Effective interactions between a CP-even scalar ϕ and SM gauge fields

$$\mathcal{L}_{\text{eff}} = \frac{1}{\Lambda}\phi(k_1 B_{\mu\nu}B^{\mu\nu} + k_2 W^a_{\mu\nu}W^{a\mu\nu} + k_3 G^a_{\mu\nu}G^{a\mu\nu})$$

$$B^{\mu\nu} \equiv \partial^\mu B^\nu - \partial^\nu B^\mu, \quad W^{a\mu\nu} \equiv \partial^\mu W^{a\nu} - \partial^\nu W^{a\mu} + g_2 \varepsilon^{abc} W^{b\mu} W^{c\nu}$$

$$B_\mu = c_W A_\mu - s_W Z_\mu, \quad W^3_\mu = s_W A_\mu + c_W Z_\mu, \quad W^\pm_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu \mp i W^2_\mu)$$

$$c_W \equiv \cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad s_W \equiv \sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = c_W A_{\mu\nu} - s_W Z_{\mu\nu}$$

$$W^3_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu - g_2 \varepsilon^{3bc} W^b_\mu W^c_\nu = s_W A_{\mu\nu} + c_W Z_{\mu\nu} - g_2 W^1_\mu W^2_\nu + g_2 W^2_\mu W^1_\nu$$

$$A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$B_{\mu\nu}B^{\mu\nu} = c_W^2 A_{\mu\nu}A^{\mu\nu} - 2 s_W c_W A_{\mu\nu} Z^{\mu\nu} + s_W^2 Z_{\mu\nu} Z^{\mu\nu}$$

$$W^3_{\mu\nu}W^{3\mu\nu} \supset s_W^2 A_{\mu\nu}A^{\mu\nu} + 2 s_W c_W A_{\mu\nu} Z^{\mu\nu} + c_W^2 Z_{\mu\nu} Z^{\mu\nu}$$

$$k_1 B_{\mu\nu}B^{\mu\nu} + k_2 W^3_{\mu\nu}W^{3\mu\nu} \supset k_{\text{AA}} A_{\mu\nu}A^{\mu\nu} + k_{\text{AZ}} A_{\mu\nu}Z^{\mu\nu} + k_{\text{ZZ}} Z_{\mu\nu}Z^{\mu\nu}$$

$$k_{\text{AA}} \equiv k_1 c_W^2 + k_2 s_W^2, \quad k_{\text{AZ}} \equiv 2 s_W c_W (k_2 - k_1), \quad k_{\text{ZZ}} \equiv k_1 s_W^2 + k_2 c_W^2$$

$$A_{\mu\nu}A^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

$$A_{\mu\nu}Z^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) = 2(\partial_\mu A_\nu \partial^\mu Z^\nu - \partial_\mu A_\nu \partial^\nu Z^\mu)$$

$$Z_{\mu\nu}Z^{\mu\nu} = (\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) = 2(\partial_\mu Z_\nu \partial^\mu Z^\nu - \partial_\mu Z_\nu \partial^\nu Z^\mu)$$

$$W^1_{\mu\nu}W^{1\mu\nu} + W^2_{\mu\nu}W^{2\mu\nu} = \frac{1}{2}(F^+_{\mu\nu} + F^-_{\mu\nu})(F^{+\mu\nu} + F^{-\mu\nu}) - \frac{1}{2}(F^+_{\mu\nu} - F^-_{\mu\nu})(F^{+\mu\nu} - F^{-\mu\nu}) = 2F^+_{\mu\nu}F^{-\mu\nu}$$

$$k_2(W^1_{\mu\nu}W^{1\mu\nu} + W^2_{\mu\nu}W^{2\mu\nu}) = 2k_2 F^+_{\mu\nu}F^{-\mu\nu} \supset 2k_2(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu})$$

$$= 4k_2(\partial_\mu W^+_\nu \partial^\mu W^{-\nu} - \partial_\mu W^+_\nu \partial^\nu W^{-\mu})$$

$$k_3 G^a_{\mu\nu}G^{a\mu\nu} \supset 2k_3(\partial_\mu G^a_\nu \partial^\mu G^{a\nu} - \partial_\mu G^a_\nu \partial^\nu G^{a\mu})$$

Feynman rules

$$\mathcal{L} \supset \frac{1}{\Lambda}\phi(k_1 B_{\mu\nu}B^{\mu\nu} + k_2 W^a_{\mu\nu}W^{a\mu\nu} + k_3 G^a_{\mu\nu}G^{a\mu\nu})$$

$$\supset \frac{\phi}{\Lambda}[2k_{\mathrm{AA}}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + 2k_{\mathrm{AZ}}(\partial_\mu A_\nu \partial^\mu Z^\nu - \partial_\mu A_\nu \partial^\nu Z^\mu)$$

$$+ 2k_{\mathrm{ZZ}}(\partial_\mu Z_\nu \partial^\mu Z^\nu - \partial_\mu Z_\nu \partial^\nu Z^\mu) + 4k_2(\partial_\mu W^+_\nu \partial^\mu W^{-\nu} - \partial_\mu W^+_\nu \partial^\nu W^{-\mu})$$

$$+ 2k_3(\partial_\mu G^a_\nu \partial^\mu G^{a\nu} - \partial_\mu G^a_\nu \partial^\nu G^{a\mu})]$$

For momenta pointing into the vertex: $\partial_\mu \to -ip_\mu$

$\phi(q) - X_{1\mu}(p_1) - X_{2\nu}(p_2)$ Feynman rules

$$\phi A_\mu(p_1)A_\nu(p_2) \to 2k_{\mathrm{AA}}\frac{\phi}{\Lambda}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

$$\to 2k_{\mathrm{AA}}\frac{\phi}{\Lambda}(g^{\rho\sigma}g^{\mu\nu}\partial_\rho A_\mu \partial_\sigma A_\nu + g^{\rho\sigma}g^{\nu\mu}\partial_\rho A_\nu \partial_\sigma A_\mu - g^{\rho\nu}g^{\mu\sigma}\partial_\rho A_\mu \partial_\sigma A_\nu - g^{\rho\mu}g^{\nu\sigma}\partial_\rho A_\nu \partial_\sigma A_\mu)$$

$$\to \frac{2ik_{\mathrm{AA}}}{\Lambda}[g^{\rho\sigma}g^{\mu\nu}(-ip_{1\rho})(-ip_{2\sigma}) + g^{\rho\sigma}g^{\nu\mu}(-ip_{2\rho})(-ip_{1\sigma}) - g^{\rho\nu}g^{\mu\sigma}(-ip_{1\rho})(-ip_{2\sigma}) - g^{\rho\mu}g^{\nu\sigma}(-ip_{2\rho})(-ip_{1\sigma})]$$

$$= -\frac{4ik_{\mathrm{AA}}}{\Lambda}(g^{\mu\nu}p_1 \cdot p_2 - p_2^\mu p_1^\nu)$$

$$\phi Z_\mu(p_1)Z_\nu(p_2) \to -\frac{4ik_{\mathrm{ZZ}}}{\Lambda}(g^{\mu\nu}p_1 \cdot p_2 - p_2^\mu p_1^\nu)$$

$$\phi A_\mu(p_1)Z_\nu(p_2) \to 2k_{\mathrm{AZ}}\frac{\phi}{\Lambda}(\partial_\mu A_\nu \partial^\mu Z^\nu - \partial_\mu A_\nu \partial^\nu Z^\mu) = 2k_{\mathrm{AZ}}\frac{\phi}{\Lambda}(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})\partial_\rho A_\mu \partial_\sigma Z_\nu$$

$$\to \frac{2ik_{\mathrm{AZ}}}{\Lambda}(g^{\rho\sigma}g^{\mu\nu} - g^{\rho\nu}g^{\mu\sigma})(-ip_{1\rho})(-ip_{2\sigma}) = -\frac{2ik_{\mathrm{AZ}}}{\Lambda}(g^{\mu\nu}p_1 \cdot p_2 - p_2^\mu p_1^\nu)$$

$$\phi W^+_\mu(p_1)W^-_\nu(p_2) \to -\frac{4ik_2}{\Lambda}(g^{\mu\nu}p_1 \cdot p_2 - p_2^\mu p_1^\nu)$$

$$\phi G^a_\mu(p_1)G^a_\nu(p_2) \to -\frac{4ik_3}{\Lambda}(g^{\mu\nu}p_1 \cdot p_2 - p_2^\mu p_1^\nu)$$

CP-even ϕ Decay widths

$\phi(q) \to X_1(p_1) + X_2(p_2)$ kinematics

$$m_\phi^2 = q^2 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2 p_1 \cdot p_2$$

$$p_1 \cdot p_2 = \frac{1}{2}(m_\phi^2 - m_1^2 - m_2^2)$$

$$|\mathbf{p}_1| = \frac{1}{2m_\phi}\sqrt{\left[m_\phi^2 - (m_1 + m_2)^2\right]\left[m_\phi^2 - (m_1 - m_2)^2\right]}$$

$$m_1 = m_2 = m_X \quad \to \quad p_1 \cdot p_2 = \frac{1}{2}(m_\phi^2 - 2m_X^2), \quad |\mathbf{p}_1| = |\mathbf{p}_2| = \frac{m_\phi}{2}\sqrt{1 - 4m_X^2 / m_\phi^2} = \frac{m_\phi}{2}\eta_X, \quad \eta_X \equiv \sqrt{1 - 4m_X^2 / m_\phi^2}$$

$$m_2 = 0 \quad \to \quad p_1 \cdot p_2 = \frac{1}{2}(m_\phi^2 - m_1^2) = \frac{m_\phi^2}{2}(1 - \xi_1^2), \quad |\mathbf{p}_1| = |\mathbf{p}_2| = \frac{1}{2m_\phi}(m_\phi^2 - m_1^2) = \frac{m_\phi}{2}(1 - \xi_1^2), \quad \xi_1 \equiv \frac{m_1}{m_\phi}$$

$$m_1 = m_2 = 0 \quad \to \quad p_1 \cdot p_2 = \frac{m_\phi^2}{2}, \quad |\mathbf{p}_1| = |\mathbf{p}_2| = \frac{m_\phi}{2}$$

$$\Gamma(\phi \to X_1 X_2) = n_{\text{id}} \frac{1}{8\pi} \frac{|\mathbf{p}_1|}{m_\phi^2} \sum_{\text{spins}} |\mathcal{M}|^2, \quad n_{\text{id}} = \begin{cases} 1, & X_1 \neq X_2 \\ \frac{1}{2}, & X_1 = X_2 \end{cases}$$

$\phi(q) \to \gamma(p_1) + \gamma(p_2)$

$$i\mathcal{M} = -\frac{4ik_{\text{AA}}}{\Lambda}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)\varepsilon_\mu^*(p_1)\varepsilon_\nu^*(p_2)$$

$$(i\mathcal{M})^* = \frac{4ik_{\text{AA}}}{\Lambda}(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)\varepsilon_\rho(p_1)\varepsilon_\sigma(p_2)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{16k_{\text{AA}}^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)\sum_{\text{spins}} \varepsilon_\mu^*(p_1)\varepsilon_\rho(p_1)\varepsilon_\nu^*(p_2)\varepsilon_\sigma(p_2)$$

$$= \frac{16k_{\text{AA}}^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)(-g_{\mu\rho})(-g_{\nu\sigma})$$

$$= \frac{16k_{\text{AA}}^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g_{\mu\nu} p_1 \cdot p_2 - p_{2\mu} p_{1\nu}) = \frac{16k_{\text{AA}}^2}{\Lambda^2}[2(p_1 \cdot p_2)^2 + p_1^2 p_2^2]$$

$$= \frac{32k_{\text{AA}}^2}{\Lambda^2}(p_1 \cdot p_2)^2 = \frac{8k_{\text{AA}}^2 m_\phi^4}{\Lambda^2}$$

$$\Gamma(\phi \to \gamma\gamma) = \frac{1}{2}\frac{1}{8\pi}\frac{|\mathbf{p}_1|}{m_\phi^2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{2}\frac{1}{8\pi}\frac{1}{m_\phi^2}\frac{m_\phi}{2}\frac{8k_{\text{AA}}^2 m_\phi^4}{\Lambda^2} = \frac{k_{\text{AA}}^2 m_\phi^3}{4\pi\Lambda^2}$$

$$\phi(q) \to Z(p_1) + Z(p_2)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{16k_{ZZ}^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)\left(-g_{\mu\rho} + \frac{p_{1\mu} p_{1\rho}}{m_Z^2}\right)\left(-g_{\nu\sigma} + \frac{p_{2\nu} p_{2\sigma}}{m_Z^2}\right)$$

$$= \frac{16k_{ZZ}^2}{\Lambda^2}[2(p_1 \cdot p_2)^2 + p_1^2 p_2^2] = \frac{16k_{ZZ}^2}{\Lambda^2}\left[2\frac{1}{4}(m_\phi^2 - 2m_Z^2)^2 + m_Z^4\right] = \frac{8k_{ZZ}^2}{\Lambda^2}(m_\phi^4 - 4m_\phi^2 m_Z^2 + 6m_Z^4) = \frac{8k_{ZZ}^2 m_\phi^4}{\Lambda^2}(1 - 4\xi_Z^2 + 6\xi_Z^4)$$

$$\Gamma(\phi \to ZZ) = \frac{1}{2}\frac{1}{8\pi}\frac{|\mathbf{p}_1|}{m_\phi^2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{2}\frac{1}{8\pi}\frac{1}{m_\phi^2}\frac{m_\phi}{2}\eta_Z \frac{8k_{ZZ}^2 m_\phi^4}{\Lambda^2}(1 - 4\xi_Z^2 + 6\xi_Z^4) = \frac{k_{ZZ}^2 m_\phi^3}{4\pi\Lambda^2}\eta_Z(1 - 4\xi_Z^2 + 6\xi_Z^4)$$

$$\eta_X \equiv \sqrt{1 - 4m_X^2 / m_\phi^2}, \quad \xi_X \equiv m_X / m_\phi$$

$$\phi(q) \to \gamma(p_1) + Z(p_2)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4k_{AZ}^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)(-g_{\mu\rho})\left(-g_{\nu\sigma} + \frac{p_{2\nu} p_{2\sigma}}{m_Z^2}\right)$$

$$= \frac{4k_{AZ}^2}{\Lambda^2}[2(p_1 \cdot p_2)^2 + p_1^2 p_2^2] = \frac{4k_{AZ}^2}{\Lambda^2}2\frac{1}{4}(m_\phi^2 - m_Z^2)^2 = \frac{2k_{AZ}^2}{\Lambda^2}(m_\phi^2 - m_Z^2)^2 = \frac{2k_{AZ}^2 m_\phi^4}{\Lambda^2}(1 - \xi_Z^2)^2$$

$$\Gamma(\phi \to \gamma Z) = \frac{1}{8\pi}\frac{|\mathbf{p}_1|}{m_\phi^2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{8\pi}\frac{1}{m_\phi^2}\frac{m_\phi}{2}(1 - \xi_Z^2)\frac{2k_{AZ}^2 m_\phi^4}{\Lambda^2}(1 - \xi_Z^2)^2 = \frac{k_{AZ}^2 m_\phi^3}{8\pi\Lambda^2}(1 - \xi_Z^2)^3$$

$$\phi(q) \to W^+(p_1) + W^-(p_2)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{16k_2^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)\left(-g_{\mu\rho} + \frac{p_{1\mu} p_{1\rho}}{m_W^2}\right)\left(-g_{\nu\sigma} + \frac{p_{2\nu} p_{2\sigma}}{m_W^2}\right) = \frac{8k_2^2 m_\phi^4}{\Lambda^2}(1 - 4\xi_W^2 + 6\xi_W^4)$$

$$\Gamma(\phi \to W^+ W^-) = \frac{1}{8\pi}\frac{|\mathbf{p}_1|}{m_\phi^2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{k_2^2 m_\phi^3}{2\pi\Lambda^2}\eta_W(1 - 4\xi_W^2 + 6\xi_W^4)$$

$$\phi(q) \to g(p_1) + g(p_2)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{16k_3^2}{\Lambda^2}(g^{\mu\nu} p_1 \cdot p_2 - p_2^\mu p_1^\nu)(g^{\rho\sigma} p_1 \cdot p_2 - p_2^\rho p_1^\sigma)(-g_{\mu\rho})(-g_{\nu\sigma}) = \frac{8k_3^2 m_\phi^4}{\Lambda^2}$$

$$\Gamma(\phi \to gg) = 8\frac{1}{2}\frac{1}{8\pi}\frac{|\mathbf{p}_1|}{m_\phi^2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2k_3^2 m_\phi^3}{\pi\Lambda^2}$$

CP-even scalar ϕ interactions with SM quarks and gluons

$$\mathcal{L} \supset \frac{k_3}{\Lambda} \phi G^a_{\mu\nu} G^{a\mu\nu} + \sum_q y_{\phi qq} \phi \bar{q} q$$

$$\sigma(pp \to \phi) = \sigma(gg \to \phi) + \sum_{q=u,d,s,c,b} \sigma(\bar{q}q \to \phi)$$

$$\sigma(gg \to \phi) \propto \frac{k_3^2}{\Lambda^2}, \quad \sigma(\bar{q}q \to \phi) \propto y_{\phi qq}^2$$

Subprocesses of 1-body production $pp \to \phi$

$\sqrt{s} = 13$ TeV

For $k_3 = 0.1$ and $\Lambda = 1$ TeV, $\quad \sigma(gg \to \phi) = 16.725$ pb

For $y_{\phi qq} = 0.1$,

$\sigma(d\bar{d} \to \phi) = 1.5482$ pb

$\sigma(u\bar{u} \to \phi) = 2.5063$ pb

$\sigma(s\bar{s} \to \phi) = 0.14375$ pb

$\sigma(c\bar{c} \to \phi) = 0.098142$ pb

$\sigma(b\bar{b} \to \phi) = 0.044350$ pb

$\sqrt{s} = 8$ TeV

For $k_3 = 0.1$ and $\Lambda = 1$ TeV, $\quad \sigma(gg \to \phi) = 3.7830$ pb

For $y_{\phi qq} = 0.1$,

$\sigma(d\bar{d} \to \phi) = 0.57390$ pb

$\sigma(u\bar{u} \to \phi) = 0.95271$ pb

$\sigma(s\bar{s} \to \phi) = 0.036763$ pb

$\sigma(c\bar{c} \to \phi) = 0.023507$ pb

$\sigma(b\bar{b} \to \phi) = 0.0095815$ pb

$$S=\int d^5x\bar{\Psi}(i\Gamma^M D_M-\tilde{y}_f\Phi)\Psi=\int d^5x\bar{\Psi}(i\Gamma^M D_M-\tilde{y}_f\langle\Phi\rangle-\tilde{y}_f\tilde{\Phi})\Psi$$

$$D_M=\partial_M-i\tilde{g}A_M,\quad M_f=\tilde{y}_f\langle\Phi\rangle=y_f v_\phi,\quad y_f=\frac{\tilde{y}_f}{\sqrt{\pi R}}$$

$$\Gamma^\mu=\gamma^\mu\ (\mu=0,1,2,3),\quad \Gamma^5=i\gamma^5,\quad i\Gamma^M D_M=\begin{pmatrix}\partial_y & i\sigma^\mu\partial_\mu\\ i\bar{\sigma}^\mu\partial_\mu & -\partial_y\end{pmatrix}$$

$$\text{Fermion EoM:}\quad \begin{pmatrix}\partial_y-M_f & i\sigma^\mu\partial_\mu\\ i\bar{\sigma}^\mu\partial_\mu & -\partial_y-M_f\end{pmatrix}\begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}=0$$

$$\Psi_L=\sum_n\chi_a^{(n)}(x)f^{(n)}(y),\quad \Psi_R=\sum_n\xi^{(n)\dagger\dot{a}}(x)g^{(n)}(y)$$

$$\text{Normalization}:\quad \int_0^{\pi R}dy[f^{(n)}(y)]^2=\int_0^{\pi R}dy[g^{(n)}(y)]^2=1$$

$$\text{Solution:}\quad \begin{pmatrix}-m_n & i\sigma^\mu\partial_\mu\\ i\bar{\sigma}^\mu\partial_\mu & -m_n\end{pmatrix}\begin{pmatrix}\chi_a^{(n)}(x)\\ \xi^{(n)\dagger\dot{a}}(x)\end{pmatrix}=0,\quad (-\partial_y+M_f)f^{(n)}(y)=m_n g^{(n)}(y),\quad (\partial_y+M_f)g^{(n)}(y)=m_n f^{(n)}(y)$$

$$\left[\begin{array}{l}(\partial_y-M_f)\Psi_L+i\sigma^\mu\partial_\mu\Psi_R\\ \quad\to\quad \chi_a^{(n)}(x)(\partial_y-M_f)f^{(n)}(y)+i\sigma^\mu\partial_\mu\xi^{(n)\dagger\dot{a}}(x)g^{(n)}(y)=-m_n\chi_a^{(n)}(x)g^{(n)}(y)+i\sigma^\mu\partial_\mu\xi^{(n)\dagger\dot{a}}(x)g^{(n)}(y)=0\\ i\bar{\sigma}^\mu\partial_\mu\Psi_L-(\partial_y+M_f)\Psi_R\\ \quad\to\quad i\bar{\sigma}^\mu\partial_\mu\chi_a^{(n)}(x)f^{(n)}(y)-\xi^{(n)\dagger\dot{a}}(x)(\partial_y+M_f)g^{(n)}(y)=i\bar{\sigma}^\mu\partial_\mu\chi_a^{(n)}(x)f^{(n)}(y)-m_n\xi^{(n)\dagger\dot{a}}(x)f^{(n)}(y)=0\end{array}\right]$$

$$m_0=0\quad\to\quad(-\partial_y+M_f)f^{(0)}(y)=0,\quad(\partial_y+M_f)g^{(0)}(y)=0$$

$$\to\quad f^{(0)}(y)=\sqrt{\frac{2M_f}{e^{2\pi RM_f}-1}}e^{M_f y}\ \text{or }0,\quad g^{(0)}(y)=\sqrt{\frac{2M_f}{1-e^{-2\pi RM_f}}}e^{-M_f y}\ \text{or }0$$

Boundary condition $\Psi_R(y=0)=\Psi_R(y=\pi R)=0$:

$$f_L^{(0)}(y)=\sqrt{\frac{2M_f}{e^{2\pi RM_f}-1}}e^{M_f y},\quad g_L^{(0)}(y)=0\quad\to\quad\text{Left-handed 0-mode}$$

$$n\ge1\quad\to\quad f_L^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n}{R}\cos\frac{ny}{R}+M_f\sin\frac{ny}{R}\right),\quad g_L^{(n)}(y)=\sqrt{\frac{2}{\pi R}}\sin\frac{ny}{R},\quad m_n^2=M_f^2+n^2M_{\mathrm{KK}}^2,\quad M_{\mathrm{KK}}\equiv\frac{1}{R}$$

$$\left[\begin{array}{l}(-\partial_y+M_f)f_L^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left[-\left(-\frac{n^2}{R^2}\sin\frac{ny}{R}+M_f\frac{n}{R}\cos\frac{ny}{R}\right)+M_f\left(\frac{n}{R}\cos\frac{ny}{R}+M_f\sin\frac{ny}{R}\right)\right]\\ =\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n^2}{R^2}+M_f^2\right)\sin\frac{ny}{R}=m_n g_L^{(n)}(y)\\ (\partial_y+M_f)g_L^{(n)}(y)=\sqrt{\frac{2}{\pi R}}(\partial_y+M_f)\sin\frac{ny}{R}=\sqrt{\frac{2}{\pi R}}\left(\frac{n}{R}\cos\frac{ny}{R}+M_f\sin\frac{ny}{R}\right)=m_n f_L^{(n)}(y)\end{array}\right]$$

$$\text{Orthogonality:}\quad \int_0^{\pi R}dyf_L^{(n)}(y)f_L^{(m)}(y)=\delta^{nm},\quad \int_0^{\pi R}dyg_L^{(n)}(y)g_L^{(m)}(y)=\delta^{nm}$$

Boundary condition $\Psi_L(y=0)=\Psi_L(y=\pi R)=0$:

$$f_R^{(0)}(y)=0,\quad g_R^{(0)}(y)=\sqrt{\frac{2M_f}{1-e^{-2\pi RM_f}}}e^{-M_f y}\quad\to\quad\text{Right-handed 0-mode}$$

$$n\ge1\quad\to\quad f_R^{(n)}(y)=-\sqrt{\frac{2}{\pi R}}\sin\frac{ny}{R},\quad g_R^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n}{R}\cos\frac{ny}{R}-M_f\sin\frac{ny}{R}\right),\quad m_n^2=M_f^2+n^2M_{\mathrm{KK}}^2$$

$$\left[\begin{array}{l}(-\partial_y+M_f)f_R^{(n)}(y)=-\sqrt{\frac{2}{\pi R}}(-\partial_y+M_f)\sin\frac{ny}{R}=-\sqrt{\frac{2}{\pi R}}\left(-\frac{n}{R}\cos\frac{ny}{R}+M_f\sin\frac{ny}{R}\right)=m_n g_R^{(n)}(y)\\ (\partial_y+M_f)g_R^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(-\frac{n^2}{R^2}\sin\frac{ny}{R}-M_f\frac{n}{R}\cos\frac{ny}{R}+M_f\frac{n}{R}\cos\frac{ny}{R}-M_f^2\sin\frac{ny}{R}\right)\\ =-\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n^2}{R^2}+M_f^2\right)\sin\frac{ny}{R}=m_n f_R^{(n)}(y)\end{array}\right]$$

$$\text{Orthogonality:}\quad \int_0^{\pi R}dyf_R^{(n)}(y)f_R^{(m)}(y)=\delta^{nm},\quad \int_0^{\pi R}dyg_R^{(n)}(y)g_R^{(m)}(y)=\delta^{nm}$$

Neumann boundary condition $\partial_y \Phi\Big|_{y=0,\pi R} = 0$:

$$\Phi(x^\mu, y) = \sum_n \phi^{(n)}(x^\mu) f_\phi^{(n)}(y) = \frac{1}{\sqrt{\pi R}}\phi^{(0)}(x^\mu) + \sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}\phi^{(n)}(x^\mu)\cos\left(\frac{ny}{R}\right)$$

$$f_\phi^{(0)}(y) = \frac{1}{\sqrt{\pi R}}; \quad n \ge 1, \quad f_\phi^{(n)}(y) = \sqrt{\frac{2}{\pi R}}\cos\left(\frac{ny}{R}\right)$$

$$\int_0^{\pi R} dy f_\phi^{(n)}(y) f_\phi^{(m)}(y) = \delta^{nm}$$

$$\phi(x^\mu) \equiv \phi^{(0)}(x^\mu), \quad F^{(n)}(x) \equiv \begin{pmatrix} \chi_a^{(n)}(x) \\ \xi^{(n)\dagger\dot{a}}(x) \end{pmatrix}$$

$$\bar{\Psi}\Psi = \bar{\Psi}P_L\Psi + \bar{\Psi}P_R\Psi = \sum_{nm}[\xi^{(n)a}(x)\chi_a^{(m)}(x)g^{(n)}(y)f^{(m)}(y) + \chi_{\dot{a}}^{(m)\dagger}(x)\xi^{(n)\dagger\dot{a}}(x)f^{(m)}(y)g^{(n)}(y)]$$

$$= \sum_{nm}[\bar{F}^{(n)}(x)P_L F^{(m)}(x)g^{(n)}(y)f^{(m)}(y) + \bar{F}^{(m)}(x)P_R F^{(n)}(x)f^{(m)}(y)g^{(n)}(y)]$$

$$\frac{1}{2}\left(\coth\frac{\pi M_f}{M_{\text{KK}}} - 1\right) = \frac{1}{e^{2\pi M_f/M_{\text{KK}}} - 1}, \quad (-1)^n = \cos(n\pi)$$

For $\Psi_R(y=0) = \Psi_R(y=\pi R) = 0, \quad \int dy g_L^{(0)}(y) f_L^{(0)}(y) = 0$

$$n \ne 0, \quad \int dy g_L^{(0)}(y) f_L^{(n)}(y) = 0, \quad \int dy g_L^{(n)}(y) f_L^{(n)}(y) = \frac{M_f}{\sqrt{M_f^2 + n^2 M_{\text{KK}}^2}}$$

$$\int dy g_L^{(n)}(y) f_L^{(0)}(y) = \sqrt{\frac{2}{\pi}}\sqrt{M_f\left(\coth\frac{\pi M_f}{M_{\text{KK}}} - 1\right)}\frac{nM_{\text{KK}}^{3/2}[1-(-1)^n e^{\pi M_f/M_{\text{KK}}}]}{M_f^2 + n^2 M_{\text{KK}}^2} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\text{KK}}} - 1}}\frac{nM_{\text{KK}}^{3/2}[1-\cos(n\pi)e^{\pi M_f/M_{\text{KK}}}]}{M_f^2 + n^2 M_{\text{KK}}^2}$$

$$n \ne 0, \quad m \ne 0, \quad n \ne m, \quad \int dy g_L^{(n)}(y) f_L^{(m)}(y) = \frac{2mn[1-(-1)^{m+n}]}{\pi(n^2 - m^2)}\frac{M_{\text{KK}}}{\sqrt{M_f^2 + m^2 M_{\text{KK}}^2}}$$

$$S \supset -\int d^4x dy \tilde{y}_f \tilde{\Phi}\bar{\Psi}\Psi \supset -\frac{\tilde{y}_f}{\sqrt{\pi R}}\int d^4x dy \phi(x)\bar{\Psi}\Psi = -y_f\int d^4x\phi(x)\int dy\bar{\Psi}\Psi$$

$$= \int d^4x\Big\{-y_f\sum_{n=1}^{\infty}\frac{M_f}{\sqrt{M_f^2 + n^2 M_{\text{KK}}^2}}\phi(x)\bar{F}^{(n)}(x)F^{(n)}(x)$$

$$-\frac{2y_f}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\text{KK}}} - 1}}\sum_{n=1}^{\infty}\frac{nM_{\text{KK}}^{3/2}[1-\cos(n\pi)e^{\pi M_f/M_{\text{KK}}}]}{M_f^2 + n^2 M_{\text{KK}}^2}\phi(x)[\bar{F}^{(n)}(x)P_L F^{(0)}(x) + h.c.]$$

$$-\frac{y_f}{\pi}\sum_{\substack{n=1,m=1 \\ m+n=\text{odd}}}^{\infty}\frac{4mn}{n^2 - m^2}\frac{M_{\text{KK}}}{\sqrt{M_f^2 + m^2 M_{\text{KK}}^2}}\phi(x)[\bar{F}^{(n)}(x)P_L F^{(m)}(x) + h.c.]\Big\}$$

$$\frac{1}{2}\left(\coth\frac{\pi M_f}{M_{\rm KK}}+1\right)=\frac{e^{2\pi M_f/M_{\rm KK}}}{e^{2\pi M_f/M_{\rm KK}}-1}$$

For $\Psi_L(y=0)=\Psi_L(y=\pi R)=0, \quad \int dy f_R^{(0)}(y)g_R^{(0)}(y)=0$

$$n\neq 0, \quad \int dy f_R^{(0)}(y)g_R^{(n)}(y)=0, \quad \int dy f_R^{(n)}(y)g_R^{(n)}(y)=\frac{M_f}{\sqrt{M_f^2+n^2M_{\rm KK}^2}}$$

$$\int dy f_R^{(n)}(y)g_R^{(0)}(y)=\sqrt{\frac{2}{\pi}}\sqrt{M_f\left(\coth\frac{\pi M_f}{M_{\rm KK}}+1\right)}\frac{nM_{\rm KK}^{3/2}e^{-\pi M_f/M_{\rm KK}}[(-1)^n-e^{\pi M_f/M_{\rm KK}}]}{M_f^2+n^2M_{\rm KK}^2}$$

$$=\frac{2}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\rm KK}}-1}}\frac{nM_{\rm KK}^{3/2}[\cos(n\pi)-e^{\pi M_f/M_{\rm KK}}]}{M_f^2+n^2M_{\rm KK}^2}$$

$$n\neq 0, \quad m\neq 0, \quad n\neq m, \quad \int dy f_R^{(n)}(y)g_R^{(m)}(y)=-\frac{2mn[1-(-1)^{m+n}]}{\pi(n^2-m^2)}\frac{M_{\rm KK}}{\sqrt{M_f^2+m^2M_{\rm KK}^2}}$$

$$S\supset -y_f\int d^4x\phi(x)\int dy\bar{\Psi}\Psi$$

$$=\int d^4x\Big\{-y_f\sum_{n=1}^{\infty}\frac{M_f}{\sqrt{M_f^2+n^2M_{\rm KK}^2}}\phi(x)\bar{F}^{(n)}(x)F^{(n)}(x)$$

$$-\frac{2y_f}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\rm KK}}-1}}\sum_{n=1}^{\infty}\frac{nM_{\rm KK}^{3/2}[\cos(n\pi)-e^{\pi M_f/M_{\rm KK}}]}{M_f^2+n^2M_{\rm KK}^2}\phi(x)[\bar{F}^{(n)}(x)P_RF^{(0)}(x)+h.c.]$$

$$+\frac{y_f}{\pi}\sum_{\substack{n=1,m=1\\ m+n=\text{odd}}}^{\infty}\frac{4mn}{n^2-m^2}\frac{M_{\rm KK}}{\sqrt{M_f^2+m^2M_{\rm KK}^2}}\phi(x)[\bar{F}^{(n)}(x)P_RF^{(m)}(x)+h.c.]\Big\}$$

Change the sign of Yukawa coupling:

$$S=\int d^5x\bar{\Psi}(i\Gamma^M D_M+\tilde{y}_f\Phi)\Psi=\int d^5x\bar{\Psi}(i\Gamma^M D_M+M_f+\tilde{y}_f\tilde{\Phi})\Psi$$

$$M_f=\tilde{y}_f\langle\Phi\rangle=y_f v_\phi,\quad \begin{pmatrix}\partial_y+M_f & i\sigma^\mu\partial_\mu\\ i\bar{\sigma}^\mu\partial_\mu & -\partial_y+M_f\end{pmatrix}\begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}=0$$

$$(-\partial_y-M_f)f^{(n)}(y)=m_n g^{(n)}(y),\quad (\partial_y-M_f)g^{(n)}(y)=m_n f^{(n)}(y)$$

$$m_0=0\quad\to\quad(-\partial_y-M_f)f^{(0)}(y)=0,\quad(\partial_y-M_f)g^{(0)}(y)=0$$

$$\to\quad f^{(0)}(y)=\sqrt{\frac{2M_f}{1-e^{-2\pi RM_f}}}e^{-M_f y}\text{ or }0,\quad g^{(0)}(y)=\sqrt{\frac{2M_f}{e^{2\pi RM_f}-1}}e^{M_f y}\text{ or }0$$

Boundary condition $\Psi_L(y=0)=\Psi_L(y=\pi R)=0$:

$$\hat{f}_R^{(0)}(y)=0,\quad \hat{g}_R^{(0)}(y)=\sqrt{\frac{2M_f}{e^{2\pi RM_f}-1}}e^{M_f y}\quad\to\quad\text{Right-handed 0-mode}$$

$$n\geq 1\quad\to\quad \hat{f}_R^{(n)}(y)=-\sqrt{\frac{2}{\pi R}}\sin\frac{ny}{R},\quad \hat{g}_R^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n}{R}\cos\frac{ny}{R}+M_f\sin\frac{ny}{R}\right),\quad m_n^2=M_f^2+n^2M_{\mathrm{KK}}^2$$

$$\left[\begin{aligned}&(-\partial_y-M_f)\hat{f}_R^{(n)}(y)=-\sqrt{\frac{2}{\pi R}}(-\partial_y-M_f)\sin\frac{ny}{R}=-\sqrt{\frac{2}{\pi R}}\left(-\frac{n}{R}\cos\frac{ny}{R}-M_f\sin\frac{ny}{R}\right)=m_n\hat{g}_R^{(n)}(y)\\ &(\partial_y-M_f)\hat{g}_R^{(n)}(y)=\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(-\frac{n^2}{R^2}\sin\frac{ny}{R}+M_f\frac{n}{R}\cos\frac{ny}{R}-M_f\frac{n}{R}\cos\frac{ny}{R}-M_f^2\sin\frac{ny}{R}\right)\\ &=-\sqrt{\frac{2/\pi R}{M_f^2+n^2/R^2}}\left(\frac{n^2}{R^2}+M_f^2\right)\sin\frac{ny}{R}=m_n\hat{f}_R^{(n)}(y)\end{aligned}\right]$$

$$\Rightarrow\quad \hat{g}_R^{(n)}(y)=f_L^{(0)}(y)\quad\Rightarrow\quad\int dy\hat{g}_R^{(n)}(y)f_L^{(m)}(y)=\delta^{nm}$$

Orthogonality: $\int_0^{\pi R}dy\hat{f}_R^{(n)}(y)\hat{f}_R^{(m)}(y)=\delta^{nm},\quad \int_0^{\pi R}dy\hat{g}_R^{(n)}(y)\hat{g}_R^{(m)}(y)=\delta^{nm}$

$$\int dy\hat{f}_R^{(0)}(y)\hat{g}_R^{(0)}(y)=0$$

$$n\neq 0,\quad\int dy\hat{f}_R^{(0)}(y)\hat{g}_R^{(n)}(y)=0,\quad\int dy\hat{f}_R^{(n)}(y)\hat{g}_R^{(n)}(y)=-\frac{M_f}{\sqrt{M_f^2+n^2M_{\mathrm{KK}}^2}}$$

$$\int dy\hat{f}_R^{(n)}(y)\hat{g}_R^{(0)}(y)=\sqrt{\frac{2}{\pi}}\sqrt{M_f\left(\coth\frac{\pi M_f}{M_{\mathrm{KK}}}-1\right)}\frac{nM_{\mathrm{KK}}^{3/2}[(-1)^n e^{\pi M_f/M_{\mathrm{KK}}}-1]}{M_f^2+n^2M_{\mathrm{KK}}^2}=-\frac{2}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\mathrm{KK}}}-1}}\frac{nM_{\mathrm{KK}}^{3/2}[1-\cos(n\pi)e^{\pi M_f/M_{\mathrm{KK}}}]}{M_f^2+n^2M_{\mathrm{KK}}^2}$$

$$n\neq 0,\quad m\neq 0,\quad n\neq m,\quad\int dy\hat{f}_R^{(n)}(y)\hat{g}_R^{(m)}(y)=-\frac{2mn[1-(-1)^{m+n}]}{\pi(n^2-m^2)}\frac{M_{\mathrm{KK}}}{\sqrt{M_f^2+m^2M_{\mathrm{KK}}^2}}$$

$$S\supset +y_f\int d^4x\phi(x)\int dy\bar{\Psi}\Psi$$

$$\begin{aligned}&=\int d^4x\Bigg\{-y_f\sum_{n=1}^{\infty}\frac{M_f}{\sqrt{M_f^2+n^2M_{\mathrm{KK}}^2}}\phi(x)\bar{F}^{(n)}(x)F^{(n)}(x)\\ &\quad-\frac{2y_f}{\sqrt{\pi}}\sqrt{\frac{M_f}{e^{2\pi M_f/M_{\mathrm{KK}}}-1}}\sum_{n=1}^{\infty}\frac{nM_{\mathrm{KK}}^{3/2}[1-\cos(n\pi)e^{\pi M_f/M_{\mathrm{KK}}}]}{M_f^2+n^2M_{\mathrm{KK}}^2}\phi(x)[\bar{F}^{(n)}(x)P_R F^{(0)}(x)+h.c.]\\ &\quad-\frac{y_f}{\pi}\sum_{\substack{n=1,m=1\\ m+n=\mathrm{odd}}}^{\infty}\frac{4mn}{n^2-m^2}\frac{M_{\mathrm{KK}}}{\sqrt{M_f^2+m^2M_{\mathrm{KK}}^2}}\phi(x)[\bar{F}^{(n)}(x)P_R F^{(m)}(x)+h.c.]\Bigg\}\end{aligned}$$

Gauge couplings

Boundary condition $A_y(y=0,\pi R)=0, \quad \partial_y A_\mu\big|_{y=0,\pi R}=0$:

$$\sum_M A_M(x^\mu,y)=\sum_{n,M} A_M^{(n)}(x^\mu)f_{A,M}^{(n)}(y)=\sum_\mu\left[\frac{1}{\sqrt{\pi R}}A_\mu^{(0)}(x^\mu)+\sqrt{\frac{2}{\pi R}}\sum_{n=1}^{\infty}A_\mu^{(n)}(x^\mu)\cos\left(\frac{ny}{R}\right)\right]$$

$$f_{A,\mu}^{(0)}(y)=\frac{1}{\sqrt{\pi R}}, \quad f_{A,5}^{(0)}(y)=0; \quad n\geq 1, \quad f_{A,\mu}^{(n)}(y)=\sqrt{\frac{2}{\pi R}}\cos\left(\frac{ny}{R}\right), \quad f_{A,5}^{(n)}(y)=0$$

$$A_\mu(x)\equiv A_\mu^{(0)}(x), \quad g\equiv\frac{\tilde{g}}{\sqrt{\pi R}}$$

$$\bar{\Psi}\gamma^\mu\Psi=\sum_{nm}\begin{pmatrix}\chi_{\dot{a}}^{(n)\dagger}(x)f^{(n)}(y) & \xi^{(n)a}(x)g^{(n)}(y)\end{pmatrix}\begin{pmatrix}\bar{\sigma}^\mu & \\ & \sigma^\mu\end{pmatrix}\begin{pmatrix}\chi_a^{(m)}(x)f^{(m)}(y)\\ \xi^{(m)\dagger\dot{a}}(x)g^{(m)}(y)\end{pmatrix}$$

$$=\sum_{nm}[\chi_{\dot{a}}^{(n)\dagger}(x)\bar{\sigma}^\mu\chi_a^{(m)}(x)f^{(n)}(y)f^{(m)}(y)+\xi^{(n)a}(x)\sigma^\mu\xi^{(m)\dagger\dot{a}}(x)g^{(n)}(y)g^{(m)}(y)]$$

$$=\sum_{nm}[\bar{F}^{(n)}(x)\gamma^\mu P_L F^{(m)}(x)f^{(n)}(y)f^{(m)}(y)+\bar{F}^{(n)}(x)\gamma^\mu P_R F^{(m)}(x)g^{(n)}(y)g^{(m)}(y)]$$

$$\int_0^{\pi R}dy f^{(n)}(y)f^{(m)}(y)=\delta^{nm}, \quad \int_0^{\pi R}dy g^{(n)}(y)g^{(m)}(y)=\delta^{nm}$$

$$S\supset\int d^5x\tilde{g}\bar{\Psi}\Gamma^M A_M\Psi\supset g\int d^4x A_\mu^a(x)\int dy\bar{\Psi}\gamma^\mu t^a\Psi=g\int d^4x A_\mu^a(x)\sum_{n=0}^{\infty}\bar{F}^{(n)}(x)\gamma^\mu t^a F^{(n)}(x)$$

Abelian: $\quad F_{MN}=\partial_M A_N-\partial_N A_M$

$$-\frac{1}{4}F_{MN}F^{MN}=-\frac{1}{2}(\partial_M A_N\partial^M A^N-\partial_M A_N\partial^N A^M)$$

$$\partial_y f_{A,\mu}^{(n)}(y)=\sqrt{\frac{2}{\pi R}}\partial_y\cos\left(\frac{ny}{R}\right)=-\sqrt{\frac{2}{\pi R}}\frac{n}{R}\sin\left(\frac{ny}{R}\right)$$

$$\int dy\partial_y f_{A,\mu}^{(n)}(y)\partial^y f_{A,\mu}^{(m)}(y)=-\frac{2}{\pi R^3}\int dy nm\sin\left(\frac{ny}{R}\right)\sin\left(\frac{my}{R}\right)=-\frac{n^2}{R^2}\delta^{nm}$$

$$S\supset-\frac{1}{4}\int d^5x F_{MN}F^{MN}=-\frac{1}{2}\int d^5x(\partial_M A_N\partial^M A^N-\partial_M A_N\partial^N A^M)$$

$$\supset-\frac{1}{2}\int d^4x dy\sum_{n,m=1}^{\infty}A_\mu^{(n)}(x)A^{(m)\mu}(x)\partial_y f_{A,\mu}^{(n)}(y)\partial^y f_{A,\mu}^{(m)}(y)=\frac{1}{2}\int d^4x\sum_{n=1}^{\infty}\frac{n^2}{R^2}A_\mu^{(n)}(x)A^{(n)\mu}(x)$$

$$m_{A^{(n)}}=\frac{n}{R}=nM_{KK}$$

Scalar sector

$$S \supset \int d^5x \left[(D^M H)^\dagger D_M H + \frac{1}{2}\partial^M \Phi \partial_M \Phi + \mu^2 |H|^2 + \frac{1}{2}M^2\Phi^2 - \tilde{\lambda}|H|^4 - \frac{1}{4!}\tilde{\lambda}_\phi \Phi^4 - \frac{1}{2}\tilde{\lambda}_{\phi h}\Phi^2 |H|^2 \right]$$

Energy density: $E = E_{\text{der}} + V(\Phi, H)$

$$E_{\text{der}} = \int dy \left[-(\partial^y H)^\dagger \partial_y H - \frac{1}{2}\partial^y \Phi \partial_y \Phi \right]$$

$$V(\Phi, H) = \int dy \left[-\mu^2 |H|^2 - \frac{1}{2}M^2\Phi^2 + \tilde{\lambda}|H|^4 + \frac{\tilde{\lambda}_\phi}{4!}\Phi^4 + \frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2 |H|^2 \right]$$

$$H(x^\mu, y) = \frac{1}{\sqrt{\pi R}} H^{(0)}(x^\mu) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} H^{(n)}(x^\mu) \cos\left(\frac{ny}{R}\right)$$

$$\Phi(x^\mu, y) = \frac{1}{\sqrt{\pi R}} \phi^{(0)}(x^\mu) + \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \phi^{(n)}(x^\mu) \cos\left(\frac{ny}{R}\right)$$

$$\int dy |H|^2 = \sum_{n=0}^{\infty} |H^{(n)}(x)|^2, \quad \int dy \Phi^2 = \sum_{n=0}^{\infty} [\phi^{(n)}(x)]^2$$

$$\int dy (\partial^y H)^\dagger \partial_y H = -\sum_{n=1}^{\infty} \frac{n^2}{R^2} |H^{(n)}(x)|^2, \quad \int dy \partial^y \Phi \partial_y \Phi = \sum_{n=1}^{\infty} \frac{n^2}{R^2} [\phi^{(n)}(x)]^2$$

$$E_{\text{der}} = \sum_{n=1}^{\infty} \left\{ \frac{n^2}{R^2} |H^{(n)}(x)|^2 + \frac{1}{2}\frac{n^2}{R^2} [\phi^{(n)}(x)]^2 \right\}$$

$$n \geq 1, \quad m^2_{\phi^{(n)}} = \frac{n^2}{R^2} - M^2 = n^2 M^2_{\text{KK}} - M^2$$

$|H^{(n)}|^2 = 0$ and $(\phi^{(n)})^2 = 0$ minimize E_{der} $\quad \Rightarrow \quad \langle H^{(n)} \rangle = 0, \quad \langle \phi^{(n)} \rangle = 0$

$$\frac{\tilde{\lambda}_\phi}{4!}\left[(\Phi^2 - \tilde{v}_\phi^2) + \frac{6\tilde{\lambda}_{\phi h}}{\tilde{\lambda}_\phi}\left(|H|^2 - \frac{\tilde{v}^2}{2} \right) \right]^2 = \frac{\tilde{\lambda}_\phi}{4!}\left[\Phi^4 - 2\tilde{v}_\phi^2 \Phi^2 + \tilde{v}_\phi^4 + \frac{36\tilde{\lambda}_{\phi h}^2}{\tilde{\lambda}_\phi^2}\left(|H|^4 - \tilde{v}^2 |H|^2 + \frac{\tilde{v}^4}{4} \right) + \frac{12\tilde{\lambda}_{\phi h}}{\tilde{\lambda}_\phi}\left(\Phi^2 |H|^2 - \frac{\tilde{v}^2}{2}\Phi^2 - \tilde{v}_\phi^2 |H|^2 + \tilde{v}_\phi^2 \frac{\tilde{v}^2}{2} \right) \right]$$

$$= \frac{\tilde{\lambda}_\phi}{4!}\Phi^4 - \frac{\tilde{\lambda}_\phi}{12}\tilde{v}_\phi^2\Phi^2 - \frac{\tilde{\lambda}_{\phi h}}{4}\tilde{v}^2\Phi^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2|H|^2 + \frac{3\tilde{\lambda}_{\phi h}^2}{2\tilde{\lambda}_\phi}|H|^4 - \frac{3\tilde{\lambda}_{\phi h}^2}{2\tilde{\lambda}_\phi}\tilde{v}^2|H|^2 - \frac{\tilde{\lambda}_{\phi h}}{2}\tilde{v}_\phi^2|H|^2 + \frac{\tilde{\lambda}_\phi}{4!}\tilde{v}_\phi^4 + \frac{3\tilde{\lambda}_{\phi h}^2}{8\tilde{\lambda}_\phi}\tilde{v}^4 + \frac{\tilde{\lambda}_{\phi h}}{4}\tilde{v}_\phi^2\tilde{v}^2$$

$$\frac{\tilde{\lambda}_\phi}{4!}\left[(\Phi^2 - \tilde{v}_\phi^2) + \frac{6\tilde{\lambda}_{\phi h}}{\tilde{\lambda}_\phi}\left(|H|^2 - \frac{\tilde{v}^2}{2} \right) \right]^2 + \left(\tilde{\lambda} - \frac{3\tilde{\lambda}_{\phi h}^2}{2\tilde{\lambda}_\phi} \right)\left(|H|^4 - \tilde{v}^2|H|^2 + \frac{\tilde{v}^4}{4} \right)$$

$$= \frac{\tilde{\lambda}_\phi}{4!}\Phi^4 - \frac{1}{2}\left(\frac{\tilde{\lambda}_\phi}{6}\tilde{v}_\phi^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\tilde{v}^2 \right)\Phi^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2|H|^2 + \tilde{\lambda}|H|^4 - \left(\tilde{\lambda}\tilde{v}^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\tilde{v}_\phi^2 \right)|H|^2 + \frac{\tilde{\lambda}_\phi}{24}\tilde{v}_\phi^4 + \frac{\tilde{\lambda}_{\phi h}}{4}\tilde{v}_\phi^2\tilde{v}^2 + \frac{\tilde{\lambda}}{4}\tilde{v}^4$$

$$V(\Phi, H) = \int dy \left\{ \frac{\tilde{\lambda}_\phi}{4!}\left[(\Phi^2 - \tilde{v}_\phi^2) + \frac{6\tilde{\lambda}_{\phi h}}{\tilde{\lambda}_\phi}\left(|H|^2 - \frac{\tilde{v}^2}{2} \right) \right]^2 + \left(\tilde{\lambda} - \frac{3\tilde{\lambda}_{\phi h}^2}{2\tilde{\lambda}_\phi} \right)\left(|H|^2 - \frac{\tilde{v}^2}{2} \right)^2 - \frac{\tilde{\lambda}_{\phi h}}{4}\tilde{v}_\phi^2\tilde{v}^2 - \frac{\tilde{\lambda}_\phi}{24}\tilde{v}_\phi^4 - \frac{\tilde{\lambda}}{4}\tilde{v}^4 \right\}$$

$$= \int dy \left[\frac{\tilde{\lambda}_\phi}{4!}\Phi^4 - \frac{1}{2}\left(\frac{\tilde{\lambda}_\phi}{6}\tilde{v}_\phi^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\tilde{v}^2 \right)\Phi^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2|H|^2 + \tilde{\lambda}|H|^4 - \left(\tilde{\lambda}\tilde{v}^2 + \frac{\tilde{\lambda}_{\phi h}}{2}\tilde{v}_\phi^2 \right)|H|^2 \right]$$

$$= \int dy \left[-\mu^2|H|^2 - \frac{1}{2}M^2\Phi^2 + \tilde{\lambda}|H|^4 + \frac{\tilde{\lambda}_\phi}{4!}\Phi^4 + \frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2|H|^2 \right]$$

$$\Rightarrow \quad \frac{1}{2}\tilde{\lambda}_{\phi h}\tilde{v}^2 + \frac{1}{6}\tilde{\lambda}_\phi\tilde{v}_\phi^2 = M^2, \quad \tilde{\lambda}\tilde{v}^2 + \frac{1}{2}\tilde{\lambda}_{\phi h}\tilde{v}_\phi^2 = \mu^2$$

$$\Rightarrow \quad \tilde{v}^2 = \frac{6\tilde{\lambda}_{\phi h}M^2 - 2\tilde{\lambda}_\phi\mu^2}{3\tilde{\lambda}_{\phi h}^2 - 2\tilde{\lambda}\tilde{\lambda}_\phi}, \quad \tilde{v}_\phi^2 = \frac{6\tilde{\lambda}_{\phi h}\mu^2 - 12\tilde{\lambda}M^2}{3\tilde{\lambda}_{\phi h}^2 - 2\tilde{\lambda}\tilde{\lambda}_\phi}$$

$\tilde{\lambda}_\phi > 0, \quad \tilde{\lambda} - \dfrac{3\tilde{\lambda}_{\phi h}^2}{2\tilde{\lambda}_\phi} > 0 \quad \Rightarrow \quad |H|^2 = \dfrac{\tilde{v}^2}{2}$ and $\Phi^2 = \tilde{v}_\phi^2$ minimize $V(\Phi, H) \quad \Rightarrow \quad \langle |H|^2 \rangle = \dfrac{\tilde{v}^2}{2}, \quad \langle \Phi \rangle = \tilde{v}_\phi$

0-mode mixing

$$\phi^{(0)}(x)=v_\phi+\phi(x),\quad H^{(0)}(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h(x)\end{pmatrix},\quad v_\phi^2\equiv\pi R\tilde{v}_\phi^2,\quad v^2\equiv\pi R\tilde{v}^2$$

$$V(\Phi,H)\supset\int dy\left[-\mu^2|H|^2-\frac{M^2}{2}\Phi^2+\tilde{\lambda}|H|^4+\frac{\tilde{\lambda}_\phi}{4!}\Phi^4+\frac{\tilde{\lambda}_{\phi h}}{2}\Phi^2|H|^2\right]$$

$$\supset-\mu^2|H^{(0)}(x)|^2-\frac{M^2}{2}[\phi^{(0)}(x)]^2+\frac{\tilde{\lambda}}{\pi R}|H^{(0)}(x)|^4+\frac{\tilde{\lambda}_\phi}{4!\pi R}[\phi^{(0)}(x)]^4+\frac{\tilde{\lambda}_{\phi h}}{2\pi R}|H^{(0)}(x)|^2[\phi^{(0)}(x)]^2$$

$$=-\frac{\mu^2}{2}(v+h)^2-\frac{M^2}{2}(v_\phi+\phi)^2+\frac{\lambda}{4}(v+h)^4+\frac{\lambda_\phi}{4!}(v_\phi+\phi)^4+\frac{\lambda_{\phi h}}{4}(v+h)^2(v_\phi+\phi)^2$$

$$\lambda\equiv\frac{\tilde{\lambda}}{\pi R},\quad \lambda_\phi\equiv\frac{\tilde{\lambda}_\phi}{\pi R},\quad \lambda_{\phi h}\equiv\frac{\tilde{\lambda}_{\phi h}}{\pi R}$$

Minimization conditions $\rightarrow\quad \frac{1}{2}\lambda_{\phi h}v^2+\frac{1}{6}\lambda_\phi v_\phi^2=M^2,\quad \lambda v^2+\frac{1}{2}\lambda_{\phi h}v_\phi^2=\mu^2$

Mass terms:

$$V(\Phi,H)\supset-\frac{\mu^2}{2}h^2-\frac{M^2}{2}\phi^2+\frac{\lambda}{4}(4v^2h^2+2v^2h^2)+\frac{\lambda_\phi}{24}(4v_\phi^2\phi^2+2v_\phi^2\phi^2)+\frac{\lambda_{\phi h}}{4}(v^2\phi^2+v_\phi^2h^2+4vv_\phi h\phi)$$

$$=\lambda v^2h^2+\frac{1}{6}\lambda_\phi v_\phi^2\phi^2+\lambda_{\phi h}vv_\phi h\phi=\frac{1}{2}\begin{pmatrix}h & \phi\end{pmatrix}\begin{pmatrix}m_h^2 & \lambda_{\phi h}vv_\phi\\ \lambda_{\phi h}vv_\phi & m_\phi^2\end{pmatrix}\begin{pmatrix}h\\ \phi\end{pmatrix}$$

$$m_h^2\equiv 2\lambda v^2,\quad m_\phi^2\equiv\frac{1}{3}\lambda_\phi v_\phi^2,\quad m_{h_{1,2}}^2=\frac{1}{2}\left[m_h^2+m_\phi^2\pm\sqrt{(m_\phi^2-m_h^2)^2+4\lambda_{\phi h}^2v^2v_\phi^2}\right]$$

$$\begin{pmatrix}h_1\\ h_2\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}h\\ \phi\end{pmatrix},\quad \tan 2\alpha=\frac{2\lambda_{\phi h}vv_\phi}{m_\phi^2-m_h^2}$$

H couplings to quarks

$$\int dy f_L^{(0)}(y) g_R^{(0)}(y) = 2\pi \frac{M_f}{M_{\rm KK}} [(1-e^{-2\pi M_f/M_{\rm KK}})(e^{2\pi M_f/M_{\rm KK}}-1)]^{-1/2}$$

$$= 2\pi \frac{M_f}{M_{\rm KK}} [(e^{\pi M_f/M_{\rm KK}} - e^{-\pi M_f/M_{\rm KK}})^2]^{-1/2} = \frac{2\pi M_f}{M_{\rm KK}(e^{\pi M_f/M_{\rm KK}} - e^{-\pi M_f/M_{\rm KK}})}$$

$$\int dy f_L^{(n)}(y) g_R^{(n)}(y) = \frac{n^2 M_{\rm KK}^2 - M_f^2}{n^2 M_{\rm KK}^2 + M_f^2}$$

$$\int dy f_L^{(n)}(y) g_R^{(0)}(y) = 2\sqrt{\frac{2}{\pi}} \sqrt{\coth\frac{\pi M_f}{M_{\rm KK}} + 1} \left(\frac{M_f M_{\rm KK}}{M_f^2 + n^2 M_{\rm KK}^2} \right)^{3/2} n[1-(-1)^n e^{-\pi M_f/M_{\rm KK}}]$$

$$= \frac{4}{\sqrt{\pi}} n \left(\frac{M_f M_{\rm KK}}{M_f^2 + n^2 M_{\rm KK}^2} \right)^{3/2} \frac{(-1)^n e^{\pi M_f/M_{\rm KK}} - 1}{\sqrt{e^{2\pi M_f/M_{\rm KK}} - 1}} (-1)^n$$

$$\int dy f_L^{(0)}(y) g_R^{(n)}(y) = 2\sqrt{\frac{2}{\pi}} \sqrt{\coth\frac{\pi M_f}{M_{\rm KK}} - 1} \left(\frac{M_f M_{\rm KK}}{M_f^2 + n^2 M_{\rm KK}^2} \right)^{3/2} n[(-1)^n e^{\pi M_f/M_{\rm KK}} - 1]$$

$$= \frac{4}{\sqrt{\pi}} n \left(\frac{M_f M_{\rm KK}}{M_f^2 + n^2 M_{\rm KK}^2} \right)^{3/2} \frac{(-1)^n e^{\pi M_f/M_{\rm KK}} - 1}{\sqrt{e^{2\pi M_f/M_{\rm KK}} - 1}}$$

$$S \supset \int d^5x (-\tilde{y}_u' \bar{Q} i\sigma_2 H^* U + h.c.) \supset \int d^5x (-y_u' \bar{Q} i\sigma_2 H^{(0)*} U + h.c.) \supset -\frac{y_u'}{\sqrt{2}}(v+h) \int d^5x (\bar{U}_2 U_1 + h.c.)$$

$$\supset -\frac{y_u'}{\sqrt{2}} \frac{2\pi M_f}{M_{\rm KK}(e^{\pi M_f/M_{\rm KK}} - e^{-\pi M_f/M_{\rm KK}})} (v+h) \int d^4x [\bar{u}_2^{(0)} P_R u_1^{(0)} + h.c.]$$

$$- \frac{y_u'}{\sqrt{2}} (v+h) \int d^5x \sum_{n=1}^{\infty} \frac{n^2 M_{\rm KK}^2 - M_f^2}{n^2 M_{\rm KK}^2 + M_f^2} [\bar{u}_2^{(n)} P_R u_1^{(n)} + h.c.]$$

$$- \frac{4 y_u'}{\sqrt{2\pi}} (v+h) \int d^4x \Big\{ \sum_{n=1}^{\infty} n \left(\frac{M_f M_{\rm KK}}{M_f^2 + n^2 M_{\rm KK}^2} \right)^{3/2} \frac{(-1)^n e^{\pi M_f/M_{\rm KK}} - 1}{\sqrt{e^{2\pi M_f/M_{\rm KK}} - 1}} [(-1)^n \bar{u}_2^{(n)} P_R u_1^{(0)} + \bar{u}_2^{(0)} P_R u_1^{(n)}] + h.c. \Big\}$$

$$\lim_{M_f/M_{KK} \to 0} \frac{2\pi M_f}{M_{KK}(e^{\pi M_f/M_{KK}} - e^{-\pi M_f/M_{KK}})} = 1$$

$\Rightarrow$ The $h\bar{t}t$ coupling is the same as in the SM for $M_f / M_{\rm KK} \to 0$,

but it is reduced as $M_f / M_{\rm KK}$ increases

Change the sign of the $\Phi \bar{U} U$ Yukawa coupling:

$$\int dy f_L^{(0)}(y) g_R^{(0)}(y) = \int dy f_L^{(n)}(y) g_R^{(n)}(y) = 1, \quad \int dy f_L^{(n)}(y) g_R^{(0)}(y) = \int dy f_L^{(0)}(y) g_R^{(n)}(y) = 0$$

$$S \supset \int d^5x (-\tilde{y}_u \bar{Q} i\sigma_2 H^* U + h.c.) \supset -\frac{y_u}{\sqrt{2}} (v+h) \int d^5x (\bar{U}_2 U_1 + h.c.)$$

$$= -\frac{y_u}{\sqrt{2}} (v+h) \int d^5x [\bar{u}_2^{(0)} P_R u_1^{(0)} + \bar{u}_2^{(n)} P_R u_1^{(n)} + h.c.]$$

$\Rightarrow$ The $h\bar{t}t$ coupling is the same as in the SM

$\phi \to f\bar{f}$ decay induced by $\langle H \rangle$

$$\phi \bar{u}_2^{(0)} P_R u_2^{(n)} - \left\langle H^{(0)} \right\rangle - \bar{u}_2^{(n)} P_R u_1^{(0)}, \quad \bar{u}_2^{(0)} P_R u_1^{(n)} - \left\langle H^{(0)} \right\rangle - \bar{u}_1^{(n)} P_R u_1^{(0)} \phi$$

$$h.c. \quad \to \quad \bar{u}_1^{(0)} P_L u_2^{(n)} - \left\langle H^{(0)} \right\rangle - \bar{u}_2^{(n)} P_L u_2^{(0)} \phi, \quad \phi \bar{u}_1^{(0)} P_L u_1^{(n)} - \left\langle H^{(0)} \right\rangle - \bar{u}_1^{(n)} P_L u_2^{(0)}$$

$$y_{\phi tt} = -\frac{4m_t}{\sqrt{\pi}} \sum_{n=1}^{\infty} n \left(\frac{M_q M_{\mathrm{KK}}}{M_q^2 + n^2 M_{\mathrm{KK}}^2} \right)^{3/2} \frac{(-1)^n e^{\pi M_q / M_{\mathrm{KK}}} - 1}{\sqrt{e^{2\pi M_q / M_{\mathrm{KK}}} - 1}} \times \frac{1}{\sqrt{M_q^2 + n^2 M_{\mathrm{KK}}^2}}$$

$$\times \left[(-1)^n \frac{-2y_q}{\sqrt{\pi}} \sqrt{\frac{M_q}{e^{2\pi M_q / M_{\mathrm{KK}}} - 1}} \frac{n M_{\mathrm{KK}}^{3/2} [1 - (-1)^n e^{\pi M_q / M_{\mathrm{KK}}}]}{M_q^2 + n^2 M_{\mathrm{KK}}^2} + \frac{-2y_q}{\sqrt{\pi}} \sqrt{\frac{M_f}{e^{2\pi M_q / M_{\mathrm{KK}}} - 1}} \frac{n M_{\mathrm{KK}}^{3/2} [(-1)^n - e^{\pi M_q / M_{\mathrm{KK}}}]}{M_q^2 + n^2 M_{\mathrm{KK}}^2} \right]$$

$$= \frac{16 y_q m_t}{\pi} \sum_{n=1}^{\infty} \frac{n^2 M_q^2 M_{\mathrm{KK}}^3}{(M_q^2 + n^2 M_{\mathrm{KK}}^2)^6} \frac{(-1)^{n+1} [(-1)^n - e^{\pi M_q / M_{\mathrm{KK}}}]^2}{e^{2\pi M_q / M_{\mathrm{KK}}} - 1}$$

$$= \frac{16 m_t}{\pi v_\phi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[1 - \frac{2(-1)^n}{e^{\pi M_q / M_{\mathrm{KK}}} + (-1)^n} \right] \left(\frac{M_q}{n M_{\mathrm{KK}}} + \frac{n M_{\mathrm{KK}}}{M_q} \right)^{-3}$$

Mass **mixing term insertion**

(a) In the viewpoint of insertion

$$\mathcal{L} \supset y_{\phi tR}\phi(\bar{R}_L t_R + \bar{t}_R R_L) + y_{\phi tL}\phi(\bar{L}_R t_L + \bar{t}_L L_R) - m_{\mathrm{DR}}(\bar{t}_L R_R + \bar{R}_R t_L) - m_{\mathrm{DL}}(\bar{L}_L t_R + \bar{t}_R L_L)$$
$$- m_t \bar{t}t - m_R \bar{R}R - m_L \bar{L}L$$
$$\to y_{\phi tt}\phi\bar{t}t$$

$$q^2 = m_t^2, \quad \bar{u}(q)\not{q} = m_t, \quad \not{q}v(-q) = -[-\not{q}v(-q)] = -(-m_t) = m_t$$

$\phi\bar{t}R - Rt:$

$$iy_{\phi tt}^{(R)}\bar{u}(q)v(-q) = \bar{u}(q)(-im_{\mathrm{DR}})P_R\frac{i(\not{q}+m_R)}{q^2-m_R^2}iy_{\phi tR}P_R v(-q) + \bar{u}(q)iy_{\phi tR}P_L\frac{i(\not{q}+m_R)}{q^2-m_R^2}(-im_{\mathrm{DR}})P_L v(-q)$$
$$= i\frac{y_{\phi tR}m_{\mathrm{DR}}}{m_t^2-m_R^2}[\bar{u}(q)m_R P_R v(-q) + \bar{u}(q)m_R P_L v(-q)] = i\frac{y_{\phi tR}m_{\mathrm{DR}}m_R}{m_t^2-m_R^2}\bar{u}(q)v(-q)$$
$$\Rightarrow \quad y_{\phi tt}^{(R)} = \frac{y_{\phi tR}m_{\mathrm{DR}}m_R}{m_t^2-m_R^2}$$

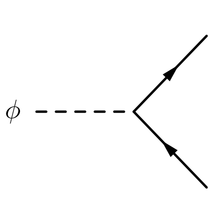
$\phi\bar{t}L - Lt:$

$$iy_{\phi tt}^{(L)}\bar{u}(q)v(-q) = \bar{u}(q)(-im_{\mathrm{DL}})P_L\frac{i(\not{q}+m_L)}{q^2-m_L^2}iy_{\phi tL}P_L v(-q) + \bar{u}(q)iy_{\phi tL}P_R\frac{i(\not{q}+m_L)}{q^2-m_L^2}(-im_{\mathrm{DL}})P_R v(-q)$$
$$= i\frac{y_{\phi tL}m_{\mathrm{DL}}}{m_t^2-m_L^2}[\bar{u}(q)m_L P_L v(-q) + \bar{u}(q)m_L P_R v(-q)] = i\frac{y_{\phi tL}m_{\mathrm{DL}}m_L}{m_t^2-m_L^2}\bar{u}(q)v(-q)$$
$$\Rightarrow \quad y_{\phi tt}^{(L)} = \frac{y_{\phi tL}m_{\mathrm{DL}}m_L}{m_t^2-m_L^2}$$

$$y_{\phi tt} = y_{\phi tt}^{(R)} + y_{\phi tt}^{(L)} = \frac{y_{\phi tR}m_{\mathrm{DR}}m_R}{m_t^2-m_R^2} + \frac{y_{\phi tL}m_{\mathrm{DL}}m_L}{m_t^2-m_L^2} \simeq -\frac{y_{\phi tR}m_{\mathrm{DR}}}{m_R} - \frac{y_{\phi tL}m_{\mathrm{DL}}}{m_L}$$

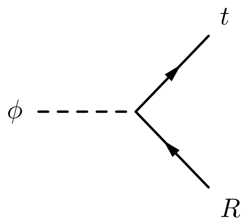
Feynman Rules:

$y_{\phi tR}\phi\bar{R}_L t_R$:



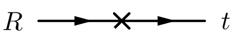
$= iy_{\phi tR}P_R$

$y_{\phi tR}\phi\bar{t}_R R_L$:



$= iy_{\phi tR}P_L$

$-m_{\rm DR}\bar{t}_L R_R$:

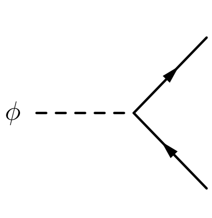


$= -im_{\rm DR}P_R$

$-m_{\rm DR}\bar{R}_R t_L$:

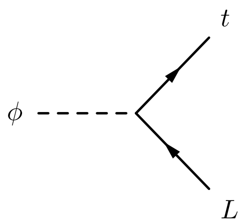
$= -im_{\rm DR}P_L$

$y_{\phi tL}\phi\bar{L}_R t_L$:



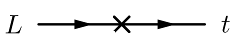
$= iy_{\phi tL}P_L$

$y_{\phi tL}\phi\bar{t}_L L_R$:



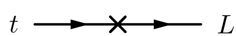
$= iy_{\phi tL}P_R$

$-m_{\rm DL}\bar{t}_R L_L$:



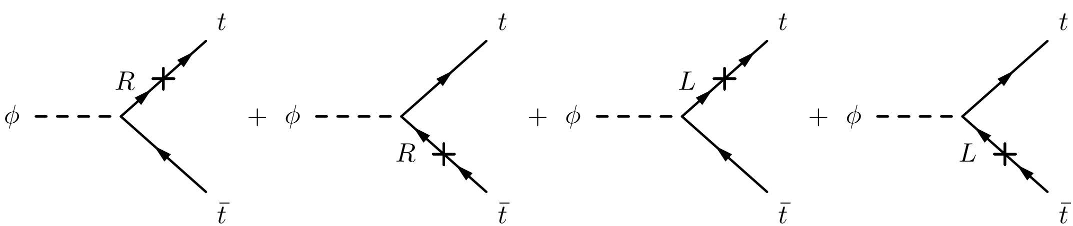
$= -im_{\rm DL}P_L$

$-m_{\rm DL}\bar{L}_L t_R$:



$= -im_{\rm DL}P_R$

$\phi \to t\bar{t}$ **Feynman diagrams:**



(b) In the viewpoint of state mixing

$$-\mathcal{L}_{\text{mass}} = m_t\bar{t}t + m_R\bar{R}R + m_L\bar{L}L + m_{\text{DR}}(\bar{t}_L R_R + \bar{R}_R t_L) + m_{\text{DL}}(\bar{L}_L t_R + \bar{t}_R L_L)$$

$$= \begin{pmatrix} \bar{t}_L & \bar{R}_L & \bar{L}_L \end{pmatrix} \begin{pmatrix} m_t & m_{\text{DR}} & 0 \\ 0 & m_R & 0 \\ m_{\text{DL}} & 0 & m_L \end{pmatrix} \begin{pmatrix} t_R \\ R_R \\ L_R \end{pmatrix} + h.c. = \begin{pmatrix} \bar{t}'_L & \bar{R}'_L & \bar{L}'_L \end{pmatrix} \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \begin{pmatrix} t'_R \\ R'_R \\ L'_R \end{pmatrix} + h.c.$$

$$\mathcal{M} = \begin{pmatrix} m_t & m_{\text{DR}} & 0 \\ 0 & m_R & 0 \\ m_{\text{DL}} & 0 & m_L \end{pmatrix}, \quad \widehat{\mathcal{M}} = \begin{pmatrix} m_{t'} & & \\ & m_{R'} & \\ & & m_{L'} \end{pmatrix}$$

$$V^\dagger \mathcal{M} U = \widehat{\mathcal{M}}, \quad \begin{pmatrix} t_R \\ R_R \\ L_R \end{pmatrix} = U \begin{pmatrix} t'_R \\ R'_R \\ L'_R \end{pmatrix}, \quad \begin{pmatrix} t_L \\ R_L \\ L_L \end{pmatrix} = V \begin{pmatrix} t'_L \\ R'_L \\ L'_L \end{pmatrix}$$

$$U^\dagger \mathcal{M}^\dagger \mathcal{M} U = V^\dagger \mathcal{M}\mathcal{M}^\dagger V = \text{diag}(m_{t'}^2, m_{R'}^2, m_{L'}^3)$$

$$\left[\begin{array}{l} \text{Eigenvalue perturbation:} \quad \mathbf{K}^{(0)}\mathbf{x}_i^{(0)} = \lambda_i^{(0)}\mathbf{x}_i^{(0)} \\ \mathbf{K} = \mathbf{K}^{(0)} + \delta\mathbf{K}, \quad \lambda_i = \lambda_i^{(0)} + \delta\lambda_i, \quad \mathbf{x}_i = \mathbf{x}_i^{(0)} + \delta\mathbf{x}_i \\ \text{Solution to } \mathbf{K}\mathbf{x}_i = \lambda_i\mathbf{x}_i : \\ \lambda_i \simeq \lambda_i^{(0)} + \mathbf{x}_i^{(0)\mathrm{T}}\delta\mathbf{K}\mathbf{x}_i^{(0)}, \quad \mathbf{x}_i \simeq \mathbf{x}_i^{(0)} + \sum_{j\neq i} \frac{\mathbf{x}_j^{(0)\mathrm{T}}\delta\mathbf{K}\mathbf{x}_i^{(0)}}{\lambda_i^{(0)} - \lambda_j^{(0)}}\mathbf{x}_j^{(0)} \end{array}\right]$$

$$\mathbf{K} = \mathcal{M}^\dagger\mathcal{M}, \quad \mathbf{K}^{(0)} = \begin{pmatrix} m_t^2 & & \\ & m_R^2 & \\ & & m_L^2 \end{pmatrix}, \quad \delta\mathbf{K} = \begin{pmatrix} m_{\text{DL}}^2 & m_{\text{DR}}m_t & m_{\text{DL}}m_L \\ m_{\text{DR}}m_t & m_{\text{DR}}^2 & 0 \\ m_{\text{DL}}m_L & 0 & 0 \end{pmatrix}$$

$$\lambda_i^{(0)} = (m_t^2, m_R^2, m_L^2), \quad \mathbf{x}_{U1}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{U2}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{U3}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_i \simeq \lambda_i^{(0)} + \mathbf{x}_i^{(0)\mathrm{T}}\delta\mathbf{K}\mathbf{x}_i^{(0)} = (m_t^2 + m_{\text{DL}}^2, m_R^2 + m_{\text{DR}}^2, m_L^2)$$

$$\mathbf{x}_{U1} \simeq \begin{pmatrix} 1 \\ \dfrac{m_{\text{DR}}m_t}{m_t^2 - m_R^2} \\ \dfrac{m_{\text{DL}}m_L}{m_t^2 - m_L^2} \end{pmatrix}, \quad \mathbf{x}_{U2} \simeq \begin{pmatrix} \dfrac{m_{\text{DR}}m_t}{m_R^2 - m_t^2} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{U3} \simeq \begin{pmatrix} \dfrac{m_{\text{DL}}m_L}{m_L^2 - m_t^2} \\ 0 \\ 1 \end{pmatrix}, \quad U = \begin{pmatrix} \dfrac{\mathbf{x}_{U1}}{N_{U1}} & \dfrac{\mathbf{x}_{U2}}{N_{U2}} & \dfrac{\mathbf{x}_{U3}}{N_{U3}} \end{pmatrix}$$

$$\mathbf{K} = \mathcal{M}\mathcal{M}^\dagger, \quad \mathbf{K}^{(0)} = \begin{pmatrix} m_t^2 & & \\ & m_R^2 & \\ & & m_L^2 \end{pmatrix}, \quad \delta\mathbf{K} = \begin{pmatrix} m_{\mathrm{DR}}^2 & m_{\mathrm{DR}} m_R & m_{\mathrm{DL}} m_t \\ m_{\mathrm{DR}} m_R & 0 & 0 \\ m_{\mathrm{DL}} m_t & 0 & m_{\mathrm{DL}}^2 \end{pmatrix}$$

$$\lambda_i^{(0)} = (m_t^2, m_R^2, m_L^2), \quad \mathbf{x}_{V1}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{v2}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{V3}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_i \simeq \lambda_i^{(0)} + \mathbf{x}_i^{(0)\mathrm{T}} \delta\mathbf{K} \mathbf{x}_i^{(0)} = (m_t^2 + m_{\mathrm{DR}}^2, m_R^2, m_L^2 + m_{\mathrm{DL}}^2)$$

$$\mathbf{x}_{V1} \simeq \begin{pmatrix} 1 \\ \dfrac{m_{\mathrm{DR}} m_R}{m_t^2 - m_R^2} \\ \dfrac{m_{\mathrm{DL}} m_t}{m_t^2 - m_L^2} \end{pmatrix}, \quad \mathbf{x}_{V2} \simeq \begin{pmatrix} \dfrac{m_{\mathrm{DR}} m_R}{m_R^2 - m_t^2} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{V3} \simeq \begin{pmatrix} \dfrac{m_{\mathrm{DL}} m_t}{m_L^2 - m_t^2} \\ 0 \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} \dfrac{\mathbf{x}_{V1}}{N_{V1}} & \dfrac{\mathbf{x}_{V2}}{N_{V2}} & \dfrac{\mathbf{x}_{V3}}{N_{V3}} \end{pmatrix}$$

$$t_R \simeq t'_R + \frac{m_{\mathrm{DR}} m_t}{m_R^2 - m_t^2} R'_R + \frac{m_{\mathrm{DL}} m_L}{m_L^2 - m_t^2} L'_R, \quad R_R \simeq \frac{m_{\mathrm{DR}} m_t}{m_t^2 - m_R^2} t'_R + R'_R, \quad L_R \simeq \frac{m_{\mathrm{DL}} m_L}{m_t^2 - m_L^2} t'_R + L'_R$$

$$t_L \simeq t'_L + \frac{m_{\mathrm{DR}} m_R}{m_R^2 - m_t^2} R'_L + \frac{m_{\mathrm{DL}} m_t}{m_L^2 - m_t^2} L'_L, \quad R_L \simeq \frac{m_{\mathrm{DR}} m_R}{m_t^2 - m_R^2} t'_L + R'_L, \quad L_L \simeq \frac{m_{\mathrm{DL}} m_t}{m_t^2 - m_L^2} t'_L + L'_L$$

$$\begin{aligned} \mathcal{L} &\supset y_{\phi tR} \phi \bar{R}_L t_R + y_{\phi tL} \phi \bar{L}_R t_L + h.c. \\ &\simeq y_{\phi tR} \phi \left(\frac{m_{\mathrm{DR}} m_R}{m_t^2 - m_R^2} \bar{t}'_L + \bar{R}'_L \right) \left(t'_R + \frac{m_{\mathrm{DR}} m_t}{m_R^2 - m_t^2} R'_R + \frac{m_{\mathrm{DL}} m_L}{m_L^2 - m_t^2} L'_R \right) \\ &\quad + y_{\phi tL} \phi \left(\frac{m_{\mathrm{DL}} m_L}{m_t^2 - m_L^2} t'_R + L'_R \right) \left(t'_L + \frac{m_{\mathrm{DR}} m_R}{m_R^2 - m_t^2} R'_L + \frac{m_{\mathrm{DL}} m_t}{m_L^2 - m_t^2} L'_L \right) + h.c. \\ &\supset y_{\phi tR} \phi \frac{m_{\mathrm{DR}} m_R}{m_t^2 - m_R^2} \bar{t}'_L t'_R + y_{\phi tL} \phi \frac{m_{\mathrm{DL}} m_L}{m_t^2 - m_L^2} \bar{t}'_R t'_L + h.c. = y_{\phi tt} \phi \bar{t}' t' \end{aligned}$$

$$y_{\phi tt} = \frac{y_{\phi tR} m_{\mathrm{DR}} m_R}{m_t^2 - m_R^2} + \frac{y_{\phi tL} m_{\mathrm{DL}} m_L}{m_t^2 - m_L^2} \simeq -\frac{y_{\phi tR} m_{\mathrm{DR}}}{m_R} - \frac{y_{\phi tL} m_{\mathrm{DL}}}{m_L}$$