AM205 HW2

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> Checked results and discussed with: **Rui Fang**
>
> Got help from **Chris** Office Hour

In [1]:

```
%matplotlib inline
from math import *
import numpy as np
import matplotlib.pyplot as plt
```

Problem 1

(a)

Let $b = \begin{bmatrix} x \\ y \end{bmatrix}$. By $||b||_2 = 1$, we get $\sqrt{x^2 + y^2} = 1$. **Plot** the parametric representation for $||b||_2 = 1$ as

> $x = cos\theta$
> $y = sin\theta$

Let $Ab = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x' - y' \\ x' \end{bmatrix}$.

Similarly, by $||Ab||_2 = 1$, we get $\sqrt{(3x' - y')^2 + x'^2} = 1$ which gives $Ab = \begin{bmatrix} 3x' - y' \\ x' \end{bmatrix} = \begin{bmatrix} cos\alpha \\ sin\alpha \end{bmatrix}$. **Plot** the parametric representation for $||Ab||_2 = 1$ as

> $x' = sin\alpha$
> $y' = 3sin\alpha - cos\alpha$

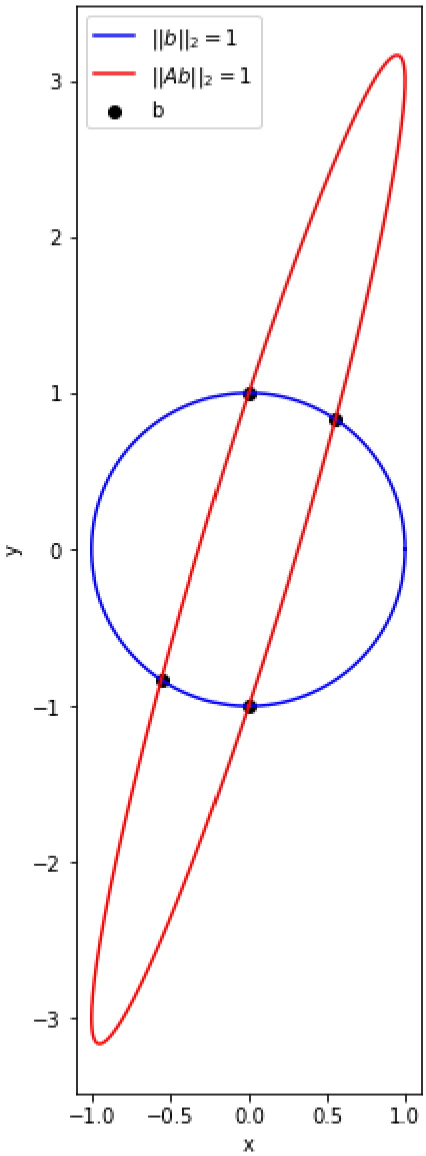
The intersection points of $||b||_2 = 1$ and $||Ab||_2 = 1$ could be solved as

$$\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cos\theta \\ sin\theta \end{bmatrix} = \begin{bmatrix} 3cos\theta - sin\theta \\ cos\theta \end{bmatrix}$$

> $(3cos\theta - sin\theta)^2 + cos\theta^2 = 1$
>
> $cos\theta = 0$ or $cos\theta = \frac{2}{3}sin\theta$
>
> $b_1 = (0, 1) \quad b_2 = (0, -1) \quad b_3 = (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}) \quad b_4 = (\frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}})$

In [2]:

```
# plot ||b||_2 = 1, b = (x, y) = (cos(t), sin(t)), t in [0, 2*pi]
t_lin = np.linspace(0, 2*pi, 1000)
b_x = np.array([cos(tt) for tt in t_lin])
b_y = np.array([sin(tt) for tt in t_lin])
plt.plot(b_x, b_y, 'b-', label='$||b||_2=1$')

# plot ||Ab|| = 1, Ab = (sin(t), 3sin(t)-cos(t))
Ab_x = np.array([sin(tt) for tt in t_lin])
Ab_y = np.array([3*sin(tt)-cos(tt) for tt in t_lin])
plt.plot(Ab_x, Ab_y, 'r-', label='$||Ab||_2=1$')

# mark points b: (0, 1), (0, -1), (2/sqrt(13), 3/sqrt(13)), (-2/sqrt(13), -3/sqr
t(13))
b_x = np.array([0, 0, 2/sqrt(13), -2/sqrt(13)])
b_y = np.array([1, -1, 3/sqrt(13), -3/sqrt(13)])
plt.scatter(x=b_x, y=b_y, marker='o', color='k', label='b')

plt.gcf().set_size_inches(6, 10)
plt.gca().set_aspect('equal')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



(b)

Let $c = \begin{bmatrix} x \\ y \end{bmatrix}$. By $||c||_\infty = max(|x|, |y|) = 1$

$$|x| \leq |y| = 1$$
$$|y| < |x| = 1$$

Equivalently, **plot** $y = y(x)$ as

$$x = 1, \quad -1 \leq y \leq 1$$
$$x = -1, \quad -1 \leq y \leq 1$$
$$y = 1, \quad -1 \leq x \leq 1$$
$$y = -1, \quad -1 \leq x \leq 1$$

Let $Ac = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x' - y' \\ x' \end{bmatrix}$. By $||Ac||_\infty = max(|3x' - y'|, |x'|) = 1$

$$|x'| \leq |3x' - y'| = 1$$
$$|3x' - y'| < |x'| = 1$$

Equivalently, **plot** $y' = y'(x')$ as

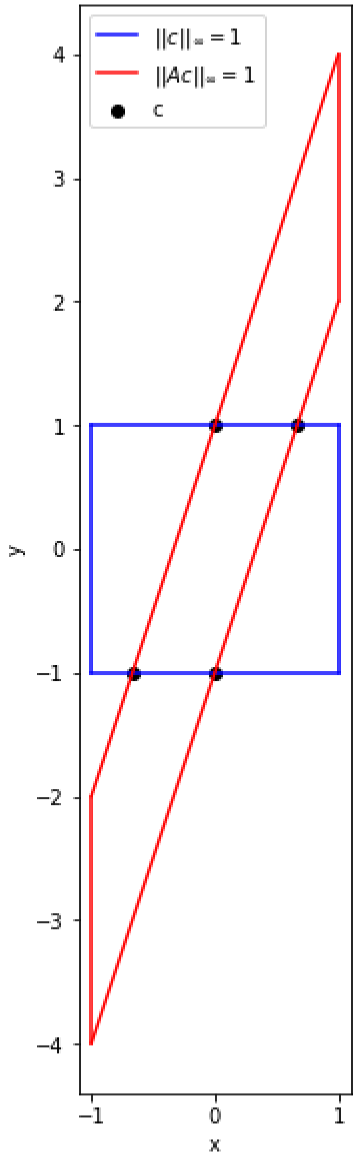
$$x' = 1, \quad 2 \leq y' \leq 4$$
$$x' = -1, \quad -4 \leq y' \leq -2$$
$$y' = 3x' + 1, \quad -1 \leq x' \leq 1$$
$$y' = 3x' - 1, \quad -1 \leq x' \leq 1$$

The intersection points of $||c||_\infty = 1$ and $||Ac||_\infty = 1$ could be solved as

$$y = 1 \quad \text{or} \quad y = 1 \quad \text{or} \quad y = -1 \quad \text{or} \quad y = -1$$
$$y = 3x + 1 \quad\quad y = 3x - 1 \quad\quad y = 3x + 1 \quad\quad y = 3x - 1$$
$$c_1 = (0, 1) \quad\quad c_2 = (\tfrac{2}{3}, 1) \quad\quad c_3 = (-\tfrac{2}{3}, -1) \quad\quad c_4 = (0, -1)$$

In [3]:

```
# plot ||c||_\\infty = 1, a square
x_range = np.linspace(-1, 1, 1000)
y_range = np.linspace(-1, 1, 1000)
plt.plot(np.ones(y_range.shape), y_range, 'b-', label='$||c||_\\infty=1$') # rig
ht side of the square
plt.plot(x_range, np.ones(x_range.shape), 'b-') # top side of the square
plt.plot(-np.ones(y_range.shape), y_range, 'b-') # left side of the square
plt.plot(x_range, -np.ones(x_range.shape), 'b-') # bottom side of the square

# plot ||Ac||_\\infty = 1, a polygon
Ac_right_y_range = y_range + 3
Ac_left_y_range = y_range - 3
Ac_top_y_range = np.array([3*xx+1 for xx in x_range])
Ac_bottom_y_range = np.array([3*xx-1 for xx in x_range])
plt.plot(np.ones(Ac_right_y_range.shape), Ac_right_y_range, 'r-', label='$||Ac||
_\\infty=1$') # right side of the polygon
plt.plot(x_range, Ac_top_y_range, 'r-') # top side of the polygon
plt.plot(-np.ones(Ac_left_y_range.shape), Ac_left_y_range, 'r-') # left side of
 the polygon
plt.plot(x_range, Ac_bottom_y_range, 'r-') #bottom side of the polygon

# mark points c:
c_x = np.array([0, 2/3, -2/3, 0])
c_y = np.array([1, 1, -1, -1])
plt.scatter(x=c_x, y=c_y, marker='o', color='k', label='c')

plt.gcf().set_size_inches(6, 10)
plt.gca().set_aspect('equal')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



$||c||_\infty = 1$
$||Ac||_\infty = 1$
c
4
3
2
1
0
−1
−2
−3
−4
y
−1
0
1
x

(c)

Let $d = \begin{bmatrix} x \\ y \end{bmatrix}$. By $||d||_4 = (x^4 + y^4)^{-\frac{1}{4}} = 1$, **plot** $y = y(x)$ as

$$y = (1 - x^4)^{-\frac{1}{4}}$$
$$y = -(1 - x^4)^{-\frac{1}{4}}$$

Let $Ad = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x' - y' \\ x' \end{bmatrix}$. By $||Ad||_4 = ((3x - y)^4 + x^4)^{-\frac{1}{4}} = 1$, **plot** $y' = y'(x')$ as

$$y' = 3x' - (1 - x'^4)^{-\frac{1}{4}}$$
$$y' = 3x' + (1 - x'^4)^{-\frac{1}{4}}$$

The intersection points of $||d||_4 = 1$ and $||Ad||_4 = 1$ could be solved as

$$F_1 = x^4 + y^4 - 1 = 0$$
$$F_2 = (3x - y)^4 + x^4 - 1 = 0$$

Using Newton's root finding method, the Jacob Matrix of (F_1, F_2) is calculated to iteratively approach the roots from our initial guess by observartion:

```
inits = (2, 2), (-2, -2), (1, 10), (-1, -10)
```

$$d_1 = (0.6373, 0.9559) \qquad d_2 = (-0.6373, -0.9559) \qquad d_3 = (0, 1) \qquad d_4 = (0, -1)$$

Implementation details see below.

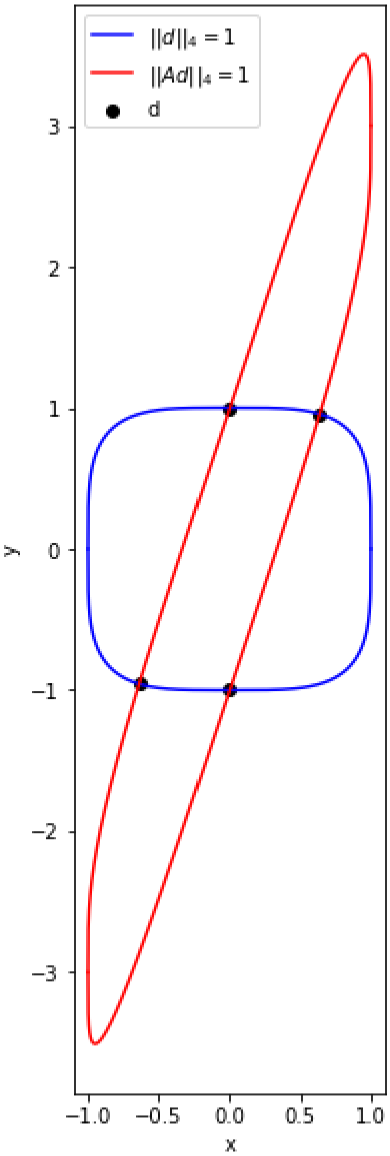
In [4]:

```
# (F1, F2)
def F_x_y(x, y):
    F1_x_y = x**4 + y**4 -1
    F2_x_y = (3*x-y)**4 + x**4 -1
    return np.array([
        [F1_x_y],
        [F2_x_y]
    ])

# Jacob Matrix for (F1, F2)
def J_x_y(x, y):
    dF1_dx = 4 * x**3
    dF1_dy = 4 * y**3
    dF2_dx = 4 * x**3 + 12 * ((3*x-y)**3)
    dF2_dy = -4 * ((3*x-y)**3)
    return np.array([
        [dF1_dx, dF1_dy],
        [dF2_dx, dF2_dy]
    ])

# Newton roots finding
inits = [(2, 2), (-2, -2), (1, 10), (-1, -10)]
d_x = []
d_y = []
for x0, y0 in inits:
    x_y_matrix = np.array([
        [x0],
        [y0]
    ])
    for i in range(1000):
        x = x_y_matrix[0, 0]
        y = x_y_matrix[1, 0]

        F = F_x_y(x, y)
        J = J_x_y(x, y)
        delta_x_y = np.linalg.solve(J, -1*F)
        x_y_matrix = np.add(x_y_matrix, delta_x_y)

    d_x.append(np.around(x_y_matrix[0, 0], decimals = 4))
    d_y.append(np.around(x_y_matrix[1, 0], decimals = 4))

print('The x_values for the 4 intersection points are:\n', d_x)
print('The y_values for the 4 intersection points are:\n', d_y)
```

```
The x_values for the 4 intersection points are:
 [0.63729999999999998, -0.63729999999999998, -0.0, 0.0]
The y_values for the 4 intersection points are:
 [0.95589999999999997, -0.95589999999999997, 1.0, -1.0]
```

In [5]:

```
def f1_1(x):
    return pow(1-x**4, 1/4)

def f1_2(x):
    return -f1_1(x)

def f2_1(x):
    return 3*x + f1_1(x)

def f2_2(x):
    return 3*x - f1_1(x)

# plot ||d||_4 = 1
x_range = np.linspace(-1, 1, 1000)
y1_1 = np.array([f1_1(xx) for xx in x_range])
y1_2 = np.array([f1_2(xx) for xx in x_range])
plt.plot(x_range, y1_1, 'b-', label='$||d||_4=1$') # upper half
plt.plot(x_range, y1_2, 'b-') # lower half

# plot ||Ad||_4 = 1
y2_1 = np.array([f2_1(xx) for xx in x_range])
y2_2 = np.array([f2_2(xx) for xx in x_range])
plt.plot(x_range, y2_1, 'r-', label='$||Ad||_4=1$') # upper half
plt.plot(x_range, y2_2, 'r-') # lower half

# mark points d:
plt.scatter(x=d_x, y=d_y, marker='o', color='k', label='d')

plt.gcf().set_size_inches(6, 10)
plt.gca().set_aspect('equal')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.show()
```



$||d||_4 = 1$
$||Ad||_4 = 1$
d
3
2
1
0
−1
−2
−3
y
−1.0
−0.5
0.0
0.5
1.0
x

(d)

We are solving the equations

$$
\begin{aligned}
|3x-y|^p + |x|^p &= 1 \\
|x|^p + |y|^p &= 1
\end{aligned}
$$

Let $f(x) = |x|^p$, then

$$
\begin{aligned}
f(3x-y) + f(x) &= 1\,. \\
f(x) + f(y) &= 1
\end{aligned}
$$

Therefore, we have $f(3x-y) = f(y)$. Since $f(x) = |x|^p = |-x|^p = f(-x)$ is an even function, then the interaction points of those norm functions line on the two lines:

- $3x - y = y$
- $3x - y = -y$

which could be equivalently written as

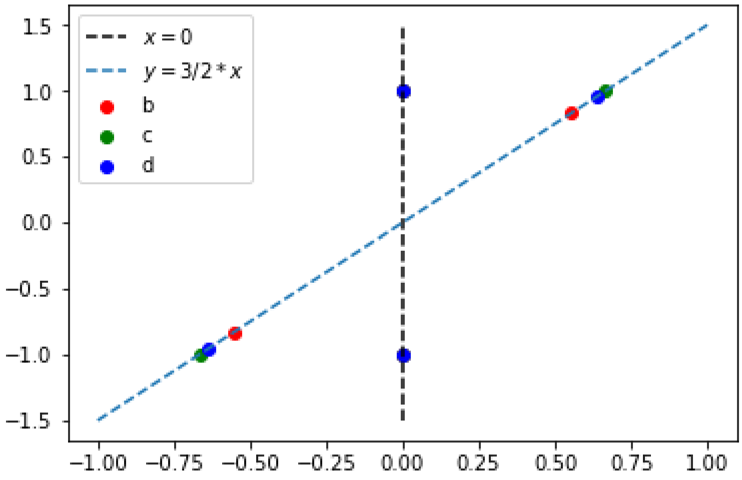
- $y = \frac{3}{2}x$
- $x = 0$

In [6]:

```
# Plot the lines and show that the family of points b, c, d lie on two lines
y_range = np.linspace(-1.5, 1.5, 300)
plt.plot(np.zeros(y_range.shape), y_range, 'k--', label='$x = 0$')
plt.plot(np.array([2/3*yy for yy in y_range]), y_range, '--', label= '$y =3/2*x
 $')

plt.scatter(x=b_x, y=b_y, marker='o', color='r', label='b')
plt.scatter(x=c_x, y=c_y, marker='o', color='g', label='c')
plt.scatter(x=d_x, y=d_y, marker='o', color='b', label='d')
plt.legend()
```

Out[6]:

```
<matplotlib.legend.Legend at 0x11b7e7e80>
```



Problem 2

(a)

In [7]:

```
def fsolve(L, b):
    (n, n) = L.shape

    # Initalize the solution vector x (shape=(n,1)) with integer zeros
    x = np.zeros((n, 1), dtype=np.int8)
    for i in range(n):
        if L[i,i] == 0:
            raise Exception('L is Singular: L has 0 on its diagonal.')

        xi = b[i,0]
        for j in range(i):
            xi = (xi ^ (L[i,j]&x[j,0]))
        x[i,0] = xi/L[i,i]
    return x
```

(b)

In [8]:

```
def rsolve(U, b):
    (n, n) = U.shape

    # Initalize the solution vector x (shape=(n,1)) with integer zeros
    x = np.zeros((n, 1), dtype=np.int8)
    for i in range(n-1, -1, -1):
        if U[i,i] == 0:
            raise Exception('U is Singular: U has 0 on its diagonal.')

        xi_back = b[i,0]
        for j in range(n-1-i):
            xi_back = (xi_back ^ (U[i,n-1-j]&x[n-1-j,0]))
        x[i,0] = xi_back/U[i,i]
    return x
```

(c)

In [9]:

```
import warnings
def LU_factorize(A):
    (n, n) = A.shape
    U = np.array(A)
    L = np.eye(n).astype(np.int8)
    P = np.eye(n).astype(np.int8)
    for j in range(n-1):
        # select i that has the maximal magnitude on U[j:n,j]
        i = j + np.argmax(U[j:n,j])
        # exchange rows of U: U[j,j:n] <-> U[i,j:n]
        u_j = np.array(U[j,j:n])
        U[j,j:n] = U[i,j:n]
        U[i,j:n] = u_j
        # exchange rows of L: L[j,0:j] <-> L[i,0:j]
        l_j = np.array(L[j,0:j])
        L[j,0:j] = L[i,0:j]
        L[i,0:j] = l_j
        # exchange rows of P: P[j,:] <-> P[i,:]
        p_j = np.array(P[j,:])
        P[j,:] = P[i,:]
        P[i,:] = p_j

        # LU_factorization works for singular A by skipping the column where enc
ountering U[j,j] = 0
        if U[j,j] == 0:
            warnings.warn("A is singular: Encountering zero on a diagonal.")
            continue
        for i in range(j+1, n):
            L[i,j] = U[i,j]/U[j,j]
            for k in range(j, n):
                U[i,k] = (U[i,k]^(L[i,j]&U[j,k]))
    return (U, L, P)
```

In [10]:

```
# Binary multiplication routine provided by the HW files
def bin_mul(c,d):

    # Check that the dimensions of the matrices are compatible
    (m,n) = c.shape
    (nn,p) = d.shape
    if n != nn:
        print("Matrix size mismatch")
        sys.exit()

    # Initalize blank matrix of integer zeros
    e=np.zeros((m, p), dtype=np.int8)

    # Calculate each term, using "&" instead of "*" and "^" instead of "+"
    for i in range(m):
        for j in range(p):
            for k in range(n):
                e[i,j]=e[i,j]^(c[i,k]&d[k,j])
    return e

# Define the example L and U matrices
l=np.array([[1,0,0,0],[0,1,0,0],[1,1,1,0],[1,0,1,1]],dtype=np.int8)
u=np.array([[1,0,1,0],[0,1,1,1],[0,0,1,0],[0,0,0,1]],dtype=np.int8)

# Carry out binary matrix multiplication and print the result
a = bin_mul(l,u)
a
```

Out[10]:

```
array([[1, 0, 1, 0],
       [0, 1, 1, 1],
       [1, 1, 1, 1],
       [1, 0, 0, 1]], dtype=int8)
```

In [11]:

```
# Validate LU_factorize() results by PA == LU
U_test, L_test, P_test = LU_factorize(a)
bin_mul(L_test, U_test) == bin_mul(P_test, a)
```

Out[11]:

```
array([[ True,  True,  True,  True],
       [ True,  True,  True,  True],
       [ True,  True,  True,  True],
       [ True,  True,  True,  True]], dtype=bool)
```

(d)

In [12]:

```
# load data
A_small = np.loadtxt('q2_small/a.txt').astype(np.int8)
b_small = np.loadtxt('q2_small/b.txt').astype(np.int8)
b_small = b_small.reshape((b_small.shape[0], 1))

A_large = np.loadtxt('q2_large/a.txt').astype(np.int8)
b_large = np.loadtxt('q2_large/b.txt').astype(np.int8)
b_large = b_large.reshape((b_large.shape[0], 1))
```

In [13]:

```
# LU_factorize data
U_small, L_small, P_small = LU_factorize(A_small)
U_large, L_large, P_large = LU_factorize(A_large)
```

In [14]:

```
bin_mul(P_small, A_small) == bin_mul(L_small, U_small)
```

Out[14]:

```
array([[ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True]], dtype=bool)
```

In [15]:

```
# Solve Ax = b --- q2_small
Pb_small = np.dot(P_small, b_small)
y_small = fsolve(L_small, Pb_small)
x_small = rsolve(U_small, y_small)
print('q2_small: Solution for Ax=b \nx_small =', x_small[:,0].T)
```

```
q2_small: Solution for Ax=b
x_small = [1 0 1 0 1 0 1 0 1 0 1 0 1 0]
```

In [16]:

```
# Validate x_small
(bin_mul(A_small, x_small) == b_small).T
```

Out[16]:

```
array([[ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True]], dtype=bool)
```

In [17]:

```
# Solve Ax = b --- q2_small
Pb_large = np.dot(P_large, b_large)
y_large = fsolve(L_large, Pb_large)
x_large = rsolve(U_large, y_large)
print('q2_large: Solution for Ax=b \nx_large =', x_large[:,0].T)
```

```
q2_large: Solution for Ax=b
x_large = [0 1 1 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1
0 1 0 0 0 0 0 1
 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0]
```

In [18]:

```
# Validate x_large
(bin_mul(A_large, x_large) == b_large).T
```

Out[18]:

```
array([[ True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True,  True,  True,  True,  True,  True,  True,  Tru
e,
         True,  True]], dtype=bool)
```

Problem 3

(a)

In [19]:

```
# Get the 1D index of the 2D board
def get_1d_index(i, j, n=7):
    return i*n+j

# Get the changes in the lights_vector (b) due to the press of any press at M[i,
j]
def get_delta_b(i, j, m=7, n=7):
    b = np.zeros((m*n, 1), dtype=np.int8)
    b[get_1d_index(i,j,n)] = 1 # center
    if i-1 >= 0:
        b[get_1d_index(i-1,j,n)] = 1 # upper
    if i+1 < m:
        b[get_1d_index(i+1,j,n)] = 1 # bottom
    if j-1 >= 0:
        b[get_1d_index(i,j-1,n)] = 1 # left
    if j+1 < n:
        b[get_1d_index(i,j+1,n)] = 1 # right
    return b

# Construct A by calculating A's column vectors
def construct_A_light(m, n):
    mn = m*n
    A_light_game = np.zeros((mn, mn), dtype=np.int8)
    for a in range(m):
        for b in range(n):
            b_delta = get_delta_b(a, b, m, n)
            A_light_game[:,get_1d_index(a,b,n)] =
b_delta.reshape((b_delta.shape[0],))
    return A_light_game

A_light_game = construct_A_light(7, 7)
```

(b)

In [20]:

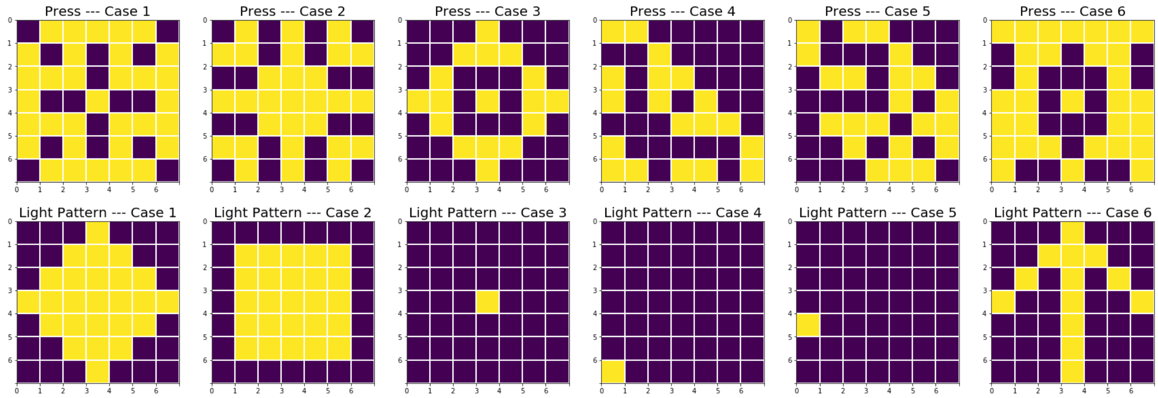
```
(U_lg, L_lg, P_lg) = LU_factorize(A_light_game)
size = 7*7

# Solve presses for case 1: center lights on
b1 = np.zeros((size, 1), dtype=np.int8)
lights_pos_1 = [
    (0,3),
    (1,2), (1,3), (1,4),
    (2,1), (2,2), (2,3), (2,4), (2,5),
    (3,0), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
    (4,1), (4,2), (4,3), (4,4), (4,5),
    (5,2), (5,3), (5,4),
    (6,3)
]
for (a,b) in lights_pos_1:
    b1[get_1d_index(a,b)] = 1
Pb1 = np.dot(P_lg, b1)
y_b1 = fsolve(L_lg, Pb1)
x_b1 = rsolve(U_lg, y_b1)

# Solve presses for case 2: center square
```

```
b2 = np.zeros((size, 1), dtype=np.int8)
lights_pos_2 = [
    (1,1), (1,2), (1,3), (1,4), (1,5),
    (2,1), (2,2), (2,3), (2,4), (2,5),
    (3,1), (3,2), (3,3), (3,4), (3,5),
    (4,1), (4,2), (4,3), (4,4), (4,5),
    (5,1), (5,2), (5,3), (5,4), (5,5)
]
for (a,b) in lights_pos_2:
    b2[get_1d_index(a,b)] = 1
Pb2 = np.dot(P_lg, b2)
y_b2 = fsolve(L_lg, Pb2)
x_b2 = rsolve(U_lg, y_b2)

# Solve presses for case3: one center light on
b3 = np.zeros((size, 1), dtype=np.int8)
lights_pos_3 = [
    (3,3)
]
for (a,b) in lights_pos_3:
    b3[get_1d_index(a,b)] = 1
Pb3 = np.dot(P_lg, b3)
y_b3 = fsolve(L_lg, Pb3)
x_b3 = rsolve(U_lg, y_b3)

# Solve presses for case4: the left bottom light on
b4 = np.zeros((size, 1), dtype=np.int8)
lights_pos_4 = [
    (6,0)
]
for (a,b) in lights_pos_4:
    b4[get_1d_index(a,b)] = 1
Pb4 = np.dot(P_lg, b4)
y_b4 = fsolve(L_lg, Pb4)
x_b4 = rsolve(U_lg, y_b4)

# Solve presses for case5: light (4,0) on
b5 = np.zeros((size, 1), dtype=np.int8)
lights_pos_5 = [
    (4,0)
]
for (a,b) in lights_pos_5:
    b5[get_1d_index(a,b)] = 1
Pb5 = np.dot(P_lg, b5)
y_b5 = fsolve(L_lg, Pb5)
x_b5 = rsolve(U_lg, y_b5)

# Solve presses for my own pattern
b6 = np.zeros((size, 1), dtype=np.int8)
lights_pos_6 = [
    (0,3),
    (1,2), (1,3), (1,4),
    (2,1), (2,3), (2,5),
    (3,0), (3,3), (3,6),
    (4,3),
    (5,3),
    (6,3)
]
for (a,b) in lights_pos_6:
    b6[get_1d_index(a,b)] = 1
Pb6 = np.dot(P_lg, b6)
```

```
y_b6 = fsolve(L_lg, Pb6)
x_b6 = rsolve(U_lg, y_b6)
```

In [21]:

```
width = 7

# Presses and light_patterns case 1-5
presses_1 = x_b1.reshape(width, width)
lights_1 = bin_mul(A_light_game, x_b1).reshape(width, width)

presses_2 = x_b2.reshape(width, width)
lights_2 = bin_mul(A_light_game, x_b2).reshape(width, width)

presses_3 = x_b3.reshape(width, width)
lights_3 = bin_mul(A_light_game, x_b3).reshape(width, width)

presses_4 = x_b4.reshape(width, width)
lights_4 = bin_mul(A_light_game, x_b4).reshape(width, width)

presses_5 = x_b5.reshape(width, width)
lights_5 = bin_mul(A_light_game, x_b5).reshape(width, width)

# Presses and light_patterns of my own
presses_6 = x_b6.reshape(width, width)
lights_6 = bin_mul(A_light_game, x_b6).reshape(width, width)

presses = list([presses_1, presses_2, presses_3, presses_4, presses_5,
presses_6])
light_patterns = list([lights_1, lights_2, lights_3, lights_4, lights_5, lights_
6])

# Plot each set of presses and light_patterns
fig, axes = plt.subplots(2, 6, figsize=(30, 10))

for j in range(6):
    axes[0, j].imshow(presses[j])
    axes[0, j].set_title('Press --- Case '+str(j+1), fontsize=20)
    axes[0, j].set_xticks(np.arange(-.5, 7, 1))
    axes[0, j].set_yticks(np.arange(-.5, 7, 1))
    axes[0, j].set_xticklabels(np.arange(0, 7, 1))
    axes[0, j].set_yticklabels(np.arange(0, 7, 1))
    axes[0, j].grid(color='w', linestyle='-', linewidth=2)

    axes[1, j].imshow(light_patterns[j])
    axes[1, j].set_title('Light Pattern --- Case '+str(j+1), fontsize=20)
    axes[1, j].set_xticks(np.arange(-.5, 7, 1))
    axes[1, j].set_yticks(np.arange(-.5, 7, 1))
    axes[1, j].set_xticklabels(np.arange(0, 7, 1))
    axes[1, j].set_yticklabels(np.arange(0, 7, 1))
    axes[1, j].grid(color='w', linestyle='-', linewidth=2)
```



(c)

In [22]:

```
# Construct f(m,n) for m,n in (1,2,3,4,5,6,7,8,9)
f = -1*np.zeros((10, 10), dtype=np.int8)
for m in range(1, 10):
    for n in range(1, 10):
        A = construct_A_light(m, n)
        U, L, P = LU_factorize(A)
        rank_A = np.count_nonzero(np.tril(np.triu(U)))
        f[m,n] = m*n - rank_A
f_mn = f[1:, 1:] # Cutting out the padding
f_mn
```

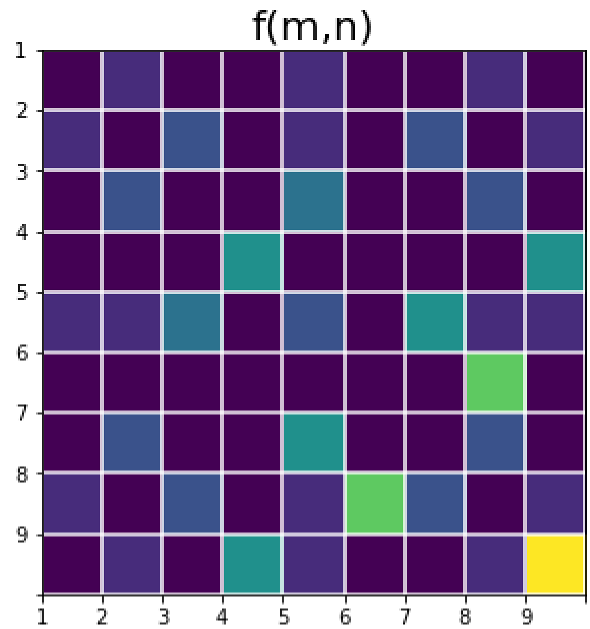
```
/Users/jasminetong/anaconda/lib/python3.6/site-packages/ipykernel_la
uncher.py:25: UserWarning: A is singular: Encountering zero on a dia
gonal.
```

Out[22]:

```
array([[0, 1, 0, 0, 1, 0, 0, 1, 0],
       [1, 0, 2, 0, 1, 0, 2, 0, 1],
       [0, 2, 0, 0, 3, 0, 0, 2, 0],
       [0, 0, 0, 4, 0, 0, 0, 0, 4],
       [1, 1, 3, 0, 2, 0, 4, 1, 1],
       [0, 0, 0, 0, 0, 0, 0, 6, 0],
       [0, 2, 0, 0, 4, 0, 0, 2, 0],
       [1, 0, 2, 0, 1, 6, 2, 0, 1],
       [0, 1, 0, 4, 1, 0, 0, 1, 8]], dtype=int8)
```

In [23]:

```
# Plot f(m,n) as image
fig, axes = plt.subplots(1, 1, figsize=(5, 5))
axes.imshow(f_mn/np.max(f))
axes.set_title('f(m,n)', fontsize=20)
axes.set_xticks(np.arange(-.5, 9, 1))
axes.set_yticks(np.arange(-.5, 9, 1))
axes.set_xticklabels(np.arange(1, 10, 1))
axes.set_yticklabels(np.arange(1, 10, 1))
axes.grid(color='w', linestyle='-', linewidth=1.5)
```



Problem 4

(a)

In [24]:

```
def generate_g(n):
    G_n = -1*np.tril(np.ones((n, n))) + 2*np.eye(n)
    G_n[:,-1] = 1
    return G_n

generate_g(6)
```

Out[24]:

```
array([[ 1.,  0.,  0.,  0.,  0.,  1.],
       [-1.,  1.,  0.,  0.,  0.,  1.],
       [-1., -1.,  1.,  0.,  0.,  1.],
       [-1., -1., -1.,  1.,  0.,  1.],
       [-1., -1., -1., -1.,  1.,  1.],
       [-1., -1., -1., -1., -1.,  1.]])
```

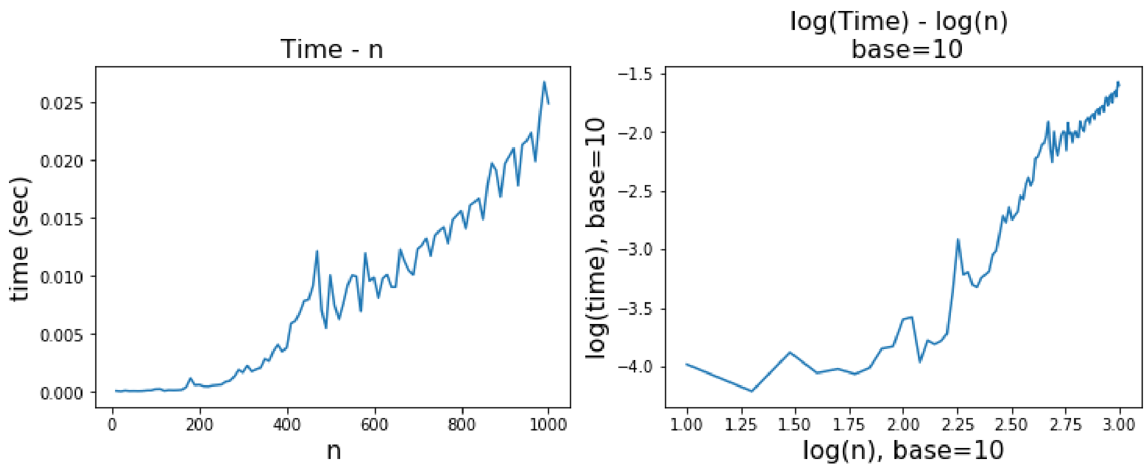
(b)

In [25]:

```
import time
# Compute an array of n
ns = range(10, 1001, 10) # original scale
log_ns = np.array([log10(nn) for nn in ns]) # log scale

# Computer an array of times
times = []
for n in ns:
    t0 = time.time()
    generate_g(n)
    t1 = time.time()
    dt = t1-t0
    times.append(dt) # original scale
log_times = np.array([log10(t) for t in times]) # log scale

fig, axes = plt.subplots(1, 2, figsize=(12, 4))

axes[0].plot(ns, times)
axes[0].set_title('Time - n',fontsize=16)
axes[0].set_xlabel('n',fontsize=16)
axes[0].set_ylabel('time (sec)',fontsize=16)

axes[1].plot(log_ns, log_times)
axes[1].set_title('log(Time) - log(n) \n base=10',fontsize=16)
axes[1].set_xlabel('log(n), base=10', fontsize=16)
axes[1].set_ylabel('log(time), base=10', fontsize=16)
```

Out[25]:

```
<matplotlib.text.Text at 0x1176a54a8>
```



In [26]:

```
# Fit the original time(n) using the degree = 2 polyfit
beta = 2 # Value of beta is determined by calculating the slope in the log10 plo
t
poly_coeffs = np.polyfit(ns, times, 2)

# Get the coefficients for the polynomial fitting
alpha = poly_coeffs[0]
c1 = poly_coeffs[1]
c2 = poly_coeffs[2]
print('alpha = ', alpha)

# Construct the polynomial fitting function
poly_fit = np.array([alpha*(nn**beta)+c1*(nn)+c2 for nn in ns])
```

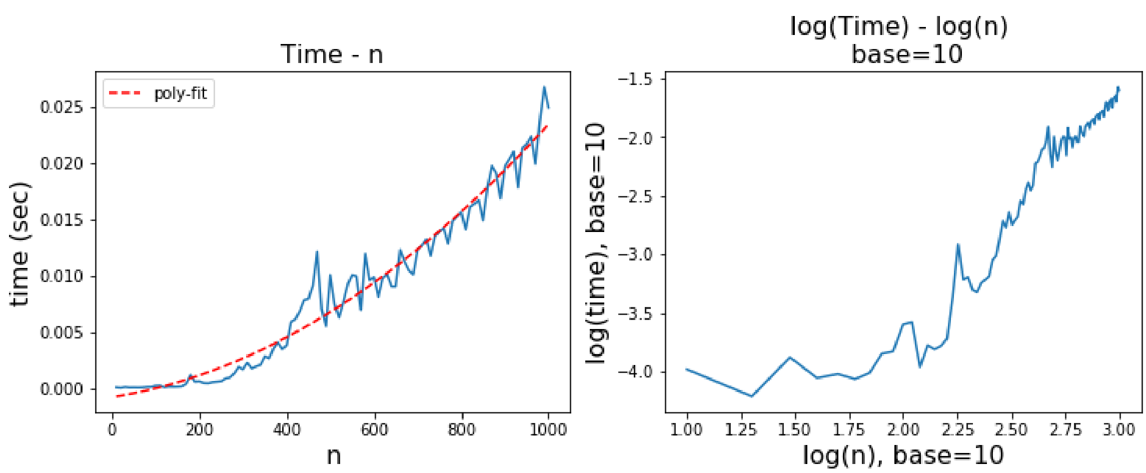
```
alpha =  1.82076904391e-08
```

In [27]:

```
# Plot the poly-fit together with the original data
fig, axes = plt.subplots(1, 2, figsize=(12, 4))

axes[0].plot(ns, times)
axes[0].plot(ns, poly_fit, 'r--', label='poly-fit')
axes[0].set_title('Time - n',fontsize=16)
axes[0].set_xlabel('n',fontsize=16)
axes[0].set_ylabel('time (sec)',fontsize=16)
axes[0].legend()

axes[1].plot(log_ns, log_times)
axes[1].set_title('log(Time) - log(n) \n base=10',fontsize=16)
axes[1].set_xlabel('log(n), base=10', fontsize=16)
axes[1].set_ylabel('log(time), base=10', fontsize=16)
```

Out[27]:

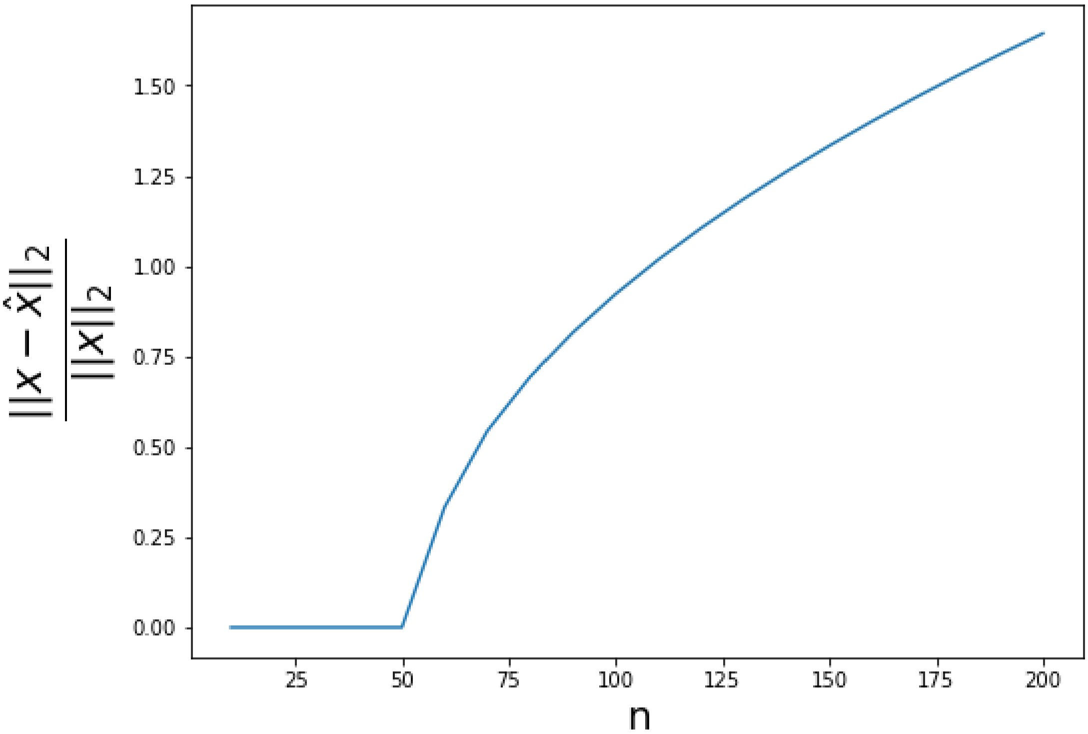
```
<matplotlib.text.Text at 0x11b3247b8>
```



(c)

In [28]:

```
ns = range(10, 201, 10)

# Calculate the ||x-x_hat||_2
rel_err_2norm = []
for n in ns:
    x = np.ones(n)
    G_n = generate_g(n)
    b = np.dot(G_n, x)
    x_hat = np.linalg.solve(G_n, b)
    rel_err_2norm.append(np.linalg.norm(x_hat-x) / np.linalg.norm(x_hat))

plt.figure(figsize=(8,6))
plt.plot(ns, rel_err_2norm)
plt.xlabel('n', fontsize=20)
plt.ylabel('$\\frac{||x-\hat{x}||_2}{||x||_2}$', fontsize=30)
plt.show()
```



In [29]:

```
import scipy.linalg
N = 4
x_g = np.ones(N)
G_g = generate_g(N)
(P_g, L_g, U_g) = scipy.linalg.lu(G_g)
b_g = np.dot(G_g, x_g)
x_g_hat = np.linalg.solve(G_g, b_g)
x_g_hat
```

Out[29]:

```
array([ 1.,  1.,  1.,  1.])
```

In [30]:

```
L_g
```

Out[30]:

```
array([[ 1.,  0.,  0.,  0.],
       [-1.,  1.,  0.,  0.],
       [-1., -1.,  1.,  0.],
       [-1., -1., -1.,  1.]])
```

In [31]:

```
U_g
```

Out[31]:

```
array([[ 1.,  0.,  0.,  1.],
       [ 0.,  1.,  0.,  2.],
       [ 0.,  0.,  1.,  4.],
       [ 0.,  0.,  0.,  8.]])
```

In [32]:

```
G_g
```

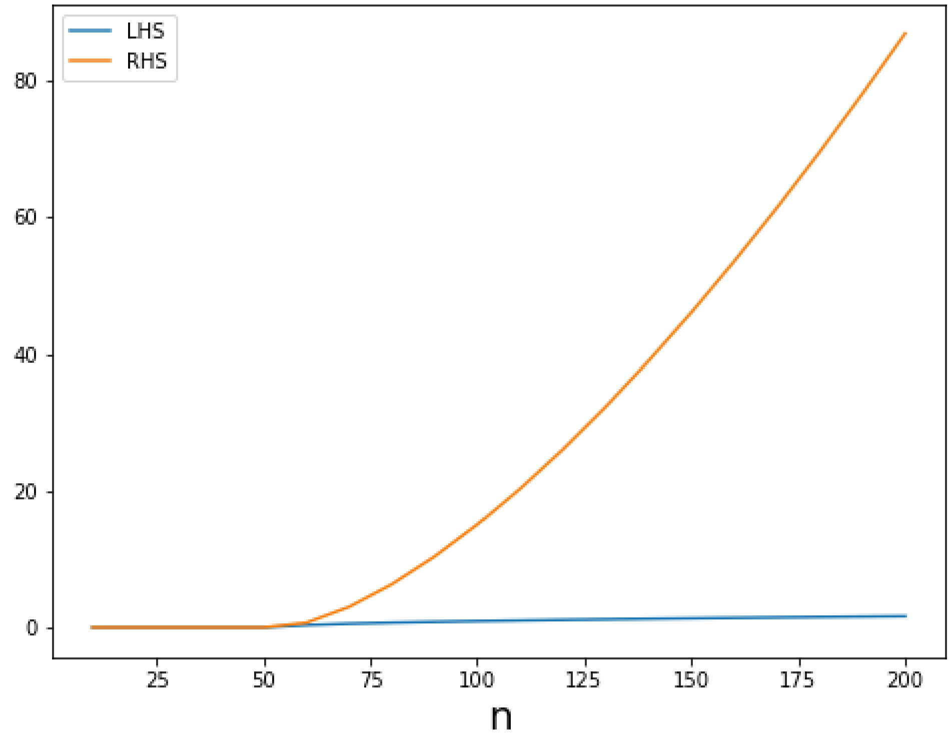
Out[32]:

```
array([[ 1.,  0.,  0.,  1.],
       [-1.,  1.,  0.,  1.],
       [-1., -1.,  1.,  1.],
       [-1., -1., -1.,  1.]])
```

(d)

In [33]:

```
ns = range(10, 201, 10)

# Calculate the ||x-x_hat||_2
LHS = []
RHS = []
for n in ns:
    x = np.ones(n)
    G_n = generate_g(n)
    b = np.dot(G_n, x)
    x_hat = np.linalg.solve(G_n, b)
    b_hat = np.dot(G_n, x_hat)

    LHS.append(np.linalg.norm(x-x_hat) / np.linalg.norm(x_hat))
    RHS.append((np.linalg.cond(G_n)*np.linalg.norm(b-b_hat)) / (np.linalg.norm(G
_n)*np.linalg.norm(x_hat)))

plt.figure(figsize=(8,6))
plt.plot(ns, LHS, label='LHS')
plt.plot(ns, RHS, label='RHS')
plt.xlabel('n', fontsize=20)
plt.legend()
plt.show()
```



Problem 6

Note: I used the smallest set of images: 356 x 280

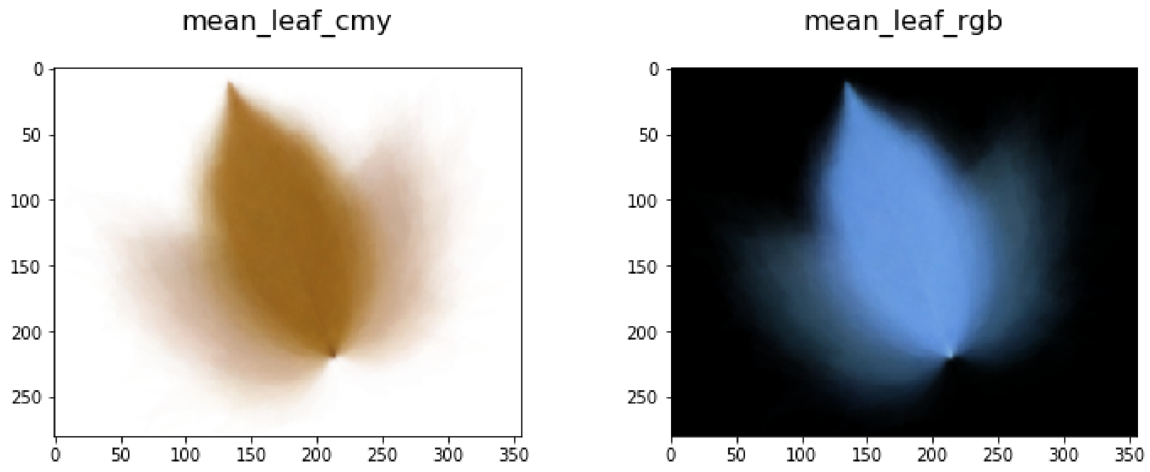
(a)

In [34]:

```
from skimage import io
from os import listdir

# List the filenames of all leaves
path = 'leaves/main/'
file_names = [f for f in listdir(path)]
L = len(file_names) # 143

# read in all leaves and compute their mean
(M, N) = (356, 280)
mean_leaf_rgb = np.zeros([N, M, 3])
for f_name in file_names:
    leaf = 1 - io.imread(path+f_name)/255
    mean_leaf_rgb += leaf

mean_leaf_cmy = 1 - mean_leaf_rgb/L
# plt.imshow(mean_leaf_cmy)

mean_leaf_rgb = mean_leaf_rgb/L
# plt.imshow(mean_leaf_rgb)

fig, axes = plt.subplots(1, 2, figsize=(12, 4))

axes[0].imshow(mean_leaf_cmy)
axes[0].set_title('mean_leaf_cmy\n', fontsize=16)

axes[1].imshow(mean_leaf_rgb)
axes[1].set_title('mean_leaf_rgb\n', fontsize=16)
```

Out[34]:

<matplotlib.text.Text at 0x11bc14630>



(b)

In [35]:

```
# Stretch the mean leaf into a long (3*M*N, 1) array
mean_r = mean_leaf_rgb[:,:,0].reshape(M*N)
mean_g = mean_leaf_rgb[:,:,1].reshape(M*N)
mean_b = mean_leaf_rgb[:,:,2].reshape(M*N)
mean_leaf_rgb_long = np.concatenate((mean_r,mean_g,mean_b), axis=0)

# Assemble the matrix A_leaf
A_leaf = np.zeros([3*M*N, L])
j = 0
for f_name in file_names:
    leaf = 1 - io.imread(path+f_name).astype(np.float64)/255

    # Stretch this leaf into a long (3*M*N, 1) array
    r = leaf[:,:,0].reshape(M*N)
    g = leaf[:,:,1].reshape(M*N)
    b = leaf[:,:,2].reshape(M*N)
    leaf_long = np.concatenate((r,g,b), axis=0)

    # Deduct the 1d mean leaf from this leaf and append this leaf to A's j-th co
lumn
    A_leaf[:,j] = leaf_long - mean_leaf_rgb_long
    j += 1

# Do reduced SVD of A_leaf
U_leaf, s_leaf, V_leaf = np.linalg.svd(A_leaf, full_matrices=False)

#print(A_leaf.shape, U_leaf.shape, s_leaf.shape, V_leaf.shape)
```

In [36]:

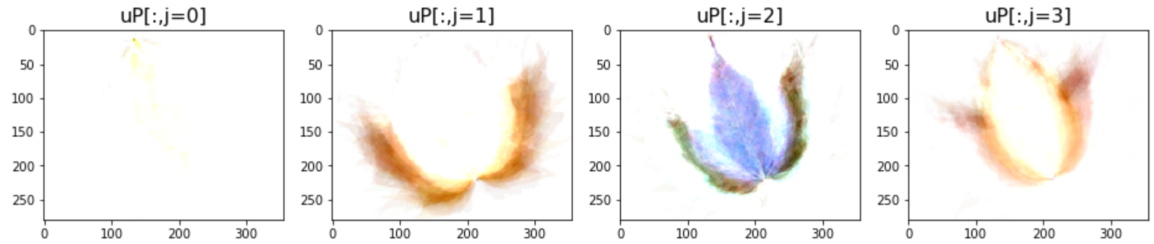
```
# For each column in U, find c[j] = min{U[:,j]} and d[j] = max{U[:,j]}
col = 4
c = []
d = []
for j in range(col):
    c.append(np.min(U_leaf[:,j]))
    d.append(np.max(U_leaf[:,j]))

uP = np.zeros((3*M*N, col))
uN = np.zeros((3*M*N, col))

# Separate positive and negative components of each column of U into uP and uN
for j in range(col):
    for i in range(3*M*N):
        uP[i,j] = max(0, U_leaf[i,j]/d[j])
        uN[i,j] = max(0, U_leaf[i,j]/c[j])


#print(uP.shape, uN.shape)
```

In [37]:

```
# uP[:,j=0]
uP_0_r = uP[:,0][0:M*N].reshape((N,M))
uP_0_g = uP[:,0][M*N:2*M*N].reshape((N,M))
uP_0_b = uP[:,0][2*M*N:3*M*N].reshape((N,M))
uP_0_rgb = np.zeros([N,M,3])
uP_0_rgb[:,:,0] = uP_0_r
uP_0_rgb[:,:,1] = uP_0_g
uP_0_rgb[:,:,2] = uP_0_b
uP_0_cmy = 1 - uP_0_rgb

# uP[:,j=1]
uP_1_r = uP[:,1][0:M*N].reshape((N,M))
uP_1_g = uP[:,1][M*N:2*M*N].reshape((N,M))
uP_1_b = uP[:,1][2*M*N:3*M*N].reshape((N,M))
uP_1_rgb = np.zeros([N,M,3])
uP_1_rgb[:,:,0] = uP_1_r
uP_1_rgb[:,:,1] = uP_1_g
uP_1_rgb[:,:,2] = uP_1_b
uP_1_cmy = 1 - uP_1_rgb

# uP[:,j=2]
uP_2_r = uP[:,2][0:M*N].reshape((N,M))
uP_2_g = uP[:,2][M*N:2*M*N].reshape((N,M))
uP_2_b = uP[:,2][2*M*N:3*M*N].reshape((N,M))
uP_2_rgb = np.zeros([N,M,3])
uP_2_rgb[:,:,0] = uP_2_r
uP_2_rgb[:,:,1] = uP_2_g
uP_2_rgb[:,:,2] = uP_2_b
uP_2_cmy = 1 - uP_2_rgb

# uP[:,j=2]
uP_3_r = uP[:,3][0:M*N].reshape((N,M))
uP_3_g = uP[:,3][M*N:2*M*N].reshape((N,M))
uP_3_b = uP[:,3][2*M*N:3*M*N].reshape((N,M))
uP_3_rgb = np.zeros([N,M,3])
uP_3_rgb[:,:,0] = uP_3_r
uP_3_rgb[:,:,1] = uP_3_g
uP_3_rgb[:,:,2] = uP_3_b
uP_3_cmy = 1 - uP_3_rgb

fig, axes = plt.subplots(1, 4, figsize=(16, 6))

axes[0].imshow(uP_0_cmy)
axes[0].set_title('uP[:,j=0]', fontsize=16)

axes[1].imshow(uP_1_cmy)
axes[1].set_title('uP[:,j=1]', fontsize=16)

axes[2].imshow(uP_2_cmy)
axes[2].set_title('uP[:,j=2]', fontsize=16)

axes[3].imshow(uP_3_cmy)
axes[3].set_title('uP[:,j=3]', fontsize=16)
```

Out[37]:

<matplotlib.text.Text at 0x11b9f2048>

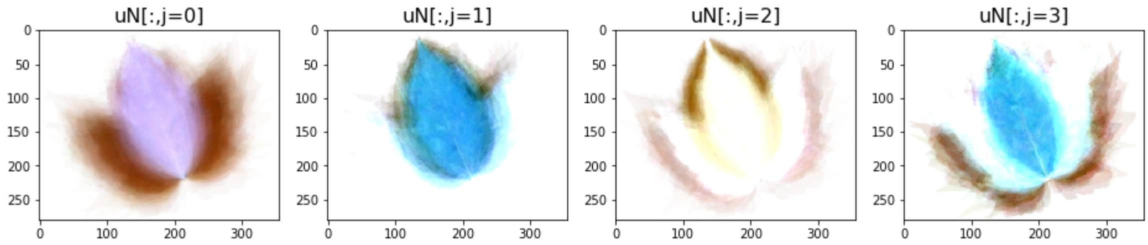


In [38]:

```
# uN[:,j=0]
uN_0_r = uN[:,0][0:M*N].reshape((N,M))
uN_0_g = uN[:,0][M*N:2*M*N].reshape((N,M))
uN_0_b = uN[:,0][2*M*N:3*M*N].reshape((N,M))
uN_0_rgb = np.zeros([N,M,3])
uN_0_rgb[:,:,0] = uN_0_r
uN_0_rgb[:,:,1] = uN_0_g
uN_0_rgb[:,:,2] = uN_0_b
uN_0_cmy = 1 - uN_0_rgb

# uN[:,j=1]
uN_1_r = uN[:,1][0:M*N].reshape((N,M))
uN_1_g = uN[:,1][M*N:2*M*N].reshape((N,M))
uN_1_b = uN[:,1][2*M*N:3*M*N].reshape((N,M))
uN_1_rgb = np.zeros([N,M,3])
uN_1_rgb[:,:,0] = uN_1_r
uN_1_rgb[:,:,1] = uN_1_g
uN_1_rgb[:,:,2] = uN_1_b
uN_1_cmy = 1 - uN_1_rgb

# uN[:,j=2]
uN_2_r = uN[:,2][0:M*N].reshape((N,M))
uN_2_g = uN[:,2][M*N:2*M*N].reshape((N,M))
uN_2_b = uN[:,2][2*M*N:3*M*N].reshape((N,M))
uN_2_rgb = np.zeros([N,M,3])
uN_2_rgb[:,:,0] = uN_2_r
uN_2_rgb[:,:,1] = uN_2_g
uN_2_rgb[:,:,2] = uN_2_b
uN_2_cmy = 1 - uN_2_rgb

# uN[:,j=3]
uN_3_r = uN[:,3][0:M*N].reshape((N,M))
uN_3_g = uN[:,3][M*N:2*M*N].reshape((N,M))
uN_3_b = uN[:,3][2*M*N:3*M*N].reshape((N,M))
uN_3_rgb = np.zeros([N,M,3])
uN_3_rgb[:,:,0] = uN_3_r
uN_3_rgb[:,:,1] = uN_3_g
uN_3_rgb[:,:,2] = uN_3_b
uN_3_cmy = 1 - uN_3_rgb

fig, axes = plt.subplots(1, 4, figsize=(16, 6))

axes[0].imshow(uN_0_cmy)
axes[0].set_title('uN[:,j=0]', fontsize=16)

axes[1].imshow(uN_1_cmy)
axes[1].set_title('uN[:,j=1]', fontsize=16)

axes[2].imshow(uN_2_cmy)
axes[2].set_title('uN[:,j=2]', fontsize=16)

axes[3].imshow(uN_3_cmy)
axes[3].set_title('uN[:,j=3]', fontsize=16)
```

Out[38]:

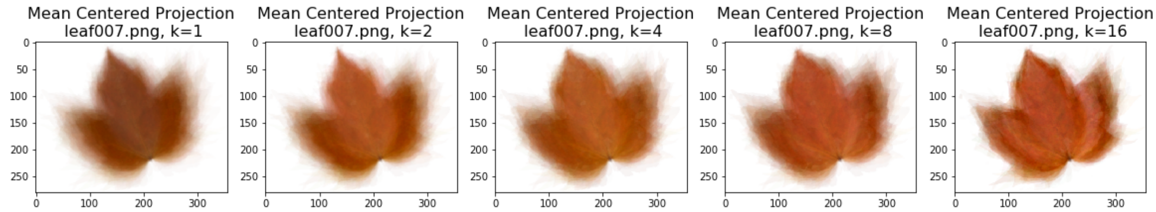
<matplotlib.text.Text at 0x11c9abf60>



(c)

In [39]:

```
def projection_left_K(k, filename):
    T = 1 - io.imread(filename)/255

    # Stretch this leaf into a long (3*M*N, 1) 1d array
    T_r = T[:,:,0].reshape(M*N)
    T_g = T[:,:,1].reshape(M*N)
    T_b = T[:,:,2].reshape(M*N)
    T_rgb_long = np.concatenate((T_r,T_g,T_b), axis=0)

    # Build the projection from the mean_leaf
    P = np.array(mean_leaf_rgb_long)
    T_centered = T_rgb_long - mean_leaf_rgb_long
    for j in range(k):
        P += np.dot(U_leaf[:,j].T, T_centered) * U_leaf[:,j]

    # Transform the (3*M*N, 1) 1d leaf projection into an image
    P_r = P[0:M*N].reshape((N,M))
    P_g = P[M*N:2*M*N].reshape((N,M))
    P_b = P[2*M*N:3*M*N].reshape((N,M))
    P_rgb = np.zeros([N,M,3])
    P_rgb[:,:,0] = P_r
    P_rgb[:,:,1] = P_g
    P_rgb[:,:,2] = P_b
    P_cmy = 1 - P_rgb

    # Return the projection in both RGB and CMY color mixing
    return (P_rgb, P_cmy)

# For k = 1, 2, 4, 8, 16, make a list of the leaf's projections using its k left
 singular vectors
P_rgb = []
P_cmy = []
k_values = [1,2,4,8,16]
for kv in k_values:
    P_rgb_k, P_cmy_k = projection_left_K(k=kv, filename=path+file_names[7]) # ch
oose leaf007.png
    P_rgb.append(P_rgb_k)
    P_cmy.append(P_cmy_k)

# Plot the projections
fig, axes = plt.subplots(1, 5, figsize=(20, 8))
for i in range(len(k_values)):
    axes[i].imshow(np.clip(P_cmy[i], 0, 1))
    axes[i].set_title('Mean Centered Projection\n leaf007.png, k=%d' %
k_values[i], fontsize=16)
```



(d)

In [40]:

```
# Construct the list of distances that a leaf is away from its mean centered pro
jection
# using its 8 left singular vectors
dis_S = []
for f_name in file_names:
    S_rgb = 1 - io.imread(path+f_name)/255
    P_rgb, P_cmy = projection_left_K(k=8, filename=path+f_name) # k = 8 specifie
d

    # Stretch leaf S into a long (3*M*N, 1) array
    S_r = S_rgb[:,:,0].reshape(M*N)
    S_g = S_rgb[:,:,1].reshape(M*N)
    S_b = S_rgb[:,:,2].reshape(M*N)
    S_rgb_long = np.concatenate((S_r,S_g,S_b), axis=0)

    # Stretch projected leaf P into a long (3*M*N, 1) array
    P_r = P_rgb[:,:,0].reshape(M*N)
    P_g = P_rgb[:,:,1].reshape(M*N)
    P_b = P_rgb[:,:,2].reshape(M*N)
    P_rgb_long = np.concatenate((P_r,P_g,P_b), axis=0)

    dis_S.append(np.sum(np.square(S_rgb_long-P_rgb_long))/(M*N))
```