AM205 HW1 Writeup

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Problem 1 By solving the b for Vandermonde matrix, I got the polynomial:

$y = -0.167x^3 + x^2 - 0.833x$

(keep coefficients with 3 decimal places). Plotted in Figure 1:

- data points plotted as blue stars
- red line: the polynomial with monomial basis
- dashed black line: the polynomial with Lagrange basis

The two polynomials match exactly in the figure, showing that the two representations are equivalent.

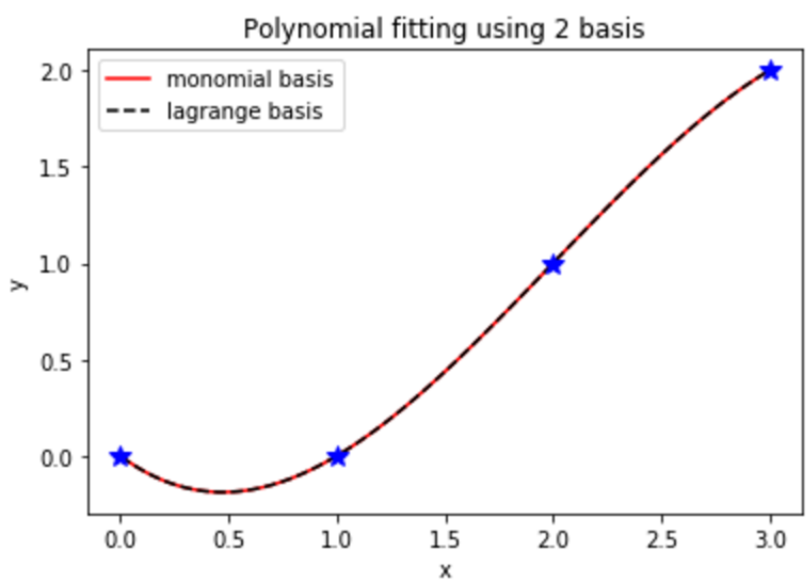

Figure 1: Polynomial Fitting with Monomial Basis and Lagrange Basis

Problem 2

(a) The chebyshev points ($x_k = cos(\frac{(2j-1)\pi}{2n})$, $y_k = f(x_k)$) for j = 1,2,3,4; n=4:

($x_0 = 0.924$, $y_0 = 40.424$)
($x_1 = 0.383$, $y_1 = 5.086$)
($x_2 = -0.383$, $y_2 = 2.366$)
($x_3 = -0.924$, $y_3 = 6.370$)
giving the Lagrange polynomial as $p_3(x) = \Sigma_{k=0}^{3} y_k L_k(x)$.

Plotted in Figure 2:

- red line: $f(x) = e^{4x} + e^{-2x}$
- dashed black line: the Lagrange polynomial $p_3(x) = \Sigma_{k=0}^{3} y_k L_k(x)$

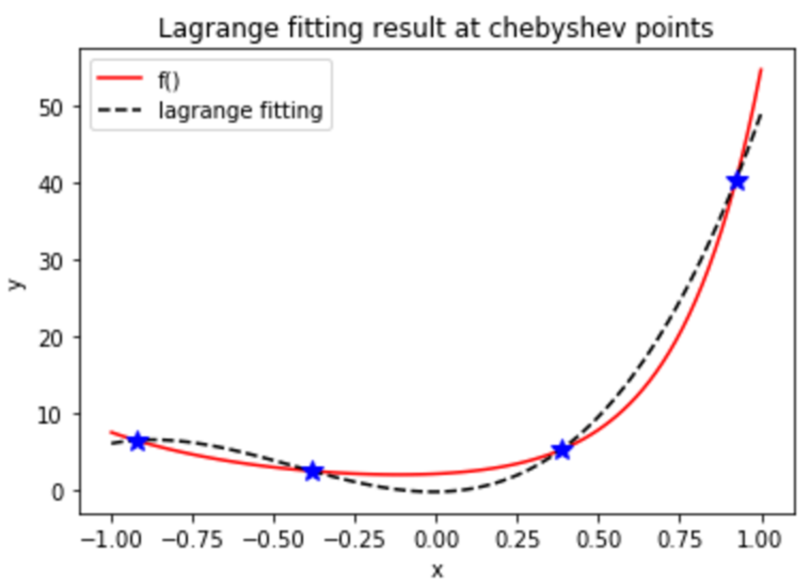

Figure 2: Lagrange Fitting on Chebyshev Points

(b) By sampling 1000 equally-spaced points over [-1, 1] from the function $|f - p_3|$, $||f - p_3||\infty$ is calculated to be **5.75190850272**.Plotted in Figure 3.

(c)

$|f^{(n)}(\theta)| = |4^n e^{4\theta} + (-2)^n e^{-2\theta}| < 4^n e^{4\theta} + 2^n e^{-2\theta}$
let $\phi(\theta) = 4^n e^{4\theta} + 2^n e^{-2\theta}$
then $\phi^{(1)}(\theta) = 4^{n+1} e^{4\theta} - 2^{n+1} e^{-2\theta} = 2^{n+1} e^{-2\theta}(2^{n+1} e^{6\theta} - 1)$
when $2^{n+1} e^{6\theta} < 1$: $\phi^{(1)}(\theta) < 0 \Rightarrow \phi(\theta)$ decreases
when $2^{n+1} e^{6\theta} \geq 1$: $\phi^{(1)}(\theta) > 0 \Rightarrow \phi(\theta)$ increases

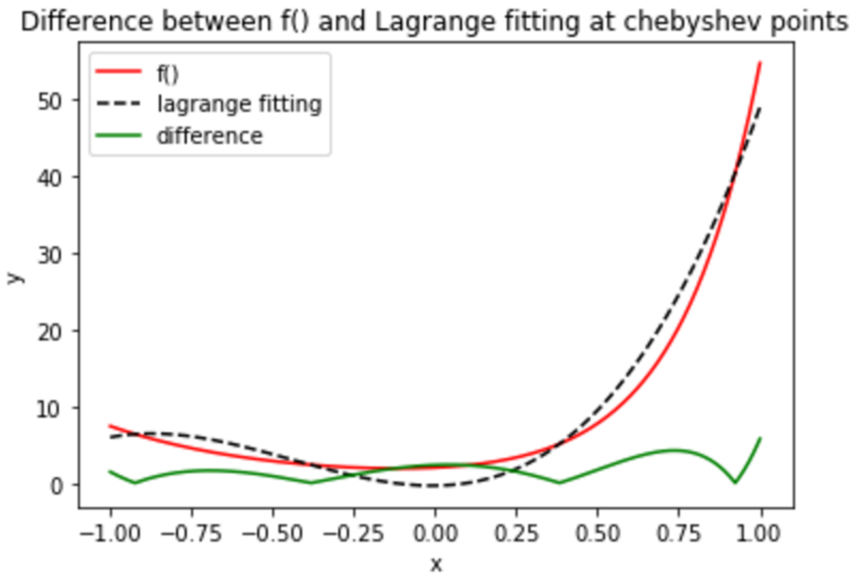

Figure 3: f, p_3 and $|f - p_3|$

Hence, $max_{\theta \in [-1,1]} \phi(\theta)$ is reached either at $\theta = 1$ or $\theta = -1$.
$\phi(1) = 4^n e^4 + 2^n e^{-2} < 4^n e^4 + 2^n e^2$
$\phi(-1) = 4^n e^{-4} + 2^n e^2 < 4^n e^4 + 2^n e^2$
Hence, $|f^{(n)}(\theta)| = \phi(\theta) < 4^n e^4 + 2^n e^2$

By approximation theory, the minimum value of $||(x - x_1)(x - x_2)...(x - x_n)||\infty$ is $\frac{1}{2^{n-1}}$, achieved by the Chebyshev polynomial.

Putting things together:
$||f(x) - p_{n-1}(x)||\infty = ||\frac{f^{(n)}(\theta)}{n!}(x - x_1)(x - x_2)...(x - x_n)||\infty$

$< \frac{4^n e^4 + 2^n e^2}{n! 2^{n-1}} = \frac{2^{n+1} e^4 + 2e^2}{n!}$

Since $n!$ grows much faster than 2^{n+1}, this term decreases as n grows. Its maximum is reached with $n = 1$. Thus, mathematically, this bound will be no larger than $4e^4 + 2e^2$.

(d) $p_3^+(x)$ using Lagrange basis on the following 4 data points makes a smaller norm:

$||f - p_3^+||\infty = \mathbf{4.48381276065}$. Plotted in Figure 4.

$(x_0 = -0.95,\ y_0 = 6.708)$
$(x_1 = -0.45,\ y_1 = 2.625)$
$(x_2 = 0.45,\ y_2 = 6.456)$
$(x_3 = 0.95,\ y_3 = 44.851)$ fits the data better.

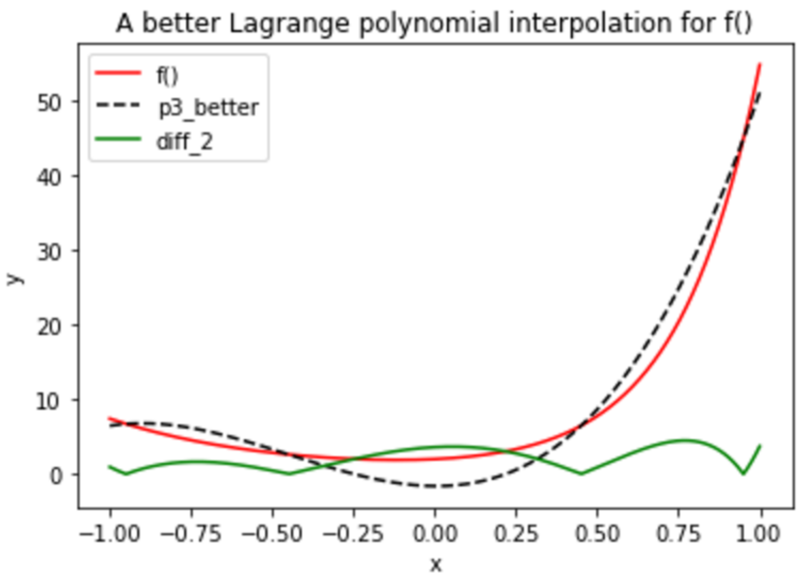

Figure 4: f, p_3^+ and $|f - p_3^+|$

Problem 3
(a)

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B + C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$\kappa(B) = 2$
$\kappa(C) = 2$
$\kappa(B + C) = 2$
$\kappa(B + C) < \kappa(B) + \kappa(C)$

(b)

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \quad B + C = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$$

$\kappa(B) = 2$
$\kappa(C) = 2$
$\kappa(B + C) = 6$

$\kappa(B+C) > \kappa(B) + \kappa(C)$

(c) Since A is invertible, there exists A^{-1} such that $AA^{-1} = A^{-1}A = I$

- $\kappa(2A) = ||2A||||(2A)^{-1}|| = 2||A||\frac{1}{2}||A^{-1}|| = ||A||||A^{-1}|| = \kappa(A)$

- By properties of matrix norm [1],
  $||Av||^2 = |(Av, Av)| = |(v, A^T Av)| \le ||v||||A^T Av|| \le ||A^T A||||v||^2$
  $||Av|| \le \sqrt{||A^T A||}||v||$ for any v
  $max(\frac{||Av||}{||v||}) = ||A|| \le \sqrt{||A^T A||}$
  $||A||^2 \le ||A^T A||^2 = ||A^2||$ (by $A^T = A$ since A is symmetric)
  $||A^2|| \le ||A||^2$ (by $||AB|| \le ||A||||B||$)
  Hence, $||A||^2 \le ||A^T A|| = ||A^2|| \le ||A||^2$
  $||A||^2 = ||A^2||$
  Similarly, $||A^{-1}||^2 = ||(A^{-1})^2||$
  $\kappa(A^2) = ||A^2||||(A^{-1})^2|| = ||A||^2||A^{-1}||^2 = \kappa(A)^2$

Problem 4
(a)
By using basis functions provided in the class note, I calculated the coefficients for them as follows:

$$\begin{array}{ccc} c_0(x) = x^2(3-2x) & c_0^{(1)}(x) = -6x^2 + 6x & c_0^{(2)}(x) = -12x + 6 \\ c_1(x) = -x^2(1-x) & c_1^{(1)}(x) = 3x^2 - 2x & c_1^{(2)}(x) = 6x - 2 \\ c_2(x) = (x-1)^2 x & c_2^{(1)}(x) = 3x^2 - 4x + 1 & c_2^{(2)}(x) = 6x - 4 \\ c_3(x) = 2x^3 - 3x^2 + 1 & c_3^{(1)}(x) = 6x^2 - 6x & c_3^{(2)}(x) = 12x - 6 \end{array}$$

$$\begin{array}{lcc} y_1(0) = 0 \rightarrow & y_1 = ac_0(x) + bc_1(x) + cc_2(x) & x \in [0,1) \\ y_2(2) = 0 \rightarrow & y_2 = dc_1(x-1) + ec_1(x-1) + fc_3(x-1) & x \in [1,2) \\ y_3(2) = 0 \rightarrow & y_2 = gc_0(x-2) + hc_1(x-2) + ic_2(x-2) & x \in [2,3) \\ y_4(4) = 0 \rightarrow & y_2 = jc_1(x-3) + kc_1(x-3) + lc_3(x-3) & x \in [3,4) \end{array}$$

$$\begin{array}{lc} y_1(1) = 1 \rightarrow & a = 1 \\ y_2(1) = 1 \rightarrow & f = 1 \\ y_3(3) = -1 \rightarrow & g = -1 \\ y_4(3) = -1 \rightarrow & l = -1 \end{array}$$

$$\begin{array}{lc} y_1^{(1)}(0) = y_4^{(1)}(4) \rightarrow & c = j \\ y_1^{(1)}(1) = y_2^{(1)}(1) \rightarrow & b = e \\ y_2^{(1)}(2) = y_3^{(1)}(2) \rightarrow & d = i \\ y_3^{(1)}(3) = y_4^{(1)}(3) \rightarrow & h = k \end{array}$$

$$
\begin{aligned}
y_1^{(2)}(0) = y_4^{(2)}(4) \rightarrow &\quad b = 0 \\
y_1^{(2)}(1) = y_2^{(2)}(1) \rightarrow &\quad c = \tfrac{3}{2} \\
y_2^{(2)}(2) = y_3^{(2)}(2) \rightarrow &\quad d = -\tfrac{3}{2} \\
y_3^{(2)}(3) = y_4^{(2)}(3) \rightarrow &\quad h = 0
\end{aligned}
$$

Hence, the spline $s_x(t)$ is:

$$
\begin{gathered}
s_x(t) = c_0(x) + \tfrac{3}{2}c_2(x) \quad t \in [0,1) \\
s_x(t) = -\tfrac{3}{2}c_1(x-1) + c_3(x-1) \quad t \in [1,2) \\
s_x(t) = -c_0(x-2) - \tfrac{3}{2}c_2(x-2) \quad t \in [2,3) \\
s_x(t) = \tfrac{3}{2}c_1(x-3) - c_3(x-3) \quad t \in [3,4)
\end{gathered}
$$

(b)
$s_x(t)$ and $sin(\frac{t\pi}{2})$ plotted in Figure 5. They are very similar.

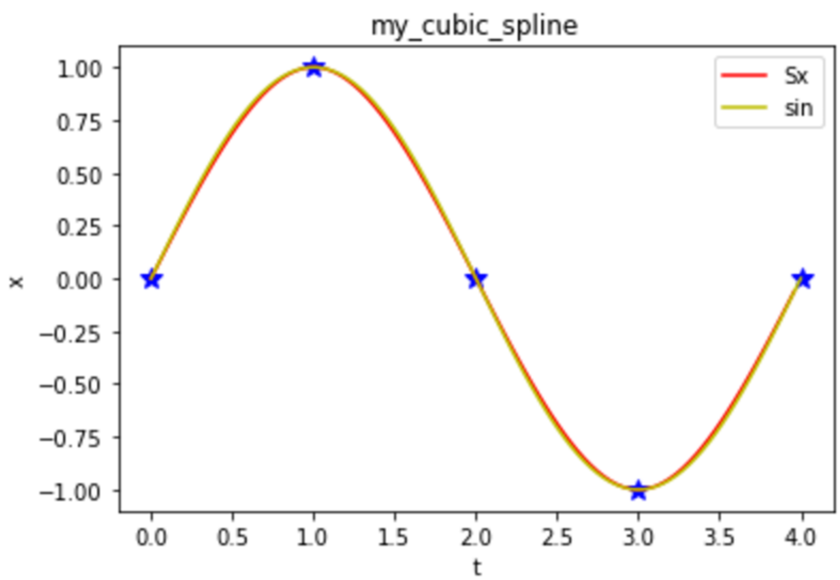

Figure 5: $s_x(t)$ and $sin(\frac{t\pi}{2})$

(c)
Using the same procedure and same basis for $s_x(t)$, I got $s_y(t)$ as:

$$
\begin{gathered}
s_y(t) = -\tfrac{3}{2}c_1(x) + c_3(x) \quad t \in [0,1) \\
s_y(t) = -c_0(x-1) - \tfrac{3}{2}c_2(x-1) \quad t \in [1,2) \\
s_y(t) = \tfrac{3}{2}c_1(x-2) - c_3(x-2) \quad t \in [2,3) \\
s_y(t) = c_0(x-3) + \tfrac{3}{2}c_2(x-3) \quad t \in [3,4)
\end{gathered}
$$

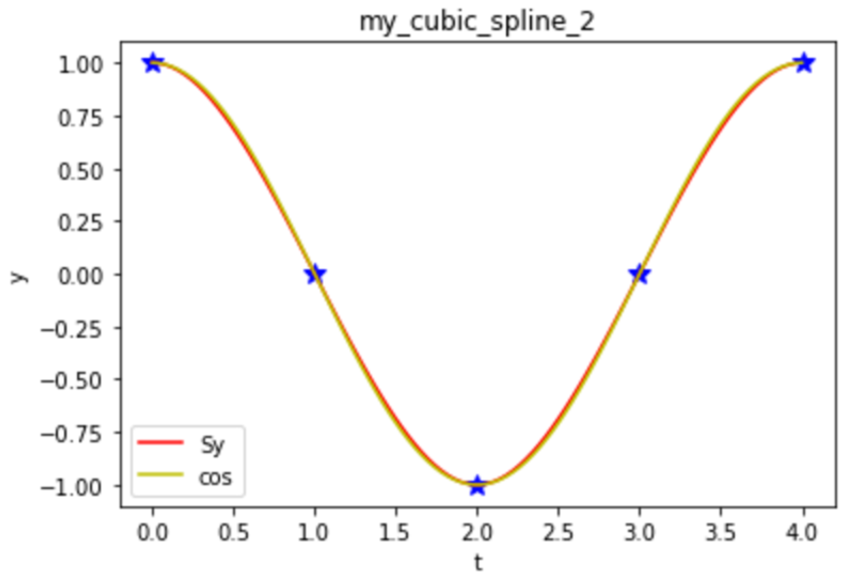

Figure 6: $s_y(t)$ and $cos(\frac{t\pi}{2})$

$s_y(t)$ and $cos(\frac{t\pi}{2})$ plotted in Figure 6. They are very similar.

(d)

Plotted in Figure 7. The area enclosed by the parametric curve is calculated by the summation of all small rectangle slices with area $= 2 * |d(Sx)| * |Sy|$.

The numerical value of π from my code is **3.0499999499900081**, with relative error: **2.91548630581%**

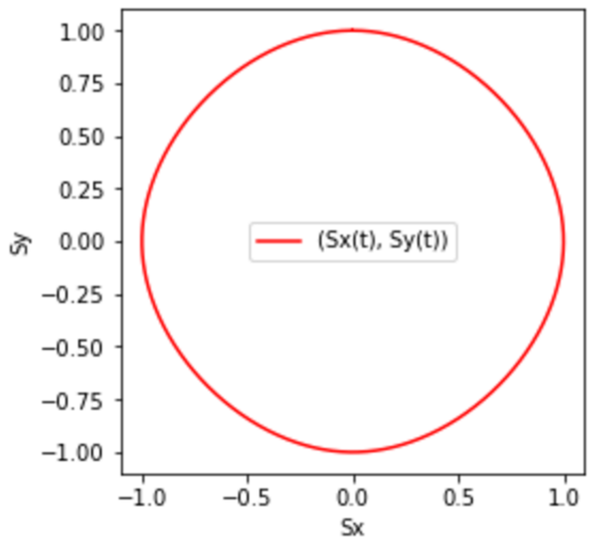

Figure 7: Parametric Curve for $s_y(t)$ and $s_x(t)$

Problem 5
(a)
Rewrite the fitting equation

$$\mathbf{p}_k^A = F^B \mathbf{p}_k^B + F^C \mathbf{p}_k^C + F^D \mathbf{p}_k^D + \mathbf{p}_{const} \tag{1}$$

to be

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ G_1 \\ B_1 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} R_2 \\ G_2 \\ B_2 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} R_3 \\ G_3 \\ B_3 \end{bmatrix} + \begin{bmatrix} r \\ g \\ b \end{bmatrix}$$

Loop through each pixel in each low-light images to least square fit (with python's numpy.linalg.lstsq) the regular R, G, B components respectively.

$L = MN - 1$

$$
\begin{bmatrix} A_0[0] \\ A_0[1] \\ \vdots \\ A_0[L] \end{bmatrix} \sim \begin{bmatrix} R_1[0] & G_1[0] & B_1[0] & R_2[0] & G_2[0] & B_2[0] & R_3[0] & G_3[0] & B_3[0] & 1 \\ R_1[1] & G_1[1] & B_1[1] & R_2[1] & G_2[1] & B_2[1] & R_3[1] & G_3[1] & B_3[1] & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_1[L] & G_1[L] & B_1[L] & R_2[L] & G_2[L] & B_2[L] & R_3[L] & G_3[L] & B_3[L] & 1 \end{bmatrix} * \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ c_{11} \\ c_{12} \\ c_{13} \\ d_{11} \\ d_{12} \\ d_{13} \\ r \end{bmatrix}
$$

$$
\begin{bmatrix} A_1[0] \\ A_1[1] \\ \vdots \\ A_1[L] \end{bmatrix} \sim \begin{bmatrix} R_1[0] & G_1[0] & B_1[0] & R_2[0] & G_2[0] & B_2[0] & R_3[0] & G_3[0] & B_3[0] & 1 \\ R_1[1] & G_1[1] & B_1[1] & R_2[1] & G_2[1] & B_2[1] & R_3[1] & G_3[1] & B_3[1] & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_1[L] & G_1[L] & B_1[L] & R_2[L] & G_2[L] & B_2[L] & R_3[L] & G_3[L] & B_3[L] & 1 \end{bmatrix} * \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \\ c_{21} \\ c_{22} \\ c_{23} \\ d_{21} \\ d_{22} \\ d_{23} \\ g \end{bmatrix}
$$

$$
\begin{bmatrix} A_2[0] \\ A_2[1] \\ \vdots \\ A_2[L] \end{bmatrix} \sim \begin{bmatrix} R_1[0] & G_1[0] & B_1[0] & R_2[0] & G_2[0] & B_2[0] & R_3[0] & G_3[0] & B_3[0] & 1 \\ R_1[1] & G_1[1] & B_1[1] & R_2[1] & G_2[1] & B_2[1] & R_3[1] & G_3[1] & B_3[1] & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_1[L] & G_1[L] & B_1[L] & R_2[L] & G_2[L] & B_2[L] & R_3[L] & G_3[L] & B_3[L] & 1 \end{bmatrix} * \begin{bmatrix} b_{31} \\ b_{32} \\ b_{33} \\ c_{31} \\ c_{32} \\ c_{33} \\ d_{31} \\ d_{32} \\ d_{33} \\ b \end{bmatrix}
$$

The fitting parameters returned by lstsq are as follows:

$$
F^B = \begin{bmatrix} 0.81197402 & 0.51473525 & -0.41834615 \\ 0.56911652 & 1.24681112 & -0.64012445 \\ 0.43841706 & -0.17996042 & -0.23558825 \end{bmatrix} F^C = \begin{bmatrix} -0.13785987 & 0.06741681 & 0.300502 \\ -0.24472206 & 0.26526897 & 0.32317837 \\ -0.06930897 & 0.25107695 & 1.1830199 \end{bmatrix}
$$

$$
F^D = \begin{bmatrix} 0.48057985 & -0.27830102 & -0.59472792 \\ -0.32582091 & 0.02593846 & -0.44420352 \\ -0.37981819 & -0.18260174 & 0.33164883 \end{bmatrix} p_{const} = \begin{bmatrix} 17.3193773 \\ 14.96314257 \\ -2.31665613 \end{bmatrix}
$$

The mean square error on the original image for the three channels are: ($S_R =$**152.536,** $S_G =$**148.380,** $S_B =$**79.609)**

Total mean square error: S**=126.842**, mean error rate: **4.417%**
See the reconstructed object image in Figure 8.

Figure 8: Reconstructed Objects 400*300

(b)

The mean square errors by using parameters to fit the "objects" image to reconstruct the "bears" image are: **(T_R =431.604, T_G =488.428, T_B =198.839)**

Total mean square error: T=**372.957**, mean error rate: **7.573%**
See constructed "bear" in Figure 9.

Figure 9: Reconstructed Bears 400*300

References

[1] Keith Conrad. *Computing the Norm of a Matrix*. URL: `http://www.math.uconn.edu/~kconrad/blurbs/linmultialg/matrixnorm.pdf`.