极化振幅

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目录

1 旋量的螺旋态

一对二分量旋量 $\xi_\lambda(p)$ $(\lambda=\pm)$ 构成一组螺旋态基底, 并满足

$$(\widehat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(p)=\lambda\xi_\lambda(p),\quad \lambda=\pm. \tag{1}$$

Dirac 方程的平面波解可用这组基底表示成 [1]

$$u(p,\lambda)=\begin{pmatrix}\omega_{-\lambda}(p)\xi_\lambda(p)\\ \omega_\lambda(p)\xi_\lambda(p)\end{pmatrix},\quad v(p,\lambda)=\begin{pmatrix}-\lambda\omega_\lambda(p)\xi_{-\lambda}(p)\\ \lambda\omega_{-\lambda}(p)\xi_{-\lambda}(p)\end{pmatrix}, \tag{2}$$

其中 $\omega_\lambda(p)=\sqrt{E_p+\lambda|\mathbf{p}|}$, $E_p=\sqrt{|\mathbf{p}|^2+m^2}$.

$$\begin{aligned}
\not{p}u(p,\lambda) &= \begin{pmatrix} & E_p-\mathbf{p}\cdot\boldsymbol{\sigma}\\ E_p+\mathbf{p}\cdot\boldsymbol{\sigma} & \end{pmatrix}\begin{pmatrix}\omega_{-\lambda}(p)\xi_\lambda(p)\\ \omega_\lambda(p)\xi_\lambda(p)\end{pmatrix}=\begin{pmatrix}\omega_\lambda(p)(E_p-\lambda|\mathbf{p}|)\xi_\lambda(p)\\ \omega_{-\lambda}(p)(E_p+\lambda|\mathbf{p}|)\xi_\lambda(p)\end{pmatrix}\\
&=\sqrt{(E_p+\lambda|\mathbf{p}|)(E_p-\lambda|\mathbf{p}|)}\begin{pmatrix}\omega_{-\lambda}(p)\xi_\lambda(p)\\ \omega_\lambda(p)\xi_\lambda(p)\end{pmatrix}=mu(p,\lambda), \qquad (3)\\
\not{p}v(p,\lambda) &= \begin{pmatrix} & E_p-\mathbf{p}\cdot\boldsymbol{\sigma}\\ E_p+\mathbf{p}\cdot\boldsymbol{\sigma} & \end{pmatrix}\begin{pmatrix}-\lambda\omega_\lambda(p)\xi_{-\lambda}(p)\\ \lambda\omega_{-\lambda}(p)\xi_{-\lambda}(p)\end{pmatrix}=\begin{pmatrix}\lambda\omega_{-\lambda}(p)(E_p+\lambda|\mathbf{p}|)\xi_{-\lambda}(p)\\ -\lambda\omega_\lambda(p)(E_p-\lambda|\mathbf{p}|)\xi_{-\lambda}(p)\end{pmatrix}
\end{aligned}$$

$$= \sqrt{(E_p + \lambda|\mathbf{p}|)(E_p - \lambda|\mathbf{p}|)} \begin{pmatrix} \lambda\omega_\lambda(p)\xi_{-\lambda}(p) \\ -\lambda\omega_{-\lambda}(p)\xi_{-\lambda}(p) \end{pmatrix} = -mv(p,\lambda), \tag{4}$$

可见, Dirac 方程 $(\not{p} - m)u(p,\lambda) = 0$ 和 $(\not{p} + m)v(p,\lambda) = 0$ 成立.

当 $E_p \gg m$ 时, $\omega_+(p) \to \sqrt{2|\mathbf{p}|}$, $\omega_-(p) \to 0$, 有

$$u(p,-) \to \begin{pmatrix} \sqrt{2|\mathbf{p}|}\xi_-(p) \\ 0 \end{pmatrix}, \quad u(p,+) \to \begin{pmatrix} 0 \\ \sqrt{2|\mathbf{p}|}\xi_+(p) \end{pmatrix}; \tag{5}$$

$$v(p,-) \to \begin{pmatrix} 0 \\ -\sqrt{2|\mathbf{p}|}\xi_+(p) \end{pmatrix}, \quad v(p,+) \to \begin{pmatrix} -\sqrt{2|\mathbf{p}|}\xi_-(p) \\ 0 \end{pmatrix}. \tag{6}$$

从而,

$$\begin{aligned} &P_L u(p,-) \to u(p,-), \quad P_R u(p,+) \to u(p,+), \quad P_R u(p,-) \to 0, \quad P_L u(p,+) \to 0; \\ &P_R v(p,-) \to v(p,-), \quad P_L v(p,+) \to v(p,+), \quad P_L v(p,-) \to 0, \quad P_R v(p,+) \to 0. \end{aligned} \tag{7}$$

这是将 P_L 和 P_R 称为左右手投影算符的原因. 注意, 正能解和负能解的投影关系相反.

将 E_p 展开为 $E_p \simeq |\mathbf{p}| + \dfrac{1}{2}m^2/|\mathbf{p}|$, 可得

$$\omega_+(p) \simeq \sqrt{2|\mathbf{p}|}\left(1 + \frac{m^2}{8|\mathbf{p}|^2}\right), \quad \omega_-(p) \simeq \frac{m}{\sqrt{2|\mathbf{p}|}}. \tag{8}$$

于是,

$$u(p,-) \simeq \begin{pmatrix} \sqrt{2|\mathbf{p}|}\xi_-(p) \\ \dfrac{m}{\sqrt{2|\mathbf{p}|}}\xi_-(p) \end{pmatrix}, \quad u(p,+) \simeq \begin{pmatrix} \dfrac{m}{\sqrt{2|\mathbf{p}|}}\xi_+(p) \\ \sqrt{2|\mathbf{p}|}\xi_+(p) \end{pmatrix}; \tag{9}$$

$$v(p,-) \simeq \begin{pmatrix} \dfrac{m}{\sqrt{2|\mathbf{p}|}}\xi_+(p) \\ -\sqrt{2|\mathbf{p}|}\xi_+(p) \end{pmatrix}, \quad v(p,+) \simeq \begin{pmatrix} -\sqrt{2|\mathbf{p}|}\xi_-(p) \\ \dfrac{m}{\sqrt{2|\mathbf{p}|}}\xi_-(p) \end{pmatrix}. \tag{10}$$

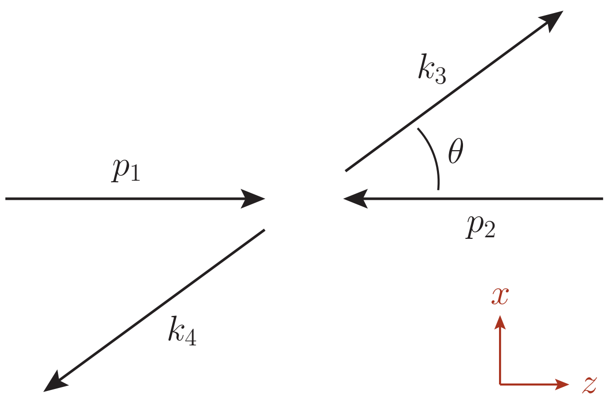

Figure 1: 两体散射示意图.

考虑费米子散射过程 $q(p_1) + \bar{q}(p_2) \to \chi(k_3) + \bar{\chi}(k_4)$, 散射角为 θ, 如 Fig. 1 所示. 记 $c_\theta \equiv \cos\theta$,

$s_\theta \equiv \sin\theta$. 参考文献 [2] 的附录 A, 初态粒子的动量和螺旋态可表示成

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,\beta_q), \quad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-\beta_q), \quad \beta_q \equiv \sqrt{1-4m_q^2/s}, \tag{11}$$

$$\xi_+(p_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_-(p_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_+(p_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \xi_-(p_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{12}$$

末态粒子的动量和螺旋态可表示成

$$k_3 = \frac{\sqrt{s}}{2}(1,\beta_\chi s_\theta,0,\beta_\chi c_\theta), \quad k_4 = \frac{\sqrt{s}}{2}(1,-\beta_\chi s_\theta,0,-\beta_\chi c_\theta), \quad \beta_\chi \equiv \sqrt{1-4m_\chi^2/s}, \tag{13}$$

$$\xi_+(k_3) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix}, \quad \xi_-(k_3) = \begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2} \end{pmatrix}, \quad \xi_+(k_4) = \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix}, \quad \xi_-(k_4) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix}. \tag{14}$$

可以验证, 这些螺旋态都满足本征值方程 (1):

$$\widehat{\mathbf{k}}_3 \cdot \boldsymbol{\sigma} = \begin{pmatrix} c_\theta & s_\theta \\ s_\theta & -c_\theta \end{pmatrix}, \quad \widehat{\mathbf{k}}_4 \cdot \boldsymbol{\sigma} = \begin{pmatrix} -c_\theta & -s_\theta \\ -s_\theta & c_\theta \end{pmatrix},$$

$$\begin{aligned}
(\widehat{\mathbf{k}}_3 \cdot \boldsymbol{\sigma})\xi_+(k_3) &= \begin{pmatrix} c_\theta & s_\theta \\ s_\theta & -c_\theta \end{pmatrix}\begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix} = \begin{pmatrix} c_\theta c_{\theta/2} + s_\theta s_{\theta/2} \\ s_\theta c_{\theta/2} - c_\theta s_{\theta/2} \end{pmatrix} = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix} = +\xi_+(k_3), \\
(\widehat{\mathbf{k}}_3 \cdot \boldsymbol{\sigma})\xi_-(k_3) &= \begin{pmatrix} c_\theta & s_\theta \\ s_\theta & -c_\theta \end{pmatrix}\begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2} \end{pmatrix} = \begin{pmatrix} -c_\theta s_{\theta/2} + s_\theta c_{\theta/2} \\ -s_\theta s_{\theta/2} - c_\theta c_{\theta/2} \end{pmatrix} = \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix} = -\xi_-(k_3), \\
(\widehat{\mathbf{k}}_4 \cdot \boldsymbol{\sigma})\xi_+(k_4) &= \begin{pmatrix} -c_\theta & -s_\theta \\ -s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix} = \begin{pmatrix} -c_\theta s_{\theta/2} + s_\theta c_{\theta/2} \\ -s_\theta s_{\theta/2} - c_\theta c_{\theta/2} \end{pmatrix} = \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix} = +\xi_+(k_4), \\
(\widehat{\mathbf{k}}_4 \cdot \boldsymbol{\sigma})\xi_-(k_4) &= \begin{pmatrix} -c_\theta & -s_\theta \\ -s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix} = \begin{pmatrix} -c_\theta c_{\theta/2} - s_\theta s_{\theta/2} \\ -s_\theta c_{\theta/2} + c_\theta s_{\theta/2} \end{pmatrix} = \begin{pmatrix} -c_{\theta/2} \\ -s_{\theta/2} \end{pmatrix} = -\xi_-(k_4).
\end{aligned}$$

2 矢量的极化态

对于无质量矢量粒子 (以光子为例), Fig. 1 中所示动量表示为

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1), \quad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1), \quad k_3 = \frac{\sqrt{s}}{2}(1,s_\theta,0,c_\theta), \quad k_4 = \frac{\sqrt{s}}{2}(1,-s_\theta,0,-c_\theta). \tag{15}$$

参考文献 [2] 的附录 A, 动量 p_1 和 p_2 对应的 (横向) 极化矢量可表示成

$$\varepsilon(p_1,\lambda) = \frac{1}{\sqrt{2}}(-\lambda\varepsilon_1 - i\varepsilon_2), \quad \varepsilon(p_2,\lambda) = \frac{1}{\sqrt{2}}(\lambda\varepsilon_1 - i\varepsilon_2), \quad \lambda = \pm, \tag{16}$$

其中

$$\varepsilon_1 = (0,1,0,0), \quad \varepsilon_2 = (0,0,1,0). \tag{17}$$

亦即

$$\varepsilon(p_1,+) = \frac{1}{\sqrt{2}}(0,-1,-i,0), \quad \varepsilon(p_1,-) = \frac{1}{\sqrt{2}}(0,1,-i,0),$$

$$\varepsilon(p_2,+) = \frac{1}{\sqrt{2}}(0,1,-i,0), \quad \varepsilon(p_2,-) = \frac{1}{\sqrt{2}}(0,-1,-i,0). \tag{18}$$

在 Lorentz 群的矢量表示中, 角动量算符的 3 个空间分量分别为

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

利用

$$\widehat{\mathbf{p}}_1 \cdot \mathbf{J} = J_3, \quad \widehat{\mathbf{p}}_2 \cdot \mathbf{J} = -J_3, \tag{20}$$

可以验证本征值方程 $(\widehat{\mathbf{p}} \cdot \mathbf{J})\varepsilon(p,\lambda) = \lambda\varepsilon(p,\lambda)$:

$$(\widehat{\mathbf{p}}_1 \cdot \mathbf{J})\varepsilon(p_1,+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} = +\varepsilon(p_1,+),$$

$$(\widehat{\mathbf{p}}_1 \cdot \mathbf{J})\varepsilon(p_1,-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix} = -\varepsilon(p_1,-),$$

$$(\widehat{\mathbf{p}}_2 \cdot \mathbf{J})\varepsilon(p_2,+) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} = +\varepsilon(p_2,+),$$

$$(\widehat{\mathbf{p}}_2 \cdot \mathbf{J})\varepsilon(p_2,-) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} = -\varepsilon(p_2,-).$$

如 Fig. 1 中所示, 动量 k_3 (k_4) 可以通过将动量 p_1 (p_2) 绕 y 轴旋转 θ 角得到, 即

$$k_3 = R(\mathbf{e}_2,\theta)p_1, \quad k_4 = R(\mathbf{e}_2,\theta)p_2, \quad R(\mathbf{e}_2,\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & 0 & s_\theta \\ 0 & 0 & 1 & 0 \\ 0 & -s_\theta & 0 & c_\theta \end{pmatrix}. \tag{21}$$

由于极化矢量与动量同处于 Lorentz 群的矢量表示, 应有

$$\varepsilon(k_3,\lambda) = R(\mathbf{e}_2,\theta)\varepsilon(p_1,\lambda), \quad \varepsilon(k_4,\lambda) = R(\mathbf{e}_2,\theta)\varepsilon(p_2,\lambda). \tag{22}$$

因此

$$\varepsilon(k_3,\lambda) = \frac{1}{\sqrt{2}}(-\lambda\varepsilon_1' - i\varepsilon_2'), \quad \varepsilon(k_4,\lambda) = \frac{1}{\sqrt{2}}(\lambda\varepsilon_1' - i\varepsilon_2'), \tag{23}$$

其中

$$\varepsilon_1' = R(\mathbf{e}_2,\theta)\varepsilon_1 = (0,c_\theta,0,-s_\theta), \quad \varepsilon_2' = R(\mathbf{e}_2,\theta)\varepsilon_2 = (0,0,1,0). \tag{24}$$

故

$$
\begin{aligned}
\varepsilon(k_3,+) &= \frac{1}{\sqrt{2}}(0,-c_\theta,-i,s_\theta), \quad \varepsilon(k_3,-) = \frac{1}{\sqrt{2}}(0,c_\theta,-i,-s_\theta),\\
\varepsilon(k_4,+) &= \frac{1}{\sqrt{2}}(0,c_\theta,-i,-s_\theta), \quad \varepsilon(k_4,-) = \frac{1}{\sqrt{2}}(0,-c_\theta,-i,s_\theta).
\end{aligned}
\tag{25}
$$

同样, 利用

$$
\widehat{\mathbf{k}}_3\cdot\mathbf{J} = s_\theta J_1 + c_\theta J_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -ic_\theta & 0\\ 0 & ic_\theta & 0 & -is_\theta\\ 0 & 0 & is_\theta & 0 \end{pmatrix} = -\widehat{\mathbf{k}}_4\cdot\mathbf{J},
\tag{26}
$$

可用验证本征值方程 $(\widehat{\mathbf{p}}\cdot\mathbf{J})\varepsilon(p,\lambda)=\lambda\varepsilon(p,\lambda)$:

$$
\begin{aligned}
(\widehat{\mathbf{k}}_3\cdot\mathbf{J})\varepsilon(k_3,+) &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -ic_\theta & 0\\ 0 & ic_\theta & 0 & -is_\theta\\ 0 & 0 & is_\theta & 0 \end{pmatrix}\begin{pmatrix} 0\\ -c_\theta\\ -i\\ s_\theta \end{pmatrix} = \begin{pmatrix} 0\\ -c_\theta\\ -i\\ s_\theta \end{pmatrix} = +\varepsilon(k_3,+),\\
(\widehat{\mathbf{k}}_3\cdot\mathbf{J})\varepsilon(k_3,-) &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -ic_\theta & 0\\ 0 & ic_\theta & 0 & -is_\theta\\ 0 & 0 & is_\theta & 0 \end{pmatrix}\begin{pmatrix} 0\\ c_\theta\\ -i\\ -s_\theta \end{pmatrix} = \begin{pmatrix} 0\\ -c_\theta\\ i\\ s_\theta \end{pmatrix} = -\varepsilon(k_3,-),\\
(\widehat{\mathbf{k}}_4\cdot\mathbf{J})\varepsilon(k_4,+) &= -\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -ic_\theta & 0\\ 0 & ic_\theta & 0 & -is_\theta\\ 0 & 0 & is_\theta & 0 \end{pmatrix}\begin{pmatrix} 0\\ c_\theta\\ -i\\ -s_\theta \end{pmatrix} = \begin{pmatrix} 0\\ c_\theta\\ -i\\ -s_\theta \end{pmatrix} = +\varepsilon(k_4,+),\\
(\widehat{\mathbf{k}}_4\cdot\mathbf{J})\varepsilon(k_4,-) &= -\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -ic_\theta & 0\\ 0 & ic_\theta & 0 & -is_\theta\\ 0 & 0 & is_\theta & 0 \end{pmatrix}\begin{pmatrix} 0\\ -c_\theta\\ -i\\ s_\theta \end{pmatrix} = \begin{pmatrix} 0\\ c_\theta\\ i\\ -s_\theta \end{pmatrix} = -\varepsilon(k_4,-),
\end{aligned}
$$

3 费米子矢量流耦合

下面讨论夸克 (q) 与 Dirac WIMP (χ) 的矢量流有效耦合

$$
\mathcal{L}_{\mathrm{V}} = \frac{1}{\Lambda^2}\bar{q}\gamma^\mu q\bar{\chi}\gamma_\mu\chi.
\tag{27}
$$

$q(p_1)+\bar{q}(p_2)\to\chi(k_3)+\bar{\chi}(k_4)$ 过程的不变振幅为

$$
\mathcal{M}(q_{\lambda_1}\bar{q}_{\lambda_2}\to\chi_{\lambda_3}\bar{\chi}_{\lambda_4}) = \frac{1}{\Lambda^2}\bar{v}_{\lambda_2}(p_2)\gamma^\mu u_{\lambda_1}(p_1)\bar{u}_{\lambda_3}(k_3)\gamma_\mu v_{\lambda_4}(k_4),
\tag{28}
$$

$$
\mathcal{M}^*(q_{\lambda_1}\bar{q}_{\lambda_2}\to\chi_{\lambda_3}\bar{\chi}_{\lambda_4}) = \frac{1}{\Lambda^2}\bar{u}_{\lambda_1}(p_1)\gamma^\nu v_{\lambda_2}(p_2)\bar{v}_{\lambda_4}(k_4)\gamma_\nu u_{\lambda_3}(k_3).
\tag{29}
$$

3.1 非极化振幅

利用

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \frac{\sqrt{s}}{2}\beta_q, \quad |\mathbf{k}_3| = |\mathbf{k}_4| = \frac{\sqrt{s}}{2}\beta_\chi, \quad |\mathbf{p}_1||\mathbf{k}_3|\cos\theta = \frac{s}{4}\beta_q\beta_\chi\cos\theta, \tag{30}$$

可得

$$p_1 \cdot p_2 = \frac{s}{2} - m_q^2, \quad k_3 \cdot k_4 = \frac{s}{2} - m_\chi^2, \tag{31}$$

$$p_1 \cdot k_3 = p_2 \cdot k_4 = \frac{s}{4}(1 - \beta_q\beta_\chi\cos\theta), \quad p_1 \cdot k_4 = p_2 \cdot k_3 = \frac{s}{4}(1 + \beta_q\beta_\chi\cos\theta), \tag{32}$$

从而, 非极化散射振幅的模方为

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(q\bar{q}\to\chi\bar{\chi})|^2 &= \frac{1}{4}\sum_{\text{spins}}\frac{1}{\Lambda^4}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(k_3)\gamma_\mu v(k_4)\bar{u}(p_1)\gamma^\nu v(p_2)\bar{v}(k_4)\gamma_\nu u(k_3) \\
&= \frac{1}{4}\sum_{\text{spins}}\frac{1}{\Lambda^4}\mathrm{Tr}[v(p_2)\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu]\mathrm{Tr}[u(k_3)\bar{u}(k_3)\gamma_\mu v(k_4)\bar{v}(k_4)\gamma_\nu] \\
&= \frac{1}{4\Lambda^4}\mathrm{Tr}[(\not{p}_2 - m_q)\gamma^\mu(\not{p}_1 + m_q)\gamma^\nu]\mathrm{Tr}[(\not{k}_3 + m_\chi)\gamma_\mu(\not{k}_4 - m_\chi)\gamma_\nu] \\
&= \frac{8}{\Lambda^4}[(p_1\cdot k_3)(p_2\cdot k_4) + (p_1\cdot k_4)(p_2\cdot k_3) + m_q^2(k_3\cdot k_4) + m_\chi^2(p_1\cdot p_2) + 2m_q^2m_\chi^2] \\
&= \frac{1}{\Lambda^4}[s^2 + 4s(m_q^2 + m_\chi^2) + 16(|\mathbf{p}_1||\mathbf{k}_3|\cos\theta)^2] \\
&= \frac{1}{\Lambda^4}[s^2(1 + \beta_q^2\beta_\chi^2\cos^2\theta) + 4s(m_q^2 + m_\chi^2)].
\end{aligned} \tag{33}$$

3.2 极化振幅

下面将会用到如下等式.

$$\omega_+(p_2)\omega_+(p_1) = \frac{\sqrt{s}}{2}(1 + \beta_q), \quad \omega_-(p_2)\omega_-(p_1) = \frac{\sqrt{s}}{2}(1 - \beta_q), \tag{34}$$

$$\omega_+(k_3)\omega_+(k_4) = \frac{\sqrt{s}}{2}(1 + \beta_\chi), \quad \omega_-(k_3)\omega_-(k_4) = \frac{\sqrt{s}}{2}(1 - \beta_\chi), \tag{35}$$

$$\omega_-(p_2)\omega_+(p_1) = \omega_+(p_2)\omega_-(p_1) = m_q, \quad \omega_-(k_3)\omega_+(k_4) = \omega_+(k_3)\omega_-(k_4) = m_\chi. \tag{36}$$

$$\gamma^0\gamma^\mu = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix}. \tag{37}$$

初态极化矢量流:

$$\begin{aligned}
q_-\bar{q}_+ \to \bar{v}_+(p_2)\gamma^\mu u_-(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix}\begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix}\begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= -\omega_+(p_2)\omega_+(p_1)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \omega_-(p_2)\omega_-(p_1)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1) \\
&= -\frac{\sqrt{s}}{2}(1 + \beta_q)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \frac{\sqrt{s}}{2}(1 - \beta_q)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1),
\end{aligned} \tag{38}$$

$$\begin{aligned}
q_+\bar{q}_- \to \bar{v}_-(p_2)\gamma^\mu u_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix}\begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix}\begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= \omega_-(p_2)\omega_-(p_1)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) - \omega_+(p_2)\omega_+(p_1)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{s}}{2}(1-\beta_q)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) - \frac{\sqrt{s}}{2}(1+\beta_q)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1), \quad (39)\\
q_-\bar{q}_- \to \bar{v}_-(p_2)\gamma^\mu u_-(p_1) &= \begin{pmatrix}\omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2)\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= \omega_-(p_2)\omega_+(p_1)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) - \omega_+(p_2)\omega_-(p_1)\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\\
&= m_q\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) - m_q\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1), \quad (40)\\
q_+\bar{q}_+ \to \bar{v}_+(p_2)\gamma^\mu u_+(p_1) &= \begin{pmatrix}-\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2)\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= -\omega_+(p_2)\omega_-(p_1)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) + \omega_-(p_2)\omega_+(p_1)\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1)\\
&= -m_q\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) + m_q\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1). \quad (41)
\end{aligned}
$$

末态极化矢量流:

$$
\begin{aligned}
\chi_-\bar{\chi}_+ \to \bar{u}_-(k_3)\gamma_\mu v_+(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix}\bar{\sigma}_\mu & \\ & \sigma_\mu\end{pmatrix}\begin{pmatrix}-\omega_+(k_4)\xi_-(k_4)\\ \omega_-(k_4)\xi_-(k_4)\end{pmatrix}\\
&= -\omega_+(k_3)\omega_+(k_4)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) + \omega_-(k_3)\omega_-(k_4)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4)\\
&= -\frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) + \frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4), \quad (42)\\
\chi_+\bar{\chi}_- \to \bar{u}_+(k_3)\gamma_\mu v_-(k_4) &= \begin{pmatrix}\omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3)\end{pmatrix}\begin{pmatrix}\bar{\sigma}_\mu & \\ & \sigma_\mu\end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\\
&= \omega_-(k_3)\omega_-(k_4)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - \omega_+(k_3)\omega_+(k_4)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4)\\
&= \frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - \frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4), \quad (43)\\
\chi_-\bar{\chi}_- \to \bar{u}_-(k_3)\gamma_\mu v_-(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix}\bar{\sigma}_\mu & \\ & \sigma_\mu\end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\\
&= \omega_+(k_3)\omega_-(k_4)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - \omega_-(k_3)\omega_+(k_4)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4)\\
&= m_\chi\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - m_\chi\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4), \quad (44)\\
\chi_+\bar{\chi}_+ \to \bar{u}_+(k_3)\gamma_\mu v_+(k_4) &= \begin{pmatrix}\omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3)\end{pmatrix}\begin{pmatrix}\bar{\sigma}_\mu & \\ & \sigma_\mu\end{pmatrix}\begin{pmatrix}-\omega_+(k_4)\xi_-(k_4)\\ \omega_-(k_4)\xi_-(k_4)\end{pmatrix}\\
&= -\omega_-(k_3)\omega_+(k_4)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) + \omega_+(k_3)\omega_-(k_4)\xi_+^\dagger(k_3)\sigma_\mu\xi_-(k_4)\\
&= -m_\chi\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) + m_\chi\xi_+^\dagger(k_3)\sigma_\mu\xi_-(k_4). \quad (45)
\end{aligned}
$$

$$
\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = \begin{pmatrix}0 & -1\end{pmatrix}\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix} = (0,-1,-i,0) = -\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1), \quad (46)
$$

$$
\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1) = \begin{pmatrix}1 & 0\end{pmatrix}\sigma^\mu\begin{pmatrix}0\\1\end{pmatrix} = (0,1,-i,0) = -\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1), \quad (47)
$$

$$
\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4) = \begin{pmatrix}c_{\theta/2} & s_{\theta/2}\end{pmatrix}\sigma^\mu\begin{pmatrix}s_{\theta/2}\\-c_{\theta/2}\end{pmatrix} = (0,-c_\theta,i,s_\theta) = -\xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4), \quad (48)
$$

$$\xi_-^\dagger(k_3)\sigma^\mu\xi_-(k_4) \;=\; \begin{pmatrix} -s_{\theta/2} & c_{\theta/2} \end{pmatrix}\sigma^\mu \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix} = (0, c_\theta, i, -s_\theta) = -\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4). \tag{49}$$

$$\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1) \;=\; \begin{pmatrix} 1 & 0 \end{pmatrix}\sigma^\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1,0,0,1), \quad \xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) = (1,0,0,-1), \tag{50}$$

$$\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1) \;=\; \begin{pmatrix} 0 & -1 \end{pmatrix}\sigma^\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1,0,0,1), \quad \xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) = (-1,0,0,-1), \tag{51}$$

$$\xi_-^\dagger(k_3)\sigma^\mu\xi_+(k_4) \;=\; \begin{pmatrix} -s_{\theta/2} & c_{\theta/2} \end{pmatrix}\sigma^\mu \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2} \end{pmatrix} = (-1, s_\theta, 0, c_\theta), \quad \xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4) = (-1, -s_\theta, 0, -c_\theta), \tag{52}$$

$$\xi_+^\dagger(k_3)\sigma^\mu\xi_-(k_4) \;=\; \begin{pmatrix} c_{\theta/2} & s_{\theta/2} \end{pmatrix}\sigma^\mu \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix} = (1, s_\theta, 0, c_\theta), \quad \xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) = (1, -s_\theta, 0, -c_\theta). \tag{53}$$

在宇称变换下,

$$q_\pm \xrightarrow{P} q_\mp, \quad \bar{q}_\pm \xrightarrow{P} \bar{q}_\mp, \quad \chi_\pm \xrightarrow{P} \chi_\mp, \quad \bar{\chi}_\pm \xrightarrow{P} \bar{\chi}_\mp. \tag{54}$$

由于矢量流耦合 (27) 保持宇称守恒, 对于任一散射过程, 作宇称变换后得到的过程相应的散射振幅模方与原过程振幅模方是一样的 (而振幅可以相差一个相位因子), 如 $|\mathcal{M}(q_+\bar{q}_- \to \chi_+\bar{\chi}_-)|^2 = |\mathcal{M}(q_-\bar{q}_+ \to \chi_-\bar{\chi}_+)|^2$, $|\mathcal{M}(q_+\bar{q}_- \to \chi_-\bar{\chi}_+)|^2 = |\mathcal{M}(q_-\bar{q}_+ \to \chi_+\bar{\chi}_-)|^2$.

下面计算各极化过程的振幅.

$$\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) \;=\; \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4) = -(1+c_\theta), \tag{55}$$

$$\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4) \;=\; \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) = 1+c_\theta, \tag{56}$$

$$\begin{aligned}
&\mathcal{M}(q_+\bar{q}_- \to \chi_+\bar{\chi}_-) = \mathcal{M}(q_-\bar{q}_+ \to \chi_-\bar{\chi}_+) = \frac{1}{\Lambda^2}\bar{v}_+(p_2)\gamma^\mu u_-(p_1)\bar{u}_-(k_3)\gamma_\mu v_+(k_4) \\
=\;& \frac{1}{\Lambda^2}\Big[-\frac{\sqrt{s}}{2}(1+\beta_q)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \frac{\sqrt{s}}{2}(1-\beta_q)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\Big] \\
&\quad \times\Big[-\frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_-(k_4) + \frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4)\Big] \\
=\;& \frac{1}{\Lambda^2}\Big[-\frac{s}{4}(1+\beta_q)(1+\beta_\chi)(1+c_\theta) - \frac{s}{4}(1+\beta_q)(1-\beta_\chi)(1+c_\theta) \\
&\quad -\frac{s}{4}(1-\beta_q)(1+\beta_\chi)(1+c_\theta) - \frac{s}{4}(1-\beta_q)(1-\beta_\chi)(1+c_\theta)\Big] \\
=\;& -\frac{1}{\Lambda^2}s(1+c_\theta).
\end{aligned} \tag{57}$$

$$\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) \;=\; \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4) = -(1-c_\theta), \tag{58}$$

$$\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4) \;=\; \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) = 1-c_\theta, \tag{59}$$

$$\begin{aligned}
&\mathcal{M}(q_+\bar{q}_- \to \chi_-\bar{\chi}_+) = \mathcal{M}(q_-\bar{q}_+ \to \chi_+\bar{\chi}_-) = \frac{1}{\Lambda^2}\bar{v}_+(p_2)\gamma^\mu u_-(p_1)\bar{u}_+(k_3)\gamma_\mu v_-(k_4) \\
=\;& \frac{1}{\Lambda^2}\Big[-\frac{\sqrt{s}}{2}(1+\beta_q)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \frac{\sqrt{s}}{2}(1-\beta_q)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\Big] \\
&\quad \times\Big[\frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_+^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - \frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4)\Big] \\
=\;& \frac{1}{\Lambda^2}\Big[\frac{s}{4}(1+\beta_q)(1-\beta_\chi)(1-c_\theta) + \frac{s}{4}(1+\beta_q)(1+\beta_\chi)(1-c_\theta)
\end{aligned}$$

$$+\frac{s}{4}(1-\beta_q)(1-\beta_\chi)(1-c_\theta)+\frac{s}{4}(1-\beta_q)(1+\beta_\chi)(1-c_\theta)\Big]$$
$$= \frac{1}{\Lambda^2}s(1-c_\theta). \tag{60}$$

$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) = 1-c_\theta, \tag{61}$$
$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4) = 1+c_\theta, \tag{62}$$

$$\mathcal{M}(q_+\bar q_+\to\chi_+\bar\chi_+) = \mathcal{M}(q_-\bar q_-\to\chi_-\bar\chi_-) = \frac{1}{\Lambda^2}\bar v_-(p_2)\gamma^\mu u_-(p_1)\bar u_-(k_3)\gamma_\mu v_-(k_4)$$
$$= \frac{1}{\Lambda^2}[m_q\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)-m_q\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)][m_\chi\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4)-m_\chi\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4)]$$
$$= \frac{1}{\Lambda^2}[m_qm_\chi(1-c_\theta)-m_qm_\chi(1+c_\theta)-m_qm_\chi(1+c_\theta)+m_qm_\chi(1-c_\theta)]$$
$$= -\frac{4}{\Lambda^2}m_qm_\chi c_\theta. \tag{63}$$

$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_-(k_4) = -(1+c_\theta), \tag{64}$$
$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_-(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4) = -(1-c_\theta), \tag{65}$$

$$\mathcal{M}(q_+\bar q_+\to\chi_-\bar\chi_-) = \mathcal{M}(q_-\bar q_-\to\chi_+\bar\chi_+) = \frac{1}{\Lambda^2}\bar v_-(p_2)\gamma^\mu u_-(p_1)\bar u_+(k_3)\gamma_\mu v_+(k_4)$$
$$= \frac{1}{\Lambda^2}[m_q\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)-m_q\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)][-m_\chi\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4)+m_\chi\xi_+^\dagger(k_3)\sigma_\mu\xi_-(k_4)]$$
$$= \frac{1}{\Lambda^2}[m_qm_\chi(1+c_\theta)-m_qm_\chi(1-c_\theta)-m_qm_\chi(1-c_\theta)+m_qm_\chi(1+c_\theta)]$$
$$= \frac{4}{\Lambda^2}m_qm_\chi c_\theta. \tag{66}$$

$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4) = s_\theta, \tag{67}$$
$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4) = -s_\theta, \tag{68}$$

$$-\mathcal{M}(q_+\bar q_+\to\chi_+\bar\chi_-) = \mathcal{M}(q_-\bar q_-\to\chi_-\bar\chi_+) = \frac{1}{\Lambda^2}\bar v_-(p_2)\gamma^\mu u_-(p_1)\bar u_-(k_3)\gamma_\mu v_+(k_4)$$
$$= \frac{1}{\Lambda^2}[m_q\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)-m_q\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)]$$
$$\times\Big[-\frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_-^\dagger(k_3)\bar\sigma_\mu\xi_-(k_4)+\frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_-^\dagger(k_3)\sigma_\mu\xi_-(k_4)\Big]$$
$$= \frac{1}{\Lambda^2}\Big[-m_q\frac{\sqrt{s}}{2}(1+\beta_\chi)s_\theta-m_q\frac{\sqrt{s}}{2}(1-\beta_\chi)s_\theta-m_q\frac{\sqrt{s}}{2}(1+\beta_\chi)s_\theta-m_q\frac{\sqrt{s}}{2}(1-\beta_\chi)s_\theta\Big]$$
$$= -\frac{2}{\Lambda^2}\sqrt{s}m_qs_\theta. \tag{69}$$

$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4) = -s_\theta, \tag{70}$$
$$\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4) = \xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4) = s_\theta, \tag{71}$$

$$-\mathcal{M}(q_+\bar q_+\to\chi_-\bar\chi_+) = \mathcal{M}(q_-\bar q_-\to\chi_+\bar\chi_-) = \frac{1}{\Lambda^2}\bar v_-(p_2)\gamma^\mu u_-(p_1)\bar u_+(k_3)\gamma_\mu v_-(k_4)$$
$$= \frac{1}{\Lambda^2}[m_q\xi_+^\dagger(p_2)\bar\sigma^\mu\xi_-(p_1)-m_q\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1)]$$
$$\times\Big[\frac{\sqrt{s}}{2}(1-\beta_\chi)\xi_+^\dagger(k_3)\bar\sigma_\mu\xi_+(k_4)-\frac{\sqrt{s}}{2}(1+\beta_\chi)\xi_+^\dagger(k_3)\sigma_\mu\xi_+(k_4)\Big]$$

$$
\begin{aligned}
&= \frac{1}{\Lambda^2}\bigg[-m_q\frac{\sqrt{s}}{2}(1-\beta_\chi)s_\theta - m_q\frac{\sqrt{s}}{2}(1+\beta_\chi)s_\theta - m_q\frac{\sqrt{s}}{2}(1-\beta_\chi)s_\theta - m_q\frac{\sqrt{s}}{2}(1+\beta_\chi)s_\theta\bigg] \\
&= -\frac{2}{\Lambda^2}\sqrt{s}m_q s_\theta. \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) &= \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) = -s_\theta, \qquad (73)\\
\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) &= \xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) = s_\theta, \qquad (74)
\end{aligned}
$$

$$
\begin{aligned}
&-\mathcal{M}(q_+\bar{q}_- \to \chi_+\bar{\chi}_+) = \mathcal{M}(q_-\bar{q}_+ \to \chi_-\bar{\chi}_-) = \frac{1}{\Lambda^2}\bar{v}_+(p_2)\gamma^\mu u_-(p_1)\bar{u}_-(k_3)\gamma_\mu v_-(k_4) \\
&= \frac{1}{\Lambda^2}\bigg[-\frac{\sqrt{s}}{2}(1+\beta_q)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \frac{\sqrt{s}}{2}(1-\beta_q)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\bigg] \\
&\qquad \times[m_\chi\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - m_\chi\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4)] \\
&= \frac{1}{\Lambda^2}\bigg[m_\chi\frac{\sqrt{s}}{2}(1+\beta_q)s_\theta + m_\chi\frac{\sqrt{s}}{2}(1+\beta_q)s_\theta + m_\chi\frac{\sqrt{s}}{2}(1-\beta_q)s_\theta + m_\chi\frac{\sqrt{s}}{2}(1-\beta_q)s_\theta\bigg] \\
&= \frac{2}{\Lambda^2}\sqrt{s}m_\chi s_\theta. \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) &= \xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) = s_\theta, \qquad (76)\\
\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1)\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4) &= \xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1)\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) = -s_\theta, \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
&-\mathcal{M}(q_-\bar{q}_+ \to \chi_+\bar{\chi}_+) = \mathcal{M}(q_+\bar{q}_- \to \chi_-\bar{\chi}_-) = \frac{1}{\Lambda^2}\bar{v}_-(p_2)\gamma^\mu u_+(p_1)\bar{u}_-(k_3)\gamma_\mu v_-(k_4) \\
&= \frac{1}{\Lambda^2}\bigg[\frac{\sqrt{s}}{2}(1-\beta_q)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) - \frac{\sqrt{s}}{2}(1+\beta_q)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1)\bigg] \\
&\qquad \times[m_\chi\xi_-^\dagger(k_3)\bar{\sigma}_\mu\xi_+(k_4) - m_\chi\xi_-^\dagger(k_3)\sigma_\mu\xi_+(k_4)] \\
&= \frac{1}{\Lambda^2}\bigg[\frac{\sqrt{s}}{2}(1-\beta_q)m_\chi s_\theta + \frac{\sqrt{s}}{2}(1-\beta_q)m_\chi s_\theta + \frac{\sqrt{s}}{2}(1+\beta_q)m_\chi s_\theta + \frac{\sqrt{s}}{2}(1+\beta_q)m_\chi s_\theta\bigg] \\
&= \frac{2}{\Lambda^2}\sqrt{s}m_\chi s_\theta. \qquad (78)
\end{aligned}
$$

结果总结如下.

$$
\begin{aligned}
\mathcal{M}(q_+\bar{q}_- \to \chi_+\bar{\chi}_-) &= \mathcal{M}(q_-\bar{q}_+ \to \chi_-\bar{\chi}_+) = -\frac{1}{\Lambda^2}s(1+\cos\theta), \qquad (79)\\
\mathcal{M}(q_+\bar{q}_- \to \chi_-\bar{\chi}_+) &= \mathcal{M}(q_-\bar{q}_+ \to \chi_+\bar{\chi}_-) = \frac{1}{\Lambda^2}s(1-\cos\theta), \qquad (80)\\
\mathcal{M}(q_+\bar{q}_+ \to \chi_+\bar{\chi}_+) &= \mathcal{M}(q_-\bar{q}_- \to \chi_-\bar{\chi}_-) = -\frac{4}{\Lambda^2}m_q m_\chi\cos\theta, \qquad (81)\\
\mathcal{M}(q_+\bar{q}_+ \to \chi_-\bar{\chi}_-) &= \mathcal{M}(q_-\bar{q}_- \to \chi_+\bar{\chi}_+) = \frac{4}{\Lambda^2}m_q m_\chi\cos\theta, \qquad (82)\\
-\mathcal{M}(q_+\bar{q}_+ \to \chi_+\bar{\chi}_-) &= \mathcal{M}(q_-\bar{q}_- \to \chi_-\bar{\chi}_+) = -\frac{2}{\Lambda^2}\sqrt{s}m_q\sin\theta, \qquad (83)\\
-\mathcal{M}(q_+\bar{q}_+ \to \chi_-\bar{\chi}_+) &= \mathcal{M}(q_-\bar{q}_- \to \chi_+\bar{\chi}_-) = -\frac{2}{\Lambda^2}\sqrt{s}m_q\sin\theta, \qquad (84)\\
-\mathcal{M}(q_+\bar{q}_- \to \chi_+\bar{\chi}_+) &= \mathcal{M}(q_-\bar{q}_+ \to \chi_-\bar{\chi}_-) = \frac{2}{\Lambda^2}\sqrt{s}m_\chi\sin\theta, \qquad (85)\\
-\mathcal{M}(q_-\bar{q}_+ \to \chi_+\bar{\chi}_+) &= \mathcal{M}(q_+\bar{q}_- \to \chi_-\bar{\chi}_-) = \frac{2}{\Lambda^2}\sqrt{s}m_\chi\sin\theta. \qquad (86)
\end{aligned}
$$

由此, 可得非极化散射振幅模方

$$
\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(q\bar{q}\to\chi\bar{\chi})|^2 = \frac{1}{4}\sum_{\lambda_1\lambda_2}\sum_{\lambda_3\lambda_4}|\mathcal{M}(q_{\lambda_1}\bar{q}_{\lambda_2}\to\chi_{\lambda_3}\bar{\chi}_{\lambda_4})|^2\\
= &\ \frac{1}{4}\frac{1}{\Lambda^4}[2s^2(1+\cos\theta)^2+2s^2(1-\cos\theta)^2+64m_q^2m_\chi^2\cos^2\theta+16sm_q^2\sin^2\theta+16sm_\chi^2\sin^2\theta]\\
= &\ \frac{1}{\Lambda^4}(s^2+s^2\cos^2\theta+16m_q^2m_\chi^2\cos^2\theta+4sm_q^2-4sm_q^2\cos^2\theta+4sm_\chi^2-4sm_\chi^2\cos^2\theta)\\
= &\ \frac{1}{\Lambda^4}[s^2+4s(m_q^2+m_\chi^2)+(s-4m_q^2)(s-4m_\chi^2)\cos^2\theta]\\
= &\ \frac{1}{\Lambda^4}[s^2(1+\beta_q^2\beta_\chi^2\cos^2\theta)+4s(m_q^2+m_\chi^2)]. \qquad (87)
\end{aligned}
$$

这一结果与 (33) 式相同.

4 正负电子湮灭到双光子

下面讨论 $e^-(p_1)+e^+(p_2)\to\gamma(k_3)+\gamma(k_4)$ 过程, 各动量可表示成

$$
p_1=\frac{\sqrt{s}}{2}(1,0,0,\beta_e),\quad p_2=\frac{\sqrt{s}}{2}(1,0,0,-\beta_e),\quad k_3=\frac{\sqrt{s}}{2}(1,s_\theta,0,c_\theta),\quad k_4=\frac{\sqrt{s}}{2}(1,-s_\theta,0,-c_\theta). \qquad (88)
$$

此过程的不变振幅为

$$
\begin{aligned}
&i\mathcal{M}(e^-e^+\to\gamma\gamma)\\
= &\ \bar{v}(p_2)(-ie\gamma^\mu)\varepsilon_\mu^*(k_4)\frac{i(\not{p}_1-\not{k}_3+m_e)}{(p_1-k_3)^2-m_e^2}(-ie\gamma^\nu)\varepsilon_\nu^*(k_3)u(p_1)\\
&+\bar{v}(p_2)(-ie\gamma^\nu)\varepsilon_\nu^*(k_3)\frac{i(\not{p}_1-\not{k}_4+m_e)}{(p_1-k_4)^2-m_e^2}(-ie\gamma^\mu)\varepsilon_\mu^*(k_4)u(p_1)\\
= &\ -ie^2\varepsilon_\mu^*(k_4)\varepsilon_\nu^*(k_3)\bar{v}(p_2)\left[\frac{\gamma^\mu(\not{p}_1-\not{k}_3+m_e)\gamma^\nu}{(p_1-k_3)^2-m_e^2}+\frac{\gamma^\nu(\not{p}_1-\not{k}_4+m_e)\gamma^\mu}{(p_1-k_4)^2-m_e^2}\right]u(p_1)\\
= &\ -ie^2\varepsilon_\mu^*(k_4)\varepsilon_\nu^*(k_3)\bar{v}(p_2)\left[\frac{\gamma^\mu\not{k}_3\gamma^\nu-2\gamma^\mu p_1^\nu}{2p_1\cdot k_3}+\frac{\gamma^\nu\not{k}_4\gamma^\mu-2\gamma^\nu p_1^\mu}{2p_1\cdot k_4}\right]u(p_1), \qquad (89)
\end{aligned}
$$

$$
[i\mathcal{M}(e^-e^+\to\gamma\gamma)]^* = +ie^2\varepsilon_\rho(k_4)\varepsilon_\sigma(k_3)\bar{u}(p_1)\left[\frac{\gamma^\sigma\not{k}_3\gamma^\rho-2\gamma^\rho p_1^\sigma}{2p_1\cdot k_3}+\frac{\gamma^\rho\not{k}_4\gamma^\sigma-2\gamma^\sigma p_1^\rho}{2p_1\cdot k_4}\right]v(p_2). \qquad (90)
$$

在上述计算过程中, 用到了 Dirac 方程平面波解的性质

$$
(\not{p}_1+m_e)\gamma^\nu u(p_1)=[2p_1^\nu-\gamma^\nu(\not{p}_1-m_e)]u(p_1)=2p_1^\nu u(p_1). \qquad (91)
$$

4.1 非极化振幅

利用

$$
\begin{aligned}
p_1\cdot p_2=\frac{s}{2}-m_e^2,&\qquad k_3\cdot k_4=\frac{s}{2},\\
p_1\cdot k_3=p_2\cdot k_4=\frac{s}{4}(1-\beta_e c_\theta),&\qquad p_1\cdot k_4=p_2\cdot k_3=\frac{s}{4}(1+\beta_e c_\theta), \qquad (92)
\end{aligned}
$$

计算非极化振幅的模方:

$$\begin{aligned}
&\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(e^-e^+\to\gamma\gamma)|^2 \\
&= \frac{1}{4}\sum_{\text{spins}} e^4\varepsilon^*_\mu(k_4)\varepsilon_\rho(k_4)\varepsilon^*_\nu(k_3)\varepsilon_\sigma(k_3)\bar{v}(p_2)\left[\frac{\gamma^\mu \not{k}_3\gamma^\nu-2\gamma^\mu p_1^\nu}{2p_1\cdot k_3}+\frac{\gamma^\nu \not{k}_4\gamma^\mu-2\gamma^\nu p_1^\mu}{2p_1\cdot k_4}\right]u(p_1) \\
&\qquad\qquad \times\bar{u}(p_1)\left[\frac{\gamma^\sigma \not{k}_3\gamma^\rho-2\gamma^\rho p_1^\sigma}{2p_1\cdot k_3}+\frac{\gamma^\rho \not{k}_4\gamma^\sigma-2\gamma^\sigma p_1^\rho}{2p_1\cdot k_4}\right]v(p_2) \\
&= \frac{e^4}{4}g_{\mu\rho}g_{\nu\sigma}\mathrm{Tr}\Bigg\{(\not{p}_2-m_e)\left[\frac{\gamma^\mu \not{k}_3\gamma^\nu-2\gamma^\mu p_1^\nu}{2p_1\cdot k_3}+\frac{\gamma^\nu \not{k}_4\gamma^\mu-2\gamma^\nu p_1^\mu}{2p_1\cdot k_4}\right] \\
&\qquad\qquad \times(\not{p}_1+m_e)\left[\frac{\gamma^\sigma \not{k}_3\gamma^\rho-2\gamma^\rho p_1^\sigma}{2p_1\cdot k_3}+\frac{\gamma^\rho \not{k}_4\gamma^\sigma-2\gamma^\sigma p_1^\rho}{2p_1\cdot k_4}\right]\Bigg\} \\
&= \frac{4e^4[8sm_e^2(1-\beta_e^2c_\theta^2)+s^2(1-\beta_e^4c_\theta^4)-32m_e^4]}{s^2(1-\beta_e^2c_\theta^2)^2} \\
&= 4e^4\left[\frac{1+\beta_e^2c_\theta^2}{1-\beta_e^2c_\theta^2}+\frac{8m_e^2}{s(1-\beta_e^2c_\theta^2)}-\frac{32m_e^4}{s^2(1-\beta_e^2c_\theta^2)^2}\right] \\
&= 64\pi^2\alpha^2\left[\frac{1+\beta_e^2c_\theta^2}{1-\beta_e^2c_\theta^2}+\frac{8m_e^2}{s(1-\beta_e^2c_\theta^2)}-\frac{32m_e^4}{s^2(1-\beta_e^2c_\theta^2)^2}\right].
\end{aligned} \tag{93}$$

4.2 极化振幅

正负电子湮灭到双光子过程的极化振幅

$$\mathcal{M}(e^-_{\lambda_1}e^+_{\lambda_2}\to\gamma_{\lambda_3}\gamma_{\lambda_4}) = -e^2\varepsilon^{*\mu}_{\lambda_4}(k_4)\varepsilon^{*\nu}_{\lambda_3}(k_3)\bar{v}_{\lambda_2}(p_2)\left[\frac{\gamma_\mu \not{k}_3\gamma_\nu-2\gamma_\mu p_{1\nu}}{2p_1\cdot k_3}+\frac{\gamma_\nu \not{k}_4\gamma_\mu-2\gamma_\nu p_{1\mu}}{2p_1\cdot k_4}\right]u_{\lambda_1}(p_1). \tag{94}$$

利用前面列出的旋量态和极化矢量表达式, 可得

$$\begin{aligned}
\mathcal{M}(e^-_+e^+_-\to\gamma_+\gamma_-) &= \mathcal{M}(e^-_-e^+_+\to\gamma_-\gamma_+) = -\frac{2e^2\beta_e s_\theta(1+c_\theta)}{1-\beta_e^2c_\theta^2}, \\
\mathcal{M}(e^-_+e^+_-\to\gamma_-\gamma_+) &= \mathcal{M}(e^-_-e^+_+\to\gamma_+\gamma_-) = \frac{2e^2\beta_e s_\theta(1-c_\theta)}{1-\beta_e^2c_\theta^2}, \\
\mathcal{M}(e^-_+e^+_-\to\gamma_+\gamma_+) &= \mathcal{M}(e^-_-e^+_+\to\gamma_-\gamma_-) = 0, \\
\mathcal{M}(e^-_+e^+_-\to\gamma_-\gamma_-) &= \mathcal{M}(e^-_-e^+_+\to\gamma_+\gamma_+) = 0, \\
\mathcal{M}(e^-_+e^+_+\to\gamma_+\gamma_-) &= -\mathcal{M}(e^-_-e^+_-\to\gamma_-\gamma_+) = \frac{4e^2m_e\beta_e s_\theta^2}{\sqrt{s}(1-\beta_e^2c_\theta^2)}, \\
\mathcal{M}(e^-_+e^+_+\to\gamma_-\gamma_+) &= -\mathcal{M}(e^-_-e^+_-\to\gamma_+\gamma_-) = \frac{4e^2m_e\beta_e s_\theta^2}{\sqrt{s}(1-\beta_e^2c_\theta^2)}, \\
\mathcal{M}(e^-_+e^+_+\to\gamma_+\gamma_+) &= -\mathcal{M}(e^-_-e^+_-\to\gamma_-\gamma_-) = -\frac{4e^2m_e(1+\beta_e)}{\sqrt{s}(1-\beta_e^2c_\theta^2)}, \\
\mathcal{M}(e^-_+e^+_+\to\gamma_-\gamma_-) &= -\mathcal{M}(e^-_-e^+_-\to\gamma_+\gamma_+) = \frac{4e^2m_e(1-\beta_e)}{\sqrt{s}(1-\beta_e^2c_\theta^2)}.
\end{aligned} \tag{95}$$

由此, 亦可以计算非极化振幅模方如下.

$$
\begin{aligned}
& \frac{1}{4}\sum_{\lambda_1\lambda_2}\sum_{\lambda_3\lambda_4}|\mathcal{M}(e^-_{\lambda_1}e^+_{\lambda_2}\to\gamma_{\lambda_3}\gamma_{\lambda_4})|^2 \\
= & \frac{4e^4\{s\beta_e^2(1-c_\theta^2)+4m_e^2[1+\beta_e^2(c_\theta^4+2c_\theta^2+2)]\}}{s(1-\beta_e^2c_\theta^2)^2} \\
= & \frac{64\pi^2\alpha^2}{s^2(1-\beta_e^2c_\theta^2)^2}\{s^2\beta_e^2(1-c_\theta^2)+4m_e^2s[1+\beta_e^2(c_\theta^4+2c_\theta^2+2)]\},
\end{aligned}
\tag{96}
$$

而

$$
\begin{aligned}
& s^2\beta_e^2(1-c_\theta^4)+4m_e^2s[1+\beta_e^2(c_\theta^4+2c_\theta^2+2)] \\
= & \beta_e^2s^2-\beta_e^2c_\theta^4s^2+4m_e^2s+4m_e^2s(\beta_e^2c_\theta^4-2\beta_e^2c_\theta^2+2\beta_e^2+2-2) \\
= & \beta_e^2s^2-\beta_e^2c_\theta^4s^2+s^2(1-\beta_e^2)+4m_e^2s[\beta_e^2c_\theta^4+2(1-\beta_e^2c_\theta^2)-2(1-\beta_e^2)] \\
= & s^2-\beta_e^2c_\theta^4s^2+4m_e^2s\beta_e^2c_\theta^4+8m_e^2s(1-\beta_e^2c_\theta^2)-8m_e^2\cdot 4m_e^2 \\
= & s^2-s\beta_e^2c_\theta^4(s-4m_e^2)+8m_e^2s(1-\beta_e^2c_\theta^2)-32m_e^4 \\
= & s^2-s^2\beta_e^4c_\theta^4+8m_e^2s(1-\beta_e^2c_\theta^2)-32m_e^4 \\
= & [s^2(1+\beta_e^2c_\theta^2)+8m_e^2s](1-\beta_e^2c_\theta^2)-32m_e^4,
\end{aligned}
\tag{97}
$$

故

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}(e^-e^+\to\gamma\gamma)|^2 = & \frac{1}{4}\sum_{\lambda_1\lambda_2}\sum_{\lambda_3\lambda_4}|\mathcal{M}(e^-_{\lambda_1}e^+_{\lambda_2}\to\gamma_{\lambda_3}\gamma_{\lambda_4})|^2 \\
= & \frac{64\pi^2\alpha^2}{s^2(1-\beta_e^2c_\theta^2)^2}\{[s^2(1+\beta_e^2c_\theta^2)+8m_e^2s](1-\beta_e^2c_\theta^2)-32m_e^4\} \\
= & 64\pi^2\alpha^2\left[\frac{1+\beta_e^2c_\theta^2}{1-\beta_e^2c_\theta^2}+\frac{8m_e^2}{s(1-\beta_e^2c_\theta^2)}-\frac{32m_e^4}{s^2(1-\beta_e^2c_\theta^2)^2}\right].
\end{aligned}
\tag{98}
$$

这一结果与 (93) 式一致.

5 旋量双线性型的螺旋态表达式

在质心系中, 考虑一对入射的正反费米子 $i(p_1)$ 和 $\bar{i}(p_2)$, 质量为 m_i, 则其动量和螺旋态可表示成

$$
p_1=\frac{\sqrt{s}}{2}(1,0,0,\beta_i),\quad p_2=\frac{\sqrt{s}}{2}(1,0,0,-\beta_i),\quad \beta_i\equiv\sqrt{1-4m_i^2/s},
\tag{99}
$$

$$
\xi_+(p_1)=\begin{pmatrix}1\\0\end{pmatrix},\quad \xi_-(p_1)=\begin{pmatrix}0\\1\end{pmatrix},\quad \xi_+(p_2)=\begin{pmatrix}0\\-1\end{pmatrix},\quad \xi_-(p_2)=\begin{pmatrix}1\\0\end{pmatrix},
\tag{100}
$$

而 $\omega_\pm(p_2)\omega_\pm(p_1)=\frac{\sqrt{s}}{2}(1\pm\beta_i)$, $\omega_\pm(p_2)\omega_\mp(p_1)=m_i$.

参考第 3 节的表达式, 对于矢量算符 $\bar{\psi}\gamma^\mu\psi$, 有

$$
\begin{aligned}
\bar{v}_+(p_2)\gamma^\mu u_-(p_1) = & -\frac{\sqrt{s}}{2}(1+\beta_i)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)+\frac{\sqrt{s}}{2}(1-\beta_i)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1) \\
= & -\sqrt{s}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1)=\sqrt{s}(0,1,-i,0)=\lim_{\beta_i\to 1}\bar{v}_+(p_2)\gamma^\mu P_L u_-(p_1),
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
\bar{v}_-(p_2)\gamma^\mu u_+(p_1) &= \frac{\sqrt{s}}{2}(1-\beta_i)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) - \frac{\sqrt{s}}{2}(1+\beta_i)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) \\
&= -\sqrt{s}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = \sqrt{s}(0,1,i,0) = \lim_{\beta_i\to 1}\bar{v}_-(p_2)\gamma^\mu P_R u_+(p_1), \qquad (102)\\
\bar{v}_-(p_2)\gamma^\mu u_-(p_1) &= m_i\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) - m_i\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1) = -2m_i(0,0,0,1), \qquad (103)\\
\bar{v}_+(p_2)\gamma^\mu u_+(p_1) &= -m_i\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) + m_i\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2m_i(0,0,0,1). \qquad (104)
\end{aligned}
$$

对于轴矢量算符 $\bar{\psi}\gamma^\mu\gamma_5\psi$, 有

$$
\begin{aligned}
\bar{v}_+(p_2)\gamma^\mu\gamma_5 u_-(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= \frac{\sqrt{s}}{2}(1+\beta_i)\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + \frac{\sqrt{s}}{2}(1-\beta_i)\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1)\\
&= \beta_i\sqrt{s}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) = \beta_i\sqrt{s}(0,-1,i,0)\\
&\xrightarrow{\beta_i\to 1} \sqrt{s}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) = \sqrt{s}(0,-1,i,0) = \lim_{\beta_i\to 1}\bar{v}_+(p_2)\gamma^\mu\gamma_5 P_L u_-(p_1), \qquad (105)\\
\bar{v}_-(p_2)\gamma^\mu\gamma_5 u_+(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= -\frac{\sqrt{s}}{2}(1-\beta_i)\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) - \frac{\sqrt{s}}{2}(1+\beta_i)\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1)\\
&= -\beta_i\sqrt{s}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = \beta_i\sqrt{s}(0,1,i,0)\\
&\xrightarrow{\beta_i\to 1} -\sqrt{s}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = \sqrt{s}(0,1,i,0) = \lim_{\beta_i\to 1}\bar{v}_-(p_2)\gamma^\mu\gamma_5 P_R u_+(p_1), \qquad (106)\\
\bar{v}_-(p_2)\gamma^\mu\gamma_5 u_-(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= -m_i\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) - m_i\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1) = 2m_i(1,0,0,0), \qquad (107)\\
\bar{v}_+(p_2)\gamma^\mu\gamma_5 u_+(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= m_i\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) + m_i\xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2m_i(1,0,0,0). \qquad (108)
\end{aligned}
$$

对于标量算符 $\bar{\psi}\psi$, 有

$$
\begin{aligned}
\bar{v}_+(p_2)u_-(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= -m_i\xi_-^\dagger(p_2)\xi_-(p_1) + m_i\xi_-^\dagger(p_2)\xi_-(p_1) = 0, \qquad (109)\\
\bar{v}_-(p_2)u_+(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= m_i\xi_+^\dagger(p_2)\xi_+(p_1) - m_i\xi_+^\dagger(p_2)\xi_+(p_1) = 0, \qquad (110)\\
\bar{v}_-(p_2)u_-(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= \frac{\sqrt{s}}{2}(1-\beta_i)\xi_+^\dagger(p_2)\xi_-(p_1) - \frac{\sqrt{s}}{2}(1+\beta_i)\xi_+^\dagger(p_2)\xi_-(p_1)\\
&= -\beta_i\sqrt{s}\xi_+^\dagger(p_2)\xi_-(p_1) = \beta_i\sqrt{s}, \qquad (111)
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}_+(p_2)u_+(p_1) \;&=\; \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&=\; -\frac{\sqrt{s}}{2}(1+\beta_i)\xi_-^\dagger(p_2)\xi_+(p_1)+\frac{\sqrt{s}}{2}(1-\beta_i)\xi_-^\dagger(p_2)\xi_+(p_1)\\
&=\; -\beta_i\sqrt{s}\xi_-^\dagger(p_2)\xi_+(p_1)=-\beta_i\sqrt{s}.
\end{aligned}\tag{112}
$$

对于赝标量算符 $\bar{\psi}i\gamma_5\psi$, 有

$$
\begin{aligned}
\bar{v}_+(p_2)i\gamma_5u_-(p_1) \;&=\; \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&=\; -im_i\xi_-^\dagger(p_2)\xi_-(p_1)-im_i\xi_-^\dagger(p_2)\xi_-(p_1)=-2im_i\xi_-^\dagger(p_2)\xi_-(p_1)=0, \qquad (113)\\
\bar{v}_-(p_2)i\gamma_5u_+(p_1) \;&=\; \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&=\; im_i\xi_+^\dagger(p_2)\xi_+(p_1)+im_i\xi_+^\dagger(p_2)\xi_+(p_1)=2im_i\xi_+^\dagger(p_2)\xi_+(p_1)=0, \qquad (114)\\
\bar{v}_-(p_2)i\gamma_5u_-(p_1) \;&=\; \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&=\; i\frac{\sqrt{s}}{2}(1-\beta_i)\xi_+^\dagger(p_2)\xi_-(p_1)+i\frac{\sqrt{s}}{2}(1+\beta_i)\xi_+^\dagger(p_2)\xi_-(p_1)\\
&=\; i\sqrt{s}\xi_+^\dagger(p_2)\xi_-(p_1)=-i\sqrt{s}, \qquad (115)\\
\bar{v}_+(p_2)i\gamma_5u_+(p_1) \;&=\; \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&=\; -i\frac{\sqrt{s}}{2}(1+\beta_i)\xi_-^\dagger(p_2)\xi_+(p_1)-i\frac{\sqrt{s}}{2}(1-\beta_i)\xi_-^\dagger(p_2)\xi_+(p_1)\\
&=\; -i\sqrt{s}\xi_-^\dagger(p_2)\xi_+(p_1)=-i\sqrt{s}. \qquad (116)
\end{aligned}
$$

对于 2 阶反对称张量算符 $\bar{\psi}\sigma^{\mu\nu}\psi$, 需要如下表达式:

$$
\xi_+^\dagger(p_2)\Sigma^{\mu\nu}\xi_+(p_1) \;=\; \begin{pmatrix} & i & -1 & \\ -i & & & i \\ 1 & & & -1 \\ & -i & 1 & \end{pmatrix}, \quad \xi_+^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_+(p_1)=\begin{pmatrix} & -i & 1 & \\ i & & & i \\ -1 & & & -1 \\ & -i & 1 & \end{pmatrix}, \tag{117}
$$

$$
\xi_-^\dagger(p_2)\Sigma^{\mu\nu}\xi_-(p_1) \;=\; \begin{pmatrix} & -i & -1 & \\ i & & & i \\ 1 & & & 1 \\ & -i & -1 & \end{pmatrix}, \quad \xi_-^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_-(p_1)=\begin{pmatrix} & i & 1 & \\ -i & & & i \\ -1 & & & 1 \\ & -i & -1 & \end{pmatrix}, \tag{118}
$$

$$
\xi_-^\dagger(p_2)\Sigma^{\mu\nu}\xi_+(p_1) \;=\; \xi_+^\dagger(p_2)\Sigma^{\mu\nu}\xi_-(p_1)=\begin{pmatrix} & & & -i \\ & & 1 & \\ & -1 & & \\ i & & & \end{pmatrix}, \tag{119}
$$

$$
\xi_-^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_+(p_1) \;=\; \xi_+^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_-(p_1)=\begin{pmatrix} & & & i \\ & & 1 & \\ & -1 & & \\ -i & & & \end{pmatrix}. \tag{120}
$$

从而,

$$\begin{aligned}
\bar{v}_+(p_2)\sigma^{\mu\nu}u_-(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix} \\
&= -m_i\xi_-^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_-(p_1) + m_i\xi_-^\dagger(p_2)\Sigma^{\mu\nu}\xi_-(p_1) = 2m_i\begin{pmatrix} & & -i & -1 \\ i & & & \\ 1 & & & \\ & & & 0\end{pmatrix}, \qquad (121) \\
\bar{v}_-(p_2)\sigma^{\mu\nu}u_+(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix} \\
&= m_i\xi_+^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_+(p_1) - m_i\xi_+^\dagger(p_2)\Sigma^{\mu\nu}\xi_+(p_1) = 2m_i\begin{pmatrix} & & -i & 1 \\ i & & & \\ -1 & & & \\ & & & 0\end{pmatrix}, \qquad (122) \\
\bar{v}_-(p_2)\sigma^{\mu\nu}u_-(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix}\begin{pmatrix}\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix} \\
&= \frac{\sqrt{s}}{2}(1-\beta_i)\xi_+^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_-(p_1) - \frac{\sqrt{s}}{2}(1+\beta_i)\xi_+^\dagger(p_2)\Sigma^{\mu\nu}\xi_-(p_1) \\
&= \sqrt{s}\begin{pmatrix} & & & i \\ & & -\beta_i & \\ & \beta_i & & \\ -i & & & \end{pmatrix}, \qquad (123) \\
\bar{v}_+(p_2)\sigma^{\mu\nu}u_+(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix} \\
&= -\frac{\sqrt{s}}{2}(1+\beta_i)\xi_-^\dagger(p_2)\bar{\Sigma}^{\mu\nu}\xi_+(p_1) + \frac{\sqrt{s}}{2}(1-\beta_i)\xi_-^\dagger(p_2)\Sigma^{\mu\nu}\xi_+(p_1) \\
&= \sqrt{s}\begin{pmatrix} & & & -i \\ & & -\beta_i & \\ & \beta_i & & \\ i & & & \end{pmatrix}. \qquad (124)
\end{aligned}$$

在质心系中, 考虑一对出射的正反费米子 $f(k_3)$ 和 $\bar{f}(k_4)$, 质量为 m_f, 则其动量和螺旋态可表示成

$$k_3 = \frac{\sqrt{s}}{2}(1, \beta_f s_\theta, 0, \beta_f c_\theta), \quad k_4 = \frac{\sqrt{s}}{2}(1, -\beta_f s_\theta, 0, -\beta_f c_\theta), \quad \beta_f \equiv \sqrt{1-4m_f^2/s}, \qquad (125)$$

$$\xi_+(k_3) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}\end{pmatrix}, \quad \xi_-(k_3) = \begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2}\end{pmatrix}, \quad \xi_+(k_4) = \begin{pmatrix} s_{\theta/2} \\ -c_{\theta/2}\end{pmatrix}, \quad \xi_-(k_4) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}\end{pmatrix}, \qquad (126)$$

而 $\omega_\pm(k_3)\omega_\pm(k_4) = \frac{\sqrt{s}}{2}(1 \pm \beta_f)$, $\omega_\pm(k_3)\omega_\mp(k_4) = m_f$.

对于矢量算符 $\bar{\psi}\gamma^\mu\psi$, 有

$$\begin{aligned}
\bar{u}_-(k_3)\gamma^\mu v_+(k_4) &= -\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) + \frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(k_3)\sigma^\mu\xi_-(k_4) \\
&= -\sqrt{s}\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) = \sqrt{s}(0, c_\theta, i, -s_\theta) = \lim_{\beta_f\to 1}\bar{u}_-(k_3)\gamma^\mu P_L v_+(k_4), \qquad (127)
\end{aligned}$$

$$\begin{aligned}\bar{u}_+(k_3)\gamma^\mu v_-(k_4) &= \frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4) - \frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4)\\ &= -\sqrt{s}\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4) = \sqrt{s}(0,c_\theta,-i,-s_\theta) = \lim_{\beta_f\to 1}\bar{u}_+(k_3)\gamma^\mu P_R v_-(k_4), \quad (128)\\ \bar{u}_-(k_3)\gamma^\mu v_-(k_4) &= m_f\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4) - m_f\xi_-^\dagger(k_3)\sigma^\mu\xi_+(k_4) = -2m_f(0,s_\theta,0,c_\theta), \quad (129)\\ \bar{u}_+(k_3)\gamma^\mu v_+(k_4) &= -m_f\xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) + m_f\xi_+^\dagger(k_3)\sigma^\mu\xi_-(k_4) = 2m_f(0,s_\theta,0,c_\theta). \quad (130)\end{aligned}$$

对于轴矢量算符 $\bar{\psi}\gamma^\mu\gamma_5\psi$, 有

$$\begin{aligned}\bar{u}_-(k_3)\gamma^\mu\gamma_5 v_+(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix}-\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}-\omega_+(k_4)\xi_-(k_4)\\ \omega_-(k_4)\xi_-(k_4)\end{pmatrix}\\ &= \frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) + \frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(k_3)\sigma^\mu\xi_-(k_4)\\ &= \beta_f\sqrt{s}\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) = \beta_f\sqrt{s}(0,-c_\theta,-i,s_\theta)\\ &\xrightarrow{\beta_f\to 1} \sqrt{s}\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) = \sqrt{s}(0,-c_\theta,-i,s_\theta)\\ &= \lim_{\beta_f\to 1}\bar{u}_-(k_3)\gamma^\mu\gamma_5 P_L v_+(k_4), \quad (131)\\ \bar{u}_+(k_3)\gamma^\mu\gamma_5 v_-(k_4) &= \begin{pmatrix}\omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3)\end{pmatrix}\begin{pmatrix}-\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\\ &= -\frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4) - \frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4)\\ &= -\beta_f\sqrt{s}\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4) = \beta_f\sqrt{s}(0,c_\theta,-i,-s_\theta)\\ &\xrightarrow{\beta_f\to 1} -\sqrt{s}\xi_+^\dagger(k_3)\sigma^\mu\xi_+(k_4) = \sqrt{s}(0,c_\theta,-i,-s_\theta)\\ &= \lim_{\beta_f\to 1}\bar{u}_+(k_3)\gamma^\mu\gamma_5 P_R v_-(k_4), \quad (132)\\ \bar{u}_-(k_3)\gamma^\mu\gamma_5 v_-(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix}-\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\\ &= -m_f\xi_-^\dagger(k_3)\bar{\sigma}^\mu\xi_+(k_4) - m_f\xi_-^\dagger(k_3)\sigma^\mu\xi_+(k_4) = 2m_f(1,0,0,0), \quad (133)\\ \bar{u}_+(k_3)\gamma^\mu\gamma_5 v_+(k_4) &= \begin{pmatrix}\omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3)\end{pmatrix}\begin{pmatrix}-\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}-\omega_+(k_4)\xi_-(k_4)\\ \omega_-(k_4)\xi_-(k_4)\end{pmatrix}\\ &= m_f\xi_+^\dagger(k_3)\bar{\sigma}^\mu\xi_-(k_4) + m_f\xi_+^\dagger(k_3)\sigma^\mu\xi_-(k_4) = 2m_f(1,0,0,0). \quad (134)\end{aligned}$$

对于标量算符 $\bar{\psi}\psi$, 有

$$\begin{aligned}\bar{u}_-(k_3)v_+(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}-\omega_+(k_4)\xi_-(k_4)\\ \omega_-(k_4)\xi_-(k_4)\end{pmatrix}\\ &= m_f\xi_-^\dagger(k_3)\xi_-(k_4) - m_f\xi_-^\dagger(k_3)\xi_-(k_4) = 0, \quad (135)\\ \bar{u}_+(k_3)v_-(k_4) &= \begin{pmatrix}\omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3)\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\\ &= -m_f\xi_+^\dagger(k_3)\xi_+(k_4) + m_f\xi_+^\dagger(k_3)\xi_+(k_4) = 0, \quad (136)\\ \bar{u}_-(k_3)v_-(k_4) &= \begin{pmatrix}\omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3)\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}\omega_-(k_4)\xi_+(k_4)\\ -\omega_+(k_4)\xi_+(k_4)\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(k_3)\xi_+(k_4) + \frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(k_3)\xi_+(k_4) \\
&= -\beta_f\sqrt{s}\xi_-^\dagger(k_3)\xi_+(k_4) = \beta_f\sqrt{s},
\end{aligned} \tag{137}
$$

$$
\begin{aligned}
\bar{u}_+(k_3)v_+(k_4) &= \Big(\omega_-(k_3)\xi_+^\dagger(k_3),\, \omega_+(k_3)\xi_+^\dagger(k_3)\Big)\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} -\omega_+(k_4)\xi_-(k_4) \\ \omega_-(k_4)\xi_-(k_4) \end{pmatrix} \\
&= \frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(k_3)\xi_-(k_4) - \frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(k_3)\xi_-(k_4) \\
&= -\beta_f\sqrt{s}\xi_+^\dagger(k_3)\xi_-(k_4) = -\beta_f\sqrt{s}.
\end{aligned} \tag{138}
$$

对于赝标量算符 $\bar{\psi}i\gamma_5\psi$, 有

$$
\begin{aligned}
\bar{u}_-(k_3)i\gamma_5 v_+(k_4) &= \Big(\omega_+(k_3)\xi_-^\dagger(k_3),\, \omega_-(k_3)\xi_-^\dagger(k_3)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix} -\omega_+(k_4)\xi_-(k_4) \\ \omega_-(k_4)\xi_-(k_4) \end{pmatrix} \\
&= im_f\xi_-^\dagger(k_3)\xi_-(k_4) + im_f\xi_-^\dagger(k_3)\xi_-(k_4) = 2im_f\xi_-^\dagger(k_3)\xi_-(k_4) = 0,
\end{aligned} \tag{139}
$$

$$
\begin{aligned}
\bar{u}_+(k_3)i\gamma_5 v_-(k_4) &= \Big(\omega_-(k_3)\xi_+^\dagger(k_3),\, \omega_+(k_3)\xi_+^\dagger(k_3)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix} \omega_-(k_4)\xi_+(k_4) \\ -\omega_+(k_4)\xi_+(k_4) \end{pmatrix} \\
&= -im_f\xi_+^\dagger(k_3)\xi_+(k_4) - im_f\xi_+^\dagger(k_3)\xi_+(k_4) = -2im_f\xi_+^\dagger(k_3)\xi_+(k_4) = 0,
\end{aligned} \tag{140}
$$

$$
\begin{aligned}
\bar{u}_-(k_3)i\gamma_5 v_-(k_4) &= \Big(\omega_+(k_3)\xi_-^\dagger(k_3),\, \omega_-(k_3)\xi_-^\dagger(k_3)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix} \omega_-(k_4)\xi_+(k_4) \\ -\omega_+(k_4)\xi_+(k_4) \end{pmatrix} \\
&= -i\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(k_3)\xi_+(k_4) - i\frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(k_3)\xi_+(k_4) \\
&= -i\sqrt{s}\xi_-^\dagger(k_3)\xi_+(k_4) = i\sqrt{s},
\end{aligned} \tag{141}
$$

$$
\begin{aligned}
\bar{u}_+(k_3)i\gamma_5 v_+(k_4) &= \Big(\omega_-(k_3)\xi_+^\dagger(k_3),\, \omega_+(k_3)\xi_+^\dagger(k_3)\Big)\begin{pmatrix} & i \\ -i & \end{pmatrix}\begin{pmatrix} -\omega_+(k_4)\xi_-(k_4) \\ \omega_-(k_4)\xi_-(k_4) \end{pmatrix} \\
&= i\frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(k_3)\xi_-(k_4) + i\frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(k_3)\xi_-(k_4) \\
&= i\sqrt{s}\xi_+^\dagger(k_3)\xi_-(k_4) = i\sqrt{s}.
\end{aligned} \tag{142}
$$

对于 2 阶反对称张量算符 $\bar{\psi}\sigma^{\mu\nu}\psi$, 需要如下表达式:

$$
\xi_+^\dagger(k_3)\Sigma^{\mu\nu}\xi_+(k_4) = \begin{pmatrix} & ic_\theta & 1 & -is_\theta \\ -ic_\theta & & s_\theta & -i \\ -1 & -s_\theta & & -c_\theta \\ is_\theta & i & c_\theta & \end{pmatrix}, \quad \xi_+^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_+(k_4) = \begin{pmatrix} & -ic_\theta & -1 & is_\theta \\ ic_\theta & & s_\theta & -i \\ 1 & -s_\theta & & -c_\theta \\ -is_\theta & i & c_\theta & \end{pmatrix}, \tag{143}
$$

$$
\xi_-^\dagger(k_3)\Sigma^{\mu\nu}\xi_-(k_4) = \begin{pmatrix} & -ic_\theta & 1 & is_\theta \\ ic_\theta & & -s_\theta & -i \\ -1 & s_\theta & & c_\theta \\ -is_\theta & i & -c_\theta & \end{pmatrix}, \quad \xi_-^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_-(k_4) = \begin{pmatrix} & ic_\theta & -1 & -is_\theta \\ -ic_\theta & & -s_\theta & -i \\ 1 & s_\theta & & c_\theta \\ is_\theta & i & -c_\theta & \end{pmatrix}, \tag{144}
$$

$$
\xi_-^\dagger(k_3)\Sigma^{\mu\nu}\xi_+(k_4) = \xi_+^\dagger(k_3)\Sigma^{\mu\nu}\xi_-(k_4) = \begin{pmatrix} & -is_\theta & & -ic_\theta \\ is_\theta & & c_\theta & \\ & -c_\theta & & s_\theta \\ ic_\theta & & -s_\theta & \end{pmatrix}, \tag{145}
$$

$$
\xi_-^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_+(k_4) = \xi_+^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_-(k_4) = \begin{pmatrix} & is_\theta & & ic_\theta \\ -is_\theta & & c_\theta & \\ & -c_\theta & & s_\theta \\ -ic_\theta & & -s_\theta & \end{pmatrix}. \tag{146}
$$

从而,

$$
\begin{aligned}
\bar{u}_-(k_3)\sigma^{\mu\nu}v_+(k_4) &= \begin{pmatrix} \omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3) \end{pmatrix} \begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix} \begin{pmatrix} -\omega_+(k_4)\xi_-(k_4) \\ \omega_-(k_4)\xi_-(k_4) \end{pmatrix} \\
&= m_f\xi_-^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_-(k_4) - m_f\xi_-^\dagger(k_3)\Sigma^{\mu\nu}\xi_-(k_4) \\
&= 2m_f \begin{pmatrix} & ic_\theta & -1 & -is_\theta \\ -ic_\theta & & & \\ 1 & & & \\ is_\theta & & & \end{pmatrix}, \qquad (147) \\
\bar{u}_+(k_3)\sigma^{\mu\nu}v_-(k_4) &= \begin{pmatrix} \omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3) \end{pmatrix} \begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix} \begin{pmatrix} \omega_-(k_4)\xi_+(k_4) \\ -\omega_+(k_4)\xi_+(k_4) \end{pmatrix} \\
&= -m_f\xi_+^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_+(k_4) + m_f\xi_+^\dagger(k_3)\Sigma^{\mu\nu}\xi_+(k_4) \\
&= 2m_f \begin{pmatrix} & ic_\theta & 1 & -is_\theta \\ -ic_\theta & & & \\ -1 & & & \\ is_\theta & & & \end{pmatrix}, \qquad (148) \\
\bar{u}_-(k_3)\sigma^{\mu\nu}v_-(k_4) &= \begin{pmatrix} \omega_+(k_3)\xi_-^\dagger(k_3), & \omega_-(k_3)\xi_-^\dagger(k_3) \end{pmatrix} \begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix} \begin{pmatrix} \omega_-(k_4)\xi_+(k_4) \\ -\omega_+(k_4)\xi_+(k_4) \end{pmatrix} \\
&= -\frac{\sqrt{s}}{2}(1+\beta_f)\xi_-^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_+(k_4) + \frac{\sqrt{s}}{2}(1-\beta_f)\xi_-^\dagger(k_3)\Sigma^{\mu\nu}\xi_+(k_4) \\
&= \sqrt{s} \begin{pmatrix} & -is_\theta & & -ic_\theta \\ is_\theta & & -\beta_f c_\theta & \\ & \beta_f c_\theta & & -\beta_f s_\theta \\ ic_\theta & & \beta_f s_\theta & \end{pmatrix}, \qquad (149) \\
\bar{u}_+(k_3)\sigma^{\mu\nu}v_+(k_4) &= \begin{pmatrix} \omega_-(k_3)\xi_+^\dagger(k_3), & \omega_+(k_3)\xi_+^\dagger(k_3) \end{pmatrix} \begin{pmatrix} & \bar{\Sigma}^{\mu\nu} \\ \Sigma^{\mu\nu} & \end{pmatrix} \begin{pmatrix} -\omega_+(k_4)\xi_-(k_4) \\ \omega_-(k_4)\xi_-(k_4) \end{pmatrix} \\
&= \frac{\sqrt{s}}{2}(1-\beta_f)\xi_+^\dagger(k_3)\bar{\Sigma}^{\mu\nu}\xi_-(k_4) - \frac{\sqrt{s}}{2}(1+\beta_f)\xi_+^\dagger(k_3)\Sigma^{\mu\nu}\xi_-(k_4) \\
&= \sqrt{s} \begin{pmatrix} & is_\theta & & ic_\theta \\ -is_\theta & & -\beta_f c_\theta & \\ & \beta_f c_\theta & & -\beta_f s_\theta \\ -ic_\theta & & \beta_f s_\theta & \end{pmatrix}. \qquad (150)
\end{aligned}
$$

可以看出, 在做完矩阵运算之后, 这些表达式均包括两项, 分别对应于如下左右手投影分解:

$$
\begin{aligned}
\bar{\psi}\gamma^\mu\psi &= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R, \quad \bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}_L\gamma^\mu\gamma_5\psi_L + \bar{\psi}_R\gamma^\mu\gamma_5\psi_R, \qquad (151) \\
\bar{\psi}\psi &= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L, \quad \bar{\psi}i\gamma_5\psi = \bar{\psi}_L i\gamma_5\psi_R + \bar{\psi}_R i\gamma_5\psi_L = i\bar{\psi}_L\psi_R - i\bar{\psi}_R\psi_L, \qquad (152) \\
\bar{\psi}\sigma^{\mu\nu}\psi &= \bar{\psi}_L\sigma^{\mu\nu}\psi_R + \bar{\psi}_R\sigma^{\mu\nu}\psi_L. \qquad (153)
\end{aligned}
$$

Table 1: 旋量双线性型螺旋态表达式小结.

operator	L	S	${}^{2S+1}L_J$	J^{PC}	$\overset{\Rightarrow}{\longrightarrow}\overset{\Rightarrow}{\longleftarrow}$ or $\overset{\Leftarrow}{\longrightarrow}\overset{\Leftarrow}{\longleftarrow}$	$\overset{\Rightarrow}{\longrightarrow}\overset{\Leftarrow}{\longleftarrow}$ or $\overset{\Leftarrow}{\longrightarrow}\overset{\Rightarrow}{\longleftarrow}$
$\bar{\psi}\psi$	1	1	${}^3P_0$	0^{++}	0	$\propto \beta\sqrt{s}$
$\bar{\psi}i\gamma_5\psi$	0	0	${}^1S_0$	0^{-+}	0	$\propto \sqrt{s}$
$\bar{\psi}\gamma^0\psi$	1	0	${}^1P_1$	1^{+-}	0	0
$\bar{\psi}\gamma^i\psi$	0	1	${}^3S_1$	1^{--}	$\propto \sqrt{s}$	$\propto m$
$\bar{\psi}\gamma^0\gamma_5\psi$	0	0	${}^1S_0$	0^{-+}	0	$\propto m$
$\bar{\psi}\gamma^i\gamma_5\psi$	1	1	${}^3P_1$	1^{++}	$\propto \beta\sqrt{s}$	0
$\bar{\psi}\sigma^{0i}\psi$	0	1	${}^3S_1$	1^{--}	$\propto m$	$\propto \sqrt{s}$
$\bar{\psi}\sigma^{ij}\psi$	1	0	${}^1P_1$	1^{+-}	0	$\propto \beta\sqrt{s}$

注: $\longrightarrow$ 表示动量方向, $\Rightarrow$ 表示自旋方向. 表中列出了入射的情况. 对于出射的情况, 结论是类似的, 只需将 $\longrightarrow\longleftarrow$ 换成 $\longleftarrow\longrightarrow$.

表 1 总结了每个旋量双线性型的各种螺旋态表达式. 对于一对正反费米子态, $P = (-)^{L+1}$, $C = (-)^{L+S}$, 角动量量子数 L 和 S 参考文献 [3, 4].

下面讨论释放出一个虚粒子对螺旋度的影响. 假设费米子 f 通过相互作用 $\bar{\psi}\Gamma_A\psi X^A$ 释放出一个虚粒子 X. 如 Fig. 2 所示, 对于无质量费米子, 初末态动量和螺旋态可以表示成

$$p_1 = E_1(1,0,0,1), \quad p_2 = E_2(1,s_\theta,0,c_\theta), \tag{154}$$

$$\xi_+(p_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_-(p_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_+(p_2) = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2} \end{pmatrix}, \quad \xi_-(p_2) = \begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2} \end{pmatrix}, \tag{155}$$

而 $\omega_+(p_1) = \sqrt{2E_1}$, $\omega_+(p_2) = \sqrt{2E_2}$, $\omega_-(p_1) = \omega_-(p_2) = 0$.

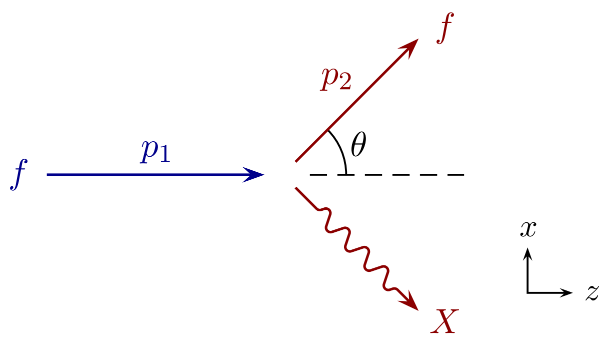

Figure 2: 费米子 f 释放一个虚粒子 X 的示意图.

利用

$$\xi_-^\dagger(p_2)\sigma^\mu\xi_-(p_1) = (c_{\theta/2}, -s_{\theta/2}, is_{\theta/2}, -c_{\theta/2}), \quad \xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = (c_{\theta/2}, s_{\theta/2}, is_{\theta/2}, c_{\theta/2}), \tag{156}$$

$$\xi_+^\dagger(p_2)\sigma^\mu\xi_-(p_1) = (s_{\theta/2}, c_{\theta/2}, -ic_{\theta/2}, -s_{\theta/2}), \quad \xi_-^\dagger(p_2)\sigma^\mu\xi_+(p_1) = (-s_{\theta/2}, c_{\theta/2}, ic_{\theta/2}, -s_{\theta/2}), \tag{157}$$

$$\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) = (c_{\theta/2}, s_{\theta/2}, -is_{\theta/2}, c_{\theta/2}), \quad \xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) = (c_{\theta/2}, -s_{\theta/2}, -is_{\theta/2}, -c_{\theta/2}), \tag{158}$$

$$\xi_+^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) = (s_{\theta/2}, -c_{\theta/2}, ic_{\theta/2}, s_{\theta/2}), \quad \xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_+(p_1) = (-s_{\theta/2}, -c_{\theta/2}, -ic_{\theta/2}, s_{\theta/2}), \tag{159}$$

对于矢量算符 $\bar{\psi}\gamma^\mu\psi$, 有

$$\begin{aligned}
\bar{u}_-(p_2)\gamma^\mu u_-(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= 2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + 0 = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, -is_{\theta/2}, c_{\theta/2}), \qquad (160)\\
\bar{u}_+(p_2)\gamma^\mu u_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, is_{\theta/2}, c_{\theta/2}), \qquad (161)\\
\bar{u}_+(p_2)\gamma^\mu u_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0, \qquad (162)\\
\bar{u}_-(p_2)\gamma^\mu u_+(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (163)\\
\bar{v}_-(p_2)\gamma^\mu v_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, is_{\theta/2}, c_{\theta/2}), \qquad (164)\\
\bar{v}_+(p_2)\gamma^\mu v_+(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= 2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + 0 = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, -is_{\theta/2}, c_{\theta/2}), \qquad (165)\\
\bar{v}_+(p_2)\gamma^\mu v_-(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (166)\\
\bar{v}_-(p_2)\gamma^\mu v_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0. \qquad (167)
\end{aligned}$$

对于轴矢量算符, 则有

$$\begin{aligned}
\bar{u}_-(p_2)\gamma^\mu\gamma_5 u_-(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= -2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + 0 = -2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, -is_{\theta/2}, c_{\theta/2}), \qquad (168)\\
\bar{u}_+(p_2)\gamma^\mu\gamma_5 u_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, is_{\theta/2}, c_{\theta/2}), \qquad (169)\\
\bar{u}_+(p_2)\gamma^\mu\gamma_5 u_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0, \qquad (170)\\
\bar{u}_-(p_2)\gamma^\mu\gamma_5 u_+(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (171)\\
\bar{v}_-(p_2)\gamma^\mu\gamma_5 v_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\sigma^\mu\xi_+(p_1) = 2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, is_{\theta/2}, c_{\theta/2}), \qquad (172)
\end{aligned}$$

$$\begin{aligned}
\bar{v}_+(p_2)\gamma^\mu\gamma_5 v_+(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= -2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\bar{\sigma}^\mu\xi_-(p_1) + 0 = -2\sqrt{E_1E_2}(c_{\theta/2}, s_{\theta/2}, -is_{\theta/2}, c_{\theta/2}), \qquad (173) \\
\bar{v}_+(p_2)\gamma^\mu\gamma_5 v_-(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (174) \\
\bar{v}_-(p_2)\gamma^\mu\gamma_5 v_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} -\bar{\sigma}^\mu & \\ & \sigma^\mu \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0. \qquad (175)
\end{aligned}$$

可见, 对于矢量流和轴矢量流相互作用, 释放出自旋为 1 的 X 粒子之后, 费米子的螺旋度不变.

对于标量算符 $\bar{\psi}\psi$, 有

$$\begin{aligned}
\bar{u}_-(p_2)u_-(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0, \qquad (176) \\
\bar{u}_+(p_2)u_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (177) \\
\bar{u}_+(p_2)u_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\xi_-(p_1) = 2\sqrt{E_1E_2}s_{\theta/2}, \qquad (178) \\
\bar{u}_-(p_2)u_+(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\xi_+(p_1) + 0 = -2\sqrt{E_1E_2}s_{\theta/2}, \qquad (179) \\
\bar{v}_-(p_2)v_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (180) \\
\bar{v}_+(p_2)v_+(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0, \qquad (181) \\
\bar{v}_+(p_2)v_-(p_1) &= \begin{pmatrix} -\omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ -\omega_+(p_1)\xi_+(p_1) \end{pmatrix} \\
&= 2\sqrt{E_1E_2}\xi_-^\dagger(p_2)\xi_+(p_1) + 0 = -2\sqrt{E_1E_2}s_{\theta/2}, \qquad (182) \\
\bar{v}_-(p_2)v_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & -\omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} -\omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} \\
&= 0 + 2\sqrt{E_1E_2}\xi_+^\dagger(p_2)\xi_-(p_1) = 2\sqrt{E_1E_2}s_{\theta/2}. \qquad (183)
\end{aligned}$$

对于赝标量算符 $\bar{\psi}i\gamma_5\psi$, 则有

$$\begin{aligned}
\bar{u}_-(p_2)i\gamma_5 u_-(p_1) &= \begin{pmatrix} \omega_+(p_2)\xi_-^\dagger(p_2), & \omega_-(p_2)\xi_-^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & i \\ -i & \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix} = 0, \qquad (184) \\
\bar{u}_+(p_2)i\gamma_5 u_+(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & i \\ -i & \end{pmatrix} \begin{pmatrix} \omega_-(p_1)\xi_+(p_1) \\ \omega_+(p_1)\xi_+(p_1) \end{pmatrix} = 0, \qquad (185) \\
\bar{u}_+(p_2)i\gamma_5 u_-(p_1) &= \begin{pmatrix} \omega_-(p_2)\xi_+^\dagger(p_2), & \omega_+(p_2)\xi_+^\dagger(p_2) \end{pmatrix} \begin{pmatrix} & i \\ -i & \end{pmatrix} \begin{pmatrix} \omega_+(p_1)\xi_-(p_1) \\ \omega_-(p_1)\xi_-(p_1) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= 0 - 2i\sqrt{E_1E_2}\xi_+^\dagger(p_2)\xi_-(p_1) = -2i\sqrt{E_1E_2}s_{\theta/2}, && (186)\\
\bar{u}_-(p_2)i\gamma_5 u_+(p_1) &= \Big(\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & i\\ -i & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ \omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= 2i\sqrt{E_1E_2}\xi_-^\dagger(p_2)\xi_+(p_1) + 0 = -2i\sqrt{E_1E_2}s_{\theta/2}, && (187)\\
\bar{v}_-(p_2)i\gamma_5 v_-(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & i\\ -i & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ -\omega_+(p_1)\xi_+(p_1)\end{pmatrix} = 0, && (188)\\
\bar{v}_+(p_2)i\gamma_5 v_+(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & i\\ -i & \end{pmatrix}\begin{pmatrix}-\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix} = 0, && (189)\\
\bar{v}_+(p_2)i\gamma_5 v_-(p_1) &= \Big(-\omega_+(p_2)\xi_-^\dagger(p_2),\ \omega_-(p_2)\xi_-^\dagger(p_2)\Big)\begin{pmatrix} & i\\ -i & \end{pmatrix}\begin{pmatrix}\omega_-(p_1)\xi_+(p_1)\\ -\omega_+(p_1)\xi_+(p_1)\end{pmatrix}\\
&= 2i\sqrt{E_1E_2}\xi_-^\dagger(p_2)\xi_+(p_1) + 0 = -2i\sqrt{E_1E_2}s_{\theta/2}, && (190)\\
\bar{v}_-(p_2)i\gamma_5 v_+(p_1) &= \Big(\omega_-(p_2)\xi_+^\dagger(p_2),\ -\omega_+(p_2)\xi_+^\dagger(p_2)\Big)\begin{pmatrix} & i\\ -i & \end{pmatrix}\begin{pmatrix}-\omega_+(p_1)\xi_-(p_1)\\ \omega_-(p_1)\xi_-(p_1)\end{pmatrix}\\
&= 0 - 2i\sqrt{E_1E_2}\xi_+^\dagger(p_2)\xi_-(p_1) = -2i\sqrt{E_1E_2}s_{\theta/2}. && (191)
\end{aligned}
$$

可见, 对于标量和赝标量相互作用, 释放出自旋为 0 的 X 粒子之后, 费米子的螺旋度与原来相反. 振幅 $\propto \sin(\theta/2)$, 出射费米子的运动方向倾向于与入射费米子相反.

A 惯例

度规张量

$$
g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 &&& \\ & -1 && \\ && -1 & \\ &&& -1 \end{pmatrix}. \tag{192}
$$

Pauli 矩阵

$$
\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{193}
$$

$$
\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \quad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}). \tag{194}
$$

手征表示下的 Dirac 矩阵

$$
\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \tag{195}
$$

$$
\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} = \begin{pmatrix} \Sigma^{\mu\nu} & \\ & \bar{\Sigma}^{\mu\nu} \end{pmatrix}, \tag{196}
$$

其中 $\Sigma^{\mu\nu} \equiv \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\bar{\Sigma}^{\mu\nu} \equiv \frac{i}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\mu\sigma^\nu)$. 左右手投影算符

$$
P_L \equiv \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad P_R \equiv \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}. \tag{197}
$$

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