AM205 HW4 Writeup

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Problem 1
(a)

$$T_j^n = \frac{u(t^{n+1}, x_j) - 2u(t^n, x_j) + u(t^{n-1}, x_j)}{\Delta t^2} - c^2 \frac{u(t^n, x_{j+1}) - 2u(t^n, x_j) + u(t^n, x_{j-1})}{\Delta x^2}$$

Expand the above truncation error T_j^n using Taylor Series at (t^n, x_j) will give:

$$u(t^{n+1}, x_j) = u(t^n, x_j) + \Delta t u_t(t^n, x_j) + \frac{\Delta t^2}{2} u_{tt}(t^n, x_j) + \frac{\Delta t^3}{6} u_{ttt}(t^n, x_j) + \frac{\Delta t^4}{24} u_{tttt}(t^n, x_j) + h.o.t$$

$$u(t^{n-1}, x_j) = u(t^n, x_j) - \Delta t u_t(t^n, x_j) + \frac{\Delta t^2}{2} u_{tt}(t^n, x_j) - \frac{\Delta t^3}{6} u_{ttt}(t^n, x_j) + \frac{\Delta t^4}{24} u_{tttt}(t^n, x_j) + h.o.t$$

$$u(t^n, x_{j+1}) = u(t^n, x_j) + \Delta x u_x(t^n, x_j) + \frac{\Delta x^2}{2} u_{xx}(t^n, x_j) + \frac{\Delta x^3}{6} u_{xxx}(t^n, x_j) + \frac{\Delta x^4}{24} u_{xxxx}(t^n, x_j) + h.o.t$$

$$u(t^n, x_{j-1}) = u(t^n, x_j) - \Delta x u_x(t^n, x_j) + \frac{\Delta x^2}{2} u_{xx}(t^n, x_j) - \frac{\Delta x^3}{6} u_{xxx}(t^n, x_j) + \frac{\Delta x^4}{24} u_{xxxx}(t^n, x_j) + h.o.t$$

$$\begin{aligned} T_j^n &= u_{tt}(t^n, x_j) + \frac{\Delta t^2}{12} u_{tttt}(t^n, x_j) - c^2 u_{xx}(t^n, x_j) - c^2 \frac{\Delta x^2}{12} u_{xxxx}(t^n, x_j) + h.o.t \\ &= \frac{1}{12}(\Delta t^2 u_{tttt}(t^n, x_j) - c^2 \Delta x^2 u_{xxxx}(t^n, x_j)) + h.o.t \end{aligned}$$

Since we have Δt, Δx to the **2nd order** in the truncation error, this numerical scheme to solve the 1D wave equation is **second-order accurate**.

(b)

$$U_j^{n+1} - 2U_j^n + U_j^{n-1} - \frac{c^2 \Delta t^2}{\Delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n) = 0$$

Let $v = \frac{c\Delta t}{\Delta x}$,

$$U_j^{n+1} = 2U_j^n - U_j^{n-1} + v^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$\lambda(k)^2 e^{ikj\Delta x} = \lambda(k)(2 - 2v^2)e^{ikj\Delta x} - e^{ikj\Delta x} + \lambda(k)v^2 e^{ik(j+1)\Delta x} + +\lambda(k)v^2 e^{ik(j-1)\Delta x}$$

$$\lambda(k)^2 = \lambda(k)(2 - 2v^2) - 1 + \lambda(k)v^2 2cos(k\Delta x)$$

$$\lambda(k)^2 - 2[1 - v^2(1 - cos(k\Delta x))]\lambda(k) + 1 = 0$$

Let $b = 1 - v^2(1 - cos(k\Delta x)) = 1 - v^2 2sin^2 \frac{k\Delta x}{2}$, we get

$$\lambda(k)^2 - 2b\lambda(k) + 1 = 0$$

The discriminant of this quadratic equations is $\Delta = 4(b^2 - 1)$.

- If $b^2 < 1$, it has two imaginary roots:

$$\lambda_1(k) = b + \sqrt{1 - b^2}i \quad \lambda_2(k) = b - \sqrt{1 - b^2}i$$

$$|\lambda_1(k)|^2 = (b + \sqrt{1 - b^2}i)(b - \sqrt{1 - b^2}i) = b^2 + (1 - b^2) = 1$$

$$|\lambda_2(k)|^2 = (b - \sqrt{1 - b^2}i)(b + \sqrt{1 - b^2}i) = b^2 + (1 - b^2) = 1$$

- If $b^2 = 1$, it has two same real roots:

$$\lambda_1(k) = \lambda_2(k) = 1 \quad or \quad \lambda_1(k) = \lambda_2(k) = -1$$

$$|\lambda_1(k)| = |\lambda_2(k)| = 1$$

- If $b^2 > 1$, $b > 1$ or $b < -1$

$$\lambda_1(k) = b + \sqrt{b^2 - 1} \quad \lambda_2(k) = b - \sqrt{b^2 - 1}$$

$$|\lambda_1(k)|^2 = 2b^2 - 1 + 2b\sqrt{b^2 - 1} > 1 \quad (b > 1)$$

$$|\lambda_2(k)|^2 = 2b^2 - 1 - 2b\sqrt{b^2 - 1} > 1 \quad (b < -1)$$

This case does not satisfy the CFL condition that $|v| \leq 1$

In conclusion, with $|v| = |\frac{c\Delta t}{\Delta x}| \leq 1$, for each k, both $\lambda_1(k)$ and $\lambda_2(k)$ have magnitude = 1. This numerical scheme is stable with $|v| = |\frac{c\Delta t}{\Delta x}| \leq 1$.

Problem 2

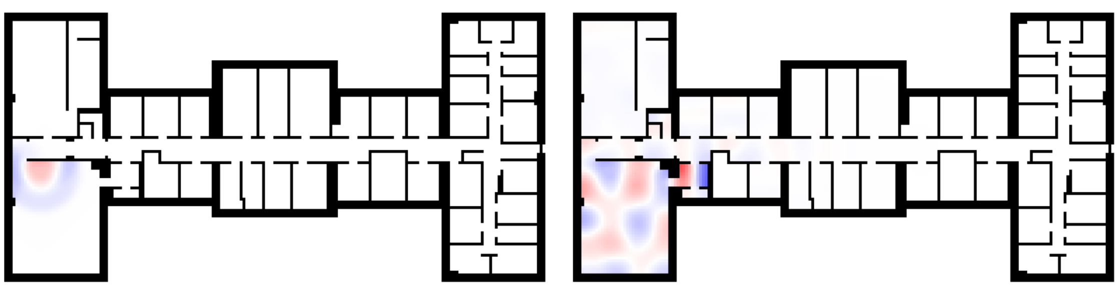
(a)

The program reads in the map of the Pierce Hall and update the pressure field over the map for each $\Delta t = \frac{h}{2c}$. The evolution of the 2D pressure field is recorded as a 3D matrix (100 x 200 x time_steps = 2001).

The duration of total iterations is approximately 10 mins on my machine. I saved the matrix to local file 'pierce_t2000.npy' so that we can simply np.load() to restore results. See implementation details in code.

(b)

See 2D plots of the pressure field at the in Figure 1 and 2.



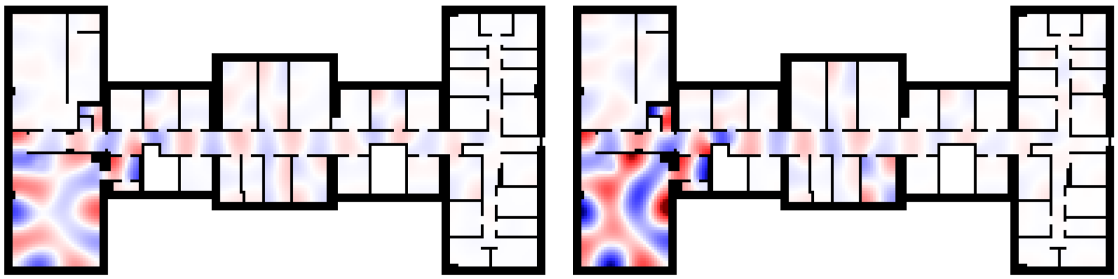
(a) t = 0.015s (b) t = 0.105s

Figure 1: Problem 2:(b) - t = 0.015s, 0.105s

(c)

Time (sec) when the three people first hear the sound, defined as when $|p(t)|$ at their location exceeds 10^{-3} Pa:

- C: 0.0736
- G: 0.1088
- M: 0.2310



(a) t = 0.505s (b) t = 1.005s

Figure 2: Problem 2:(b) - t = 0.505s, 1.005s

The results are reasonable: the closer the person is to the source region, the earlier he would first hear the sound.

(d)

See the plot p(t) at the three people's location over the interval $0 \leq t \leq 1$s in Figure 3. Since **G**'s location has the largest magnitude of the sound, **G** is most likely to be disturbed.

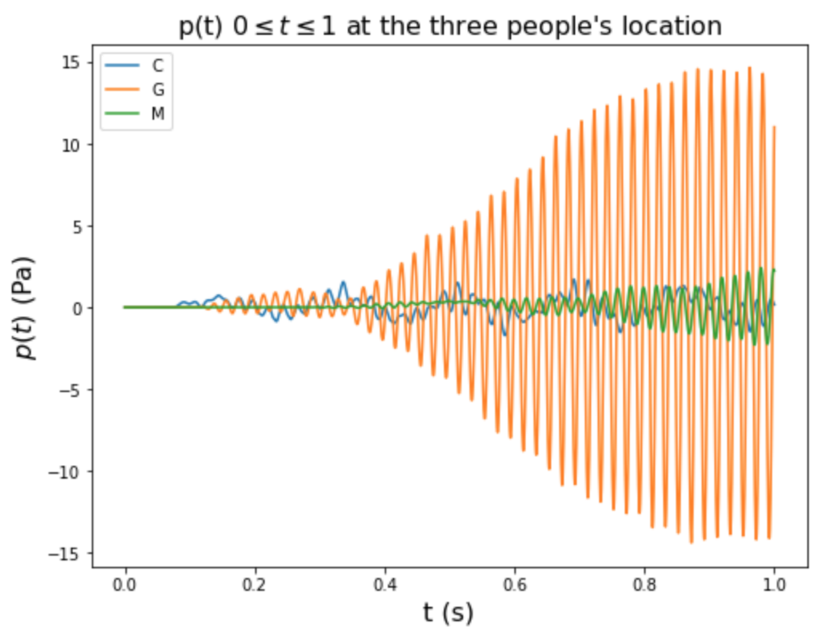


Figure 3: Problem 2: (d)

Problem 3

(a)

$$J_F = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_{n-2}} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_{n-2}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_{n-3}}{\partial u_1} & \frac{\partial F_{n-3}}{\partial u_2} & \cdots & \frac{\partial F_{n-3}}{\partial u_{n-2}} \\ \frac{\partial F_{n-2}}{\partial u_1} & \frac{\partial F_{n-2}}{\partial u_2} & \cdots & \frac{\partial F_{n-2}}{\partial u_{n-2}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{h^2} - e^{U_1} & \frac{1}{h^2} & 0 & 0 & \cdots & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} - e^{U_2} & \frac{1}{h^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{h^2} & -\frac{2}{h^2} - e^{U_{n-3}} & \frac{1}{h^2} \\ 0 & 0 & \cdots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} - e^{U_{n-2}} \end{bmatrix}$$

Sparcity of J_F is high. Only the main diagonal and the 2 sub-diagonals have non-zero values:

- number of all entries: $(n-2)^2$
- number of non-zero entries: $n-2+2(n-3) = 3n-8$

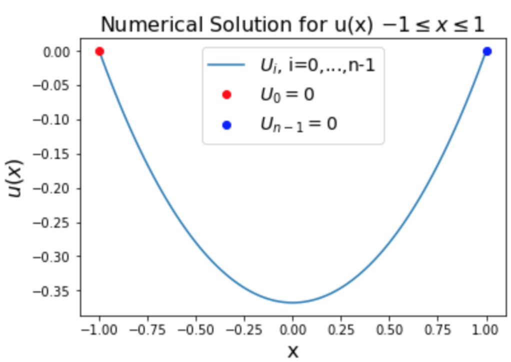
(b)

Numerically solved this ODE BVP problem by transforming it into a non-linear root finding problem.

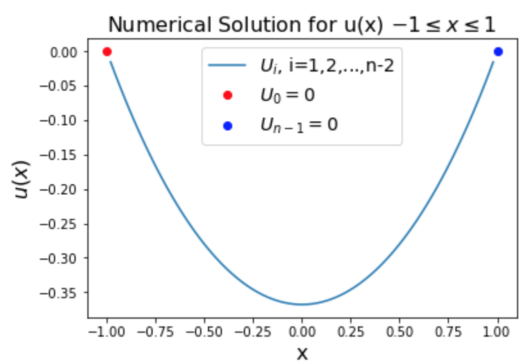
To solve the function $u(x)$, we suppose $U_i \sim u(x_i)$. The numerical solution could be found using Newton's Method to find the $u(x)$ for the system $F(U) = 0$. Update U_{k+1} as follows until a relative step size $||\Delta U^k||_2/||U^k||_2 \leq 10^{-10}$.

$$J_F(U_k)\Delta U_k = -F(U_k)$$
$$U_{k+1} = U_k + \Delta U$$

See plot of the approximate solution for u(x) $(-1 \leq x \leq 1)$ in Figure 4. Difference of the two plots below is just appending U_0 and U_{n-1} to the list of U_i before or after plotting.



(a) pad U_0, U_{n-1} BEFORE plot



(b) pad U_0, U_{n-1} AFTER plot

Figure 4: Problem 3:(b)

My **approximation to u(0) is -0.368** (to three significant digits)