暗物质与标准模型费米子耦合的有效场论

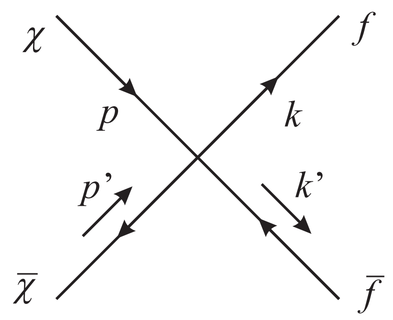
Ref: Beltran, *et al.*, **PRD 80**, 043509(2009)

（一）湮灭方面

一、Dirac fermionic WIMP

设暗物质粒子 χ 和 $\bar{\chi}$ 是 Dirac 旋量，

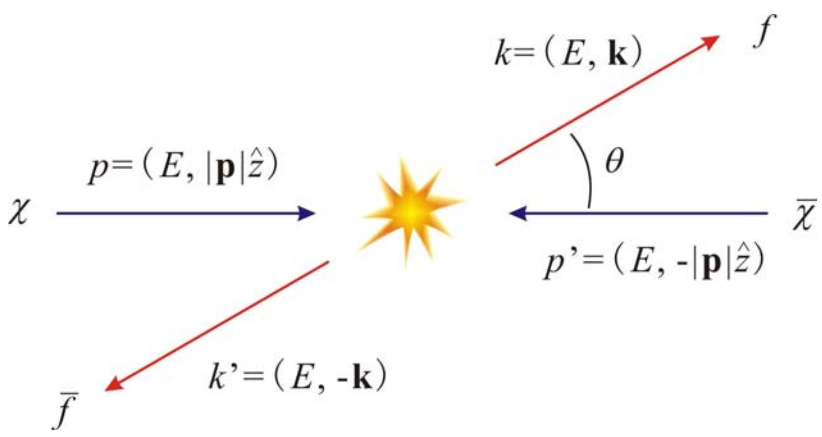
f 和 $\bar{f}$ 是标准模型中的费米子



Mandelstam 变量

$$
\begin{aligned}
s &= (p+p')^2 = 2p\cdot p' + 2m_\chi^2 \\
&= (k+k')^2 = 2k\cdot k' + 2m_f^2 \\
t &= (k-p)^2 = -2k\cdot p + m_\chi^2 + m_f^2 \\
&= (k'-p')^2 = -2k'\cdot p' + m_\chi^2 + m_f^2 \\
u &= (k'-p)^2 = -2k'\cdot p + m_\chi^2 + m_f^2 \\
&= (k-p')^2 = -2k\cdot p' + m_\chi^2 + m_f^2
\end{aligned}
$$

$$s+t+u = 2m_\chi^2 + 2m_f^2$$



质心系下，

$$E_{\rm cm} = 2E_p = 2E_k = 2E$$

$$s = (p+p')^2 = E_{\rm cm}^2 = (2E_p)^2 = (2E_k)^2 = (2E)^2$$

$$s = (k+k')^2 = 4E^2 = 4\left(|\mathbf{k}|^2 + m_f^2\right)$$

$$p\cdot k = E^2 - \mathbf{p}\cdot\mathbf{k} = \frac{s}{4} - |\mathbf{p}||\mathbf{k}|\cos\theta$$

$$|v-v'|E_{\rm cm} = \frac{|\mathbf{p}-\mathbf{p}'|}{\gamma m_\chi}2\gamma m_\chi = 4|\mathbf{p}| = 4\sqrt{\frac{s}{4} - m_\chi^2}$$

$$16|\mathbf{p}|^2|\mathbf{k}|^2 = \left(s-4m_\chi^2\right)\left(s-4m_f^2\right)$$

$$\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\rm cm}} = \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}$$

$$\int_0^\pi \cos^2\theta\sin\theta d\theta = \frac{2}{3}，\quad \int_0^\pi \cos\theta\sin\theta d\theta = 0$$

$$t=(k-p)^2=-2k\cdot p+m_\chi^2+m_f^2=-2\left(E^2-|\mathbf{p}||\mathbf{k}|\cos\theta\right)+m_\chi^2+m_f^2=-2\left(\frac{s}{4}-|\mathbf{p}||\mathbf{k}|\cos\theta\right)+m_\chi^2+m_f^2$$

$$u=(k-p')^2=-2k\cdot p'+m_\chi^2+m_f^2=-2\left(E^2+|\mathbf{p}||\mathbf{k}|\cos\theta\right)+m_\chi^2+m_f^2=-2\left(\frac{s}{4}+|\mathbf{p}||\mathbf{k}|\cos\theta\right)+m_\chi^2+m_f^2$$

质心系下的非相对论近似，

v 是两暗物质粒子间的相对速度，

两暗物质粒子的速度分别是 $\frac{\mathbf{v}}{2}$ 和 $-\frac{\mathbf{v}}{2}$，能量均为 $E_\chi=m_\chi+\frac{1}{2}m_\chi\left(\frac{v}{2}\right)^2$，

$$s=(p+p')^2=E_{\text{cm}}^2=(2E_\chi)^2\simeq 4\left[m_\chi+\frac{1}{2}m_\chi\left(\frac{v}{2}\right)^2\right]^2\simeq 4m_\chi^2+m_\chi^2v^2$$

$$\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\simeq\sqrt{\frac{4m_\chi^2+m_\chi^2v^2-4m_f^2}{4m_\chi^2+m_\chi^2v^2-4m_\chi^2}}=\frac{\sqrt{4+v^2-4m_f^2/m_\chi^2}}{v}$$

$$v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\simeq\sqrt{4+v^2-4m_f^2/m_\chi^2}$$

$$\simeq\sqrt{4-4m_f^2/m_\chi^2}+\left[\frac{1}{2\sqrt{4+v^2-4m_f^2/m_\chi^2}}\right]_{v=0}v^2$$

$$=2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}$$

$$\frac{1}{s}\simeq\frac{1}{4m_\chi^2+m_\chi^2v^2}\simeq\frac{1}{4m_\chi^2}-\left[\frac{m_\chi^2}{(4m_\chi^2+m_\chi^2v^2)^2}\right]_{v=0}v^2$$

$$=\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}$$

1. Scalar 耦合： $\mathcal{L}=\frac{G_S}{\sqrt{2}}\bar{\chi}\chi\bar{f}f$

$$\left[\bar{u}(k)v(k')\right]^*=\left[u^\dagger(k)\gamma^0v(k')\right]^*$$
$$=v^\dagger(k')\gamma^0u(k)=\bar{v}(k')u(k)$$
$$\sum_{\text{spins}}u(p)\bar{u}(p)=\not{p}+m,\quad \sum_{\text{spins}}v(p)\bar{v}(p)=\not{p}-m$$

$$i\mathcal{M}=i\frac{G_S}{\sqrt{2}}\bar{u}(k)v(k')\bar{v}(p')u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\frac{G_S^2}{2}\left[\bar{u}(k)v(k')\bar{v}(p')u(p)\right]\left[\bar{u}(k)v(k')\bar{v}(p')u(p)\right]^*$$
$$=\frac{1}{4}\frac{G_S^2}{2}\sum_{\text{spins}}\left[\bar{u}(k)v(k')\bar{v}(k')u(k)\right]\left[\bar{v}(p')u(p)\bar{u}(p)v(p')\right]$$
$$=\frac{1}{4}\frac{G_S^2}{2}\mathrm{tr}\left[v(k')\bar{v}(k')u(k)\bar{u}(k)\right]\mathrm{tr}\left[u(p)\bar{u}(p)v(p')\bar{v}(p')\right]$$
$$=\frac{1}{4}\frac{G_S^2}{2}\mathrm{tr}\left[(\not{k}'-m_f)(\not{k}+m_f)\right]\mathrm{tr}\left[(\not{p}+m_\chi)(\not{p}'-m_\chi)\right]$$
$$=\frac{1}{4}\frac{G_S^2}{2}\left(4s^2-16(m_f^2+m_\chi^2)s+64m_f^2m_\chi^2\right)$$
$$=\frac{G_S^2}{2}\left[s^2-4(m_f^2+m_\chi^2)s+16m_f^2m_\chi^2\right]$$

$$\left(\frac{d\sigma_S}{d\Omega}\right)_{\text{CM}} = \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{1}{s}\frac{\sqrt{\frac{s}{4}-m_f^2}}{16\pi^2 4\sqrt{\frac{s}{4}-m_\chi^2}}\frac{G_S^2}{2}\left[s^2-4\left(m_f^2+m_\chi^2\right)s+16m_f^2 m_\chi^2\right]$$

$$= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_S^2}{2}\left[s^2-4\left(m_f^2+m_\chi^2\right)s+16m_f^2 m_\chi^2\right]$$

$$\sigma_S^{\text{CM}} = \frac{1}{16\pi s}\frac{G_S^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s^2-4\left(m_f^2+m_\chi^2\right)s+16m_f^2 m_\chi^2\right]$$

$$= \frac{1}{16\pi}\frac{G_S^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{s}\right]$$

$$\sigma_S^{\text{CM}} v = \frac{1}{16\pi}\frac{G_S^2}{2} v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{s}\right]$$

$$\simeq \frac{1}{16\pi}\frac{G_S^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\left(\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}\right)\left(4m_\chi^2+m_\chi^2 v^2-4m_\chi^2\right)\left(4m_\chi^2+m_\chi^2 v^2-4m_f^2\right)\right]$$

$$= \frac{1}{16\pi}\frac{G_S^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\left(\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}\right)\left(m_\chi^2 v^2\right)\left(4m_\chi^2-4m_f^2+m_\chi^2 v^2\right)\right]$$

$$\simeq \frac{1}{16\pi}\frac{G_S^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\left(1-m_f^2/m_\chi^2\right)\left(m_\chi^2 v^2\right)\right]$$

$$\simeq \frac{1}{16\pi}\frac{G_S^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}\right)\left[\left(1-m_f^2/m_\chi^2\right)\left(m_\chi^2 v^2\right)\right]$$

$$= \frac{1}{2\pi}\frac{G_S^2}{2}m_\chi^2\left(\sqrt{1-m_f^2/m_\chi^2}\right)\left[\frac{1}{4}\left(1-m_f^2/m_\chi^2\right)v^2\right]$$

2. Pseudoscalar 耦合： $\mathcal{L} = \frac{G_P}{\sqrt{2}}\bar{\chi}\gamma^5\chi\bar{f}\gamma_5 f$

$$i\mathcal{M} = i\frac{G_P}{\sqrt{2}}\bar{u}(k)\gamma^5 v(k')\bar{v}(p')\gamma_5 u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}}\frac{G_P^2}{2}\left[\bar{u}(k)\gamma^5 v(k')\bar{v}(p')\gamma_5 u(p)\right]\left[\bar{u}(k)\gamma^5 v(k')\bar{v}(p')\gamma_5 u(p)\right]^*$$

$$= \frac{1}{4}\sum_{\text{spins}}\frac{G_P^2}{2}\left[\bar{u}(k)\gamma^5 v(k')\bar{v}(k')\gamma^5 u(k)\right]\left[\bar{v}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5 v(p')\right]$$

$$= \frac{1}{4}\frac{G_P^2}{2}\text{tr}\left[u(k)\bar{u}(k)\gamma^5 v(k')\bar{v}(k')\gamma^5\right]\text{tr}\left[v(p')\bar{v}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5\right]$$

$$= \frac{1}{4}\frac{G_P^2}{2}\text{tr}\left[\left(\not{k}+m_f\right)\gamma^5\left(\not{k}'-m_f\right)\gamma^5\right]\text{tr}\left[\left(\not{p}'-m_\chi\right)\gamma_5\left(\not{p}+m_\chi\right)\gamma_5\right]$$

$$= \frac{1}{4}\frac{G_P^2}{2}\left[16m_\chi^2 m_f^2+16(k\cdot k')(p\cdot p')+16(k\cdot k')m_\chi^2+16(p\cdot p')m_f^2\right]$$

$$= \frac{1}{4}\frac{G_P^2}{2}\left(4s^2\right) = \frac{G_P^2}{2}s^2$$

$$\left(\frac{d\sigma_P}{d\Omega}\right)_{\text{CM}} = \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{1}{s}\frac{\sqrt{\frac{s}{4}-m_f^2}}{16\pi^2 4\sqrt{\frac{s}{4}-m_\chi^2}}\frac{G_P^2}{2}s^2$$

$$= \frac{1}{16\pi^2}\frac{1}{4}\frac{G_P^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}s$$

$$\sigma_P^{\mathrm{CM}}=\frac{1}{16\pi}\frac{G_P^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}s$$

$$\begin{aligned}\sigma_P^{\mathrm{CM}}v&=\frac{1}{16\pi}\frac{G_P^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}s\\&=\frac{1}{16\pi}\frac{G_P^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}s\end{aligned}$$

$$\begin{aligned}\sigma_P^{\mathrm{CM}}v&=\frac{1}{16\pi}\frac{G_P^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}s\\&\simeq\frac{1}{16\pi}\frac{G_P^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left(4m_\chi^2+m_\chi^2v^2\right)\\&\simeq\frac{1}{16\pi}\frac{G_P^2}{2}\left(8m_\chi^2\sqrt{1-m_f^2/m_\chi^2}+\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{1}{\sqrt{1-m_f^2/m_\chi^2}}\right)m_\chi^2v^2\right)\\&\simeq\frac{1}{2\pi}\frac{G_P^2}{2}\left(m_\chi^2\sqrt{1-m_f^2/m_\chi^2}+\frac{1}{8}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{1}{\sqrt{1-m_f^2/m_\chi^2}}\right)m_\chi^2v^2\right)\\&=\frac{1}{2\pi}\frac{G_P^2}{2}\left(m_\chi^2\sqrt{1-m_f^2/m_\chi^2}+\frac{3-2\left(m_f^2/m_\chi^2\right)}{8\sqrt{1-m_f^2/m_\chi^2}}m_\chi^2v^2\right)\end{aligned}$$

3. Vector 耦合： $\mathcal{L}=\frac{G_V}{\sqrt{2}}\bar{\chi}\gamma^\mu\chi\bar{f}\gamma_\mu f$

$$i\mathcal{M}=i\frac{G_V}{\sqrt{2}}\bar{u}(k)\gamma^\mu v(k')\bar{v}(p')\gamma_\mu u(p)$$

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2&=\frac{1}{4}\sum_{\text{spins}}\frac{G_V^2}{2}\left[\bar{u}(k)\gamma^\mu v(k')\bar{v}(p')\gamma_\mu u(p)\right]\left[\bar{u}(k)\gamma^\nu v(k')\bar{v}(p')\gamma_\nu u(p)\right]^*\\&=\frac{1}{4}\sum_{\text{spins}}\frac{G_V^2}{2}\left[\bar{u}(k)\gamma^\mu v(k')\bar{v}(p')\gamma_\mu u(p)\right]\left[\bar{v}(k')\gamma^\nu u(k)\bar{u}(p)\gamma_\nu v(p')\right]\\&=\frac{1}{4}\frac{G_V^2}{2}\mathrm{tr}\left[u(k)\bar{u}(k)\gamma^\mu v(k')\bar{v}(k')\gamma^\nu\right]\mathrm{tr}\left[v(p')\bar{v}(p')\gamma_\mu u(p)\bar{u}(p)\gamma_\nu\right]\\&=\frac{1}{4}\frac{G_V^2}{2}\mathrm{tr}\left[\left(\not{k}+m_f\right)\gamma^\mu\left(\not{k}'-m_f\right)\gamma^\nu\right]\mathrm{tr}\left[\left(\not{p}'-m_\chi\right)\gamma_\mu\left(\not{p}+m_\chi\right)\gamma_\nu\right]\\&=\frac{1}{4}\frac{G_V^2}{2}\left[64m_\chi^2m_f^2+32(k\cdot k')m_\chi^2+32(k\cdot p)(k'\cdot p')+32(k\cdot p')(k'\cdot p)+32(p_1\cdot p_2)m_f^2\right]\\&=\frac{1}{4}\frac{G_V^2}{2}\left[64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+16\left(m_\chi^2+m_f^2\right)s+4s^2\right]\\&=\frac{G_V^2}{2}\left[16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+4\left(m_\chi^2+m_f^2\right)s+s^2\right]\end{aligned}$$

$$\begin{aligned}\left(\frac{d\sigma_V}{d\Omega}\right)_{\mathrm{CM}}&=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\mathrm{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\\&=\frac{1}{s}\frac{1}{64\pi^2}\frac{\sqrt{\frac{s}{4}-m_f^2}}{\sqrt{\frac{s}{4}-m_\chi^2}}\frac{G_V^2}{2}\left[16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+4\left(m_\chi^2+m_f^2\right)s+s^2\right]\\&=\frac{1}{64\pi^2}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}+4\left(m_\chi^2+m_f^2\right)+s\right]\end{aligned}$$

$$
\begin{aligned}
\sigma_V^{\text{CM}} &= \int d\Omega \left(\frac{d\sigma_V}{d\Omega}\right)_{\text{CM}} = \int \frac{1}{32\pi}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}+4\left(m_\chi^2+m_f^2\right)+s\right]\sin\theta d\theta \\
&= \frac{1}{32\pi}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{32|\mathbf{p}|^2|\mathbf{k}|^2}{3s}+8\left(m_\chi^2+m_f^2\right)+2s\right] \\
&= \frac{1}{16\pi}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{32\left(\frac{s}{4}-m_\chi^2\right)\left(\frac{s}{4}-m_f^2\right)}{3s}+8\left(m_\chi^2+m_f^2\right)+2s\right] \\
&= \frac{1}{16\pi}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s}+4\left(m_\chi^2+m_f^2\right)+s\right] \\
&= \frac{1}{12\pi}\frac{G_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+2\left(m_\chi^2+m_f^2\right)+4\frac{m_\chi^2 m_f^2}{s}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_V^{\text{CM}} v &= \frac{1}{16\pi}\frac{G_V^2}{2} v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s}+4\left(m_\chi^2+m_f^2\right)+s\right] \\
&\simeq \frac{1}{16\pi}\frac{G_V^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\frac{1}{3}\left(\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}\right)\left(m_\chi^2v^2\right)\left(4m_\chi^2+m_\chi^2v^2-4m_f^2\right)+4\left(m_\chi^2+m_f^2\right)+\left(4m_\chi^2+m_\chi^2v^2\right)\right] \\
&\simeq \frac{1}{16\pi}\frac{G_V^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\left(8m_\chi^2+4m_f^2\right)+\frac{1}{3}\left(1-m_f^2/m_\chi^2\right)m_\chi^2v^2+m_\chi^2v^2\right] \\
&\simeq \frac{1}{16\pi}\frac{G_V^2}{2}\left[\left(8m_\chi^2+4m_f^2\right)\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)+2\sqrt{1-m_f^2/m_\chi^2}\left(\frac{1}{3}\left(1-m_f^2/m_\chi^2\right)m_\chi^2v^2+m_\chi^2v^2\right)\right] \\
&= \frac{1}{16\pi}\frac{G_V^2}{2}\left[\left(8\left(2+m_f^2/m_\chi^2\right)m_\chi^2\sqrt{1-m_f^2/m_\chi^2}+\frac{2+m_f^2/m_\chi^2}{\sqrt{1-m_f^2/m_\chi^2}}m_\chi^2v^2\right)+\frac{2}{3}\sqrt{1-m_f^2/m_\chi^2}\left(4-m_f^2/m_\chi^2\right)m_\chi^2v^2\right] \\
&= \frac{1}{16\pi}\frac{G_V^2}{2}\left\{8\left(2+m_f^2/m_\chi^2\right)m_\chi^2\sqrt{1-m_f^2/m_\chi^2}+\left[\frac{2+m_f^2/m_\chi^2}{\sqrt{1-m_f^2/m_\chi^2}}+\frac{2}{3}\sqrt{1-m_f^2/m_\chi^2}\left(4-m_f^2/m_\chi^2\right)\right]m_\chi^2v^2\right\} \\
&= \frac{1}{2\pi}\frac{G_V^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[\left(2+m_f^2/m_\chi^2\right)+\frac{1}{8}\frac{2+m_f^2/m_\chi^2}{1-m_f^2/m_\chi^2}v^2+\frac{1}{12}\left(4-m_f^2/m_\chi^2\right)v^2\right] \\
&= \frac{1}{2\pi}\frac{G_V^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[\left(2+m_f^2/m_\chi^2\right)+\frac{1}{24}\frac{14-7m_f^2/m_\chi^2+2\left(m_f^2/m_\chi^2\right)^2}{1-m_f^2/m_\chi^2}v^2\right]
\end{aligned}
$$

4. Axial Vector 耦合： $\mathcal{L}=\frac{G_A}{\sqrt{2}}\bar{\chi}\gamma^\mu\gamma^5\chi\bar{f}\gamma_\mu\gamma_5 f$

$$
i\mathcal{M}=i\frac{G_A}{\sqrt{2}}\bar{u}(k)\gamma^\mu\gamma^5 v(k')\bar{v}(p')\gamma_\mu\gamma_5 u(p)
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\frac{G_A^2}{2}\left[\bar{u}(k)\gamma^\mu\gamma^5 v(k')\bar{v}(p')\gamma_\mu\gamma_5 u(p)\right]\left[\bar{u}(k)\gamma^\nu\gamma^5 v(k')\bar{v}(p')\gamma_\nu\gamma_5 u(p)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\frac{G_A^2}{2}\left[\bar{u}(k)\gamma^\mu\gamma^5 v(k')\bar{v}(p')\gamma_\mu\gamma_5 u(p)\right]\left[\bar{v}(k')\gamma^\nu\gamma^5 u(k)\bar{u}(p)\gamma_\nu\gamma_5 v(p')\right] \\
&= \frac{1}{4}\frac{G_A^2}{2}\text{tr}\left[u(k)\bar{u}(k)\gamma^\mu\gamma^5 v(k')\bar{v}(k')\gamma^\nu\gamma^5\right]\text{tr}\left[v(p')\bar{v}(p')\gamma_\mu\gamma_5 u(p)\bar{u}(p)\gamma_\nu\gamma_5\right] \\
&= \frac{1}{4}\frac{G_A^2}{2}\text{tr}\left[\left(\not{k}+m_f\right)\gamma^\mu\gamma^5\left(\not{k}'-m_f\right)\gamma^\nu\gamma^5\right]\text{tr}\left[\left(\not{p}'-m_\chi\right)\gamma_\mu\gamma_5\left(\not{p}+m_\chi\right)\gamma_\nu\gamma_5\right] \\
&= \frac{1}{4}\frac{G_A^2}{2}\left[64m_\chi^2m_f^2-32(k\cdot k')m_\chi^2+32(k\cdot p)(k'\cdot p')+32(k\cdot p')(k'\cdot p)-32(p\cdot p')m_f^2\right] \\
&= \frac{1}{4}\frac{G_A^2}{2}\left[128m_\chi^2m_f^2+64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16\left(m_\chi^2+m_f^2\right)s+4s^2\right] \\
&= \frac{G_A^2}{2}\left[32m_\chi^2m_f^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-4\left(m_\chi^2+m_f^2\right)s+s^2\right]
\end{aligned}
$$

$$\left(\frac{d\sigma_A}{d\Omega}\right)_{\mathrm{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\mathrm{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\frac{\sqrt{\frac{s}{4}-m_f^2}}{\sqrt{\frac{s}{4}-m_\chi^2}}\frac{G_A^2}{2}\left[32m_\chi^2m_f^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-4\left(m_\chi^2+m_f^2\right)s+s^2\right]$$

$$=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_A^2}{2}\left[\frac{32m_\chi^2m_f^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}-4\left(m_\chi^2+m_f^2\right)+s\right]$$

$$\sigma_A^{\mathrm{CM}}=\int d\Omega\left(\frac{d\sigma_A}{d\Omega}\right)_{\mathrm{CM}}=\int\frac{1}{32\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_A^2}{2}\left[\frac{32m_\chi^2m_f^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}-4\left(m_\chi^2+m_f^2\right)+s\right]\sin\theta d\theta$$

$$=\frac{1}{32\pi}\frac{G_A^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{64m_\chi^2m_f^2}{s}+\frac{32|\mathbf{p}|^2|\mathbf{k}|^2}{3s}-8\left(m_\chi^2+m_f^2\right)+2s\right]$$

$$=\frac{1}{32\pi}\frac{G_A^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{64m_\chi^2m_f^2}{s}+\frac{32\left(\frac{s}{4}-m_\chi^2\right)\left(\frac{s}{4}-m_f^2\right)}{3s}-8\left(m_\chi^2+m_f^2\right)+2s\right]$$

$$=\frac{1}{16\pi}\frac{G_A^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{96m_\chi^2m_f^2+\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s}-4\left(m_\chi^2+m_f^2\right)+s\right]$$

$$=\frac{1}{12\pi}\frac{G_A^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-4\left(m_\chi^2+m_f^2\right)+28\frac{m_\chi^2m_f^2}{s}\right]$$

$$\sigma_A^{\mathrm{CM}}v=\frac{1}{16\pi}\frac{G_A^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[\frac{96m_\chi^2m_f^2+\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s}-4\left(m_\chi^2+m_f^2\right)+s\right]$$

$$\simeq\frac{1}{16\pi}\frac{G_A^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[\frac{1}{3}\left(\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}\right)\left[96m_\chi^2m_f^2+m_\chi^2v^2\left(4m_\chi^2+m_\chi^2v^2-4m_f^2\right)\right]-4m_f^2+m_\chi^2v^2\right]$$

$$\simeq\frac{1}{16\pi}\frac{G_A^2}{2}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left[4m_f^2+\frac{1}{3}\left(4-7m_f^2/m_\chi^2\right)m_\chi^2v^2\right]$$

$$\simeq\frac{1}{16\pi}\frac{G_A^2}{2}\left[8m_f^2\sqrt{1-m_f^2/m_\chi^2}+m_f^2\frac{v^2}{\sqrt{1-m_f^2/m_\chi^2}}+\frac{2}{3}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left(4-7m_f^2/m_\chi^2\right)v^2\right]$$

$$=\frac{1}{2\pi}\frac{G_A^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left\{\left(m_f^2/m_\chi^2\right)+\frac{1}{8}\left[\frac{m_f^2/m_\chi^2}{1-m_f^2/m_\chi^2}+\frac{2}{3}\left(4-7m_f^2/m_\chi^2\right)\right]v^2\right\}$$

$$=\frac{1}{2\pi}\frac{G_A^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[\left(m_f^2/m_\chi^2\right)+\frac{1}{24}\frac{8-19m_f^2/m_\chi^2+14\left(m_f^2/m_\chi^2\right)^2}{1-m_f^2/m_\chi^2}v^2\right]$$

$$=\frac{1}{2\pi}\frac{G_A^2}{2}m_f^2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{1}{24}\frac{8m_\chi^2/m_f^2-19+14\left(m_f^2/m_\chi^2\right)}{1-m_f^2/m_\chi^2}v^2\right]$$

5. Tensor 耦合： $\mathcal{L}=\frac{G_T}{\sqrt{2}}\bar{\chi}\sigma^{\mu\nu}\chi\bar{f}\sigma_{\mu\nu}f$

$$\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$$

$$\left[\bar{u}(k)\sigma^{\mu\nu}v(k')\right]^*=\left[u^\dagger(k)\gamma^0\frac{i}{2}\left(\gamma^\mu\gamma^\nu-\gamma^\mu\gamma^\nu\right)v(k')\right]^*$$

$$=-\frac{i}{2}\left[v^\dagger(k')\left(\gamma^{\nu\dagger}\gamma^{\mu\dagger}-\gamma^{\nu\dagger}\gamma^{\mu\dagger}\right)\gamma^0u(k)\right]$$

$$=-\frac{i}{2}\left[v^\dagger(k')\gamma^0\left(\gamma^\nu\gamma^\mu-\gamma^\nu\gamma^\mu\right)u(k)\right]$$

$$=-\left[\bar{v}(k')\frac{i}{2}\left(\gamma^\nu\gamma^\mu-\gamma^\nu\gamma^\mu\right)u(k)\right]$$

$$=-\left[\bar{v}(k')\sigma^{\nu\mu}u(k)\right]=\bar{v}(k')\sigma^{\mu\nu}u(k)$$

$$i\mathcal{M} = i\frac{G_T}{\sqrt{2}}\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)$$

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\frac{G_T^2}{2}\left[\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)\right]\left[\bar{u}(k)\sigma^{\rho\sigma}v(k')\bar{v}(p')\sigma_{\rho\sigma}u(p)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\frac{G_T^2}{2}\left[\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)\right]\left[\bar{v}(k')\sigma^{\rho\sigma}u(k)\bar{u}(p)\sigma_{\rho\sigma}v(p')\right] \\
&= \frac{1}{4}\frac{G_T^2}{2}\mathrm{tr}\left[u(k)\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(k')\sigma^{\rho\sigma}\right]\mathrm{tr}\left[v(p')\bar{v}(p')\sigma_{\mu\nu}u(p)\bar{u}(p)\sigma_{\rho\sigma}\right] \\
&= \frac{1}{4}\frac{G_T^2}{2}\mathrm{tr}\left[\left(\not{k}+m_f\right)\sigma^{\mu\nu}\left(\not{k}'-m_f\right)\sigma^{\rho\sigma}\right]\mathrm{tr}\left[\left(\not{p}'-m_\chi\right)\sigma_{\mu\nu}\left(\not{p}+m_\chi\right)\sigma_{\rho\sigma}\right] \\
&= \frac{1}{4}\frac{G_T^2}{2}\mathrm{tr}\left[\left(\not{k}+m_f\right)\frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]\left(\not{k}'-m_f\right)\frac{1}{2}\left[\gamma^\rho,\gamma^\sigma\right]\right]\mathrm{tr}\left[\left(\not{p}'-m_\chi\right)\frac{1}{2}\left[\gamma_\mu,\gamma_\nu\right]\left(\not{p}+m_\chi\right)\frac{1}{2}\left[\gamma_\rho,\gamma_\sigma\right]\right] \\
&= \frac{1}{4}\frac{G_T^2}{2}\left[384m_\chi^2m_f^2 - 128(k\cdot k')(p\cdot p') + 256(k\cdot p)(k'\cdot p') + 256(k\cdot p')(k'\cdot p)\right] \\
&= \frac{1}{4}\frac{G_T^2}{2}\left[256m_\chi^2m_f^2 + 512\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 64\left(m_\chi^2+m_f^2\right)s\right] \\
&= \frac{G_T^2}{2}\left[64m_\chi^2m_f^2 + 128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 16\left(m_\chi^2+m_f^2\right)s\right]
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d\sigma_T}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{s}\frac{1}{64\pi^2}\frac{\sqrt{\frac{s}{4}-m_f^2}}{\sqrt{\frac{s}{4}-m_\chi^2}}\frac{G_T^2}{2}\left[64m_\chi^2m_f^2 + 128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 16\left(m_\chi^2+m_f^2\right)s\right] \\
&= \frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{64m_\chi^2m_f^2 + 128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s} + 16\left(m_\chi^2+m_f^2\right)\right]
\end{aligned}$$

$$\begin{aligned}
\sigma_T^{\text{CM}} = \int d\Omega\left(\frac{d\sigma_T}{d\Omega}\right)_{\text{CM}} &= \int\frac{1}{32\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{64m_\chi^2m_f^2 + 128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s} + 16\left(m_\chi^2+m_f^2\right)\right]\sin\theta d\theta \\
&= \frac{1}{32\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{384m_\chi^2m_f^2 + 256|\mathbf{p}|^2|\mathbf{k}|^2}{3s} + 32\left(m_\chi^2+m_f^2\right)\right] \\
&= \frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{192m_\chi^2m_f^2 + 128\left(\frac{s}{4}-m_\chi^2\right)\left(\frac{s}{4}-m_f^2\right)}{3s} + 16\left(m_\chi^2+m_f^2\right)\right] \\
&= \frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{192m_\chi^2m_f^2 + 8\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s} + 16\left(m_\chi^2+m_f^2\right)\right] \\
&= \frac{1}{6\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[s + 2\left(m_\chi^2+m_f^2\right) + 40\frac{m_\chi^2m_f^2}{s}\right]
\end{aligned}$$

$$\sigma_T^{\rm CM}v=\frac{1}{16\pi}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\frac{G_T^2}{2}\left[\frac{192m_\chi^2m_f^2+8\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)}{3s}+16\left(m_\chi^2+m_f^2\right)\right]$$

$$=\frac{1}{16\pi}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\frac{G_T^2}{2}\left\{\frac{1}{3}\left(\frac{1}{4m_\chi^2}-\frac{v^2}{16m_\chi^2}\right)\left[192m_\chi^2m_f^2+8m_\chi^2v^2\left(4m_\chi^2-4m_f^2+m_\chi^2v^2\right)\right]+16\left(m_\chi^2+m_f^2\right)\right\}$$

$$\simeq\frac{1}{16\pi}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\frac{G_T^2}{2}\left\{16\left(m_\chi^2+2m_f^2\right)-\frac{1}{3}12m_f^2v^2+\frac{1}{3}8m_\chi^2v^2\left(1-m_f^2/m_\chi^2\right)\right\}$$

$$=\frac{1}{16\pi}\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\frac{G_T^2}{2}\left\{16\left(m_\chi^2+2m_f^2\right)+\frac{4}{3}\left(2-5m_f^2/m_\chi^2\right)m_\chi^2v^2\right\}$$

$$\simeq\frac{1}{16\pi}\frac{G_T^2}{2}\left\{32\left(m_\chi^2+2m_f^2\right)\sqrt{1-m_f^2/m_\chi^2}+4\left(m_\chi^2+2m_f^2\right)\frac{v^2}{\sqrt{1-m_f^2/m_\chi^2}}+\frac{8}{3}\left(\sqrt{1-m_f^2/m_\chi^2}\right)\left(2-5m_f^2/m_\chi^2\right)m_\chi^2v^2\right\}$$

$$=\frac{1}{2\pi}\frac{G_T^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left\{4\left(1+2m_f^2/m_\chi^2\right)+\frac{1}{2}\frac{1+2m_f^2/m_\chi^2}{1-m_f^2/m_\chi^2}v^2+\frac{1}{3}\left(2-5m_f^2/m_\chi^2\right)v^2\right\}$$

$$=\frac{1}{2\pi}\frac{G_T^2}{2}m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[4\left(1+2m_f^2/m_\chi^2\right)+\frac{1}{6}\frac{7-8m_f^2/m_\chi^2+10\left(m_f^2/m_\chi^2\right)^2}{1-m_f^2/m_\chi^2}v^2\right]$$

6. Scalar-Pseudoscalar 耦合： $\mathcal{L}=\frac{G_{SP}}{\sqrt{2}}\bar{\chi}\chi\bar{f}i\gamma_5 f$

$$i\mathcal{M}=i\frac{G_{SP}}{\sqrt{2}}\bar{u}(k)i\gamma_5 v(k')\bar{v}(p')u(p)$$

$$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2=\frac{1}{4}\sum_{\rm spins}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_5v(k')\bar{v}(p')u(p)\right]\left[\bar{u}(k)\gamma_5v(k')\bar{v}(p')u(p)\right]^*$$

$$=-\frac{1}{4}\sum_{\rm spins}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_5v(k')\bar{v}(p')u(p)\right]\left[\bar{v}(k')\gamma_5u(k)\bar{u}(p)v(p')\right]$$

$$=-\frac{1}{4}\sum_{\rm spins}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_5v(k')\bar{v}(k')\gamma_5u(k)\right]\left[\bar{v}(p')u(p)\bar{u}(p)v(p')\right]$$

$$=-\frac{1}{4}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2{\rm tr}\left[u(k)\bar{u}(k)\gamma_5v(k')\bar{v}(k')\gamma_5\right]{\rm tr}\left[v(p')\bar{v}(p')u(p)\bar{u}(p)\right]$$

$$=-\frac{1}{4}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2{\rm tr}\left[\left(\not{k}+m_f\right)\gamma_5\left(\not{k}'-m_f\right)\gamma_5\right]{\rm tr}\left[\left(\not{p}'-m_\chi\right)\left(\not{p}+m_\chi\right)\right]$$

$$=\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(s^2-4m_\chi^2s\right)$$

$$\left(\frac{d\sigma_{SP}}{d\Omega}\right)_{\rm CM}=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\rm cm}}\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\frac{\sqrt{\frac{s}{4}-m_f^2}}{\sqrt{\frac{s}{4}-m_\chi^2}}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(s^2-4m_\chi^2s\right)$$

$$=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(s-4m_\chi^2\right)$$

$$\sigma_{SP}^{\rm CM}=\int d\Omega\left(\frac{d\sigma_{SP}}{d\Omega}\right)_{\rm CM}=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(s-4m_\chi^2\right)$$

$$\sigma_{SP}^{\rm CM}v=\frac{1}{16\pi}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(s-4m_\chi^2\right)$$

$$\simeq\frac{1}{16\pi}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2\left(2\sqrt{1-m_f^2/m_\chi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\chi^2}}\right)\left(m_\chi^2v^2\right)$$

$$\simeq\frac{1}{8\pi}\left(\frac{G_{SP}}{\sqrt{2}}\right)^2m_\chi^2\sqrt{1-m_f^2/m_\chi^2}v^2$$

7. Pseudoscalar-Scalar 耦合： $\mathcal{L}=\frac{G_{PS}}{\sqrt{2}}\bar{\chi}i\gamma_5\chi\bar{f}f$

$$i\mathcal{M}=i\frac{G_{PS}}{\sqrt{2}}\bar{u}(k)v(k')\bar{v}(p')i\gamma_5 u(p)$$

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2&=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left[\bar{u}(k)v(k')\bar{v}(p')\gamma_5 u(p)\right]\left[\bar{u}(k)v(k')\bar{v}(p')\gamma_5 u(p)\right]^*\\
&=-\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left[\bar{u}(k)v(k')\bar{v}(p')\gamma_5 u(p)\right]\left[\bar{v}(k')u(k)\bar{u}(p)\gamma_5 v(p')\right]\\
&=-\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left[\bar{u}(k)v(k')\bar{v}(k')u(k)\right]\left[\bar{v}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5 v(p')\right]\\
&=-\frac{1}{4}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(k)\bar{u}(k)v(k')\bar{v}(k')\right]\operatorname{tr}\left[v(p')\bar{v}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5\right]\\
&=-\frac{1}{4}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{k}+m_f\right)\left(\not{k}'-m_f\right)\right]\operatorname{tr}\left[\left(\not{p}'-m_\chi\right)\gamma_5\left(\not{p}+m_\chi\right)\gamma_5\right]\\
&=\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(s^2-4m_f^2 s\right)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d\sigma_{PS}}{d\Omega}\right)_{\text{CM}}&=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\\
&=\frac{1}{s}\frac{1}{64\pi^2}\frac{\sqrt{\frac{s}{4}-m_f^2}}{\sqrt{\frac{s}{4}-m_\chi^2}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(s^2-4m_f^2 s\right)\\
&=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(s-4m_f^2\right)
\end{aligned}$$

$$\sigma_{PS}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{PS}}{d\Omega}\right)_{\text{CM}}=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(s-4m_f^2\right)$$

$$\begin{aligned}
\sigma_{PS}^{\text{CM}}v&=\frac{1}{16\pi}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(s-4m_f^2\right)\\
&\simeq\frac{1}{16\pi}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left(1+\frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)}\right)\left(4m_\chi^2-4m_f^2+m_\chi^2 v^2\right)\\
&\simeq\frac{1}{16\pi}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left(4\left(m_\chi^2-m_f^2\right)+m_\chi^2 v^2+\frac{m_\chi^2-m_f^2}{2\left(1-m_f^2/m_\chi^2\right)}v^2\right)\\
&=\frac{1}{16\pi}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(m_\chi^2-m_f^2\right)\sqrt{1-m_f^2/m_\chi^2}\left(8+\frac{3}{\left(1-m_f^2/m_\chi^2\right)}v^2\right)\\
&=\frac{1}{2\pi}\left(\frac{G_{PS}}{\sqrt{2}}\right)^2\left(m_\chi^2-m_f^2\right)\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{3}{8\left(1-m_f^2/m_\chi^2\right)}v^2\right]
\end{aligned}$$

8. Vector-Axial Vector 耦合： $\mathcal{L}=\frac{G_{VA}}{\sqrt{2}}\bar{\chi}\gamma^\mu\chi\bar{f}\gamma_\mu\gamma_5 f$

$$i\mathcal{M}=i\frac{G_{VA}}{\sqrt{2}}\bar{u}(k)\gamma_\mu\gamma_5 v(k')\bar{v}(p')\gamma^\mu u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\gamma_5 v(k')\bar{v}(p')\gamma^\mu u(p)\right]\left[\bar{u}(k)\gamma_\nu\gamma_5 v(k')\bar{v}(p')\gamma^\nu u(p)\right]^*$$
$$=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\gamma_5 v(k')\bar{v}(p')\gamma^\mu u(p)\right]\left[\bar{u}(p)\gamma^\nu v(p')\bar{v}(k')\gamma_\nu\gamma_5 u(k)\right]$$
$$=\frac{1}{4}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(k)\bar{u}(k)\gamma_\mu\gamma_5 v(k')\bar{v}(k')\gamma_\nu\gamma_5\right]\operatorname{tr}\left[v(p')\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu\right]$$
$$=\frac{1}{4}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{k}+m_f\right)\gamma_\mu\gamma_5\left(\not{k}'-m_f\right)\gamma_\nu\gamma_5\right]\operatorname{tr}\left[\left(\not{p}'-m_\chi\right)\gamma^\mu\left(\not{p}+m_\chi\right)\gamma^\nu\right]$$
$$=\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s^2-4m_f^2 s+4m_\chi^2 s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16m_\chi^2 m_f^2\right]$$

$$\left(\frac{d\sigma_{VA}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s^2-4m_f^2 s+4m_\chi^2 s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16m_\chi^2 m_f^2\right]$$
$$=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s-4m_f^2+4m_\chi^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\frac{1}{s}-16m_\chi^2 m_f^2\frac{1}{s}\right]$$

$$\sigma_{VA}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{VA}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{32\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s-4m_f^2+4m_\chi^2+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\frac{1}{s}-16m_\chi^2 m_f^2\frac{1}{s}\right]\sin\theta d\theta$$
$$=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s-4m_f^2+4m_\chi^2+\frac{16}{3}|\mathbf{p}|^2|\mathbf{k}|^2\frac{1}{s}-16m_\chi^2 m_f^2\frac{1}{s}\right]$$
$$=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s-4m_f^2+4m_\chi^2+\frac{1}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)-\frac{16}{s}m_\chi^2 m_f^2\right]$$
$$=\frac{1}{12\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s+2\left(m_\chi^2-2m_f^2\right)-8\frac{m_\chi^2 m_f^2}{s}\right]$$

$$\sigma_{VA}^{\text{CM}}v=\frac{1}{12\pi}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left[s+2\left(m_\chi^2-2m_f^2\right)-8\frac{m_\chi^2 m_f^2}{s}\right]$$
$$\simeq\frac{1}{12\pi}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)}\right]\left[6m_\chi^2-6m_f^2+\left(\frac{1}{2}m_f^2+m_\chi^2\right)v^2\right]$$
$$\simeq\frac{1}{12\pi}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left[6\left(m_\chi^2-m_f^2\right)+\frac{3\left(m_\chi^2-m_f^2\right)v^2}{4\left(1-m_f^2/m_\chi^2\right)}+\left(\frac{1}{2}m_f^2+m_\chi^2\right)v^2\right]$$
$$=\frac{1}{\pi}\left(\frac{G_{VA}}{\sqrt{2}}\right)^2\left(m_\chi^2-m_f^2\right)\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{7+2m_f^2/m_\chi^2}{24\left(1-m_f^2/m_\chi^2\right)}v^2\right]$$

9. Axial Vector-Vector 耦合： $\mathcal{L}=\frac{G_{AV}}{\sqrt{2}}\bar{\chi}\gamma^\mu\gamma_5\chi\bar{f}\gamma_\mu f$

$$i\mathcal{M}=i\frac{G_{AV}}{\sqrt{2}}\bar{u}(k)\gamma_\mu v(k')\bar{v}(p')\gamma^\mu\gamma_5 u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu v(k')\bar{v}(p')\gamma^\mu\gamma_5 u(p)\right]\left[\bar{u}(k)\gamma_\nu v(k')\bar{v}(p')\gamma^\nu\gamma_5 u(p)\right]^*$$
$$=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu v(k')\bar{v}(p')\gamma^\mu\gamma_5 u(p)\right]\left[\bar{u}(p)\gamma^\nu\gamma_5 v(p')\bar{v}(k')\gamma_\nu u(k)\right]$$
$$=\frac{1}{4}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(k)\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu\right]\operatorname{tr}\left[v(p')\bar{v}(p')\gamma^\mu\gamma_5 u(p)\bar{u}(p)\gamma^\nu\gamma_5\right]$$
$$=\frac{1}{4}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{k}+m_f\right)\gamma_\mu\left(\not{k}'-m_f\right)\gamma_\nu\right]\operatorname{tr}\left[\left(\not{p}'-m_\chi\right)\gamma^\mu\gamma_5\left(\not{p}+m_\chi\right)\gamma^\nu\gamma_5\right]$$
$$=\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s^2+4m_f^2 s-4m_\chi^2 s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16m_\chi^2 m_f^2\right]$$

$$\left(\frac{d\sigma_{AV}}{d\Omega}\right)_{\mathrm{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\mathrm{cm}}}\frac{1}{4}\sum_{\mathrm{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s^2+4m_f^2 s-4m_\chi^2 s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16m_\chi^2 m_f^2\right]$$

$$=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+4m_f^2-4m_\chi^2+16\frac{1}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16\frac{m_\chi^2 m_f^2}{s}\right]$$

$$\sigma_{AV}^{\mathrm{CM}}=\int d\Omega\left(\frac{d\sigma_{AV}}{d\Omega}\right)_{\mathrm{CM}}=\int\frac{1}{32\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+4m_f^2-4m_\chi^2+16\frac{1}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-16\frac{m_\chi^2 m_f^2}{s}\right]\sin\theta d\theta$$

$$=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+4m_f^2-4m_\chi^2+\frac{16}{3}\frac{1}{s}|\mathbf{p}|^2|\mathbf{k}|^2-16\frac{m_\chi^2 m_f^2}{s}\right]$$

$$=\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+4m_f^2-4m_\chi^2+\frac{1}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)-16\frac{m_\chi^2 m_f^2}{s}\right]$$

$$=\frac{1}{12\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+2\left(m_f^2-2m_\chi^2\right)-8\frac{m_\chi^2 m_f^2}{s}\right]$$

$$\sigma_{AV}^{\mathrm{CM}}v=\frac{1}{12\pi}v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\left[s+2\left(m_f^2-2m_\chi^2\right)-8\frac{m_\chi^2 m_f^2}{s}\right]$$

$$\simeq\frac{1}{12\pi}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)}\right]\left[4m_\chi^2+m_\chi^2 v^2+2\left(m_f^2-2m_\chi^2\right)-8m_\chi^2 m_f^2\frac{1}{4m_\chi^2}\left(1-\frac{v^2}{4}\right)\right]$$

$$\simeq\frac{1}{6\pi}\left(\frac{G_{AV}}{\sqrt{2}}\right)^2\sqrt{1-m_f^2/m_\chi^2}\left(m_\chi^2+\frac{1}{2}m_f^2\right)v^2$$

10. Alternative Tensor 耦合： $\mathcal{L}=\frac{\tilde{G}_T}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\bar{\chi}\sigma_{\mu\nu}\chi\bar{f}\sigma_{\rho\sigma}f$

$$i\mathcal{M}=i\frac{\tilde{G}_T}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\bar{u}(k)\sigma_{\rho\sigma}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)$$

$$\frac{1}{4}\sum_{\mathrm{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\mathrm{spins}}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\left[\varepsilon^{\mu\nu\rho\sigma}\bar{u}(k)\sigma_{\rho\sigma}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)\right]\left[\varepsilon^{\alpha\beta\gamma\delta}\bar{u}(k)\sigma_{\gamma\delta}v(k')\bar{v}(p')\sigma_{\alpha\beta}u(p)\right]^*$$

$$=\frac{1}{4}\sum_{\mathrm{spins}}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\left[\varepsilon^{\mu\nu\rho\sigma}\bar{u}(k)\sigma_{\rho\sigma}v(k')\bar{v}(p')\sigma_{\mu\nu}u(p)\right]\left[\varepsilon^{\alpha\beta\gamma\delta}\bar{u}(p)\sigma_{\alpha\beta}v(p')\bar{v}(k')\sigma_{\gamma\delta}u(k)\right]$$

$$=\frac{1}{4}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\mathrm{tr}\left[\varepsilon^{\mu\nu\rho\sigma}u(k)\bar{u}(k)\sigma_{\rho\sigma}v(k')\bar{v}(k')\sigma_{\gamma\delta}\right]\mathrm{tr}\left[\varepsilon^{\alpha\beta\gamma\delta}v(p')\bar{v}(p')\sigma_{\mu\nu}u(p)\bar{u}(p)\sigma_{\alpha\beta}\right]$$

$$=\frac{1}{4}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\mathrm{tr}\left[\varepsilon^{\mu\nu\rho\sigma}\left(\not{k}+m_f\right)\sigma_{\rho\sigma}\left(\not{k}'-m_f\right)\sigma_{\gamma\delta}\right]\mathrm{tr}\left[\varepsilon^{\alpha\beta\gamma\delta}\left(\not{p}'-m_\chi\right)\sigma_{\mu\nu}\left(\not{p}+m_\chi\right)\sigma_{\alpha\beta}\right]$$

$$=\frac{1}{4}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\mathrm{tr}\left[\left(\not{k}+m_f\right)\frac{1}{2}\left[\gamma_\rho,\gamma_\sigma\right]\left(\not{k}'-m_f\right)\frac{1}{2}\left[\gamma_\gamma,\gamma_\delta\right]\right]\mathrm{tr}\left[\left(\not{p}'-m_\chi\right)\frac{1}{2}\left[\gamma_\mu,\gamma_\nu\right]\left(\not{p}+m_\chi\right)\frac{1}{2}\left[\gamma_\alpha,\gamma_\beta\right]\right]$$

$$=-\frac{1}{4}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2 2\left[1024m_\chi^2 m_f^2-1024\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-128\left(m_\chi^2+m_f^2\right)s\right]$$

$$=\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\left[64\left(m_\chi^2+m_f^2\right)s+512\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-512m_\chi^2 m_f^2\right]$$

$$\left(\frac{d\sigma_{\tilde{T}}}{d\Omega}\right)_{\mathrm{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\mathrm{cm}}}\frac{1}{4}\sum_{\mathrm{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\left[64\left(m_\chi^2+m_f^2\right)s+512\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-512m_\chi^2 m_f^2\right]$$

$$=\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{\tilde{G}_T}{\sqrt{2}}\right)^2\left[64\left(m_\chi^2+m_f^2\right)+\frac{512}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-512\frac{m_\chi^2 m_f^2}{s}\right]$$

$$
\begin{aligned}
\sigma_{\tilde{T}}^{\mathrm{CM}} &= \int d\Omega \left(\frac{d\sigma_{\tilde{T}}}{d\Omega} \right)_{\mathrm{CM}} = \int \frac{1}{32\pi} \sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 \left[64\left(m_\chi^2+m_f^2\right) + \frac{512}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 - 512\frac{m_\chi^2 m_f^2}{s} \right] \sin\theta d\theta \\
&= \frac{1}{16\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 \sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}} \left[64\left(m_\chi^2+m_f^2\right) + \frac{512}{3s}|\mathbf{p}|^2|\mathbf{k}|^2 - 512\frac{m_\chi^2 m_f^2}{s} \right] \\
&= \frac{1}{16\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 \sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}} \left[64\left(m_\chi^2+m_f^2\right) + \frac{32}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right) - 512\frac{m_\chi^2 m_f^2}{s} \right] \\
&= \frac{2}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 \sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}} \left[s + 2\left(m_\chi^2+m_f^2\right) - 32\frac{m_\chi^2 m_f^2}{s} \right]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\tilde{T}}^{\mathrm{CM}} v &= \frac{2}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 v \sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}} \left[s + 2\left(m_\chi^2+m_f^2\right) - 32\frac{m_\chi^2 m_f^2}{s} \right] \\
&\simeq \frac{2}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 2\sqrt{1-m_f^2/m_\chi^2} \left[1 + \frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)} \right] \left[4m_\chi^2 + m_\chi^2 v^2 + 2\left(m_\chi^2+m_f^2\right) - 32\frac{1}{4m_\chi^2}\left(1-\frac{v^2}{4}\right) m_\chi^2 m_f^2 \right] \\
&= \frac{2}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 2\sqrt{1-m_f^2/m_\chi^2} \left[1 + \frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)} \right] \left[6\left(m_\chi^2-m_f^2\right) + \left(2m_f^2+m_\chi^2\right)v^2 \right] \\
&\simeq \frac{2}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 2\sqrt{1-m_f^2/m_\chi^2} \left[6\left(m_\chi^2-m_f^2\right) + \frac{3\left(m_\chi^2-m_f^2\right)v^2}{4\left(1-m_f^2/m_\chi^2\right)} + \left(2m_f^2+m_\chi^2\right)v^2 \right] \\
&= \frac{4}{3\pi} \left(\frac{\tilde{G}_T}{\sqrt{2}} \right)^2 \sqrt{1-m_f^2/m_\chi^2}\left(m_\chi^2-m_f^2\right) \left[6 + \frac{7+8m_f^2/m_\chi^2}{4\left(1-m_f^2/m_\chi^2\right)} v^2 \right]
\end{aligned}
$$

11. 左手-左手（L-L）耦合：$\mathcal{L} = \frac{G_{LL}}{\sqrt{2}} \bar{\chi}\gamma^\mu\left(1-\gamma_5\right)\chi \bar{f}\gamma_\mu\left(1-\gamma_5\right) f$

$$
i\mathcal{M} = i\frac{G_{LL}}{\sqrt{2}} \bar{u}(k)\gamma_\mu\left(1-\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)
$$

$$
\begin{aligned}
&\left[\bar{u}(k)\gamma^\mu\left(1-\gamma_5\right)v(k')\right]^* = \left[u^\dagger(k)\gamma^0\gamma^\mu\left(1-\gamma_5\right)v(k')\right]^* \\
&= v^\dagger(k')\left(1-\gamma_5\right)^\dagger \gamma^{\mu\dagger}\gamma^0 u(k) = v^{\square}(k')\left(1-\gamma_5\right)\gamma\ \gamma^\mu u(k) \\
&= v^\dagger(k')\gamma^0\left(1+\gamma_5\right)\gamma^\mu u(k) = \bar{v}(k')\left(1+\gamma_5\right)\gamma^\mu u(k) \\
&= \bar{v}(k')\gamma^\mu\left(1-\gamma_5\right)u(k)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 \left[\bar{u}(k)\gamma_\mu\left(1-\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\right]\left[\bar{u}(k)\gamma_\nu\left(1-\gamma_5\right)v(k')\bar{v}(p')\gamma^\nu\left(1-\gamma_5\right)u(p)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 \left[\bar{u}(k)\gamma_\mu\left(1-\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\right]\left[\bar{u}(p)\gamma^\nu\left(1-\gamma_5\right)v(p')\bar{v}(k')\gamma_\nu\left(1-\gamma_5\right)u(k)\right] \\
&= \frac{1}{4}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 \operatorname{tr}\left[u(k)\bar{u}(k)\gamma_\mu\left(1-\gamma_5\right)v(k')\bar{v}(k')\gamma_\nu\left(1-\gamma_5\right)\right]\operatorname{tr}\left[v(p')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\bar{u}(p)\gamma^\nu\left(1-\gamma_5\right)\right] \\
&= \frac{1}{4}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 \operatorname{tr}\left[\left(\not{k}+m_f\right)\gamma_\mu\left(1-\gamma_5\right)\left(\not{k}'-m_f\right)\gamma_\nu\left(1-\gamma_5\right)\right]\operatorname{tr}\left[\left(\not{p}'-m_\chi\right)\gamma^\mu\left(1-\gamma_5\right)\left(\not{p}+m_\chi\right)\gamma^\nu\left(1-\gamma_5\right)\right] \\
&= \frac{1}{4}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 2\left[8s^2 + 64s|\mathbf{p}||\mathbf{k}|\cos\theta + 128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right] \\
&= \left(\frac{G_{LL}}{\sqrt{2}}\right)^2 4\left[s^2 + 8s|\mathbf{p}||\mathbf{k}|\cos\theta + 16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\sigma_{LL}}{d\Omega}\right)_{\mathrm{CM}} &= \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\mathrm{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 4\left[s^2 + 8s|\mathbf{p}||\mathbf{k}|\cos\theta + 16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right] \\
&= \frac{1}{16\pi^2}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s + 8|\mathbf{p}||\mathbf{k}|\cos\theta + \frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]
\end{aligned}
$$

$$\sigma_{LL}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{LL}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{8\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]\sin\theta d\theta$$

$$=\frac{1}{8\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{16}{3s}|\mathbf{p}|^2|\mathbf{k}|^2\right]$$

$$=\frac{1}{8\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{1}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)\right]$$

$$=\frac{1}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-\left(m_\chi^2+m_f^2\right)+4\frac{m_\chi^2m_f^2}{s}\right]$$

$$\sigma_{LL}^{\text{CM}}v=\frac{1}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2v\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-\left(m_\chi^2+m_f^2\right)+4\frac{m_\chi^2m_f^2}{s}\right]$$

$$\simeq\frac{1}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)}\right]\left[4m_\chi^2+m_\chi^2v^2-\left(m_\chi^2+m_f^2\right)+4\frac{1}{4m_\chi^2}\left(1-\frac{v^2}{4}\right)m_\chi^2m_f^2\right]$$

$$=\frac{1}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{v^2}{8\left(1-m_f^2/m_\chi^2\right)}\right]\left[3m_\chi^2+\left(m_\chi^2-\frac{1}{4}m_f^2\right)v^2\right]$$

$$\simeq\frac{1}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 2\sqrt{1-m_f^2/m_\chi^2}\left\{3m_\chi^2+\left[\frac{3m_\chi^2}{8\left(1-m_f^2/m_\chi^2\right)}+\left(m_\chi^2-\frac{1}{4}m_f^2\right)\right]v^2\right\}$$

$$=\frac{2}{3\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[3+\frac{11-10m_f^2/m_\chi^2+2m_f^4/m_\chi^4}{8\left(1-m_f^2/m_\chi^2\right)}v^2\right]$$

$$=\frac{2}{\pi}\left(\frac{G_{LL}}{\sqrt{2}}\right)^2 m_\chi^2\sqrt{1-m_f^2/m_\chi^2}\left[1+\frac{11-10m_f^2/m_\chi^2+2m_f^4/m_\chi^4}{24\left(1-m_f^2/m_\chi^2\right)}v^2\right]$$

12. 右手-右手（R-R）耦合：$\mathcal{L}=\frac{G_{RR}}{\sqrt{2}}\bar{\chi}\gamma^\mu\left(1+\gamma_5\right)\chi\bar{f}\gamma_\mu\left(1+\gamma_5\right)f$

$$i\mathcal{M}=i\frac{G_{RR}}{\sqrt{2}}\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1+\gamma_5\right)u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1+\gamma_5\right)u(p)\right]\left[\bar{u}(k)\gamma_\nu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\nu\left(1+\gamma_5\right)u(p)\right]^*$$

$$=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1+\gamma_5\right)u(p)\right]\left[\bar{u}(p)\gamma^\nu\left(1+\gamma_5\right)v(p')\bar{v}(k')\gamma_\nu\left(1+\gamma_5\right)u(k)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\text{tr}\left[u(k)\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(k')\gamma_\nu\left(1+\gamma_5\right)\right]\text{tr}\left[v(p')\bar{v}(p')\gamma^\mu\left(1+\gamma_5\right)u(p)\bar{u}(p)\gamma^\nu\left(1+\gamma_5\right)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\text{tr}\left[\left(\not{k}+m_f\right)\gamma_\mu\left(1+\gamma_5\right)\left(\not{k}'-m_f\right)\gamma_\nu\left(1+\gamma_5\right)\right]\text{tr}\left[\left(\not{p}'-m_\chi\right)\gamma^\mu\left(1+\gamma_5\right)\left(\not{p}+m_\chi\right)\gamma^\nu\left(1+\gamma_5\right)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2 2\left[8s^2+64s|\mathbf{p}||\mathbf{k}|\cos\theta+128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$=\left(\frac{G_{RR}}{\sqrt{2}}\right)^2 4\left[s^2+8s|\mathbf{p}||\mathbf{k}|\cos\theta+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$\left(\frac{d\sigma_{RR}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2 4\left[s^2+8s|\mathbf{p}||\mathbf{k}|\cos\theta+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$=\frac{1}{16\pi^2}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$\sigma_{RR}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{RR}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{8\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]\sin\theta d\theta$$

$$=\frac{1}{4\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{16}{3s}|\mathbf{p}|^2|\mathbf{k}|^2\right]$$

$$=\frac{1}{4\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{1}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)\right]$$

$$=\frac{1}{3\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-\left(m_\chi^2+m_f^2\right)+4\frac{m_\chi^2 m_f^2}{s}\right]$$

13. 左手-右手（L-R）耦合：$\mathcal{L}=\frac{G_{LR}}{\sqrt{2}}\bar{\chi}\gamma^\mu\left(1-\gamma_5\right)\chi\bar{f}\gamma_\mu\left(1+\gamma_5\right)f$

$$i\mathcal{M}=i\frac{G_{LR}}{\sqrt{2}}\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\right]\left[\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\right]^*$$

$$=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\right]\left[\bar{u}(p)\gamma^\nu\left(1-\gamma_5\right)v(p')\bar{v}(k')\gamma_\nu\left(1+\gamma_5\right)u(k)\right]$$

$$=\frac{1}{4}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(k)\bar{u}(k)\gamma_\mu\left(1+\gamma_5\right)v(k')\bar{v}(k')\gamma_\nu\left(1+\gamma_5\right)\right]\operatorname{tr}\left[v(p')\bar{v}(p')\gamma^\mu\left(1-\gamma_5\right)u(p)\bar{u}(p)\gamma^\nu\left(1-\gamma_5\right)\right]$$

$$=\frac{1}{4}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{k}+m_f\right)\gamma_\mu\left(1+\gamma_5\right)\left(\not{k}'-m_f\right)\gamma_\nu\left(1+\gamma_5\right)\right]\operatorname{tr}\left[\left(\not{p}'-m_\chi\right)\gamma^\mu\left(1-\gamma_5\right)\left(\not{p}+m_\chi\right)\gamma^\nu\left(1-\gamma_5\right)\right]$$

$$=\frac{1}{4}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2 2\left[8s^2-64s|\mathbf{p}||\mathbf{k}|\cos\theta+128\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$=\left(\frac{G_{LR}}{\sqrt{2}}\right)^2 4\left[s^2-8s|\mathbf{p}||\mathbf{k}|\cos\theta+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$\left(\frac{d\sigma_{LR}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2 4\left[s^2-8s|\mathbf{p}||\mathbf{k}|\cos\theta+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$=\frac{1}{16\pi^2}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$\sigma_{LR}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{LR}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{8\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]\sin\theta d\theta$$

$$=\frac{1}{4\pi}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{16}{3s}|\mathbf{p}|^2|\mathbf{k}|^2\right]$$

$$=\frac{1}{4\pi}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{1}{3s}\left(s-4m_\chi^2\right)\left(s-4m_f^2\right)\right]$$

$$=\frac{1}{3\pi}\left(\frac{G_{LR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-\left(m_\chi^2+m_f^2\right)+4\frac{m_\chi^2 m_f^2}{s}\right]$$

14. 右手-左手（R-L）耦合：$\mathcal{L}=\frac{G_{RL}}{\sqrt{2}}\bar{\chi}\gamma^\mu\left(1+\gamma_5\right)\chi\bar{f}\gamma_\mu\left(1-\gamma_5\right)f$

$$i\mathcal{M}=i\frac{G_{RL}}{\sqrt{2}}\bar{u}(k)\gamma_\mu\left(1-\gamma_5\right)v(k')\bar{v}(p')\gamma^\mu\left(1+\gamma_5\right)u(p)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu(1-\gamma_5)v(k')\bar{v}(p')\gamma^\mu(1+\gamma_5)u(p)\right]\left[\bar{u}(k)\gamma_\mu(1-\gamma_5)v(k')\bar{v}(p')\gamma^\mu(1+\gamma_5)u(p)\right]^*$$

$$=\frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\left[\bar{u}(k)\gamma_\mu(1-\gamma_5)v(k')\bar{v}(p')\gamma^\mu(1+\gamma_5)u(p)\right]\left[\bar{u}(p)\gamma^\nu(1+\gamma_5)v(p')\bar{v}(k')\gamma_\nu(1-\gamma_5)u(k)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\mathrm{tr}\left[u(k)\bar{u}(k)\gamma_\mu(1-\gamma_5)v(k')\bar{v}(k')\gamma_\nu(1-\gamma_5)\right]\mathrm{tr}\left[v(p')\bar{v}(p')\gamma^\mu(1+\gamma_5)u(p)\bar{u}(p)\gamma^\nu(1+\gamma_5)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\mathrm{tr}\left[(\not{k}+m_f)\gamma_\mu(1-\gamma_5)(\not{k}'-m_f)\gamma_\nu(1-\gamma_5)\right]\mathrm{tr}\left[(\not{p}'-m_\chi)\gamma^\mu(1+\gamma_5)(\not{p}+m_\chi)\gamma^\nu(1+\gamma_5)\right]$$

$$=\frac{1}{4}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2 2\left[8s^2-64s|\mathbf{p}||\mathbf{k}|\cos\theta+128(|\mathbf{p}||\mathbf{k}|\cos\theta)^2\right]$$

$$=\left(\frac{G_{RL}}{\sqrt{2}}\right)^2 4\left[s^2-8s|\mathbf{p}||\mathbf{k}|\cos\theta+16(|\mathbf{p}||\mathbf{k}|\cos\theta)^2\right]$$

$$\left(\frac{d\sigma_{RL}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{s}\frac{1}{64\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2 4\left[s^2-8s|\mathbf{p}||\mathbf{k}|\cos\theta+16(|\mathbf{p}||\mathbf{k}|\cos\theta)^2\right]$$

$$=\frac{1}{16\pi^2}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}(|\mathbf{p}||\mathbf{k}|\cos\theta)^2\right]$$

$$\sigma_{RL}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{RL}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{8\pi}\left(\frac{G_{RR}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-8|\mathbf{p}||\mathbf{k}|\cos\theta+\frac{16}{s}(|\mathbf{p}||\mathbf{k}|\cos\theta)^2\right]\sin\theta d\theta$$

$$=\frac{1}{4\pi}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{16}{3s}|\mathbf{p}|^2|\mathbf{k}|^2\right]$$

$$=\frac{1}{4\pi}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s+\frac{1}{3s}(s-4m_\chi^2)(s-4m_f^2)\right]$$

$$=\frac{1}{3\pi}\left(\frac{G_{RL}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\chi^2}}\left[s-(m_\chi^2+m_f^2)+4\frac{m_\chi^2m_f^2}{s}\right]$$

Maxwell 速度分布律

（此处所用的记号 v 与上面不同！）

$$\int\exp\left(-\frac{mv^2}{2kT}\right)d^3v=4\pi\int_0^\infty\exp\left(-\frac{mv^2}{2kT}\right)v^2dv=4\pi\frac{\sqrt{\pi}}{4\left(\frac{m}{2kT}\right)^{3/2}}=\left(\frac{2\pi kT}{m}\right)^{3/2}$$

$$f(v)=\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{mv^2}{2kT}\right)$$

$$\langle Q\rangle=\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty Q\exp\left(-\frac{mv^2}{2kT}\right)4\pi v^2d^3v$$

$$\langle v^2\rangle=4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^4\exp\left(-\frac{mv^2}{2kT}\right)d^3v=\frac{3kT}{m}$$

$$\text{Velocity dispersion}\quad \bar{v}\equiv\sqrt{\langle v^2\rangle}=\sqrt{\frac{3kT}{m}}$$

对暗物质，可取 $\bar{v}=270$ km/s [Jungman *et al.* **Phys. Rep.** 267, (1996) 195-373]

设两暗物质粒子的速度分别为 $\mathbf{v}_1$ 和 $\mathbf{v}_2$，则它们之间相对速度平方的热平均值

$$\left\langle|\mathbf{v}_1-\mathbf{v}_2|^2\right\rangle=\left\langle\mathbf{v}_1^2-2\mathbf{v}_1\cdot\mathbf{v}_2+\mathbf{v}_2^2\right\rangle=\left\langle v_1^2\right\rangle-2\left\langle\mathbf{v}_1\cdot\mathbf{v}_2\right\rangle+\left\langle v_2^2\right\rangle=2\left\langle v^2\right\rangle=2\bar{v}^2,$$

即是单粒子 Velocity dispersion $\bar{v}$ 平方的两倍，故可取

$$\left\langle|\mathbf{v}_1-\mathbf{v}_2|^2\right\rangle=2\bar{v}^2=2\cdot(270\text{ km/s})^2=1.458\times10^{15}\text{cm}^2/\text{s}^2$$

Dirac 费米子型暗物质给出的遗迹密度

设 $\sigma_{\rm ann} v = a + bv^2$（计上标准模型中所有费米子的贡献），则

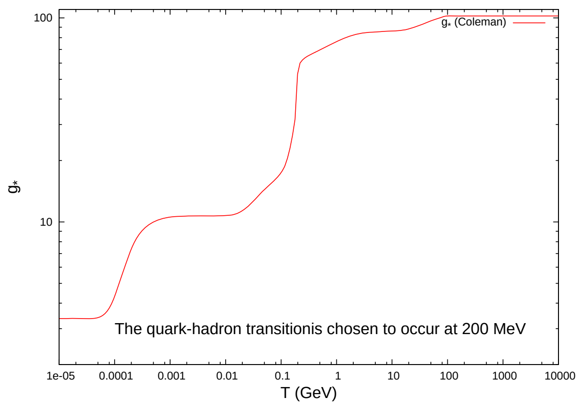
$$\Omega_\chi h^2 = 1.0665207\times10^9\ \text{GeV}^{-1}\left(\frac{T_0}{2.75\ \text{K}}\right)^3 \frac{x_f}{\left[g_{*s}\left(T_f\right)/\sqrt{g_*\left(T_f\right)}\right]M_{\rm pl}\left(a+3b/x_f\right)}$$

$$\simeq 1.0665207\times10^9\ \text{GeV}^{-1}\left(\frac{T_0}{2.75\ \text{K}}\right)^3 \frac{x_f}{M_{\rm pl}\sqrt{g_*\left(T_f\right)}\left(a+3b/x_f\right)}$$

其中 $x_f = m_\chi / T_f$，T_f 是冻结温度，Planck 质量 $M_{\rm pl} = 1.2209\times10^{19}$ GeV

$T_0 = 2.725 \pm 0.002$ K (Ref: Mather *et al.* ApJ (1999) 512:511-520)

g_* 随温度关系如下图所示（Coleman & Roos, PRD 68, 027702 (2003) 中 Fig. 1，夸克-强子转变温度取在 200 MeV）



与 freeze out 相关的 x_f 可由方程

$$x_f = \ln\left[c\left(c+2\right)\sqrt{\frac{45}{8}}\frac{g m_\chi M_{\rm pl}\left(a+6b/x_f\right)}{2\pi^3\sqrt{x_f g_*}}\right]$$

解得，其中 c 是一个量级为 1 的数。

因上式的对数依赖关系，c 的数值对结果的影响不大，通常可取 $c = \frac{1}{2}$。

二、Complex Scalar WIMP

设暗物质粒子 ϕ 和 ϕ^* 是复标量场玻色子，

f 和 $\overline{f}$ 是标准模型中的费米子

1. Scalar 耦合： $\mathcal{L} = \frac{F_S}{\sqrt{2}}\phi^*\phi\overline{f}f$

$$i\mathcal{M} = i\frac{F_S}{\sqrt{2}}\overline{u}(k)v(k')$$

$$\begin{aligned}
\sum_{\text{spins}}|\mathcal{M}|^2 &= \sum_{\text{spins}}\frac{F_S^2}{2}\left[\overline{u}(k)v(k')\right]\left[\overline{u}(k)v(k')\right]^* \\
&= \sum_{\text{spins}}\frac{F_S^2}{2}\left[\overline{u}(k)v(k')\overline{v}(k')u(k)\right] \\
&= \frac{F_S^2}{2}\mathrm{tr}\left[u(k)\overline{u}(k)v(k')\overline{v}(k')\right] \\
&= \frac{F_S^2}{2}\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{k}'-m_f\right)\right] \\
&= \frac{F_S^2}{2}\left(4k\cdot k'-2m_f^2\right) \\
&= 2\frac{F_S^2}{2}\left(s-4m_f^2\right)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d\sigma_S}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\,2\frac{F_S^2}{2}\left(s-4m_f^2\right)
\end{aligned}$$

$$\begin{aligned}
\sigma_S^{\text{CM}} &= \frac{1}{16\pi s}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\,2\frac{F_S^2}{2}\left(s-4m_f^2\right) \\
&= 2\frac{1}{16\pi}\left(\frac{F_S}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{s-4m_f^2}{s} \\
&= \frac{1}{8\pi}\left(\frac{F_S}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{s-4m_f^2}{s}
\end{aligned}$$

$$\begin{aligned}
\sigma_S^{\text{CM}}v &= 2\frac{1}{16\pi}\frac{F_S^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left(\frac{s-4m_f^2}{s}\right) \\
&\simeq 2\frac{1}{16\pi}\frac{F_S^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\left(\frac{1}{4m_\phi^2}-\frac{v^2}{16m_\phi^2}\right)\left(4m_\phi^2+m_\phi^2v^2-4m_f^2\right) \\
&\simeq 2\frac{1}{16\pi}\frac{F_S^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\left[1-m_f^2/m_\phi^2+\frac{1}{4}\left(m_f^2/m_\phi^2\right)v^2\right] \\
&\simeq 2\frac{1}{16\pi}\frac{F_S^2}{2}\left\{2\left(1-m_f^2/m_\phi^2\right)^{3/2}+\frac{1}{4}\sqrt{1-m_f^2/m_\phi^2}\left[1+2\left(m_f^2/m_\phi^2\right)\right]v^2\right\} \\
&= \frac{1}{8\pi}\frac{F_S^2}{2}\left\{2\left(1-m_f^2/m_\phi^2\right)^{3/2}+\frac{1}{4}\sqrt{1-m_f^2/m_\phi^2}\left[1+2\left(m_f^2/m_\phi^2\right)\right]v^2\right\} \\
&= \frac{1}{8\pi}\left(\frac{F_S}{\sqrt{2}}\right)^2\sqrt{1-\frac{m_f^2}{m_\phi^2}}\left[2\left(1-\frac{m_f^2}{m_\phi^2}\right)+\frac{1}{4}\left(1+2\frac{m_f^2}{m_\phi^2}\right)v^2\right]
\end{aligned}$$

2. Vector 耦合： $\mathcal{L} = \frac{F_V}{\sqrt{2}}\phi^* i\overleftrightarrow{\partial}_\mu\phi\overline{f}\gamma^\mu f = \frac{F_V}{\sqrt{2}}i\left[\phi^*\partial_\mu\phi-\left(\partial_\mu\phi^*\right)\phi\right]\overline{f}\gamma^\mu f$

$$i\mathcal{M} = i\frac{F_S}{\sqrt{2}}\left(p_\mu-p'_\mu\right)\overline{u}(k)\gamma^\mu v(k')$$

$$\sum_{\text{spins}}|\mathcal{M}|^2=\sum_{\text{spins}}\frac{F_V^2}{2}\left[\left(p_\mu-p'_\mu\right)\bar{u}(k)\gamma^\mu v(k')\right]\left[\left(p_\nu-p'_\nu\right)\bar{u}(k)\gamma^\nu v(k')\right]^*$$

$$=\sum_{\text{spins}}\frac{F_V^2}{2}\left(p_\mu-p'_\mu\right)\left(p_\nu-p'_\nu\right)\left[\bar{u}(k)\gamma^\mu v(k')\right]\left[\bar{v}(k')\gamma^\nu u(k)\right]$$

$$=\frac{F_V^2}{2}\left(p_\mu-p'_\mu\right)\left(p_\nu-p'_\nu\right)\mathrm{tr}\left[u(k)\bar{u}(k)\gamma^\mu v(k')\bar{v}(k')\gamma^\nu\right]$$

$$=\frac{F_V^2}{2}\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{p}-\not{p}'\right)\left(\not{k}'-m_f\right)\left(\not{p}-\not{p}'\right)\right]$$

$$=2\frac{F_V^2}{2}\left[s^2-4m_\phi^2 s-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$\left(\frac{d\sigma_V}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\,2\frac{F_V^2}{2}\left[s^2-4m_\phi^2 s-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$$

$$=\frac{1}{32\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{F_V^2}{2}\left[s-4m_\phi^2-\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right]$$

$$\sigma_V^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_V}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{F_V^2}{2}\left[s-4m_\phi^2-\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right]\sin\theta d\theta$$

$$=\frac{1}{16\pi}\frac{F_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left(2s-8m_\phi^2-\frac{32|\mathbf{p}|^2|\mathbf{k}|^2}{3s}\right)$$

$$=\frac{1}{8\pi}\frac{F_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[s-4m_\phi^2-\frac{16\left(\frac{s}{4}-m_\phi^2\right)\left(\frac{s}{4}-m_f^2\right)}{3s}\right]$$

$$=\frac{1}{8\pi}\frac{F_V^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[s-4m_\phi^2-\frac{\left(s-4m_\phi^2\right)\left(s-4m_f^2\right)}{3s}\right]$$

$$=\frac{1}{8\pi}\left(\frac{F_V}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[\frac{2\left(s-4m_\phi^2\right)\left(s+2m_f^2\right)}{3s}\right]$$

$$=\frac{1}{12\pi}\left(\frac{F_V}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[s+2\left(m_f^2-2m_\phi^2\right)-8\frac{m_\phi^2 m_f^2}{s}\right]$$

$$\sigma_V^{\text{CM}}v=\frac{1}{8\pi}\frac{F_V^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[\frac{2\left(s-4m_\phi^2\right)\left(s+2m_f^2\right)}{3s}\right]$$

$$\simeq\frac{1}{8\pi}\frac{F_V^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\left[\frac{2}{3}\left(\frac{1}{4m_\phi^2}-\frac{v^2}{16m_\phi^2}\right)\left(m_\phi^2 v^2\right)\left(4m_\phi^2+2m_f^2+m_\phi^2 v^2\right)\right]$$

$$\simeq\frac{1}{8\pi}\frac{F_V^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\left[\frac{1}{3}\left(2m_\phi^2+m_f^2\right)v^2\right]$$

$$\simeq\frac{1}{8\pi}\frac{F_V^2}{2}\frac{2}{3}\sqrt{1-m_f^2/m_\phi^2}\left(2m_\phi^2+m_f^2\right)v^2$$

$$=\frac{1}{8\pi}\left(\frac{F_V}{\sqrt{2}}\right)^2\sqrt{1-m_f^2/m_\phi^2}\,\frac{2}{3}\left(2+m_f^2/m_\phi^2\right)m_\phi^2 v^2$$

$$=\frac{1}{12\pi}\left(\frac{F_V}{\sqrt{2}}\right)^2 m_\phi^2\sqrt{1-\frac{m_f^2}{m_\phi^2}}\left(2+\frac{m_f^2}{m_\phi^2}\right)v^2$$

3. Scalar-Pseudoscalar 耦合： $\mathcal{L}=\frac{F_{\text{SP}}}{\sqrt{2}}\phi^*\phi\bar{f}i\gamma_5 f$

$$i\mathcal{M}=i\frac{F_{\text{SP}}}{\sqrt{2}}\bar{u}(k)i\gamma_5 v(k')$$

$$\sum_{\text{spins}}|\mathcal{M}|^2=\sum_{\text{spins}}\frac{F_{\text{SP}}^2}{2}\left[\bar{u}(k)i\gamma_5 v(k')\right]\left[\bar{u}(k)i\gamma_5 v(k')\right]^*$$

$$=-\sum_{\text{spins}}\frac{F_{\text{SP}}^2}{2}\left[\bar{u}(k)\gamma_5 v(k')\bar{v}(k')\gamma_5 u(k)\right]$$

$$=-\frac{F_{\text{SP}}^2}{2}\text{tr}\left[u(k)\bar{u}(k)\gamma_5 v(k')\bar{v}(k')\gamma_5\right]$$

$$=-\frac{F_{\text{SP}}^2}{2}\text{tr}\left[\left(\not{k}+m_f\right)\gamma_5\left(\not{k}'-m_f\right)\gamma_5\right]$$

$$=2\frac{F_{\text{SP}}^2}{2}s$$

$$\left(\frac{d\sigma_{\text{SP}}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}2\frac{F_{\text{SP}}^2}{2}s$$

$$=\frac{1}{8\pi^2}\frac{1}{4}\frac{F_{\text{SP}}^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}$$

$$\sigma_{\text{SP}}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{\text{SP}}}{d\Omega}\right)_{\text{CM}}=\frac{1}{8\pi}\left(\frac{F_{SP}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}$$

$$\sigma_{\text{SP}}^{\text{CM}}v=\frac{1}{8\pi}\frac{F_{\text{SP}}^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}$$

$$\simeq\frac{1}{8\pi}\frac{F_{\text{SP}}^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)$$

$$=\frac{1}{8\pi}\left(\frac{F_{SP}}{\sqrt{2}}\right)^2\sqrt{1-\frac{m_f^2}{m_\phi^2}}\left(2+\frac{v^2}{4\left(1-m_f^2/m_\phi^2\right)}\right)$$

$$=\frac{1}{8\pi}\left(\frac{F_{SP}}{\sqrt{2}}\right)^2\sqrt{1-\frac{m_f^2}{m_\phi^2}}\left[2+\frac{v^2}{4\left(1-m_f^2/m_\phi^2\right)}\right]$$

4. Vector-Axial vector 耦合： $\mathcal{L}=\frac{F_{\text{VA}}}{\sqrt{2}}\phi^* i\overleftrightarrow{\partial}_\mu\phi\bar{f}\gamma^\mu\gamma_5 f=\frac{F_{\text{V}}}{\sqrt{2}}i\left[\phi^*\partial_\mu\phi-\left(\partial_\mu\phi^*\right)\phi\right]\bar{f}\gamma^\mu\gamma_5 f$

$$i\mathcal{M}=i\frac{F_{\text{VA}}}{\sqrt{2}}\left(p_\mu-p'_\mu\right)\bar{u}(k)\gamma^\mu\gamma_5 v(k')$$

$$\sum_{\text{spins}}|\mathcal{M}|^2=\sum_{\text{spins}}\frac{F_{\text{VA}}^2}{2}\left[\left(p_\mu-p'_\mu\right)\bar{u}(k)\gamma^\mu\gamma_5 v(k')\right]\left[\left(p_\nu-p'_\nu\right)\bar{u}(k)\gamma^\nu\gamma_5 v(k')\right]^*$$

$$=\sum_{\text{spins}}\frac{F_{\text{VA}}^2}{2}\left(p_\mu-p'_\mu\right)\left(p_\nu-p'_\nu\right)\left[\bar{u}(k)\gamma^\mu\gamma_5 v(k')\right]\left[\bar{v}(k')\gamma^\nu\gamma_5 u(k)\right]$$

$$=\frac{F_{\text{VA}}^2}{2}\left(p_\mu-p'_\mu\right)\left(p_\nu-p'_\nu\right)\text{tr}\left[u(k)\bar{u}(k)\gamma^\mu\gamma_5 v(k')\bar{v}(k')\gamma^\nu\gamma_5\right]$$

$$=\frac{F_{\text{VA}}^2}{2}\text{tr}\left[\left(\not{k}+m_f\right)\left(\not{p}-\not{p}'\right)\gamma_5\left(\not{k}'-m_f\right)\left(\not{p}-\not{p}'\right)\gamma_5\right]$$

$$=2\frac{F_{\text{VA}}^2}{2}\left[s^2-4\left(m_f^2+m_\phi^2\right)s-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+16m_f^2m_\phi^2\right]$$

$$\left(\frac{d\sigma_{\text{VA}}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}2\frac{F_{\text{VA}}^2}{2}\left[s^2-4\left(m_f^2+m_\phi^2\right)s-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+16m_f^2m_\phi^2\right]$$

$$=\frac{1}{16\pi^2}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}2\frac{F_{\text{VA}}^2}{2}\left[s-4\left(m_f^2+m_\phi^2\right)+\frac{16m_f^2m_\phi^2-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right]$$

$$=\frac{1}{32\pi^2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{F_{\text{VA}}^2}{2}\left[s-4\left(m_f^2+m_\phi^2\right)+\frac{16m_f^2m_\phi^2-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right]$$

$$\sigma_{\mathrm{VA}}^{\mathrm{CM}}=\int d\Omega\left(\frac{d\sigma_{\mathrm{VA}}}{d\Omega}\right)_{\mathrm{CM}}=\int\frac{1}{16\pi}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{F_{\mathrm{VA}}^2}{2}\left[s-4\left(m_f^2+m_\phi^2\right)+\frac{16m_f^2m_\phi^2-16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right]\sin\theta d\theta$$

$$=\frac{1}{16\pi}\frac{F_{\mathrm{VA}}^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[2s-8\left(m_f^2+m_\phi^2\right)+\frac{32m_f^2m_\phi^2}{s}-\frac{32|\mathbf{p}|^2|\mathbf{k}|^2}{3s}\right]$$

$$=\frac{1}{16\pi}\frac{F_{\mathrm{VA}}^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[2s-8\left(m_f^2+m_\phi^2\right)+\frac{32m_f^2m_\phi^2}{s}-\frac{32\left(\frac{s}{4}-m_\phi^2\right)\left(\frac{s}{4}-m_f^2\right)}{3s}\right]$$

$$=\frac{1}{8\pi}\frac{F_{\mathrm{VA}}^2}{2}\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[\frac{s^2-4\left(m_f^2+m_\phi^2\right)s+16m_f^2m_\phi^2}{s}-\frac{\left(s-4m_\phi^2\right)\left(s-4m_f^2\right)}{3s}\right]$$

$$=\frac{1}{8\pi}\left(\frac{F_{VA}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{2\left(s-4m_\phi^2\right)\left(s-4m_f^2\right)}{3s}$$

$$=\frac{1}{12\pi}\left(\frac{F_{VA}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\left[s-4\left(m_\phi^2+m_f^2\right)+16\frac{m_\phi^2m_f^2}{s}\right]$$

$$\sigma_{\mathrm{VA}}^{\mathrm{CM}}v=\frac{1}{8\pi}\frac{F_{\mathrm{VA}}^2}{2}v\sqrt{\frac{s-4m_f^2}{s-4m_\phi^2}}\frac{2\left(s-4m_\phi^2\right)\left(s-4m_f^2\right)}{3s}$$

$$\simeq\frac{1}{8\pi}\frac{F_{\mathrm{VA}}^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\frac{2}{3}\left(\frac{1}{4m_\phi^2}-\frac{v^2}{16m_\phi^2}\right)\left(m_\phi^2v^2\right)\left(4m_\phi^2-4m_f^2+m_\phi^2v^2\right)$$

$$\simeq\frac{1}{8\pi}\frac{F_{\mathrm{VA}}^2}{2}\left(2\sqrt{1-m_f^2/m_\phi^2}+\frac{v^2}{4\sqrt{1-m_f^2/m_\phi^2}}\right)\frac{2}{3}\left(1-m_f^2/m_\phi^2\right)m_\phi^2v^2$$

$$\simeq\frac{1}{8\pi}\frac{F_{\mathrm{VA}}^2}{2}\frac{4}{3}\sqrt{1-m_f^2/m_\phi^2}\left(1-m_f^2/m_\phi^2\right)m_\phi^2v^2$$

$$=\frac{1}{6\pi}\left(\frac{F_{VA}}{\sqrt{2}}\right)^2m_\phi^2\left(1-\frac{m_f^2}{m_\phi^2}\right)^{3/2}v^2$$

三、Complex Vector WIMP

设暗物质粒子 X 和 X^* 是复矢量场玻色子，

f 和 $\overline{f}$ 是标准模型中的费米子

$$\sum_i \epsilon^{i\mu}(p)\epsilon^{i\nu*}(p) \to -g^{\mu\nu} + \frac{p^\mu p^\nu}{M_X^2}$$

$$v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}} \simeq 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]$$

$$s \simeq 4M_X^2 + M_X^2 v^2$$

$$\frac{1}{s} \simeq \frac{1}{4M_X^2}\left(1-\frac{v^2}{4}\right)$$

1. Scalar 耦合： $\mathcal{L}_{\text{eff}} = \sum_f \frac{K_{S,f}}{\sqrt{2}} X_\mu^* X^\mu \overline{f} f$

$$i\mathcal{M} = i\frac{K_{S,f}}{\sqrt{2}}\epsilon_\mu^*(p')\epsilon^\mu(p)\overline{u}(k)v(k')$$

$$\begin{aligned}
\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 &= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 \epsilon_\mu^{j*}(p')\epsilon^{i\mu}(p)\overline{u}(k)v(k')\left[\epsilon_\nu^{j*}(p')\epsilon^{i\nu}(p)\overline{u}(k)v(k')\right]^* \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 \epsilon^{i\mu}(p)\epsilon^{i\nu*}(p)\epsilon_\mu^{j*}(p')\epsilon_\nu^{j}(p')\overline{u}(k)v(k')\overline{v}(k')u(k) \\
&= \frac{1}{9}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M_X^2}\right)\left(-g_{\mu\nu}+\frac{p'_\mu p'_\nu}{M_X^2}\right)\mathrm{tr}\left[u(k)\overline{u}(k)v(k')\overline{v}(k')\right] \\
&= \frac{1}{9}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M_X^2}\right)\left(-g_{\mu\nu}+\frac{p'_\mu p'_\nu}{M_X^2}\right)\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{k}'-m_f\right)\right] \\
&= \frac{1}{9}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left(4-\frac{p^2+p'^2}{M_X^2}+\frac{(p'\cdot p)^2}{M_X^4}\right)\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{k}'-m_f\right)\right] \\
&= \frac{1}{9}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left[2+\frac{(p'\cdot p)^2}{M_X^4}\right]\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{k}'-m_f\right)\right] \\
&= \frac{1}{9}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 2\left[-12m_f^2+\left(3+4\frac{m_f^2}{M_X^2}\right)s-\left(\frac{m_f^2}{M_X^4}+\frac{1}{M_X^2}\right)s^2+\frac{1}{4M_X^4}s^3\right] \\
&= \frac{1}{18M_X^4}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left[s^3-4\left(m_f^2+M_X^2\right)s^2+\left(12M_X^4+16m_f^2M_X^2\right)s-48m_f^2M_X^4\right]
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d\sigma_{S,f}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{18M_X^4}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left[s^3-4\left(m_f^2+M_X^2\right)s^2+\left(12M_X^4+16m_f^2M_X^2\right)s-48m_f^2M_X^4\right] \\
&= \frac{1}{1152\pi^2 M_X^4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_f^2+M_X^2\right)s+\left(12M_X^4+16m_f^2M_X^2\right)-48\frac{m_f^2M_X^4}{s}\right]
\end{aligned}$$

$$\begin{aligned}
\sigma_{S,\text{ann}} &= \sum_f c_f\int d\Omega\left(\frac{d\sigma_{S,f}}{d\Omega}\right)_{\text{CM}} \\
&= \sum_f c_f\int\frac{1}{576\pi M_X^4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_f^2+M_X^2\right)s+\left(12M_X^4+16m_f^2M_X^2\right)-48\frac{m_f^2M_X^4}{s}\right]\sin\theta d\theta \\
&= \frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left[s^2-4\left(m_f^2+M_X^2\right)s+\left(12M_X^4+16m_f^2M_X^2\right)-48\frac{m_f^2M_X^4}{s}\right]
\end{aligned}$$

$$\sigma_{S,\mathrm{ann}}v=\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left[s^2-4\left(m_f^2+M_X^2\right)s+\left(12M_X^4+16m_f^2M_X^2\right)-48\frac{m_f^2M_X^4}{s}\right]$$

$$\simeq\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left[\left(4M_X^2+M_X^2v^2\right)^2-4\left(m_f^2+M_X^2\right)\left(4M_X^2+M_X^2v^2\right)+\left(12M_X^4+16m_f^2M_X^2\right)-12m_f^2M_X^2\left(1-\frac{v^2}{4}\right)\right]$$

$$\simeq\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left[12\left(M_X^2-m_f^2\right)M_X^2+\left(4M_X^2-m_f^2\right)M_X^2v^2\right]$$

$$\simeq\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[12\left(M_X^2-m_f^2\right)M_X^2+\frac{12\left(M_X^2-m_f^2\right)M_X^2v^2}{8\left(1-m_f^2/M_X^2\right)}+\left(4M_X^2-m_f^2\right)M_X^2v^2\right]$$

$$\simeq\frac{1}{12\pi}\sum_f\left(\frac{K_{S,f}}{\sqrt{2}}\right)^2 c_f\sqrt{1-\frac{m_f^2}{M_X^2}}\left[\left(1-\frac{m_f^2}{M_X^2}\right)+\left(\frac{11}{24}-\frac{1}{12}\frac{m_f^2}{M_X^2}\right)v^2\right]$$

2. Vector 耦合： $\mathcal{L}_{\mathrm{eff}}=\sum_f\frac{K_{V,f}}{\sqrt{2}}\left(X_\nu^* i\overleftrightarrow{\partial}_\mu X^\nu\right)\bar{f}\gamma^\mu f$

$$i\mathcal{M}=i\frac{K_{V,f}}{\sqrt{2}}\left(p_\mu-p'_\mu\right)\epsilon_\nu^*\left(p'\right)\epsilon^\nu\left(p\right)\bar{u}\left(k\right)\gamma^\mu v\left(k'\right)$$

$$\frac{1}{9}\sum_{\mathrm{spins}}\sum_{i,j}|\mathcal{M}|^2=\frac{1}{9}\sum_{\mathrm{spins}}\sum_{i,j}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left(p_\mu-p'_\mu\right)\epsilon_\nu^{j*}\left(p'\right)\epsilon^{i\nu}\left(p\right)\bar{u}\left(k\right)\gamma^\mu v\left(k'\right)\left[\left(p_\rho-p'_\rho\right)\epsilon_\sigma^{j*}\left(p'\right)\epsilon^{i\sigma}\left(p\right)\bar{u}\left(k\right)\gamma^\rho v\left(k'\right)\right]^*$$

$$=\frac{1}{9}\sum_{\mathrm{spins}}\sum_{i,j}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left(p_\mu-p'_\mu\right)\left(p_\rho-p'_\rho\right)\epsilon^{i\sigma*}\left(p\right)\epsilon^{i\nu}\left(p\right)\epsilon_\sigma^{j}\left(p'\right)\epsilon_\nu^{j*}\left(p'\right)\bar{u}\left(k\right)\gamma^\mu v\left(k'\right)\bar{v}\left(k'\right)\gamma^\rho u\left(k\right)$$

$$=\frac{1}{9}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left(p_\mu-p'_\mu\right)\left(p_\rho-p'_\rho\right)\left(-g^{\sigma\nu}+\frac{p^\sigma p^\nu}{M_X^2}\right)\left(-g_{\sigma\nu}+\frac{p'_\sigma p'_\nu}{M_X^2}\right)\mathrm{tr}\left[u\left(k\right)\bar{u}\left(k\right)\gamma^\mu v\left(k'\right)\bar{v}\left(k'\right)\gamma^\rho\right]$$

$$=\frac{1}{9}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left[2+\frac{\left(p'\cdot p\right)^2}{M_X^4}\right]\mathrm{tr}\left[\left(\not{k}+m_f\right)\left(\not{p}-\not{p}'\right)\left(\not{k}'-m_f\right)\left(\not{p}-\not{p}'\right)\right]$$

$$=\frac{1}{9}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 2\left\{\frac{1}{4M_X^4}s^4-\frac{2}{M_X^2}s^3+\left[7-\frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2+\left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-12M_X^2\right]s-48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right\}$$

$$=\frac{2}{9}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{4M_X^4}s^4-\frac{2}{M_X^2}s^3+\left[7-\frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2+\left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-12M_X^2\right]s-48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right\}$$

$$\left(\frac{d\sigma_{V,f}}{d\Omega}\right)_{\mathrm{CM}}=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{\left(2\pi\right)^2 4E_{\mathrm{cm}}}\frac{1}{9}\sum_{\mathrm{spins}}\sum_{i,j}|\mathcal{M}|^2$$

$$=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{2}{9}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{4M_X^4}s^4-\frac{2}{M_X^2}s^3+\left[7-\frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2+\left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-12M_X^2\right]s-48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right\}$$

$$=\frac{1}{288\pi^2}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{4M_X^4}s^3-\frac{2}{M_X^2}s^2+\left[7-\frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s+\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-12M_X^2-48\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right\}$$

$$
\begin{aligned}
\sigma_{V,\text{ann}} &= \sum_f c_f \int d\Omega \left(\frac{d\sigma_{V,f}}{d\Omega}\right)_{\text{CM}} \\
&= \sum_f c_f \int \frac{1}{144\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{4M_X^4}s^3 - \frac{2}{M_X^2}s^2 + \left[7 - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s + \frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 12M_X^2 - 48\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}\right\}\sin\theta d\theta \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{4M_X^4}s^3 - \frac{2}{M_X^2}s^2 + \left[7 - \frac{4|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^4}\right]s + \frac{16|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2} - 12M_X^2 - 48\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3s}\right\} \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{4M_X^4}s^3 - \frac{2}{M_X^2}s^2 + \left[7 - \frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{12M_X^4}\right]s + \frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{3M_X^2} - 12M_X^2 - \frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{s}\right\} \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \frac{1}{4M_X^4 s} \\
&\quad \times\left\{s^4 - 8M_X^2 s^3 + 28M_X^4 s^2 - \frac{1}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)s^2 + \frac{4}{3}M_X^2\left(s-4M_X^2\right)\left(s-4m_f^2\right)s - 48M_X^6 s - 4M_X^4\left(s-4M_X^2\right)\left(s-4m_f^2\right)\right\} \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \frac{1}{12M_X^4 s}\left[3s\left(s^3 - 8M_X^2 s^2 + 28M_X^4 s - 48M_X^6\right) - \left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right)\right] \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \frac{1}{12M_X^4 s}\left[3s\left(s-4M_X^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right) - \left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right)\right] \\
&= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 \frac{1}{6M_X^4 s}\left(s+2m_f^2\right)\left(s-4M_X^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right) \\
&= \frac{1}{432\pi M_X^4}\sum_f \left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{\left(s+2m_f^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right)}{s}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{V,\text{ann}} v &= \frac{1}{432\pi M_X^4}\sum_f \left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{\left(s-4M_X^2\right)\left(s+2m_f^2\right)\left(s^2 - 4M_X^2 s + 12M_X^4\right)}{s} \\
&\simeq \frac{1}{432\pi M_X^4}\sum_f \left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1 + \frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\frac{1}{4M_X^2}\left(1-\frac{v^2}{4}\right) \\
&\qquad \times\left(4M_X^2 + M_X^2 v^2 - 4M_X^2\right)\left(4M_X^2 + M_X^2 v^2 + 2m_f^2\right)\left[\left(4M_X^2 + M_X^2 v^2\right)^2 - 4M_X^2\left(4M_X^2 + M_X^2 v^2\right) + 12M_X^4\right] \\
&\simeq \frac{1}{432\pi M_X^4}\sum_f \left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\frac{1}{4}v^2\left(4M_X^2 + 2m_f^2\right)12M_X^4 \\
&= \frac{1}{36\pi}\sum_f \left(\frac{K_{V,f}}{\sqrt{2}}\right)^2 c_f M_X^2\sqrt{1-\frac{m_f^2}{M_X^2}}\left(2 + \frac{m_f^2}{M_X^2}\right)v^2
\end{aligned}
$$

3. Tensor 耦合： $\mathcal{L}_{\text{eff}} = \sum_f \frac{K_{T,f}}{\sqrt{2}} i\left(X_\mu^* X_\nu - X_\nu^* X_\mu\right)\bar{f}\sigma^{\mu\nu} f$

$$
i\mathcal{M} = i\frac{K_{T,f}}{\sqrt{2}} i\left[\epsilon_\mu^*(p')\epsilon_\nu(p) - \epsilon_\nu^*(p')\epsilon_\mu(p)\right]\bar{u}(k)\sigma^{\mu\nu}v(k')
$$

$$
\begin{aligned}
\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 &= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 \left[\epsilon_\mu^{j*}(p')\epsilon_\nu^{i}(p)-\epsilon_\nu^{j*}(p')\epsilon_\mu^{i}(p)\right]\bar{u}(k)\sigma^{\mu\nu}v(k')\left\{\left[\epsilon_\rho^{j*}(p')\epsilon_\sigma^{i}(p)-\epsilon_\sigma^{j*}(p')\epsilon_\rho^{i}(p)\right]\bar{u}(k)\sigma^{\rho\sigma}v(k')\right\}^* \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 \left[\epsilon_\mu^{j*}(p')\epsilon_\nu^{i}(p)-\epsilon_\nu^{j*}(p')\epsilon_\mu^{i}(p)\right]\bar{u}(k)\sigma^{\mu\nu}v(k')\left[\epsilon_\rho^{j}(p')\epsilon_\sigma^{i*}(p)-\epsilon_\sigma^{j}(p')\epsilon_\rho^{i*}(p)\right]\bar{v}(k')\sigma^{\rho\sigma}u(k) \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 \left[\epsilon_\sigma^{i*}(p)\epsilon_\nu^{i}(p)\epsilon_\mu^{j*}(p')\epsilon_\rho^{j}(p')-\epsilon_\rho^{i*}(p)\epsilon_\nu^{i}(p)\epsilon_\mu^{j*}(p')\epsilon_\sigma^{j}(p')-\epsilon_\sigma^{i*}(p)\epsilon_\mu^{i}(p)\epsilon_\nu^{j*}(p')\epsilon_\rho^{j}(p')+\epsilon_\rho^{i*}(p)\epsilon_\mu^{i}(p)\epsilon_\nu^{j*}(p')\epsilon_\sigma^{j}(p')\right] \\
&\qquad \times\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(k')\sigma^{\rho\sigma}u(k) \\
&= \frac{1}{9}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\left[\left(-g_{\sigma\nu}+\frac{p_\sigma p_\nu}{M_X^2}\right)\left(-g_{\mu\rho}+\frac{p'_\mu p'_\rho}{M_X^2}\right)-\left(-g_{\rho\nu}+\frac{p_\rho p_\nu}{M_X^2}\right)\left(-g_{\mu\sigma}+\frac{p'_\mu p'_\sigma}{M_X^2}\right)-\left(-g_{\sigma\mu}+\frac{p_\sigma p_\mu}{M_X^2}\right)\left(-g_{\nu\rho}+\frac{p'_\nu p'_\rho}{M_X^2}\right)+\left(-g_{\rho\mu}+\frac{p_\rho p_\mu}{M_X^2}\right)\left(-g_{\nu\sigma}+\frac{p'_\nu p'_\sigma}{M_X^2}\right)\right] \\
&\qquad \times\mathrm{tr}\left[u(k)\bar{u}(k)\sigma^{\mu\nu}v(k')\bar{v}(k')\sigma^{\rho\sigma}\right] \\
&= -\frac{1}{9}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\left[\left(-g_{\sigma\nu}+\frac{p_\sigma p_\nu}{M_X^2}\right)\left(-g_{\mu\rho}+\frac{p'_\mu p'_\rho}{M_X^2}\right)-\left(-g_{\rho\nu}+\frac{p_\rho p_\nu}{M_X^2}\right)\left(-g_{\mu\sigma}+\frac{p'_\mu p'_\sigma}{M_X^2}\right)-\left(-g_{\sigma\mu}+\frac{p_\sigma p_\mu}{M_X^2}\right)\left(-g_{\nu\rho}+\frac{p'_\nu p'_\rho}{M_X^2}\right)+\left(-g_{\rho\mu}+\frac{p_\rho p_\mu}{M_X^2}\right)\left(-g_{\nu\sigma}+\frac{p'_\nu p'_\sigma}{M_X^2}\right)\right] \\
&\qquad \times\mathrm{tr}\left\{\left(\not{k}+m_f\right)\frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]\left(\not{k}'-m_f\right)\frac{1}{2}\left[\gamma^\rho,\gamma^\sigma\right]\right\} \\
&= \frac{1}{9}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 2\left\{4\left(\frac{m_f^2}{M_X^4}+\frac{1}{M_X^2}\right)s^2+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-8-16\frac{m_f^2}{M_X^2}\right]s-64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-32m_f^2\right\} \\
&= \frac{8}{9}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\left\{\left(\frac{m_f^2}{M_X^4}+\frac{1}{M_X^2}\right)s^2+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2-4\frac{m_f^2}{M_X^2}\right]s-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-8m_f^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\sigma_{T,f}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{8}{9}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\left\{\left(\frac{m_f^2}{M_X^4}+\frac{1}{M_X^2}\right)s^2+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2-4\frac{m_f^2}{M_X^2}\right]s-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-8m_f^2\right\} \\
&= \frac{1}{72\pi^2 M_X^4}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\left(m_f^2+M_X^2\right)s+\left[4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-2M_X^4-4m_f^2M_X^2\right]-16\frac{M_X^2}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-8\frac{M_X^4 m_f^2}{s}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{T,\text{ann}} &= \sum_f c_f\int d\Omega\left(\frac{d\sigma_{T,f}}{d\Omega}\right)_{\text{CM}} \\
&= \sum_f c_f\int\frac{1}{36\pi M_X^4}\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\left(m_f^2+M_X^2\right)s+4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-2M_X^4-4m_f^2M_X^2-16\frac{M_X^2}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-8\frac{M_X^4m_f^2}{s}\right\}\sin\theta d\theta \\
&= \frac{1}{18\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\left(m_f^2+M_X^2\right)s+\frac{4}{3}|\mathbf{p}|^2|\mathbf{k}|^2-2M_X^4-4m_f^2M_X^2-16\frac{M_X^2}{3s}|\mathbf{p}|^2|\mathbf{k}|^2-8\frac{M_X^4m_f^2}{s}\right\} \\
&= \frac{1}{18\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\left(m_f^2+M_X^2\right)s+\frac{1}{12}\left(s-4M_X^2\right)\left(s-4m_f^2\right)-2M_X^4-4m_f^2M_X^2-\frac{M_X^2}{3s}\left(s-4M_X^2\right)\left(s-4m_f^2\right)-8\frac{M_X^4m_f^2}{s}\right\} \\
&= \frac{1}{216\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{12\left(m_f^2+M_X^2\right)s+\left(s-4M_X^2\right)\left(s-4m_f^2\right)-24M_X^4-48m_f^2M_X^2-4\frac{M_X^2}{s}\left(s-4M_X^2\right)\left(s-4m_f^2\right)-96\frac{M_X^4m_f^2}{s}\right\} \\
&= \frac{1}{216\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left[s^2+\left(8m_f^2+4M_X^2\right)s-16m_f^2M_X^2-8M_X^4-160\frac{m_f^2M_X^4}{s}\right]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{T,\text{ann}}v &= \frac{1}{216\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left[s^2+\left(8m_f^2+4M_X^2\right)s-16m_f^2M_X^2-8M_X^4-160\frac{m_f^2M_X^4}{s}\right] \\
&\simeq \frac{1}{216\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left[\left(4M_X^2+M_X^2v^2\right)^2+\left(8m_f^2+4M_X^2\right)\left(4M_X^2+M_X^2v^2\right)-16m_f^2M_X^2-8M_X^4-40m_f^2M_X^2\left(1-\frac{v^2}{4}\right)\right] \\
&\simeq \frac{1}{216\pi M_X^4}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left[24M_X^4-24m_f^2M_X^2+\left(18m_f^2+12M_X^2\right)M_X^2v^2\right] \\
&= \frac{1}{216\pi}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left[24-24\frac{m_f^2}{M_X^2}+\left(12+18\frac{m_f^2}{M_X^2}\right)v^2\right] \\
&\simeq \frac{1}{36\pi}\sum_f\left(\frac{K_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{1-\frac{m_f^2}{M_X^2}}\left[8\left(1-\frac{m_f^2}{M_X^2}\right)+\left(5+6\frac{m_f^2}{M_X^2}\right)v^2\right]
\end{aligned}
$$

4. Scalar-Pseudoscalar 耦合： $\mathcal{L}_{\rm eff}=\sum_f \frac{K_{SP,f}}{\sqrt{2}}X_\mu^* X^\mu \bar{f} i\gamma_5 f$

$$i\mathcal{M}=i\frac{K_{SP,f}}{\sqrt{2}}\epsilon_\mu^*(p')\epsilon^\mu(p)\bar{u}(k)i\gamma_5 v(k')$$

$$\begin{aligned}
\frac{1}{9}\sum_{\rm spins}\sum_{i,j}|\mathcal{M}|^2&=\frac{1}{9}\sum_{\rm spins}\sum_{i,j}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\epsilon_\mu^{j*}(p')\epsilon^{i\mu}(p)\bar{u}(k)i\gamma_5 v(k')\left[\epsilon_\nu^{j*}(p')\epsilon^{i\nu}(p)\bar{u}(k)i\gamma_5 v(k')\right]^*\\
&=-\frac{1}{9}\sum_{\rm spins}\sum_{i,j}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\epsilon^{i\mu}(p)\epsilon^{i\nu*}(p)\epsilon_\mu^{j*}(p')\epsilon_\nu^{j}(p')\bar{u}(k)\gamma_5 v(k')\bar{v}(k')\gamma_5 u(k)\\
&=-\frac{1}{9}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M_X^2}\right)\left(-g_{\mu\nu}+\frac{p'_\mu p'_\nu}{M_X^2}\right)\mathrm{Tr}\left[u(k)\bar{u}(k)\gamma_5 v(k')\bar{v}(k')\gamma_5\right]\\
&=-\frac{1}{9}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\left[2+\frac{(p'\cdot p)^2}{M_X^4}\right]\mathrm{Tr}\left[(\not{k}+m_f)\gamma_5(\not{k}'-m_f)\gamma_5\right]\\
&=\frac{2}{9}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\left(\frac{1}{4M_X^4}s^3-\frac{1}{M_X^2}s^2+3s\right)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d\sigma_{SP,f}}{d\Omega}\right)_{\rm CM}&=\frac{1}{2E_p 2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\rm cm}}\frac{1}{9}\sum_{\rm spins}\sum_{i,j}|\mathcal{M}|^2\\
&=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{2}{9}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\left(\frac{1}{4M_X^4}s^3-\frac{1}{M_X^2}s^2+3s\right)\\
&=\frac{1}{288\pi^2 M_X^4}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{1}{4}s^2-M_X^2 s+3M_X^4\right)
\end{aligned}$$

$$\begin{aligned}
\sigma_{SP,\rm ann}&=\sum_f c_f\int d\Omega\left(\frac{d\sigma_{SP,f}}{d\Omega}\right)_{\rm CM}\\
&=\sum_f c_f\int\frac{1}{144\pi M_X^4}\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{1}{4}s^2-M_X^2 s+3M_X^4\right)\sin\theta d\theta\\
&=\frac{1}{144\pi M_X^4}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{1}{2}s^2-2M_X^2 s+6M_X^4\right)\\
&=\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(s^2-4M_X^2 s+12M_X^4\right)
\end{aligned}$$

$$\begin{aligned}
\sigma_{SP,\rm ann}v&=\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(s^2-4M_X^2 s+12M_X^4\right)\\
&\simeq\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8(1-m_f^2/M_X^2)}\right]\left[\left(4M_X^2+M_X^2v^2\right)^2-4M_X^2\left(4M_X^2+M_X^2v^2\right)+12M_X^4\right]\\
&\simeq\frac{1}{288\pi M_X^4}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8(1-m_f^2/M_X^2)}\right]\left[12M_X^4+4M_X^4v^2\right]\\
&\simeq\frac{1}{36\pi}\sum_f\left(\frac{K_{SP,f}}{\sqrt{2}}\right)^2 c_f\sqrt{1-\frac{m_f^2}{M_X^2}}\left\{3+\left[1+\frac{3}{8(1-m_f^2/M_X^2)}\right]v^2\right\}
\end{aligned}$$

5. Vector-Axial Vector 耦合： $\mathcal{L}_{\rm eff}=\sum_f \frac{K_{VA,f}}{\sqrt{2}}\left(X_\nu^* i\overleftrightarrow{\partial}_\mu X^\nu\right)\bar{f}\gamma^\mu\gamma_5 f$

$$i\mathcal{M}=i\frac{K_{VA,f}}{\sqrt{2}}\left(p_\mu-p'_\mu\right)\epsilon_\nu^*(p')\epsilon^\nu(p)\bar{u}(k)\gamma^\mu\gamma_5 v(k')$$

$$
\begin{aligned}
\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 &= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \left(p_\mu - p'_\mu\right)\epsilon^{*}_{\nu}\left(p'\right)\epsilon^{\nu}\left(p\right)\bar{u}\left(k\right)\gamma^\mu\gamma_5 v\left(k'\right)\left[\left(p_\rho - p'_\rho\right)\epsilon^{j*}_{\sigma}\left(p'\right)\epsilon^{i\sigma}\left(p\right)\bar{u}\left(k\right)\gamma^\rho\gamma_5 v\left(k'\right)\right]^* \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \left(p_\mu - p'_\mu\right)\left(p_\rho - p'_\rho\right)\epsilon^{i\sigma*}\left(p\right)\epsilon^{i\nu}\left(p\right)\epsilon^{j}_{\sigma}\left(p'\right)\epsilon^{j*}_{\nu}\left(p'\right)\bar{u}\left(k\right)\gamma^\mu\gamma_5 v\left(k'\right)\bar{v}\left(k'\right)\gamma^\rho\gamma_5 u\left(k\right) \\
&= \frac{1}{9}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \left(p_\mu - p'_\mu\right)\left(p_\rho - p'_\rho\right)\left(-g^{\sigma\nu} + \frac{p^\sigma p^\nu}{M_X^2}\right)\left(-g_{\sigma\nu} + \frac{p'_\sigma p'_\nu}{M_X^2}\right)\text{tr}\left[u\left(k\right)\bar{u}\left(k\right)\gamma^\mu\gamma_5 v\left(k'\right)\bar{v}\left(k'\right)\gamma^\rho\gamma_5\right] \\
&= \frac{1}{9}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2\left[2 + \frac{\left(p'\cdot p\right)^2}{M_X^4}\right]\text{tr}\left[\left(\not{k} + m_f\right)\left(\not{p} - \not{p}'\right)\gamma_5\left(\not{k}' - m_f\right)\left(\not{p} - \not{p}'\right)\gamma_5\right] \\
&= \frac{1}{9}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 2\left\{\frac{1}{4M_X^4}s^4 - \left(\frac{2}{M_X^2} + \frac{m_f^2}{M_X^4}\right)s^3 + \left[7 + 8\frac{m_f^2}{M_X^2} - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2 + \left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 28m_f^2 - 12M_X^2\right]s - 48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 48m_f^2M_X^2\right\} \\
&= \frac{2}{9}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{4M_X^4}s^4 - \left(\frac{2}{M_X^2} + \frac{m_f^2}{M_X^4}\right)s^3 + \left[7 + 8\frac{m_f^2}{M_X^2} - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2 + \left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 28m_f^2 - 12M_X^2\right]s - 48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 48m_f^2M_X^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\sigma_{\text{VA},f}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}\left|v - v'\right|}\frac{|\mathbf{k}|}{\left(2\pi\right)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{2}{9}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \\
&\quad\times\left\{\frac{1}{4M_X^4}s^4 - \left(\frac{2}{M_X^2} + \frac{m_f^2}{M_X^4}\right)s^3 + \left[7 + 8\frac{m_f^2}{M_X^2} - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s^2 + \left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 28m_f^2 - 12M_X^2\right]s - 48\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 48m_f^2M_X^2\right\} \\
&= \frac{1}{288\pi^2}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{\text{VA},f}}{\sqrt{2}}\right)^2 \\
&\quad\times\left\{\frac{1}{4M_X^4}s^3 - \left(\frac{2}{M_X^2} + \frac{m_f^2}{M_X^4}\right)s^2 + \left[7 + 8\frac{m_f^2}{M_X^2} - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s + \left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 28m_f^2 - 12M_X^2\right] - \frac{48}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 48\frac{m_f^2M_X^2}{s}\right\}
\end{aligned}
$$

$$\sigma_{VA,\text{ann}} = \sum_f c_f \int d\Omega \left(\frac{d\sigma_{VA,f}}{d\Omega}\right)_{\text{CM}}$$

$$= \sum_f c_f \int \frac{1}{144\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2$$

$$\times\left\{\frac{1}{4M_X^4}s^3 - \left(\frac{2}{M_X^2}+\frac{m_f^2}{M_X^4}\right)s^2 + \left[7+8\frac{m_f^2}{M_X^2} - \frac{4\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}\right]s + \left[\frac{16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} - 28m_f^2 - 12M_X^2\right] - \frac{48}{s}\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2 + 48\frac{m_f^2M_X^2}{s}\right\}\sin\theta d\theta$$

$$= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{4M_X^4}s^3 - \left(\frac{2}{M_X^2}+\frac{m_f^2}{M_X^4}\right)s^2 + \left[7+8\frac{m_f^2}{M_X^2} - \frac{4|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^4}\right]s + \left[\frac{16|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2} - 28m_f^2 - 12M_X^2\right] - \frac{48}{3s}|\mathbf{p}|^2|\mathbf{k}|^2 + 48\frac{m_f^2M_X^2}{s}\right\}$$

$$= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2$$

$$\times\left\{\frac{1}{4M_X^4}s^3 - \left(\frac{2}{M_X^2}+\frac{m_f^2}{M_X^4}\right)s^2 + \left[7+8\frac{m_f^2}{M_X^2} - \frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{12M_X^4}\right]s + \left[\frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{3M_X^2} - 28m_f^2 - 12M_X^2\right] - \frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)}{s} + 48\frac{m_f^2M_X^2}{s}\right\}$$

$$= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2$$

$$\times\frac{1}{4M_X^4 s}\left\{\begin{array}{l} s^4 - 4\left(2M_X^2+m_f^2\right)s^3 + \left[28M_X^4 + 32M_X^2m_f^2 - \frac{1}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)\right]s^2 \\ + \left[\frac{4}{3}M_X^2\left(s-4M_X^2\right)\left(s-4m_f^2\right) - 112M_X^4m_f^2 - 48M_X^6\right]s - 4M_X^4\left(s-4M_X^2\right)\left(s-4m_f^2\right) + 192m_f^2M_X^6 \end{array}\right\}$$

$$= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2$$

$$\times\frac{1}{4M_X^4 s}\left\{s^4 - 4\left(2M_X^2+m_f^2\right)s^3 + \left(28M_X^4+32M_X^2m_f^2\right)s^2 - 112M_X^4m_f^2s - 48M_X^6s + 192m_f^2M_X^6 - \frac{1}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)\right\}$$

$$= \sum_f c_f \frac{1}{72\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2\frac{1}{4M_X^4 s}\left\{\left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right) - \frac{1}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)\right\}$$

$$= \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{\left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)}{s}$$

$$= \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)}{s}$$

$$\sigma_{VA,\text{ann}}v = \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)}{s}$$

$$= \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left(s-4M_X^2\right)\left(s-4m_f^2\right)\left(s^2-4M_X^2s+12M_X^4\right)$$

$$\simeq \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\frac{1}{4M_X^2}\left(1-\frac{v^2}{4}\right)\left(4M_X^2+M_X^2v^2-4M_X^2\right)\left(4M_X^2+M_X^2v^2-4m_f^2\right)$$

$$\times\left[\left(4M_X^2+M_X^2v^2\right)^2 - 4M_X^2\left(4M_X^2+M_X^2v^2\right)+12M_X^4\right]$$

$$\simeq \frac{1}{432\pi M_X^4}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\frac{1}{4M_X^2}\left(M_X^2v^2\right)\left(4M_X^2-4m_f^2\right)12M_X^4$$

$$= \frac{1}{18\pi}\sum_f\left(\frac{K_{VA,f}}{\sqrt{2}}\right)^2 c_f M_X^2\left(1-\frac{m_f^2}{M_X^2}\right)^{3/2}v^2$$

6. Alternative Vector 耦合： $\mathcal{L}_{\text{eff}} = \sum_f \frac{\tilde{K}_{V,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\left(X_\mu^* i\overleftrightarrow{\partial}_\nu X_\rho\right)\bar{f}\gamma_\sigma f$

$$i\mathcal{M} = i\frac{\tilde{K}_{V,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\epsilon_\mu^*\left(p'\right)\left(p_\nu-p'_\nu\right)\epsilon_\rho\left(p\right)\bar{u}\left(k\right)\gamma_\sigma v\left(k'\right)$$

$$\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2=\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\varepsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{j*}(p')(p_\nu-p'_\nu)\epsilon_\rho^i(p)\bar{u}(k)\gamma_\sigma v(k')\left[\varepsilon^{\alpha\beta\gamma\delta}\epsilon_\alpha^{j*}(p')(p_\beta-p'_\beta)\epsilon_\gamma^i(p)\bar{u}(k)\gamma_\delta v(k')\right]^*$$

$$=\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}(p_\nu-p'_\nu)(p_\beta-p'_\beta)\epsilon_\gamma^{i*}(p)\epsilon_\rho^i(p)\epsilon_\mu^{j*}(p')\epsilon_\alpha^j(p')\bar{u}(k)\gamma_\sigma v(k')\bar{v}(k')\gamma_\delta u(k)$$

$$=\frac{1}{9}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}(p_\nu-p'_\nu)(p_\beta-p'_\beta)\left(-g_{\gamma\rho}+\frac{p_\gamma p_\rho}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)\operatorname{tr}\left[u(k)\bar{u}(k)\gamma_\sigma v(k')\bar{v}(k')\gamma_\delta\right]$$

$$=\frac{1}{9}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}(p_\nu-p'_\nu)(p_\beta-p'_\beta)\left(-g_{\gamma\rho}+\frac{p_\gamma p_\rho}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)\operatorname{tr}\left[(\not{k}+m_f)\gamma_\sigma(\not{k}'-m_f)\gamma_\delta\right]$$

$$=\frac{1}{9}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 2\left\{\frac{1}{2}\frac{1}{M_X^2}s^3+\left(2\frac{m_f^2}{M_X^2}-4\right)s^2+\left[8\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-16m_f^2+8M_X^2\right]s-32\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_f^2M_X^2\right\}$$

$$=\frac{1}{9}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^3+\left(4\frac{m_f^2}{M_X^2}-8\right)s^2+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-32m_f^2+16M_X^2\right]s-64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+64m_f^2M_X^2\right\}$$

$$\left(\frac{d\tilde{\sigma}_{V,f}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p2E_{p'}|v-v'|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}|\mathcal{M}|^2$$

$$=\frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{9}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^3+\left(4\frac{m_f^2}{M_X^2}-8\right)s^2+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-32m_f^2+16M_X^2\right]s-64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+64m_f^2M_X^2\right\}$$

$$=\frac{1}{576\pi^2}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^2+\left(4\frac{m_f^2}{M_X^2}-8\right)s+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-32m_f^2+16M_X^2\right]-64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}+64\frac{m_f^2M_X^2}{s}\right\}$$

$$\tilde{\sigma}_{V,\text{ann}}=\sum_f c_f\int d\Omega\left(\frac{d\tilde{\sigma}_{V,f}}{d\Omega}\right)_{\text{CM}}$$

$$=\sum_f c_f\int\frac{1}{288\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^2+\left(4\frac{m_f^2}{M_X^2}-8\right)s+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}-32m_f^2+16M_X^2\right]-64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}+64\frac{m_f^2M_X^2}{s}\right\}\sin\theta d\theta$$

$$=\sum_f\frac{1}{144\pi}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\frac{1}{M_X^2}s^2+\left(4\frac{m_f^2}{M_X^2}-8\right)s+\left[16\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2}-32m_f^2+16M_X^2\right]-64\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3s}+64\frac{m_f^2M_X^2}{s}\right\}$$

$$=\sum_f\frac{1}{144\pi M_X^2}\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{s^2+\left(4m_f^2-8M_X^2\right)s+\left[16\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3}-32m_f^2M_X^2+16M_X^4\right]-64M_X^2\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3s}+64\frac{m_f^2M_X^4}{s}\right\}$$

$$=\frac{1}{144\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s^3+\left(4m_f^2-8M_X^2\right)s^2+\frac{1}{3}s\left(s-4M_X^2\right)\left(s-4m_f^2\right)-32m_f^2M_X^2s+16M_X^4s-\frac{4M_X^2}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)+64m_f^2M_X^4\right\}$$

$$=\frac{1}{144\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s^3+\left(4m_f^2-8M_X^2\right)s^2-32m_f^2M_X^2s+16M_X^4s+64m_f^2M_X^4+\frac{1}{3}\left(s-4M_X^2\right)^2\left(s-4m_f^2\right)\right\}$$

$$=\frac{1}{144\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s\left(s-4M_X^2\right)^2+4m_f^2\left(s-4M_X^2\right)^2+\frac{1}{3}\left(s-4M_X^2\right)^2\left(s-4m_f^2\right)\right\}$$

$$=\frac{1}{144\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{4}{3s}\left(s-4M_X^2\right)^2\left(s+2m_f^2\right)$$

$$=\frac{1}{108\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{\left(s-4M_X^2\right)\left(s+2m_f^2\right)}{s}$$

$$
\begin{aligned}
\tilde{\sigma}_{V,\text{ann}} v &= \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{\left(s-4M_X^2\right)\left(s+2m_f^2\right)}{s} \\
&= \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{\left(s-4M_X^2\right)^2\left(s+2m_f^2\right)}{s} \\
&\simeq \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\frac{1}{4M_X^2}\left(1-\frac{v^2}{4}\right)\left(4M_X^2+M_X^2v^2-4M_X^2\right)^2\left(4M_X^2+M_X^2v^2+2m_f^2\right) \\
&\simeq \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f M_X^4\sqrt{1-\frac{m_f^2}{M_X^2}}\left(2+\frac{m_f^2}{M_X^2}\right)v^4 \\
&= \frac{1}{108\pi}\sum_f \left(\frac{\tilde{K}_{V,f}}{\sqrt{2}}\right)^2 c_f M_X^2\sqrt{1-\frac{m_f^2}{M_X^2}}\left(2+\frac{m_f^2}{M_X^2}\right)v^4
\end{aligned}
$$

7. Alternative Vector-Axial Vector 耦合： $\mathcal{L}_{\text{eff}} = \sum_f \frac{\tilde{K}_{VA,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\left(X_\mu^* i\overleftrightarrow{\partial}_\nu X_\rho\right)\bar{f}\gamma_\sigma\gamma_5 f$

$$
i\mathcal{M} = i\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma}\epsilon_\mu^*\left(p'\right)\left(p_\nu-p'_\nu\right)\epsilon_\rho\left(p\right)\bar{u}\left(k\right)\gamma_\sigma\gamma_5 v\left(k'\right)
$$

$$
\begin{aligned}
\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left|\mathcal{M}\right|^2 &= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{j*}\left(p'\right)\left(p_\nu-p'_\nu\right)\epsilon_\rho^i\left(p\right)\bar{u}\left(k\right)\gamma_\sigma\gamma_5 v\left(k'\right)\left[\varepsilon^{\alpha\beta\gamma\delta}\epsilon_\alpha^{j*}\left(p'\right)\left(p_\beta-p'_\beta\right)\epsilon_\gamma^i\left(p\right)\bar{u}\left(k\right)\gamma_\delta\gamma_5 v\left(k'\right)\right]^* \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left(p_\nu-p'_\nu\right)\left(p_\beta-p'_\beta\right)\epsilon_\gamma^{i*}\left(p\right)\epsilon_\rho^i\left(p\right)\epsilon_\mu^{j*}\left(p'\right)\epsilon_\alpha^j\left(p'\right)\bar{u}\left(k\right)\gamma_\sigma\gamma_5 v\left(k'\right)\bar{v}\left(k'\right)\gamma_\delta\gamma_5 u\left(k\right) \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left(p_\nu-p'_\nu\right)\left(p_\beta-p'_\beta\right)\left(-g_{\gamma\rho}+\frac{p_\gamma p_\rho}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)\text{tr}\left[u\left(k\right)\bar{u}\left(k\right)\gamma_\sigma\gamma_5 v\left(k'\right)\bar{v}\left(k'\right)\gamma_\delta\gamma_5\right] \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left(p_\nu-p'_\nu\right)\left(p_\beta-p'_\beta\right)\left(-g_{\gamma\rho}+\frac{p_\gamma p_\rho}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)\text{tr}\left[\left(\not{k}+m_f\right)\gamma_\sigma\gamma_5\left(\not{k}'-m_f\right)\gamma_\delta\gamma_5\right] \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 2\left\{\frac{1}{2}\frac{1}{M_X^2}s^3-\left(4+2\frac{m_f^2}{M_X^2}\right)s^2+\left[8\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+24m_f^2+8M_X^2\right]s-32\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-64m_f^2M_X^2\right\} \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{M_X^2}s^3-\left(8+4\frac{m_f^2}{M_X^2}\right)s^2+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+48m_f^2+16M_X^2\right]s-64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-128m_f^2M_X^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\tilde{\sigma}_{VA,f}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}\left|v-v'\right|}\frac{|\mathbf{k}|}{\left(2\pi\right)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left|\mathcal{M}\right|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{9}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^3-\left(8+4\frac{m_f^2}{M_X^2}\right)s^2+\left[16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+48m_f^2+16M_X^2\right]s-64\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2-128m_f^2M_X^2\right\} \\
&= \frac{1}{576\pi^2}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^2-\left(8+4\frac{m_f^2}{M_X^2}\right)s+16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+48m_f^2+16M_X^2-64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s}-128\frac{m_f^2M_X^2}{s}\right\}
\end{aligned}
$$

$$\tilde{\sigma}_{VA,\mathrm{ann}} = \sum_f c_f \int d\Omega \left(\frac{d\tilde{\sigma}_{VA,f}}{d\Omega}\right)_{\mathrm{CM}}$$

$$= \sum_f c_f \int \frac{1}{288\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{M_X^2}s^2 - \left(8+4\frac{m_f^2}{M_X^2}\right)s + 16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2} + 48m_f^2 + 16M_X^2 - 64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{s} - 128\frac{m_f^2 M_X^2}{s}\right\}\sin\theta d\theta$$

$$= \frac{1}{144\pi}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\frac{1}{M_X^2}s^2 - \left(8+4\frac{m_f^2}{M_X^2}\right)s + 16\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2} + 48m_f^2 + 16M_X^2 - 64\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3s} - 128\frac{m_f^2 M_X^2}{s}\right\}$$

$$= \frac{1}{144\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s^3 - \left(8M_X^2+4m_f^2\right)s^2 + 48m_f^2M_X^2 s + 16M_X^4 s - 128m_f^2M_X^4 + \frac{16}{3}|\mathbf{p}|^2|\mathbf{k}|^2\left(s-4M_X^2\right)\right\}$$

$$= \frac{1}{144\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{\left(s-4M_X^2\right)s^2 - 4M_X^2 s\left(s-4M_X^2\right) - 4m_f^2 s\left(s-4M_X^2\right) + 32m_f^2M_X^2\left(s-4M_X^2\right) + \frac{1}{3}\left(s-4M_X^2\right)^2\left(s-4m_f^2\right)\right\}$$

$$= \frac{1}{144\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{3s}\left\{\left(s-4M_X^2\right)\left(3s^2 - 12M_X^2 s - 12m_f^2 s + 96m_f^2M_X^2\right) + \left(s-4M_X^2\right)^2\left(s-4m_f^2\right)\right\}$$

$$= \frac{1}{144\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\frac{4}{3s}\left[s^2 - 4\left(M_X^2+m_f^2\right)s + 28m_f^2M_X^2\right]$$

$$= \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f \sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\left[s - 4\left(M_X^2+m_f^2\right) + 28\frac{m_f^2M_X^2}{s}\right]$$

$$\tilde{\sigma}_{VA,\mathrm{ann}}v = \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\left(s-4m_f^2\right)\left(s-4M_X^2\right)}\left[s - 4\left(M_X^2+m_f^2\right) + 28\frac{m_f^2M_X^2}{s}\right]$$

$$= \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f v\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(s-4M_X^2\right)\left[s - 4\left(M_X^2+m_f^2\right) + 28\frac{m_f^2M_X^2}{s}\right]$$

$$\simeq \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right]\left(4M_X^2 + M_X^2v^2 - 4M_X^2\right)\left[4M_X^2 + M_X^2v^2 - 4\left(M_X^2+m_f^2\right) + 28\frac{1}{4M_X^2}\left(1-\frac{v^2}{4}\right)m_f^2M_X^2\right]$$

$$\simeq \frac{1}{108\pi M_X^2}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f 2\sqrt{1-\frac{m_f^2}{M_X^2}}\left[4M_X^2 - 4\left(M_X^2+m_f^2\right) + 28\frac{1}{4M_X^2}m_f^2M_X^2\right]M_X^2v^2$$

$$= \frac{1}{18\pi}\sum_f \left(\frac{\tilde{K}_{VA,f}}{\sqrt{2}}\right)^2 c_f m_f^2\sqrt{1-\frac{m_f^2}{M_X^2}}v^2$$

8. Alternative Tensor 耦合： $\mathcal{L}_{\mathrm{eff}} = \sum_f \frac{\tilde{K}_{T,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma} i\left(X_\mu^* X_\nu - X_\nu^* X_\mu\right)\bar{f}\sigma_{\rho\sigma}f$

$$i\mathcal{M} = i\frac{\tilde{K}_{T,f}}{\sqrt{2}}\varepsilon^{\mu\nu\rho\sigma} i\left[\epsilon_\mu^*(p')\epsilon_\nu(p) - \epsilon_\nu^*(p')\epsilon_\mu(p)\right]\bar{u}(k)\sigma_{\rho\sigma}v(k')$$

$$
\begin{aligned}
\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left|\mathcal{M}\right|^2 &= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma} i\left[\epsilon_\mu^{j*}(p')\epsilon_\nu^{i}(p)-\epsilon_\nu^{j*}(p')\epsilon_\mu^{i}(p)\right]\overline{u}(k)\sigma_{\rho\sigma}v(k')\left\{\varepsilon^{\alpha\beta\gamma\delta} i\left[\epsilon_\alpha^{j*}(p')\epsilon_\beta^{i}(p)-\epsilon_\beta^{j*}(p')\epsilon_\alpha^{i}(p)\right]\overline{u}(k)\sigma_{\gamma\delta}v(k')\right\}^* \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\left[\epsilon_\mu^{j*}(p')\epsilon_\nu^{i}(p)-\epsilon_\nu^{j*}(p')\epsilon_\mu^{i}(p)\right]\varepsilon^{\alpha\beta\gamma\delta}\left[\epsilon_\alpha^{j}(p')\epsilon_\beta^{i*}(p)-\epsilon_\beta^{j}(p')\epsilon_\alpha^{i*}(p)\right]\overline{u}(k)\sigma_{\rho\sigma}v(k')\overline{v}(k')\sigma_{\gamma\delta}u(k) \\
&= \frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left[\epsilon_\beta^{i*}(p)\epsilon_\nu^{i}(p)\epsilon_\mu^{j*}(p')\epsilon_\alpha^{j}(p')-\epsilon_\alpha^{i*}(p)\epsilon_\nu^{i}(p)\epsilon_\mu^{j*}(p')\epsilon_\beta^{j}(p')-\epsilon_\beta^{i*}(p)\epsilon_\mu^{i}(p)\epsilon_\nu^{j*}(p')\epsilon_\alpha^{j}(p')+\epsilon_\alpha^{i*}(p)\epsilon_\mu^{i}(p)\epsilon_\nu^{j*}(p')\epsilon_\beta^{j}(p')\right] \\
&\quad \times\overline{u}(k)\sigma_{\rho\sigma}v(k')\overline{v}(k')\sigma_{\gamma\delta}u(k) \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left[\left(-g_{\beta\nu}+\frac{p_\beta p_\nu}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)-\left(-g_{\alpha\nu}+\frac{p_\alpha p_\nu}{M_X^2}\right)\left(-g_{\mu\beta}+\frac{p'_\mu p'_\beta}{M_X^2}\right)-\left(-g_{\beta\mu}+\frac{p_\beta p_\mu}{M_X^2}\right)\left(-g_{\nu\alpha}+\frac{p'_\nu p'_\alpha}{M_X^2}\right)+\left(-g_{\alpha\mu}+\frac{p_\alpha p_\mu}{M_X^2}\right)\left(-g_{\nu\beta}+\frac{p'_\nu p'_\beta}{M_X^2}\right)\right] \\
&\quad \times\operatorname{tr}\left[u(k)\overline{u}(k)\sigma_{\rho\sigma}v(k')\overline{v}(k')\sigma_{\gamma\delta}\right] \\
&= -\frac{1}{9}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\left[\left(-g_{\beta\nu}+\frac{p_\beta p_\nu}{M_X^2}\right)\left(-g_{\mu\alpha}+\frac{p'_\mu p'_\alpha}{M_X^2}\right)-\left(-g_{\alpha\nu}+\frac{p_\alpha p_\nu}{M_X^2}\right)\left(-g_{\mu\beta}+\frac{p'_\mu p'_\beta}{M_X^2}\right)-\left(-g_{\beta\mu}+\frac{p_\beta p_\mu}{M_X^2}\right)\left(-g_{\nu\alpha}+\frac{p'_\nu p'_\alpha}{M_X^2}\right)+\left(-g_{\alpha\mu}+\frac{p_\alpha p_\mu}{M_X^2}\right)\left(-g_{\nu\beta}+\frac{p'_\nu p'_\beta}{M_X^2}\right)\right] \\
&\quad \times\operatorname{tr}\left[\left(\not{k}+m_f\right)\frac{1}{2}\left[\gamma_\rho,\gamma_\sigma\right]\left(\not{k}'-m_f\right)\frac{1}{2}\left[\gamma_\gamma,\gamma_\delta\right]\right] \\
&= \frac{1}{9}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 2\left\{16\frac{1}{M_X^2}s^2+\left[64\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-32\right]s-256\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+256m_f^2\right\} \\
&= \frac{32}{9}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \left\{\frac{1}{M_X^2}s^2+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2\right]s-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+16m_f^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{d\tilde{\sigma}_{T,f}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_{p'}\left|v-v'\right|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{9}\sum_{\text{spins}}\sum_{i,j}\left|\mathcal{M}\right|^2 \\
&= \frac{1}{16\pi^2 s}\frac{1}{4}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{32}{9}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s^2+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2\right]s-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2}+16m_f^2\right\} \\
&= \frac{1}{18\pi^2}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2\right]-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2 s}+16\frac{m_f^2}{s}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma}_{T,\text{ann}} &= \sum_f c_f\int d\Omega\left(\frac{d\tilde{\sigma}_{T,f}}{d\Omega}\right)_{\text{CM}} \\
&= \sum_f c_f\int\frac{1}{9\pi}\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2\left\{\frac{1}{M_X^2}s+\left[4\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^4}-2\right]-16\frac{\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2}{M_X^2 s}+16\frac{m_f^2}{s}\right\}\sin\theta d\theta \\
&= \frac{2}{9\pi}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{\frac{1}{M_X^2}s+\left(4\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^4}-2\right)-16\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2 s}+16\frac{m_f^2}{s}\right\} \\
&= \frac{2}{9\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s^2+\left(4\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3M_X^2}-2M_X^2\right)s-16\frac{|\mathbf{p}|^2|\mathbf{k}|^2}{3}+16m_f^2M_X^2\right\} \\
&= \frac{2}{9\pi M_X^2}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{s}\left\{s^2+\frac{1}{12M_X^2}s\left(s-4M_X^2\right)\left(s-4m_f^2\right)-2M_X^2 s-\frac{1}{3}\left(s-4M_X^2\right)\left(s-4m_f^2\right)+16m_f^2M_X^2\right\} \\
&= \frac{2}{9\pi M_X^4}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{12s}\left\{12M_X^2s^2+s\left(s-4M_X^2\right)\left(s-4m_f^2\right)-24M_X^4s-4M_X^2\left(s-4M_X^2\right)\left(s-4m_f^2\right)+192m_f^2M_X^4\right\} \\
&= \frac{2}{9\pi M_X^4}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{12s}\left\{12M_X^2s^2-24M_X^4s+192m_f^2M_X^4+\left(s-4M_X^2\right)^2\left(s-4m_f^2\right)\right\} \\
&= \frac{2}{9\pi M_X^4}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\frac{1}{12s}\left\{s^3+4\left(M_X^2-m_f^2\right)s^2+32m_f^2M_X^2s-8M_X^4s+128m_f^2M_X^4\right\} \\
&= \frac{1}{54\pi M_X^4}\sum_f\left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f\sqrt{\frac{s-4m_f^2}{s-4M_X^2}}\left\{s^2+4\left(M_X^2-m_f^2\right)s+32m_f^2M_X^2-8M_X^4+128\frac{m_f^2M_X^4}{s}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\sigma}_{T,\mathrm{ann}} v &= \frac{1}{54\pi M_X^4}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 c_f v \sqrt{\frac{s-4m_f^2}{s-4M_X^2}} \left\{ s^2 + 4\left(M_X^2 - m_f^2\right)s + 32 m_f^2 M_X^2 - 8M_X^4 + 128\frac{m_f^2 M_X^4}{s} \right\} \\
&\simeq \frac{1}{54\pi M_X^4}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 2\sqrt{1-\frac{m_f^2}{M_X^2}} \left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right] \left\{ \left(4M_X^2 + M_X^2 v^2\right)^2 + 4\left(M_X^2 - m_f^2\right)\left(4M_X^2 + M_X^2 v^2\right) + 32 m_f^2 M_X^2 - 8M_X^4 + 32\left(1-\frac{v^2}{4}\right) m_f^2 M_X^2 \right\} \\
&\simeq \frac{1}{54\pi M_X^4}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 2\sqrt{1-\frac{m_f^2}{M_X^2}} \left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right] \left[24M_X^4 + 48 m_f^2 M_X^2 + 12\left(1-\frac{m_f^2}{M_X^2}\right) M_X^4 v^2 \right] \\
&= \frac{1}{54\pi}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 2\sqrt{1-\frac{m_f^2}{M_X^2}} \left[1+\frac{v^2}{8\left(1-m_f^2/M_X^2\right)}\right] \left[24\left(1+2\frac{m_f^2}{M_X^2}\right) + 12\left(1-\frac{m_f^2}{M_X^2}\right) v^2 \right] \\
&\simeq \frac{1}{9\pi}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \sqrt{1-\frac{m_f^2}{M_X^2}} \left[8\left(1+2\frac{m_f^2}{M_X^2}\right) + \frac{1+2m_f^2/M_X^2}{1-m_f^2/M_X^2} v^2 + 4\left(1-\frac{m_f^2}{M_X^2}\right) v^2 \right] \\
&= \frac{1}{9\pi}\sum_f \left(\frac{\tilde{K}_{T,f}}{\sqrt{2}}\right)^2 \sqrt{1-\frac{m_f^2}{M_X^2}} \left[8\left(1+2\frac{m_f^2}{M_X^2}\right) + \frac{5-6m_f^2/M_X^2 + 4m_f^4/M_X^4}{1-m_f^2/M_X^2} v^2 \right]
\end{aligned}
$$

设$\sigma v=\sum_{i=0}^{\infty}a_{(i)}v^{2i}$，则$\left\langle\sigma v\right\rangle=\dfrac{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}\sum_{i=0}^{\infty}a_{(i)}v^{2i}}{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}}=\sum_{i=0}^{\infty}\dfrac{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}a_{(i)}v^{2i}}{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}}$

非相对论近似下，$d\Pi\equiv\dfrac{d^3\mathbf{p}}{(2\pi)^3 2E}=\dfrac{d^3\mathbf{p}}{(2\pi)^3 2m}$，$s\simeq 4m^2+m^2v^2=4m^2\left(1+v^2/4\right)$，$E_1+E_2=\sqrt{s}\simeq 2m\left(1+v^2/4\right)^{1/2}\simeq 2m+mv^2/4$；

在替换$\mathbf{p}_2'\equiv\mathbf{p}_2-\mathbf{p}_1$下，$\int d^3\mathbf{p}_1 d^3\mathbf{p}_2=\int d^3\mathbf{p}_1 d^3\mathbf{p}_2'$，而$\left|\mathbf{p}_2'\right|\simeq mv$；并注意到$\int_0^{\infty}y^{2n}e^{-ay^2}dy=\dfrac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\dfrac{\pi}{a}}$；故

$$\frac{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}a_{(i)}v^{2i}}{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}}=a_{(i)}\frac{\int\frac{d^3\mathbf{p}_1}{(2\pi)^3 2m}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2m}e^{-\left(2m+mv^2/4\right)/T}v^{2i}}{\int\frac{d^3\mathbf{p}_1}{(2\pi)^3 2m}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2m}e^{-\left(2m+mv^2/4\right)/T}}=a_{(i)}\frac{\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 e^{-mv^2/4T}v^{2i}}{\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 e^{-mv^2/4T}}$$

$$=a_{(i)}\frac{\int d^3\mathbf{p}_1 d^3\mathbf{p}_2' e^{-xv^2/4}v^{2i}}{\int d^3\mathbf{p}_1 d^3\mathbf{p}_2' e^{-xv^2/4}}=a_{(i)}\frac{\int d^3\mathbf{p}_2' e^{-xv^2/4}v^{2i}}{\int d^3\mathbf{p}_2' e^{-xv^2/4}}=a_{(i)}\frac{4\pi m^3\int dvv^2e^{-xv^2/4}v^{2i}}{4\pi m^3\int dvv^2e^{-xv^2/4}}$$

$$=a_{(i)}\frac{\int dvv^{2i+2}e^{-xv^2/4}}{\int dvv^2e^{-xv^2/4}}=a_{(i)}\frac{\int dvv^{2i+2}e^{-xv^2/4}}{\int dvv^2e^{-xv^2/4}}=a_{(i)}\frac{(4/x)^{i+2}\int dyy^{2i+2}e^{-y^2}}{(4/x)^2\int dyy^2e^{-y^2}}$$

$$=a_{(i)}\left(\frac{4}{x}\right)^i\frac{\int dyy^{2i+2}e^{-y^2}}{\int dyy^2e^{-y^2}}=a_{(i)}\left(\frac{4}{x}\right)^i\frac{\frac{(2i+2-1)!!}{2^{i+1+1}}\sqrt{\pi}}{\frac{(2-1)!!}{2^{1+1}}\sqrt{\pi}}=a_{(i)}\left(\frac{4}{x}\right)^i\frac{(2i+1)!!}{2^i}$$

$$=a_{(i)}\frac{2^i(2i+1)!!}{x^i}$$

于是$\left\langle\sigma v\right\rangle=\sum_{i=0}^{\infty}\dfrac{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}a_{(i)}v^{2i}}{\int d\Pi_1 d\Pi_2 e^{-(E_1+E_2)/T}}=\sum_{i=0}^{\infty}a_{(i)}\dfrac{2^i(2i+1)!!}{x^i}$

$$x_f=\ln\left[c(c+2)\sqrt{\frac{45}{8}}\frac{gm_\chi M_{\rm pl}\left\langle\sigma v\right\rangle}{2\pi^3\sqrt{x_f g_*}}\right]=\ln\left[c(c+2)\sqrt{\frac{45}{8}}\frac{gm_\chi M_{\rm pl}\sum_{i=0}^{\infty}a_{(i)}2^i(2i+1)!!/x_f^i}{2\pi^3\sqrt{x_f g_*}}\right]$$

$$\frac{d\Delta}{\Delta^2}=-c_0\frac{\left\langle\sigma v\right\rangle}{x^2}dx$$

$$-\frac{1}{\Delta_\infty}+\frac{1}{\Delta_f}=\int_{x=x_f}^{x=\infty}\frac{d\Delta}{\Delta^2}=-c_0\int_{x_f}^{\infty}\frac{\left\langle\sigma v\right\rangle}{x^2}dx=-c_0\int_{x_f}^{\infty}\frac{\sum_{i=0}^{\infty}a_{(i)}\frac{2^i(2i+1)!!}{x^i}}{x^2}dx$$

$$=-c_0\sum_{i=0}^{\infty}a_{(i)}2^i(2i+1)!!\int_{x_f}^{\infty}x^{-(i+2)}dx=-c_0\sum_{i=0}^{\infty}a_{(i)}2^i(2i+1)!!\left.\frac{x^{-(i+1)}}{-(i+1)}\right|_{x_f}^{\infty}$$

$$=-c_0\sum_{i=0}^{\infty}a_{(i)}2^i(2i+1)!!\frac{x_f^{-(i+1)}}{(i+1)}=-c_0\frac{\sum_{i=0}^{\infty}a_{(i)}\frac{2^i(2i+1)!!}{(i+1)x_f^i}}{x_f}$$

$$\Omega_\chi h^2\simeq 1.0665207\times 10^9\ \mathrm{GeV}^{-1}\left(\frac{T_0}{2.75\ \mathrm{K}}\right)^3\frac{x_f}{M_{\rm pl}\sqrt{g_*\left(T_f\right)}\sum_{i=0}^{\infty}a_{(i)}\frac{2^i(2i+1)!!}{(i+1)x_f^i}}$$

将σv展开至$O\left(v^4\right)$，有$x_f=\ln\left[c(c+2)\sqrt{\dfrac{45}{8}}\dfrac{gm_\chi M_{\rm pl}\left(a_{(0)}+6a_{(1)}/x_f+60a_{(2)}/x_f^2\right)}{2\pi^3\sqrt{x_f g_*}}\right]$，

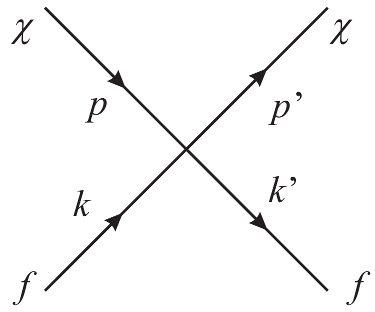
$\Omega_\chi h^2\simeq 1.0665207\times 10^9\ \mathrm{GeV}^{-1}\left(\dfrac{T_0}{2.75\ \mathrm{K}}\right)^3\dfrac{x_f}{M_{\rm pl}\sqrt{g_*\left(T_f\right)}\left(a_{(0)}+3a_{(1)}/x_f+20a_{(1)}/x_f^2\right)}$。

（二）散射方面

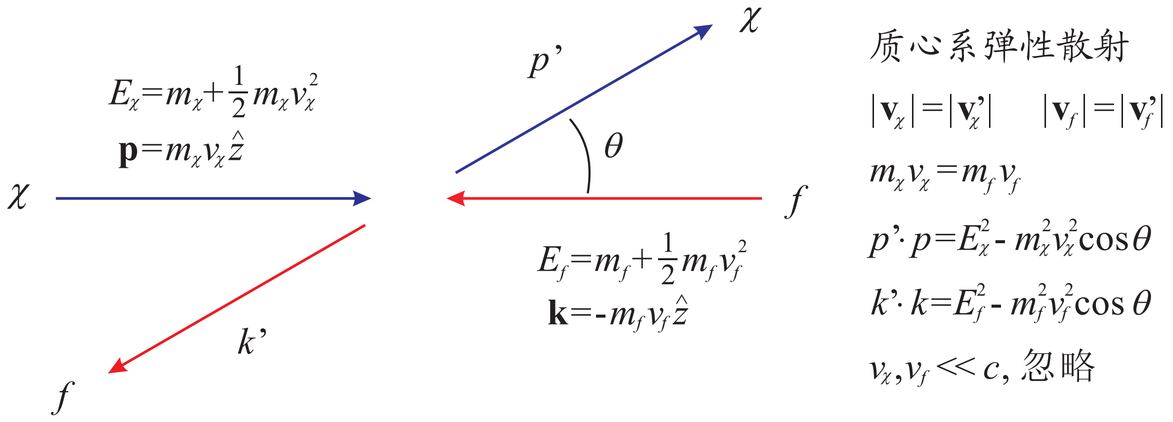
一、Dirac fermionic WIMP

设暗物质粒子 χ 和 $\bar{\chi}$ 是 Dirac 旋量，

f 和 $\bar{f}$ 是标准模型中的费米子



考虑质心系下的弹性散射，如下图所示，



质心系下的非相对论近似，

v 是两暗物质粒子间的相对速度，

两暗物质粒子的速度分别是 $\frac{\mathbf{v}}{2}$ 和 $-\frac{\mathbf{v}}{2}$，能量均为 $E_\chi = m_\chi + \frac{1}{2} m_\chi \left(\frac{v}{2}\right)^2$，

$v << c$ 时，$p \cdot p' \simeq E_\chi^2 \simeq m_\chi^2$，$k \cdot k' \simeq E_f^2 \simeq m_f^2$

$$s = (p + p')^2 = E_{\mathrm{cm}}^2 = (2E_\chi)^2 \simeq 4m_\chi^2,\quad (s - 4m_\chi^2) \sim \mathcal{O}(v^2)$$

$$|\mathbf{p}|^2 = m_\chi^2 v_\chi^2 = \frac{1}{4} m_\chi^2 v^2,\quad |\mathbf{k}|^2 = m_f^2 v_f^2 = m_\chi^2 v_\chi^2 = \frac{1}{4} m_\chi^2 v^2$$

1. Scalar 耦合：$\mathcal{L}_{\mathrm{int}} = \frac{G_{S,f}}{\sqrt{2}} \bar{\chi}\chi \bar{f} f$

$$i\mathcal{M} = i\frac{G_{S,f}}{\sqrt{2}} \bar{u}(p') u(p) \bar{u}(k') u(k)$$

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}} \frac{G_{S,f}^2}{2} \left[\bar{u}(p') u(p) \bar{u}(k') u(k)\right]\left[\bar{u}(p') u(p) \bar{u}(k') u(k)\right]^* \\
&= \frac{1}{4}\frac{G_S^2}{2}\sum_{\text{spins}} \left[\bar{u}(p') u(p) \bar{u}(p) u(p')\right]\left[\bar{u}(k') u(k) \bar{u}(k) u(k')\right] \\
&= \frac{1}{4}\frac{G_S^2}{2} \mathrm{tr}\left[u(p')\bar{u}(p') u(p)\bar{u}(p)\right] \mathrm{tr}\left[u(k')\bar{u}(k') u(k)\bar{u}(k)\right] \\
&= \frac{1}{4}\frac{G_S^2}{2} \mathrm{tr}\left[(\not{p}' + m_\chi)(\not{p} + m_\chi)\right] \mathrm{tr}\left[(\not{k}' + m_f)(\not{k} + m_f)\right] \\
&= \frac{1}{4}\frac{G_S^2}{2} 16\left[m_\chi^2 m_f^2 + (k \cdot k')(p \cdot p') + m_\chi^2 (k \cdot k') + m_f^2 (p \cdot p')\right] \\
&\simeq \frac{1}{4}\frac{G_S^2}{2} 16\left[m_\chi^2 m_f^2 + m_\chi^2 m_f^2 + m_\chi^2 m_f^2 + m_\chi^2 m_f^2\right] \\
&= \frac{G_S^2}{2} 16 m_\chi^2 m_f^2
\end{aligned}$$

（1）对于核子 N（n,p），拉氏量相应项 $\frac{G_{S,N}}{2} \bar{\chi}\chi \bar{\psi}_N \psi_N$，

对应着 $\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\frac{G_{S,N}^2}{2}16m_\chi^2 m_N^2$，其中

$$
\begin{aligned}
G_{S,N}&=\sum_q G_{S,q} f_q^N \frac{m_N}{m_q} \\
&=\sum_{q=u,t,s} G_{S,q} f_q^N \frac{m_N}{m_q}+\sum_{q=c,b,t} G_{S,q} f_Q^N \frac{m_N}{m_q}
\end{aligned},
$$

$$
\text{而 } f_Q^N=\frac{2}{27}\left(1-\sum_{q=u,d,s} f_q^N\right),
$$

依照 Beltran 所引 J. R. Ellis, A. Ferstl, and K. A. Olive, Phys. Lett. B 481, 304(2000)，form factor f_q^N 的值为

$f_u^p=0.020\pm0.004$， $f_d^p=0.026\pm0.005$， $f_s^p=0.118\pm0.062$，

$f_u^n=0.014\pm0.003$， $f_d^p=0.036\pm0.008$， $f_s^p=0.118\pm0.062$。

$$
\begin{aligned}
\left(\frac{d\sigma_{\chi N}}{d\Omega}\right)_{\text{CM}}&=\frac{1}{2E_p 2E_k|v_p-v_k|}\frac{|\mathbf{k}'|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&\simeq\frac{1}{2m_\chi 2m_N\left(\frac{m_N}{m_\chi}v_N+v_N\right)}\frac{m_N v_N}{(2\pi)^2 4(m_\chi+m_N)}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&=\frac{1}{4(m_N+m_\chi)^2 16\pi^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&=\frac{1}{4(m_N+m_\chi)^2 16\pi^2}\frac{G_{S,N}^2}{2}16m_\chi^2 m_N^2 \\
&=\frac{m_\chi^2 m_N^2}{4\pi^2(m_N+m_\chi)^2}\frac{G_{S,N}^2}{2}
\end{aligned}
$$

$$
\text{故 } \sigma_{\chi N}^{\text{CM}}=\frac{m_\chi^2 m_N^2}{\pi(m_N+m_\chi)^2}\frac{G_{S,N}^2}{2}
$$

（2）对于原子数为 A，原子序数为 Z 的原子核与 χ 散射，

$$
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=16m_\chi^2 m_A^2\left[Z\frac{G_{S,p}}{\sqrt{2}}+(A-Z)\frac{G_{S,n}}{\sqrt{2}}\right]^2 \quad \text{（SI 通用）}
$$

$$
\begin{aligned}
\left(\frac{d\sigma_{\chi A}}{d\Omega}\right)_{\text{CM}}&=\frac{1}{2E_p 2E_k|v_p-v_k|}\frac{|\mathbf{k}'|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&\simeq\frac{1}{4(m_A+m_\chi)^2 16\pi^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&=\frac{1}{4(m_A+m_\chi)^2 16\pi^2}16m_\chi^2 m_A^2\left[Z\frac{G_{S,p}}{\sqrt{2}}+(A-Z)\frac{G_{S,n}}{\sqrt{2}}\right]^2 \\
&=\frac{m_\chi^2 m_A^2}{4\pi^2(m_A+m_\chi)^2}\left[Z\frac{G_{S,p}}{\sqrt{2}}+(A-Z)\frac{G_{S,n}}{\sqrt{2}}\right]^2
\end{aligned}
$$

$$
\text{故 } \sigma_{\chi A}^{\text{CM}}=\frac{m_\chi^2 m_A^2}{\pi(m_A+m_\chi)^2}\left[Z\frac{G_{S,p}}{\sqrt{2}}+(A-Z)\frac{G_{S,n}}{\sqrt{2}}\right]^2
$$

2. Pseudoscalar 耦合： $\mathcal{L}_{\text{int}}=\frac{G_{P,f}}{\sqrt{2}}\bar{\chi}\gamma_5\chi\bar{f}\gamma_5 f$

$$
i\mathcal{M}=i\frac{G_{P,f}}{\sqrt{2}}\bar{u}(p')\gamma_5 u(p)\bar{u}(k')\gamma_5 u(k)
$$

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{P,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma_5 u(p)\bar{u}(k')\gamma_5 u(k)\right]\left[\bar{u}(p')\gamma_5 u(p)\bar{u}(k')\gamma_5 u(k)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{P,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5 u(p')\right]\left[\bar{u}(k')\gamma_5 u(k)\bar{u}(k)\gamma_5 u(k')\right] \\
&= \frac{1}{4}\left(\frac{G_{P,f}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(p')\bar{u}(p')\gamma_5 u(p)\bar{u}(p)\gamma_5\right]\operatorname{tr}\left[u(k')\bar{u}(k')\gamma_5 u(k)\bar{u}(k)\gamma_5\right] \\
&= \frac{1}{4}\left(\frac{G_{P,f}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{p}'+m_\chi\right)\gamma_5\left(\not{p}+m_\chi\right)\gamma_5\right]\operatorname{tr}\left[\left(\not{k}'+m_f\right)\gamma_5\left(\not{k}+m_f\right)\gamma_5\right] \\
&= \frac{1}{4}\left(\frac{G_{P,f}}{\sqrt{2}}\right)^2 2\left[2s^2-8\left(m_\chi^2+m_f^2\right)s+32m_\chi^2 m_f^2\right] \\
&= \left(\frac{G_{P,f}}{\sqrt{2}}\right)^2\left(s-4m_\chi^2\right)\left(s-4m_f^2\right) \\
&\propto s-4m_\chi^2 \\
&\sim \mathcal{O}\left(v^2\right)
\end{aligned}$$

故 $\sigma_{\chi N}^{\text{CM}} \sim \mathcal{O}\left(v^2\right)$， $\sigma_{\chi A}^{\text{CM}} \sim \mathcal{O}\left(v^2\right)$

3. Vector 耦合： $\mathcal{L}_{\text{int}} = \frac{G_{V,f}}{\sqrt{2}}\bar{\chi}\gamma^\mu\chi\bar{f}\gamma_\mu f$

$$i\mathcal{M} = i\frac{G_{V,f}}{\sqrt{2}}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k)$$

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{V,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k)\right]\left[\bar{u}(p')\gamma^\nu u(p)\bar{u}(k')\gamma_\nu u(k)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{V,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu u(p')\right]\left[\bar{u}(k')\gamma_\mu u(k)\bar{u}(k)\gamma_\nu u(k')\right] \\
&= \frac{1}{4}\left(\frac{G_{V,f}}{\sqrt{2}}\right)^2\operatorname{tr}\left[u(p')\bar{u}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu\right]\operatorname{tr}\left[u(k')\bar{u}(k')\gamma_\mu u(k)\bar{u}(k)\gamma_\nu\right] \\
&= \frac{1}{4}\left(\frac{G_{V,f}}{\sqrt{2}}\right)^2\operatorname{tr}\left[\left(\not{p}'+m_\chi\right)\gamma^\mu\left(\not{p}+m_\chi\right)\gamma^\nu\right]\operatorname{tr}\left[\left(\not{k}'+m_f\right)\gamma_\mu\left(\not{k}+m_f\right)\gamma_\nu\right] \\
&= \frac{1}{4}\left(\frac{G_{V,f}}{\sqrt{2}}\right)^2 2\left[2s^2-8\left(m_\chi^2+m_f^2\right)s+32\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+64m_\chi^2 m_f^2\right] \\
&= \left(\frac{G_{V,f}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_f^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_f^2\right]
\end{aligned}$$

（1）对于核子 N（n,p），拉氏量相应项 $\frac{G_{V,N}}{2}\bar{\chi}\gamma^\mu\chi\bar{\psi}_N\gamma_\mu\psi_N$，

对应着 $\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]$，其中

$$G_{V,p} = 2G_{V,u}+G_{V,d},\quad G_{V,n} = G_{V,u}+2G_{V,d},$$

$$\left(\frac{d\sigma_{\chi N}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_k\left|v_p-v_k\right|}\frac{|\mathbf{k}'|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$\simeq\frac{1}{2m_\chi 2m_N\left(\frac{m_N}{m_\chi}v_N+v_N\right)}\frac{m_N v_N}{(2\pi)^2 4\left(m_\chi+m_N\right)}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$=\frac{1}{4\left(m_N+m_\chi\right)^2 16\pi^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$=\frac{1}{4\left(m_N+m_\chi\right)^2 16\pi^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]$$
$$=\frac{1}{64\pi^2\left(m_N+m_\chi\right)^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]$$

$$\sigma_{\chi N}^{\text{CM}}=\int d\Omega\left(\frac{d\sigma_{\chi N}}{d\Omega}\right)_{\text{CM}}=\int\frac{1}{32\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]\sin\theta d\theta$$
$$=\frac{1}{16\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[s^2-4\left(m_\chi^2+m_N^2\right)s+\frac{16}{3}|\mathbf{p}|^2|\mathbf{k}|^2+32m_\chi^2 m_N^2\right]$$
$$\simeq\frac{1}{16\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2\left[16m_\chi^4-16m_\chi^2\left(m_\chi^2+m_N^2\right)+\frac{16}{3}\frac{1}{4}m_\chi^2 v^2\frac{1}{4}m_\chi^2 v^2+32m_\chi^2 m_N^2\right]$$
$$\simeq\frac{m_\chi^2 m_N^2}{\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{V,N}}{\sqrt{2}}\right)^2$$

（2）对于原子数为 A，原子序数为 Z 的原子核与 χ 散射，

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2=\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]\left[Z\frac{G_{V,p}}{\sqrt{2}}+(A-Z)\frac{G_{V,n}}{\sqrt{2}}\right]^2$$
$$\left(\frac{d\sigma_{\chi A}}{d\Omega}\right)_{\text{CM}}=\frac{1}{2E_p 2E_k\left|v_p-v_k\right|}\frac{|\mathbf{k}'|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$\simeq\frac{1}{4\left(m_A+m_\chi\right)^2 16\pi^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$
$$=\frac{1}{4\left(m_A+m_\chi\right)^2 16\pi^2}\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]\left[Z\frac{G_{V,p}}{\sqrt{2}}+(A-Z)\frac{G_{V,n}}{\sqrt{2}}\right]^2$$
$$=\frac{m_\chi^2 m_A^2}{64\pi^2\left(m_A+m_\chi\right)^2}\left[s^2-4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2+32m_\chi^2 m_N^2\right]\left[Z\frac{G_{V,p}}{\sqrt{2}}+(A-Z)\frac{G_{V,n}}{\sqrt{2}}\right]^2$$

故 $\sigma_{\chi A}^{\text{CM}}=\dfrac{m_\chi^2 m_A^2}{\pi\left(m_A+m_\chi\right)^2}\left[Z\dfrac{G_{V,p}}{\sqrt{2}}+(A-Z)\dfrac{G_{V,n}}{\sqrt{2}}\right]^2$

（结果与 scalar 耦合的情况一样，除了耦合常数的联系方式不同）

3. Axial-Vector 耦合： $\mathcal{L}_{\text{int}}=\dfrac{G_{A,f}}{\sqrt{2}}\bar{\chi}\gamma^\mu\gamma_5\chi\bar{f}\gamma_\mu\gamma_5 f$

$$i\mathcal{M}=i\frac{G_{A,f}}{\sqrt{2}}\bar{u}(p')\gamma^\mu\gamma_5 u(p)\bar{u}(k')\gamma_\mu\gamma_5 u(k)$$

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{A,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma^\mu\gamma_5 u(p)\bar{u}(k')\gamma_\mu\gamma_5 u(k)\right]\left[\bar{u}(p')\gamma^\nu\gamma_5 u(p)\bar{u}(k')\gamma_\nu\gamma_5 u(k)\right]^* \\
&= \frac{1}{4}\sum_{\text{spins}}\left(\frac{G_{A,f}}{\sqrt{2}}\right)^2\left[\bar{u}(p')\gamma^\mu\gamma_5 u(p)\bar{u}(p)\gamma^\nu\gamma_5 u(p')\right]\left[\bar{u}(k')\gamma_\mu\gamma_5 u(k)\bar{u}(k)\gamma_\nu\gamma_5 u(k')\right] \\
&= \frac{1}{4}\left(\frac{G_{A,f}}{\sqrt{2}}\right)^2 \operatorname{tr}\left[u(p')\bar{u}(p')\gamma^\mu\gamma_5 u(p)\bar{u}(p)\gamma^\nu\gamma_5\right]\operatorname{tr}\left[u(k')\bar{u}(k')\gamma_\mu\gamma_5 u(k)\bar{u}(k)\gamma_\nu\gamma_5\right] \\
&= \frac{1}{4}\left(\frac{G_{A,f}}{\sqrt{2}}\right)^2 \operatorname{tr}\left[\left(\not{p}'+m_\chi\right)\gamma^\mu\gamma_5\left(\not{p}+m_\chi\right)\gamma^\nu\gamma_5\right]\operatorname{tr}\left[\left(\not{k}'+m_f\right)\gamma_\mu\gamma_5\left(\not{k}+m_f\right)\gamma_\nu\gamma_5\right] \\
&= \frac{1}{4}\left(\frac{G_{A,f}}{\sqrt{2}}\right)^2 2\left[2s^2+8\left(m_\chi^2+m_f^2\right)s+32\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right] \\
&= \left(\frac{G_{A,f}}{\sqrt{2}}\right)^2\left[s^2+4\left(m_\chi^2+m_f^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]
\end{aligned}
$$

（1）对于核子 N（n,p），拉氏量相应项 $\frac{G_{A,N}}{2}\bar{\chi}\gamma^\mu\gamma_5\chi\bar{\psi}_N\gamma_\mu\gamma_5\psi_N$，

对应着 $\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \left(\frac{G_{A,N}}{\sqrt{2}}\right)^2\left[s^2+4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]$，其中

$$G_{A,N} = \sum_{q=u,d,s} G_{A,q}\Delta_q^N \text{ ,}$$

$\Delta_u^p = \Delta_d^n = 0.78\pm 0.02,\quad \Delta_d^p = \Delta_u^n = -0.48\pm 0.02,\quad \Delta_s^p = \Delta_s^n = -0.15\pm 0.02$（Beltran *et al.* 采用）

$\Delta_u^p = \Delta_d^n = 0.842\pm 0.012,\quad \Delta_d^p = \Delta_u^n = -0.427\pm 0.013,\quad \Delta_s^p = \Delta_s^n = -0.085\pm 0.018$（Belanger *et al.* micrOMEGAs 2.2, arXiv:0803.2360 提到）

$$
\begin{aligned}
\left(\frac{d\sigma_{\chi N}}{d\Omega}\right)_{\text{CM}} &= \frac{1}{2E_p 2E_k\left|v_p-v_k\right|}\frac{|\mathbf{k}'|}{(2\pi)^2 4E_{\text{cm}}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&\simeq \frac{1}{2m_\chi 2m_N\left(\frac{m_N}{m_\chi}v_N+v_N\right)}\frac{m_N v_N}{(2\pi)^2 4\left(m_\chi+m_N\right)}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2\left(m_N+m_\chi\right)^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{64\pi^2\left(m_N+m_\chi\right)^2}\left(\frac{G_{A,N}}{\sqrt{2}}\right)^2\left[s^2+4\left(m_\chi^2+m_N^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\chi N}^{\text{CM}} &= \int d\Omega\left(\frac{d\sigma_{\chi N}}{d\Omega}\right)_{\text{CM}} = \int\frac{1}{32\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{A,N}}{\sqrt{2}}\right)^2\left[s^2+4\left(m_\chi^2+m_f^2\right)s+16\left(|\mathbf{p}||\mathbf{k}|\cos\theta\right)^2\right]\sin\theta d\theta \\
&= \frac{1}{16\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{A,N}}{\sqrt{2}}\right)^2\left[s^2+4\left(m_\chi^2+m_f^2\right)s+\frac{16}{3}|\mathbf{p}|^2|\mathbf{k}|^2\right] \\
&\simeq \frac{1}{16\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{A,N}}{\sqrt{2}}\right)^2\left[16m_\chi^4+16m_\chi^2\left(m_\chi^2+m_N^2\right)+\frac{16}{3}\frac{1}{4}m_\chi^2v^2\frac{1}{4}m_\chi^2v^2\right] \\
&\simeq \frac{2m_\chi^4+m_\chi^2m_N^2}{\pi\left(m_N+m_\chi\right)^2}\left(\frac{G_{A,N}}{\sqrt{2}}\right)^2
\end{aligned}
$$