p_i + \frac{d}{dt}(p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i \right] \\
&= \int_{t_1}^{t_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] + p_i \delta q_i \bigg|_{t_1}^{t_2}.
\end{aligned} \tag{1.108}$$

根据前面的假设，上式最后一行第二项为零，于是，$\delta S = 0$ 给出

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \cdots, n. \tag{1.109}$$

这是 **Hamilton 正则运动方程**，相当于用 $2n$ 个一阶方程代替原来的 n 个二阶方程 (1.105).

1.6.2 经典场论中的作用量原理

场是时空坐标的函数。在经典场论中，场 $\phi(\mathbf{x}, t)$ 是系统的广义坐标，每一个空间点 $\mathbf{x}$ 都是一个自由度，因此场论相当于具有无穷多自由度的质点力学。在局域场论中，拉格朗日量 $L = \int d^3x\, \mathcal{L}(x)$，其中 $\mathcal{L}(x)$ 称为**拉格朗日量密度**（下文将它简称为**拉氏量**）。$\mathcal{L}$ 是系统中 n 个场 $\phi_a(\mathbf{x}, t)$ $(a = 1, \cdots, n)$ 及其时空导数 $\partial_\mu \phi_a$ 的函数。现在，作用量可以表达为

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{1.110}$$

(1.44) 式告诉我们，时空体积元 d^4x 是 Lorentz 不变的，如果拉氏量 $\mathcal{L}$ 也是 Lorentz 不变的，则作用量 S 就是 Lorentz 不变的，从而，由作用量原理得到的运动方程满足狭义相对性原理。因此，构建相对论性场论的关键在于使用 Lorentz 不变的拉氏量 $\mathcal{L}$，即要求 $\mathcal{L}$ 是一个 *Lorentz* 标量。

类似于前面质点力学的处理方式，假设时空坐标的变分为零，则对场的时空导数的变分等于场变分的时空导数，即

$$\delta(\partial_\mu\phi_a) = \partial_\mu(\delta\phi_a). \tag{1.111}$$

于是，利用分部积分可得

$$\begin{aligned}
\delta S &= \int d^4x\,\delta\mathcal{L} = \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\right] = \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)\right]\\
&= \int d^4x\left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right] - \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a\right\}\\
&= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right].
\end{aligned} \tag{1.112}$$

上式最后一行第二项的积分项是关于时空坐标的全散度，利用 Stokes 定理，可以将它转化为积分区域边界面 $\mathcal{S}$ 上的积分：

$$\int d^4x\,\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right] = \int_{\mathcal{S}} d\mathcal{S}_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a, \tag{1.113}$$

其中 $d\mathcal{S}_\mu$ 是 $\mathcal{S}$ 上的面元。进一步假设在边界面 $\mathcal{S}$ 上 $\delta\phi_a = 0$，则上式为零。我们通常讨论整个时空区域上的场，从而这里相当于假设 ϕ_a 在无穷远时空边界上的变分为零，是很合理的。这样一来，$\delta S = 0$ 给出

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{1.114}$$

这就是场的 *Euler-Lagrange* 方程，它给出场的经典运动方程。

引入场的**共轭动量密度** (conjugate momentum density)

$$\pi_a(\mathbf{x},t) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}, \tag{1.115}$$

则可以用 Legendre 变换将哈密顿量定义为

$$H \equiv \int d^3x\,\pi_a\dot{\phi}_a - L \equiv \int d^3x\,\mathcal{H}, \tag{1.116}$$

其中，**哈密顿量密度**

$$\mathcal{H}(\phi_a,\pi_a,\nabla\phi_a) = \pi_a\dot{\phi}_a - \mathcal{L}. \tag{1.117}$$

作用量变分为

$$\begin{aligned}
\delta S &= \int d^4x\,\delta\mathcal{L} = \int d^4x\,\delta(\pi_a\dot{\phi}_a - \mathcal{H})\\
&= \int d^4x\left[\dot{\phi}_a\delta\pi_a + \pi_a\delta\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\cdot\delta(\nabla\phi_a)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int d^4x \left[\dot{\phi}_a \delta\pi_a + \pi_a \frac{d}{dt}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a - \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\cdot\nabla(\delta\phi_a)\right] \\
&= \int d^4x \left\{\dot{\phi}_a \delta\pi_a + \frac{d}{dt}(\pi_a\delta\phi_a) - \dot{\pi}_a\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a \right. \\
&\qquad\qquad \left. -\nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right] + \left[\nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\delta\phi_a\right\} \\
&= \int d^4x \left\{\left(\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\right)\delta\pi_a - \left[\dot{\pi}_a + \frac{\partial\mathcal{H}}{\partial\phi_a} - \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\delta\phi_a\right\} \\
&\quad + \int d^4x \frac{d}{dt}(\pi_a\delta\phi_a) - \int d^4x\,\nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right]. \qquad (1.118)
\end{aligned}
$$

与前面一样，假设在时空区域边界面上 $\delta\phi_a = 0$，则上式最后一行的两项均为零，于是，$\delta S = 0$ 给出场的正则运动方程

$$\dot{\phi}_a = \frac{\partial\mathcal{H}}{\partial\pi_a}, \quad \dot{\pi}_a = -\frac{\partial\mathcal{H}}{\partial\phi_a} + \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}. \qquad (1.119)$$

1.7 Noether 定理、对称性与守恒定律

若一种对称变换可以用一组连续变化的参数来描述，则它是一种连续变换，连续变换对应的对称性称为连续对称性。**Noether 定理**指出，如果一个系统具有某种不显含时间的连续对称性，就必然存在一种对应的守恒定律。Noether 定理首先是在经典物理中给出的，但实际上它对所有物理行为由作用量原理决定的系统都成立。因此，可以将它推广到量子物理中。

1.7.1 场论中的 Noether 定理

下面在场论中证明 Noether 定理。在时空区域 R 中的作用量为

$$S = \int_R d^4x\,\mathcal{L}(\phi_a, \partial_\mu\phi_a). \qquad (1.120)$$

考虑一个连续变换，使得

$$\phi_a(x) \to \phi'_a(x'), \qquad (1.121)$$

其中已包含了坐标的变换

$$x^\mu \to x'^\mu, \qquad (1.122)$$

它引起的拉氏量变换为

$$\mathcal{L}(x) \to \mathcal{L}'(x'). \qquad (1.123)$$

记这个变换的无穷小变换形式为

$$\phi'_a(x') = \phi_a(x) + \delta\phi_a, \quad x'^\mu = x^\mu + \delta x^\mu, \quad \mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}, \qquad (1.124)$$

如果在此变换下

$$\delta S = \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) = 0, \qquad (1.125)$$

则系统具有相应的连续对称性。

体积元的变化为

$$d^4x' = |\mathcal{J}|d^4x, \quad \mathcal{J} = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) \simeq \det\left[\delta^\mu{}_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right], \tag{1.126}$$

上式中约等于号表示只展开到一阶小量，下同。若方阵 $\mathbf{A}$ 满足 $\det(\mathbf{A}) \ll 1$，则有如下表达式：

$$\det(\mathbf{1} + \mathbf{A}) \simeq 1 + \mathrm{tr}(\mathbf{A}). \tag{1.127}$$

利用上式可以将 Jacobi 行列式 $\mathcal{J}$ 化为

$$\mathcal{J} \simeq 1 + \mathrm{tr}\left[\frac{\partial(\delta x^\mu)}{\partial x^\nu}\right] = 1 + \partial_\mu(\delta x^\mu), \tag{1.128}$$

从而，体积元的无穷小变换形式为

$$d^4x' \simeq [1 + \partial_\mu(\delta x^\mu)]d^4x. \tag{1.129}$$

作用量在此无穷小变换下的变分为

$$\begin{aligned}
\delta S &= \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) \\
&= \int_{R'} d^4x'\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}'(x') + \int_R d^4x\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}(x) \\
&\simeq \int_R d^4x[1 + \partial_\mu(\delta x^\mu)]\,\mathcal{L}'(x') - \int_R d^4x\,\mathcal{L}'(x') + \int_R d^4x\,\delta\mathcal{L} \\
&\simeq \int_R d^4x\,\mathcal{L}'(x')\partial_\mu(\delta x^\mu) + \int_R d^4x\,\delta\mathcal{L} \;\simeq \int_R d^4x\,[\delta\mathcal{L} + \mathcal{L}(x)\partial_\mu(\delta x^\mu)] \\
&= \int_R d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) + \mathcal{L}\partial_\mu(\delta x^\mu)\right].
\end{aligned} \tag{1.130}$$

记 x^μ 固定时的变分算符为 $\bar{\delta}$，使得

$$\bar{\delta}\phi_a(x) = \phi'_a(x) - \phi_a(x). \tag{1.131}$$

$\bar{\delta}$ 算符可以与时空导数交换，

$$\bar{\delta}(\partial_\mu\phi_a) = \partial_\mu(\bar{\delta}\phi_a), \tag{1.132}$$

δ 算符则不能。$\delta\phi_a$ 与 $\bar{\delta}\phi_a$ 的关系为

$$\begin{aligned}
\delta\phi_a &= \phi'_a(x') - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \phi'_a(x) - \phi_a(x) = \phi'_a(x') - \phi'_a(x) + \bar{\delta}\phi_a \\
&\simeq \bar{\delta}\phi_a + (\partial_\mu\phi'_a)\delta x^\mu \simeq \bar{\delta}\phi + (\partial_\mu\phi_a)\delta x^\mu,
\end{aligned} \tag{1.133}$$

即

$$\bar{\delta}\phi = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu. \tag{1.134}$$

同理，

$$\delta(\partial_\mu\phi_a) = \bar{\delta}(\partial_\mu\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu = \partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu. \tag{1.135}$$

将 (1.133) 和 (1.135) 式代入 (1.130) 式，得到

$$\begin{aligned}
\delta S &= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}[\bar{\delta}\phi_a + (\partial_\mu\phi_a)\delta x^\mu] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}[\partial_\mu(\bar{\delta}\phi_a) + \partial_\nu(\partial_\mu\phi_a)\delta x^\nu] + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} \\
&= \int_R d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\bar{\delta}\phi_a + \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\bar{\delta}\phi_a \right. \\
&\qquad \left. + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu) \right\} \\
&= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a\right] \right. \\
&\qquad \left. + \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu)\right] \right\} \\
&= \int_R d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\bar{\delta}\phi_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] \right\}.
\end{aligned} \tag{1.136}$$

第二步用到分部积分，最后一步用到求导关系式

$$\frac{\partial}{\partial x^\mu}(\mathcal{L}\delta x^\mu) = \frac{\partial\mathcal{L}}{\partial\phi_a}\frac{\partial\phi_a}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\frac{\partial(\partial_\nu\phi_a)}{\partial x^\mu}\delta x^\mu + \mathcal{L}\frac{\partial}{\partial x^\mu}(\delta x^\mu). \tag{1.137}$$

根据 Euler-Lagrange 方程 (1.114)，(1.136) 式最后一行花括号中第一项为零。由于积分区域 R 可以是任意的，$\delta S = 0$ 等价于第二项为零，即

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu\right] = 0. \tag{1.138}$$

定义 **Noether 守恒流** (conserved current)

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu, \tag{1.139}$$

则有守恒流方程

$$\partial_\mu j^\mu = 0. \tag{1.140}$$

方程 (1.140) 左边对整个三维空间积分，运用 Stokes 定理，得

$$\int d^3x\,\partial_\mu j^\mu = \int d^3x\,\partial_0 j^0 + \int d^3x\,\partial_i j^i = \frac{d}{dt}\int d^3x\,j^0 + \int_{\mathcal{S}} d\mathcal{S}_i\,j^i, \tag{1.141}$$

其中 $i = 1, 2, 3$。对于整个三维空间而言，边界面 $\mathcal{S}$ 位于无穷远处。通常假设场 ϕ_a 在无穷远处消失，从而，在无穷远处 $j^i \to 0$，所以上式最后一项为零。定义**守恒荷** (conserved charge)

$$Q \equiv \int d^3x\,j^0, \tag{1.142}$$

则由 (1.141) 和 (1.140) 式可得

$$\frac{dQ}{dt} = \frac{d}{dt}\int d^3x\,j^0 = \int d^3x\,\partial_\mu j^\mu = 0. \tag{1.143}$$

可见，Q 不随时间变化，是守恒的。

综上，在场论中，如果一个系统具有某种连续对称性，则存在相应的守恒流 (1.139)，它满足守恒流方程 (1.140)，而守恒荷 (1.142) 不随时间变化。下面举一些应用 Noether 定理的例子。

1.7.2　时空平移对称性

考虑时空坐标的无穷小平移变换

$$x'^{\mu} = x^{\mu} - \varepsilon^{\mu}, \tag{1.144}$$

其中 ε^{μ} 是常数。要求场 ϕ_a 具有时空平移对称性，则

$$\phi'_a(x') = \phi'_a(x - \varepsilon) = \phi_a(x). \tag{1.145}$$

现在，$\delta x^{\mu} = -\varepsilon^{\mu}$，由 (1.134) 式可得

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^{\mu} = \phi'_a(x') - \phi_a(x) + \varepsilon^{\mu}\partial_\mu\phi_a = 0 + \varepsilon^{\mu}\partial_\mu\phi_a = \varepsilon^{\rho}\partial_\rho\phi_a, \tag{1.146}$$

代入到 Noether 守恒流表达式 (1.139)，得

$$j^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\varepsilon^{\rho}\partial_\rho\phi_a - \mathcal{L}\varepsilon^{\mu} = \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^{\mu}{}_{\rho}\mathcal{L}\right]\varepsilon^{\rho}. \tag{1.147}$$

从而，$\partial_\mu j^{\mu} = 0$ 给出

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\rho\phi_a - \delta^{\mu}{}_{\rho}\mathcal{L}\right] = 0, \tag{1.148}$$

各项乘以 $g^{\rho\nu}$，缩并，得

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^{\nu}\phi_a - g^{\mu\nu}\mathcal{L}\right] = 0. \tag{1.149}$$

上式方括号部分是场的**能动张量** (energy-momentum tensor)

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial^{\nu}\phi_a - g^{\mu\nu}\mathcal{L}, \tag{1.150}$$

它满足

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.151}$$

因此，对 $T^{0\nu}$ $(\nu = 0, 1, 2, 3)$ 作全空间积分，就可以得到 4 个守恒荷。

$T^{\mu\nu}$ 的 00 分量为

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^{0}\phi_a - \mathcal{L}, \tag{1.152}$$

与 (1.117) 和 (1.115) 式比较，可以看出 T^{00} 就是哈密顿量密度 $\mathcal{H}$。T^{00} 的全空间积分

$$H = \int d^3x\, T^{00} = \int d^3x\, \mathcal{H} \tag{1.153}$$

是场的哈密顿量，或者说**总能量**。$T^{\mu\nu}$ 的 $0i$ 分量

$$T^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_a)}\partial^i\phi_a = \pi_a\partial^i\phi_a \tag{1.154}$$

是场的动量密度，它的全空间积分

$$P^i = \int d^3x\, T^{0i} = \int d^3x\, \pi_a\partial^i\phi_a \tag{1.155}$$

是场的**总动量**。根据 (1.55) 式，上式也可以写成

$$\mathbf{P} = -\int d^3x\, \pi_a\nabla\phi_a. \tag{1.156}$$

H 和 P^i 都是守恒荷，可见，时间平移对称性对应于**能量守恒定律**，空间平移对称性对应于**动量守恒定律**。

1.7.3 Lorentz 对称性

考虑无穷小固有保时向 Lorentz 变换

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{1.157}$$

其中 $\omega^\mu{}_\nu$ 是变换的无穷小参数。由保度规条件 (1.30)，有

$$\begin{aligned} g_{\alpha\beta} &= g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\mu\nu}(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha)(\delta^\nu{}_\beta + \omega^\nu{}_\beta) \simeq g_{\mu\nu}\delta^\mu{}_\alpha\delta^\nu{}_\beta + g_{\mu\nu}\delta^\mu{}_\alpha\omega^\nu{}_\beta + g_{\mu\nu}\omega^\mu{}_\alpha\delta^\nu{}_\beta \\ &= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha}, \end{aligned} \tag{1.158}$$

可见，

$$\omega_{\mu\nu} \equiv g_{\mu\rho}\omega^\rho{}_\nu \tag{1.159}$$

关于两个指标反对称：

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{1.160}$$

因此，$\omega_{\mu\nu}$ 只有 6 个独立分量。

下面举两个例子说明 $\omega_{\mu\nu}$ 的具体形式。对于绕 z 轴旋转 θ 角的变换 (1.31)，利用三角函数展开式 $\cos\theta = 1 + \mathcal{O}(\theta^2)$ 和 $\sin\theta = \theta + \mathcal{O}(\theta^3)$，可得

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & \theta & \\ & -\theta & 0 & \\ & & & 0 \end{pmatrix}, \quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}. \tag{1.161}$$

对于沿 x 的增速变换 (1.28)，可以先定义快度 (rapidity)

$$\xi \equiv \tanh^{-1}\beta, \tag{1.162}$$

再利用双曲函数公式 $\tanh\xi = \sinh\xi/\cosh\xi$ 和 $\cosh^2\xi - \sinh^2\xi = 1$ 得

$$\begin{aligned}\gamma &= (1-\beta^2)^{-1/2} = (1-\tanh^2\xi)^{-1/2} = \left(\frac{\cosh^2\xi - \sinh^2\xi}{\cosh^2\xi}\right)^{-1/2} = \cosh\xi,\\ \beta\gamma &= \tanh\xi\cosh\xi = \sinh\xi,\end{aligned}\tag{1.163}$$

从而将 (1.28) 式改写成

$$\Lambda^\mu{}_\nu = \begin{pmatrix}\cosh\xi & -\sinh\xi & & \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ & & & 1\end{pmatrix}.\tag{1.164}$$

根据双曲函数展开式 $\cosh\xi = 1 + \mathcal{O}(\xi^2)$ 和 $\sinh\xi = \xi + \mathcal{O}(\xi^3)$，有

$$\omega^\mu{}_\nu = \begin{pmatrix}0 & -\xi & & \\ -\xi & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix},\quad \omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu = \begin{pmatrix}0 & -\xi & & \\ \xi & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}.\tag{1.165}$$

在无穷小 Lorentz 变换 (1.157) 的作用下，一般地，场的变换可以写成

$$\phi'_a(x') = \left[\delta_{ab} - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\right]\phi_b(x) = \phi_a(x) - \frac{i}{2}\omega_{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x),\tag{1.166}$$

其中 $I^{\mu\nu}$ 是 ϕ_a 所属 Lorentz 群线性表示的**生成元** (generator)。由于 $\omega_{\mu\nu}$ 是反对称的，有

$$\omega_{\mu\nu}(I^{\mu\nu})_{ab} = \omega_{\nu\mu}(I^{\nu\mu})_{ab} = -\omega_{\mu\nu}(I^{\nu\mu})_{ab},\tag{1.167}$$

因而 $(I^{\mu\nu})_{ab}$ 也应该关于 μ 和 ν 反对称：

$$(I^{\mu\nu})_{ab} = -(I^{\nu\mu})_{ab}.\tag{1.168}$$

现在，$\delta x^\mu = \omega^\mu{}_\nu x^\nu$，而

$$\bar{\delta}\phi_a = \delta\phi_a - (\partial_\mu\phi_a)\delta x^\mu = \phi'_a(x') - \phi_a(x) - (\partial_\mu\phi_a)\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}(I^{\nu\rho})_{ab}\phi_b - (\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho,\tag{1.169}$$

故 Noether 流为

$$\begin{aligned}j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu = -\frac{i}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(I^{\nu\rho})_{ab}\phi_b - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a)\omega^\nu{}_\rho x^\rho + \mathcal{L}\omega^\mu{}_\rho x^\rho\\ &= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(\partial_\nu\phi_a) - \delta^\mu{}_\nu\mathcal{L}\right]\omega^\nu{}_\rho x^\rho\\ &= \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - T^\mu{}_\nu\omega^\nu{}_\rho x^\rho,\end{aligned}\tag{1.170}$$

其中

$$T^\mu{}_\nu \equiv T^{\mu\rho}g_{\rho\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L}\tag{1.171}$$

是能动张量的另一种写法。利用度规可以进行如下指标升降操作：

$$T^\mu{}_\nu\omega^\nu{}_\rho = T^\mu{}_\nu\delta^\nu{}_\sigma\omega^\sigma{}_\rho = T^\mu{}_\nu g^{\nu\alpha}g_{\alpha\sigma}\omega^\sigma{}_\rho = T^{\mu\alpha}\omega_{\alpha\rho} = T^{\mu\nu}\omega_{\nu\rho}, \tag{1.172}$$

即参与缩并的指标一升一降不会改变表达式的结果。再利用 $\omega_{\mu\nu}$ 的反对称性可得

$$\begin{aligned} T^\mu{}_\nu\omega^\nu{}_\rho x^\rho &= T^{\mu\nu}\omega_{\nu\rho}x^\rho = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\nu}\omega_{\rho\nu}x^\rho) = \frac{1}{2}(T^{\mu\nu}\omega_{\nu\rho}x^\rho - T^{\mu\rho}\omega_{\nu\rho}x^\nu) \\ &= \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu). \end{aligned} \tag{1.173}$$

于是，Noether 流 (1.170) 可化为

$$j^\mu = \frac{1}{2}\omega_{\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b - \frac{1}{2}\omega_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu) = \frac{1}{2}J^{\mu\nu\rho}\omega_{\nu\rho} \tag{1.174}$$

其中

$$J^{\mu\nu\rho} \equiv T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b. \tag{1.175}$$

$\partial_\mu j^\mu = 0$ 给出

$$\partial_\mu J^{\mu\nu\rho} = 0, \tag{1.176}$$

守恒荷为

$$J^{\nu\rho} \equiv \int d^3x\, J^{0\nu\rho} = \int d^3x \left[T^{0\rho}x^\nu - T^{0\nu}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b\right]. \tag{1.177}$$

易见 $J^{\nu\rho} = -J^{\rho\nu}$，因而一共有 6 个独立的守恒荷，满足 $dJ^{\nu\rho}/dt = 0$。

为明确物理含义，可将 $J^{\nu\rho}$ 分解成两项：

$$J^{\nu\rho} = L^{\nu\rho} + S^{\nu\rho}. \tag{1.178}$$

第一项为

$$\begin{aligned} L^{\nu\rho} &\equiv \int d^3x\,(T^{0\rho}x^\nu - T^{0\nu}x^\rho) \\ &= \int d^3x \left[\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\rho\phi_a - g^{0\rho}\mathcal{L}\right)x^\nu - \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}\partial^\nu\phi_a - g^{0\nu}\mathcal{L}\right)x^\rho\right] \\ &= \int d^3x\,[(\pi_a\partial^\rho\phi_a - g^{0\rho}\mathcal{L})x^\nu - (\pi_a\partial^\nu\phi_a - g^{0\nu}\mathcal{L})x^\rho] \\ &= \int d^3x\,[\pi_a(x^\nu\partial^\rho - x^\rho\partial^\nu)\phi_a + (g^{0\nu}x^\rho - g^{0\rho}x^\nu)\mathcal{L}]. \end{aligned} \tag{1.179}$$

它的纯空间分量 L^{jk} 中只有 3 个是独立的，可以等价地定义成

$$L^i \equiv \frac{1}{2}\varepsilon^{ijk}L^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\,\pi_a(x^j\partial^k - x^k\partial^j)\phi_a, \tag{1.180}$$

这是场的**轨道角动量**。第二项为

$$S^{\nu\rho} \equiv \int d^3x\,\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)}(-iI^{\nu\rho})_{ab}\phi_b = \int d^3x\,\pi_a(-iI^{\nu\rho})_{ab}\phi_b, \tag{1.181}$$

同样，3 个独立的等价纯空间分量是

$$S^i \equiv \frac{1}{2}\varepsilon^{ijk}S^{jk} = \frac{1}{2}\varepsilon^{ijk}\int d^3x\,\pi_a(-iI^{jk})_{ab}\phi_b, \tag{1.182}$$

这是场的**自旋角动量**。因此，$J^{\nu\rho}$ 的纯空间分量等价于

$$J^i \equiv \frac{1}{2}\varepsilon^{ijk}J^{jk} = L^i + S^i, \tag{1.183}$$

这是场的**总角动量**。固有保时向 Lorentz 群的纯空间部分就是空间旋转群 SO(3)，而空间旋转对称性对应于**角动量守恒定律**。

另一方面，$L^{\nu\rho}$ 的 $i0$ 分量为

$$L^{i0} = \int d^3x\,(T^{00}x^i - T^{0i}x^0) = \int d^3x\,(x^i\mathcal{H} - x^0\pi_a\partial^i\phi_a) = \int d^3x\,x^i\mathcal{H} - tP^i. \tag{1.184}$$

若 $dS^{i0}/dt = 0$，则有 $dL^{i0}/dt = 0$，从而

$$L^{i0}(t) = L^{i0}\Big|_{t=0} = \int d^3x\,x^i\mathcal{H}(t=0), \tag{1.185}$$

这是场在 $t=0$ 时刻的能量中心。在低速极速下，能量密度相当于质量密度，则 L^{i0} 是 $t=0$ 时刻的质心 (即质量中心，center of mass)。L^{i0} 的守恒在经典力学中对应于质心运动守恒定律：当没有外力存在时，质心的加速度为零，质心保持静止或作匀速直线运动。

1.7.4　U(1) 整体对称性

考虑一个包含复场 $\phi(x)$ 及其复共轭 $\phi^*(x)$ 的拉氏量

$$\mathcal{L} = (\partial^\mu\phi^*)\partial_\mu\phi - m^2\phi^*\phi. \tag{1.186}$$

对 ϕ 作 U(1) 整体变换

$$\phi'(x) = e^{iq\theta}\phi(x), \tag{1.187}$$

其中 θ 是不依赖于 x^μ 的连续变换实参数，q 是一个常数。这里不包含坐标的变换。$e^{iq\theta}$ 是个纯相位因子，可以看成是一个 1 维幺正 (unitary) 矩阵，形式为 $e^{iq\theta}$ 的所有变换组成的群称为 **U(1) 群**。**整体** (global) 指的是变换参数不依赖于时空坐标。相应地，ϕ^* 的 U(1) 整体变换形式为

$$[\phi^*(x)]' = [\phi'(x)]^* = e^{-iq\theta}\phi^*(x). \tag{1.188}$$

容易看出，由 (1.186) 式定义的 $\mathcal{L}$ 在这种变换下不变，即具有 U(1) 整体对称性。与前面叙述的两种对称性不同，这里的对称性出现在由场组成的抽象空间中，与时间和空间相对独立 ($\delta x^\mu = 0$)，因而是一种**内部对称性**。

U(1) 整体变换的无穷小形式为

$$\phi'(x) = \phi(x) + iq\theta\phi(x), \quad [\phi^*(x)]' = \phi^*(x) - iq\theta\phi^*(x), \tag{1.189}$$

结合 $\delta x^\mu = 0$，有

$$\bar{\delta}\phi = \delta\phi = iq\theta\phi, \quad \bar{\delta}\phi^* = \delta\phi^* = -iq\theta\phi^*, \tag{1.190}$$

于是，Noether 流为

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\bar{\delta}\phi^* = \partial^\mu\phi^*(iq\theta\phi) + \partial^\mu\phi(-iq\theta\phi^*) \\ &= iq\theta[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*] = -q\theta\phi^* i\overleftrightarrow{\partial^\mu}\phi, \end{aligned} \tag{1.191}$$

其中，$\overleftrightarrow{\partial^\mu}$ 符号通过下式定义：

$$\phi^*\overleftrightarrow{\partial^\mu}\phi \equiv \phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi. \tag{1.192}$$

扔掉无穷小参数 $-\theta$，定义

$$J^\mu \equiv q\phi^* i\overleftrightarrow{\partial^\mu}\phi, \tag{1.193}$$

则 Noether 定理给出 $\partial_\mu J^\mu = 0$，相应的守恒荷为

$$Q = \int d^3x\, J^0 = q\int d^3x\, \phi^* i\overleftrightarrow{\partial^0}\phi. \tag{1.194}$$

在实际情况中，q 是由 ϕ 场描述的粒子所携带的某种荷，如电荷、重子数、轻子数、奇异数、粲数、底数、顶数等。因此，一种 U(1) 整体对称性对应于一条荷数守恒定律，比如，电磁 U(1) 整体对称性就对应于**电荷守恒定律**。

第 2 章　标量场

本章讲述标量场的**正则量子化** (canonical quantization) 方法。标量场的量子化可以看作简谐振子量子化的推广，因此，我们先来回顾一下简谐振子的正则量子化程序。

2.1　简谐振子的正则量子化

一维简谐振子 (simple harmonic oscillator) 的哈密顿量可以表达为

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{2.1}$$

其中 m 是质量，ω 是角频率。第一项是动能，第二项是势能。在量子力学中，把坐标 x 和动量 p 当成厄米算符，满足**正则对易关系**

$$[x, p] = xp - px = i. \tag{2.2}$$

可以用 x 和 p 构造两个非厄米的无量纲算符

$$a = \frac{1}{\sqrt{2m\omega}}(m\omega x + ip), \quad a^\dagger = \frac{1}{\sqrt{2m\omega}}(m\omega x - ip). \tag{2.3}$$

a 称为**湮灭算符** (annihilation operator)，$a^\dagger$ 称为**产生算符** (creation operator)，两者互为厄米共轭 (Hermitian conjugate)。它们的对易关系为

$$\begin{aligned}
[a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] = \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\
&= \frac{1}{2}(-i[x, p] + i[p, x]) = -i[x, p] = 1.
\end{aligned} \tag{2.4}$$

根据 (2.3) 式，可以反过来用 a 和 $a^\dagger$ 表示 x 和 p：

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger). \tag{2.5}$$

从而，哈密顿量表示成

$$\begin{aligned}
H &= -\frac{1}{2m}\frac{m\omega}{2}(a - a^\dagger)^2 + \frac{1}{2}m\omega^2\frac{1}{2m\omega}(a + a^\dagger)^2 \\
&= -\frac{\omega}{4}(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger) + \frac{\omega}{4}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger) = \frac{\omega}{2}(aa^\dagger + a^\dagger a).
\end{aligned} \tag{2.6}$$

由对易关系 (2.4) 可得 $aa^\dagger = a^\dagger a + 1$，于是

$$H = \frac{\omega}{2}(2a^\dagger a + 1) = \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right), \tag{2.7}$$

其中，$N \equiv a^\dagger a$ 是个厄米算符，称为**粒子数算符**。N 还是个正定算符，对于任意态 $|\psi\rangle$，N 的平均值非负：

$$\langle\psi| N |\psi\rangle = \langle\psi| a^\dagger a |\psi\rangle = \langle a\psi|a\psi\rangle \geq 0. \tag{2.8}$$

设 $|n\rangle$ 是 N 的本征态，满足本征方程

$$N |n\rangle = n |n\rangle . \tag{2.9}$$

由 $n = \langle n| N |n\rangle = \langle n| a^\dagger a |n\rangle \geq 0$ 可知，n 是个非负实数。利用对易子公式

$$[AB, C] = ABC - ACB + ACB - CAB = A[B,C] + [A,C]B, \tag{2.10}$$

$$[A, BC] = ABC - BAC + BAC - BCA = [A,B]C + B[A,C], \tag{2.11}$$

可得

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = a^\dagger, \quad [N, a] = [a^\dagger a, a] = [a^\dagger, a]a = -a, \tag{2.12}$$

从而，有

$$Na^\dagger |n\rangle = ([N, a^\dagger] + a^\dagger N) |n\rangle = (a^\dagger + a^\dagger n) |n\rangle = (n+1)a^\dagger |n\rangle , \tag{2.13}$$

$$Na |n\rangle = ([N, a] + aN) |n\rangle = (-a + an) |n\rangle = (n-1)a |n\rangle . \tag{2.14}$$

可见，$a^\dagger |n\rangle$ 和 $a |n\rangle$ 都是 N 的本征态，本征值分别为 $n+1$ 和 $n-1$，也就是说，

$$a^\dagger |n\rangle = c_1 |n+1\rangle , \quad a |n\rangle = c_2 |n-1\rangle , \tag{2.15}$$

其中 c_1 和 c_2 是两个归一化常数。$a^\dagger$ 将本征值为 n 的态变成本征值为 $n+1$ 的态，因而也称为升算符 (raising operator)；a 将本征值为 n 的态变成本征值为 $n-1$ 的态，因而也称为降算符 (lowering operator)。为确定归一化常数的值，可作如下计算：

$$n+1 = \langle n| (N+1) |n\rangle = \langle n| (a^\dagger a + 1) |n\rangle = \langle n| aa^\dagger |n\rangle = |c_1|^2 \langle n+1|n+1\rangle = |c_1|^2, \tag{2.16}$$

$$n = \langle n| N |n\rangle = \langle n| a^\dagger a |n\rangle = |c_2|^2 \langle n+1|n+1\rangle = |c_2|^2. \tag{2.17}$$

将 c_1 和 c_2 都取为实数，则有 $c_1 = \sqrt{n+1}$ 和 $c_2 = \sqrt{n}$，故

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \quad a |n\rangle = \sqrt{n} |n-1\rangle . \tag{2.18}$$

从 N 的某个本征态 $|n\rangle$ 出发，用降算符 a 逐步操作，可得本征值逐次减小的一系列本征态

$$a |n\rangle , \ a^2 |n\rangle , \ a^3 |n\rangle , \ \cdots , \tag{2.19}$$

本征值分别为

$$n-1,\ n-2,\ n-3,\ \cdots \tag{2.20}$$

由于 $n \geq 0$，必定存在一个最小本征值 n_0，它的本征态 $|n_0\rangle$ 满足

$$a\,|n_0\rangle = 0. \tag{2.21}$$

于是，有

$$N\,|n_0\rangle = a^\dagger a\,|n_0\rangle = 0 = 0\,|n_0\rangle\,, \tag{2.22}$$

可见，$n_0 = 0$，即

$$|n_0\rangle = |0\rangle\,. \tag{2.23}$$

反过来，从 $|0\rangle$ 出发，用升算符 $a^\dagger$ 逐步操作，可得本征值逐次增加的一系列本征态

$$a^\dagger\,|0\rangle\,,\ (a^\dagger)^2\,|0\rangle\,,\ (a^\dagger)^3\,|0\rangle\,,\ \cdots, \tag{2.24}$$

本征值分别为

$$1,\ 2,\ 3,\ \cdots \tag{2.25}$$

综上，本征值 n 的取值是非负整数，是量子化的；本征态 $|n\rangle$ 可以用 $a^\dagger$ 和 $|0\rangle$ 表示为

$$|n\rangle = c_3(a^\dagger)^n\,|0\rangle\,. \tag{2.26}$$

为确定归一化常数 c_3，可作如下运算：

$$\begin{aligned}\langle n|n\rangle &= |c_3|^2\,\langle 0|\,a^n(a^\dagger)^n\,|0\rangle = |c_3|^2\,\langle 1|\,a^{n-1}(a^\dagger)^{n-1}\,|1\rangle = 1\cdot 2\,|c_3|^2\,\langle 2|\,a^{n-2}(a^\dagger)^{n-2}\,|2\rangle = \cdots \\ &= (n-1)!\,|c_3|^2\,\langle n-1|\,aa^\dagger\,|n-1\rangle = n!\,|c_3|^2\,\langle n|n\rangle\,,\end{aligned} \tag{2.27}$$

故 $|c_3|^2 = 1/n!$。取 c_3 为实数，可得 $c_3 = 1/\sqrt{n!}$，于是

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle\,. \tag{2.28}$$

从 (2.7) 式容易看出，$|n\rangle$ 也是 H 的本征态：

$$H\,|n\rangle = \omega\left(N+\frac{1}{2}\right)|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle = E_n\,|n\rangle\,, \tag{2.29}$$

相应的能量本征值为

$$E_n = \omega\left(n+\frac{1}{2}\right). \tag{2.30}$$

基态 $|0\rangle$ 的能量本征值不是零，而是 $E_0 = \omega/2$，称为零点能 (zero-point energy)，这是量子力学的特有结果。我们可以将 $|0\rangle$ 看作真空态，将 $n>0$ 的 $|n\rangle$ 看作包含 n 个声子 (phonon) 的激发态，每个声子具有一份能量 ω。这样一来，n 表示声子的数目，故粒子数算符 N 描述的是声子数。$a^\dagger$ 的作用是产生一个声子，从而增加一份能量；a 的作用是湮灭一个声子，从而减少一份能量。这是将 $a^\dagger$ 和 a 称为产生算符和湮灭算符的原因。

2.2　场论中的正则对易关系

在量子力学中，Schrödinger 绘景和 Heisenberg 绘景提供了两种等价的描述方法，它们之间可以通过一个幺正变换联系起来。在 Schrödinger 绘景中，波函数 $\psi_\mathrm{S}(t)$ 代表随时间演化的物理态，而任意算符 O_S 不依赖于时间。在 Heisenberg 绘景中，波函数 ψ_H 代表不随时间演化的物理态，它与 $\psi_\mathrm{S}(t)$ 的关系为

$$\psi_\mathrm{H} = e^{iHt}\psi_\mathrm{S}(t), \tag{2.31}$$

其中 H 是哈密顿量；而算符 $O_\mathrm{H}(t)$ 依赖于时间，通过下式与 O_S 联系起来：

$$O_\mathrm{H}(t) = e^{iHt}O_\mathrm{S}(t)e^{-iHt}. \tag{2.32}$$

上一节的量子化可以认为是在 Schrödinger 绘景中实现的，因为我们没有考虑坐标算符 x 和动量算符 p 的时间依赖性。将正则对易关系 (2.2) 改记为 $[x_\mathrm{S}, p_\mathrm{S}] = i$，它在 Heisenberg 绘景中的形式为

$$\begin{aligned}
[x_\mathrm{H}(t), p_\mathrm{H}(t)] &= [e^{iHt}x_\mathrm{S}e^{-iHt}, e^{iHt}p_\mathrm{S}e^{-iHt}] = e^{iHt}x_\mathrm{S}e^{-iHt}e^{iHt}p_\mathrm{S}e^{-iHt} - e^{iHt}p_\mathrm{S}e^{-iHt}e^{iHt}x_\mathrm{S}e^{-iHt}\\
&= e^{iHt}x_\mathrm{S}p_\mathrm{S}e^{-iHt} - e^{iHt}p_\mathrm{S}x_\mathrm{S}e^{-iHt} = e^{iHt}[x_\mathrm{S}, p_\mathrm{S}]e^{-iHt} = e^{iHt}ie^{-iHt} = i.
\end{aligned} \tag{2.33}$$

可见，正则对易关系的形式不依赖于绘景。(2.33) 式是在同一时刻 t 成立的，称为**等时** (equal time) 对易关系。

将讨论推广到自由度为 n 的系统，记 $q_i(t)$ 为系统在 Heisenberg 绘景中的广义坐标算符，$p_i(t)$ 为相应的广义动量算符。由于不同自由度不应该相互影响，这些算符需要满足如下等时对易关系：

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [q_i(t), q_j(t)] = 0, \quad [p_i(t), p_j(t)] = 0. \tag{2.34}$$

1.1 节提到，在量子场论中，为了平等地处理时间和空间，空间坐标 $\mathbf{x}$ 应该与时间坐标 t 一样作为量子场算符 $\phi(\mathbf{x}, t)$ 的参数。由于这里量子场作为算符是依赖于时间的，使用 Heisenberg 绘景会比较合适。接下来的讨论在 Heisenberg 绘景中进行，省略绘景的标志性下标 H。

场论讨论的是无穷多自由度的系统，每一个空间点 $\mathbf{x}$ 上的 $\phi(\mathbf{x}, t)$ 都是一个广义坐标。为了从有限可数个自由度过渡到无穷多个自由度，我们可以先将空间离散化，划分成 n 个小体积元 V_i，然后再取 $V_i \to 0$ 的极限来得到 $n \to \infty$ 的结果。在体积元 V_i 中，定义相应的广义坐标为

$$\phi_i(t) \equiv \frac{1}{V_i}\int_{V_i} d^3x\, \phi(\mathbf{x}, t), \tag{2.35}$$

它是场 $\phi(\mathbf{x}, t)$ 在 V_i 中的平均值。将拉格朗日量密度 $\mathcal{L}(\phi, \partial_\mu\phi)$ 在小体积元 V_i 中的平均值记为

$$\mathcal{L}_i \equiv \frac{1}{V_i}\int_{V_i} d^3x\, \mathcal{L}(\phi, \partial_\mu\phi), \tag{2.36}$$

当体积元取得足够小时，它就成为 ϕ_i 和 $\partial_0\phi_i$ 的函数 $\mathcal{L}_i(\phi_i, \partial_0\phi_i)$。拉格朗日量可表达为

$$L = \int d^3x\, \mathcal{L} = \sum_i \int_{V_i} d^3x\, \mathcal{L} = \sum_i V_i \frac{1}{V_i}\int_{V_i} d^3x\, \mathcal{L} = \sum_i V_i\, \mathcal{L}_i(\phi_i, \partial_0\phi_i). \tag{2.37}$$

于是，由 (1.106) 式定义的广义动量为

$$\Pi_i(t) = \frac{\partial L}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \frac{\partial \mathcal{L}_j}{\partial[\partial_0\phi_i(t)]} = \sum_j V_j \delta_{ji} \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]} = V_i \pi_i(t), \quad (2.38)$$

其中，

$$\pi_i(t) \equiv \frac{\partial \mathcal{L}_i}{\partial[\partial_0\phi_i(t)]}. \quad (2.39)$$

现在，等时对易关系变成

$$[\phi_i(t), \Pi_j(t)] = i\delta_{ij}, \quad [\phi_i(t), \phi_j(t)] = 0, \quad [\Pi_i(t), \Pi_j(t)] = 0. \quad (2.40)$$

第一条和第三条关系可以用 $\pi_i(t)$ 表达为

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{V_j}, \quad [\pi_i(t), \pi_j(t)] = 0. \quad (2.41)$$

对于任意连续函数 $f(x)$，**Dirac δ 函数** $\delta(x)$ 使下式成立：

$$f(x) = \int dy\, f(y)\delta(x-y). \quad (2.42)$$

函数 $\delta(x)$ 只在 $x=0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即

$$\delta(x) = \delta(-x), \quad (2.43)$$

而且满足

$$\int dx\, \delta(x) = 1. \quad (2.44)$$

定义三维 δ 函数为

$$\delta^{(3)}(\mathbf{x}) = \delta(x^1)\delta(x^2)\delta(x^3), \quad (2.45)$$

则对于任意连续函数 $f(\mathbf{x})$，下式成立：

$$f(\mathbf{x}) = \int d^3y\, f(\mathbf{y})\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.46)$$

类似地，函数 $\delta^{(3)}(\mathbf{x})$ 只在 $\mathbf{x}=0$ 处非零，是关于 $\mathbf{x}$ 的偶函数，即 $\delta^{(3)}(\mathbf{x}) = \delta^{(3)}(-\mathbf{x})$，而且满足 $\int d^3x\, \delta^{(3)}(\mathbf{x}) = 1$。

设 f_i 是 $f(\mathbf{x})$ 在 V_i 上的平均值，则它会满足

$$f_i = \sum_j f_j\, \delta_{ij} = \sum_j V_j\, f_j\, \frac{\delta_{ij}}{V_j}. \quad (2.47)$$

(2.46) 式是 (2.47) 式在 $V_i \to 0$ 时的极限。可见，在 $V_i \to 0$ 极限下，

$$\frac{\delta_{ij}}{V_j} \to \delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (2.48)$$

另一方面，在此极限下，$\phi_i(t) \to \phi(\mathbf{x}, t)$，而 $\pi_i(t)$ 变成由 (1.115) 式定义的共轭动量密度：

$$\pi_i(t) = \frac{\partial \mathcal{L}_i}{\partial[\partial_0 \phi_i(t)]} \to \frac{\partial \mathcal{L}}{\partial[\partial_0 \phi(\mathbf{x}, t)]} = \pi(\mathbf{x}, t). \tag{2.49}$$

因此，等时对易关系化为

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \tag{2.50}$$

推广到包含若干个场 ϕ_a 的系统，假设不同的场不会相互影响，则有

$$[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = i\delta_{ab}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi_a(\mathbf{x}, t), \phi_b(\mathbf{y}, t)] = 0, \quad [\pi_a(\mathbf{x}, t), \pi_b(\mathbf{y}, t)] = 0. \tag{2.51}$$

这就是**场论中的正则对易关系**。

2.3　实标量场的正则量子化

如果场 $\phi(x)$ 是一个 Lorentz 标量，就称它为**标量场**。在 Lorentz 变换下，若时空坐标的变换为 $x' = \Lambda x$，则标量场 $\phi(x)$ 的变换形式是

$$\phi'(x') = \phi(x). \tag{2.52}$$

在本节中，我们讨论**实**标量场 $\phi(x)$，它满足自共轭 (self-conjugate) 条件

$$\phi^\dagger(x) = \phi(x), \tag{2.53}$$

即 $\phi(x)$ 是个厄米算符。

假设 $\phi(x)$ 是不参与相互作用的自由实标量场，相应的 **Lorentz 不变拉氏量**可以写成

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi - \frac{1}{2}m^2\phi^2. \tag{2.54}$$

注意到

$$\frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi = \frac{1}{2}g^{\mu\nu}(\partial_\mu \phi)\partial_\nu \phi = \frac{1}{2}[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2], \tag{2.55}$$

可得

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \partial^0 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} = -\partial_i \phi = \partial^i \phi, \tag{2.56}$$

归纳起来，有

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi. \tag{2.57}$$

因此，Euler-Lagrange 方程 (1.114) 给出

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi, \tag{2.58}$$

也就是说，$\phi(x)$ 满足 *Klein-Gordon 方程*

$$(\partial^2 + m^2)\phi(x) = 0. \tag{2.59}$$

2.3.1 平面波展开

设 Klein-Gordon 方程具有平面波解 (plane-wave solution)

$$\varphi(x) = \exp(-ik\cdot x) = \exp(-ik_\mu x^\mu) = \exp(-ik^\mu x_\mu), \tag{2.60}$$

则有

$$\partial^2\varphi = \partial^\mu\partial_\mu\varphi = \partial^\mu(-ik_\mu\varphi) = -ik_\mu\partial^\mu\varphi = (-i)^2k_\mu k^\mu = -k^2, \tag{2.61}$$

从而，

$$0 = (\partial^2 + m^2)\varphi = -(k^2 - m^2)\varphi = -[(k^0)^2 - |\mathbf{k}|^2 - m^2]\varphi. \tag{2.62}$$

这就要求 $(k^0)^2 = |\mathbf{k}|^2 + m^2$，即 $k^0 = \pm E_\mathbf{k}$，其中 $E_\mathbf{k} \equiv \sqrt{|\mathbf{k}|^2 + m^2}$。因此，有两种平面波解。

(1) $k^0 = E_\mathbf{k}$ 对应于正能解

$$\varphi^{(+)}_\mathbf{k}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[-i(E_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x})]. \tag{2.63}$$

(2) $k^0 = -E_\mathbf{k}$ 对应于负能解

$$\varphi^{(-)}_\mathbf{k}(x) = \exp[-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})] = \exp[i(E_\mathbf{k}t + \mathbf{k}\cdot\mathbf{x})]. \tag{2.64}$$

从而，$\phi(\mathbf{x},t)$ 的通解可以写成如下形式：

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}\varphi^{(+)}_\mathbf{k}(x) + \tilde{a}_\mathbf{k}\varphi^{(-)}_\mathbf{k}(x)\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}_\mathbf{k}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right],\end{aligned} \tag{2.65}$$

其中 $a_\mathbf{k}$ 和 $\tilde{a}_\mathbf{k}$ 是两个只依赖于 $\mathbf{k}$ 的算符。这是一种 Fourier 变换，把 $\phi(\mathbf{x},t)$ 展开成三维动量空间中的无穷多个动量模式 (mode)。取上式的厄米共轭，得

$$\begin{aligned}\phi^\dagger(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a^\dagger_\mathbf{k}e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + \tilde{a}^\dagger_\mathbf{k}e^{-i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a^\dagger_{-\mathbf{k}}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})} + \tilde{a}^\dagger_{-\mathbf{k}}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.66}$$

第二步利用了如下性质：对整个三维动量空间进行积分时，将积分项中的 $\mathbf{k}$ 换成 $-\mathbf{k}$ 不会改变积分的结果。于是，由自共轭条件 $\phi^\dagger(\mathbf{x},t) = \phi(\mathbf{x},t)$ 可得

$$\tilde{a}_\mathbf{k} = a^\dagger_{-\mathbf{k}}. \tag{2.67}$$

（注意：由上式可以推出 $\tilde{a}^\dagger_\mathbf{k} = a_{-\mathbf{k}}$ 和 $\tilde{a}^\dagger_{-\mathbf{k}} = a_\mathbf{k}$。）因而，有

$$\phi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})} + a^\dagger_{-\mathbf{k}}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{-i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger_{\mathbf{k}} e^{i(E_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \right]. \tag{2.68}$$

替换一下动量记号，可以把 $\phi(\mathbf{x}, t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right), \tag{2.69}$$

其中，p^0 应该满足

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{|\mathbf{p}|^2 + m^2}, \tag{2.70}$$

而 $a_{\mathbf{p}}$ 是湮灭算符，$a^\dagger_{\mathbf{p}}$ 是产生算符。$\phi(\mathbf{x}, t)$ 对应的共轭动量密度为

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (-ip_0) \left(a_{\mathbf{p}} e^{-ip\cdot x} - a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right). \tag{2.71}$$

正则量子化程序要求它们满足等时对易关系

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0, \quad [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \tag{2.72}$$

2.3.2 产生湮灭算符的对易关系

利用 Fourier 变换公式

$$\int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}} = \int d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{p}), \tag{2.73}$$

可得

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x} \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \left[a_{\mathbf{p}} e^{-i(p-q)\cdot x} + a^\dagger_{\mathbf{p}} e^{i(p+q)\cdot x} \right] \\
&= \int d^3p\, \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a^\dagger_{\mathbf{p}} e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}} e^{2iq^0 t} \right),
\end{aligned} \tag{2.74}$$

以及

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x} \partial_0 \phi &= \int \frac{d^3p}{(2\pi)^3} \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \left[a_{\mathbf{p}} e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p}} e^{i(p+q)\cdot x} \right] \\
&= \int d^3p\, \frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-i(p^0 - q^0)t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) - a^\dagger_{\mathbf{p}} e^{i(p^0+q^0)t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}} - a^\dagger_{-\mathbf{q}} e^{2iq^0 t} \right).
\end{aligned} \tag{2.75}$$

从而，有

$$-i\sqrt{2E_{\mathbf{q}}}\, a_{\mathbf{q}} = \frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}} a_{\mathbf{q}} = \int d^3x\, e^{iq\cdot x} \partial_0 \phi - iq_0 \int d^3x\, e^{iq\cdot x} \phi = \int d^3x\, e^{iq\cdot x} (\partial_0 \phi - iq_0 \phi), \tag{2.76}$$

亦即

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{ip\cdot x} \left[\partial_0\phi(x) - ip_0\phi(x)\right]. \tag{2.77}$$

上式取厄米共轭，并使用自共轭条件 $\phi^\dagger = \phi$，得

$$a_{\mathbf{p}}^\dagger = \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x\, e^{-ip\cdot x} \left[\partial_0\phi(x) + ip_0\phi(x)\right]. \tag{2.78}$$

利用上面两个表达式和等时对易关系 (2.72)，可得

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y\ \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) + iq_0\phi(\mathbf{y},t)\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left(iq_0[\pi(\mathbf{x},t), \phi(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t), \pi(\mathbf{y},t)]\right) \\
&= \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{2.79}$$

最后一行中的 $\delta^{(3)}(\mathbf{p}-\mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = \mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$，则 $E_{\mathbf{p}} + E_{\mathbf{q}} = 2E_{\mathbf{p}} = \sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}$，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{2.80}$$

类似地，

$$\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x d^3y\ \left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t) - iq_0\phi(\mathbf{y},t)\}\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \left[\pi(\mathbf{x},t) - ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t) - iq_0\phi(\mathbf{y},t)\right] \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left(-iq_0[\pi(\mathbf{x},t), \phi(\mathbf{y},t)] - ip_0[\phi(\mathbf{x},t), \pi(\mathbf{y},t)]\right) \\
&= \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, d^3y\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
&= \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \int d^3x\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{q}}-E_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} (2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}).
\end{aligned} \tag{2.81}$$

最后一行中的 $\delta^{(3)}(\mathbf{p}+\mathbf{q})$ 因子说明上式只有可能在 $\mathbf{p} = -\mathbf{q}$ 时非零，此时有 $E_{\mathbf{p}} = E_{\mathbf{q}}$，故

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \tag{2.82}$$

此外，

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}} a_{\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \tag{2.83}$$

综上，产生湮灭算符满足如下对易关系：

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \tag{2.84}$$

这可以看成是对易关系 (2.4) 在量子场论中的推广。

2.3.3 哈密顿量和总动量

根据定义式 (1.117)，实标量场的哈密顿量密度为

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = (\partial_0\phi)^2 - \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi + \frac{1}{2}m^2\phi^2 = \frac{1}{2}[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2]. \tag{2.85}$$

对全空间积分以得到哈密顿量：

$$\begin{aligned}
H &= \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\,[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2] \\
&= \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\left(-ip_0 a_{\mathbf{p}} e^{-ip\cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(-iq_0 a_{\mathbf{q}} e^{-iq\cdot x} + iq_0 a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad + \left(i\mathbf{p}\,a_{\mathbf{p}} e^{-ip\cdot x} - i\mathbf{p}\,a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\,a_{\mathbf{q}} e^{-iq\cdot x} - i\mathbf{q}\,a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad + m^2\left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}} e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)\Big] \\
&= \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \\
&\qquad + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2) a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2) a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= \frac{1}{2}\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right] \\
&\qquad + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right]\Big\} \\
&= \frac{1}{2}\int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})(p_0q_0 + \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t}\right] \\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})(-p_0q_0 - \mathbf{p}\cdot\mathbf{q} + m^2)\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}\right]\Big\} \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \Big[\left(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2\right)\left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&\qquad + \left(-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2\right)\left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\Big].
\end{aligned} \tag{2.86}$$

由 (2.70) 式可得 $-E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2 = 0$，故上式最后两行方括号中第二项没有贡献。从而，

$$\begin{aligned}
H &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 + m^2\right)\left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3\, 2E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p})\right]
\end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2}, \tag{2.87}$$

其中第四步用到对易关系 (2.84)。

这个结果可以看作是一维简谐振子哈密顿量 (2.7) 向无穷多自由度的推广。$a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ 是动量为 $\mathbf{p}$ 的模式对应的**粒子数密度算符**（动量空间中的密度），相应的能量是 $E_{\mathbf{p}}$。在 (2.87) 式最后一行中，第一项代表所有动量模式所有粒子贡献的能量之和。由 (2.73) 式可得

$$(2\pi)^3\delta^{(3)}(0) = \int d^3x = V, \tag{2.88}$$

其中 V 是进行积分的空间体积，对于全空间而言是无穷大的。因此，(2.87) 式最后一行的第二项是一个无穷大 c 数，是真空的零点能，是所有动量模式在全空间贡献的零点能之和。2.1 节末尾的讨论表明，一维简谐振子的零点能为 $E_0 = \omega/2$。这是自由度为 1 时的结果，推广到无穷多自由度自然会得到无穷大的零点能。如果不讨论引力现象，这个零点能通常并不重要，因为实验上只能测量两个能量之差。经过正则量子化之后，实标量场的哈密顿量 H 是正定的，不存在负能量困难。

哈密顿量 H 与产生算符和湮灭算符的对易子分别为

$$[H, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q}-\mathbf{p}) = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \tag{2.89}$$

$$[H, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int d^3q\, E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \tag{2.90}$$

设 $|E\rangle$ 是 H 的本征态，本征值为 E，则

$$H\,|E\rangle = E\,|E\rangle\,. \tag{2.91}$$

从而，有

$$Ha_{\mathbf{p}}^\dagger\,|E\rangle = (a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}} a_{\mathbf{p}}^\dagger)\,|E\rangle = (E + E_{\mathbf{p}})a_{\mathbf{p}}^\dagger\,|E\rangle\,. \tag{2.92}$$

$$Ha_{\mathbf{p}}\,|E\rangle = (a_{\mathbf{p}} H - E_{\mathbf{p}} a_{\mathbf{p}})\,|E\rangle = (E - E_{\mathbf{p}})a_{\mathbf{p}}\,|E\rangle\,. \tag{2.93}$$

可见，产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是使能量本征值增加 $E_{\mathbf{k}}$，湮灭算符 $a_{\mathbf{p}}$ 的作用是使能量本征值减少 $E_{\mathbf{k}}$。

根据 (1.156) 式，实标量场的总动量是

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi\nabla\phi = -\int d^3x (\partial_0\phi)\nabla\phi \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \left(-ip_0 a_{\mathbf{p}} e^{-ip\cdot x} + ip_0 a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(i\mathbf{q}\, a_{\mathbf{q}} e^{-iq\cdot x} - i\mathbf{q}\, a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[- p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} - p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + p_0\mathbf{q}\, a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + p_0\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big]\\
&= -\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{ - p_0\mathbf{q}\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right]\\
&\qquad + p_0\mathbf{q}\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p_0+q_0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right]\Big\}\\
&= -\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{ - p_0\mathbf{q}\,\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p_0-q_0)t}\right]\\
&\qquad + p_0\mathbf{q}\,\delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p^0+q^0)t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p^0+q^0)t}\right]\Big\}\\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}(-E_{\mathbf{p}}\mathbf{p})\left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right).
\end{aligned}
\tag{2.94}
$$

先作 $\mathbf{p}\to-\mathbf{p}$ 的替换，再利用对易关系 (2.84)，可得

$$
\begin{aligned}
&\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(-\mathbf{p})\left(a_{-\mathbf{p}} a_{\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right)\\
&\quad = -\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2iE_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2iE_{\mathbf{p}}t}\right).
\end{aligned}
\tag{2.95}
$$

可见，(2.94) 式最后一行圆括号中最后两项没有贡献。从而，

$$
\begin{aligned}
\mathbf{P} &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left[2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(0)\right]\\
&= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}\delta^{(3)}(0)\int d^3p\,\mathbf{p}.
\end{aligned}
\tag{2.96}
$$

由于 $\int d^3p\,\mathbf{p} = \int d^3p\,(-\mathbf{p}) = -\int d^3p\,\mathbf{p}$，上式最后一行第二项没有贡献。于是，

$$
\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\, a_{\mathbf{p}}^\dagger a_{\mathbf{p}},
\tag{2.97}
$$

即总动量是所有动量模式所有粒子贡献的动量之和。

$\mathbf{P}$ 与产生湮灭算符的对易子为

$$
[\mathbf{P}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\, a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = \int d^3q\,\mathbf{q}\, a_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{q}-\mathbf{p}) = \mathbf{p}\, a_{\mathbf{p}}^\dagger,
\tag{2.98}
$$

$$
[\mathbf{P}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{d^3q}{(2\pi)^3}\,\mathbf{q}\,[a_{\mathbf{q}}^\dagger, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int d^3q\,\mathbf{q}\, a_{\mathbf{q}} \delta^{(3)}(\mathbf{q}-\mathbf{p}) = -\mathbf{p}\, a_{\mathbf{p}}.
\tag{2.99}
$$

2.3.4 粒子态

真空态 $|0\rangle$ 是能量最低的态，对于任意动量 $\mathbf{p}$ 对应的湮灭算符 $a_{\mathbf{p}}$，满足

$$
a_{\mathbf{p}}\,|0\rangle = 0,
\tag{2.100}
$$

归一化为

$$\langle 0|0\rangle = 1. \tag{2.101}$$

由哈密顿量的表达式 (2.87)可得

$$H\,|0\rangle = E_{\text{vac}}\,|0\rangle\,, \quad E_{\text{vac}} = \delta^{(3)}(0)\int d^3p\,\frac{E_{\mathbf{p}}}{2}, \tag{2.102}$$

可见，这样定义的真空态的能量本征值 E_{vac} 确实是能量最低的零点能。此外，由 (2.97) 式可知，$|0\rangle$ 的总动量本征值是零：

$$\mathbf{P}\,|0\rangle = 0\,|0\rangle\,, \tag{2.103}$$

即真空态不具有动量。

接着，定义动量为 $\mathbf{p}$ 的**单粒子态**为

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}}\,a_{\mathbf{p}}^\dagger\,|0\rangle\,. \tag{2.104}$$

从而，利用 (2.89) 和 (2.98) 式可得

$$H\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,Ha_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^\dagger H + E_{\mathbf{p}}a_{\mathbf{p}}^\dagger)\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(E_{\text{vac}} + E_{\mathbf{p}})a_{\mathbf{p}}^\dagger\,|0\rangle = (E_{\text{vac}} + E_{\mathbf{p}})\,|\mathbf{p}\rangle\,, \tag{2.105}$$

$$\mathbf{P}\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{P}\,a_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(a_{\mathbf{p}}^\dagger\,\mathbf{P} + \mathbf{p}\,a_{\mathbf{p}}^\dagger)\,|0\rangle = \sqrt{2E_{\mathbf{p}}}\,\mathbf{p}\,a_{\mathbf{p}}^\dagger\,|0\rangle = \mathbf{p}\,|\mathbf{p}\rangle\,. \tag{2.106}$$

可以看出，相比于真空态 $|0\rangle$，单粒子态 $|\mathbf{p}\rangle$ 多了一份能量 $E_{\mathbf{p}}$，也多了一份动量 $\mathbf{p}$。因此，$|\mathbf{p}\rangle$ 描述的是一个动量为 $\mathbf{p}$ 的粒子，这个粒子的能量为 $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$，满足狭义相对论中的能量一动量关系 (1.52)，而拉氏量 (2.54) 中的参数 m 就是粒子的**质量**。可以看出，产生算符 $a_{\mathbf{p}}^\dagger$ 的作用是产生一个动量为 $\mathbf{p}$ 的粒子。

此外，可作如下计算：

$$a_{\mathbf{p}}\,|\mathbf{q}\rangle = \sqrt{2E_{\mathbf{q}}}\,a_{\mathbf{p}}a_{\mathbf{q}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{q}}}\,[a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle = \sqrt{2E_{\mathbf{p}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle\,. \tag{2.107}$$

如果 $\mathbf{p} \neq \mathbf{q}$，则上式为零；如果 $\mathbf{p} = \mathbf{q}$，则单粒子态 $|\mathbf{q}\rangle = |\mathbf{p}\rangle$ 在 $a_{\mathbf{p}}$ 的作用下变成真空态 $|0\rangle$。可见，湮灭算符 $a_{\mathbf{p}}$ 的作用是湮灭一个动量为 $\mathbf{p}$ 的粒子。

单粒子态的内积关系为

$$\begin{aligned}\langle\mathbf{q}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\,a_{\mathbf{q}}a_{\mathbf{p}}^\dagger\,|0\rangle = \sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\,[a_{\mathbf{p}}^\dagger a_{\mathbf{q}} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\,|0\rangle \\ &= 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{2.108}$$

上式是 *Lorentz* 不变的，这是 (2.104) 式中归一化因子取成 $\sqrt{2E_{\mathbf{p}}}$ 的原因。相关证明如下。

证明 若实函数 $f(x)$ 连续且方程 $f(x) = 0$ 具有若干个分立的根 x_i，则如下等式成立：

$$\delta\,(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}. \tag{2.109}$$

引入**阶跃函数** (step function)

$$\theta(x)=\begin{cases}1, & x\geq 0,\\ 0, & x<0,\end{cases} \tag{2.110}$$

则任意 Lorentz 标量函数 $F(p)$ 在四维动量 p^μ 满足质壳条件 $p^2-m^2=0$ 且能量为正 $(p^0>0)$ 的动量空间区域上的 Lorentz 不变积分为

$$\begin{aligned}&\int d^4p\,\delta(p^2-m^2)\theta(p^0)F(p)=\int d^3p\,dp^0\,\delta\left((p^0)^2-|\mathbf{p}|^2-m^2\right)\theta(p^0)F(p^0,\mathbf{p})\\ &=\int d^3p\,\frac{1}{2\sqrt{|\mathbf{p}|^2+m^2}}\,F\left(\sqrt{|\mathbf{p}|^2+m^2},\mathbf{p}\right)=\int\frac{d^3p}{2E_{\mathbf{p}}}\,F\left(\sqrt{|\mathbf{p}|^2+m^2},\mathbf{p}\right).\end{aligned} \tag{2.111}$$

这里第二步用到 (2.109) 式。可见，

$$\frac{d^3p}{2E_{\mathbf{p}}} \tag{2.112}$$

是 Lorentz 不变的体积元。对任意 Lorentz 标量函数 $g(\mathbf{q})$，按照 δ 函数定义，有

$$g(\mathbf{q})=\int d^3p\,\delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p})=\int\frac{d^3p}{2E_{\mathbf{p}}}\,2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})g(\mathbf{p}). \tag{2.113}$$

由于上式最左边和最右边都是 Lorentz 不变的，

$$2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{2.114}$$

必定是 Lorentz 不变的。**证毕**。

进一步，可以定义动量分别为 $\mathbf{p}_1,\cdots,\mathbf{p}_n$ 的 n 个粒子对应的**多粒子态**为

$$|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\equiv\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle\,. \tag{2.115}$$

H 对它的作用给出

$$\begin{aligned}&H\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,Ha^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle\\ &=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,(a^\dagger_{\mathbf{p}_1}H+E_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_1})\cdots a^\dagger_{\mathbf{p}_n}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ &=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}Ha^\dagger_{\mathbf{p}_2}\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle+E_{\mathbf{p}_1}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ &=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}H\cdots a^\dagger_{\mathbf{p}_n}\,|0\rangle+(E_{\mathbf{p}_1}+E_{\mathbf{p}_2})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ &=\cdots=\sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}\cdots a^\dagger_{\mathbf{p}_n}H\,|0\rangle+(E_{\mathbf{p}_1}+E_{\mathbf{p}_2}+\cdots+E_{\mathbf{p}_n})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\\ &=(E_{\mathrm{vac}}+E_{\mathbf{p}_1}+E_{\mathbf{p}_2}+\cdots+E_{\mathbf{p}_n})\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,,\end{aligned} \tag{2.116}$$

同理，$\mathbf{P}$ 对它的作用给出

$$\mathbf{P}\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=(\mathbf{p}_1+\mathbf{p}_2+\cdots+\mathbf{p}_n)\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle\,. \tag{2.117}$$

也就是说，多粒子态 $|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle$ 的能量本征值和动量本征值直接由各个粒子的能量和动量叠加贡献。

由对易关系 (2.84) 可得

$$
\begin{aligned}
|\mathbf{p}_1,\cdots,\mathbf{p}_i,\cdots,\mathbf{p}_j,\cdots,\mathbf{p}_n\rangle &= \sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_i}\cdots a^\dagger_{\mathbf{p}_j}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle \\
&= \sqrt{2E_{\mathbf{p}_1}}\cdots\sqrt{2E_{\mathbf{p}_n}}\,a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_j}\cdots a^\dagger_{\mathbf{p}_i}\cdots a^\dagger_{\mathbf{p}_n}|0\rangle \\
&= |\mathbf{p}_1,\cdots,\mathbf{p}_j,\cdots,\mathbf{p}_i,\cdots,\mathbf{p}_n\rangle\,. \qquad (2.118)
\end{aligned}
$$

可以看出，对调多粒子态中的任意两个粒子，得到的态相同，即多粒子态对于全同粒子交换是对称的。这说明实标量场描述的粒子是**玻色子** (boson)，服从 Bose-Einstein 统计。得到这个结论的关键是两个产生算符相互对易。

双粒子态的内积关系为

$$
\begin{aligned}
&\langle\mathbf{q}_1,\mathbf{q}_2|\mathbf{p}_1,\mathbf{p}_2\rangle = \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\,\langle 0|\,a_{\mathbf{q}_2}a_{\mathbf{q}_1}a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}|0\rangle \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_1}a_{\mathbf{q}_1}a^\dagger_{\mathbf{p}_2}|0\rangle + (2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_2}|0\rangle\right] \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[(2\pi)^3\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_1}|0\rangle + (2\pi)^3\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\,\langle 0|\,a_{\mathbf{q}_2}a^\dagger_{\mathbf{p}_2}|0\rangle\right] \\
&= \sqrt{16E_{\mathbf{p}_1}E_{\mathbf{p}_2}E_{\mathbf{q}_1}E_{\mathbf{q}_2}}\left[(2\pi)^6\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2) + (2\pi)^6\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\right] \\
&= 4E_{\mathbf{p}_1}E_{\mathbf{p}_2}(2\pi)^6\left[\delta^{(3)}(\mathbf{p}_1-\mathbf{q}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_1) + \delta^{(3)}(\mathbf{p}_1-\mathbf{q}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{q}_2)\right]. \qquad (2.119)
\end{aligned}
$$

此外，还可以定义动量均为 $\mathbf{p}$ 的 n 个粒子对应的多粒子态为

$$
|n_\mathbf{p}\rangle \equiv (2E_\mathbf{p})^{n_\mathbf{p}/2}\left(a^\dagger_\mathbf{p}\right)^{n_\mathbf{p}}|0\rangle\,, \qquad (2.120)
$$

则粒子数密度算符

$$
N_\mathbf{p} \equiv a^\dagger_\mathbf{p}a_\mathbf{p} \qquad (2.121)
$$

对它的作用为

$$
\begin{aligned}
N_\mathbf{p}|n_\mathbf{q}\rangle &= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}|0\rangle = (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left[a^\dagger_\mathbf{q}a_\mathbf{p} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle \\
&= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}a^\dagger_\mathbf{q}a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle \\
&= (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^2a_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-2}|0\rangle + 2(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle \\
&= \cdots = (2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}a_\mathbf{p}|0\rangle + n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle \\
&= n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle\,. \qquad (2.122)
\end{aligned}
$$

在动量空间对粒子数密度算符进行积分，得到的是**粒子数算符**

$$
N \equiv \int\frac{d^3p}{(2\pi)^3}\,N_\mathbf{p} = \int\frac{d^3p}{(2\pi)^3}\,a^\dagger_\mathbf{p}a_\mathbf{p}. \qquad (2.123)
$$

由 (2.122) 式，可得

$$
\begin{aligned}
N|n_\mathbf{q}\rangle &= \int\frac{d^3p}{(2\pi)^3}\,N_\mathbf{p}|n_\mathbf{q}\rangle = \int\frac{d^3p}{(2\pi)^3}\,n_\mathbf{q}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})(2E_\mathbf{q})^{n_\mathbf{q}/2}a^\dagger_\mathbf{p}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}-1}|0\rangle \\
&= n_\mathbf{q}(2E_\mathbf{q})^{n_\mathbf{q}/2}\left(a^\dagger_\mathbf{q}\right)^{n_\mathbf{q}}|0\rangle = n_\mathbf{q}|n_\mathbf{q}\rangle\,. \qquad (2.124)
\end{aligned}
$$

因此，$|n_{\mathbf{q}}\rangle$ 是 N 的本征态，本征值为粒子数 $n_{\mathbf{q}}$。

更一般地，可以定义多粒子态

$$|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle \equiv \prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle \tag{2.125}$$

来描述动量为 $\mathbf{p}_1,\cdots,\mathbf{p}_m$ 的粒子分别有 $n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}$ 个的情况。此时，有

$$\begin{aligned}
&N|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\,a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]a^\dagger_{\mathbf{p}}a_{\mathbf{p}}(a^\dagger_{\mathbf{p}_1})^{n_{\mathbf{p}_1}}\cdots(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right]\Big[a^\dagger_{\mathbf{p}}(a^\dagger_{\mathbf{p}_1})^{n_{\mathbf{p}_1}}a_{\mathbf{p}}(a^\dagger_{\mathbf{p}_2})^{n_{\mathbf{p}2}}\cdots(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle\\
&\qquad+n_{\mathbf{p}_1}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}_1)a^\dagger_{\mathbf{p}}(a^\dagger_{\mathbf{p}_1})^{n_{\mathbf{p}_1}-1}(a^\dagger_{\mathbf{p}_2})^{n_{\mathbf{p}2}}\cdots(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle\Big]\\
=&\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right][a^\dagger_{\mathbf{p}}(a^\dagger_{\mathbf{p}_1})^{n_{\mathbf{p}_1}}a_{\mathbf{p}}(a^\dagger_{\mathbf{p}_2})^{n_{\mathbf{p}2}}\cdots(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}|0\rangle]+n_{\mathbf{p}_1}|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&\cdots=\int\frac{d^3p}{(2\pi)^3}\left[\prod_{i=1}^{m}(2E_{\mathbf{p}_i})^{n_{\mathbf{p}_i}/2}\right][a^\dagger_{\mathbf{p}}(a^\dagger_{\mathbf{p}_1})^{n_{\mathbf{p}_1}}\cdots(a^\dagger_{\mathbf{p}_i})^{n_{\mathbf{p}_i}}a_{\mathbf{p}}|0\rangle]\\
&\quad+(n_{\mathbf{p}_1}+\cdots+n_{\mathbf{p}_m})|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle\\
=&(n_{\mathbf{p}_1}+\cdots+n_{\mathbf{p}_m})|n_{\mathbf{p}_1},\cdots,n_{\mathbf{p}_m}\rangle.
\end{aligned}\tag{2.126}$$

可见，N 确实是描述总粒子数的算符。

2.4 复标量场的正则量子化

在本节中，我们讨论复标量场 $\phi(x)$，它不满足自共轭条件 (2.53)，即

$$\phi^\dagger(x)\neq\phi(x). \tag{2.127}$$

自由复标量场的拉氏量具有 1.7.4 小节中 (1.186) 式的形式。不过，由于 $\phi(x)$ 是量子场算符，需要把那里的复共轭记号 $*$ 改成厄米共轭记号 $\dagger$，故 **Lorentz** 不变拉氏量为

$$\mathcal{L}=(\partial^\mu\phi^\dagger)\partial_\mu\phi-m^2\phi^\dagger\phi. \tag{2.128}$$

把 $\phi(x)$ 和 $\phi^\dagger(x)$ 当成两个独立的场变量，注意到

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}=\partial^\mu\phi,\quad \frac{\partial\mathcal{L}}{\partial\phi^\dagger}=-m^2\phi,\quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=\partial^\mu\phi^\dagger,\quad \frac{\partial\mathcal{L}}{\partial\phi}=-m^2\phi^\dagger, \tag{2.129}$$

则 Euler-Lagrange 方程 (1.114) 给出

$$(\partial^2+m^2)\phi(x)=0,\quad (\partial^2+m^2)\phi^\dagger(x)=0. \tag{2.130}$$

也就是说，$\phi(x)$ 和 $\phi^\dagger(x)$ 均满足 Klein-Gordon 方程.

可以将复标量场 ϕ 分解为两个实标量场 ϕ_1 和 ϕ_2 的线性组合：

$$\phi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2),\quad \phi^\dagger=\frac{1}{\sqrt{2}}(\phi_1-i\phi_2). \tag{2.131}$$

从而，拉氏量 (2.128) 化为

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}[\partial^\mu(\phi_1-i\phi_2)]\partial_\mu(\phi_1+i\phi_2)-\frac{1}{2}m^2(\phi_1-i\phi_2)(\phi_1+i\phi_2)\\&=\frac{1}{2}(\partial^\mu\phi_1)\partial_\mu\phi_1-\frac{1}{2}m^2\phi_1^2+\frac{1}{2}(\partial^\mu\phi_2)\partial_\mu\phi_2-\frac{1}{2}m^2\phi_2^2.\end{aligned} \tag{2.132}$$

与 (2.54) 式比较可知，复标量场的拉氏量相当于两个质量相同的实标量场的拉氏量。

2.4.1 平面波展开

对于复标量场，我们可以遵循 2.3.1 小节中的方法讨论它的平面波解展开，但不能够应用自共轭条件。因此，$\phi(\mathbf{x},t)$ 的通解也具有的 (2.65) 式的形式：

$$\begin{aligned}\phi(\mathbf{x},t)&=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_\mathbf{k}e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]\\&=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_{-\mathbf{k}}e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.133}$$

由于不满足自共轭条件 (2.53)，算符 $\tilde{a}_{-\mathbf{k}}$ 与 $a_\mathbf{k}$ 没有什么关系，改记为

$$b_\mathbf{k}^\dagger=\tilde{a}_{-\mathbf{k}}, \tag{2.134}$$

则展开式变成

$$\phi(\mathbf{x},t)=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+b_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right]. \tag{2.135}$$

替换一下动量记号，可以把 $\phi(\mathbf{x},t)$ 的平面波解展开式整理成

$$\phi(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip\cdot x}+b_\mathbf{p}^\dagger e^{ip\cdot x}\right), \tag{2.136}$$

其中，p^0 应该满足

$$p^0=E_\mathbf{p}\equiv\sqrt{|\mathbf{p}|^2+m^2}. \tag{2.137}$$

取厄米共轭，就得到 $\phi^\dagger(\mathbf{x},t)$ 的平面波解展开式

$$\phi^\dagger(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(b_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right). \tag{2.138}$$

现在，$a_{\mathbf{p}}$ 和 $b_{\mathbf{p}}$ 是两个相互独立的湮灭算符，而 $a_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}^\dagger$ 是两个相互独立的产生算符。

$\phi(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi(\mathbf{x},t)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}=\partial_0\phi^\dagger=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(-ip_0)\left(b_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right), \tag{2.139}$$

$\phi^\dagger(\mathbf{x},t)$ 对应的共轭动量密度是

$$\pi^\dagger(\mathbf{x},t)=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)}=\partial_0\phi=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(-ip_0)\left(a_{\mathbf{p}}e^{-ip\cdot x}-b_{\mathbf{p}}^\dagger e^{ip\cdot x}\right). \tag{2.140}$$

根据 (2.51) 式，等时对易关系为

$$\begin{aligned}
[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi(\mathbf{x},t),\phi(\mathbf{y},t)]=[\pi(\mathbf{x},t),\pi(\mathbf{y},t)]=0,\\
[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [\phi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]=[\pi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)]=0,\\
[\phi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)] &= [\phi^\dagger(\mathbf{x},t),\pi(\mathbf{y},t)]=[\phi(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]=[\pi(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)]=0.
\end{aligned} \tag{2.141}$$

2.4.2 产生湮灭算符的对易关系

由

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}\phi &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[a_{\mathbf{p}}e^{-i(p-q)\cdot x}+b_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})+b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}}+b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right)
\end{aligned} \tag{2.142}$$

和

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}\partial_0\phi &= \int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[a_{\mathbf{p}}e^{-i(p-q)\cdot x}-b_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})-b_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}}-b_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),
\end{aligned} \tag{2.143}$$

可得

$$-i\sqrt{2E_{\mathbf{q}}}\,a_{\mathbf{q}}=\frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}a_{\mathbf{q}}=\int d^3x\,e^{iq\cdot x}\partial_0\phi-iq_0\int d^3x\,e^{iq\cdot x}\phi=\int d^3x\,e^{iq\cdot x}\left(\partial_0\phi-iq_0\phi\right). \tag{2.144}$$

于是，

$$a_{\mathbf{p}}=\frac{i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\,e^{ip\cdot x}\left(\partial_0\phi-ip_0\phi\right),\quad a_{\mathbf{p}}^\dagger=\frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\,e^{-ip\cdot x}\left(\partial_0\phi^\dagger+ip_0\phi^\dagger\right). \tag{2.145}$$

从而，有

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \\
=& \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t)+iq_0\phi^\dagger(\mathbf{y},t)\}\right] \\
=& \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\left[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t)+iq_0\phi^\dagger(\mathbf{y},t)\right] \\
=& \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]-ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right) \\
=& \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right] \\
=& \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} = \frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
=& (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned}
\tag{2.146}
$$

以及

$$
\begin{aligned}
&[a_{\mathbf{p}}, a_{\mathbf{q}}] \\
=& \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\}\right] \\
=& \frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\left[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)-iq_0\phi(\mathbf{y},t)\right]=0.
\end{aligned}
\tag{2.147}
$$

另一方面，由

$$
\begin{aligned}
\int d^3x\,e^{iq\cdot x}\phi^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}+a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})+a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}+a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right)
\end{aligned}
\tag{2.148}
$$

和

$$
\begin{aligned}
\int d^3x\,e^{iq\cdot x}\partial_0\phi^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\left[b_{\mathbf{p}}e^{-i(p-q)\cdot x}-a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&= \int d^3p\,\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left[b_{\mathbf{p}}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})-a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right] \\
&= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\left(b_{\mathbf{q}}-a_{-\mathbf{q}}^\dagger e^{2iq^0t}\right),
\end{aligned}
\tag{2.149}
$$

可得

$$
-i\sqrt{2E_{\mathbf{q}}}\,b_{\mathbf{q}}=\frac{-2iq_0}{\sqrt{2E_{\mathbf{q}}}}b_{\mathbf{q}}=\int d^3x\,e^{iq\cdot x}\partial_0\phi^\dagger-iq_0\int d^3x\,e^{iq\cdot x}\phi^\dagger=\int d^3x\,e^{iq\cdot x}\left(\partial_0\phi^\dagger-iq_0\phi^\dagger\right).
\tag{2.150}
$$

于是，

$$b_{\mathbf{p}}=\frac{i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{ip\cdot x}\left(\partial_0\phi^\dagger-ip_0\phi^\dagger\right),\quad b_{\mathbf{p}}^\dagger=\frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\, e^{-ip\cdot x}\left(\partial_0\phi+ip_0\phi\right). \tag{2.151}$$

从而，有

$$\begin{aligned}
&[b_{\mathbf{p}},b_{\mathbf{q}}^\dagger]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\}\right]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\left[\pi(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\right]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left(iq_0[\pi(\mathbf{x},t),\phi(\mathbf{y},t)]-ip_0[\phi^\dagger(\mathbf{x},t),\pi^\dagger(\mathbf{y},t)]\right)\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\left[-i(p_0+q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right]\\
=&\frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}=\frac{E_{\mathbf{p}}+E_{\mathbf{q}}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
=&(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),
\end{aligned} \tag{2.152}$$

以及

$$\begin{aligned}
&[b_{\mathbf{p}},b_{\mathbf{q}}]\\
=&\frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi^\dagger(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\}\right]\\
=&\frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\left[\pi(\mathbf{x},t)-ip_0\phi^\dagger(\mathbf{x},t),\ \pi(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\right]=0.
\end{aligned} \tag{2.153}$$

此外，还有

$$\begin{aligned}
&[a_{\mathbf{p}},b_{\mathbf{q}}^\dagger]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{-iq\cdot y}\{\partial_0\phi(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\}\right]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x-q\cdot y)}\left[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi^\dagger(\mathbf{y},t)+iq_0\phi(\mathbf{y},t)\right]=0,
\end{aligned} \tag{2.154}$$

以及

$$\begin{aligned}
&[a_{\mathbf{p}},b_{\mathbf{q}}]\\
=&\frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3xd^3y\left[e^{ip\cdot x}\{\partial_0\phi(\mathbf{x},t)-ip_0\phi(\mathbf{x},t)\},\ e^{iq\cdot y}\{\partial_0\phi^\dagger(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\}\right]\\
=&\frac{-1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p\cdot x+q\cdot y)}\left[\pi^\dagger(\mathbf{x},t)-ip_0\phi(\mathbf{x},t),\ \pi(\mathbf{y},t)-iq_0\phi^\dagger(\mathbf{y},t)\right]\\
=&\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\left(-iq_0[\pi^\dagger(\mathbf{x},t),\phi^\dagger(\mathbf{y},t)]-ip_0[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,d^3y\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}\,[-i(p_0-q_0)i\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\,e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} = \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}e^{i(E_\mathbf{p}+E_\mathbf{q})t}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0. \qquad (2.155)
\end{aligned}
$$

归纳起来，产生湮灭算符的对易关系如下：

$$
\begin{aligned}
[a_\mathbf{p}, a_\mathbf{q}^\dagger] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_\mathbf{p}, a_\mathbf{q}] = [a_\mathbf{p}^\dagger, a_\mathbf{q}^\dagger] = 0, \\
[b_\mathbf{p}, b_\mathbf{q}^\dagger] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_\mathbf{p}, b_\mathbf{q}] = [b_\mathbf{p}^\dagger, b_\mathbf{q}^\dagger] = 0, \qquad (2.156) \\
[a_\mathbf{p}, b_\mathbf{q}^\dagger] &= [b_\mathbf{p}, a_\mathbf{q}^\dagger] = [a_\mathbf{p}, b_\mathbf{q}] = [a_\mathbf{p}^\dagger, b_\mathbf{q}^\dagger] = 0.
\end{aligned}
$$

这说明 $a_\mathbf{p}^\dagger, a_\mathbf{p}$ 与 $b_\mathbf{p}^\dagger, b_\mathbf{p}$ 是两套不同的产生湮灭算符，描述两种不同的玻色子。

2.4.3 U(1) 整体对称性

对复标量场作 U(1) 整体变换

$$
\phi'(x) = e^{iq\theta}\phi(x), \quad [\phi^\dagger(x)]' = e^{-iq\theta}\phi^\dagger(x), \qquad (2.157)
$$

则拉氏量 (2.128) 不变。依照 1.7.4 小节的讨论，相应的守恒流为

$$
J^\mu = q\phi^\dagger i\overleftrightarrow{\partial^\mu}\phi, \qquad (2.158)
$$

相应的守恒荷为

$$
\begin{aligned}
Q &= q\int d^3x\,\phi^\dagger i\overleftrightarrow{\partial^0}\phi = iq\int d^3x\,[\phi^\dagger\partial^0\phi - (\partial^0\phi^\dagger)\phi] \\
&= iq\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{k}}}\Big[\left(b_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right)\partial^0\left(a_\mathbf{k}e^{-ik\cdot x}+b_\mathbf{k}^\dagger e^{ik\cdot x}\right) \\
&\qquad - \partial^0\left(b_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right)\left(a_\mathbf{k}e^{-ik\cdot x}+b_\mathbf{k}^\dagger e^{ik\cdot x}\right)\Big] \\
&= iq\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{k}}}\Big[\left(b_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right)(-ik^0)\left(a_\mathbf{k}e^{-ik\cdot x}-b_\mathbf{k}^\dagger e^{ik\cdot x}\right) \\
&\qquad - (-ip^0)\left(b_\mathbf{p}e^{-ip\cdot x}-a_\mathbf{p}^\dagger e^{ip\cdot x}\right)\left(a_\mathbf{k}e^{-ik\cdot x}+b_\mathbf{k}^\dagger e^{ik\cdot x}\right)\Big] \\
&= iq\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{k}}}\Big[(ik^0+ip^0)b_\mathbf{p}b_\mathbf{k}^\dagger e^{-i(p-k)\cdot x}+(-ik^0-ip^0)a_\mathbf{p}^\dagger a_\mathbf{k}e^{i(p-k)\cdot x} \\
&\qquad +(-ik^0+ip^0)b_\mathbf{p}a_\mathbf{k}e^{-i(p+k)\cdot x}+(ik^0-ip^0)a_\mathbf{p}^\dagger b_\mathbf{k}^\dagger e^{i(p+k)\cdot x}\Big] \\
&= q\int\frac{d^3x\,d^3p\,d^3k}{(2\pi)^6\sqrt{2E_\mathbf{p}2E_\mathbf{k}}}\Big[-(E_\mathbf{k}+E_\mathbf{p})b_\mathbf{p}b_\mathbf{k}^\dagger e^{-i(p-k)\cdot x}+(E_\mathbf{k}+E_\mathbf{p})a_\mathbf{p}^\dagger a_\mathbf{k}e^{i(p-k)\cdot x} \\
&\qquad +(E_\mathbf{k}-E_\mathbf{p})b_\mathbf{p}a_\mathbf{k}e^{-i(p+k)\cdot x}+(-E_\mathbf{k}+E_\mathbf{p})a_\mathbf{p}^\dagger b_\mathbf{k}^\dagger e^{i(p+k)\cdot x}\Big] \\
&= q\int\frac{d^3p\,d^3k}{(2\pi)^3\sqrt{2E_\mathbf{p}2E_\mathbf{k}}}\Big\{(E_\mathbf{k}+E_\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{k})\left[-b_\mathbf{p}b_\mathbf{k}^\dagger e^{-i(E_\mathbf{p}-E_\mathbf{k})t}+a_\mathbf{p}^\dagger a_\mathbf{k}e^{i(E_\mathbf{p}-E_\mathbf{k})t}\right]
\end{aligned}
$$

$$+ (E_{\mathbf{k}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{k})\left[b_{\mathbf{p}}a_{\mathbf{k}}e^{-i(E_{\mathbf{p}}+E_{\mathbf{k}})t} - a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{k}}e^{i(E_{\mathbf{p}}+E_{\mathbf{k}})t}\right]\Big\}$$

$$= q\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\,2E_{\mathbf{p}}\left(-b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right) = q\int\frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right). \tag{2.159}$$

利用对易关系 (2.156)，可得

$$Q = \int\frac{d^3p}{(2\pi)^3}\left(q\,a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - q\,b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) - (2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}\,q. \tag{2.160}$$

上式第二项是零点荷。在第一项的圆括号中，粒子数密度算符 $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ 的系数是 q，而粒子数密度算符 $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$ 的系数是 $-q$。可见，$a^\dagger_{\mathbf{p}}, a_{\mathbf{p}}$ 描述的粒子具有的荷为 q，习惯上称为**正粒子**；另一方面，$b^\dagger_{\mathbf{p}}, b_{\mathbf{p}}$ 描述的粒子具有相反的荷 $-q$，习惯上称为**反粒子**。除去零点荷，总荷 Q 是所有动量模式所有正反粒子贡献的荷之和。注意到 Q/q 的表达式与 (1.3) 式类似，但 Q/q 被解释为正粒子数与反粒子数之差，因而不存在负概率困难。

这里单个粒子的荷 q 或 $-q$ 对总荷 Q 的贡献是相加性的，并且来自于一种内部对称性，因而是一种**内部相加性量子数**。实际上，反粒子的所有内部相加性量子数都与正粒子相反。

如果对实标量场作类似的 U(1) 整体变换，则自共轭条件 (2.53) 使得

$$e^{iq\theta}\phi(x) = \phi'(x) = [\phi'(x)]^\dagger = [e^{iq\theta}\phi(x)]^\dagger = e^{-iq\theta}\phi^\dagger(x) = e^{-iq\theta}\phi(x). \tag{2.161}$$

上式要求 $q = 0$。因此，对实标量场不能进行非平庸的 U(1) 整体变换。实际上，自共轭条件使实标量场描述的粒子不能具有任何非零的内部相加性量子数，也就是说，正粒子与反粒子是相同的，实标量场描述的是一种纯中性粒子。

2.4.4 哈密顿量和总动量

根据 (1.117) 式，复标量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L} = (\partial^0\phi^\dagger)\partial_0\phi + (\partial^0\phi)\partial_0\phi^\dagger - (\partial^\mu\phi^\dagger)\partial_\mu\phi + m^2\phi^\dagger\phi\\ &= (\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger)\cdot\nabla\phi + m^2\phi^\dagger\phi.\end{aligned} \tag{2.162}$$

于是，哈密顿量可以写成

$$\begin{aligned}H &= \int d^3x\,\mathcal{H} = \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi + (\nabla\phi^\dagger)\cdot\nabla\phi + m^2\phi^\dagger\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi + \nabla\cdot(\phi^\dagger\nabla\phi) - \phi^\dagger\nabla^2\phi + m^2\phi^\dagger\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^0\partial_0 - \nabla^2 + m^2)\phi]\\ &= \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi + \phi^\dagger(\partial^2 + m^2)\phi].\end{aligned} \tag{2.163}$$

上式第三步用了分部积分，第四步扔掉了一个全散度，最后一行方括号里第三项可以通过 ϕ 的运动方程 (2.130) 消去。从而，得到

$$H = \int d^3x\,[(\partial^0\phi^\dagger)\partial_0\phi - \phi^\dagger\partial^0\partial_0\phi]$$

$$
\begin{aligned}
&= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\partial^0 \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \partial^0\partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \Big] \\
&= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{(-ip^0) \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) (-iq_0) \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad - \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \left[(-iq^0)(-iq_0)a_{\mathbf{q}}e^{-iq\cdot x} + iq^0 iq_0 b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right] \Big\} \\
&= \int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(p^0q_0 + q^0q_0) b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(p-q)\cdot x} + (p^0q_0 + q^0q_0) a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} \\
&\qquad + (-p^0q_0 + q^0q_0) b_{\mathbf{p}}a_{\mathbf{q}} e^{-i(p+q)\cdot x} + (-p^0q_0 + q^0q_0) a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\Big] \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} E_{\mathbf{q}} \Big\{(E_{\mathbf{p}} + E_{\mathbf{q}})\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right] \\
&\qquad + (E_{\mathbf{q}} - E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[b_{\mathbf{p}}a_{\mathbf{q}} e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right] \Big\} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} 2E_{\mathbf{p}}^2 \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(b_{\mathbf{p}}b_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) + (2\pi)^3\delta^{(3)}(0) \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}.
\end{aligned}
\tag{2.164}
$$

除了零点能，哈密顿量是所有动量模式所有正反粒子的能量之和。对于相同的动量模式 $\mathbf{p}$，正粒子与反粒子具有相同的能量 $E_{\mathbf{p}}$，因而它们具有相同的质量 m。

根据 (1.156) 式，复标量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\,(\pi\nabla\phi + \pi^\dagger\nabla\phi^\dagger) = -\int d^3x\,[(\partial_0\phi^\dagger)\nabla\phi + (\partial_0\phi)\nabla\phi^\dagger] \\
&= -\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\partial_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \nabla \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad + \partial_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} + b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \nabla \left(b_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \Big] \\
&= -\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[-ip_0 \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) i\mathbf{q} \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\
&\qquad - iq_0 \left(a_{\mathbf{q}}e^{-iq\cdot x} - b_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) i\mathbf{p} \left(b_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \Big] \\
&= -\int \frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}}) e^{-i(p-q)\cdot x} \\
&\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger) e^{i(p-q)\cdot x} \\
&\qquad + (E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}b_{\mathbf{p}}) e^{-i(p+q)\cdot x} \\
&\qquad + (E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger + E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger) e^{i(p+q)\cdot x} \Big] \\
&= -\int \frac{d^3p\,d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big[(-E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}}b_{\mathbf{q}}^\dagger - E_{\mathbf{q}}\mathbf{p}\, b_{\mathbf{q}}^\dagger b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \\
&\qquad\qquad + (-E_{\mathbf{p}}\mathbf{q}\, a_{\mathbf{p}}^\dagger a_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}}a_{\mathbf{p}}^\dagger) e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[(E_{\mathbf{p}}\mathbf{q}\, b_{\mathbf{p}} a_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, a_{\mathbf{q}} b_{\mathbf{p}}) e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t} \\
&\qquad\qquad\qquad + (E_{\mathbf{p}}\mathbf{q}\, a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{q}} + E_{\mathbf{q}}\mathbf{p}\, b^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{p}}) e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big]\Big\} \\
&= -\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[-E_{\mathbf{p}}\mathbf{p}\,(b_{\mathbf{p}} b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}) \\
&\qquad - E_{\mathbf{p}}\mathbf{p}\,(b_{\mathbf{p}} a_{-\mathbf{p}} - a_{-\mathbf{p}} b_{\mathbf{p}}) e^{-2iE_{\mathbf{p}}t} - E_{\mathbf{p}}\mathbf{p}\,(a^\dagger_{\mathbf{p}} b^\dagger_{-\mathbf{p}} - b^\dagger_{-\mathbf{p}} a^\dagger_{\mathbf{p}}) e^{2iE_{\mathbf{p}}t}\Big] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}(b_{\mathbf{p}} b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}) = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\,(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) + \delta^{(3)}(0)\int d^3p\,\mathbf{p} \\
&= \int \frac{d^3p}{(2\pi)^3}\mathbf{p}\,(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}).
\end{aligned} \tag{2.165}
$$

总动量是所有动量模式所有正反粒子的动量之和。

第 3 章　矢量场

3.1　量子 Lorentz 变换

设 Lorentz 变换 Λ 在物理 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换

$$|\Psi'\rangle = U(\Lambda)\,|\Psi\rangle\,, \tag{3.1}$$

其中 $U(\Lambda)$ 是一个线性幺正算符，描述**量子 Lorentz 变换**，满足

$$U^\dagger(\Lambda)U(\Lambda) = U(\Lambda)U^\dagger(\Lambda) = 1, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda). \tag{3.2}$$

先作 Lorentz 变换 Λ_1，再作 Lorentz 变换 Λ_2，相当于作 Lorentz 变换 $\Lambda_2\Lambda_1$，故以下同态 (homomorphic) 关系成立：

$$U(\Lambda_2\Lambda_1) = U(\Lambda_2)U(\Lambda_1). \tag{3.3}$$

从而，由

$$U^{-1}(\Lambda)U(\Lambda) = 1 = U(1) = U(\Lambda^{-1}\Lambda) = U(\Lambda^{-1})U(\Lambda) \tag{3.4}$$

可得

$$U^{-1}(\Lambda) = U(\Lambda^{-1}). \tag{3.5}$$

将无穷小 Lorentz 变换 (1.157) 记为 $\Lambda_\omega = \mathbf{1} + \omega$，它诱导的无穷小幺正算符可表达为

$$U(\mathbf{1}+\omega) = 1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}. \tag{3.6}$$

这里只展开到 ω 的一阶项。$J^{\mu\nu}$ 是量子 Lorentz 变换的**生成元算符**[1]。根据 1.7.3 小节的讨论，实参数 $\omega_{\mu\nu}$ 是反对称的，因而 $J^{\mu\nu}$ 也是反对称的：

$$J^{\mu\nu} = -J^{\nu\mu}. \tag{3.7}$$

由 $U(\mathbf{1}+\omega)$ 的幺正性可得

$$1 = U^\dagger(\mathbf{1}+\omega)U(\mathbf{1}+\omega) = \left[1 + \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\dagger\right]\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = 1 + \frac{i}{2}\omega_{\mu\nu}[(J^{\mu\nu})^\dagger - J^{\mu\nu}], \tag{3.8}$$

[1]虽然用了相同的符号，这里的算符 $J^{\mu\nu}$ 不同于守恒荷 (1.177)。

最后一步忽略了 ω 的二阶项。可见，$J^{\mu\nu}$ 是厄米算符：

$$(J^{\mu\nu})^\dagger = J^{\mu\nu}. \tag{3.9}$$

对算符乘积

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda). \tag{3.10}$$

的左边和右边分别展开，得

$$U^{-1}(\Lambda)U(\mathbf{1}+\omega)U(\Lambda) = U^{-1}(\Lambda)\left(1-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)U(\Lambda) = 1-\frac{i}{2}U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda), \tag{3.11}$$

$$U(\Lambda^{-1}(\mathbf{1}+\omega)\Lambda) = U(\mathbf{1}+\Lambda^{-1}\omega\Lambda) = 1-\frac{i}{2}(\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu}. \tag{3.12}$$

因此，有

$$\begin{aligned} U^{-1}(\Lambda)\omega_{\mu\nu}J^{\mu\nu}U(\Lambda) &= (\Lambda^{-1}\omega\Lambda)_{\mu\nu}J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1}\omega\Lambda)^\alpha{}_\nu J^{\mu\nu} = g_{\mu\alpha}(\Lambda^{-1})^\alpha{}_\beta\omega^\beta{}_\gamma\Lambda^\gamma{}_\nu J^{\mu\nu} \\ &= g_{\mu\alpha}\Lambda_\beta{}^\alpha\omega^\beta{}_\gamma\Lambda^\gamma{}_\nu J^{\mu\nu} = \Lambda^\beta{}_\mu\omega_{\beta\gamma}\Lambda^\gamma{}_\nu J^{\mu\nu} = \omega_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma}, \end{aligned} \tag{3.13}$$

第四步用到 (1.34) 式。上式对任意 $\omega_{\mu\nu}$ 成立，于是，

$$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma}. \tag{3.14}$$

因此，$J^{\mu\nu}$ 在 $|\Psi'\rangle$ 中的期待值 (expectation value) 与它在 $|\Psi\rangle$ 中的期待值有如下关系：

$$\langle\Psi'|\, J^{\mu\nu}\, |\Psi'\rangle = \langle\Psi|\, U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)\, |\Psi\rangle = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \langle\Psi|\, J^{\rho\sigma}\, |\Psi\rangle\,. \tag{3.15}$$

也就是说，$U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda)$ 可以看作量子 Lorentz 变换诱导出来的 $J^{\mu\nu}$ 算符的 Lorentz 变换：

$$J'^{\mu\nu} \equiv U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma}. \tag{3.16}$$

可见，$J^{\mu\nu}$ 是一个 *2* 阶 *Lorentz* 张量。

接着，考虑 Λ 的无穷小形式 $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu$，则

$$U(\Lambda) = 1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}, \quad U^{-1}(\Lambda) = U^\dagger(\Lambda) = 1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}. \tag{3.17}$$

忽略二阶小量，(3.14) 式左边为

$$\begin{aligned} U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) &= \left(1+\frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}\right)J^{\mu\nu}\left(1-\frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\alpha\beta}\right) \\ &= J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\alpha\beta}J^{\mu\nu}J^{\alpha\beta} + \frac{i}{2}\tilde{\omega}_{\gamma\delta}J^{\gamma\delta}J^{\mu\nu} = J^{\mu\nu} - \frac{i}{2}\tilde{\omega}_{\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}], \end{aligned} \tag{3.18}$$

右边为

$$\begin{aligned} \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma} &= (\delta^\mu{}_\rho + \tilde{\omega}^\mu{}_\rho)(\delta^\nu{}_\sigma + \tilde{\omega}^\nu{}_\sigma)J^{\rho\sigma} = \delta^\mu{}_\rho\delta^\nu{}_\sigma J^{\rho\sigma} + \delta^\mu{}_\rho\tilde{\omega}^\nu{}_\sigma J^{\rho\sigma} + \tilde{\omega}^\mu{}_\rho\delta^\nu{}_\sigma J^{\rho\sigma} \\ &= J^{\mu\nu} + \tilde{\omega}^\nu{}_\sigma J^{\mu\sigma} + \tilde{\omega}^\mu{}_\rho J^{\rho\nu} = J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \tilde{\omega}_{\sigma\rho}g^{\mu\sigma}J^{\rho\nu} \end{aligned}$$

$$
\begin{aligned}
&= J^{\mu\nu} + \tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho}) + \frac{1}{2}\tilde{\omega}_{\sigma\rho}(g^{\nu\sigma}J^{\mu\rho} + g^{\mu\rho}J^{\nu\sigma}) \\
&= J^{\mu\nu} + \frac{1}{2}\tilde{\omega}_{\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma}),
\end{aligned} \tag{3.19}
$$

最后三步用到 $J^{\mu\nu}$ 和 $\tilde{\omega}_{\mu\nu}$ 的反对称性。比较上面两式，可得

$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= i[g^{\nu\rho}J^{\mu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma).
\end{aligned} \tag{3.20}
$$

这是 $J^{\mu\nu}$ 满足的对易关系。以 $J^{\mu\nu}$ 作为基底张成线性空间，通过 (3.20) 式定义线性空间中的矢量乘积，则称此线性空间为 **Lorentz 代数**。

Lie 群是一类特殊的连续群，n 维 Lie 群的群空间由 n 个独立的连续实参数描述，具有 n 维微分流形的结构。Lie 群的任何线性表示的生成元均满足共同的对易关系，这些对易关系定义了生成元的 *Lie 乘积*，而生成元张成的线性空间关于 Lie 乘积是封闭的，构成代数，称为 **Lie 代数**。Lie 代数描述 Lie 群在恒元附近的局域结构。

Lorentz 群是一个 6 维 Lie 群，它对应的 Lie 代数就是 Lorentz 代数。Lorentz 群的任何线性表示的生成元都要满足 (3.20) 式。反过来，可以通过构造满足 (3.20) 式的生成元矩阵，来得到 Lorentz 群的线性表示。

我们可以把算符 $J^{\mu\nu}$ 的 6 个独立分量组合成 2 个三维矢量算符：

$$
J^i \equiv \frac{1}{2}\varepsilon^{ijk}J^{jk}, \quad K^i \equiv J^{0i}, \tag{3.21}
$$

即

$$
\mathbf{J} = (J^{23}, J^{31}, J^{12}), \quad \mathbf{K} = (J^{01}, J^{02}, J^{03}). \tag{3.22}
$$

J^i 与 J^j 的对易关系为

$$
\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\varepsilon^{ikl}\varepsilon^{jmn}[J^{kl}, J^{mn}] = \frac{i}{4}\varepsilon^{ikl}\varepsilon^{jmn}\{[g^{lm}J^{kn} - (k \leftrightarrow l)] - (m \leftrightarrow n)\} \\
&= \frac{i}{2}\varepsilon^{ikl}\varepsilon^{jmn}[g^{lm}J^{kn} - (k \leftrightarrow l)] = i\varepsilon^{ikl}\varepsilon^{jmn}g^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jmn}\delta^{lm}J^{kn} = -i\varepsilon^{ikl}\varepsilon^{jln}J^{kn} \\
&= i\varepsilon^{ikl}\varepsilon^{jnl}J^{kn} = i(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj})J^{kn} = -iJ^{ji} = iJ^{ij},
\end{aligned} \tag{3.23}
$$

第二、三步用到三维 Levi-Civita 符号的反对称性，第八步用到 (1.82) 式。由 (1.96) 式，有

$$
J^{ij} = \frac{1}{2}2\delta^{il}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{ljk}J^{lj} = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{klj}J^{lj} = \varepsilon^{ijk}J^k, \tag{3.24}
$$

从而推出

$$
[J^i, J^j] = i\varepsilon^{ijk}J^k. \tag{3.25}
$$

在量子力学中，轨道角动量算符 $\mathbf{L} = \mathbf{x} \times \mathbf{p}$，写成分量的形式是 $L^i = \varepsilon^{ijk}x^j p^k$，从而，

$$
\varepsilon^{ijk}L^k = \varepsilon^{ijk}\varepsilon^{klm}x^l p^m = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})x^l p^m = x^i p^j - x^j p^i. \tag{3.26}
$$

由 (2.10) 式、(2.11) 式及对易关系 $[x^i,p^j]=i\delta^{ij}$ 可得

$$\begin{aligned}[L^i,L^j] &= \varepsilon^{ikl}\varepsilon^{jmn}[x^kp^l,x^mp^n]=\varepsilon^{ikl}\varepsilon^{jmn}\{x^k[p^l,x^m]p^n+x^m[x^k,p^n]p^l\}\\ &= \varepsilon^{ikl}\varepsilon^{jmn}(-i\delta^{lm}x^kp^n+i\delta^{kn}x^mp^l)=i(-\varepsilon^{ikl}\varepsilon^{jln}x^kp^n+\varepsilon^{ikl}\varepsilon^{jmk}x^mp^l)\\ &= i(\varepsilon^{ikl}\varepsilon^{jnl}x^kp^n-\varepsilon^{ilk}\varepsilon^{jmk}x^mp^l)=i[(\delta^{ij}\delta^{kn}-\delta^{in}\delta^{kj})x^kp^n-(\delta^{ij}\delta^{lm}-\delta^{im}\delta^{lj})x^mp^l]\\ &= i[\delta^{ij}x^kp^k-x^jp^i-\delta^{ij}x^lp^l+x^ip^j]=i(x^ip^j-x^jp^i)=i\varepsilon^{ijk}L^k.\end{aligned}\tag{3.27}$$

可见，**J** 与 **L** 具有相同的对易关系，**J** 也是一个角动量算符。实际上，**J** 描述**总角动量**，不止可以包含轨道角动量 **L**，也可以包含*自旋角动量*。

满足

$$O^{\mathrm{T}}O=\mathbf{1}\tag{3.28}$$

的实方阵 O 称为*实正交矩阵* (real orthogonal matrix)。对上式取行列式，得

$$1=\det O^{\mathrm{T}}\cdot\det O=(\det O)^2.\tag{3.29}$$

可见，实正交矩阵 O 的行列式为 $\det O=\pm1$。由行列式为 $+1$ 的 3 维实正交矩阵按照矩阵乘法构成的群，称为**空间旋转群 SO(3)**，描述三维空间中的旋转变换。1.7.3 小节提到，SO(3) 群是 Lorentz 群的*子群*，J^i 可以看作 SO(3) 群的生成元算符，而 (3.25) 式是 SO(3) 群的 Lie 代数关系。

另一方面，**K** 是**增速算符**。**J** 与 **K** 的对易关系为

$$\begin{aligned}[J^i,K^j] &= \frac{1}{2}\varepsilon^{ikl}[J^{kl},J^{0j}]=\frac{i}{2}\varepsilon^{ikl}\{[g^{l0}J^{kj}-(k\leftrightarrow l)]-(0\leftrightarrow j)\}\\ &= i\varepsilon^{ikl}[g^{l0}J^{kj}-(0\leftrightarrow j)]=i\varepsilon^{ikl}(g^{l0}J^{kj}-g^{lj}J^{k0})=-i\varepsilon^{ikl}g^{lj}J^{k0}=i\varepsilon^{ikl}\delta^{lj}J^{k0}\\ &= i\varepsilon^{ikj}J^{k0}=i\varepsilon^{ijk}J^{0k}=i\varepsilon^{ijk}K^k,\end{aligned}\tag{3.30}$$

而 **K** 自身的对易关系为

$$\begin{aligned}[K^i,K^j] &= [J^{0i},J^{0j}]=i(g^{i0}J^{0j}-g^{00}J^{ij}-g^{ij}J^{00}+g^{0j}J^{i0})\\ &= -i(g^{00}J^{ij}+g^{ij}J^{00})=-iJ^{ij}=-i\varepsilon^{ijk}J^k.\end{aligned}\tag{3.31}$$

归纳起来，有

$$[J^i,J^j]=i\varepsilon^{ijk}J^k,\quad [J^i,K^j]=i\varepsilon^{ijk}K^k,\quad [K^i,K^j]=-i\varepsilon^{ijk}J^k.\tag{3.32}$$

3.2　量子矢量场的 Lorentz 变换

3.2.1　Lorentz 群矢量表示的生成元

Lorentz 变换的无穷小参数 $\omega^\alpha{}_\beta$ 可以转化为

$$\omega^\alpha{}_\beta=g^{\alpha\mu}\omega_{\mu\beta}=\frac{1}{2}(g^{\alpha\mu}\omega_{\mu\beta}-g^{\alpha\mu}\omega_{\beta\mu})=\frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta-g^{\alpha\mu}\omega_{\nu\mu}\delta^\nu{}_\beta)=\frac{1}{2}(g^{\alpha\mu}\omega_{\mu\nu}\delta^\nu{}_\beta-g^{\alpha\nu}\omega_{\mu\nu}\delta^\mu{}_\beta)$$

$$= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}i(g^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta g^{\nu\alpha}) = -\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta, \tag{3.33}$$

其中 $(\mathcal{J}^{\mu\nu})^\alpha{}_\beta$ 定义为

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta \equiv i(g^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta g^{\nu\alpha}) = i(g^{\mu\alpha}\delta^\nu{}_\beta - g^{\nu\alpha}\delta^\mu{}_\beta). \tag{3.34}$$

容易看出，$\mathcal{J}^{\mu\nu}$ 是反对称的：

$$\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}. \tag{3.35}$$

它的另一种写法是

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = g_{\alpha\gamma}(\mathcal{J}^{\mu\nu})^\gamma{}_\beta = ig_{\alpha\gamma}(g^{\mu\gamma}\delta^\nu{}_\beta - \delta^\mu{}_\beta g^{\nu\gamma}) = i(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha). \tag{3.36}$$

这样的话，可以把无穷小 Lorentz 变换 Λ_ω 写成

$$(\Lambda_\omega)^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta. \tag{3.37}$$

$\mathcal{J}^{\mu\nu}$ 与 $\mathcal{J}^{\rho\sigma}$ 的对易关系为

$$\begin{aligned}
&[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha{}_\beta = (\mathcal{J}^{\mu\nu})^\alpha{}_\gamma(\mathcal{J}^{\rho\sigma})^\gamma{}_\beta - (\mathcal{J}^{\rho\sigma})^\alpha{}_\gamma(\mathcal{J}^{\mu\nu})^\gamma{}_\beta \\
=\ & i^2(g^{\mu\alpha}\delta^\nu{}_\gamma - \delta^\mu{}_\gamma g^{\nu\alpha})(g^{\rho\gamma}\delta^\sigma{}_\beta - \delta^\rho{}_\beta g^{\sigma\gamma}) - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
=\ & -g^{\mu\alpha}\delta^\nu{}_\gamma g^{\rho\gamma}\delta^\sigma{}_\beta + g^{\mu\alpha}\delta^\nu{}_\gamma\delta^\rho{}_\beta g^{\sigma\gamma} + \delta^\mu{}_\gamma g^{\nu\alpha}g^{\rho\gamma}\delta^\sigma{}_\beta - \delta^\mu{}_\gamma g^{\nu\alpha}\delta^\rho{}_\beta g^{\sigma\gamma} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
=\ & -g^{\mu\alpha}g^{\rho\nu}\delta^\sigma{}_\beta + g^{\mu\alpha}\delta^\rho{}_\beta g^{\sigma\nu} + g^{\nu\alpha}g^{\rho\mu}\delta^\sigma{}_\beta - g^{\nu\alpha}\delta^\rho{}_\beta g^{\sigma\mu} - (\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma) \\
=\ & -g^{\nu\rho}g^{\mu\alpha}\delta^\sigma{}_\beta + g^{\mu\rho}g^{\nu\alpha}\delta^\sigma{}_\beta + g^{\nu\sigma}g^{\mu\alpha}\delta^\rho{}_\beta - g^{\mu\sigma}g^{\nu\alpha}\delta^\rho{}_\beta \\
& -[-g^{\sigma\mu}g^{\rho\alpha}\delta^\nu{}_\beta + g^{\rho\mu}g^{\sigma\alpha}\delta^\nu{}_\beta + g^{\sigma\nu}g^{\rho\alpha}\delta^\mu{}_\beta - g^{\rho\nu}g^{\sigma\alpha}\delta^\mu{}_\beta] \\
=\ & g^{\nu\rho}(g^{\sigma\alpha}\delta^\mu{}_\beta - g^{\mu\alpha}\delta^\sigma{}_\beta) + g^{\mu\rho}(g^{\nu\alpha}\delta^\sigma{}_\beta - g^{\sigma\alpha}\delta^\nu{}_\beta) + g^{\nu\sigma}(g^{\mu\alpha}\delta^\rho{}_\beta - g^{\rho\alpha}\delta^\mu{}_\beta) + g^{\mu\sigma}(g^{\rho\alpha}\delta^\nu{}_\beta - g^{\nu\alpha}\delta^\rho{}_\beta) \\
=\ & -ig^{\nu\rho}(\mathcal{J}^{\sigma\mu})^\alpha{}_\beta - ig^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha{}_\beta - ig^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha{}_\beta - ig^{\mu\sigma}(\mathcal{J}^{\rho\nu})^\alpha{}_\beta \\
=\ & i[g^{\nu\rho}(\mathcal{J}^{\mu\sigma})^\alpha{}_\beta - g^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha{}_\beta - g^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha{}_\beta + g^{\mu\sigma}(\mathcal{J}^{\nu\rho})^\alpha{}_\beta],
\end{aligned} \tag{3.38}$$

即

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}). \tag{3.39}$$

可见，$\mathcal{J}^{\mu\nu}$ 满足 Lorentz 代数关系 (3.20)。$\Lambda^\alpha{}_\beta$ 属于 Lorentz 群的矢量表示，因而 $\mathcal{J}^{\mu\nu}$ 就是**矢量表示的生成元**。

无穷小 Lorentz 变换 (3.37) 的矩阵记法为

$$\Lambda_\omega = \mathbf{1} + \omega = \mathbf{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}, \tag{3.40}$$

它可以看作矩阵级数

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) = e^\omega = \sum_{n=0}^\infty \frac{\omega^n}{n!} \tag{3.41}$$

只展开到 ω 一阶项的结果。矩阵 ω 与度规矩阵 $\mathbf{g}$ 有如下关系：

$$(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^{\alpha}{}_{\beta} = g^{\alpha\gamma}(\omega^{\mathrm{T}})_{\gamma}{}^{\delta}g_{\delta\beta} = g^{\alpha\gamma}\omega^{\delta}{}_{\gamma}g_{\delta\beta} = g^{\alpha\gamma}\omega_{\beta\gamma} = -g^{\alpha\gamma}\omega_{\gamma\beta} = -\omega^{\alpha}{}_{\beta}, \tag{3.42}$$

即

$$\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g} = -\omega. \tag{3.43}$$

从而，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g} = \mathbf{g}^{-1}\left[\sum_{n=0}^{\infty}\frac{(\omega^{\mathrm{T}})^n}{n!}\right]\mathbf{g} = \sum_{n=0}^{\infty}\frac{(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g})^n}{n!} = \exp(\mathbf{g}^{-1}\omega^{\mathrm{T}}\mathbf{g}) = e^{-\omega}. \tag{3.44}$$

若两个同阶方阵 A 和 B 相互对易，即 $[A,B]=0$，则**二项式定理**成立：

$$(A+B)^n = \sum_{j=0}^{n}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.45}$$

阶乘的定义可以推广到负整数：对于整数 $m<0$，定义

$$m! \to \infty, \qquad \frac{1}{m!} \to 0. \tag{3.46}$$

从而，对于 $j>n$，有 $[(n-j)!]^{-1}\to 0$。这样一来，我们可以将 (3.45) 式右边的级数化成无穷级数：

$$(A+B)^n = \sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j}. \tag{3.47}$$

利用上式，可得

$$e^{A+B} = \sum_{n=0}^{\infty}\frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{\infty}\frac{n!}{j!(n-j)!}A^jB^{n-j} = \sum_{j=0}^{\infty}\frac{A^j}{j!}\sum_{n=0}^{\infty}\frac{B^{n-j}}{(n-j)!} = e^Ae^B. \tag{3.48}$$

值得注意的是，上式不仅对相互对易的方阵成立，也对相互对易的算符成立。

根据 (3.44) 和 (3.48) 式，有

$$\mathbf{g}^{-1}\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = e^{-\omega}e^{\omega} = e^{-\omega+\omega} = e^{\mathbf{0}} = \mathbf{1}. \tag{3.49}$$

于是，

$$\Lambda^{\mathrm{T}}\mathbf{g}\,\Lambda = \mathbf{g}, \tag{3.50}$$

即 Λ 满足保度规条件 (1.41)。因此，由 (3.41) 式定义的 Λ 确实是 Lorentz 变换。此时，变换参数 $\omega_{\mu\nu}$ 不是无穷小量，而具有有限的数值，所以

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) \tag{3.51}$$

是用 Lorentz 群矢量表示生成元 $\mathcal{J}^{\mu\nu}$ 表达出来的**有限变换**。由于变换参数 $\omega_{\mu\nu}$ 可以连续地变化到 $\omega_{\mu\nu}=0$，用 (3.51) 式表达的 Lorentz 变换在群空间中与恒等变换是连通着的，因而它属于固有保时向 Lorentz 群。

3.2.2 量子标量场的 Lorentz 变换形式

经过量子化程序之后，标量场 $\phi(x)$ 是物理 Hilbert 空间中的算符，类似于 (3.16) 式，$\phi(x)$ 的 Lorentz 变换关系 (2.52) 可以表示为

$$\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x). \tag{3.52}$$

上式表明，变换后的标量场在变换后的时空点上的值等于变换前的标量场在变换前的时空点上的值。图 3.1(a) 以空间旋转变换为例说明这种情况。由于 $x' = \Lambda x$ 等价于 $x = \Lambda^{-1}x'$，(3.52) 式可以通过改变记号写作

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{3.53}$$

相应地，$\phi(x)$ 在变换后的态 $|\Psi'\rangle$ 中的期待值为

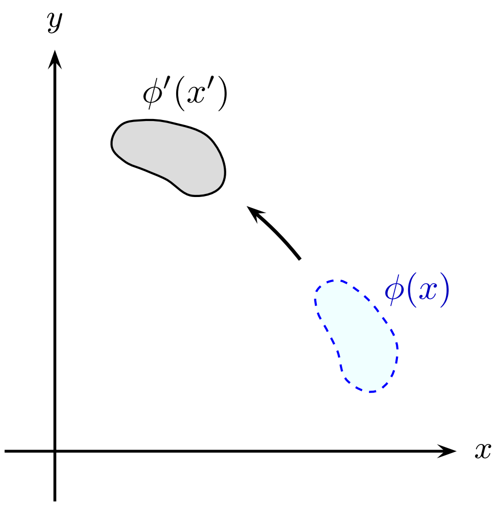
$$\langle\Psi'|\,\phi(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)\phi(x)U(\Lambda)\,|\Psi\rangle = \langle\Psi|\,\phi(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.54}$$

另一方面，由 (1.57) 式可得 $\partial^\mu\phi(x)$ 的 Lorentz 变换形式为

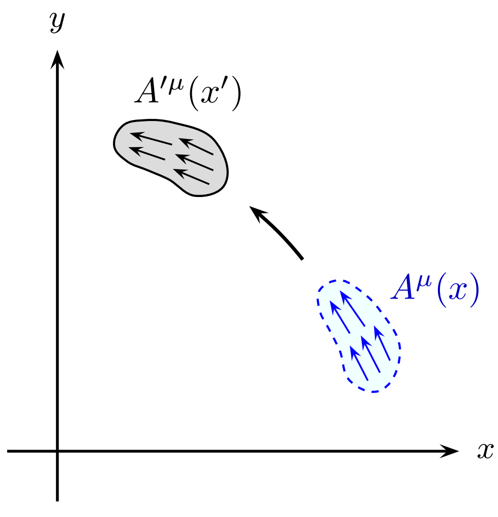
$$\partial'^\mu\phi'(x') = U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda) = \partial'^\mu[U^{-1}(\Lambda)\phi(x')U(\Lambda)] = \partial'^\mu\phi(x) = \Lambda^\mu{}_\nu\partial^\nu\phi(x). \tag{3.55}$$

于是，在 Lorentz 变换下，自由实标量场的拉氏量 (2.54) 的变换形式为

$$\begin{aligned}
\mathcal{L}'(x') &= U^{-1}(\Lambda)\mathcal{L}(x')U(\Lambda) = \frac{1}{2}U^{-1}(\Lambda)[\partial'^\mu\phi(x')\partial'_\mu\phi(x') - m^2\phi^2(x')]U(\Lambda)\\
&= \frac{1}{2}\{g_{\mu\nu}U^{-1}(\Lambda)\partial'^\mu\phi(x')U(\Lambda)U^{-1}(\Lambda)\partial'^\nu\phi(x')U(\Lambda) - m^2[U^{-1}(\Lambda)\phi(x')U(\Lambda)]^2\}\\
&= \frac{1}{2}[g_{\mu\nu}\Lambda^\mu{}_\rho\partial^\rho\phi(x)\Lambda^\nu{}_\sigma\partial^\sigma\phi(x) - m^2\phi^2(x)] = \frac{1}{2}[g_{\rho\sigma}\partial^\rho\phi(x)\partial^\sigma\phi(x) - m^2\phi^2(x)]\\
&= \mathcal{L}(x),
\end{aligned} \tag{3.56}$$



(a) 标量场 $\phi(x)$



(b) 矢量场 $A^\mu(x)$

图 3.1: 在绕 z 轴空间旋转变换下，标量场 $\phi(x)$ 和矢量场 $A^\mu(x)$ 的变换示意图。

倒数第二步用到保度规条件 (1.30)。从而，

$$U^{-1}(\Lambda)\mathcal{L}(x)U(\Lambda)=\mathcal{L}(\Lambda^{-1}x). \tag{3.57}$$

可见，拉氏量 (2.54) 确实是个 Lorentz 标量。

对于无穷小 Lorentz 变换 $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu$，可得

$$\begin{aligned}(\Lambda^{-1})^\mu{}_\nu &= \Lambda_\nu{}^\mu=g_{\nu\alpha}g^{\mu\beta}\Lambda^\alpha{}_\beta=g_{\nu\alpha}g^{\mu\beta}(\delta^\alpha{}_\beta+\omega^\alpha{}_\beta)=g_{\nu\beta}g^{\mu\beta}+g^{\mu\beta}\omega_{\nu\beta}=\delta^\mu{}_\nu-g^{\mu\beta}\omega_{\beta\nu}\\ &= \delta^\mu{}_\nu-\omega^\mu{}_\nu,\end{aligned} \tag{3.58}$$

从而，有

$$(\Lambda^{-1}x)^\mu=(\delta^\mu{}_\nu-\omega^\mu{}_\nu)x^\nu=x^\mu-\omega^\mu{}_\nu x^\nu. \tag{3.59}$$

将 (3.53) 式右边在 x 处展开到 ω 的一阶项，得

$$\begin{aligned}\phi(\Lambda^{-1}x) &= \phi(x)-\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x)=\phi(x)-\omega_{\mu\nu}x^\nu\partial^\mu\phi(x)=\phi(x)-\frac{1}{2}(\omega_{\mu\nu}x^\nu\partial^\mu+\omega_{\nu\mu}x^\mu\partial^\nu)\phi(x)\\ &= \phi(x)-\frac{1}{2}\omega_{\mu\nu}(x^\nu\partial^\mu-x^\mu\partial^\nu)\phi(x)=\phi(x)+\frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu-x^\nu\partial^\mu)\phi(x)\\ &= \phi(x)-\frac{i}{2}\omega_{\mu\nu}i(x^\mu\partial^\nu-x^\nu\partial^\mu)\phi(x).\end{aligned} \tag{3.60}$$

根据 (3.6) 式，将 (3.53) 式左边展开到 ω 的一阶项，得

$$\begin{aligned}U^{-1}(\Lambda)\phi(x)U(\Lambda) &= \left(1+\frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)\phi(x)\left(1-\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\\ &= \phi(x)-\frac{i}{2}\omega_{\alpha\beta}\phi(x)J^{\alpha\beta}+\frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\phi(x)=\phi(x)-\frac{i}{2}\omega_{\mu\nu}[\phi(x),J^{\mu\nu}].\end{aligned} \tag{3.61}$$

两相比较，给出

$$[\phi(x),J^{\mu\nu}]=i(x^\mu\partial^\nu-x^\nu\partial^\mu)\phi(x)=L^{\mu\nu}\phi(x), \tag{3.62}$$

其中 $L^{\mu\nu}$ 定义为

$$L^{\mu\nu}\equiv i(x^\mu\partial^\nu-x^\nu\partial^\mu). \tag{3.63}$$

对于空间分量 L^{ij}，可以等价地定义

$$L^i\equiv\frac{1}{2}\varepsilon^{ijk}L^{jk}=\frac{i}{2}\varepsilon^{ijk}(x^j\partial^k-x^k\partial^j)=\frac{i}{2}(\varepsilon^{ijk}x^j\partial^k-\varepsilon^{ikj}x^j\partial^k)=i\varepsilon^{ijk}x^j\partial^k, \tag{3.64}$$

写成空间矢量的形式是

$$\mathbf{L}=-i\,\mathbf{x}\times\nabla. \tag{3.65}$$

可见，$\mathbf{L}$ 就是微分算符形式的轨道角动量算符。根据 (3.21) 式，(3.62) 式的空间分量部分可以改写为

$$[\phi(x),\mathbf{J}]=\mathbf{L}\,\phi(x). \tag{3.66}$$

上式表明，总角动量算符 $\mathbf{J}$ 生成了轨道角动量，但没有生成自旋角动量。这说明标量场没有自旋，对应于**零自旋**粒子。

3.2.3 量子矢量场的 Lorentz 变换形式

$\partial^\mu\phi(x)$ 是通过对标量场 $\phi(x)$ 取时空导数得到的 Lorentz 矢量。自身就是 Lorentz 矢量的场 $A^\mu(x)$ 也应该具有 (3.55) 式的变换形式，即

$$A'^\mu(x') = U^{-1}(\Lambda)A^\mu(x')U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \tag{3.67}$$

或者写成

$$U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{3.68}$$

这就是**量子矢量场的 Lorentz 变换形式**。相应地，$A^\mu(x)$ 在 $|\Psi'\rangle$ 中的期待值为

$$\langle\Psi'|\,A^\mu(x)\,|\Psi'\rangle = \langle\Psi|\,U^{-1}(\Lambda)A^\mu(x)U(\Lambda)\,|\Psi\rangle = \Lambda^\mu{}_\nu\,\langle\Psi|\,A^\nu(\Lambda^{-1}x)\,|\Psi\rangle\,. \tag{3.69}$$

根据矢量表示的无穷小形式 (3.40)，(3.67) 式的无穷小形式为

$$A'^\mu(x') = \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]A^\nu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.70}$$

将上式与 (1.166) 式比较，可以发现，1.7.3 小节中的 $I^{\mu\nu}$ 在矢量表示中对应于 $\mathcal{J}^{\mu\nu}$。图 3.1(b) 以空间旋转变换为例说明矢量场的变换情况。可以看出，在 Lorentz 变换下，除了矢量场的分布区域发生变化之外，矢量场的分量也要以 Lorentz 矢量分量的身份发生变化。

利用 (3.59) 式，在 x 处将 $A^\nu(\Lambda^{-1}x)$ 展开到 ω 的一阶项，得

$$\begin{aligned}A^\nu(\Lambda^{-1}x) &= A^\nu(x) - \omega^\alpha{}_\beta x^\beta\partial_\alpha A^\nu(x) = A^\nu(x) - \omega_{\alpha\beta}x^\beta\partial^\alpha A^\nu(x)\\ &= A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x).\end{aligned} \tag{3.71}$$

从而，(3.68) 式右边可展开为

$$\begin{aligned}\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right]\left[A^\nu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\nu(x)\right]\\ &= A^\mu(x) + \frac{1}{2}\omega_{\alpha\beta}(x^\alpha\partial^\beta - x^\beta\partial^\alpha)A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)\\ &= A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)].\end{aligned} \tag{3.72}$$

另一方面，(3.68) 式左边的无穷小展开式为

$$\begin{aligned}&U^{-1}(\Lambda)A^\mu(x)U(\Lambda) = \left(1 + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}\right)A^\mu(x)\left(1 - \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)\\ &= A^\mu(x) - \frac{i}{2}\omega_{\alpha\beta}A^\mu(x)J^{\alpha\beta} + \frac{i}{2}\omega_{\gamma\delta}J^{\gamma\delta}A^\mu(x) = A^\mu(x) - \frac{i}{2}\omega_{\rho\sigma}[A^\mu(x), J^{\rho\sigma}].\end{aligned} \tag{3.73}$$

由此可得

$$[A^\mu(x), J^{\rho\sigma}] = L^{\rho\sigma}A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{3.74}$$

生成元 $\mathcal{J}^{\mu\nu}$ 的空间分量等价于三维矢量

$$\mathcal{J}^i \equiv \frac{1}{2}\varepsilon^{ijk}\mathcal{J}^{jk}, \quad \boldsymbol{\mathcal{J}} = (\mathcal{J}^{23}, \mathcal{J}^{31}, \mathcal{J}^{12}). \tag{3.75}$$

再根据 (3.21) 和 (3.64) 式，(3.74) 式的空间分量部分可以改写为

$$[A^\mu(x), \mathbf{J}] = \mathbf{L}\, A^\mu(x) + (\boldsymbol{\mathcal{J}})^\mu{}_\nu A^\nu(x). \tag{3.76}$$

上式表明，总角动量算符 $\mathbf{J}$ 不仅生成了轨道角动量，还生成了由 $\boldsymbol{\mathcal{J}}$ 描述的**自旋角动量**。$\mathcal{J}^i$ 的具体矩阵形式为

$$(\mathcal{J}^1)^\mu{}_\nu = (\mathcal{J}^{23})^\mu{}_\nu = i(g^{2\mu}\delta^3{}_\nu - g^{3\mu}\delta^2{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix}, \tag{3.77}$$

$$(\mathcal{J}^2)^\mu{}_\nu = (\mathcal{J}^{31})^\mu{}_\nu = i(g^{3\mu}\delta^1{}_\nu - g^{1\mu}\delta^3{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{pmatrix}, \tag{3.78}$$

$$(\mathcal{J}^3)^\mu{}_\nu = (\mathcal{J}^{12})^\mu{}_\nu = i(g^{1\mu}\delta^2{}_\nu - g^{2\mu}\delta^1{}_\nu) = \begin{pmatrix} 0 & & & \\ & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{pmatrix}. \tag{3.79}$$

只关注空间分量，可得

$$(\mathcal{J}^1\mathcal{J}^1)^i{}_j = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^2\mathcal{J}^2)^i{}_j = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad (\mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \tag{3.80}$$

因此，有

$$(\boldsymbol{\mathcal{J}}^2)^i{}_j = (\mathcal{J}^1\mathcal{J}^1 + \mathcal{J}^2\mathcal{J}^2 + \mathcal{J}^3\mathcal{J}^3)^i{}_j = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} = 2\delta^i{}_j. \tag{3.81}$$

根据量子力学的角动量理论，$\boldsymbol{\mathcal{J}}^2$ 的本征值为 $s(s+1)$，即 $(\boldsymbol{\mathcal{J}}^2)^i{}_j = s(s+1)\delta^i{}_j$，其中 s 为**自旋量子数**。可见，矢量场 $A^\mu(x)$ 的自旋量子数为

$$s = 1. \tag{3.82}$$

经过量子化程序之后，矢量场 $A^\mu(x)$ 应当描述**自旋为 1** 的粒子。

3.3　有质量矢量场的正则量子化

类似于电磁场，对任意的矢量场 A^μ 可以定义反对称的**场强张量**

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3.83}$$

对于一个自由的**有质量**的实矢量场 A^μ，用场强张量可以将它的 **Lorentz 不变拉氏量**写为

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.84}$$

上式右边第一项是动能项，第二项是质量项。动能项可以用 A^μ 表达成

$$\begin{aligned}
-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu A^\nu - (\partial_\mu A_\nu)\partial^\nu A^\mu - (\partial_\nu A_\mu)\partial^\mu A^\nu + (\partial_\nu A_\mu)\partial^\nu A^\mu] \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu.
\end{aligned} \tag{3.85}$$

从而，有

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu}, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu. \tag{3.86}$$

Euler-Lagrange 方程 (1.114) 给出

$$0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu, \tag{3.87}$$

即

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0. \tag{3.88}$$

上式称为 **Proca 方程**，是自由的有质量矢量场的相对论性运动方程。

由 $\partial_\nu\partial_\mu F^{\mu\nu} = -\partial_\nu\partial_\mu F^{\nu\mu} = -\partial_\mu\partial_\nu F^{\nu\mu} = -\partial_\nu\partial_\mu F^{\mu\nu}$ 可知

$$\partial_\nu\partial_\mu F^{\mu\nu} = 0. \tag{3.89}$$

于是，从 Proca 方程 (3.88) 可得

$$0 = \partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu = m^2\partial_\nu A^\nu. \tag{3.90}$$

这意味着，质量 $m \neq 0$ 时，矢量场 A^μ 应当满足 **Lorenz 条件**

$$\partial_\mu A^\mu = 0. \tag{3.91}$$

从而，有

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu = \partial^2 A^\nu. \tag{3.92}$$

因此，Proca 方程 (3.88) 可化为 *Klein-Gordon 方程*

$$(\partial^2 + m^2)A^\mu(x) = 0. \tag{3.93}$$

A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 = -F_{0\mu}. \tag{3.94}$$

时间分量和空间分量分别是

$$\pi_0 = -F_{00} = 0, \quad \pi_i = -\partial_0 A_i + \partial_i A_0 = -F_{0i}. \tag{3.95}$$

由于 $\pi_0 = 0$，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。实际上，由于 Lorenz 条件 (3.91) 的存在，A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于其它 3 个分量的量。因此，正则量子化程序要求*独立*的正则变量满足等时对易关系

$$[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = i\delta^i{}_j\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^i(\mathbf{x},t), A^j(\mathbf{y},t)] = [\pi_i(\mathbf{x},t), \pi_j(\mathbf{y},t)] = 0. \tag{3.96}$$

3.3.1 极化矢量与平面波展开

$A^\mu(x)$ 既然满足 Klein-Gordon 方程，应该具有两个平面波解，即正能解 $\exp(-ip\cdot x)$ 和负能解 $\exp(ip\cdot x)$。由于 $A^\mu(x)$ 带有一个 Lorentz 矢量指标，平面波展开式的系数也必须具有一个这样的指标。一般地，对于确定的动量 $\mathbf{p}$，矢量场的正能解模式具有如下形式：

$$A^\mu(x;\mathbf{p},\sigma) = e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \quad p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}. \tag{3.97}$$

这里的系数 $e^\mu(\mathbf{p},\sigma)$ 是 Lorentz 矢量，称为**极化矢量** (polarization vector)，它依赖于动量 p，而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而，可以用它们来展开一个任意的 Lorentz 矢量。为了做到这一点，一组极化矢量应当是*线性独立且正交完备*的。Lorentz 矢量空间是一个 4 维空间，因而这样的极化矢量应该有 4 个，包括 1 个类时的极化矢量 $e^\mu(\mathbf{p},0)$ 与 3 个类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$。在*没有额外约束*的情况下，我们要求这 4 个极化矢量是**实**的，而且满足 Lorentz 矢量空间中的**正交归一关系**

$$e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma') = g_{\sigma\sigma'} \tag{3.98}$$

和**完备性关系**

$$\sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu}. \tag{3.99}$$

上面这两个关系都是 *Lorentz 协变*的。只要在某个惯性参考系中取定一组符合这两个关系的极化矢量，通过 Lorentz 变换就可以在其它惯性参考系中得到依然满足这两个关系的一组极化矢量。

我们可以根据与动量 p^μ 的关系来选择一组极化矢量。首先，选取 2 个只有空间分量的类空**横向**极化矢量

$$e^\mu(\mathbf{p},1) = (0,\mathbf{e}(\mathbf{p},1)), \quad e^\mu(\mathbf{p},2) = (0,\mathbf{e}(\mathbf{p},2)). \tag{3.100}$$

此处，

$$\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}(p^1p^3, p^2p^3, -|\mathbf{p}_\mathrm{T}|^2), \quad \mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^2, p^1, 0), \tag{3.101}$$

其中

$$|\mathbf{p}_\mathrm{T}| \equiv \sqrt{(p^1)^2+(p^2)^2}. \tag{3.102}$$

“横向”指的是它们在三维空间中与 $\mathbf{p}$ 垂直，即

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},1) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}[(p^1)^2p^3+(p^2)^2p^3-p^3|\mathbf{p}_\mathrm{T}|^2]=0, \tag{3.103}$$

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^1p^2+p^2p^1)=0. \tag{3.104}$$

此外，存在如下关系：

$$\begin{aligned}\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},1) &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}[(p^1)^2(p^3)^2+(p^2)^2(p^3)^2+|\mathbf{p}_\mathrm{T}|^4]\\ &= \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2[(p^3)^2+|\mathbf{p}_\mathrm{T}|^2] = \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2|\mathbf{p}|^2=1,\end{aligned} \tag{3.105}$$

$$\mathbf{e}(\mathbf{p},2)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}[(p^2)^2+(p^1)^2] = \frac{1}{|\mathbf{p}_\mathrm{T}|^2}|\mathbf{p}_\mathrm{T}|^2=1, \tag{3.106}$$

$$\mathbf{e}(\mathbf{p},1)\cdot\mathbf{e}(\mathbf{p},2) = \frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|^2}(-p^1p^3p^2+p^2p^3p^1)=0. \tag{3.107}$$

也就是说，它们在三维空间中是正交归一的：

$$\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j)=\delta_{ij},\quad i,j=1,2. \tag{3.108}$$

因此，这两个横向极化矢量可以满足四维时空中的横向条件

$$p_\mu e^\mu(\mathbf{p},1)=p_\mu e^\mu(\mathbf{p},2)=0, \tag{3.109}$$

和正交归一关系

$$e_\mu(\mathbf{p},i)e^\mu(\mathbf{p},j)=-\mathbf{e}(\mathbf{p},i)\cdot\mathbf{e}(\mathbf{p},j)=-\delta_{ij}=g_{ij}. \tag{3.110}$$

接着，要求第 3 个类空极化矢量 $e^\mu(\mathbf{p},3)$ 是**纵向**的，即在三维空间中与 $\mathbf{p}$ 平行。这样还不能确定它的时间分量，为此，我们进一步要求它满足四维时空的横向条件 $p_\mu e^\mu(\mathbf{p},3)=0$，而正交归一关系 (3.98) 将决定它的归一化。于是，纵向极化矢量的形式为

$$e^\mu(\mathbf{p},3)=\left(\frac{|\mathbf{p}|}{m},\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}\right). \tag{3.111}$$

可以验证，它确实满足四维时空的横向条件

$$p_\mu e^\mu(\mathbf{p},3)=p^0\frac{|\mathbf{p}|}{m}-\mathbf{p}\cdot\frac{p^0\,\mathbf{p}}{m|\mathbf{p}|}=\frac{p^0|\mathbf{p}|}{m}-\frac{p^0|\mathbf{p}|}{m}=0, \tag{3.112}$$

和正交归一关系

$$e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3) = \frac{|\mathbf{p}|}{m}\frac{|\mathbf{p}|}{m}-\frac{(p^0)^2\,\mathbf{p}\cdot\mathbf{p}}{m^2|\mathbf{p}|^2}=\frac{|\mathbf{p}|^2}{m^2}-\frac{(p^0)^2}{m^2}=-\frac{(p^0)^2-|\mathbf{p}|^2}{m^2}=-1=g_{33}; \tag{3.113}$$

$$e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},i) = -\frac{p^0}{m|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2. \tag{3.114}$$

最后，我们可以将**类时**极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p},0)=\frac{1}{m}\,p^\mu=\frac{1}{m}(p^0,\mathbf{p}). \tag{3.115}$$

它满足正交归一关系 (3.98)：

$$\begin{aligned}
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0) &= \frac{p^2}{m^2}=1=g_{00}; && (3.116)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},i) &= -\frac{1}{m}\,\mathbf{p}\cdot\mathbf{e}(\mathbf{p},i)=0,\quad i=1,2; && (3.117)\\
e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},3) &= \frac{1}{m^2}\,p^0|\mathbf{p}|-\frac{p^0}{m^2|\mathbf{p}|}\,\mathbf{p}\cdot\mathbf{p}=0. && (3.118)
\end{aligned}$$

不过，它不满足四维时空的横向条件：

$$p_\mu e^\mu(\mathbf{p},0)=\frac{p^2}{m}=m. \tag{3.119}$$

可以验证，由 (3.100)、(3.101)、(3.111) 和 (3.115) 式定义的这组极化矢量确实满足完备性关系 (3.99)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)\\
&= e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)-e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)-e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2)-e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\
&= \frac{1}{m^2}\begin{pmatrix} p^0p^0 & -p^0p^1 & -p^0p^2 & -p^0p^3\\ -p^1p^0 & p^1p^1 & p^1p^2 & p^1p^3\\ -p^2p^0 & p^2p^1 & p^2p^2 & p^2p^3\\ -p^3p^0 & p^3p^1 & p^3p^2 & p^3p^3 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & p^1p^3p^1p^3 & p^1p^3p^2p^3 & -p^1p^3|\mathbf{p}_\mathrm{T}|^2\\ 0 & p^2p^3p^1p^3 & p^2p^3p^2p^3 & -p^2p^3|\mathbf{p}_\mathrm{T}|^2\\ 0 & -|\mathbf{p}_\mathrm{T}|^2p^1p^3 & -|\mathbf{p}_\mathrm{T}|^2p^2p^3 & |\mathbf{p}_\mathrm{T}|^4 \end{pmatrix}\\
&\quad -\frac{1}{|\mathbf{p}_\mathrm{T}|^2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & p^2p^2 & -p^2p^1 & 0\\ 0 & -p^1p^2 & p^1p^1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{m^2}\begin{pmatrix} |\mathbf{p}|^2 & -p^0p^1 & -p^0p^2 & -p^0p^3\\ -p^0p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^1p^3\\ -p^0p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^2p^3\\ -p^0p^3 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^1 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^2 & \frac{(p^0)^2}{|\mathbf{p}|^2}p^3p^3 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 & 0 & 0\\
0 & \frac{(p^1)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^1p^3)^2+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2}\\
0 & \frac{p^1p^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{(p^2)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{(p^2p^3)^2+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2}\\
0 & \frac{p^1p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^1p^3}{|\mathbf{p}|^2} & \frac{p^2p^3}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]+\frac{p^2p^3}{|\mathbf{p}|^2} & \frac{(p^3)^2}{m^2}\left[1-\frac{(p^0)^2}{|\mathbf{p}|^2}\right]-\frac{|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2}
\end{pmatrix}
\end{aligned}$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{(p^1)^2|\mathbf{p}_\mathrm{T}|^2+(p^1)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^2)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2} \\
0 & -\frac{p^1p^2}{|\mathbf{p}|^2}+\frac{p^1p^2}{|\mathbf{p}|^2} & -\frac{(p^2)^2|\mathbf{p}_\mathrm{T}|^2+(p^2)^2(|\mathbf{p}|^2-|\mathbf{p}_\mathrm{T}|^2)+(p^1)^2|\mathbf{p}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2} \\
0 & -\frac{p^1p^3}{|\mathbf{p}|^2}+\frac{p^1p^3}{|\mathbf{p}|^2} & -\frac{p^2p^3}{|\mathbf{p}|^2}+\frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{(p^3)^2+|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} = g_{\mu\nu}. \tag{3.120}
$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件 (3.91)，正能解模式 (3.97) 应满足

$$
0 = \partial_\mu A^\mu(x;\mathbf{p},\sigma) = -ip_\mu e^\mu(\mathbf{p},\sigma)\exp(-ip\cdot x), \tag{3.121}
$$

即

$$
p_\mu e^\mu(\mathbf{p},\sigma) = 0. \tag{3.122}
$$

也就是说，描述有质量矢量场的极化矢量必须满足四维时空的横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p},0)$ 不能用于描述有质量矢量场 A^μ。这说明 A^μ 只有 3 个物理的极化状态，由类空的极化矢量 $e^\mu(\mathbf{p},1)$、$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$ 描述。根据完备性关系 (3.99)，这 3 个物理的极化矢量满足

$$
-\sum_{\sigma=1}^{3} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = \sum_{\sigma=1}^{3} g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = g_{\mu\nu}-g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0) = g_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}, \tag{3.123}
$$

即具有求和关系

$$
\sum_{\sigma=1}^{3} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma) = -g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}. \tag{3.124}
$$

通过如下线性组合，我们可以定义另一套物理的极化矢量 $\varepsilon^\mu(p,\lambda)$，其中 $\lambda=+,0,-$：

$$
\varepsilon^\mu(\mathbf{p},\pm) \equiv \frac{1}{\sqrt{2}}\left[\mp e^\mu(\mathbf{p},1) - ie^\mu(\mathbf{p},2)\right], \tag{3.125}
$$

$$
\varepsilon^\mu(\mathbf{p},0) \equiv e^\mu(\mathbf{p},3). \tag{3.126}
$$

这样定义的 $\varepsilon^\mu(p,\pm)$ 是复的，而 $\varepsilon^\mu(p,0)$ 是实的。它们都满足**四维横向条件**

$$
p_\mu\varepsilon^\mu(\mathbf{p},\lambda) = 0. \tag{3.127}
$$

它们还满足

$$
\begin{aligned}
\varepsilon^*_\mu(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\pm) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][\mp e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)] \\
&= \frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1)+\frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2) = \frac{1}{2}(g_{11}+g_{22}) = -1,
\end{aligned} \tag{3.128}
$$

$$\begin{aligned}
\varepsilon_\mu^*(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},\mp) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][\pm e^\mu(\mathbf{p},1)-ie^\mu(\mathbf{p},2)] \\
&= -\frac{1}{2}e_\mu(\mathbf{p},1)e^\mu(\mathbf{p},1)+\frac{1}{2}e_\mu(\mathbf{p},2)e^\mu(\mathbf{p},2)=\frac{1}{2}(-g_{11}+g_{22})=0, \quad (3.129)\\
\varepsilon_\mu^*(\mathbf{p},0)\varepsilon^\mu(\mathbf{p},0) &= e_\mu(\mathbf{p},3)e^\mu(\mathbf{p},3)=-1, \quad (3.130)\\
\varepsilon_\mu^*(\mathbf{p},\pm)\varepsilon^\mu(\mathbf{p},0) &= \frac{1}{2}[\mp e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)]e^\mu(\mathbf{p},3)=0, \quad (3.131)
\end{aligned}$$

即具有**正交归一关系**

$$\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon^\mu(\mathbf{p},\lambda')=-\delta_{\lambda\lambda'}. \quad (3.132)$$

极化矢量求和关系则是

$$\begin{aligned}
\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) &= \frac{1}{2}[e_\mu(p,1)+ie_\mu(p,2)][e_\nu(p,1)-ie_\nu(p,2)] \\
&\quad +\frac{1}{2}[-e_\mu(p,1)+ie_\mu(p,2)][-e_\nu(p,1)-ie_\nu(p,2)]+e_\mu(p,3)e_\nu(p,3)\\
&= e_\mu(p,1)e_\nu(p,1)+e_\mu(p,2)e_\nu(p,2)+e_\mu(p,3)e_\nu(p,3)\\
&= \sum_{\sigma=1}^{3}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma), \quad (3.133)
\end{aligned}$$

与 (3.124) 式左边相等，故

$$\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}. \quad (3.134)$$

四维横向条件 (3.127) 在上式中体现为

$$p^\nu\sum_{\lambda=\pm,0}\varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-p_\mu+\frac{p_\mu p^2}{m^2}=-p_\mu+p_\mu=0. \quad (3.135)$$

粒子的自旋角动量在动量方向上的归一化投影称为**螺旋度** (helicity)。动量 $\mathbf{p}$ 的方向由 $\hat{\mathbf{p}}\equiv\mathbf{p}/|\mathbf{p}|$ 表征，于是，在 Lorentz 群矢量表示中，螺旋度定义为

$$\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}}=\frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\boldsymbol{\mathcal{J}}=\frac{1}{|\mathbf{p}|}\begin{pmatrix}0&&&\\&0&-ip^3&ip^2\\&ip^3&0&-ip^1\\&-ip^2&ip^1&0\end{pmatrix}. \quad (3.136)$$

将 (3.101) 和 (3.111) 式代入 (3.125) 和 (3.126) 式，得到 $\varepsilon^\mu(p,\lambda)$ 的列矢量形式为

$$\varepsilon^\mu(p,0)=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}|\mathbf{p}|^2\\p^0p^1\\p^0p^2\\p^0p^3\end{pmatrix},\quad \varepsilon^\mu(p,+)=\frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\-p^1p^3+ip^2|\mathbf{p}|\\-p^2p^3-ip^1|\mathbf{p}|\\|\mathbf{p}_\mathrm{T}|^2\end{pmatrix},$$

$$\varepsilon^\mu(p,-) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ p^1p^3+ip^2|\mathbf{p}| \\ p^2p^3-ip^1|\mathbf{p}| \\ -|\mathbf{p}_\mathrm{T}|^2 \end{pmatrix}. \tag{3.137}$$

从而，可得

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,0) = \frac{1}{m|\mathbf{p}|^2}\begin{pmatrix} 0 \\ -ip^3p^0p^2+ip^2p^0p^3 \\ ip^3p^0p^1-ip^1p^0p^3 \\ -ip^2p^0p^1+ip^1p^0p^2 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix} = 0\,\varepsilon^\mu(p,0), \tag{3.138}$$

$$\begin{aligned}(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,+) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ ip^2(p^3)^2-p^1p^3|\mathbf{p}|+ip^2|\mathbf{p}_\mathrm{T}|^2 \\ -ip^1(p^3)^2-p^2p^3|\mathbf{p}|-ip^1|\mathbf{p}_\mathrm{T}|^2 \\ ip^1p^2p^3+(p^2)^2|\mathbf{p}|-ip^1p^2p^3+(p^1)^2|\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -p^1p^3|\mathbf{p}|+ip^2|\mathbf{p}|^2 \\ -p^2p^3|\mathbf{p}|-ip^1|\mathbf{p}|^2 \\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}| \end{pmatrix} = +\varepsilon^\mu(p,+), \end{aligned} \tag{3.139}$$

$$\begin{aligned}(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,-) &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -ip^2(p^3)^2-p^1p^3|\mathbf{p}|-ip^2|\mathbf{p}_\mathrm{T}|^2 \\ ip^1(p^3)^2-p^2p^3|\mathbf{p}|+ip^1|\mathbf{p}_\mathrm{T}|^2 \\ -ip^1p^2p^3+(p^2)^2|\mathbf{p}|+ip^1p^2p^3+(p^1)^2|\mathbf{p}| \end{pmatrix} \\ &= \frac{1}{\sqrt{2}|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|}\begin{pmatrix} 0 \\ -p^1p^3|\mathbf{p}|-ip^2|\mathbf{p}|^2 \\ -p^2p^3|\mathbf{p}|+ip^1|\mathbf{p}|^2 \\ |\mathbf{p}_\mathrm{T}|^2|\mathbf{p}| \end{pmatrix} = -\varepsilon^\mu(p,-). \end{aligned} \tag{3.140}$$

归纳起来，有

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\mathcal{J}})\varepsilon^\mu(p,\lambda) = \lambda\,\varepsilon^\mu(p,\lambda). \tag{3.141}$$

上式说明极化矢量 $\varepsilon^\mu(p,\lambda)$ 是螺旋度的本征态，本征值为 λ。因此，$\varepsilon^\mu(p,\lambda)$ 描述动量为 $\mathbf{p}$、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda=\pm1$ 对应于两种**横向极化**，$\lambda=0$ 对应于**纵向极化**。

有质量的实矢量场 $A^\mu(\mathbf{x},t)$ 的平面波展开应当包含正能解和负能解的所有动量模式的所有极化态，形式为

$$A^\mu(\mathbf{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right], \tag{3.142}$$

其中 $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$，产生算符 $a^\dagger_{\mathbf{p},\lambda}$ 和湮灭算符 $a_{\mathbf{p},\lambda}$ 带着极化指标 λ。容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t). \tag{3.143}$$

根据 (3.95) 式，共轭动量密度为

$$\begin{aligned}\pi_i \;=\; -\partial_0 A_i + \partial_i A_0 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\Big\{&[ip_0\varepsilon_i(\mathbf{p},\lambda) - ip_i\varepsilon_0(\mathbf{p},\lambda)]a_{\mathbf{p},\lambda}e^{-ip\cdot x}\\ &+[-ip_0\varepsilon_i^*(\mathbf{p},\lambda) + ip_i\varepsilon_0^*(\mathbf{p},\lambda)]a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big\},\end{aligned} \tag{3.144}$$

引入

$$\tilde{\varepsilon}_i(\mathbf{p},\lambda) \equiv \varepsilon_i(\mathbf{p},\lambda) - \frac{p_i}{p_0}\varepsilon_0(\mathbf{p},\lambda), \tag{3.145}$$

则有

$$p_0\varepsilon_i(\mathbf{p},\lambda) - p_i\varepsilon_0(\mathbf{p},\lambda) = p_0\tilde{\varepsilon}_i(\mathbf{p},\lambda), \tag{3.146}$$

从而，可以将共轭动量密度的平面波展开式写得更加紧凑：

$$\pi_i(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\Big]. \tag{3.147}$$

3.3.2 产生湮灭算符的对易关系

利用

$$\begin{aligned}&\int d^3x\, e^{iq\cdot x}A^\mu\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int d^3x \sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p-q)\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p+q)\cdot x}\Big]\\ &= \int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big]\\ &= \frac{1}{\sqrt{2E_{\mathbf{q}}}}\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda} + \varepsilon^{\mu*}(-\mathbf{q},\lambda)a^\dagger_{-\mathbf{q},\lambda}e^{2iq^0t}\Big]\end{aligned} \tag{3.148}$$

和

$$\begin{aligned}\int d^3x\, e^{iq\cdot x}\partial_0 A^\mu \;&=\; \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\int d^3x\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p-q)\cdot x} - \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p+q)\cdot x}\Big]\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q})\\ &\qquad\qquad - \varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big]\\ &= \frac{-iq_0}{\sqrt{2E_{\mathbf{q}}}}\sum_{\lambda=\pm,0}\Big[\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda} - \varepsilon^{\mu*}(-\mathbf{q},\lambda)a^\dagger_{-\mathbf{q},\lambda}e^{2iq^0t}\Big],\end{aligned} \tag{3.149}$$

以及正交归一关系 (3.132)，可得

$$
\begin{aligned}
\varepsilon_\mu^*(\mathbf{q},\lambda')\int d^3x\, e^{iq\cdot x}\,(\partial_0 A^\mu - iq_0 A^\mu) &= \varepsilon_\mu(\mathbf{q},\lambda')\,\frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\lambda=\pm,0}\varepsilon^\mu(\mathbf{q},\lambda)a_{\mathbf{q},\lambda}\\
&= -i\sqrt{2E_\mathbf{q}}\sum_{\lambda=\pm,0}\delta_{\lambda'\lambda}a_{\mathbf{q},\lambda} = -i\sqrt{2E_\mathbf{q}}\,a_{\mathbf{q},\lambda'}. \qquad (3.150)
\end{aligned}
$$

由 Lorenz 条件 (3.91) 可得

$$
\partial_0 A^0 = -\partial_i A^i, \qquad (3.151)
$$

根据 (3.95) 式，有

$$
\partial_0 A^i = -\partial_0 A_i = \pi_i - \partial_i A_0 = \pi_i - \partial_i A^0. \qquad (3.152)
$$

于是，湮灭算符 $a_{\mathbf{p},\lambda}$ 可表达为

$$
\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{i}{\sqrt{2E_\mathbf{p}}}\varepsilon_\mu^*(\mathbf{p},\lambda)\int d^3x\, e^{ip\cdot x}\,(\partial_0 A^\mu - ip_0 A^\mu)\\
&= \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\,[\varepsilon_0^*(\mathbf{p},\lambda)\partial_0 A^0 + \varepsilon_i^*(\mathbf{p},\lambda)\partial_0 A^i - ip_0\varepsilon_\mu^*(\mathbf{p},\lambda)A^\mu]\\
&= \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\,\big[-\varepsilon_0^*(\mathbf{p},\lambda)\partial_i A^i + \varepsilon_i^*(\mathbf{p},\lambda)\pi_i - \varepsilon_i^*(\mathbf{p},\lambda)\partial_i A^0\\
&\qquad - ip_0\varepsilon_0^*(\mathbf{p},\lambda)A^0 - ip_0\varepsilon_i^*(\mathbf{p},\lambda)A^i\big]. \qquad (3.153)
\end{aligned}
$$

上式最后两行方括号中的第一项和第三项可以通过分部积分化为

$$
\begin{aligned}
\int d^3x\, e^{ip\cdot x}[-\varepsilon_0^*(\mathbf{p},\lambda)\partial_i A^i - \varepsilon_i^*(\mathbf{p},\lambda)\partial_i A^0] &= \int d^3x\,[\varepsilon_0^*(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x})A^i + \varepsilon_i^*(\mathbf{p},\lambda)(\partial_i e^{ip\cdot x})A^0]\\
&= \int d^3x\,[ip_i\varepsilon_0^*(\mathbf{p},\lambda)e^{ip\cdot x}A^i + ip_i\varepsilon_i^*(\mathbf{p},\lambda)e^{ip\cdot x}A^0]\\
&= \int d^3x\, e^{ip\cdot x}[i\varepsilon_0^*(\mathbf{p},\lambda)p_i A^i + ip_i\varepsilon_i^*(\mathbf{p},\lambda)A^0], \qquad (3.154)
\end{aligned}
$$

从而，有

$$
\begin{aligned}
a_{\mathbf{p},\lambda} &= \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\,\big[i\varepsilon_0^*(\mathbf{p},\lambda)p_i A^i + \varepsilon_i^*(\mathbf{p},\lambda)\pi_i + ip_i\varepsilon_i^*(\mathbf{p},\lambda)A^0\\
&\qquad - ip_0\varepsilon_0^*(\mathbf{p},\lambda)A^0 - ip_0\varepsilon_i^*(\mathbf{p},\lambda)A^i\big]\\
&= \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\,\{\varepsilon_i^*(\mathbf{p},\lambda)\pi_i - ip^\mu\varepsilon_\mu^*(\mathbf{p},\lambda)A^0 - i[p_0\varepsilon_i^*(\mathbf{p},\lambda) - p_i\varepsilon_0^*(\mathbf{p},\lambda)]A^i\}. \qquad (3.155)
\end{aligned}
$$

再利用四维横向条件 (3.127) 和 (3.146) 式，得到

$$
a_{\mathbf{p},\lambda} = \frac{i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{ip\cdot x}\,\left[-\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(x) - ip_0\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)A^i(x)\right]. \qquad (3.156)
$$

对上式取厄米共轭，得

$$
a_{\mathbf{p},\lambda}^\dagger = \frac{-i}{\sqrt{2E_\mathbf{p}}}\int d^3x\, e^{-ip\cdot x}\,\left[-\varepsilon^i(\mathbf{p},\lambda)\pi_i(x) + ip_0\tilde{\varepsilon}_i(\mathbf{p},\lambda)A^i(x)\right]. \qquad (3.157)
$$

利用等时对易关系 (3.96)，可得湮灭算符与产生算符的对易关系为

$$
\begin{aligned}
&[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x-q\cdot y)}\big[-\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t) - ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)A^i(\mathbf{x},t), \\
&\qquad\qquad -\varepsilon^j(\mathbf{q},\lambda')\pi_j(\mathbf{y},t) + iq_0\tilde{\varepsilon}_j(\mathbf{q},\lambda')A^j(\mathbf{y},t)\big] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x-q\cdot y)}\big\{-iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t), A^j(\mathbf{y},t)] \\
&\qquad\qquad + ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)]\big\} \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x-q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,[-q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_j(\mathbf{q},\lambda')\delta^j{}_i - p_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^j(\mathbf{q},\lambda')\delta^i{}_j] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, e^{i(p^0-q^0)t}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\,[-E_\mathbf{q}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{q},\lambda') - E_\mathbf{p}\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')] \\
&= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') + \tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda')].
\end{aligned}
\tag{3.158}
$$

根据定义式 (3.145)、四维横向条件 (3.127) 和正交归一关系 (3.132)，有

$$
\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i(\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') - \frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i(\mathbf{p},\lambda') + \frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0(\mathbf{p},\lambda') \\
&= \varepsilon^{\mu*}(\mathbf{p},\lambda)\varepsilon_\mu(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'},
\end{aligned}
\tag{3.159}
$$

取复共轭，可得

$$
\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda') = -\delta_{\lambda\lambda'}. \tag{3.160}
$$

于是，

$$
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,(-\delta_{\lambda\lambda'} - \delta_{\lambda\lambda'}) = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}). \tag{3.161}
$$

另一方面，

$$
\begin{aligned}
&[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\big[-\varepsilon^{i*}(\mathbf{p},\lambda)\pi_i(\mathbf{x},t) - ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)A^i(\mathbf{x},t), \\
&\qquad\qquad -\varepsilon^{j*}(\mathbf{q},\lambda')\pi_j(\mathbf{y},t) - iq_0\tilde{\varepsilon}^*_j(\mathbf{q},\lambda')A^j(\mathbf{y},t)\big] \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\big\{iq_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_j(\mathbf{q},\lambda')[\pi_i(\mathbf{x},t), A^j(\mathbf{y},t)] \\
&\qquad\qquad + ip_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')[A^i(\mathbf{x},t), \pi_j(\mathbf{y},t)]\big\} \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, d^3y\, e^{i(p\cdot x+q\cdot y)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,[q_0\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_j(\mathbf{q},\lambda')\delta^j{}_i - p_0\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{j*}(\mathbf{q},\lambda')\delta^i{}_j] \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\int d^3x\, e^{i(p^0+q^0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\,[E_\mathbf{q}\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}^*_i(\mathbf{q},\lambda') - E_\mathbf{p}\tilde{\varepsilon}^*_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')]
\end{aligned}
$$

$$= -\frac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{2iE_{\mathbf{p}}t}\left[\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda') - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')\right]. \tag{3.162}$$

对四维横向条件 (3.127) 取复共轭，得

$$p_\mu\varepsilon^{\mu*}(\mathbf{p},\lambda) = p_0\varepsilon^{0*}(\mathbf{p},\lambda) + p_i\varepsilon^{i*}(\mathbf{p},\lambda) = 0. \tag{3.163}$$

将上式中的 $\mathbf{p}$ 替换成 $-\mathbf{p}$，得

$$p_0\varepsilon^{0*}(-\mathbf{p},\lambda) - p_i\varepsilon^{i*}(-\mathbf{p},\lambda) = 0. \tag{3.164}$$

因此，有

$$p_i\varepsilon^{i*}(\mathbf{p},\lambda) = -p_0\varepsilon^{0*}(\mathbf{p},\lambda), \quad -p_i\varepsilon^{i*}(-\mathbf{p},\lambda) = -p_0\varepsilon^{0*}(-\mathbf{p},\lambda), \tag{3.165}$$

或者写成

$$\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda) = p_0\varepsilon^{0*}(\mathbf{p},\lambda), \quad -\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda) = p_0\varepsilon^{0*}(-\mathbf{p},\lambda). \tag{3.166}$$

从而，可得

$$\begin{aligned}
\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda') &= \varepsilon^{i*}(\mathbf{p},\lambda)\left[\varepsilon_i^*(-\mathbf{p},\lambda) + \frac{p_i}{p_0}\varepsilon_0^*(-\mathbf{p},\lambda)\right] \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda') + \frac{1}{p_0}p_i\varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda') \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda') - \frac{1}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda') \\
&= \varepsilon^{i*}(\mathbf{p},\lambda)\varepsilon_i^*(-\mathbf{p},\lambda') - \varepsilon^{0*}(\mathbf{p},\lambda)\varepsilon_0^*(-\mathbf{p},\lambda'),
\end{aligned} \tag{3.167}$$

$$\begin{aligned}
\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') &= \left[\varepsilon_i^*(\mathbf{p},\lambda) - \frac{p_i}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)\right]\varepsilon^{i*}(-\mathbf{p},\lambda') \\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \frac{1}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)p_i\varepsilon^{i*}(-\mathbf{p},\lambda') \\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \frac{1}{p_0}\varepsilon_0^*(\mathbf{p},\lambda)p_0\varepsilon^{0*}(-\mathbf{p},\lambda') \\
&= \varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \varepsilon_0^*(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda').
\end{aligned} \tag{3.168}$$

可见，$\varepsilon^{i*}(\mathbf{p},\lambda)\tilde{\varepsilon}_i^*(-\mathbf{p},\lambda') - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') = 0$，故

$$[a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = 0. \tag{3.169}$$

综上，产生湮灭算符的对易关系为

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0. \tag{3.170}$$

3.3.3 哈密顿量和总动量

由 (3.95) 式有

$$\pi^i = -\pi_i = \partial_0 A_i - \partial_i A_0 = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}, \tag{3.171}$$

写成空间矢量的形式为

$$\boldsymbol{\pi} = -\dot{\mathbf{A}} - \nabla A_0, \tag{3.172}$$

故

$$\dot{\mathbf{A}} = -\boldsymbol{\pi} - \nabla A_0. \tag{3.173}$$

Proca 方程 (3.88) 在 $\nu = 0$ 时的形式是 $\partial_\mu F^{\mu 0} + m^2 A^0 = 0$，因此，

$$A^0 = -\frac{1}{m^2}\partial_\mu F^{\mu 0} = -\frac{1}{m^2}\partial_i F^{i0} = -\frac{1}{m^2}\partial_i \pi^i = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}. \tag{3.174}$$

从而，可得

$$-\boldsymbol{\pi}\cdot\dot{\mathbf{A}} = \boldsymbol{\pi}\cdot(\boldsymbol{\pi} + \nabla A_0) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) - A_0(\nabla\cdot\boldsymbol{\pi}) = \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2. \tag{3.175}$$

另一方面，

$$\frac{1}{2}F_{0i}F^{0i} = \frac{1}{2}\pi_i\pi^i = -\frac{1}{2}\boldsymbol{\pi}^2. \tag{3.176}$$

利用 (1.82) 式可得

$$F^{ij} = \partial^i A^j - \partial^j A^i = (\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm})\partial^m A^n = \varepsilon^{ijk}\varepsilon^{kmn}\partial^m A^n = -\varepsilon^{ijk}\varepsilon^{kmn}\partial_m A^n, \tag{3.177}$$

从而，

$$\begin{aligned}\frac{1}{4}F_{ij}F^{ij} &= \frac{1}{4}F^{ij}F^{ij} = \frac{1}{4}\varepsilon^{ijk}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{ijl}\varepsilon^{lpq}\partial_p A^q = \frac{1}{4}2\delta^{kl}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{lpq}\partial_p A^q\\ &= \frac{1}{2}\varepsilon^{kmn}(\partial_m A^n)\varepsilon^{kpq}\partial_p A^q = \frac{1}{2}(\nabla\times\mathbf{A})^2.\end{aligned} \tag{3.178}$$

于是，有

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2. \tag{3.179}$$

根据 (1.117) 式，有质量矢量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi_i\partial_0 A^i - \mathcal{L} = \pi_i\partial_0 A^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\\ &= -\boldsymbol{\pi}\cdot\dot{\mathbf{A}} - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2}m^2(A_0^2 - \mathbf{A}^2)\\ &= \boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2 - \frac{1}{2}\boldsymbol{\pi}^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 - \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}m^2\mathbf{A}^2\\ &= \frac{1}{2}\boldsymbol{\pi}^2 + \nabla\cdot(A_0\boldsymbol{\pi}) + \frac{1}{2m^2}(\nabla\cdot\boldsymbol{\pi})^2 + \frac{1}{2}(\nabla\times\mathbf{A})^2 + \frac{1}{2}m^2\mathbf{A}^2.\end{aligned} \tag{3.180}$$

上式最后一行第二项是一个全散度，对全空间积分时它没有贡献。于是，哈密顿量为

$$H = \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\left[\boldsymbol{\pi}^2 + \frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2 + (\nabla\times\mathbf{A})^2 + m^2\mathbf{A}^2\right]. \tag{3.181}$$

下面逐项进行计算。

哈密顿量的第一项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,\boldsymbol{\pi}^2\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(ip_0)(iq_0)\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\qquad\cdot\left[\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\quad-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}+\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\quad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{-\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&\qquad+\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
&=\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}E^2_{\mathbf{p}}\Big[\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}+\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&\quad-\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}-\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}
\tag{3.182}
$$

第二项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\,\frac{1}{m^2}(\nabla\cdot\boldsymbol{\pi})^2\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\frac{(ip_0)(iq_0)}{m^2}\left[i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x}+i\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right]\\
&\qquad\times\left[i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-iq\cdot x}+i\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{iq\cdot x}\right]\\
&=-\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3x\,d^3p\,d^3q\;p_0q_0}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}\,m^2}\Big\{-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x}\\
&\quad-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x}-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x}\\
&\quad-[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big\}\\
&=\frac{1}{2}\sum_{\lambda\lambda'}\int\frac{d^3p\,d^3q\;p_0q_0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}\,m^2}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad+[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big)\\
&\qquad+\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big([\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{q},\lambda')]a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad+[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{q}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{q},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \frac{E_{\mathbf{p}}^2}{m^2} \Big\{ [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda'} \\
&\quad + [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda')] a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'} - [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\quad - [\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(-\mathbf{p},\lambda')] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t} \Big\}. \qquad (3.183)
\end{aligned}
$$

第三项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\, (\nabla\times\mathbf{A})^2 \\
&= \frac{1}{2}\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[i\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} - i\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x} \Big] \\
&\qquad \cdot \Big[i\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda') a_{\mathbf{q},\lambda'} e^{-ip\cdot x} - i\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda') a^\dagger_{\mathbf{q},\lambda'} e^{ip\cdot x} \Big] \\
&= \frac{1}{2}\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')] a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{-i(p-q)\cdot x} \\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')] a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{i(p-q)\cdot x} - [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} \\
&\quad - [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{i(p+q)\cdot x} \Big\} \\
&= \frac{1}{2}\sum_{\lambda\lambda'} \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big\{ \delta^{(3)}(\mathbf{p}-\mathbf{q}) \Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')] a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{-i(p_0-q_0)t} \\
&\qquad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')] a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{i(p_0-q_0)t} \Big) \\
&\qquad - \delta^{(3)}(\mathbf{p}+\mathbf{q}) \Big([\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}(\mathbf{q},\lambda')] a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p_0+q_0)t} \\
&\qquad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{q}\times\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{i(p_0+q_0)t} \Big) \Big\} \\
&= \sum_{\lambda\lambda'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}} \Big\{ [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{p},\lambda'} \\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda')] a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda'} + [\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] a_{\mathbf{p},\lambda} a_{-\mathbf{p},\lambda'} e^{-2iE_{\mathbf{p}}t} \\
&\quad + [\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')] a^\dagger_{\mathbf{p},\lambda} a^\dagger_{-\mathbf{p},\lambda'} e^{2iE_{\mathbf{p}}t} \Big\}. \qquad (3.184)
\end{aligned}
$$

第四项是

$$
\begin{aligned}
&\frac{1}{2}\int d^3x\, m^2 \mathbf{A}^2 \\
&= \frac{1}{2}\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q\ m^2}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} e^{-ip\cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x} \Big] \\
&\qquad \cdot \Big[\boldsymbol{\varepsilon}(\mathbf{q},\lambda') a_{\mathbf{q},\lambda'} e^{-ip\cdot x} + \boldsymbol{\varepsilon}^*(\mathbf{q},\lambda') a^\dagger_{\mathbf{q},\lambda'} e^{ip\cdot x} \Big] \\
&= \frac{1}{2}\sum_{\lambda\lambda'} \int \frac{d^3x\, d^3p\, d^3q\ m^2}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{-i(p-q)\cdot x} \\
&\quad + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{i(p-q)\cdot x} + \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda') a_{\mathbf{p},\lambda} a_{\mathbf{q},\lambda'} e^{-i(p+q)\cdot x} \\
&\quad + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda') a^\dagger_{\mathbf{p},\lambda} a^\dagger_{\mathbf{q},\lambda'} e^{i(p+q)\cdot x} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{\lambda\lambda'}\int \frac{d^3p\,d^3q\;m^2}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t}\\
&\qquad\qquad + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big]\\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t}\\
&\qquad\qquad + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\}\\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}m^2\Big[\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'} + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&\qquad + \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} + \boldsymbol{\varepsilon}^*(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big].
\end{aligned}
\tag{3.185}
$$

综合起来，哈密顿量化为

$$
\begin{aligned}
H = \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}\Big[&f_1(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'} + f_1^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\\
&+ f_2(\mathbf{p},\lambda,\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} + f_2^*(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big],
\end{aligned}
\tag{3.186}
$$

其中，

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda') \equiv\; & E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda') + \frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda'),
\end{aligned}
\tag{3.187}
$$

$$
\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda') \equiv\; & -E_{\mathbf{p}}^2\,\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda') - \frac{E_{\mathbf{p}}^2}{m^2}[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]\\
&+[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda').
\end{aligned}
\tag{3.188}
$$

现在，我们计算 $f_1(\mathbf{p},\lambda,\lambda')$。由 (3.145)、(3.166) 和 (3.132) 式，可得

$$
\begin{aligned}
&\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda') = \left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{\mathbf{p}}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]\\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) + \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^{0*}(\mathbf{p},\lambda') - \frac{\varepsilon_0^*(\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda) + \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= -\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda') + \left(\frac{|\mathbf{p}|^2}{p_0^2} - 1\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda')\\
&= \delta_{\lambda\lambda'} - \frac{m^2}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}
\tag{3.189}
$$

另一方面，

$$
\begin{aligned}
&[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}^*(\mathbf{p},\lambda')]\\
&= \left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0^*(\mathbf{p},\lambda')\right]
\end{aligned}
$$

$$
\begin{aligned}
&= [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] - \frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0^*(\mathbf{p},\lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] + \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') \\
&= p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda') - \frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') \\
&\quad - \frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda) + \frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') \\
&= \left(p_0^2 - 2|\mathbf{p}|^2 + \frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') = \left[p_0^2 - |\mathbf{p}|^2 + \frac{|\mathbf{p}|^2}{p_0^2}(|\mathbf{p}|^2 - p_0^2)\right]\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') \\
&= \left(m^2 - m^2\frac{|\mathbf{p}|^2}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') = \frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda').
\end{aligned}
\tag{3.190}
$$

对于任意空间矢量 $\mathbf{a}$ 和 $\mathbf{b}$，利用 (1.82) 式，有

$$
\begin{aligned}
(\mathbf{p}\times\mathbf{a})\cdot(\mathbf{p}\times\mathbf{b}) &= \varepsilon^{ijk}p^j a^k \varepsilon^{imn}p^m b^n = (\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km})p^j a^k p^m b^n \\
&= p^j a^k p^j b^k - p^j a^k p^k b^j = |\mathbf{p}|^2\,\mathbf{a}\cdot\mathbf{b} - (\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b}),
\end{aligned}
\tag{3.191}
$$

从而，可得

$$
\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] &= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - [\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda')] \\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^{0*}(\mathbf{p},\lambda') \\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda').
\end{aligned}
\tag{3.192}
$$

于是，(3.187) 式化为

$$
\begin{aligned}
f_1(\mathbf{p},\lambda,\lambda') &= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} - m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') + m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0^*(\mathbf{p},\lambda') \\
&\quad + |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda') + m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') \\
&= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} + E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}^*(\mathbf{p},\lambda') - E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^{0*}(\mathbf{p},\lambda') \\
&= E_{\mathbf{p}}^2\delta_{\lambda\lambda'} - E_{\mathbf{p}}^2\varepsilon_\mu(\mathbf{p},\lambda)\varepsilon^{\mu*}(\mathbf{p},\lambda') = 2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\end{aligned}
\tag{3.193}
$$

因此，

$$
f_1(\mathbf{p},\lambda,\lambda') = f_1^*(\mathbf{p},\lambda,\lambda') = 2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}.
\tag{3.194}
$$

接着，我们计算 $f_2(\mathbf{p},\lambda,\lambda')$。由 (3.145) 和 (3.166) 式，可得

$$
\begin{aligned}
&\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda') = \left[\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{\mathbf{p}}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\cdot\left[\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\mathbf{p}}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right] \\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^0(-\mathbf{p},\lambda') + \frac{\varepsilon_0(-\mathbf{p},\lambda')}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda) - \frac{|\mathbf{p}|^2}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda') \\
&= \boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda') + \frac{1}{E_{\mathbf{p}}^2}(2E_{\mathbf{p}}^2 - |\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda').
\end{aligned}
\tag{3.195}
$$

另一方面，

$$
\begin{aligned}
&[\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{p},\lambda)][\mathbf{p}\cdot\tilde{\boldsymbol{\varepsilon}}(-\mathbf{p},\lambda')]\\
=&\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)\right]\left[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')+\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(-\mathbf{p},\lambda')\right]\\
=&[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]+\frac{|\mathbf{p}|^2}{p_0}[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\varepsilon_0(-\mathbf{p},\lambda')\\
&-\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)[\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]-\frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&-p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda')+\frac{|\mathbf{p}|^2}{p_0}p_0\varepsilon^0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
&+\frac{|\mathbf{p}|^2}{p_0}\varepsilon_0(\mathbf{p},\lambda)p_0\varepsilon^0(\mathbf{p},\lambda')-\frac{|\mathbf{p}|^4}{p_0^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&\left(-p_0^2+2|\mathbf{p}|^2-\frac{|\mathbf{p}|^4}{p_0^2}\right)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')=-\frac{1}{E_{\mathbf{p}}^2}(E_{\mathbf{p}}^2-|\mathbf{p}|^2)^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
=&-\frac{m^4}{E_{\mathbf{p}}^2}\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda'),
\end{aligned}\tag{3.196}
$$

而

$$
\begin{aligned}
[\mathbf{p}\times\boldsymbol{\varepsilon}(\mathbf{p},\lambda)]\cdot[\mathbf{p}\times\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')] &= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')-[\mathbf{p}\cdot\boldsymbol{\varepsilon}(\mathbf{p},\lambda)][\mathbf{p}\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')]\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')+p_0\varepsilon^0(\mathbf{p},\lambda)p_0\varepsilon^0(-\mathbf{p},\lambda')\\
&= |\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')+E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda').
\end{aligned}\tag{3.197}
$$

于是，(3.188) 式化为

$$
\begin{aligned}
f_2(\mathbf{p},\lambda,\lambda') &= -E_{\mathbf{p}}^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')-(2E_{\mathbf{p}}^2-|\mathbf{p}|^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')+m^2\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')\\
&\quad+|\mathbf{p}|^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')+E_{\mathbf{p}}^2\varepsilon^0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda')+m^2\boldsymbol{\varepsilon}(\mathbf{p},\lambda)\cdot\boldsymbol{\varepsilon}(-\mathbf{p},\lambda')\\
&= (-2E_{\mathbf{p}}^2+|\mathbf{p}|^2+m^2+E_{\mathbf{p}}^2)\varepsilon_0(\mathbf{p},\lambda)\varepsilon_0(-\mathbf{p},\lambda')=0.
\end{aligned}\tag{3.198}
$$

因此，

$$
f_2(\mathbf{p},\lambda,\lambda')=f_2^*(\mathbf{p},\lambda,\lambda')=0.\tag{3.199}
$$

将 (3.194) 和 (3.199) 式代入 (3.186) 式，再利用产生湮灭算符的对易关系 (3.170)，可得有质量矢量场的哈密顿量为

$$
\begin{aligned}
H &= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}}2E_{\mathbf{p}}^2\delta_{\lambda\lambda'}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}^\dagger+a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda'}\right)=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\left(a_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}^\dagger+a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}\right)\\
&= \sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\,a_{\mathbf{p},\lambda}^\dagger a_{\mathbf{p},\lambda}+(2\pi)^3\delta^{(3)}(0)\int\frac{d^3p}{(2\pi)^3}\frac{3}{2}E_{\mathbf{p}}.
\end{aligned}\tag{3.200}
$$

上式第二行第一项是所有动量模式所有极化态所有粒子贡献的能量之和，第二项是零点能。

根据 (1.156) 式，有质量矢量场的总动量为

$$
\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi_i \nabla A^i \\
&= -\sum_{\lambda\lambda'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}(ip_0)\left[\tilde{\varepsilon}_i(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} - \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ip\cdot x}\right] \\
&\qquad \times\left[i\mathbf{q}\varepsilon^i(\mathbf{q},\lambda')a_{\mathbf{q},\lambda'}e^{-ip\cdot x} - i\mathbf{q}\varepsilon^{i*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}e^{ip\cdot x}\right] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3x\, d^3p\, d^3q\ p_0\mathbf{q}}{(2\pi)^6\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big[-\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p-q)\cdot x} \\
&\qquad -\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p-q)\cdot x} + \tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p+q)\cdot x} \\
&\qquad +\tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p+q)\cdot x}\Big] \\
&= \sum_{\lambda\lambda'}\int \frac{d^3p\, d^3q\ p_0\mathbf{q}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\Big\{-\delta^{(3)}(\mathbf{p}-\mathbf{q})\Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{-i(p_0-q_0)t} \\
&\qquad\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{i(p_0-q_0)t}\Big] \\
&\qquad + \delta^{(3)}(\mathbf{p}+\mathbf{q})\Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(\mathbf{q},\lambda')a_{\mathbf{p},\lambda}a_{\mathbf{q},\lambda'}e^{-i(p_0+q_0)t} \\
&\qquad\qquad + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{\mathbf{q},\lambda'}e^{i(p_0+q_0)t}\Big]\Big\} \\
&= -\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\Big[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^{i*}(\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^i(\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'} \\
&\qquad + \tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big]. \qquad (3.201)
\end{aligned}
$$

由 (3.145) 和 (3.165) 式可得

$$
\begin{aligned}
\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') &= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_i\varepsilon^i(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \frac{\varepsilon_0(\mathbf{p},\lambda)}{p_0}p_0\varepsilon^0(-\mathbf{p},\lambda') \\
&= \varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda'), \qquad (3.202)
\end{aligned}
$$

从而，有

$$
\begin{aligned}
&-\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\left[\tilde{\varepsilon}_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda')a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} + \tilde{\varepsilon}_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda')a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\right] \\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\Big\{[\varepsilon_i(\mathbf{p},\lambda)\varepsilon^i(-\mathbf{p},\lambda') - \varepsilon_0(\mathbf{p},\lambda)\varepsilon^0(-\mathbf{p},\lambda')]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(\mathbf{p},\lambda)\varepsilon^{i*}(-\mathbf{p},\lambda') - \varepsilon_0^*(\mathbf{p},\lambda)\varepsilon^{0*}(-\mathbf{p},\lambda')]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big\} \\
&= \sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{-\mathbf{p}}{2}\Big\{[\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda) - \varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)]a_{-\mathbf{p},\lambda'}a_{\mathbf{p},\lambda}e^{-2iE_{\mathbf{p}}t} \\
&\qquad + [\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda) - \varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)]a^\dagger_{-\mathbf{p},\lambda'}a^\dagger_{\mathbf{p},\lambda}e^{2iE_{\mathbf{p}}t}\Big\}
\end{aligned}
$$

$$= -\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\Big\{[\varepsilon_i(-\mathbf{p},\lambda')\varepsilon^i(\mathbf{p},\lambda)-\varepsilon_0(-\mathbf{p},\lambda')\varepsilon^0(\mathbf{p},\lambda)]a_{\mathbf{p},\lambda}a_{-\mathbf{p},\lambda'}e^{-2iE_{\mathbf{p}}t}$$
$$+[\varepsilon_i^*(-\mathbf{p},\lambda')\varepsilon^{i*}(\mathbf{p},\lambda)-\varepsilon_0^*(-\mathbf{p},\lambda')\varepsilon^{0*}(\mathbf{p},\lambda)]a^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}e^{2iE_{\mathbf{p}}t}\Big\}. \tag{3.203}$$

上式第二步进行了 $\mathbf{p}\to-\mathbf{p}$ 的替换和 $\lambda\leftrightarrow\lambda'$ 的互换，由于要对整个三维动量空间积分且对 λ 和 λ' 进行求和，这两种操作都不会改变结果。第三步用到产生湮灭算符的对易关系 (3.170)。留意到第一步与第三步的结果互为相反数，可知上式为零。因此，(3.201) 式最后两行方括号中最后两项没有贡献。再利用 (3.160) 式，可得

$$\mathbf{P} = -\sum_{\lambda\lambda'}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\left[-\delta_{\lambda\lambda'}a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda'}-\delta_{\lambda\lambda'}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda'}\right]=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}}{2}\left[a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}+a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}\right]$$
$$=\sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}+\frac{3}{2}\delta^{(3)}(0)\int d^3p\,\mathbf{p}=\sum_{\lambda=\pm,0}\int\frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}. \tag{3.204}$$

这表明总动量是所有动量模式所有极化态所有粒子贡献的动量之和。

3.4 无质量矢量场的正则量子化

3.4.1 无质量情况下的极化矢量

当质量 $m=0$ 时，由 (3.100) 和 (3.101) 式定义的两个横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 的形式不变，但 (3.111) 式显然不是纵向极化矢量 $e^\mu(\mathbf{p},3)$ 的良好定义。实际上，在满足正确归一化的条件下，$m=0$ 时不能构造第 3 个符合四维横向条件的极化矢量。另一方面，由于无质量矢量粒子的动量 p^μ 的内积为 $p^2=0$，也不能像 (3.115) 式那样将类时极化矢量 $e^\mu(\mathbf{p},0)$ 取为正比于 p^μ 的矢量，否则将出现 $e_\mu(\mathbf{p},0)e^\mu(\mathbf{p},0)=0$ 而不能得到正确的归一化。因此，我们需要重新定义 $e^\mu(\mathbf{p},3)$ 和 $e^\mu(\mathbf{p},0)$。

在用 (3.100) 和 (3.101) 式定义 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 时，我们已经选取了一个特定的惯性参考系。在这个参考系中，可以定义一个类时单位矢量

$$n^\mu=(1,0,0,0), \tag{3.205}$$

它的 Lorentz 不变内积是

$$n^2=1. \tag{3.206}$$

然后，将类时极化矢量 $e^\mu(\mathbf{p},0)$ 在此参考系中的形式就取为 n^μ，即

$$e^\mu(\mathbf{p},0)=n^\mu. \tag{3.207}$$

$e^\mu(\mathbf{p},0)$ 在其它惯性参考系中的形式可通过 Lorentz 变换得到。另一方面，纵向极化矢量 $e^\mu(\mathbf{p},3)$ 可以用 p^μ 和 n^μ 定义成如下 Lorentz 协变的形式：

$$e^\mu(\mathbf{p},3)=\frac{p^\mu-(p\cdot n)n^\mu}{p\cdot n}. \tag{3.208}$$

$p^2 = (p^0)^2 - |\mathbf{p}|^2 = 0$ 表明

$$p^0 = |\mathbf{p}|, \tag{3.209}$$

从而，$e^\mu(\mathbf{p}, 3)$ 在我们选取的参考系中化为

$$e^\mu(\mathbf{p}, 3) = \frac{p^\mu - (p \cdot n)n^\mu}{p \cdot n} = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right). \tag{3.210}$$

这样定义的 $e^\mu(\mathbf{p}, 0)$ 和 $e^\mu(\mathbf{p}, 3)$ 满足正交归一关系 (3.98)：

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 0) = n^2 = 1, \quad e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, 3) = -\frac{\mathbf{p} \cdot \mathbf{p}}{|\mathbf{p}|^2} = -1; \tag{3.211}$$

$$e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 1) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 2) = e_\mu(\mathbf{p}, 0)e^\mu(\mathbf{p}, 3) = 0; \tag{3.212}$$

$$e_\mu(\mathbf{p}, 3)e^\mu(\mathbf{p}, i) = -\frac{1}{|\mathbf{p}|}\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, i) = 0, \quad i = 1, 2. \tag{3.213}$$

此外，可以验证，由 (3.100)、(3.101)、(3.207) 和 (3.208) 式定义的这组极化矢量确实满足完备性关系 (3.99)：

$$\begin{aligned}
&\sum_{\sigma=0}^{3} g_{\sigma\sigma} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma) \\
&= e_\mu(\mathbf{p}, 0)e_\nu(\mathbf{p}, 0) - e_\mu(\mathbf{p}, 1)e_\nu(\mathbf{p}, 1) - e_\mu(\mathbf{p}, 2)e_\nu(\mathbf{p}, 2) - e_\mu(\mathbf{p}, 3)e_\nu(\mathbf{p}, 3) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1p^3p^1p^3 & p^1p^3p^2p^3 & -p^1p^3|\mathbf{p}_\mathrm{T}|^2 \\ 0 & p^2p^3p^1p^3 & p^2p^3p^2p^3 & -p^2p^3|\mathbf{p}_\mathrm{T}|^2 \\ 0 & -|\mathbf{p}_\mathrm{T}|^2p^1p^3 & -|\mathbf{p}_\mathrm{T}|^2p^2p^3 & |\mathbf{p}_\mathrm{T}|^4 \end{pmatrix} \\
&\quad - \frac{1}{|\mathbf{p}_\mathrm{T}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^2p^2 & -p^2p^1 & 0 \\ 0 & -p^1p^2 & p^1p^1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{|\mathbf{p}|^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p^1p^1 & p^1p^2 & p^1p^3 \\ 0 & p^2p^1 & p^2p^2 & p^2p^3 \\ 0 & p^3p^1 & p^3p^2 & p^3p^3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(p^1)^2(p^3)^2+(p^2)^2|\mathbf{p}|^2+(p^1)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^1p^3}{|\mathbf{p}|^2} - \frac{p^1p^3}{|\mathbf{p}|^2} \\ 0 & -\frac{p^1p^2[(p^3)^2-|\mathbf{p}|^2]+p^1p^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & -\frac{(p^2)^2(p^3)^2+(p^1)^2|\mathbf{p}|^2+(p^2)^2|\mathbf{p}_\mathrm{T}|^2}{|\mathbf{p}|^2|\mathbf{p}_\mathrm{T}|^2} & \frac{p^2p^3}{|\mathbf{p}|^2} - \frac{p^2p^3}{|\mathbf{p}|^2} \\ 0 & \frac{p^1p^3}{|\mathbf{p}|^2} - \frac{p^1p^3}{|\mathbf{p}|^2} & \frac{p^2p^3}{|\mathbf{p}|^2} - \frac{p^2p^3}{|\mathbf{p}|^2} & -\frac{|\mathbf{p}_\mathrm{T}|^2+(p^3)^2}{|\mathbf{p}|^2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}.
\end{aligned} \tag{3.214}$$

不过，$e^\mu(\mathbf{p},0)$ 和 $e^\mu(\mathbf{p},3)$ 都不满足四维横向条件：

$$p_\mu e^\mu(\mathbf{p},0)=p\cdot n=p^0=|\mathbf{p}|,\quad p_\mu e^\mu(\mathbf{p},3)=-\frac{\mathbf{p}\cdot\mathbf{p}}{|\mathbf{p}|}=-|\mathbf{p}|=-p\cdot n. \tag{3.215}$$

横向极化矢量 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$ 具有求和关系

$$\begin{aligned}-\sum_{\sigma=1}^{2}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)&=\sum_{\sigma=1}^{2}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=g_{\mu\nu}-g_{00}e_\mu(\mathbf{p},0)e_\nu(\mathbf{p},0)-g_{33}e_\mu(\mathbf{p},3)e_\nu(\mathbf{p},3)\\&=g_{\mu\nu}-n_\mu n_\nu+\frac{p_\mu-(p\cdot n)n_\mu}{p\cdot n}\frac{p_\nu-(p\cdot n)n_\nu}{p\cdot n}\\&=g_{\mu\nu}-n_\mu n_\nu+\frac{p_\mu p_\nu-(p\cdot n)p_\mu n_\nu-(p\cdot n)p_\nu n_\mu+(p\cdot n)^2n_\mu n_\nu}{(p\cdot n)^2}\\&=g_{\mu\nu}+\frac{p_\mu p_\nu}{(p\cdot n)^2}-\frac{p_\mu n_\nu+p_\nu n_\mu}{p\cdot n},\end{aligned}\tag{3.216}$$

即

$$\sum_{\sigma=1}^{2}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=-g_{\mu\nu}-\frac{p_\mu p_\nu}{(p\cdot n)^2}+\frac{p_\mu n_\nu+p_\nu n_\mu}{p\cdot n}. \tag{3.217}$$

根据 (3.125) 式，作为螺旋度本征态的极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 满足

$$\begin{aligned}\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)&=\frac{1}{2}[e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)]\\&\quad+\frac{1}{2}[-e_\mu(\mathbf{p},1)+ie_\mu(\mathbf{p},2)][-e_\nu(\mathbf{p},1)-ie_\nu(\mathbf{p},2)]\\&=e_\mu(\mathbf{p},1)e_\nu(\mathbf{p},1)+e_\mu(\mathbf{p},2)e_\nu(\mathbf{p},2)=\sum_{\sigma=1}^{2}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma),\end{aligned}\tag{3.218}$$

因而具有**求和关系**

$$\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-g_{\mu\nu}-\frac{p_\mu p_\nu}{(p\cdot n)^2}+\frac{p_\mu n_\nu+p_\nu n_\mu}{p\cdot n}. \tag{3.219}$$

四维横向条件 $p_\mu\varepsilon^\mu(\mathbf{p},\pm)=0$ 在上式中体现为

$$p^\nu\sum_{\lambda=\pm}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-p_\mu-\frac{p_\mu p^2}{(p\cdot n)^2}+\frac{p_\mu(p\cdot n)+p^2n_\mu}{p\cdot n}=-p_\mu+p_\mu=0. \tag{3.220}$$

3.4.2 无质量矢量场与规范对称性

在自由有质量矢量场的拉氏量 (3.84) 中，令参数 $m=0$，就得到自由无质量实矢量场 $A^\mu(x)$ 的**拉氏量**

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.221}$$

其中 $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$。同理，令 Proca 方程中 $m=0$，就得到自由无质量矢量场的运动方程

$$\partial_\mu F^{\mu\nu}=0. \tag{3.222}$$

根据 1.5 节的讨论，这个方程就是无源的 **Maxwell 方程**。电磁场是一种无质量矢量场。作为电磁场的量子，**光子**是一种无质量矢量粒子。

可以对 $A^\mu(x)$ 作**规范变换** (gauge transformation)

$$A'^\mu(x) = A^\mu(x) + \partial^\mu\chi(x), \tag{3.223}$$

其中，作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数，依赖于时空坐标，因而这样的变换是**局域** (local) 变换。在此规范变换下，场强张量不变：

$$\begin{aligned} F'^{\mu\nu}(x) &= \partial^\mu[A^\nu(x) + \partial^\nu\chi(x)] - \partial^\nu[A^\mu(x) + \partial^\mu\chi(x)] \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \partial^\mu\partial^\nu\chi(x) - \partial^\nu\partial^\mu\chi(x) \\ &= \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = F^{\mu\nu}(x). \end{aligned} \tag{3.224}$$

因而，拉氏量 (3.221) 和无源 Maxwell 方程 (3.222) 都不会改变，这称为**规范对称性** (gauge symmetry)。

在经典电动力学中，这种对称性广为人知，它表明四维矢势 $A^\mu(x)$ 不能被唯一地确定，因而不是直接观测量。电动力学中的直接观测量都不依赖于 $\chi(x)$，也就是说，不依赖于**规范**的选取。规范对称性的存在对研究无质量矢量场带来了不便。为了便于计算，常常将规范固定下来，使得计算过程依赖于选取的规范，不过，最后得出的可观测量必须是规范不变 (gauge invariant) 的。

一种常用的规范是 **Lorenz 规范**，规范条件为

$$\partial_\mu A^\mu = 0. \tag{3.225}$$

它具有明显的 Lorentz 协变性。虽然这个规范条件看起来与有质量矢量场的 Lorenz 条件 (3.91) 相同，但是，在研究有质量矢量场时它是从运动方程推导出来的必须满足的条件，而在研究无质量矢量场时它只是一种人为选择。

对于任意的 $A^\mu(x)$，令规范变换函数 $\chi(x)$ 满足方程

$$\partial^2\chi(x) = -\partial_\mu A^\mu(x), \tag{3.226}$$

那么，作规范变换之后的场 $A'^\mu(x)$ 就会满足 Lorenz 规范条件：

$$\partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) + \partial^2\chi(x) = \partial_\mu A^\mu(x) - \partial_\mu A^\mu(x) = 0. \tag{3.227}$$

但是，经过这种变换之后，矢量场仍然没有被唯一地确定：对于满足 Lorenz 规范条件的矢量场 $A^\mu(x)$，取满足齐次波动方程

$$\partial^2\tilde{\chi}(x) = 0 \tag{3.228}$$

的任意规范变换函数 $\tilde{\chi}(x)$ 再作一次规范变换，都能得到满足 Lorenz 规范条件的另一个矢量场 $A'^\mu(x)$。可见，存在无穷多个规范等价的矢量场，它们描述相同的物理，而且全都满足 Lorenz 规范条件 (3.225)。

矢量场 $A^\mu(x)$ 有 4 个分量，因而在没有任何约束的情况下可以具有 4 个独立的自由度。要求 Lorenz 规范条件成立将减少 1 个独立自由度。但是，上述规范等价性表明，$A^\mu(x)$ 并没有 3 个独立的自由度，否则它在强加 Lorenz 规范条件之后就必须唯一地确定下来。实际上，无质量矢量场 $A^\mu(x)$ 只具有 *2 个独立的自由度*，也就是说，有 2 个虚假 (spurious) 的自由度。这在电动力学中是一个熟知的结论：电磁波具有 2 种独立的极化态，以螺旋度 λ 来表征的话，就是 $\lambda = +1$ (右旋极化) 和 $\lambda = -1$ (左旋极化) 的态。

在上一节讨论有质量矢量场 $A^\mu(x)$ 的量子化程序时，由于场的第 0 分量 $A^0(x)$ 不拥有非零的共轭动量密度，因而没有将它作为独立的正则运动变量。但这种情况并没有使正则量子化出现困难，因为 Proca 方程要求 $A^0(x)$ 不是独立变量，而是由 (3.174) 式决定的：

$$A^0 = -\frac{1}{m^2}\nabla \cdot \boldsymbol{\pi}. \tag{3.229}$$

于是，以场的空间分量 $A^i(x)$ 作为 3 个独立正则变量进行量子化是足够的，自由度恰好与有质量矢量粒子的 3 种物理极化态 (螺旋度 $\lambda = +1, 0, -1$) 相符。

当 $m = 0$ 时，(3.229) 式显然不能成立。因此，对于无质量矢量场，最好把 $A^0(x)$ 也当作独立的正则变量。为了使 $A^0(x)$ 拥有非零的共轭动量密度，可以在拉氏量中增加一个不会影响最终物理结果的项：

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.230}$$

其中 ξ 是一个可以自由选取的实参数。可以看出，在 $A^\mu(x)$ 满足 Lorenz 规范条件 (3.225) 的情况下，由 (3.230) 式定义的 $\mathcal{L}_1$ 等价于由 (3.221) 式定义的 $\mathcal{L}$。新增的项 $-\frac{1}{2}\xi(\partial_\mu A^\mu)^2$ 破坏了规范对称性，相当于把规范固定下来，因而称为**规范固定项** (gauge-fixing term)。可以将 $\mathcal{L}_1$ 展开为

$$\mathcal{L}_1 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\xi(\partial_\mu A^\mu)^2, \tag{3.231}$$

从而，A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_1}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \xi(\partial_\nu A^\nu)\frac{\partial(\partial_\sigma A^\sigma)}{\partial(\partial_0 A^\mu)} = -F_{0\mu} - \xi g_{\mu 0}\partial_\nu A^\nu, \tag{3.232}$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\xi\partial_\mu A^\mu. \tag{3.233}$$

因此，$\xi \neq 0$ 时 A^0 可以拥有非零的共轭动量密度 π_0。

现在，正则量子化程序要求 A^μ 和 π_μ 满足如下等时对易关系：

$$[A^\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)] = i\delta^\mu{}_\nu\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] = [\pi_\mu(\mathbf{x},t), \pi_\nu(\mathbf{y},t)] = 0. \tag{3.234}$$

但是，这样的等时对易关系与 Lorenz 规范条件相互矛盾。计算 A^0 与 $\partial_\mu A^\mu$ 的对易子，利用 (3.233) 式，可得

$$[A^0(\mathbf{x},t), \partial_\mu A^\mu(\mathbf{y},t)] = -\frac{1}{\xi}[A^0(\mathbf{x},t), \pi_0(\mathbf{y},t)] = -\frac{i}{\xi}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.235}$$

上式在 $\mathbf{x} = \mathbf{y}$ 处非零，因而必有 $\partial_\mu A^\mu \neq 0$。所以，$A^\mu$ 作为场算符在满足等时对易关系的同时不能满足 Lorenz 规范条件 (3.225)。这说明 Lorenz 规范条件虽然适用于经典场 $A^\mu(x)$，但对于量子场 $A^\mu(x)$ 来说限制太强了，下面会采用一个弱化的 Lorenz 规范条件。

由

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - \xi g^{\mu\nu}(\partial_\rho A^\rho), \quad \frac{\partial \mathcal{L}_1}{\partial A_\nu} = 0, \tag{3.236}$$

可得，与 $\mathcal{L}_1$ 对应的 Euler-Lagrange 方程为

$$0 = \partial_\mu \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu \partial_\mu A^\mu - \xi g^{\mu\nu}\partial_\mu(\partial_\rho A^\rho) = -\partial^2 A^\nu + (1-\xi)\partial^\nu(\partial_\rho A^\rho), \tag{3.237}$$

即

$$\partial^2 A^\mu - (1-\xi)\partial^\mu(\partial_\nu A^\nu) = 0. \tag{3.238}$$

若取 $\xi = 1$，则上式化为 **d'Alembert 方程**

$$\partial^2 A^\mu(x) = 0, \tag{3.239}$$

可以看作无质量情况下的 Klein-Gordon 方程。可见，将规范固定参数取为

$$\xi = 1 \tag{3.240}$$

将有利于简化计算，这种取法称为 **Feynman 规范**，本节后续计算采用这个规范。在 Feynman 规范下，拉氏量化为

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\nu A_\mu)\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A_\mu(\partial_\nu A^\nu) \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\nu(A_\mu \partial^\mu A^\nu) - \frac{1}{2}A_\mu \partial_\nu \partial^\mu A^\nu - \frac{1}{2}\partial^\mu(A_\mu \partial_\nu A^\nu) + \frac{1}{2}A_\mu \partial^\mu \partial_\nu A^\nu \\
&= -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu + \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu - A_\mu \partial_\nu A^\nu).
\end{aligned} \tag{3.241}$$

上式最后一行第二项是一个全散度，它不会影响作用量和运动方程，可以舍弃。因此，可以采用更加简化的拉氏量

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu. \tag{3.242}$$

此时，共轭动量密度为

$$\pi_\mu = \frac{\partial \mathcal{L}_2}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu. \tag{3.243}$$

对于 d'Alembert 方程 (3.239)，平面波解的正能解和负能解分别正比于 $\exp(-ip\cdot x)$ 和 $\exp(ip\cdot x)$，其中

$$p^0 = E_\mathbf{p} = |\mathbf{p}|. \tag{3.244}$$

使用上一小节讨论的实极化矢量组 $e^\mu(\mathbf{p},\sigma)$，可以对无质量矢量场 $A^\mu(\mathbf{x},t)$ 作如下平面波展开：

$$A^\mu(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) \left(a_{\mathbf{p};\sigma} e^{-ip\cdot x} + a^\dagger_{\mathbf{p};\sigma} e^{ip\cdot x}\right). \tag{3.245}$$

容易验证，这个展开式满足自共轭条件

$$[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t). \tag{3.246}$$

相应的共轭动量展开式为

$$\pi_\mu(\mathbf{x},t) = -\partial_0 A_\mu = \int \frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e_\mu(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right). \tag{3.247}$$

3.4.3 产生湮灭算符的对易关系

利用

$$\begin{aligned}
\int d^3x\, e^{iq\cdot x}A^\mu &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} + a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right]\\
&= \int d^3p\, \frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{1}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} + e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right]
\end{aligned} \tag{3.248}$$

和

$$\begin{aligned}
&\int d^3x\, e^{iq\cdot x}\partial_0 A^\mu\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\int d^3x \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{i(p+q)\cdot x}\right]\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left[a_{\mathbf{p};\sigma}e^{-i(p^0-q^0)t}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - a^\dagger_{\mathbf{p};\sigma}e^{i(p^0+q^0)t}\delta^{(3)}(\mathbf{p}+\mathbf{q})\right]\\
&= \frac{-iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3}\left[e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma} - e^\mu(-\mathbf{q},\sigma)a^\dagger_{-\mathbf{q};\sigma}e^{2iq^0t}\right],
\end{aligned} \tag{3.249}$$

以及正交归一关系 (3.98)，可得

$$\begin{aligned}
e_\mu(\mathbf{q},\sigma')\int d^3x\, e^{iq\cdot x}\left(\partial_0A^\mu - iq_0A^\mu\right) &= e_\mu(\mathbf{q},\sigma')\,\frac{-2iq_0}{\sqrt{2E_\mathbf{q}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{q},\sigma)a_{\mathbf{q};\sigma}\\
&= -i\sqrt{2E_\mathbf{q}}\sum_{\sigma=0}^{3}\delta_{\sigma'\sigma}a_{\mathbf{q};\sigma'} = -i\sqrt{2E_\mathbf{q}}\,a_{\mathbf{q};\sigma'}.
\end{aligned} \tag{3.250}$$

于是，有

$$a_{\mathbf{p};\sigma} = \frac{i}{\sqrt{2E_\mathbf{p}}}\,e_\mu(\mathbf{p},\sigma)\int d^3x\, e^{ip\cdot x}\left(\partial_0A^\mu - ip_0A^\mu\right). \tag{3.251}$$

对上式取厄米共轭，得

$$a^\dagger_{\mathbf{p};\sigma} = \frac{-i}{\sqrt{2E_\mathbf{p}}}\,e_\mu(\mathbf{p},\sigma)\int d^3x\, e^{-ip\cdot x}\left(\partial_0A^\mu + ip_0A^\mu\right). \tag{3.252}$$

根据等时对易关系 (3.234)，湮灭算符与产生算符的对易关系为

$$
\begin{aligned}
[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] &= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\qquad \times [\partial_0 A^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\ \partial_0 A^\nu(\mathbf{y},t) + iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\qquad \times [-\pi^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\ -\pi^\nu(\mathbf{y},t) + iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)} \\
&\qquad \times \{-iq_0\, [\pi^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] + ip_0\, [A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)]\} \\
&= \frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x - q\cdot y)}\, [-(p_0+q_0)g^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \int d^3x\, e^{i(p^0-q^0)t} e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&= -\frac{E_\mathbf{p}+E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') e^{i(E_\mathbf{p}-E_\mathbf{q})t} (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -(2\pi)^3 e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{3.253}
$$

最后一步用到正交归一关系 (3.98)。另一方面，两个湮灭算符之间的对易关系为

$$
\begin{aligned}
[a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] &= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\qquad \times [\partial_0 A^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\ \partial_0 A^\nu(\mathbf{y},t) - iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\qquad \times [-\pi^\mu(\mathbf{x},t) - ip_0 A^\mu(\mathbf{x},t),\ -\pi^\nu(\mathbf{y},t) - iq_0 A^\nu(\mathbf{y},t)] \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)} \\
&\qquad \times \{iq_0\, [\pi^\mu(\mathbf{x},t), A^\nu(\mathbf{y},t)] + ip_0\, [A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)]\} \\
&= \frac{-1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{q},\sigma') \int d^3x\, d^3y\, e^{i(p\cdot x + q\cdot y)}\, [(q_0-p_0)g^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})] \\
&= \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \int d^3x\, e^{i(p^0+q^0)t} e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{E_\mathbf{p}-E_\mathbf{q}}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') e^{i(E_\mathbf{p}+E_\mathbf{q})t} (2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q}) = 0.
\end{aligned} \tag{3.254}
$$

归纳起来，产生湮灭算符的对易关系为

$$
[a_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p};\sigma}, a_{\mathbf{q};\sigma'}] = [a^\dagger_{\mathbf{p};\sigma}, a^\dagger_{\mathbf{q};\sigma'}] = 0. \tag{3.255}
$$

3.4.4 哈密顿量和总动量

根据 (1.117)、(3.243) 和 (3.242) 式，无质量矢量场的哈密顿量密度是

$$
\begin{aligned}
\mathcal{H} &= \pi_\mu \partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu \\
&= -\frac{1}{2}(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_i A_\mu)\partial^i A^\mu = -\frac{1}{2}\left[\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right]. \qquad (3.256)
\end{aligned}
$$

于是，哈密顿量表达为

$$
\begin{aligned}
H &= \int d^3x\, \mathcal{H} = -\frac{1}{2}\int d^3x \left[\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p} 2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma') \\
&\quad \times\Big[(ip_0)(iq_0)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\left(a_{\mathbf{q},\sigma'}e^{-iq\cdot x} - a^\dagger_{\mathbf{q},\sigma'}e^{iq\cdot x}\right) \\
&\qquad + \left(i\mathbf{p}\, a_{\mathbf{p};\sigma}e^{-ip\cdot x} - i\mathbf{p}\, a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}\right)\cdot\left(i\mathbf{q}\, a_{\mathbf{q};\sigma'}e^{-iq\cdot x} - i\mathbf{q}\, a^\dagger_{\mathbf{q};\sigma'}e^{iq\cdot x}\right)\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6\sqrt{2E_\mathbf{p} 2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')\Big[(p_0q_0 + \mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p-q)\cdot x} \\
&\quad + (p_0q_0 + \mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p-q)\cdot x} + (p_0q_0 - \mathbf{p}\cdot\mathbf{q})a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p+q)\cdot x} \\
&\quad + (-p_0q_0 - \mathbf{p}\cdot\mathbf{q})a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p+q)\cdot x}\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{2E_\mathbf{p} 2E_\mathbf{q}}}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{q},\sigma')(p_0q_0 + \mathbf{p}\cdot\mathbf{q}) \\
&\quad \times\Big\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\left[a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{i(p_0-q_0)t}\right] \\
&\qquad - \delta^{(3)}(\mathbf{p}+\mathbf{q})\left[a_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{\mathbf{q};\sigma'}e^{i(p_0+q_0)t}\right]\Big\} \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3 2E_\mathbf{p}}\Big[e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E_\mathbf{p}^2 + |\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) \\
&\qquad - e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')(E_\mathbf{p}^2 - |\mathbf{p}|^2)\left(a_{\mathbf{p};\sigma}a_{-\mathbf{p};\sigma'}e^{-2iE_\mathbf{p}t} + a^\dagger_{\mathbf{p};\sigma}a^\dagger_{-\mathbf{p};\sigma'}e^{2iE_\mathbf{p}t}\right)\Big] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3}\, E_\mathbf{p}\, e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) \\
&= -\sum_{\sigma\sigma'}\int \frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\, g_{\sigma\sigma'}\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma'}\right) = \int \frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})\left(a_{\mathbf{p};\sigma}a^\dagger_{\mathbf{p};\sigma} + a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\, E_\mathbf{p}\sum_{\sigma=0}^{3}\left(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) + (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\frac{E_\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2 \\
&= \int \frac{d^3p}{(2\pi)^3}\, E_\mathbf{p}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right) + (2\pi)^3\delta^{(3)}(0)\int \frac{d^3p}{(2\pi)^3}\, 2E_\mathbf{p}. \qquad (3.257)
\end{aligned}
$$

上式最后一行第二项是零点能。第一项中类时极化态的贡献为负，与类空极化态的贡献不一样。造成这种情况的原因是 Minkowski 度规 $g_{\sigma\sigma'}$ 是一个不定度规，时间对角元 g_{00} 与空间对角元 g_{ii} 具有相反的符号。

仿照 2.3.4 小节的讨论，将真空态定义为被任意 $a_{\mathbf{p};\sigma}$ 湮灭的态，满足

$$a_{\mathbf{p};\sigma}|0\rangle=0,\quad \langle 0|0\rangle=1,\quad H|0\rangle=E_{\mathrm{vac}}|0\rangle,\quad E_{\mathrm{vac}}=2\delta^{(3)}(0)\int d^3p\,E_{\mathbf{p}}. \tag{3.258}$$

动量为 p、极化态为 σ 的单粒子态定义为

$$|\mathbf{p};\sigma\rangle\equiv\sqrt{2E_{\mathbf{p}}}\,a^{\dagger}_{\mathbf{p};\sigma}|0\rangle. \tag{3.259}$$

从而，由

$$\begin{aligned}[H,a^{\dagger}_{\mathbf{p};\sigma}]&=\int\frac{d^3q}{(2\pi)^3}E_{\mathbf{q}}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})[a^{\dagger}_{\mathbf{q};\sigma'}a_{\mathbf{q};\sigma'},a^{\dagger}_{\mathbf{p};\sigma}]=\int\frac{d^3q}{(2\pi)^3}E_{\mathbf{q}}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})a^{\dagger}_{\mathbf{q};\sigma'}[a_{\mathbf{q};\sigma'},a^{\dagger}_{\mathbf{p};\sigma}]\\&=\int\frac{d^3q}{(2\pi)^3}E_{\mathbf{q}}\sum_{\sigma'=0}^{3}(-g_{\sigma'\sigma'})a^{\dagger}_{\mathbf{q};\sigma'}(2\pi)^3(-g_{\sigma'\sigma})\delta^{(3)}(\mathbf{q}-\mathbf{p})\\&=E_{\mathbf{p}}\sum_{\sigma'=0}^{3}g_{\sigma'\sigma'}g_{\sigma'\sigma}a^{\dagger}_{\mathbf{p};\sigma'}=E_{\mathbf{p}}a^{\dagger}_{\mathbf{p};\sigma}\end{aligned} \tag{3.260}$$

可得

$$\begin{aligned}H|\mathbf{p};\sigma\rangle&=\sqrt{2E_{\mathbf{p}}}\,Ha^{\dagger}_{\mathbf{p};\sigma}|0\rangle=\sqrt{2E_{\mathbf{p}}}\,(E_{\mathbf{p}}a^{\dagger}_{\mathbf{p};\sigma}+a^{\dagger}_{\mathbf{p};\sigma}H)|0\rangle\\&=\sqrt{2E_{\mathbf{p}}}\,(E_{\mathbf{p}}+E_{\mathrm{vac}})a^{\dagger}_{\mathbf{p};\sigma}|0\rangle=(E_{\mathbf{p}}+E_{\mathrm{vac}})|\mathbf{p};\sigma\rangle.\end{aligned} \tag{3.261}$$

这似乎是一个正常的结果，说明单粒子态 $|\mathbf{p};\sigma\rangle$ 比真空多了一份能量 $E_{\mathbf{p}}$。

利用产生湮灭算符的对易关系 (3.255)，可以计算单粒子态的内积：

$$\begin{aligned}\langle\mathbf{q};\sigma'|\mathbf{p};\sigma\rangle&=\sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|a_{\mathbf{q};\sigma'}a^{\dagger}_{\mathbf{p};\sigma}|0\rangle=\sqrt{2E_{\mathbf{q}}2E_{\mathbf{p}}}\,\langle 0|\left[a^{\dagger}_{\mathbf{p};\sigma}a_{\mathbf{q};\sigma'}-(2\pi)^3g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]|0\rangle\\&=-\sqrt{2E_{\mathbf{p}}}\,g_{\sigma\sigma'}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned} \tag{3.262}$$

于是，有

$$\langle\mathbf{p};0|\mathbf{p};0\rangle=-\sqrt{2E_{\mathbf{p}}}\,(2\pi)^3\delta^{(3)}(0),\qquad\langle\mathbf{p};i|\mathbf{p};i\rangle=\sqrt{2E_{\mathbf{p}}}\,(2\pi)^3\delta^{(3)}(0),\qquad i=1,2,3. \tag{3.263}$$

上式表明，单粒子态 $|\mathbf{p};0\rangle$ 的自我内积是负的，从而导致它的能量期待值也是负的：

$$\langle\mathbf{p};0|H|\mathbf{p};0\rangle=(E_{\mathbf{p}}+E_{\mathrm{vac}})\langle\mathbf{p};0|\mathbf{p};0\rangle=-(E_{\mathbf{p}}+E_{\mathrm{vac}})\sqrt{2E_{\mathbf{p}}}\,(2\pi)^3\delta^{(3)}(0)<0. \tag{3.264}$$

这个负能量结果在物理上看起来是不可接受的，它的根源在于不定度规。

不过，如前所述，无质量矢量场只有 2 种独立的极化态，对应于 2 种横向极化矢量 $e^{\mu}(\mathbf{p},1)$ 和 $e^{\mu}(\mathbf{p},2)$，纵向极化和类时极化都应该是非物理的。选取一定的规范条件，应该可以除去非物理的极化态。由于 Lorenz 规范条件 (3.225) 与正则量子化程序不相容，我们不能直接使用这个条件，而需要将它转换到物理 Hilbert 空间中的态的期待值上，要求任意物理态 $|\Psi\rangle$ 应满足

$$\langle\Psi|\partial_{\mu}A^{\mu}(x)|\Psi\rangle=0. \tag{3.265}$$

上式称为**弱 Lorenz 规范条件**。

$A^\mu(x)$ 的平面波展开式 (3.245) 可以分解成正能解和负能解两个部分：

$$A^\mu(x)=A^{(+)\mu}(x)+A^{(-)\mu}(x). \tag{3.266}$$

其中，正能解部分为

$$A^{(+)\mu}(\mathbf{x},t)\equiv\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3}e^\mu(\mathbf{p},\sigma)\,a_{\mathbf{p};\sigma}e^{-ip\cdot x}, \tag{3.267}$$

上式的厄米共轭即是负能解部分

$$A^{(-)\mu}(\mathbf{x},t)\equiv[A^{(+)\mu}(\mathbf{x},t)]^\dagger=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3}e^\mu(\mathbf{p},\sigma)\,a^\dagger_{\mathbf{p};\sigma}e^{ip\cdot x}. \tag{3.268}$$

如果要求

$$\partial_\mu A^{(+)\mu}(x)\left|\Psi\right\rangle=0 \tag{3.269}$$

对任意物理态 $|\Psi\rangle$ 成立，则伴随有

$$\left\langle\Psi\right|\partial_\mu A^{(-)\mu}(x)=\left\langle\Psi\right|[\partial_\mu A^{(+)\mu}(x)]^\dagger=0, \tag{3.270}$$

从而，弱 Lorenz 规范条件 (3.265) 得到满足：

$$\left\langle\Psi\right|\partial_\mu A^\mu(x)\left|\Psi\right\rangle=\left\langle\Psi\right|\partial_\mu A^{(+)\mu}(x)\left|\Psi\right\rangle+\left\langle\Psi\right|\partial_\mu A^{(-)\mu}(x)\left|\Psi\right\rangle=0. \tag{3.271}$$

利用 (3.109) 和 (3.215) 式，规范条件 (3.269) 可化为

$$\begin{aligned}0&=\partial_\mu A^{(+)\mu}(x)\left|\Psi\right\rangle\\&=\int\frac{d^3p}{(2\pi)^3}\frac{-ie^{-ip\cdot x}}{\sqrt{2E_\mathbf{p}}}\left[p_\mu e^\mu(\mathbf{p},0)a_{\mathbf{p};0}+p_\mu e^\mu(\mathbf{p},1)a_{\mathbf{p};1}+p_\mu e^\mu(\mathbf{p},2)a_{\mathbf{p};2}+p_\mu e^\mu(\mathbf{p},3)a_{\mathbf{p};3}\right]\left|\Psi\right\rangle\\&=\int\frac{d^3p}{(2\pi)^3}\frac{-ie^{-ip\cdot x}}{\sqrt{2E_\mathbf{p}}}\,p\cdot n\,(a_{\mathbf{p};0}-a_{\mathbf{p};3})\left|\Psi\right\rangle.\end{aligned} \tag{3.272}$$

这意味着

$$(a_{\mathbf{p};0}-a_{\mathbf{p};3})\left|\Psi\right\rangle=0 \tag{3.273}$$

对任意物理态 $|\Psi\rangle$ 和任意动量 $\mathbf{p}$ 成立。从而，也有

$$\left\langle\Psi\right|(a^\dagger_{\mathbf{p};0}-a^\dagger_{\mathbf{p};3})=0. \tag{3.274}$$

于是，

$$\left\langle\Psi\right|a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}\left|\Psi\right\rangle=\left\langle\Psi\right|a^\dagger_{\mathbf{p};3}a_{\mathbf{p};3}\left|\Psi\right\rangle. \tag{3.275}$$

这样一来，根据 (3.257) 式计算，$|\Psi\rangle$ 的能量期待值为

$$\left\langle\Psi\right|H\left|\Psi\right\rangle=\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\left\langle\Psi\right|\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0}+\sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)\left|\Psi\right\rangle+E_{\text{vac}}\left\langle\Psi|\Psi\right\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \sum_{\sigma=1}^{2} \langle\Psi|\, a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}\ |\Psi\rangle + E_{\text{vac}}\, \langle\Psi|\Psi\rangle\,. \tag{3.276}$$

也就是说，非物理的类时极化与纵向极化对能量的贡献总是相互抵消的，除了零点能，只有两种物理的横向极化才对能量有净贡献 (net contribution)。因此，要求弱 Lorenz 规范条件成立可以除去非物理的极化态。

另一方面，由 (1.156) 式可得无质量矢量场的总动量为

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\, \pi_\mu \nabla A^\mu \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times (ip_0) \left(a_{\mathbf{p};\sigma} e^{-ip\cdot x} - a^\dagger_{\mathbf{p};\sigma} e^{ip\cdot x}\right) \left(i\mathbf{q}\, a_{\mathbf{q};\sigma'} e^{-iq\cdot x} - i\mathbf{q}\, a^\dagger_{\mathbf{q};\sigma'} e^{iq\cdot x}\right) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, p_0 \mathbf{q}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \left[-a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{-i(p-q)\cdot x} - a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{i(p-q)\cdot x} + a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p+q)\cdot x} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{i(p+q)\cdot x}\right] \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p\, d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}}\, p_0 \mathbf{q}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\qquad \times \Big\{ -\delta^{(3)}(\mathbf{p}-\mathbf{q}) \left[a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{-i(p_0-q_0)t} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{i(p_0-q_0)t}\right] \\
&\qquad\quad + \delta^{(3)}(\mathbf{p}+\mathbf{q}) \left[a_{\mathbf{p};\sigma} a_{\mathbf{q};\sigma'} e^{-i(p_0+q_0)t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{\mathbf{q};\sigma'} e^{i(p_0+q_0)t}\right] \Big\} \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \Big[-e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) \\
&\qquad\qquad - e_\mu(\mathbf{p},\sigma) e^\mu(-\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right) \Big].
\end{aligned} \tag{3.277}$$

对上式最后两行方括号内第二项的积分及求和作 $\mathbf{p} \to -\mathbf{p}$ 的替换和 $\lambda \leftrightarrow \lambda'$ 的互换，可得

$$\begin{aligned}
&-\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(\mathbf{p},\sigma) e^\mu(-\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right) \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{-\mathbf{p}}{2}\, e_\mu(-\mathbf{p},\sigma') e^\mu(\mathbf{p},\sigma) \left(a_{-\mathbf{p};\sigma'} a_{\mathbf{p};\sigma} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{-\mathbf{p};\sigma'} a^\dagger_{\mathbf{p};\sigma} e^{2iE_{\mathbf{p}}t}\right) \\
&= \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(-\mathbf{p},\sigma') e^\mu(\mathbf{p},\sigma) \left(a_{\mathbf{p};\sigma} a_{-\mathbf{p};\sigma'} e^{-2iE_{\mathbf{p}}t} + a^\dagger_{\mathbf{p};\sigma} a^\dagger_{-\mathbf{p};\sigma'} e^{2iE_{\mathbf{p}}t}\right).
\end{aligned} \tag{3.278}$$

可以看出，上式为零。于是，总动量化为

$$\begin{aligned}
\mathbf{P} &= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{p},\sigma') \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) \\
&= -\sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2}\, g_{\sigma\sigma'} \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma'} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma'}\right) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{p}}{2} \sum_{\sigma=0}^{3} (-g_{\sigma\sigma}) \left(a_{\mathbf{p};\sigma} a^\dagger_{\mathbf{p};\sigma} + a^\dagger_{\mathbf{p};\sigma} a_{\mathbf{p};\sigma}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}) + \delta^{(3)}(0)\int d^3p\,\frac{\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2\\
&= \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right). \qquad (3.279)
\end{aligned}
$$

根据 (3.275) 式，物理态 $|\Psi\rangle$ 的动量期待值为

$$
\langle\Psi|\,\mathbf{P}\,|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\,\langle\Psi|\left(-a^\dagger_{\mathbf{p};0}a_{\mathbf{p};0} + \sum_{\sigma=1}^{3}a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\right)|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3}\,\mathbf{p}\sum_{\sigma=1}^{2}\langle\Psi|\,a^\dagger_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma}\,|\Psi\rangle. \qquad (3.280)
$$

同样，只有两种物理的横向极化才对动量有净贡献。

通过线性组合，可以用湮灭算符 $a_{\mathbf{p};1}$ 和 $a_{\mathbf{p};2}$ 定义另一组等价的湮灭算符

$$
a_{\mathbf{p},\pm} \equiv \frac{1}{\sqrt{2}}(\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2}), \qquad (3.281)
$$

相应的产生算符可以通过取厄米共轭得到。反过来，有

$$
a_{\mathbf{p};1} = -\frac{1}{\sqrt{2}}(a_{\mathbf{p},+} - a_{\mathbf{p},-}), \quad a_{\mathbf{p};2} = -\frac{i}{\sqrt{2}}(a_{\mathbf{p},+} + a_{\mathbf{p},-}). \qquad (3.282)
$$

利用对易关系 (3.255)，可得

$$
\begin{aligned}
[a_{\mathbf{p},\pm}, a^\dagger_{\mathbf{q},\pm}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2},\, \mp a^\dagger_{\mathbf{q};1} - i a^\dagger_{\mathbf{q};2}] = \frac{1}{2}[a_{\mathbf{p};1},\, a^\dagger_{\mathbf{q};1}] + \frac{1}{2}[a_{\mathbf{p};2},\, a^\dagger_{\mathbf{q};2}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),\\
[a_{\mathbf{p},\pm}, a^\dagger_{\mathbf{q},\mp}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2},\, \pm a^\dagger_{\mathbf{q};1} - i a^\dagger_{\mathbf{q};2}] = -\frac{1}{2}[a_{\mathbf{p};1},\, a^\dagger_{\mathbf{q};1}] + \frac{1}{2}[a_{\mathbf{p};2},\, a^\dagger_{\mathbf{q};2}] = 0,\\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\pm}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2},\, \mp a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0,\\
[a_{\mathbf{p},\pm}, a_{\mathbf{q},\mp}] &= \frac{1}{2}[\mp a_{\mathbf{p};1} + i a_{\mathbf{p};2},\, \pm a_{\mathbf{q};1} + i a_{\mathbf{q};2}] = 0. \qquad (3.283)
\end{aligned}
$$

于是，这组产生湮灭算符的对易关系可以整理为

$$
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0, \quad \lambda,\lambda' = \pm. \qquad (3.284)
$$

根据 (3.125) 式，可以用对应着螺旋度的横向极化矢量 $\varepsilon^\mu(\mathbf{p},\pm)$ 表示 $e^\mu(\mathbf{p},1)$ 和 $e^\mu(\mathbf{p},2)$：

$$
e^\mu(\mathbf{p},1) = -\frac{1}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)], \quad e^\mu(\mathbf{p},2) = \frac{i}{\sqrt{2}}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)]. \qquad (3.285)
$$

从而，有

$$
\begin{aligned}
\sum_{\sigma=1}^{2} e^\mu(\mathbf{p},\sigma)a_{\mathbf{p};\sigma} &= e^\mu(\mathbf{p},1)a_{\mathbf{p};1} + e^\mu(\mathbf{p},2)a_{\mathbf{p};2}\\
&= \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) - \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}[\varepsilon^\mu(\mathbf{p},+) + \varepsilon^\mu(\mathbf{p},-)](a_{\mathbf{p},+} + a_{\mathbf{p},-})\\
&= \varepsilon^\mu(\mathbf{p},+)a_{\mathbf{p},+} + \varepsilon^\mu(\mathbf{p},-)a_{\mathbf{p},-} = \sum_{\lambda=\pm}\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}, \qquad (3.286)
\end{aligned}
$$

取厄米共轭，得

$$\sum_{\sigma=1}^{2} e^{\mu}(\mathbf{p},\sigma)a^{\dagger}_{\mathbf{p};\sigma} = \sum_{\lambda=\pm} \varepsilon^{\mu*}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}. \tag{3.287}$$

于是，可以把 $A^{\mu}(x)$ 的平面波展开式 (3.245) 改写成

$$\begin{aligned} A^{\mu}(\mathbf{x},t) =& \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0,3} e^{\mu}(\mathbf{p},\sigma)\left(a_{\mathbf{p};\sigma}e^{-ip\cdot x} + a^{\dagger}_{\mathbf{p};\sigma}e^{ip\cdot x}\right) \\ &+ \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} \left[\varepsilon^{\mu}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ip\cdot x} + \varepsilon^{\mu*}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}e^{ip\cdot x}\right], \end{aligned} \tag{3.288}$$

第一行对应于非物理极化态，第二行对应于两种物理的螺旋度本征极化态。可见，(3.281) 式定义的湮灭算符 $a_{\mathbf{p},\pm}$ 正是螺旋度 $\lambda=\pm$ 对应的湮灭算符。

此外，由 (3.282) 式可得

$$\begin{aligned} \sum_{\sigma=1}^{2} a^{\dagger}_{\mathbf{p};\sigma}a_{\mathbf{p};\sigma} &= a^{\dagger}_{\mathbf{p};1}a_{\mathbf{p};1} + a^{\dagger}_{\mathbf{p};2}a_{\mathbf{p};2} = \frac{1}{2}(a^{\dagger}_{\mathbf{p},+} - a^{\dagger}_{\mathbf{p},-})(a_{\mathbf{p},+} - a_{\mathbf{p},-}) + \frac{1}{2}(a^{\dagger}_{\mathbf{p},+} + a^{\dagger}_{\mathbf{p},-})(a_{\mathbf{p},+} + a_{\mathbf{p},-}) \\ &= a^{\dagger}_{\mathbf{p},+}a_{\mathbf{p},+} + a^{\dagger}_{\mathbf{p},-}a_{\mathbf{p},-} = \sum_{\lambda=\pm} a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}, \end{aligned} \tag{3.289}$$

故物理态 $|\Psi\rangle$ 的能量期待值和动量期待值可以用螺旋度对应的产生湮灭算符表示为

$$\langle\Psi| H |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_{\lambda=\pm} \langle\Psi| a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} |\Psi\rangle + E_{\text{vac}}\langle\Psi|\Psi\rangle, \tag{3.290}$$

$$\langle\Psi| \mathbf{P} |\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_{\lambda=\pm} \langle\Psi| a^{\dagger}_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda} |\Psi\rangle. \tag{3.291}$$

附录 A　英汉对照

Annihilation operator: 湮灭算符
Antichronous: 反时向
Axial vector: 轴矢量
Boost: 增速
Boson: 玻色子
Canonical quantization: 正则量子化
Conjugate momentum density: 共轭动量密度
Conserved charge: 守恒荷
Conserved current: 守恒流
Contraction: 缩并
Contravariant vector: 逆变矢量
Covariant vector: 协变矢量
Creation operator: 产生算符
Electron: 电子
Energy-momentum tensor: 能动张量
Expectation value: 期待值
Fermion: 费米子
Field strength tensor: 场强张量
Gauge-fixing term: 规范固定项
Gauge invariant: 规范不变量
Gauge symmetry: 规范对称性
Gauge transformation: 规范变换
Generalized coordinate: 广义坐标
Generator: 生成元
Global: 整体
Helicity: 螺旋度
Hermitian conjugate: 厄米共轭
Hermitian operator: 厄米算符
Homomorphic: 同态
Improper: 非固有的
Local: 局域
Lowering operator: 降算符
Metric: 度规
Mode: 模式
Orthochronous: 保时向
Parity: 宇称
Phonon: 声子
Picture: 绘景
Plane-wave solution: 平面波解
Polarization vector: 极化矢量
Positron: 正电子
Proper: 固有的
Pseudoscalar: 赝标量
Raising operator: 升算符
Real orthogonal matrix: 实正交矩阵
Scalar: 标量
Self-conjugate: 自共轭
Simple harmonic oscillator: 简谐振子
Space inversion: 空间反射
Spinor: 旋量
Spinor bilinear: 旋量双线性型
Spinor representation: 旋量表示
Step function: 阶跃函数
Tensor: 张量
Time reversal: 时间反演
Unitary: 幺正
Vacuum: 真空
Vector: 矢量
Zero-point energy: 零点能