數位訊號處理 Digital Signal Processing (DSP)

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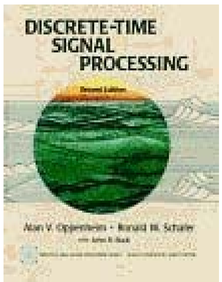
綜科館 **311-4**

Textbook:

Discrete-time Signal Processing, 1999, 2nd edition
A. V. Oppenheim, R. W. Schafer, and J. R. Buck
Prentice Hall



Resource:

網路學園

Grading:

Midterm exam	40%
Final exam	40%
Quiz	20%

Syllabus:

- Analog Signals: Continuous in Time & Frequency
- Digital Signals (I): Discrete in Time, and Continuous in Frequency
- Digital Signals (II): Discrete in Time, Discrete in Frequency
- Digital Systems
- Sampling of Continuous-Time Signals
- Structures for Discrete-Time Systems
- Filter Design Techniques

Chapter 0 Introduction

0.1 What is a signal?

- A function of independent variables such as time, distance, position, temperature, etc.
- A signal carries information. Examples: speech, music, seismic, image and video.
- A signal can be a function of one, two or *N* independent variables. For example,

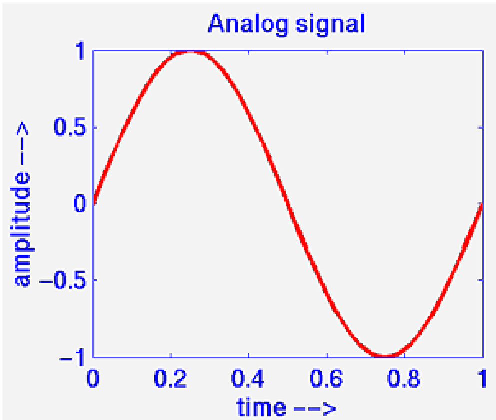
  Speech is a 1-D signal as a function of time

  Image is a 2-D signal as a function of space

  Video is a 3-D signal as a function of space and time
- Types of Signals

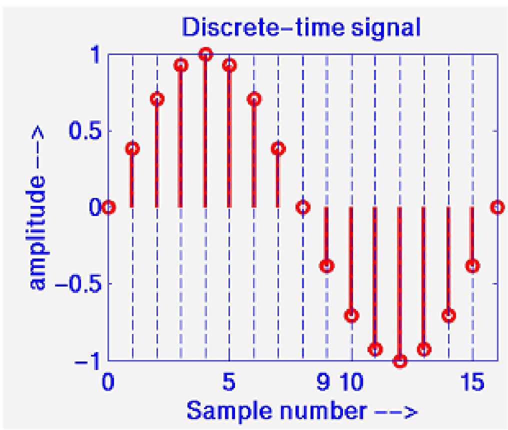
  Analog Signals (Continuous-Time Signals):

  Signals that are continuous in both the dependent and independent variable (e.g., amplitude and time). Most environmental signals are continuous-time signals.



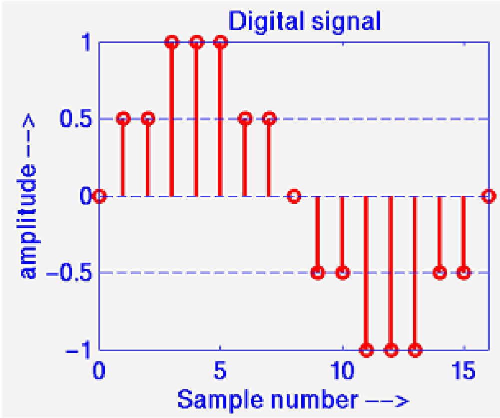
  Discrete Sequences (Discrete-Time Signals):

  Signals that are continuous in the dependent variable (e.g., amplitude) but discrete in the independent variable (e.g., time). They are typically associated with sampling of continuous-time signals.



  Digital Signals:

  Signals that are discrete in both the dependent and independent variable (e.g., amplitude and time) are digital signals. These are created by *quantizing* and *sampling* continuous-time signals or as data signals (e.g., stock market price fluctuations).



0.2 What is digital signal processing (DSP)?

- Changing or analyzing information that is measured as discrete sequences of numbers.
- Why go digital?

Analogue signal processing is achieved by using analogue components, such as resistors, capacitors, inductors. The inherent tolerances associated with these components, temperature, voltage changes and mechanical vibrations can dramatically affect the effectiveness of the analogue circuitry.

With DSP, it is easy to:

Change applications.

Correct applications.

Update applications.

In addition, DSP reduces:

Noise susceptibility.

Chip count.

Development time.

Cost.

Power consumption.

- How to process real-world signals by DSP?

Most of the signals in our environment are analog such as sound, temperature and light. To process those signals, we need to

1. Converting the analog signals into electrical signals, e.g., using a transducer such as a microphone to convert sound into electrical signal;

2. Digitizing these signals, or convert them from analog to digital, using an ADC (Analog to Digital Converter).

Processed signal may need to be converted back to an analog signal before being passed to an actuator (e.g., a loudspeaker).

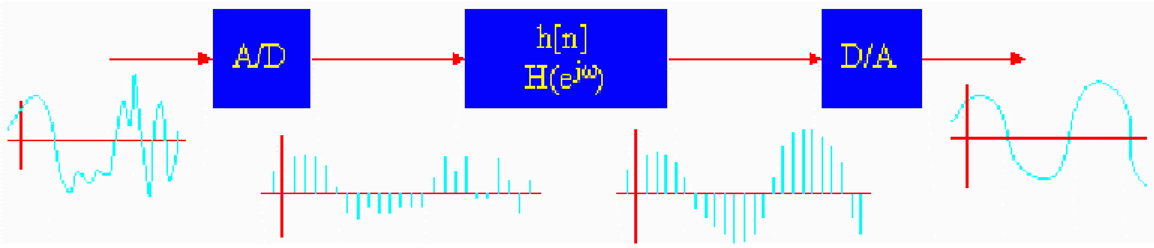
Digital to analog conversion and can be done by a DAC (Digital to Analog Converter)

- Typical DSP system components

Input lowpass filter (anti-aliasing filter)

Analog to digital converter (ADC)

Digital computer or digital signal processor
Digital to analog converter (DAC)
Output lowpass filter (anti-imaging filter)



- Applications of DSP
  - Speech Processing: noise filtering, coding, compression, recognition, synthesis, etc.
  - Music: recording, playback and manipulation (mixing, special effects), synthesis, etc.
  - Image Processing: enhancement, coding, compression, pattern recognition, etc.
  - Multimedia: motion pictures, digital TV, video conferencing, etc.
  - Communication: encoding and decoding of digital communication signals, detection, etc.
  - Radar and Sonar: target detection, position and velocity estimation, tracking
  - Biomedical Engineering: diagnosis, patient monitoring, preventive health care, etc.

0.3 Digital Signal Processors (beyond the scope of this course)

- Why Digital signal processors? Why not General Purpose Processor (GPP), e.g. Pentium®?
  - Use a DSP processor when the following are required:
    - Cost saving.
    - Smaller size.
    - Low power consumption.
    - Processing of many "high" frequency signals in real-time.
  - Use a GPP processor when the following are required:
    - Large memory.
    - Advanced operating systems.
- Floating vs. Fixed Point Processors
  - Using floating point processors when the applications require:
    - High precision.
    - Wide dynamic range.
    - High signal-to-noise ratio.
    - Ease of use.
  - However, floating point processors usually suffer from the following drawbacks:
    - Higher power consumption
    - Higher cost
    - Slower than fixed-point counterparts
    - Larger in size.
- Texas Instruments (TI) TMS320Cx DSPs

C1x, C2x

Fixed-point devices with 16-bit data bus width

Used in toys, hard disk drives, modems and active car suspensions

C3x

Floating-point devices with 32-bit data bus width, which provides much wider dynamic range as compared to fixed-point devices.

Used in hi-fi systems, voice mail systems and 3D graphic processing.

C4x

32-bit floating-point device designed for parallel processing

Optimized on-chip communication channel enables several devices to be put together to form a parallel cluster.

Used in virtual reality, recognition and parallel processing systems

C5x

Low power fixed-point DSPs

Used for personal and portable electronics such as cell phones, digital music players, and digital cameras

C6x

High performance DSPs, with speeds up to 1 GHz

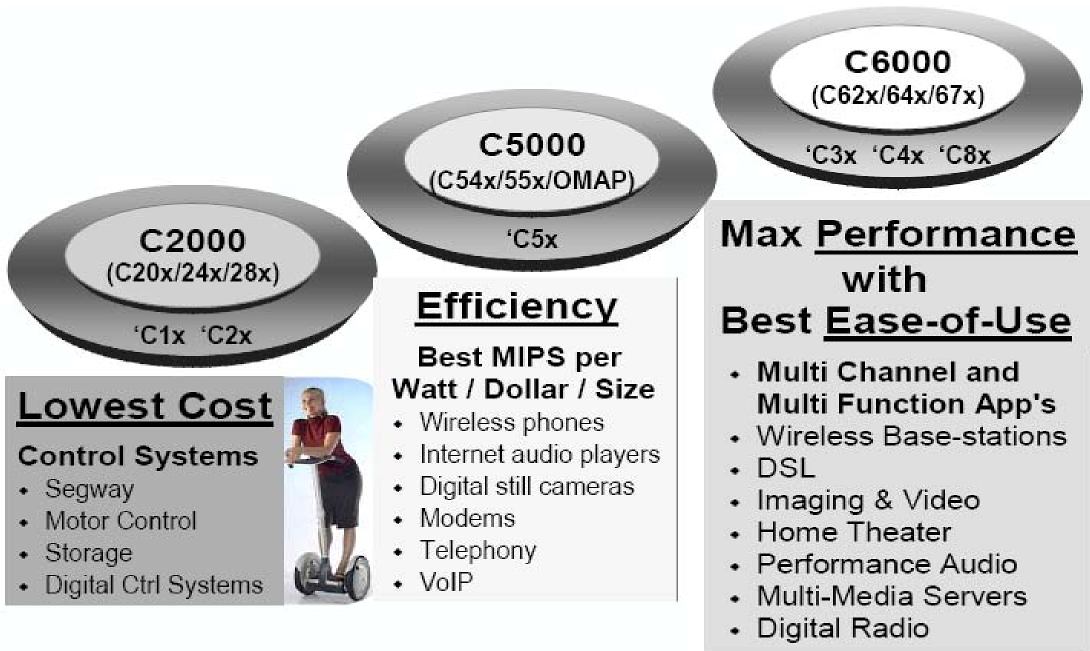
Both fixed and floating-point devices

Used in wired and wireless broadband networks, imaging applications and professional audio

C8x

Multimedia processors, with parallel processing on a single chip with advanced DSPs and a controlling RISC processor;

Used in high performance telephony, 3D computer graphics, virtual reality and a number of multimedia applications

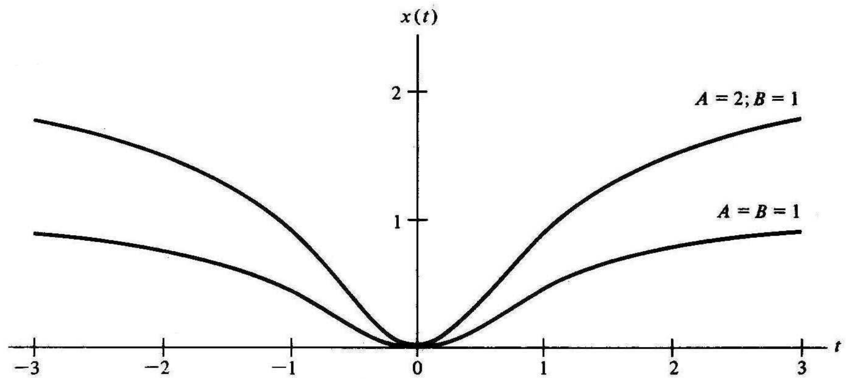


Chapter 1 Analog Signals: Continuous in Time & Frequency

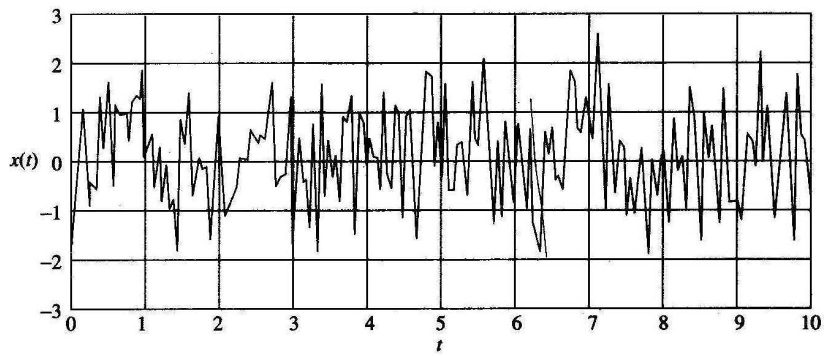
1.1 Continuous-time (Analog) Signals

- Definition – the signal $x(t)$ is a function of the real-value time variable t.
  Note that the signal is not necessary a mathematically continuous function.

- Deterministic signals vs. random signals
  - A deterministic signal can be represented by a completely specified function of time. For example,
    $x(t) = \dfrac{At^2}{B+t^2}$, $-\infty < t < \infty$, where A and B are constants.



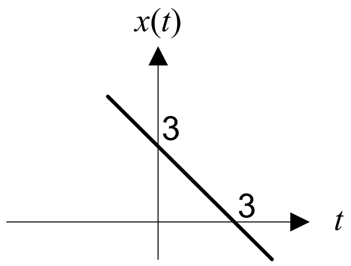
  - A random signal takes on random values at any given time instant, which can not be represented as specified functions of time. For example,



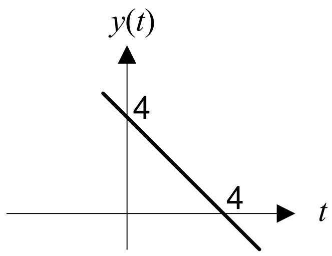
- Periodic signals vs. aperiodic signals
  - A deterministic signal $x(t)$ is periodic if $x(t) = x(t + T)$, $-\infty < t < \infty$, where T is a constant period. For example, $x(t) = A\sin(2\pi t/T + \theta)$, $-\infty < t < \infty$,
  - Any deterministic signal not satisfying $x(t) = x(t + T)$ for a finite value of T is called aperiodic.

- Shifting and scaling

  Let $x(t) = 3 - t$, observe the following signals:

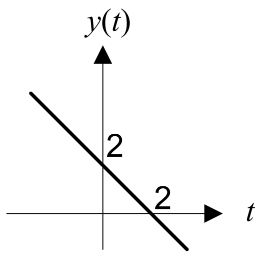


$y(t) = x(t-1)$



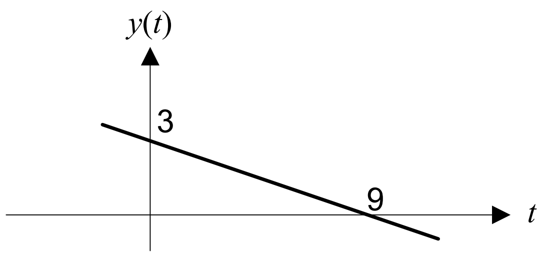
(toward right-hand side)

$y(t) = x(t+1)$



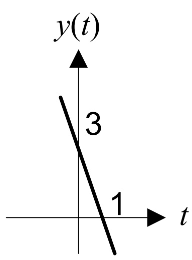
(toward left-hand side)

$y(t) = x(t/3)$



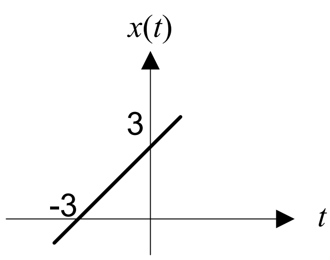
(expansion)

$y(t) = x(t \times 3)$



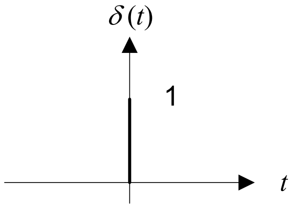
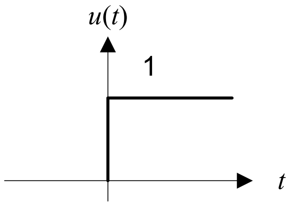
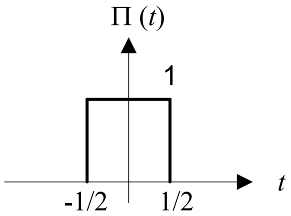
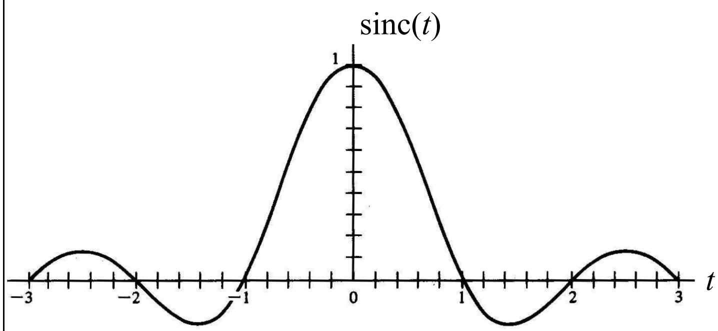
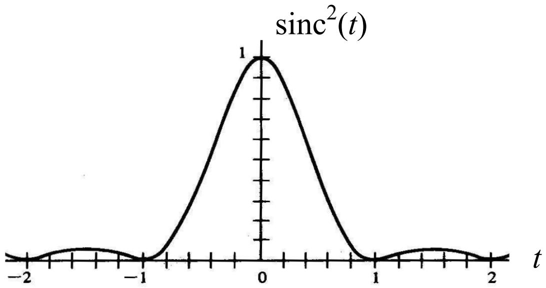
(compression)

$y(t) = x(-t)$



(flip)

- Some basic signals

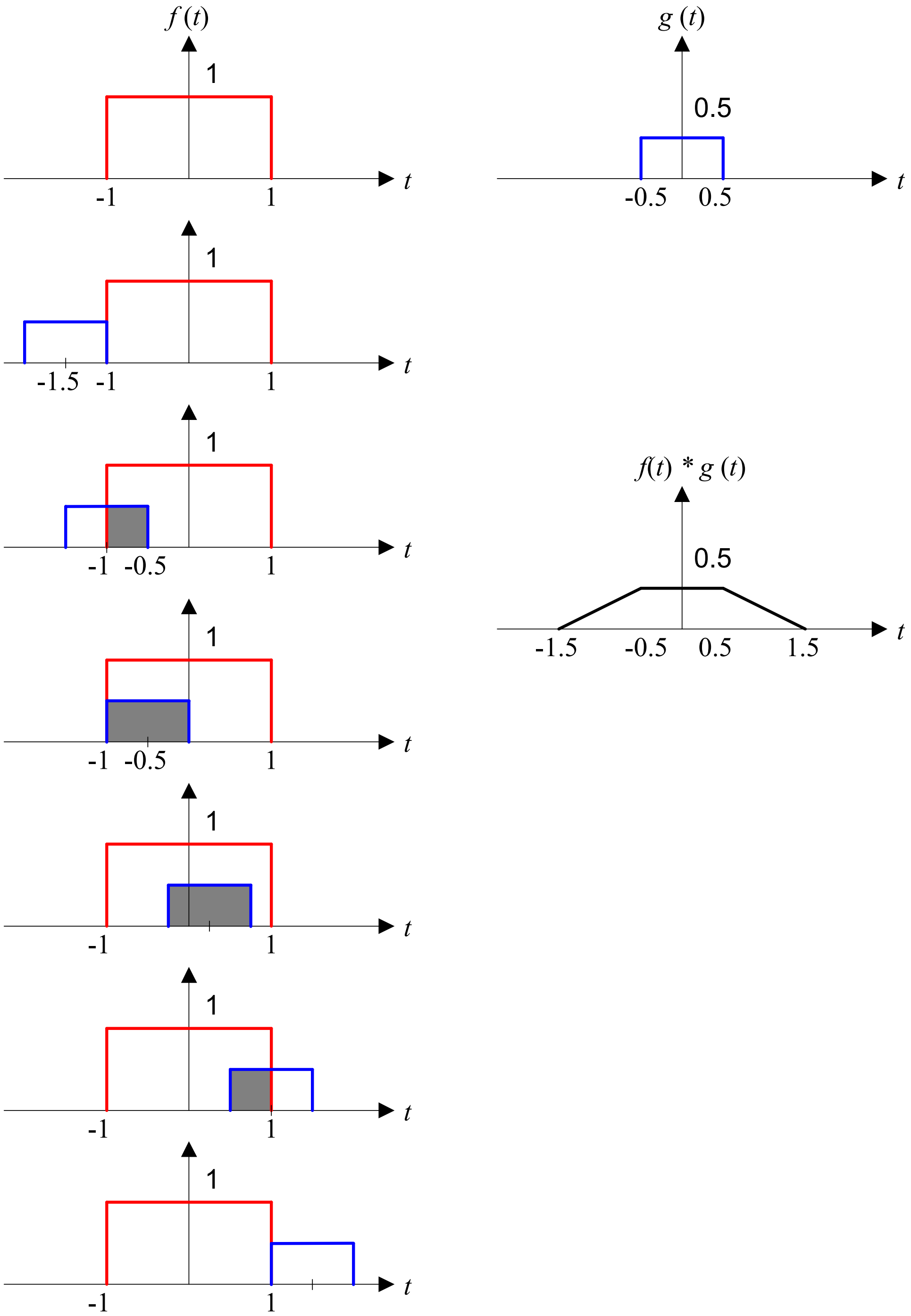
Unit impulse function (delta function), $\delta(t) = \begin{cases} 1, t = 0 \\ 0, t \neq 0 \end{cases}$	
Unit step function $u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$	
Unit pulse function $\Pi(t) = u(t + ½) - u(t - ½)$	
Sinc function $\text{sinc}(t) = \dfrac{\sin \pi t}{\pi t}$	
$\text{sinc}^2(t)$	

- Convolution

A convolution is an integral that expresses the amount of overlap of one signal $f(t)$ as it is shifted over another function $g(t)$. Specifically,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\theta) g(t - \theta) d\theta .$$

For example,



1.2 Fourier Series Expansion

If signal $x(t)$ is **periodic**, i.e., $x(t) = x(t + T)$, $-\infty < t < \infty$, then $x(t)$ can be represented by Fourier series expansion.

- Trigonometric Fourier Series

  Any signal can be viewed as a linear combination of cosine and sine waves:

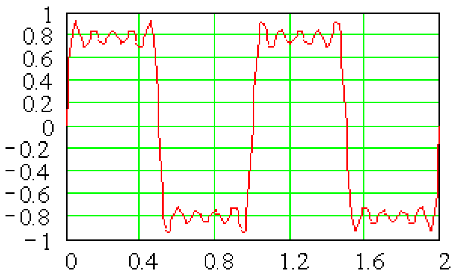
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t),$$

where $\Omega_0 = \dfrac{2\pi}{T}$ is called the fundamental frequency, and

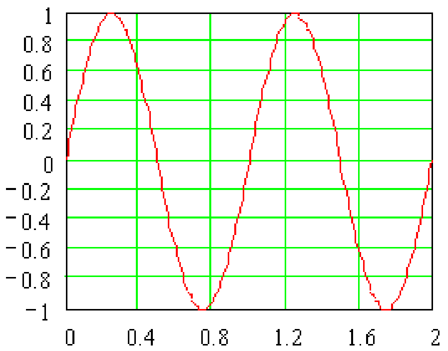
$$a_0 = \frac{1}{T}\int_T x(t)dt \,,$$

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\Omega_0 t)dt \,,\ 1 \le n < \infty$$

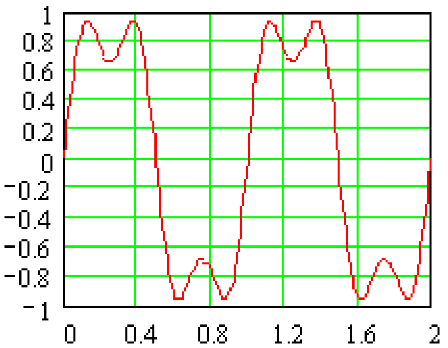
$$b_n = \frac{2}{T}\int_T x(t)\sin(n\Omega_0 t)dt \,,\ 1 \le n < \infty$$



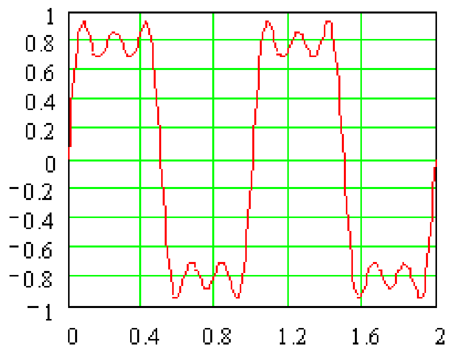
A signal $x(t)$ with period $T = 1$ sec



$\sin(\frac{2\pi}{T}t)$



$\sin(\frac{2\pi}{T}t) + \frac{1}{3}\sin(\frac{6\pi}{T}t)$



$\sin(\frac{2\pi}{T}t) + \frac{1}{3}\sin(\frac{6\pi}{T}t) + \frac{1}{5}\sin(\frac{10\pi}{T}t)$

- Complex Exponential Fourier Series

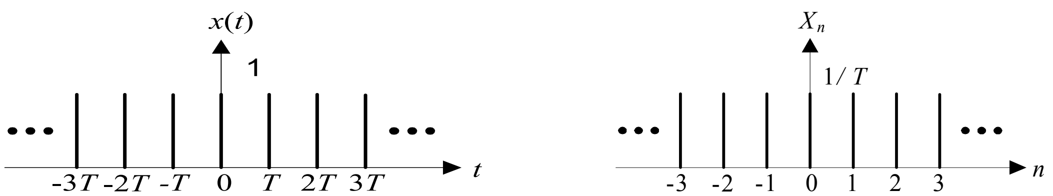
Letting $\cos(n\Omega_0 t) = \frac{1}{2}\left(e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}\right)$, and $\sin(n\Omega_0 t) = \frac{1}{2j}\left(e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}\right)$, then

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t) \\ &= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}\right) + \frac{1}{2j}\sum_{n=1}^{\infty} b_n\left(e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}\right) \\ &= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - jb_n)e^{jn\Omega_0 t} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + jb_n)e^{-jn\Omega_0 t} \\ &= \cdots + X_{-2}e^{-j2\Omega_0 t} + X_{-1}e^{-j\Omega_0 t} + X_0 + X_1 e^{j\Omega_0 t} + X_2 e^{j2\Omega_0 t} + \cdots \\ &= \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t} \end{aligned}$$

and it can be shown that $X_n = \frac{1}{T}\int_T x(t)e^{-jn\Omega_0 t}dt$, $-\infty < n < \infty$.

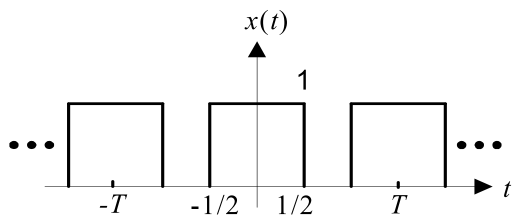
Example: Consider the periodic sequence of an impulse train $x(t) = \sum_{m=-\infty}^{\infty}\delta(t - mT)$, whose period is T. The exponential Fourier series coefficients are calculated by evaluating the integral

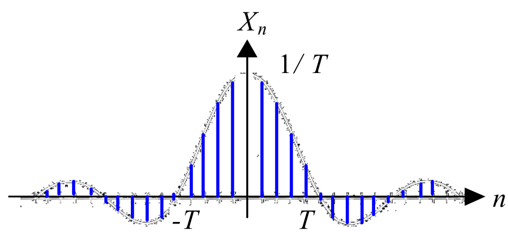
$$X_n = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jn\Omega_0 t}dt = \frac{1}{T}, \quad -\infty < n < \infty$$



Example: Consider the periodic sequence of a pulse train $x(t) = \sum_{m=-\infty}^{\infty}\Pi(t - mT)$, $T > 1/2$, whose period is T. The exponential Fourier series coefficients are

$$\begin{aligned} X_n &= \frac{1}{T}\int_{-1/2}^{1/2} e^{-jn\Omega_0 t}dt \\ &= \frac{1}{-jn\Omega_0 T}e^{-jn\Omega_0 t}\Big|_{-1/2}^{1/2} \\ &= \frac{2}{n\Omega_0 T}\left(\frac{e^{jn\Omega_0/2} - e^{-jn\Omega_0/2}}{2j}\right) \\ &= \frac{2}{n\Omega_0 T}\sin(n\Omega_0/2) \\ &= \frac{1}{T}\frac{\sin(n\pi/T)}{n\pi/T} = \frac{1}{T}\mathrm{sinc}\left(\frac{n}{T}\right), \quad -\infty < n < \infty \end{aligned}$$





1.3 Fourier Transform

- Definition

If signal $x(t)$ is not periodic (**aperiodic**), we can consider $x(t) = \lim_{T\to\infty} x_T(t)$, where $x_T(t)$ is a periodic signal, i.e., $x_T(t) = x_T(t + T)$, which can be expressed by the Fourier series expansion:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}\text{, and } X_n = \frac{1}{T}\int_T x(t)e^{-jn\Omega_0 t}dt.$$

Since $T \to \infty$, $\Omega_0 \to 0$, it can be shown that the Fourier series expansion becomes an integral over the continuous variable $\Omega = n\Omega_0$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \triangleq \mathcal{F}^{-1}\{X(j\Omega)\}$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \triangleq \mathcal{F}\{x(t)\}.$$

This operation, referred to as Fourier transform, decomposes a function into a continuous spectrum of its frequency components, and the inverse transform synthesizes a function from its spectrum of frequency components.

- Properties

	$\delta(t)$	$\leftrightarrow$	1
	$\Pi(t)$	$\leftrightarrow$	$\frac{2\sin(\Omega/2)}{\Omega} = \text{sinc } f$, where $\Omega = 2\pi f$
Superposition	$a_1 x_1(t) + a_2 x_2(t)$	$\leftrightarrow$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time delay	$x(t - t_0)$	$\leftrightarrow$	$X(j\Omega)\, e^{-j\Omega t_0}$
	$\delta(t-t_0)$	$\leftrightarrow$	$e^{-j\Omega t_0}$
Frequency translation	$x(t)\, e^{j\Omega_0 t}$	$\leftrightarrow$	$X(j(\Omega-\Omega_0))$
Time scaling	$x(at)$	$\leftrightarrow$	$\frac{1}{a}X(j\frac{\Omega}{a})$
	$\Pi(\frac{t}{a})$	$\leftrightarrow$	$a\frac{2\sin(a\Omega/2)}{a\Omega} = a\,\text{sinc}\, af$
Time reversal	$x(-t)$	$\leftrightarrow$	$X(-j\Omega)$
Duality	$X(t)$	$\leftrightarrow$	$2\pi x(-\Omega)$
	1	$\leftrightarrow$	$2\pi\delta(\Omega)$
	$e^{-j\Omega_0 t}$	$\leftrightarrow$	$2\pi\delta(\Omega+\Omega_0)$
	$\frac{\sin\Omega_c t}{\pi t}$	$\leftrightarrow$	$\Pi(\frac{\Omega}{2\Omega_c}) = \begin{cases} 1, & \lvert\Omega\rvert \le \Omega_c \\ 0, & \text{otherwise} \end{cases}$
Convolution	$x(t) * y(t)$	$\leftrightarrow$	$X(j\Omega)\, Y(j\Omega)$
Multiplication	$x(t)\, y(t)$	$\leftrightarrow$	$\frac{1}{2\pi}X(j\Omega) * Y(j\Omega)$
			$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\Omega-\theta))d\theta$

- Fourier transform can also be applied to a **periodic** signal $x(t)$ by

  Step 1: representing $x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}$ (Fourier series expansion)

  Step 2: computing $\mathcal{F}\{x(t)\}=\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}\right\}=2\pi\sum_{n=-\infty}^{\infty} X_n \delta(\Omega-n\Omega_0)$

  For example: find the Fourier transform of $x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$

  Let $x(t)=\sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$, where X_k is the Fourier series coefficients,

  $$X_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\Omega_0 t}dt=\frac{1}{T}$$

  Therefore, $X(j\Omega)=\mathcal{F}\left\{\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\Omega_0 t}\right\}=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0)$.

- Fourier series coefficients of a periodic signal $x(t)$ can be computed by using the Fourier transform of $x(t)$ within one period, that is,

  $$X_n=\frac{1}{T}\hat{X}(j\Omega)\Big|_{\Omega=n\Omega_0}, \text{ where } \hat{X}(j\Omega)=\mathcal{F}\left\{x(t)\big|_{\text{one period}}\right\}$$

  For example: We have shown that the Fourier series coefficients of $x(t)=\sum_{m=-\infty}^{\infty}A\Pi(t-mT)$ is $X_n=\frac{A}{T}\text{sinc}\left(\frac{n}{T}\right)$, $-\infty<n<\infty$. The coefficients can also be obtained by

  (i) computing the Fourier transform of $\Pi(t)$: $\mathcal{F}\{\Pi(t)\}=\frac{2\sin(\Omega/2)}{\Omega}$

  (ii) $X_n=A\frac{1}{T}\mathcal{F}\{\Pi(t)\}\Big|_{\Omega=n\Omega_0}=A\frac{1}{T}\frac{2\sin(n\Omega_0/2)}{n\Omega_0}=A\frac{1}{T}\frac{2\sin(n\pi/T)}{n2\pi/T}=\frac{A}{T}\text{sinc}\left(\frac{n}{T}\right)$

1.4 Laplace Transform

- **Definition**

  The Laplace transform and inverse Laplance transform for signal $x(t)$ is defined by

  $$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt,$$

  $$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$

  where s is a complex, i.e., $s=\sigma+j\Omega$. If we let $s=j\Omega$, then Laplace transform is equivalent to Fourier transform.

- Why Laplace transform?
  - A generalization of Fourier transform. Some signals that do not converge for Fourier transform have valid Laplace transforms.
  - Better notation (compared to Fourier transform) in analytical problems (complex variable theory)
  - Solving differential equation.

- Unilateral or one-sided Laplace transform
  For convergence, Laplace transform is usually computed by setting the low limit of t as zero, that is, $X(s)=\int_0^{\infty} x(t)e^{-st}dt$.
  For example, $x(t) = 1$, $X(s)=\int_0^{\infty} e^{-st}dt=\frac{1}{s}$.

- Properties

	Operation in Time Domain	**Operation in Frequency Domain**
1. Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
2. Differentiation	$\frac{d^nx(t)}{dt^n}$	$s^nX(s)-s^{n-1}x(0^-)-\cdots-x^{(n-1)}(0^-)$
3. Integration	$\int_{-\infty}^{t} x(\lambda)\,d\lambda$	$\frac{X(s)}{s}+\frac{x^{(-1)}(0^-)}{s}$
4. s-shift	$x(t)\exp(-\alpha t)$	$X(s+\alpha)$
5. Delay	$x(t-t_0)u(t-t_0)$	$X(s)\exp(-st_0)$
6. Convolution	$x_1(t)*x_2(t)$ $=\int_0^{\infty} x_1(\lambda)x_2(t-\lambda)\,d\lambda$	$X_1(s)X_2(s)$
7. Product	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X_1(s-\lambda)X_2(\lambda)\,d\lambda$
8. Initial value (provided limits exist)	$\lim_{t\to 0^+} x(t)$	$\lim_{s\to\infty} sX(s)$
9. Final value (provided limits exist)	$\lim_{t\to\infty} x(t)$	$\lim_{s\to 0} sX(s)$
10. Time scaling	$x(at),\quad a>0$	$a^{-1}X\left(\frac{s}{a}\right)$

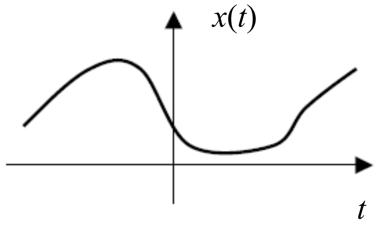
Signal	Laplace Transform
1. $\delta^{(n)}(t)$	s^n
2. 1 or $u(t)$	$\dfrac{1}{s}$
3. $\dfrac{t^n \exp(-\alpha t)u(t)}{n!}$	$\dfrac{1}{(s+\alpha)^{n+1}}$
4. $\cos \omega_0 t\, u(t)$	$\dfrac{s}{s^2+\omega_0^2}$
5. $\sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$
6. $\exp(-\alpha t)\cos \omega_0 t\, u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$
7. $\exp(-\alpha t)\sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(s+\alpha)^2+\omega_0^2}$
8. Square wave: $u(t) - 2u\left(t - \dfrac{T_0}{2}\right) + 2u(t - T_0) - \cdots$	$\dfrac{1}{s}\dfrac{1-e^{-sT_0/2}}{1+e^{-sT_0/2}}$
9. $(\sin \omega_0 t - \omega_0 t \cos \omega_0 t)u(t)$	$\dfrac{2\omega_0^3}{(s^2+\omega_0^2)^2}$
10. $(\omega_0 t \sin \omega_0 t)u(t)$	$\dfrac{2\omega_0^2 s}{(s^2+\omega_0^2)^2}$
11. $\omega_0 t \exp(-\alpha t)\sin \omega_0 t\, u(t)$	$\dfrac{2\omega_0^2(s+\alpha)}{[(s+\alpha)^2+\omega_0^2]^2}$
12. $\exp(-\alpha t)(\sin \omega_0 t - \omega_0 t \cos \omega_0 t)u(t)$	$\dfrac{2\omega_0^3}{[(s+\alpha)^2+\omega_0^2]^2}$

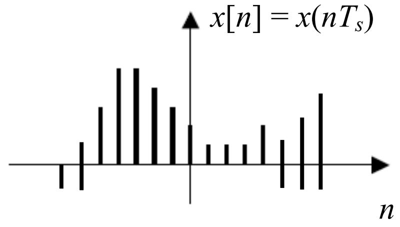
Chapter 2 Digital Signals (I):

Discrete in Time, and Continuous in Frequency

2.1 Discrete-time Signals: Sequences

- A discrete-time signal $x[n]$ is obtained by sampling a continuous-time signal $x(t)$.
  For example:

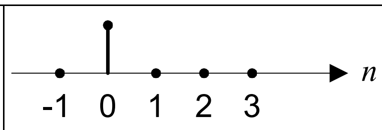
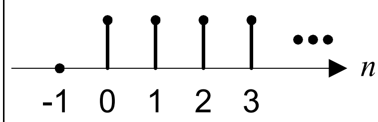
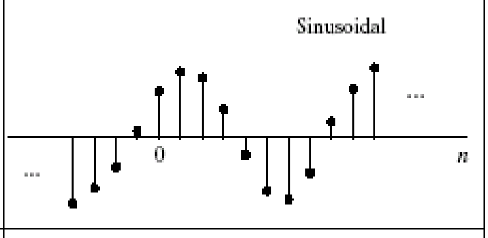
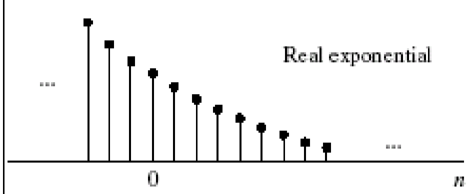




T_s is the sampling period.

Remark: Digital signals usually refer to the quantized discrete-time signal. In this course, we are mostly dealing with discrete-time signals with continuous values.

- **Some basic sequences**

Unit impulse sequence, $\delta[n] = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$	
Unit step sequence, $u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$ Note: $u[n] = \sum_{m=0}^{\infty} \delta[n-m]$, $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, and $\delta[n] = u[n] - u[n-1]$	
Sinusoidal Sequences, $x[n] = A\cos(w_0 n + \phi)$, A: amplitude; w_0: frequency; ϕ: phase Note: sinusoidal sequences are not always periodic.	
Exponential sequences, $x[n] = A\alpha^n$ $x[n] = A\alpha^n u[n]$	

2.2 Convolution

The convolution of two sequences $x[n]$ and $h[n]$ is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \triangleq x[n] * h[n]$$

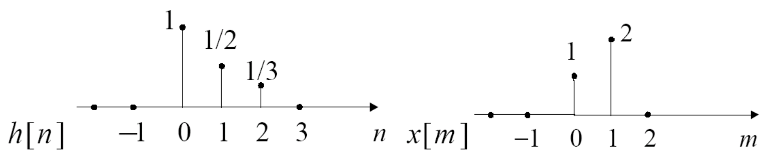
Note: $h[n] * \delta[n] = h[n]$.

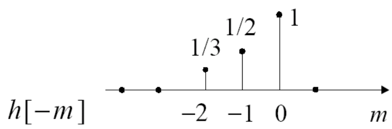
- **Procedure of convolution**

1. Time-reverse: $h[m]$ → $h[-m]$
2. Choose an n value
3. Shift $h[-m]$ by n: $h[n-m]$
4. Multiplication: $x[n] \cdot h[n-m]$
5. Summation over m: $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$

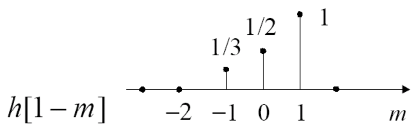
Choose another n value, go to Step 3.

Example:

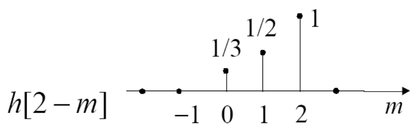




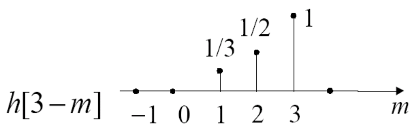
$y[0] = \sum_m x[m]h[-m] = 1$,



$y[1] = \sum_m x[m]h[1-m] = 5/2$,



$y[2] = \sum_m x[m]h[2-m] = 4/3$,



$y[3] = \sum_m x[m]h[3-m] = 2/3$.

Remark: if $x[n]$ has M non-zero samples, and $h[n]$ has N non-zero samples, then $y[n]$ has $(M+N-1)$ samples.

Alternative solution 1:

$x[n]$ \ $h[n]$	1	1/2	1/3
1	1	1/2	1/3
2	2	1	2/3

Alternative solution 2:

$$x[n] * h[n] = (\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{3}\delta[n-2]) * (\delta[n] + 2\delta[n-1])$$

$$= \delta[n] + 2\delta[n-1] + \frac{1}{2}\delta[n-1] + \delta[n-1-1] + \frac{1}{3}\delta[n-2] + \frac{2}{3}\delta[n-2-1]$$

$$= \delta[n] + \frac{5}{2}\delta[n-1] + \frac{4}{3}\delta[n-2] + \frac{2}{3}\delta[n-3]$$

2.3 Discrete-Time Fourier Transform (DTFT)

The Discrete-time Fourier transform decomposes a sequence into sinusoidal components of different frequencies, in which the frequency is continuous.

- **Definition**

Analysis: $F\{x[n]\} = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$.

Synthesis: $F^{-1}\{X(e^{jw})\} = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn} dw.$

Remark: w is often chosen as between $-\pi$ and π, but any interval of length 2π can be used. Why length 2π? The reason is that $X(e^{jw})$ is periodic with period 2π.

$$X(e^{j(w+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(w+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = X(e^{jw}).$$

$X(e^{jw})$ is a complex value, which can be represented by $X(e^{jw}) = |X(e^{jw})|\, e^{j\angle X(e^{jw})}$, where $|X(e^{jw})|$ is called *magnitude*, and $\angle X(e^{jw})$ is called *phase*.

$X(e^{jw})$ does not necessary exist (converge), unless $|X(e^{jw})| < \infty$.

The sufficient conditions (not necessary condition) for the convergence of $X(e^{jw})$ are

1. $x[n]$ is absolutely summable

$$|X(e^{jw})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}\right| \le \sum_{n=-\infty}^{\infty} |x[n]||e^{-jwn}| \le \sum_{n=-\infty}^{\infty} |x[n]|. \Rightarrow \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2. $x[n]$ is square summable $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

Example:

Let $x[n] = a^n u[n]$. $X(e^{jw}) = \sum_{n=0}^{\infty} a^n e^{-jwn} = \sum_{n=0}^{\infty} \left(ae^{-jw}\right)^n = \frac{1}{1-ae^{-jw}}$, if $|ae^{-jw}| < 1$, or $|a| < 1$

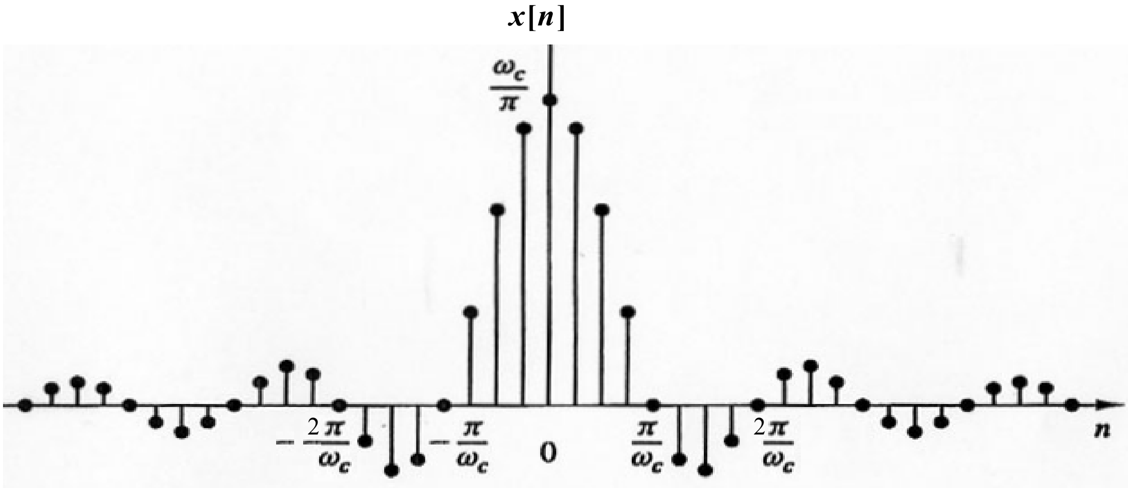
The condition for absolute summability of $x[n]$ is $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$, which means

$\sum_{n=-\infty}^{\infty} |a|^n < \infty \Rightarrow |a| < 1$

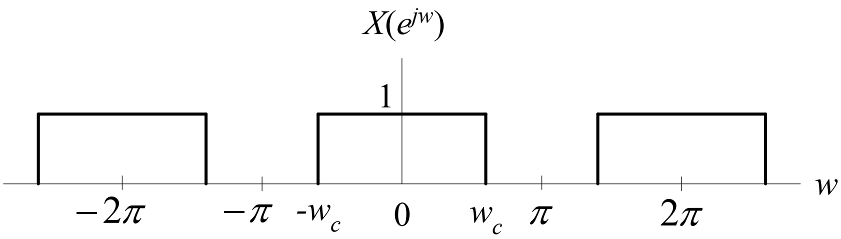
- **Special functions of DTFT:**

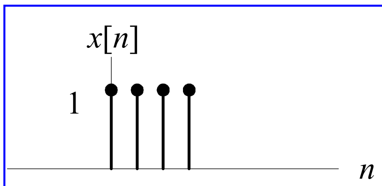
TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

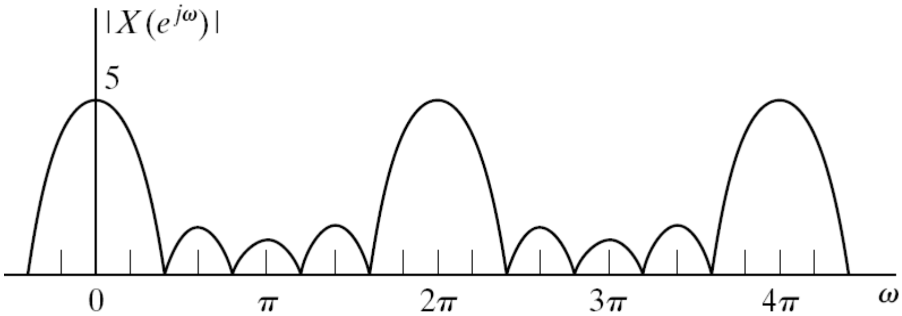


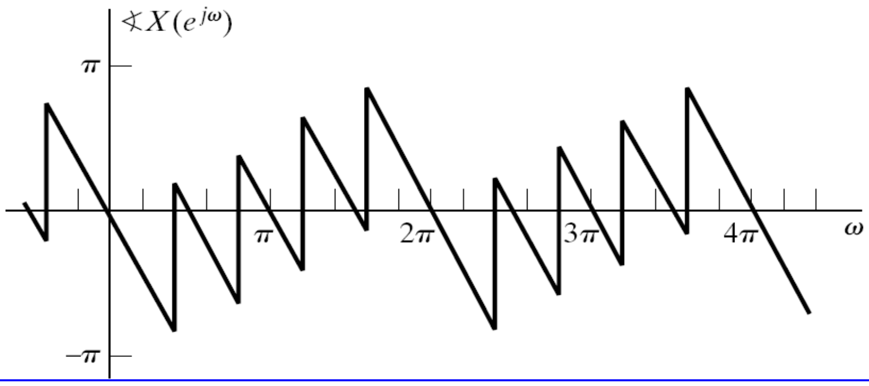
$$x[n]=\frac{\sin w_c n}{\pi n},\ -\infty\le n\le\infty \leftrightarrow X(e^{jw})=\begin{cases}1, & |w|\le w_c\\ 0, & w_c<|w|<\pi\end{cases}$$





$$x[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases} \leftrightarrow X(e^{jw})=\frac{\sin(wN/2)}{\sin(w/2)}e^{-jw(N-1)/2}$$



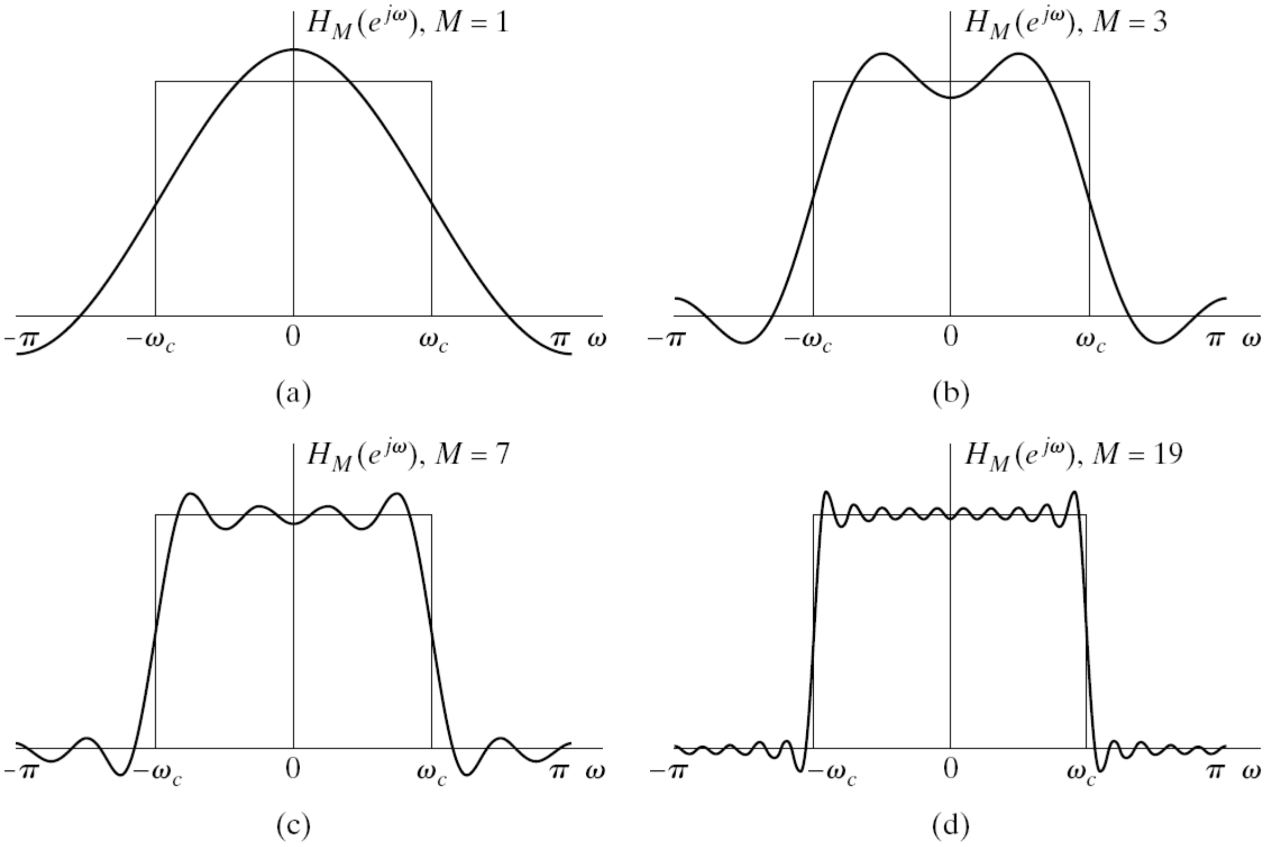


- **Gibbs phenomenon**

As we shown if $X(e^{jw}) = \begin{cases} 1, & |w| \le w_c \\ 0, & w_c < |w| < \pi \end{cases}$, then $x[n] = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn}dw = \frac{\sin w_c n}{\pi n}$, $-\infty \le n \le \infty$.

However, DTFT of $x[n]$, i.e., $\sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n} e^{-jwn}$, is not absolutely summable.

When considering the sum of a finite number of terms: $\sum_{n=-M}^{M} \frac{\sin w_c n}{\pi n} e^{-jwn}$, we obtain the following results.



As M increases, the oscillatory behavior at $w = w_c$ is more rapid, but the size of the ripples does not decrease. When $M \to \infty$, the maximum amplitude of the oscillations does not approach zero. This is called the Gibbs phenomenon.

- **Properties of DTFT:**

-- Linearity

If $x[n] \leftrightarrow X(e^{j\omega})$ and $y[n] \leftrightarrow Y(e^{j\omega})$

then $ax[n]+by[n] \leftrightarrow aX(e^{j\omega})+bY(e^{j\omega})$

-- Time Shift

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[n-n_d] \leftrightarrow X(e^{j\omega})e^{-j\omega n_d}$

-- Frequency Modulation

If $x[n] \leftrightarrow X(e^{j\omega})$

then $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$

--Time Reversal

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[-n] \leftrightarrow X(e^{-j\omega})$

-- Complex Conjugation

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x^*[n] \leftrightarrow X^*(e^{-j\omega})$

-- Differentiation in frequency

If $x[n] \leftrightarrow X(e^{jw})$, then $nx[n] \leftrightarrow j\dfrac{dX(e^{jw})}{dw}$,

because $$\frac{dX(e^{jw})}{dw} = \frac{d}{dw}\left(\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}\right) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-jwn} = -jF\{nx[n]\}$$

-- Convolution

If $x[n] \leftrightarrow X(e^{jw})$, and $y[n] \leftrightarrow Y(e^{jw})$, then $x[n]*y[n] \leftrightarrow X(e^{jw})Y(e^{jw})$,

because

$$F\{x[n]*y[n]\} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]y[n-k]\right)e^{-jwn} = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y[m]e^{-jw(m+k)}$$

$$= \sum_{m=-\infty}^{\infty} X(e^{jw})y[m]e^{-jwm} = X(e^{jw})Y(e^{jw})$$

-- Multiplication

If $x[n] \leftrightarrow X(e^{jw})$, and $y[n] \leftrightarrow Y(e^{jw})$,

then $x[n]y[n] \leftrightarrow \dfrac{1}{2\pi}X(e^{jw})*Y(e^{jw}) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(w-\theta)})d\theta$,

because

$$\frac{1}{2\pi}X(e^{jw})*Y(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(w-\theta)})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\left(\sum_{n=-\infty}^{\infty} y[n]e^{-j(w-\theta)n}\right)d\theta$$

$$= \sum_{n=-\infty}^{\infty} y[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta n}d\theta\right)e^{-jwn} = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-jwn} = F\{x[n]y[n]\}$$

- **Symmetric Properties of DTFT**

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{J}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{J}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

1. $$\sum_{n=-\infty}^{\infty} x^*[n]e^{-jwn} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{jwn}\right)^* = X^*(e^{-jw})$$

2. $$\sum_{n=-\infty}^{\infty} x^*[-n]e^{-jwn} = \left(\sum_{m=-\infty}^{\infty} x[m]e^{-jwm}\right)^* = X^*(e^{jw})$$

3. $$\sum_{n=-\infty}^{\infty} \mathrm{Re}\{x[n]\}e^{-jwn} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\{x[n]+x^*[n]\}e^{-jwn} = \frac{1}{2}\{X(e^{jw})+X^*(e^{-jw})\} = X_e(e^{jw})$$

4. $$\sum_{n=-\infty}^{\infty} j\,\mathrm{Im}\{x[n]\}e^{-jwn} = \sum_{n=-\infty}^{\infty} j\frac{1}{2j}\{x[n]-x^*[n]\}e^{-jwn} = \frac{1}{2}\{X(e^{jw})-X^*(e^{-jw})\} = X_o(e^{jw})$$

5. $$\sum_{n=-\infty}^{\infty} x_e[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\{x[n]+x^*[-n]\}e^{-jwn} = \frac{1}{2}\{X(e^{jw})+X^*(e^{jw})\} = X_R(e^{jw})$$

6. $$\sum_{n=-\infty}^{\infty} x_o[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\{x[n]-x^*[-n]\}e^{-jwn} = \frac{1}{2}\{X(e^{jw})-X^*(e^{jw})\} = jX_I(e^{jw})$$

For real $x[n]$,

7. $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} x^*[n]e^{-jwn} = X^*(e^{-jw})$

8. $X_R(e^{jw}) = \frac{1}{2}\{X(e^{jw}) + X^*(e^{jw})\}$

$$X_R(e^{-jw}) = \frac{1}{2}\{X(e^{-jw}) + X^*(e^{-jw})\} = \frac{1}{2}\{X^*(e^{jw}) + X(e^{jw})\} = X_R(e^{jw}),$$

where $X(e^{-jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{jwn} = \left(\sum_{n=-\infty}^{\infty} x^*[n]e^{-jwn}\right)^* = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}\right)^* = X^*(e^{jw})$

This tells us that "the real part of $X(e^{jw})$ is even".

9. $X_I(e^{jw}) = \frac{1}{2j}\{X(e^{jw}) - X^*(e^{jw})\}$

$$X_I(e^{-jw}) = \frac{1}{2j}\{X(e^{-jw}) - X^*(e^{-jw})\} = \frac{1}{2j}\{X^*(e^{jw}) - X(e^{jw})\} = -X_I(e^{jw}),$$

This tells us that "the imaginary part of $X(e^{jw})$ is even".

10. $|X(e^{jw})|^2 = X(e^{jw})X^*(e^{jw}) = X^*(e^{-jw})X(e^{-jw}) = |X(e^{-jw})|^2$

This tells us that "the magnitude of $X(e^{jw})$ is even".

11. $X(e^{jw}) = |X(e^{jw})|\, e^{\angle X(e^{jw})} = |X(e^{-jw})|\, e^{\angle X(e^{jw})}$

$$X(e^{-jw}) = |X(e^{-jw})|\, e^{\angle X(e^{-jw})} \Rightarrow X^*(e^{-jw}) = |X(e^{-jw})|\, e^{-\angle X(e^{-jw})}$$

Since $X(e^{jw}) = X^*(e^{-jw})$, $\angle X(e^{jw}) = -\angle X(e^{-jw})$

This tells us that "the phase of $X(e^{jw})$ is odd".

- **A Note on Sequence Decomposition**
  - ■ Any sequence can be represented by $x[n] = x_e[n] + x_o[n]$, where

$x_e[n] = \{x[n] + x^*[-n]\} / 2$

$x_o[n] = \{x[n] - x^*[-n]\} / 2$

$x_e[n]$ is called "conjugate-symmetric sequence", because $x_e[n] = x^*_e[-n]$, and $x_o[n]$ is called "conjugate-antisymmetric sequence", because $x_o[n] = -x^*_o[-n]$.

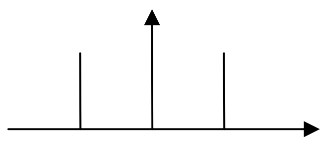
Example: $x[n] = 6n + j(n + 8)$,

$x_e[n] = (6n + jn + 8j - 6n + jn - 8j)/2 = jn$,

$x_o[n] = (6n + jn + 8j + 6n - jn + 8j)/2 = 6n + 8j$

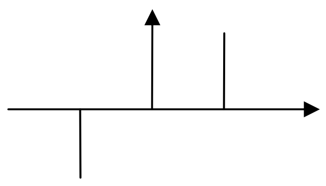
If $x_e[n]$ is real, $x_e[n]$ is called "even sequence", because $x_e[n] = x_e[-n]$.

Example:



If $x_o[n]$ is real, $x_o[n]$ is called "odd sequence", because $x_o[n] = -x_o[-n]$.

Example:



- Any sequence can be represented by $x[n] = x_R[n] + j\, x_I[n]$, where

  $x_R[n] = \{x[n] + x^*[n]\} / 2$

  $x_I[n] = \{x[n] - x^*[n]\} / 2j$

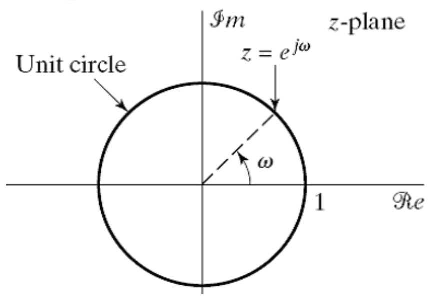
  Example: $x[n] = \cos(wn) + j\sin(wn)$.

2.4 Z Transform

- **Overview of the related transformations**

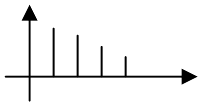
For a continuous signal $x(t)$	For a discrete-time signal $x[n]$
Fourier series (if $x(t)$ is periodic) $X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\Omega_0 kt}dt,\ \ \Omega_0 = \frac{2\pi}{T_0}$ $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_0 kt}$	**Discrete Fourier series** (if $x[n]$ is periodic) $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$, $x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j2\pi kn/N}$
Fourier transform (FT) $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt,$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$	**Discrete-Time Fourier transform (DTFT)** $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn},$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})e^{jwn}dw$ **Discrete Fourier transform** $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$, $x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j2\pi kn/N}$
Laplace transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt,$ $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$ where $s = \sigma + j\Omega$, which is a complex. If we let $s = j\Omega$, then Laplace transform is equivalent to FT, but note that s is not limited to $j\Omega$.	**z-transform** $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$ $x[n] = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$, where z is a complex, and where C is a counterclockwise closed path encircling the origin and entirely in the region of convergence (ROC). If we let $z = e^{jw}$, then z transform is equivalent to DTFT, but note that z is not limited to e^{jw}

- Why z-transform?
  - A generalization of DTFT. Some sequences that do not converge for DTFT have valid z-transforms.
  - Better notation (compared to FT) in analytical problems (complex variable theory)
  - Solving difference equation (to be discussed in Chapter 4).



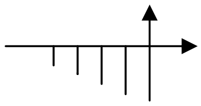
Example:

1. $x[n] = a^n\, u[n]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}}, \text{ if } |az^{-1}| < 1 \quad (|z| > |a|)$$

2. $x[n] = -a^n\, u[-n-1]$



$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}, \text{ if } |a^{-1}z| < 1 \quad (|z| < |a|)$$

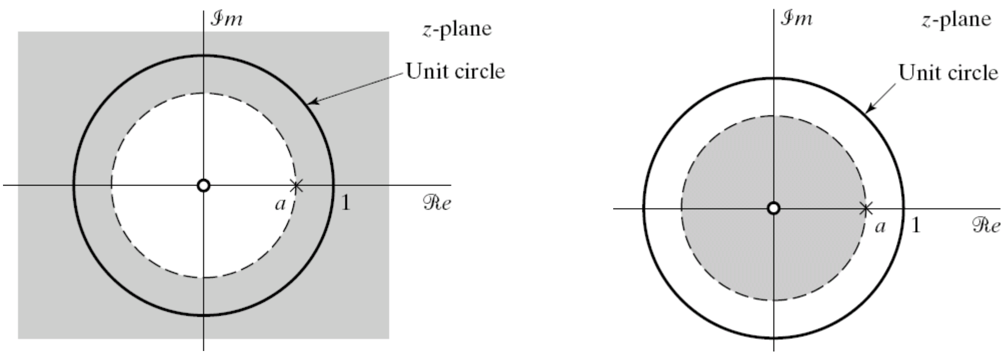
Note that in the above example, if $a > 1$, $X(e^{jw})$ does not exist, because $\sum_{n=0}^{\infty} a^n e^{-jwn}$ is not absolutely summable. However, $X(z)$ exist, if $|z| > |a|$.

- Region of convergence (ROC)

  For any given $X(z)$, the set of values of z for which the z-transform converges is called the region of convergence (ROC)

  The ROCs of the above examples are depicted as follows.

(i) (ii)



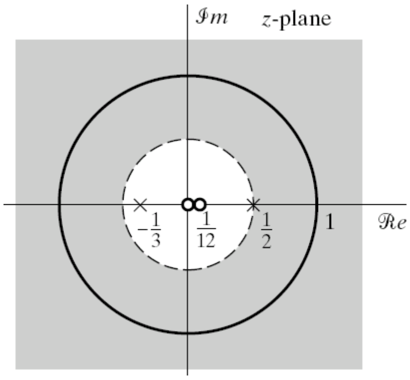
Note: if ROC contains the unit circle ($|z| = 1$), then DTFT exists.

Example: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \right\} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}, \text{ if } \left|\frac{1}{2}z^{-1}\right| < 1, \text{and} \left|\frac{1}{3}z^{-1}\right| < 1$$

$$\Rightarrow |z| > \frac{1}{2}, \text{and } |z| > \frac{1}{3} \quad \Rightarrow |z| > \frac{1}{2}$$



Zeros: $X(z) = 0 \Rightarrow z = 0,\ 1/12$

Poles: $X(z) = \infty \Rightarrow z = 1/2,\ -1/3$

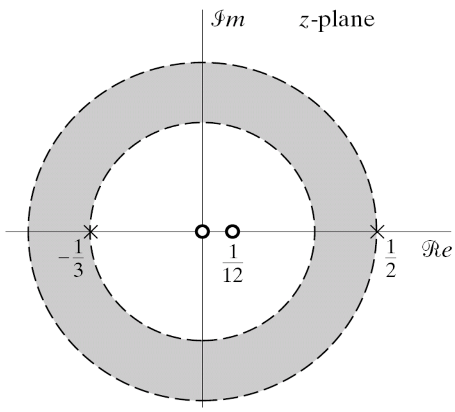
Note: the number of pole equals the number of zeros.

Example: $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} \quad (\text{if } |z| > \frac{1}{3})$$

$$+\frac{1}{1-\frac{1}{2}z^{-1}} \quad (\text{if } |z| < \frac{1}{2})$$

$$= \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})},\ |z| < \frac{1}{2}, \text{and } |z| > \frac{1}{3}$$

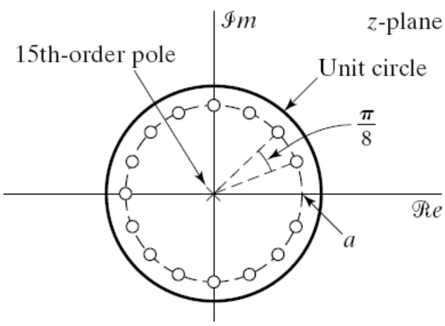


Example: $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}}\frac{z^N - a^N}{z-a}$$

ploes: $z = a$, z = 0, 0, …,0 (a total of $N-1$ ‘0’)

zeros: $z^N = a^N$, $\Rightarrow z = ae^{j2\pi k/N}$, $k = 0, 1, 2, \ldots, N-1$.

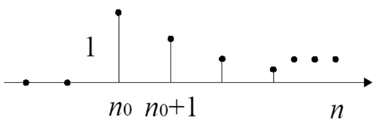


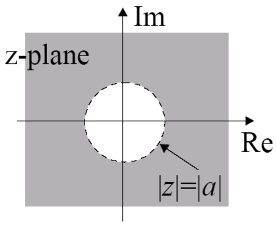
($N = 16$)

Properties of the z-transform

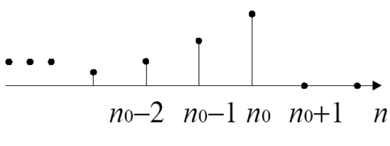
(1) The ROC is a ring or disk in the z-plane centered at the origin.
(2) The F.T. of $x[n]$ converges absolutely iff its ROC includes the unit circle.
(3) The ROC cannot contain any poles.
(4) If $x[n]$ is *finite-duration*, then the ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.
(5) If $x[n]$ is *right-sided*, the ROC, if exists, must be of the form $|z|>r_{\max}$ except possibly $z=\infty$, where $r_{\max}$ is the magnitude of the largest pole.
(6) If $x[n]$ is *left-sided*, the ROC, if exists, must be of the form $|z|<r_{\min}$ except possibly $z=0$, where $r_{\min}$ is the magnitude of the smallest pole.
(7) If $x[n]$ is *two-sided*, the ROC must be of the form $r_1<|z|<r_2$ if exists, where r_1 and r_2 are the magnitudes of the interior and exterior poles.
(8) The ROC must be a connected region.

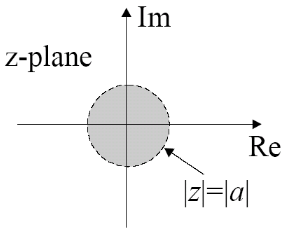
* Right-sided sequence



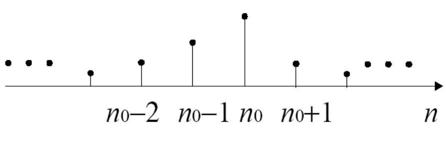


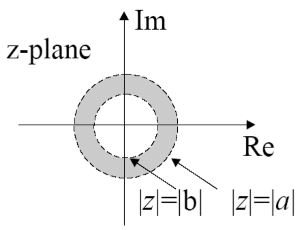
* Left-sided sequence





* Two-sided sequence





- Some frequently-used function of z-transform: (Table 3.1)

Sequence		transform	ROC
$\delta[n]$	$\leftrightarrow$	1	all z
$\delta[n\text{ -}1]$	$\leftrightarrow$	z^{-m}	all z except 0 (if $m > 0$) or ∞ (if $m < 0$)
$a^n u[n]$	$\leftrightarrow$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\leftrightarrow$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(\cos w_0 n)\, u[n]$	$\leftrightarrow$	$\frac{1-(\cos w_0 n)z^{-1}}{1-(2\cos w_0 n)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$

$$\sum_{n=-\infty}^{\infty}(\cos w_0 n)u[n]z^{-n} = \sum_{n=0}^{\infty}(\cos w_0 n)z^{-n} = \sum_{n=0}^{\infty}\frac{1}{2}\left(e^{jw_0 n}+e^{-jw_0 n}\right)z^{-n}$$
$$=\frac{1}{2}\frac{1}{1-e^{jw_0}z^{-1}}+\frac{1}{2}\frac{1}{1-e^{-jw_0}z^{-1}} \quad , \quad |e^{jw_0 n}z^{-1}|<1 \quad \Rightarrow |z|>1$$
$$=\frac{1}{2}\;\frac{1-e^{-jw_0}z^{-1}+1-e^{jw_0}z^{-1}}{1+(e^{jw_0}+e^{-jw_0})z^{-1}+z^{-2}}$$
$$=\frac{\frac{1}{2}\left[2-(e^{jw_0 n}+e^{-jw_0 n})z^{-1}\right]}{1-(2\cos w_0 n)z^{-1}+z^{-2}}=\frac{1-(\cos w_0 n)z^{-1}}{1-(2\cos w_0 n)z^{-1}+z^{-2}}$$

- Useful formulae (Table 3.2)

	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$
1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$
2	$x[n-n_0]$	$z^{-n_0}X(z)$
3	$z_0^n x[n]$	$X(z/z_0)$
4	$nx[n]$	$-z\frac{dX(z)}{dz}$
5	$x^*[n]$	$X^*(z^*)$
6	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$
7	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$
8	$x^*[-n]$	$X^*(1/z^*)$
9	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$
10	**Initial-value theorem:** $x[n]=0,\quad n<0 \qquad \lim_{z\to\infty}X(z)=x[0]$	

4. $n\,x[n] \quad \leftrightarrow \quad -z\dfrac{dX(z)}{dz}$

because $\dfrac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z^{-1}\mathcal{Z}\{nx[n]\}$

10. Initial value theorem: If $x[n] = 0$, $n < 0$, $\lim_{z\to\infty} X(z) = x[0]$

because $\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \lim_{z\to\infty}\sum_{n=1}^{\infty} x[n]z^{-n} = x[0] + 0 = x[0]$

• Inverse z-transform

Example 1: If $X(z) = z^2(1-0.5z^{-1})\,(1+z^{-1})\,(1-z^{-1})$, find $x[n]$.

Because $X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$,

$x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1]$.

Example 2: If $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-1.5z^{-1}+0.5z^{-2}}$, find $x[n]$.

Because $X(z) = 2 - \dfrac{9}{1-0.5z^{-1}} + \dfrac{8}{1-z^{-1}}$,

$x[n] = 2\delta[n] - 9(0.5)^n\,u[n] + 8u[n]$

ROC: $|z| > 0.5$ and $|z| > 1 \Rightarrow |z| > 1$

----

long division

$$
\begin{array}{r}
2 \\
0.5z^{-2} - 1.5z^{-1} + 1 \overline{\big) z^{-2} + 2z^{-1} + 1} \\
z^{-2} - 3z^{-1} + 2 \\
\hline
5z^{-1} - 1
\end{array}
$$

Hence,

$$
\begin{aligned}
X(z) &= 2 + \frac{5z^{-1}-1}{0.5z^{-2}-1.5z^{-1}+1} \\
&= 2 + \frac{5z^{-1}-1}{(1-0.5z^{-1})(1-z^{-1})} \\
&= 2 + \frac{A_1}{1-0.5z^{-1}} + \frac{A_2}{1-z^{-1}}
\end{aligned}
$$

since $A_1(1-z^{-1}) + A_2(1-0.5z^{-1}) = 5z^{-1} - 1$,

let $z^{-1} = 1 \quad \Rightarrow \quad A_2 = 8$

let $z^{-1} = 2 \quad \Rightarrow \quad A_1 = -9$

Example 3: If $X(z) = \ln(1 + az^{-1})$, $|z| > |a|$, find $x[n]$.

<Method 1> By $\boxed{n\,x[n] \quad \leftrightarrow \quad -z\dfrac{dX(z)}{dz}}$

Since $\dfrac{dX(z)}{dz} = \dfrac{-az^{-2}}{1+az^{-1}}$, $\quad -z\dfrac{dX(z)}{dz} = \dfrac{az^{-1}}{1+az^{-1}}$.

And because $\dfrac{1}{1+az^{-1}}, |z|>|a| \quad \leftrightarrow \quad (-a)^n u[n]$,

$$\frac{az^{-1}}{1+az^{-1}} \quad \leftrightarrow \quad a(-a)^{n-1}u[n-1] = (-1)^{n-1}a^n u[n-1].$$

Hence, $x[n] = \dfrac{1}{n}(-1)^{n-1}a^n u[n-1]$.

<Method 2> By power series expansion: $\log(1+x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$

$$X(z) = \log(1+az^{-1}) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^n z^{-n}}{n} = \sum_{n=-\infty}^{\infty}\frac{(-1)^{n-1}a^n}{n}u[n-1]z^{-n}$$

$$\Rightarrow x[n] = \frac{1}{n}(-1)^{n-1}a^n u[n-1]$$

Example 4: If $X(z) = \ln(1 - 4z)$, $|z| < \dfrac{1}{4}$, find $x[n]$.

By power series expansion: $\log(1+x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$

$$X(z) = \log(1-4z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-4z)^n}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Since ROC is $|z| < \dfrac{1}{4}$, it follows that $x[n]$ is of the form: $x[n] = -a^n u[-n-1]$.

That is $X(z) = -\displaystyle\sum_{n=-\infty}^{-1}a^n z^{-n} = -\sum_{n=1}^{\infty}a^{-n}z^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

By comparing (1) and (2), we have $-a^{-n} = \dfrac{(-1)^{n-1}(-4)^n}{n}$

Then, $-a^n = \dfrac{(-1)^{-n-1}(-4)^{-n}}{-n} = \dfrac{4^{-n}}{n}$

Hence, $x[n] = -a^n u[-n-1] = \dfrac{4^{-n}}{n}u[-n-1]$

Chapter 3 Digital Signals (II):

Discrete in Time, Discrete in Frequency

3.1 Discrete Fourier Series

- If discrete signal $\tilde{x}[n]$ is **periodic** with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer r, the discrete Fourier series (DFS) of $\tilde{x}[n]$ is defined by

  $$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}, \ -\infty \le k \le \infty, \text{ and}$$

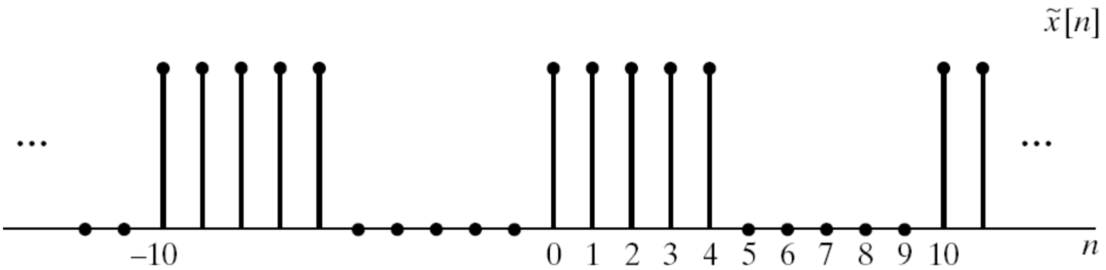
  $$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn}, \ -\infty \le n \le \infty.$$

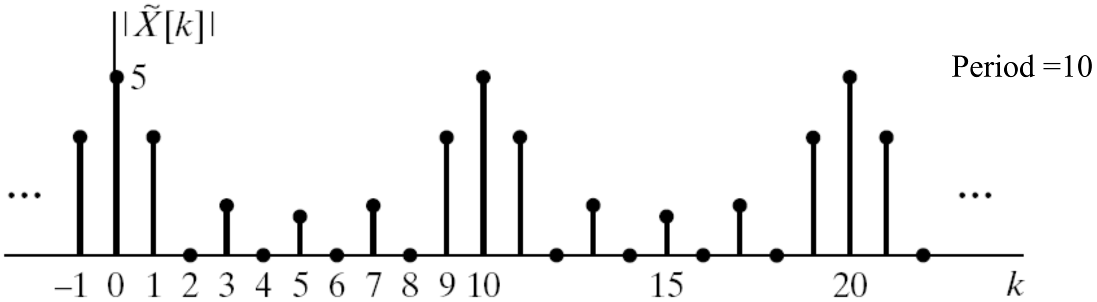
  Note that $\tilde{X}[k]$ is also periodic.

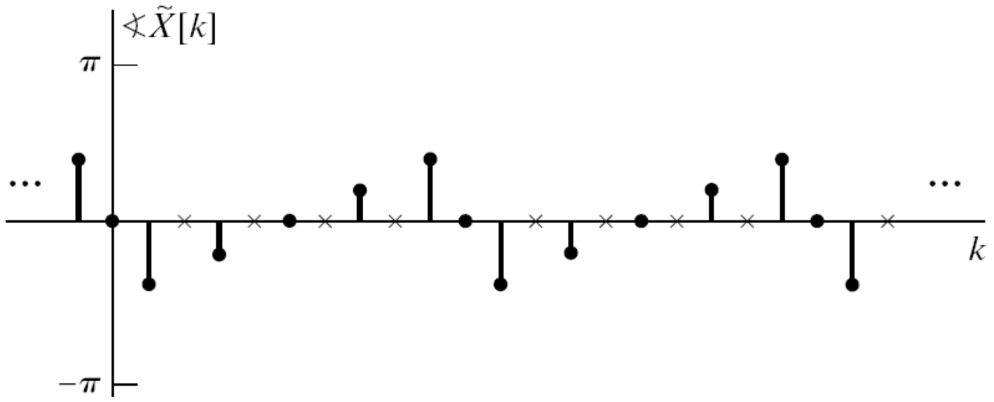
  Example:

  $\tilde{x}[n]$ is the sequence shown below, whose period is $N = 10$.

  $$\tilde{X}[k] = \sum_{n=0}^{9} \tilde{x}[n] e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^{9} e^{-j\frac{2\pi}{10}kn} = \frac{\sin(\pi k/2)}{\sin(\pi k/10)} e^{-j\frac{2\pi}{10}kn}$$







- DFS has the property of duality

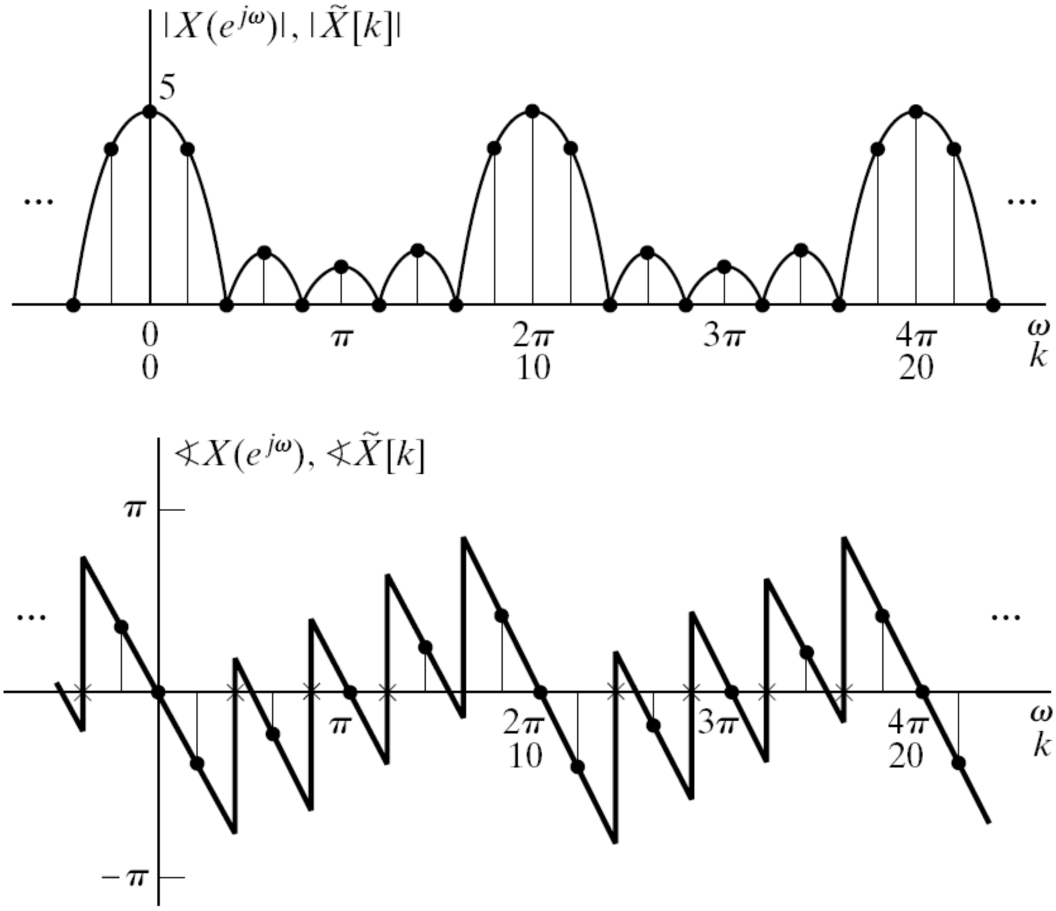
  $$\tilde{x}[n] \quad \xleftrightarrow{\text{DFS}} \quad \tilde{X}[k]$$

  $$\tilde{X}[n] \quad \xleftrightarrow{\text{DFS}} \quad N\ \tilde{x}[-k]$$

  Note that the discrete-time Fourier transform (DTFT) has no property of duality.

- $\tilde{X}[k]$ can be computed using the DTFT of $\tilde{x}[n]$ within one period:

  $$\tilde{X}[k] = \hat{X}(e^{jw})\Big|_{w=2\pi k/N} \quad \text{, where} \quad \hat{X}(e^{jw}) = F\left\{\tilde{x}[n]\,\big|_{\text{one period}}\right\} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jwn}$$



- Other properties of DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n]$, $\tilde{x}_2[n]$	$\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$ ($W_N^{km} = e^{-j2\pi km/N}$)
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

Proof for property 5:

$$DFS\{\tilde{x}[n-M]\} = \sum_{n=0}^{N-1} \tilde{x}[n-M]e^{-j\frac{2\pi}{N}kn} = \sum_{n=-M}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}k(m+M)} = e^{-j\frac{2\pi}{N}kM} \sum_{n=-M}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km}$$

$$\begin{aligned}
\sum_{m=-M}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} &= \sum_{m=-M}^{-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} + \sum_{m=0}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} \\
&= \sum_{m=-M}^{-1} \tilde{x}[m+N]e^{-j\frac{2\pi}{N}km} + \sum_{m=0}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} \\
&= \sum_{r=N-M}^{N-1} \tilde{x}[r]e^{-j\frac{2\pi}{N}k(r-N)} + \sum_{m=0}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} \\
&= \sum_{r=N-M}^{N-1} \tilde{x}[r]e^{-j\frac{2\pi}{N}kr} + \sum_{m=0}^{N-M-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} \\
&= \sum_{m=0}^{N-1} \tilde{x}[m]e^{-j\frac{2\pi}{N}km} = \tilde{X}[k].
\end{aligned}$$

Thus, $DFS\{\tilde{x}[n-M]\} = e^{-j\frac{2\pi}{N}kM}\tilde{X}[k]$

Proof for property 7:

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]\right)e^{-j\frac{2\pi}{N}kn} = \sum_{m=0}^{N-1} \tilde{x}_1[m]\left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m]e^{-j\frac{2\pi}{N}kn}\right) \\
&= \sum_{m=0}^{N-1} \tilde{x}_1[m]\left(e^{-j\frac{2\pi}{N}km}\tilde{X}_2[k]\right) = \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]e^{-j\frac{2\pi}{N}km}\right)\left(\tilde{X}_2[k]\right) = \tilde{X}_1[k]\tilde{X}_2[k]
\end{aligned}$$

3.2 Discrete Fourier Transform

- **Definition**

If a discrete signal $x[n]$ is aperiodic, $0 \le n \le N-1$, we can define a periodic signal $\tilde{x}[n]$ by making the periodic replica of $x[n]$:

$$\tilde{x}[n] = \sum_{n=-\infty}^{\infty} x[n+rN] = x\big[((n))_N\big].$$

Then, $\tilde{x}[n]$ can be represented by DFS expansion:

$$\tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn}, \text{ and } \tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}.$$

Let $X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$. This can also be viewed as $\tilde{X}[k] = X\big[((k))_N\big]$.

The coefficient $X[k]$ is the so-called discrete Fourier transform (DFT) of $x[n]$.

Thus, specific computation of DFT consists of:

$$DFT\{x[n]\} = X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, 0 \le k \le N-1, \text{ and}$$

$$DFT^{-1}(X[k]) = x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}, 0 \le n \le N-1.$$

$$\begin{tikzcd}
\text{aperiodic signal } x[n] \arrow[rr, "\text{making periodic replica}"] \arrow[dr, "\text{DTFT}"] \arrow[dd, "\text{DFT}"'] & & \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] = x\big[((n))_N\big] \arrow[dd, "\text{DFS}"] \\
& X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \arrow[dr, "\text{Let } \tilde{X}[k] = X(e^{jw})\big|_{w=2\pi k/N}"] & \\
X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, 0 \le k \le N-1 & & \begin{array}{c} \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn} \\ -\infty \le k \le \infty \end{array} \arrow[ll, "\text{taking one period } (0 \le k \le N-1)"]
\end{tikzcd}$$

- **DFT has a property of duality**

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$X[k] \xleftrightarrow{\text{DFT}} N\,x[((-k))_N]$$

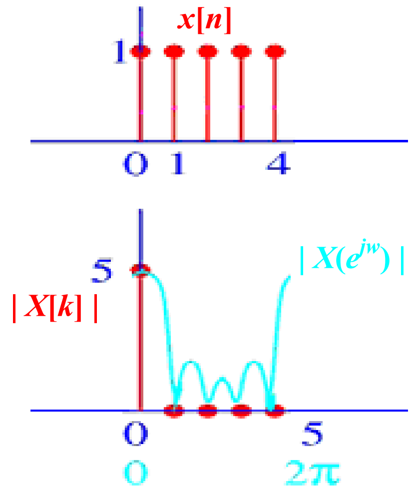
- **DFT of a rectangular window**

If $x[n]=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$, then

$$X(e^{jw})=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn}=\sum_{n=0}^{N-1}e^{-jwn}=\frac{1-e^{-jwN}}{1-e^{-jw}}=\frac{\sin(wN/2)}{\sin(w/2)}e^{-jw(N-1)/2}$$

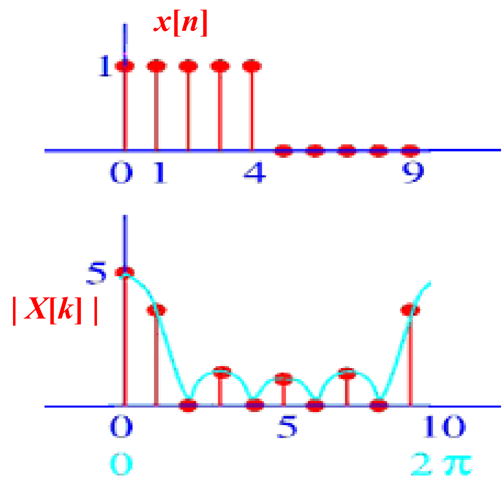
$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}e^{-j\frac{2\pi}{N}kn}=\begin{cases}N, & k=0\\ \dfrac{1-e^{-j\frac{2\pi}{N}kN}}{1-e^{-j\frac{2\pi}{N}k}}=0, & k\neq 0\end{cases}$$

Example: $N=5$



Zero padding: If we perform M-point DFT by appending $M-N$ zeros with $x[n]$, then

$$X[k]=\sum_{n=0}^{M-1}x[n]e^{-j\frac{2\pi}{M}kn}=\sum_{n=0}^{N-1}e^{-j\frac{2\pi}{M}kn}=\begin{cases}N, & k=0\\ \dfrac{1-e^{-j\frac{2\pi}{M}kN}}{1-e^{-j\frac{2\pi}{M}k}}=\dfrac{\sin(\pi kN/M)}{\sin(\pi k/M)}e^{-j\pi k(N-1)/M}, & k\neq 0\end{cases}$$



$\Rightarrow$ Zero padding of analyzed sequence results in "approximating" its DTFT better.

- **DFT has a property of symmetry:**

  • If $x[n]$ is real, then $X[k] = X^*[N-k]$, because

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}x^*[n]e^{-j\frac{2\pi}{N}kn}=\left(\sum_{n=0}^{N-1}x[n]e^{j\frac{2\pi}{N}kn}\right)^*=\left(\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}(N-k)n}\right)^*.$$
$$=X^*[N-k]$$

  • If $x[n]$ is real, and $x[n] = x[N-n]$, then $X[k]$ is also real, and $X[k] = X[N-k]$, because

$$X^*[k]=\left(\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}\right)^*=\sum_{n=0}^{N-1}x^*[n]e^{j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}x[n]e^{j\frac{2\pi}{N}kn}=\sum_{n=N}^{1}x[N-n]e^{j\frac{2\pi}{N}kn}$$
$$=\sum_{n=N}^{1}x[N-n]e^{-j\frac{2\pi}{N}k(N-n)}=\sum_{m=0}^{N-1}x[m]e^{-j\frac{2\pi}{N}km}=X[k]$$

Example 1:
$\mathbf{x}$ = {1, 2, 3, 4, 5, 6, 7, 8}
$DFT(\mathbf{x})$ = {36, -4+9.66j, -4+4j, -4+1.66j, -4, -4–1.66j, -4–4j, -4–9.66j}

$X[3] = X^*[8\text{-}3]$

Example 2:
$\mathbf{x}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9}
$DFT(\mathbf{x})$ = {45, -4.5+12.36j, -4.5+5.36j, -4.5+2.6j, -4.5+0.79j, -4.5–0.79j, -4.5–2.6j, -4.5–5.36j, -4.5–12.36j}

$X[4] = X^*[9\text{-}4]$

Example 3:
$\mathbf{x}$ = {33, 1, 2, 3, 4, 4, 3, 2, 1}
$DFT(\mathbf{x})$ = {53, 24.71, 32.72, 32, 32.57, 32.57, 32, 32.72, 24.71}
$DFT^{-1}(\mathbf{x})$ = {5.89, 2.75, 3.64, 3.56, 3.62, 3.62, 3.56, 3.64, 2.75}
We can see that $\mathbf{x}$, $DFT(\mathbf{x})$, and $DFT^{-1}(\mathbf{x})$ are all real and symmetric.

- **Matrix representation of DFT**

Let $\mathbf{x}_N=\begin{bmatrix}x[0]\\x[1]\\\vdots\\x[N-1]\end{bmatrix}$, $\mathbf{X}_N=\begin{bmatrix}X[0]\\X[1]\\\vdots\\X[N-1]\end{bmatrix}$, and

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$, where $W_N^{km} = e^{-j2\pi km/N}$

Then,

$$\begin{aligned} \mathbf{X}_N &= \mathbf{W}_N \mathbf{x}_N && N\text{ - point DFT} \\ \mathbf{x}_N &= \mathbf{W}_N^{-1} \mathbf{X}_N && N\text{ - point IDFT} \\ &= \frac{1}{N} \mathbf{W}_N^{*} \mathbf{X}_N \end{aligned}$$

Because the matrix (transformation) $\mathbf{W}_N$ has a specific structure and because W_N^k has particular values (for some k and n), we can reduce the number of arithmetic operations for computing this transform.

Example:

$x[n] = [0 \ 1 \ 2 \ 3]$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Thus, the DFT of $x[n]$ is $\mathbf{X}_4 = \mathbf{W}_4 \mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

We can see that only additions are needed to compute this specific transform.

- **Circular shift**

Recall that: if $x[n] \xleftrightarrow{\text{DTFT}} X(e^{jw})$

then $x[n-m] \xleftrightarrow{\text{DTFT}} e^{-jwm} X(e^{jw})$.

if $\tilde{x}[n] \xleftrightarrow{\text{DFS}} \tilde{X}[k]$

then $\tilde{x}[n-m] \xleftrightarrow{\text{DFS}} e^{-j\frac{2\pi}{N}km}\tilde{X}[k]$.

However, if $x[n] \xleftrightarrow{\text{DFT}} X[k]$

then $x[n-m] \underset{\cancel{\xleftarrow{\text{IDFT}}}}{\xrightarrow{\text{DFT}}} e^{-j\frac{2\pi}{N}km}X[k]$

The inverse DFT of $e^{-j\frac{2\pi}{N}km}X[k]$ is $x[((n-m))_N]$, $0 \le n \le N-1$, rather than $x[n-m]$, i.e.,

$$x[((n-m))_N],\ 0 \le n \le N-1 \xleftrightarrow{\text{DFT}} e^{-j\frac{2\pi}{N}km}X[k]$$

Proof:

Let $X_1[k] = e^{-j\frac{2\pi}{N}km}X[k]$, and the inverse DFT of $X_1[k]$ be $x_1[n]$.

We can define $\tilde{X}_1[k] = X_1[((k))_N]$ by making replica of $X_1[k]$. Then,

$$\tilde{X}_1[k] = X_1[((k))_N] = e^{-j\frac{2\pi}{N}((k))_N m}X[((k))_N] = e^{-j\frac{2\pi}{N}km}X[((k))_N]$$
$$= e^{-j\frac{2\pi}{N}km}\tilde{X}[k]$$

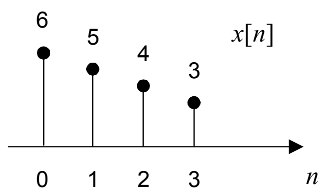
By inversing the DFS of $X_1[k]$, we have

$$\tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N].$$

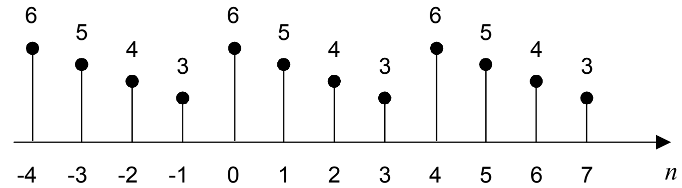
Since $x_1[n]$ belongs to one period of $\tilde{x}_1[n]$, we can conclude

$$x_1[n] = x[((n-m))_N],\ 0 \le n \le N-1.$$

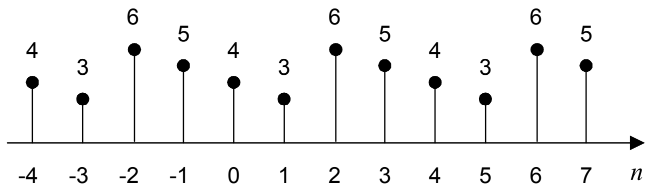
Example: Given $x[n]$ as the figure, sketch $x_1[n]$ whose 4-point DFT is $e^{-j\frac{2\pi}{4}2k}X[k]$.



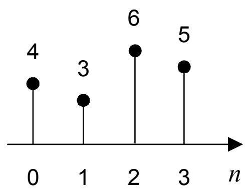
step 1: $\tilde{x}[n] = x[((n))_4]$



step 2: shifting by 2



step 3: taking one period



- **Circular convolution**

Linear convolution

$$x_1[n] * x_2[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \xleftrightarrow{\text{DTFT}} X_1(e^{jw}) \cdot X_2(e^{jw})$$

$$x_1[n] \cdot x_2[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} X_1(e^{jw}) * X_2(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta .$$

Periodic convolution

$$\tilde{x}_1[n] ⊛ \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \xleftrightarrow{\text{DFS}} \tilde{X}_1[k] \cdot \tilde{X}_2[k]$$

$$\tilde{x}_1[n] \cdot \tilde{x}_2[n] \xleftrightarrow{\text{DFS}} \frac{1}{N} \tilde{X}_1[k] ⊛ \tilde{X}_2[k] = \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\tilde{X}_2[k-m] .$$

Circular convolution

$$x_1[n] Ⓝ x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N] \xleftrightarrow{\text{DFT}} X_1[k] \cdot X_2[k]$$

$$x_1[n] \cdot x_2[n] \xleftrightarrow{\text{DFT}} \frac{1}{N} X_1[k] Ⓝ X_2[k] = \frac{1}{N}\sum_{m=0}^{N-1} X_1[m]X_2[((k-m))_N] .$$

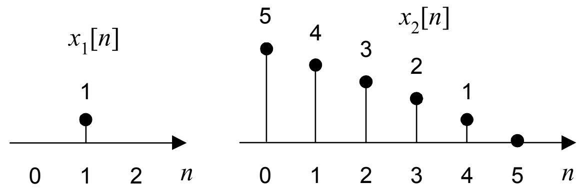
Note that in this case, the lengths of $x_1[n]$ and $x_2[n]$are both N points.

Proof:

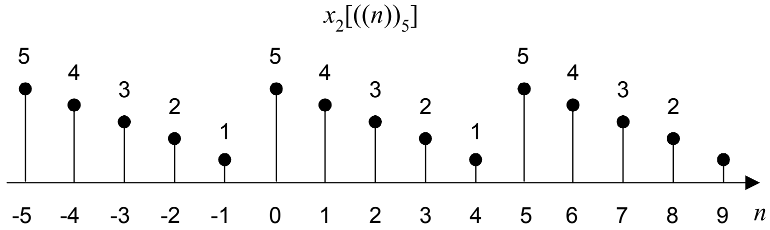
Let $X_3[k] = X_1[k] \cdot X_2[k]$, and $x_3[n]$ be the inverse DFT of $X_3[k]$.

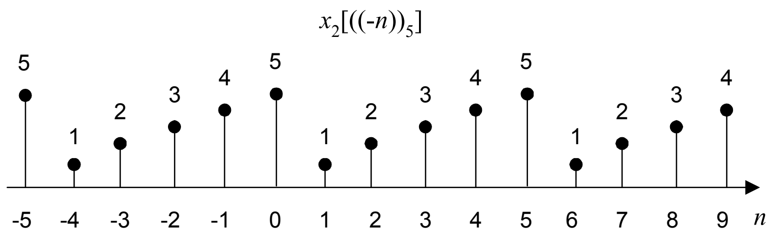
$$\begin{aligned} x_3[n] &= \tilde{x}_3[n],\ 0 \le n \le N-1 \\ &= \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N] \\ &= x_1[n] Ⓝ x_2[n] \end{aligned}$$

Example: Given $x_1[n]$ and $x_2[n]$ as the figures, sketch $x_3[n] = x_1[n] \circledS x_2[n]$.

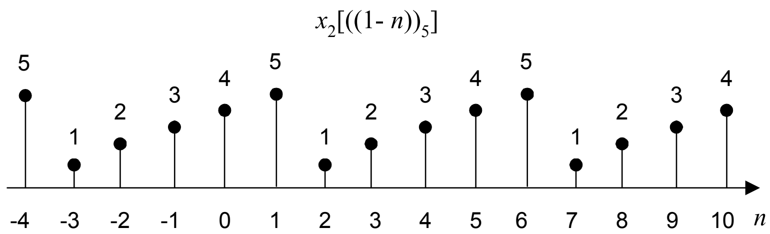


We will compute $x_3[n]$ using $x_3[n] = \sum_{m=0}^{4} x_1[m] x_2[((n-m))_5]$.

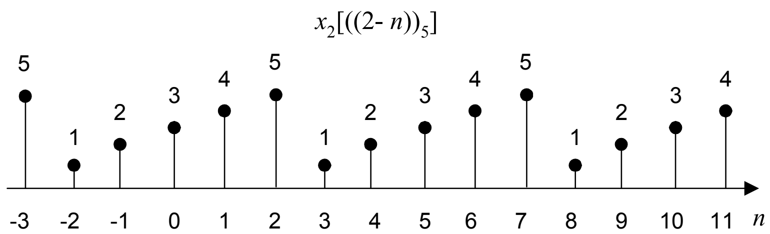




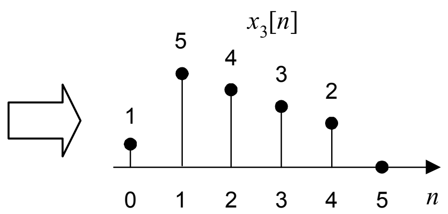
$\times\, x_1[n] = 1 = x_3[0]$



$\times\, x_1[n] = 5 = x_3[1]$



$\times\, x_1[n] = 4 = x_3[2]$



Note:

$x_3[n] = x_1[n]$ Ⓝ $x_2[n]$

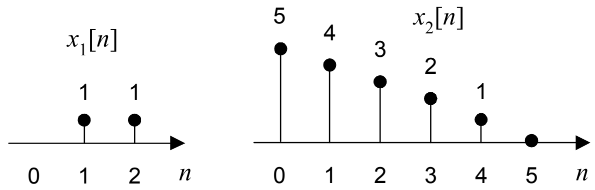
L points P points

N points

$N \geq \max(L, P)$

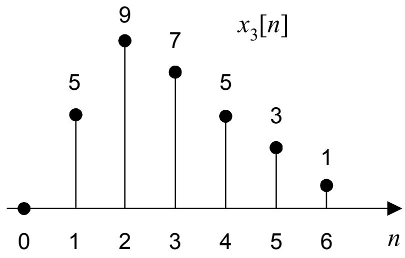
Example: Given $x_1[n]$ and $x_2[n]$ as the figures, sketch

(i) $x_3[n] = x_1[n] * x_2[n]$; (ii) $x_4[n] = x_1[n] ⑥ x_2[n]$; (iii) $x_5[n] = x_1[n] ⑦ x_2[n]$.



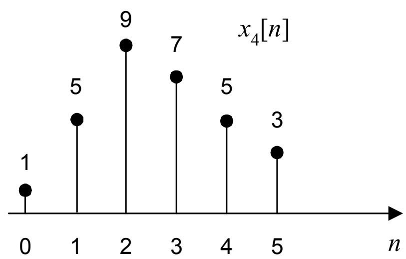
(i)

	n	0	1	2	3	4
n	$x_2[n]\rightarrow$ $x_1[n]\downarrow$	5	4	3	2	1
0	0	0	0	0	0	0
1	1	5	4	3	2	1
2	1	5	4	3	2	1



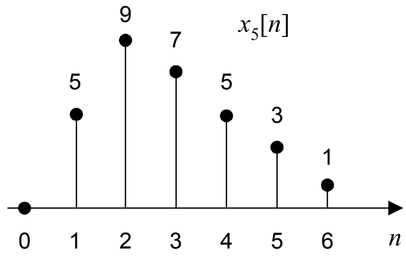
(ii)

	n	-3	-2	-1	0	1	2	3	4	5	6	7
n	$x_2[((n))_6]\rightarrow$ $x_1[n]\downarrow$	2	1	0	5	4	3	2	1	0	5	4
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	1	0	5	4	3	2	1	0	5	4
2	1	2	1	0	5	4	3	2	1	0	5	4



(iii)

	n	-3	-2	-1	0	1	2	3	4	5	6	7
n	$x_2[((n))_7]\rightarrow$ $x_1[n]\downarrow$	1	0	0	5	4	3	2	1	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	5	4	3	2	1	0	0	5
2	1	1	0	0	5	4	3	2	1	0	0	5



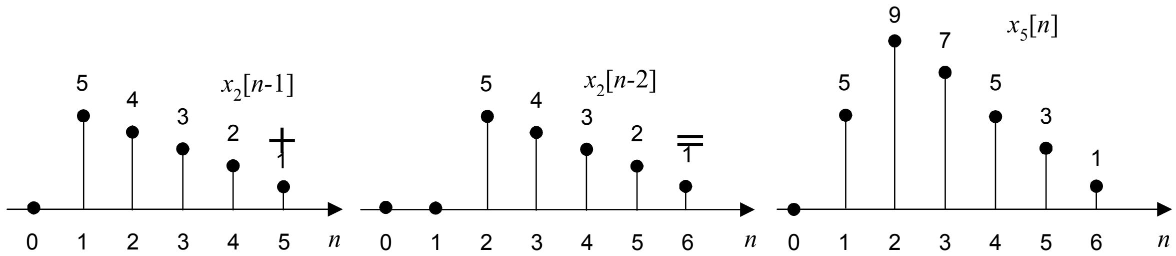
✱ Given two signals, $x_1[n]$ with length L, and $x_2[n]$ with length P, if

$x_3[n] = x_1[n] * x_2[n]$ equals $x_4[n] = x_1[n] \textcircled{N} x_2[n]$, then N must satisfy $N \geq L + P - 1$.

In the above example, $L = 3$, $P = 5$, $L + P - 1 = 7$. Thus,

$x_5[n] = x_1[n] \textcircled{7} x_2[n] = x_1[n] * x_2[n] = (\delta[n-1] + \delta[n-2]) * x_2[n] = x_2[n-1] + x_2[n-2]$.

We can sketch $x_5[n]$ in the following way:



Example:

If $\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 4 \end{bmatrix}$, and $\mathbf{q} = \mathbf{x} \textcircled{4} \mathbf{y} = \begin{bmatrix} 29 \\ 28 \\ 27 \\ 28 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} x[0] \cdot y[0] \\ x[1] \cdot y[1] \\ x[2] \cdot y[2] \\ x[3] \cdot y[3] \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \\ 6 \\ 4 \end{bmatrix}$, then

$$\mathbf{X} = DFT(\mathbf{x}) = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{Y} = DFT(\mathbf{y}) = \begin{bmatrix} Y[0] \\ Y[1] \\ Y[2] \\ Y[3] \end{bmatrix} = \begin{bmatrix} 16 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \text{ and}$$

$$\mathbf{Q} = DFT(\mathbf{q}) = \begin{bmatrix} Q[0] \\ Q[1] \\ Q[2] \\ Q[3] \end{bmatrix} = \begin{bmatrix} 112 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{B} = DFT(\mathbf{b}) = \begin{bmatrix} B[0] \\ B[1] \\ B[2] \\ B[3] \end{bmatrix} = \begin{bmatrix} 29 \\ 9 \\ 13 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} X[0] \cdot Y[0] \\ X[1] \cdot Y[1] \\ X[2] \cdot Y[2] \\ X[3] \cdot Y[3] \end{bmatrix} = \begin{bmatrix} 112 \\ 2 \\ 0 \\ 2 \end{bmatrix}$. We can see that $\mathbf{A} = \mathbf{Q}$.

Let $\mathbf{D} = \frac{1}{4} \mathbf{X} \textcircled{4} \mathbf{Y}$. We can see that $\mathbf{D} = \frac{1}{4} [116\ 36\ 52\ 36]' = \mathbf{B}$.

x **y**	3	1	2	1	3	1	2	1
5		5	10	5	15	5	10	5
4		4	8	4	12	4	8	4
3		3	6	3	9	3	6	3
4		4	8	4	12	4	8	4

↓ ↓ ↓ ↓

29 28 27 28

X **Y**	7	1	3	1	7	1	3	1
16					112	16	48	16
2				2	14	2	6	
0			0	0	0	0	0	0
2		2	6	2	14	2	6	2

↓ ↓ ↓ ↓

116 36 52 36

Example:

If $\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 4 \\ 7 \\ 4 \\ 3 \\ 4 \end{bmatrix}$, we can show that $DFT(\mathbf{z}) = \frac{1}{8} DFT(\mathbf{x}_1) \textcircled{8} DFT(\mathbf{y})$, where

$\mathbf{x}_1 = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} x[0] \cdot y[0] \\ x[1] \cdot y[1] \\ x[2] \cdot y[2] \\ x[3] \cdot y[3] \\ 0 \cdot y[4] \\ 0 \cdot y[5] \\ 0 \cdot y[6] \\ 0 \cdot y[7] \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \\ 6 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3.3 Fast Fourier Transform — Efficient algorithms for computing DFT

- **Principle**

  Divide-and-conquer

  - Decimation-in-time algorithm

  - Decimation-in-frequency algorithm

  Specific properties of $W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$

  $$W_N^{-kn} = (W_N^{kn})^*,\quad W_N^{(k+N/2)n} = -W_N^{kn},\quad W_N^{(k+N)n} = W_N^{kn},\ \text{and}\ W_{N/2}^{kn} = W_N^{2kn}$$

  If we compute DFT directly,

  $$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]\cdot W_N^{kn},\quad k = 0,1,\ldots,N-1 \\ &= \sum_{n=0}^{N-1} \left\{ \begin{array}{l} \left[\mathrm{Re}(x[n])\cdot \mathrm{Re}(W_N^{kn}) - \mathrm{Im}(x[n])\cdot \mathrm{Im}(W_N^{kn})\right] + \\ j\left[\mathrm{Re}(x[n])\cdot \mathrm{Im}(W_N^{kn}) + \mathrm{Im}(x[n])\mathrm{Re}(W_N^{kn})\right] \end{array} \right\} \end{aligned}$$

  For each k, we need N complex multiplications and N-1 complex additions. → $4N$ real multiplications and $4N$-2 real additions.

  there are around N^2 complex multiplications and N^2 complex adds for N-DFT. We will show how to use the properties of W_N^{kn} to reduce the computations.

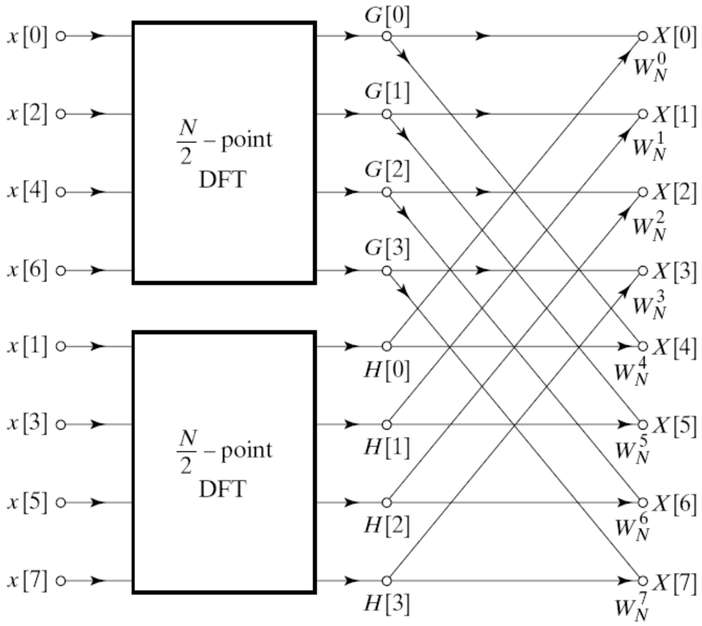
- **Decimation-in-time algorithm**

Assume $N = 2^v$, then

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn},\quad k = 0,1,\ldots,N-1 \\ &= \underbrace{\sum_{n\ \text{even}} x[n]W_N^{kn}}_{n=2r} + \underbrace{\sum_{n\ \text{odd}} x[n]W_N^{kn}}_{n=2r+1} \\ &= \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{(2r+1)k} \end{aligned}$$

$$\because W_N^2 = e^{-2j\left(\frac{2\pi}{N}\right)} = e^{-2j\left(\frac{\pi}{N/2}\right)} = W_{N/2}$$

$$\begin{aligned} X[k] &= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} + W_N^k \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} \\ &= G[k] + W_N^k H[k] \end{aligned}$$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N/2 – point DFT
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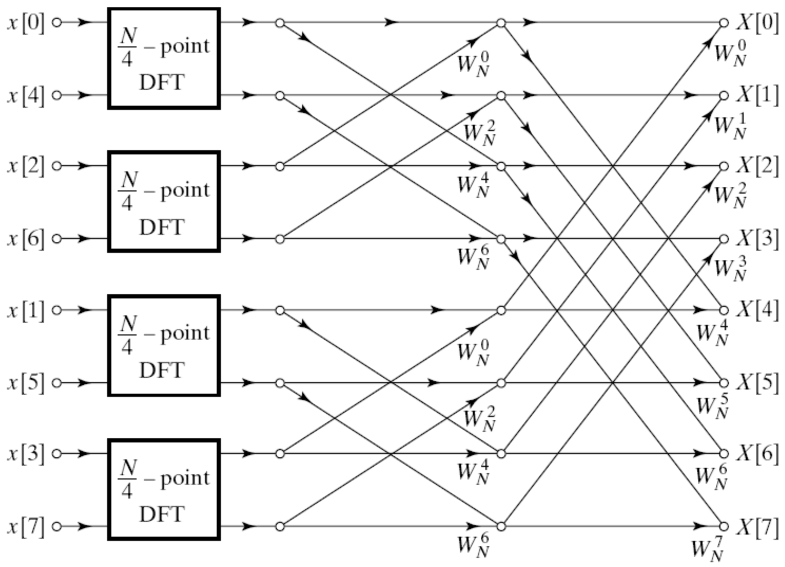
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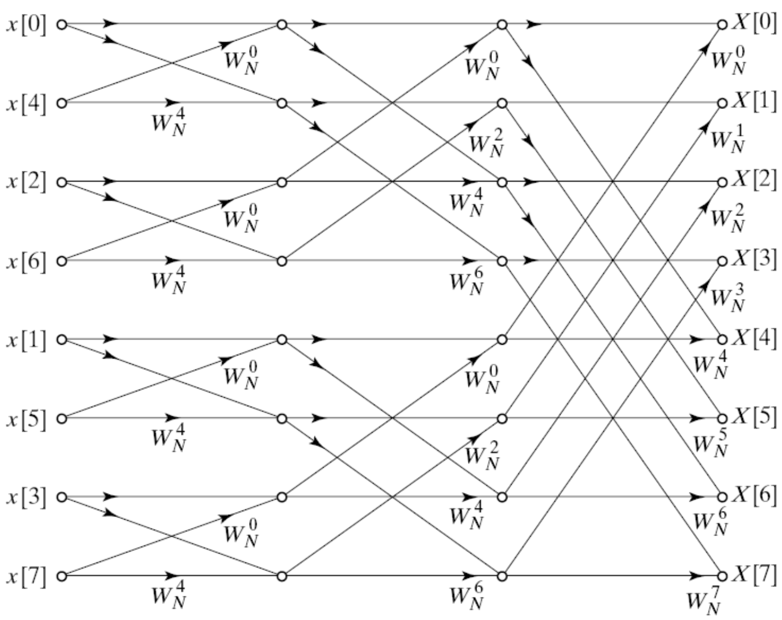
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W_N^7

Comparisons with direct computation of DFT:

(a) Direct computation of N-point DFT (N frequency samples):

$\sim N^2$ complex multiplications and N^2 complex adds

(b) Direct computation of $N/2$-point DFT:

$\sim \left(\frac{N}{2}\right)^2$ complex multiplications and $\left(\frac{N}{2}\right)^2$ complex adds

+ additional N complex multis and N complex adds

$\sim$ (Total:) $N + 2\left(\frac{N}{2}\right)^2 = N + \frac{N^2}{2}$ complex multis and adds

(c) $\log_2 N$-stage FFT

Since $N = 2^\nu$, we can further break $N/2$-point DFT into two $N/4$-point DFT and so on. (Fig.9.5, P.638) (Fig.9.7, P.639)

At each stage: $\sim N$ complex multis and adds

Total: $\sim N \log_2 N$ complex multis and adds (-->$\frac{N}{2}\log_2 N$)

Number of points, N	Direct Computation: Complex Multis	FFT: Complex Multis	Speed Improvement Factor
4	16	4	4.0
8	64	12	5.3
16	256	32	8
64	4,096	192	21.3
256	65,536	1,024	64.0
1024	1,048,576	5,120	204.8

- **Decimation-in-frequency algorithm**

Dividing $X[k]$ into smaller pieces

$$X[k]=\sum_{n=0}^{N-1}x[n]\cdot W_N^{kn},\quad k=0,1,\ldots,N-1$$

If k is even, $k=2r$.

$$\begin{aligned}
X[2r]&=\sum_{n=0}^{N-1}x[n]W_N^{2rn},\qquad r=0,1,\cdots,\frac{N}{2}-1\\
&=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2nr}+\sum_{n=\frac{N}{2}}^{N-1}x[n]W_N^{2nr}\qquad n\leftarrow(n+N/2)\\
&=\sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{2nr}+\sum_{n=0}^{\frac{N}{2}-1}x\left[n+\frac{N}{2}\right]\cdot W_N^{2r\left(n+\frac{N}{2}\right)}\\
&\because W_N^{2r[n+N/2]}=W_N^{2rn}W_N^{rN}=W_N^{2rn}\\
&=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{2nr}\\
&=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x\left[n+\frac{N}{2}\right]\right)\cdot W_{N/2}^{nr}
\end{aligned}$$

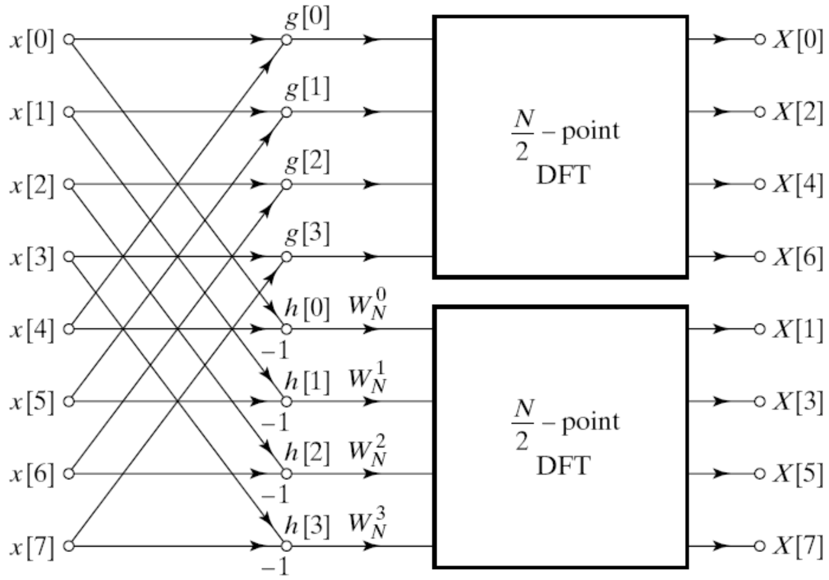
Similarly, if k is odd, $k=2r+1$.

$$X[2r+1]=\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]-x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{n}\cdot W_{N/2}^{nr}$$

$$\begin{cases}
X[2r]=\displaystyle\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]+x\left[n+\frac{N}{2}\right]\right)\cdot W_{N/2}^{nr}\\
X[2r+1]=\displaystyle\sum_{n=0}^{\frac{N}{2}-1}\left(x[n]-x\left[n+\frac{N}{2}\right]\right)\cdot W_N^{n}\cdot W_{N/2}^{nr}
\end{cases}$$

Let $\begin{cases} g[n]=x[n]+x\left[n+\dfrac{N}{2}\right]\\ h[n]=x[n]-x\left[n+\dfrac{N}{2}\right]\end{cases}$

We can further break $X[2r]$ into even and odd groups …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h[0] W_N^0
h[1] W_N^1
h[2] W_N^2
h[3] W_N^3
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$\frac{N}{2}$ – point DFT
$\frac{N}{2}$ – point DFT
X[0]
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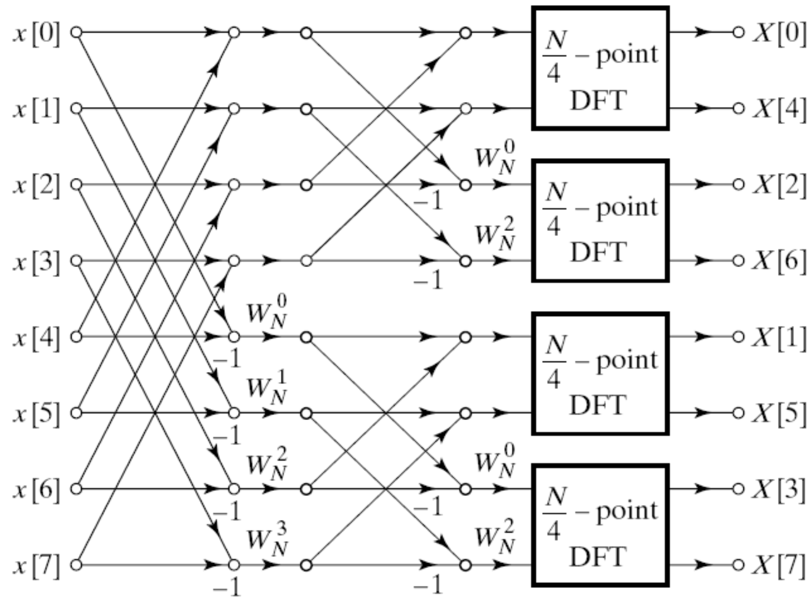
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W_N^2
−1
$\frac{N}{4}$ – point DFT
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X[2]
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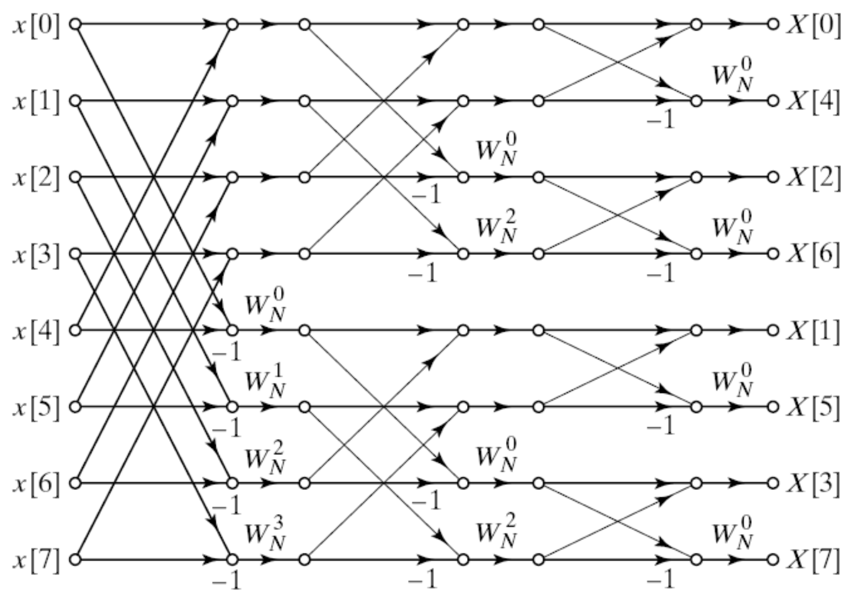
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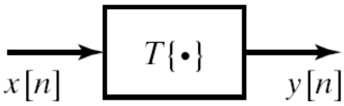
Chapter 4 Digital Systems

A digital system is defined mathematically as a transformation or operator $T\{\cdot\}$ that maps an input sequence with values $x[n]$ into an output sequence with values $y[n] = T\{x[n]\}$.



4.1 System Classes

1. **Memoryless**: if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n. For example, $y[n] = \{x[n]\}^2$.
2. **Linear**: if the system satisfies the principle of superposition, i.e., If $y_1[n] = T\{x_1[n]\}$, and $y_2[n] = T\{x_2[n]\}$, then $T\{ax_1[n] + bx_2[n]\} = ay_1[n] + by_2[n]$.
   Note that a linear system also satisfy $y[n] = 0$ for $x[n] = 0$, $\forall n$, because:
   if $y_1[n] = T\{0\} \neq 0$, and let $y_2[n] = T\{x_2[n]\}$, then $y_1[n] + y_2[n] \neq y_2[n] = T\{x_2[n]\} = T\{0+x_2[n]\}$
3. **Time-invariant**: a time shift or delay of the input sequence causes a corresponding shift in the output sequence, i.e., If $x_1[n] = x[n - n_d]$, then $y_1[n] = y[n - n_d]$.
4. **Causal**: $y[n]$ does not depend on the future value of $x[n]$, e.g., $x[n+1]$.
5. **Stable**: if $|\, x[n] \,| < \infty$, then $|\, y[n] \,| < \infty$ for all n.

Example 1: $y[n] = \sum_{k=n_0}^{n} x[k]$ is

(i) memory, because $y[n]$ depends not only on $x[n]$ but also on $x[n_0]$, $x[n_0 + 1]$, …

(ii) linear, because if $y_1[n] = \sum_{k=n_0}^{n} x_1[k]$ and $y_2[n] = \sum_{k=n_0}^{n} x_2[k]$ are the outputs for inputs $x_1[n]$ and $x_2[n]$, respectively, then for an input $x_3[n] = ax_1[n] + bx_2[n]$, the output is

$$y_3[n] = \sum_{k=n_0}^{n} x_3[k] = \sum_{k=n_0}^{n} \left(ax_1[k] + bx_2[k]\right) = ay_1[n] + by_2[n].$$

(iii) time-variant, because if let $x_1[n] = x[n - n_d]$ be an input, then the output is

$$y_1[n] = \sum_{k=n_0}^{n} x_1[k] = \sum_{k=n_0}^{n} x[k - n_d] = \sum_{m=n_0-n_d}^{n} x_1[m] \neq y[n - n_d]$$

(iv) non-causal, because $y[n]$ depends on the future value of $x[n]$ for $n < n_0$

(v) non-stable, because $|\, y[n] \,| = \infty$ for $n = \infty$.

Example 2: $y[n] = cx[n] + d$ is

(i) memoryless, because $y[n]$ depends only on $x[n]$.

(ii) non-linear, because if $y_1[n] = cx_1[n] + d$ and $y_2[n] = cx_2[n] + d$ are the outputs for inputs $x_1[n]$ and $x_2[n]$, respectively, then for an input $x_3[n] = ax_1[n] + bx_2[n]$, the output is

$$y_3[n] = cx_3[n] + d = c(ax_1[n] + bx_2[n]) + d \neq ay_1[n] + by_2[n].$$

(iii) time-invariant, because if let $x_1[n] = x[n - n_d]$ be an input, then the output is

$$y_1[n] = cx_1[n] + d = cx[n - n_d] + d = y[n - n_d].$$

(iv) causal, because $y[n]$ does not depend on the future value of $x[n]$.

(v) stable, because if $| x[n] | < \infty$, then $| y[n] | < \infty$ for all n.

4.2 Linear Constant-coefficients Difference Equations

A digital system can be characterized by a difference equation in the form of

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

For example, $y[n] = y[n\text{-}1] + x[n] + x[n\text{-}1]$.

Given an input to a system, how can we compute the output using the difference equation? See the following example.

Let $y[n] = a\, y[n-1] + x[n]$. Consider an input $x[n] = k\delta[n]$, and the auxiliary condition $y[\text{-}1] = c$. Solve $y[n]$.

$y[0] = a\, y[\text{-}1] + x[0] = ac + k$

$y[1] = a\, y[0] + 0 = a(ac + k)$

…

$y[n] = a^{n+1}c + a^n k$, for $n \geq 0$

$y[\text{-}1] = a\, y[\text{-}2] + 0$

$y[\text{-}2] = a^{-1} c$

…

$y[n] = a^{n+1}c$, for $n < 0$

Therefore, $y[n] = a^{n+1}c + a^n k\, u[n]$

Note that:

(i) the input is zero for $k = 0$, but the output $y[n] = a^{n+1}c \neq 0$. The system is therefore nonlinear.

(ii) if $x_1[n] = k\delta[n - n_0]$, then $y_1[n] = a^{n+1}c + a^{n-n0}k\, u[n - n_0] \neq y[n - n_0]$. Thus, the system is time-variant.

4.3 Linear Time-invariant (LTI) Systems

- Definition

  An LTI system can be completely characterized by its impulse response $h[n]$.

  What's the impulse response?

  Let $x[n] = \delta[n]$ be an input to a system, then the output is $y[n] = h[n]$.

  More generally, $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = x[n] * h[n]$ (**convolution**)

- **Properties of LTI Systems**

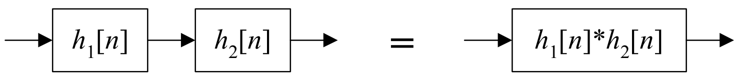
Stability: an LTI system is stable, if $h[n]$ is absolutely summable, i.e., $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Causality: an LTI system is causal, if $h[n] = 0$, for $n < 0$.

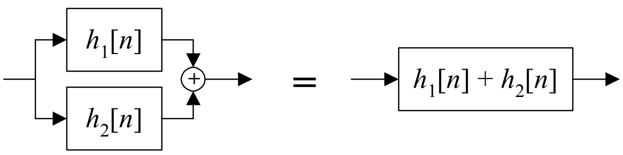
> Because $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, and if the system is causal,
>
> then $y[n]$ does not depend on any $x[n - k]$ for $k < 0$, $h[k]$ must be 0 for $k < 0$.

Memoryless: an LTI system is memoryless, if $h[n] = k\delta[n]$.

Cascade connection:



Parallel connection:



- **Eigen function of an LTI system**

If $x[n]$ is an eigen-function of an LTI system, then output $y[n] = x[n] \cdot f(\theta)$, where $f(\theta)$ is the associated eigenvalue, which is independent of n.

Example: $x[n] = e^{jwn}$, $y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{jw(n-k)} = e^{jwn}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-jwk}\right) = e^{jwn}H(e^{jw})$.

Since $H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jwk}$ is independent of n, e^{jwn} is an eigen-function.

Example: $x[n] = 5^n u[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]5^{n-k}u[n-k] = \sum_{k=-\infty}^{n} h[k]5^{n} = 5^n\left(\sum_{k=-\infty}^{n} h[k]5^{-k}\right).$$

Since $\sum_{k=-\infty}^{n} h[k]5^{-k}$ depends on n, $5^n u[n]$ is not an eigen-function.

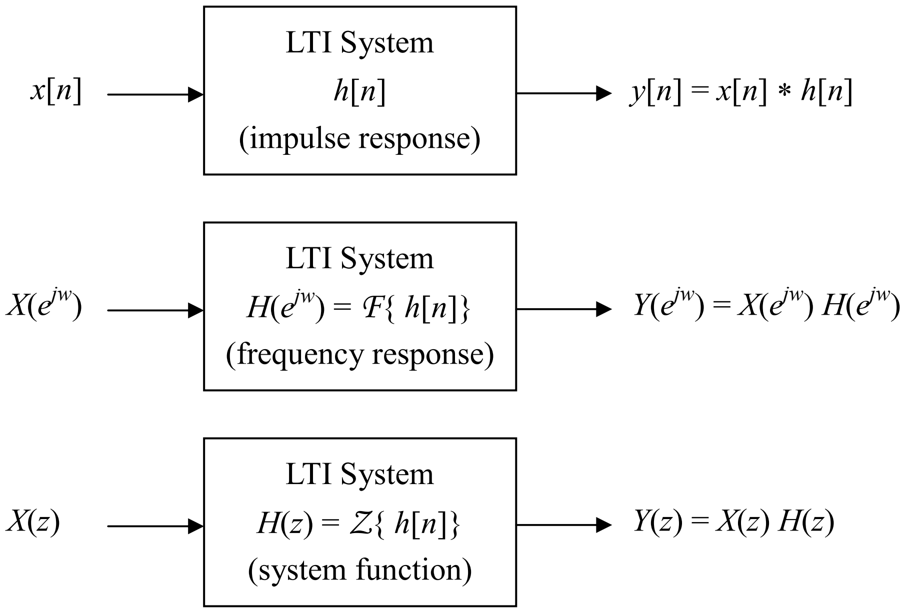
Example: If $x[n] = e^{-jwn}$ is input to an LTI system, and the output is $y[n] = e^{-jw(n-3)} + e^{-jw(n-6)}$, find the impulse response of the system, $h[n]$.

Because $x[n] = e^{-jwn}$ is an eigen function, $y[n] = x[n]H(e^{-jw})$.

$$\Rightarrow H(e^{-jw}) = \frac{y[n]}{x[n]} = e^{j3w} + e^{j6w}. \Rightarrow H(e^{jw}) = e^{-j3w} + e^{-j6w}.$$

Thus, $h[n] = \delta[n-3] + \delta[n-6]$.

4.4 Frequency Response of An LTI System



- A linear constant-coefficient difference equation can be represented by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] .$$

Taking z-transform on both sides of the equation, we have

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) .$$

Thus, the system function is of the form: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$.

- Some frequently-used systems

-- Ideal Delay

$y[n] = x[n-n_d]$ $\qquad$ $h[n] = \delta[n-n_d]$

-- Moving Average

$$y[n] = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} x[n-k] \qquad h[n] = \begin{cases} \dfrac{1}{M_1+M_2+1}, & -M_1 \le n \le M_2 \\ 0, & \text{otherwise} \end{cases}$$

-- Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad h[n] = u[n], \text{ unit step}$$

-- Forward Difference

$y[n] = x[n+1] - x[n]$ $\qquad$ $h[n] = \delta[n+1] - \delta[n]$

-- Backward Difference

$y[n] = x[n] - x[n-1]$ $\qquad$ $h[n] = \delta[n] - \delta[n-1]$

- If a system is characterized by a linear, constant-coefficient difference equation and is further specified to be LTI and causal, the solution to the equation is unique, which must satisfies the initial-rest conditions:

  If $x[n] = 0$, for $n < n_0$, then $y[n] = 0$, for $n < n_0$.

  Example: Let $y[n] = a\, y[n-1] + x[n]$. Consider an input $x[n] = k\delta[n]$, and the auxiliary condition $y[0] = 0$. Solve $y[n]$.

  In this case, $y[n]$ can be solved by using the impulse response $h[n]$.
  Taking DTFT on both sides of the difference equation, we have $Y(e^{jw}) = aY(e^{jw})\, e^{-jw} + X(e^{jw})$

  $\Rightarrow (1 - a\, e^{-jw})\, Y(e^{jw}) = X(e^{jw})$

  $$\Rightarrow \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1 - ae^{-jw}} = H(e^{jw})$$

  $\Rightarrow h[n] = a^n\, u[n]$

  Therefore, $y[n] = x[n] * h[n] = k\delta[n] * a^n\, u[n] = k\, a^n\, u[n]$

  We can verify this result as follows:

  $y[0] = a\, y[-1] + x[0] = 0 + k\delta[0] = k$

  $y[1] = a\, y[0] + x[1] = ak + 0 = ak$

  $y[2] = a\, y[1] + x[2] = a(ak) + 0 = a^2k$

  …

  $y[n] = a^n k.$

- The system response $H(e^{jw})$ can be represented by $H(e^{jw}) = |H(e^{jw})|\, e^{j\angle H(e^{jw})}$,

  where $|H(e^{jw})|$ is called magnitude, and $\angle H(e^{jw})$ is called phase or angle.

  $\angle H(e^{jw})$ is not unique, because $e^{j\angle H(e^{jw})} = e^{j(\angle H(e^{jw}) + 2\pi k)}$, where k is an integer.

- Principle phase and continuous phase function

  ARG[$H(e^{jw})$] means $-\pi \le \angle H(e^{jw}) \le \pi$, and is called the *principle phase*

  arg[$H(e^{jw})$] refers to a continuous phase function of w for $0 \le w \le \pi$, that is

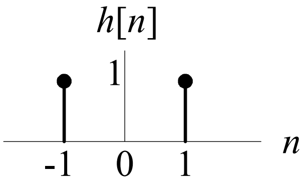
  arg[$H(e^{jw})$] $= \angle H(e^{jw})$, $0 \le w \le \pi$.

- Group delay

  $$\mathrm{grd}[H(e^{jw})] = -\frac{d}{dw}\left\{\arg[H(e^{jw})]\right\},$$

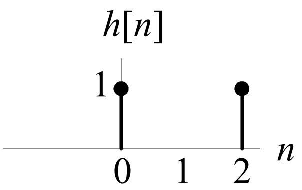
which is used to measure the linearity of the phase.

Example:



$$H(e^{jw}) = \sum_{n=-1}^{1} h[n]e^{-jwn} = e^{-jw} + e^{jw} = 2\cos w .$$

Thus, $\text{grd}[H(e^{jw})] = 0$.



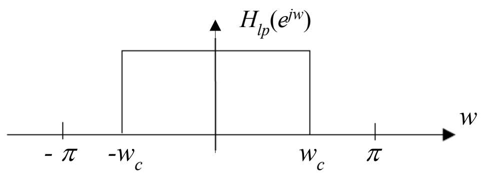
$$H(e^{jw}) = \sum_{n=0}^{2} h[n]e^{-jwn} = 1 + e^{-j2w} = e^{-jw}(e^{jw} + e^{-jw}) = 2\cos w \cdot e^{-jw} .$$

Thus, $\text{grd}[H(e^{jw})] = -\frac{d}{dw}(-w) = 1$.

- Frequency selective filters

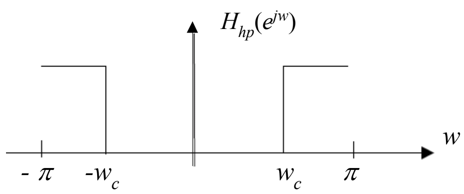
$$H(e^{jw}) = \begin{cases} 1, & w_1 \le |w| \le w_2 \\ 0, & \text{otherwise} \end{cases}$$

Lowpass filter



$$h_{lp}[n] = \frac{\sin w_c n}{\pi n}$$

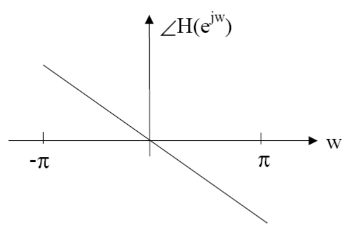
Highpass filter



$$h_{lp}[n] = \delta[n] - \frac{\sin w_c n}{\pi n}$$

They all have zero phase, which means “no delay”.

- An ideal delay system has linear phase



An ideal delay system can be expressed by $h_{id}[n] = \delta[n-n_d]$.

Taking DTFT on both sides of the equation, we have

$$H_{id}(e^{jw}) = e^{-jwn_d}.$$

Hence, the group delay of $H_{id}(e^{jw})$ is

$$-\frac{d}{dw}\{\arg[H_{id}(e^{jw})]\} = n_d ,$$

which is a constant (linear function of w). Thus, $H_{id}(e^{jw})$ has linear phase.

For example, if $x[n] = u[n]$ is input to a system, and the output is $y[n] = u[n-n_d]$, then the system has linear phase.

4.5 Characteristics of An LTI System in Z-plane

- By factorization, a system function $H(z)$ can also be represented by

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})},$$

in which the zeros of $H(z)$ are located at: $z = c_k$, $k = 1, 2, \ldots, M$
and the poles of $H(z)$ are located at: $z = d_k$, $k = 1, 2, \ldots, N$

- Causality: the ROC of $H(z)$ must be outside the outermost pole
For example, if $h[n] = (1/2)^n u[n] + (-1/3)^n u[n]$, we know that $h[n]$ is causal.
Taking the z-transform of $h[n]$, we have

$$H(z) = \frac{1}{1-(1/2)z^{-1}} + \frac{1}{1+(1/3)z^{-1}} = \frac{2z(z-1/12)}{(z-1/2)(z+1/3)}, \text{ ROC: } |z| > 1/2$$

The poles of $H(z)$ are 1/2 and -1/3. We can see that ROC of $H(z)$ is outside the outermost pole, 1/2.

- Stability: the ROC of $H(z)$ includes the unit circle. Why?

Recall that a stable system is defined as:
if input $|x[n]| < \infty$, then output $|y[n]| < \infty$ for all n.
Since $|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$, a stable system satisfies $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$,
which is equivalent to the condition $\sum_{k=-\infty}^{\infty} |h[k]z^{-k}| < \infty$ for $|z| = 1$. This means that the ROC of $H(z)$ includes the unit circle.

- An LTI system $H(z)$ is causal and stable, if all the poles of $H(z)$ are inside the unit circle.

- Inverse system
  - An inverse system $H_i(z)$ of the system $H(z)$ is defined as $H(z)\,H_i(z) = 1$, that is, $H_i(z) = 1/H(z)$
  - Since $H_i(z) = 1/H(z)$, the ROCs of $H_i(z)$ and $H(z)$ must overlap
  - Example:

    If the system $H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}}$ is stable, what is the impulse response of its inverse system?

    Apparently, $H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}} = \frac{1}{1-0.5z^{-1}} - \frac{0.9z^{-1}}{1-0.5z^{-1}}$. There are two possibilities for the ROC of $H_i(z)$, one is $|z| < 0.5$, and another is $|z| > 0.5$.

Since $H(z)$ is stable, the ROC of $H(z)$ is $|z| > 0.9$. To ensure that the ROCs of $H_i(z)$ and $H(z)$ overlap, we choose $|z| > 0.5$ as the ROC of $H_i(z)$.

Therefore, the impulse response of $H_i(z)$ is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1],$$

which is both causal and stable, or alternatively,

using $H_i(z) = 1 - \dfrac{0.4z^{-1}}{1-0.5z^{-1}}$, we have $h_i[n] = \delta[n] - 0.4(0.5)^{n-1} u[n-1]$,

which is also causal and stable.

- Minimum phase system

  If an LTI system is causal and stable, and its inverse system is also causal and stable, then this system is called "minimum phase system".

  How to check if a system is minimum phase?

  $\Rightarrow$ The poles and zeros of a minimum phase system are all inside the unit circle.

- All-pass system

  - A system of the form $H_{ap}(z) = \dfrac{z^{-1} - a^*}{1 - az^{-1}}$ is called an allpass system.
  - An allpass system has constant magnitude that is independent of frequency w, because

    $$|H_{ap}(e^{jw})| = \left| \frac{e^{-jw} - a^*}{1 - ae^{-jw}} \right| = \left| e^{-jw} \frac{1 - a^* e^{jw}}{1 - ae^{-jw}} \right| = 1.$$

  - The group delay of a causal allpass system is always positive.

    Proof:

    Let $a = r\,e^{j\theta}$, where $r \le 1$, then $H_{ap}(e^{jw}) = e^{-jw} \dfrac{1 - a^* e^{jw}}{1 - ae^{-jw}} = e^{-jw} \dfrac{1 - re^{-j\theta} e^{jw}}{1 - re^{j\theta} e^{-jw}}$

    Thus, $\angle H_{ap}(e^{jw}) = -w + \angle\left(1 - re^{-j\theta} e^{jw}\right) - \angle\left(1 - re^{j\theta} e^{-jw}\right)$, where

    $$\angle\left(1 - re^{-j\theta} e^{jw}\right) = \angle\left[1 - r\cos(w-\theta) - jr\sin(w-\theta)\right] = \tan^{-1}\left[\frac{-r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right],$$

    $$\angle\left(1 - re^{j\theta} e^{-jw}\right) = \angle\left[1 - r\cos(w-\theta) + jr\sin(w-\theta)\right] = \tan^{-1}\left[\frac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right],$$

    that is, $\angle H_{ap}(e^{jw}) = -w - 2\tan^{-1}\left[\dfrac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right]$.

    The group delay of $H_{ap}(e^{jw})$ is

    $$\mathrm{grd}[H_{ap}(e^{jw})] = -\frac{d}{dw} \angle H_{ap}(e^{jw}) = 1 + 2\frac{d}{dw} \tan^{-1}\left[\frac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right],\ 0 \le w \le \pi$$

$$= 1 + 2\frac{1}{1+\left[\dfrac{r\sin(w-\theta)}{1-r\cos(w-\theta)}\right]^2} \cdot \frac{r\cos(w-\theta)[1-r\cos(w-\theta)]-[r\sin(w-\theta)]^2}{[1-r\cos(w-\theta)]^2}$$

$$= \frac{1-r^2}{1+r^2-2r\cos(w-\theta)} = \frac{1-r^2}{\left|1-re^{-j(w-\theta)}\right|^2} \geq 0$$

- The continuous phase of a causal allpass system is non-positive (that is, $\arg[H_{ap}(e^{jw})] \leq 0$).
  Proof:

  $$\arg[H_{ap}(e^{jw})] = -\int_0^w \mathrm{grd}[H_{ap}(e^{j\phi})]d\phi + \arg[H_{ap}(e^{j0})]$$

  Since $H_{ap}(e^{j0}) = 1$, $\arg[H_{ap}(e^{j0})] = 0$. Thus, $\arg[H_{ap}(e^{jw})] = -\int_0^w \tau_{ap}(w)dw$.

  And since it has been shown that $\mathrm{grd}[H_{ap}(e^{jw})] \geq 0$, we can see that $\arg[H_{ap}(e^{jw})] \leq 0$.

- Any rational system function $H(z)$ can be decomposed as the form of

  $H(z) = H_{ap}(z)\, H_{min}(z)$

  Why?

  Suppose $H(z)$ has one zero $z = 1/c^*$, where $|c| < 1$, so that the zero is outside the unit circle. Then, we can express $H(z)$ as

  $$H(z) = H_1(z)(z^{-1} - c^*),$$

  where $H_1(z)$ is composed of the remaining poles and zeros of $H(z)$ inside the unit circle. We can also rewrite $H(z)$ as

  $$H(z) = \underbrace{H_1(z)(1-cz^{-1})}_{\text{minimum phase}} \underbrace{\frac{z^{-1}-c^*}{1-cz^{-1}}}_{\text{allpass}}.$$

  Example:

  Let $H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$. Find $H_{min}(z)$ and $H_{ap}(z)$, such that $H(z) = H_{min}(z)\, H_{ap}(z)$.

  $$H(z) = (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1}) \cdot \frac{(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})}$$

  $$= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1}) \cdot (1.25)^2\frac{(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})}$$

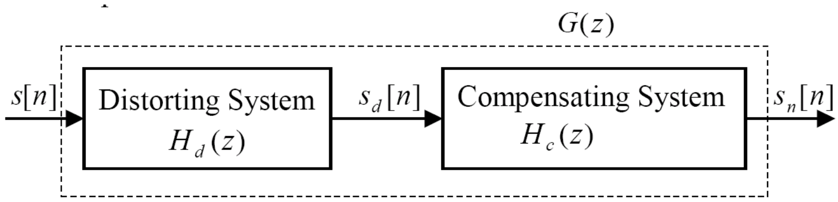
  Thus,

  $H_{min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})$

$$H_{ap}(z) = \frac{(z^{-1} - 0.8e^{-j0.8\pi})(z^{-1} - 0.8e^{j0.8\pi})}{(1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1})}$$

- Frequency-response compensation

  In many applications, signals are distorted by an LTI system with an undesirable frequency response. It may be of interest to process the distorted signal with a compensating system.



  Institutively, $H_c(z) = 1/\ H_d\ (z)$, so that the distortion can be eliminated. However, because H_d (z) is sometimes not causal and stable, it may not be possible to realize $H_c(z)$. To solve this problem, we can decompose $H_d(z)$ into $H_d(z) = H_{d,min}(z)H_{ap}(z)$, and then design $H_c(z) = 1/$ $H_{d,min}(z)$. This guarantees that $H_c(z)$ is causal and stable. In addition, we note that the overall system function $G(z) = H_d(z)H_c(z) = H_{ap}(z)$.

- Properties of minimum phase systems (compared to any other causal and stable systems)
  - (i) minimum phase lag

    The "phase lag" is defined as the amount of negative phase.

    Taking arg[·] on both sides of $H(e^{jw}) = H_{ap}(e^{jw})H_{min}(e^{jw})$, we have

    $\arg[H(e^{jw})] = \arg[H_{ap}(e^{jw})] + \arg[H_{min}(e^{jw})]$.

    Since $\arg[H_{ap}(e^{jw})] \leq 0$, we can see that $\arg[H(e^{jw})] \leq \arg[H_{min}(e^{jw})]$ for any non-minimum phase system $H(e^{jw})$.

    This means $H_{min}(e^{jw})$ has the minimal amount of negative phase among all the systems having the same magnitude response.
  - (ii) minimum group delay

    Taking grd[·] on both sides of $H(e^{jw}) = H_{ap}(e^{jw})H_{min}(e^{jw})$, we have

    $\mathrm{grd}[H(e^{jw})] = \mathrm{grd}[H_{ap}(e^{jw})] + \mathrm{grd}[H_{min}(e^{jw})]$.

    Since $\mathrm{grd}[H_{ap}(e^{jw})] \geq 0$, we can see that $\mathrm{grd}[H(e^{jw})] \geq \mathrm{grd}[H_{ap}(e^{jw})]$ for any non-minimum phase system $H(e^{jw})$.

    This means $H_{min}(e^{jw})$ has the minimal group delay among all the systems having the same magnitude response.
  - (iii) minimum energy delay

    If $h[n]$ is an arbitrary non-minimum phase sequence, then $|\ h[0]\ | < |\ h_{min}[0]\ |$

    Proof:

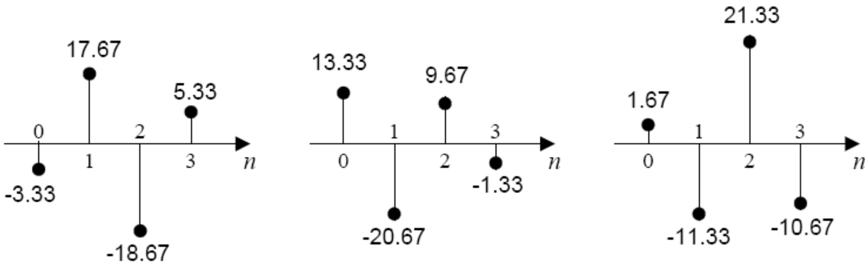
Using the initial value theorem $h[0] = \lim_{z\to\infty} H(z)$, we have

$$|h[0]| = \left|\lim_{z\to\infty} H_{ap}(z)H_{min}(z)\right| = \left|\lim_{z\to\infty} \prod_{k=1}^{M} \frac{z^{-1}-a_k^*}{1-a_k z^{-1}} \cdot H_{min}(z)\right|, \text{ where } |a_k| < 1,$$

$$\le \prod_{k=1}^{M} |a_k| \left|\lim_{z\to\infty} H_{min}(z)\right| \le \left|\lim_{z\to\infty} H_{min}(z)\right| = |h_{min}[0]|$$

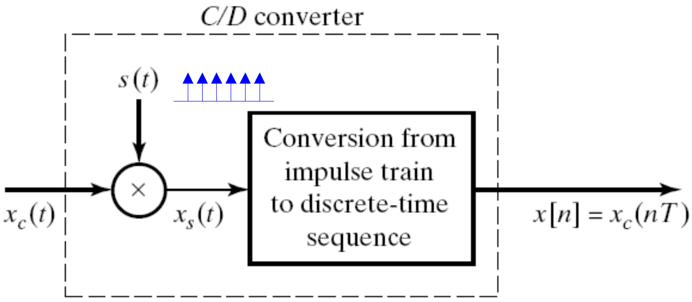
Example:

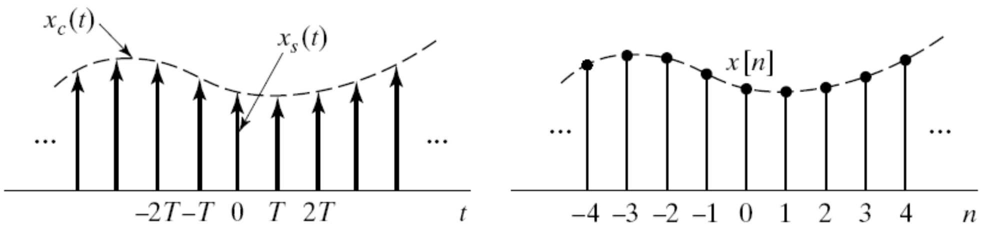
There are three systems, whose impulse responses are shown as follows. Determine which one of the three systems is most likely a minimum phase system.



Chapter 5 Sampling of Continuous-Time Signals

5.1 Analog to discrete conversion





- In the time domain

  Modulating signal $s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$

  Sampled signal $x_s(t)=x_c(t)s(t)=x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$

  Discrete-time signal $x[n]=x_c(nT)$, which is the non-zero terms of $x_s(t)$.

- In the frequency domain

  By using $x_s(t)=x_c(t)s(t)$, we can obtain

$$X_s(j\Omega)=\frac{1}{2\pi}X_c(j\Omega)*S(j\Omega)$$

$$=\frac{1}{2\pi}X_c(j\Omega)*\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(j\Omega-jk\Omega_s)\right]$$

$$=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j\Omega-jk\Omega_s)$$

$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)=\sum_{k=-\infty}^{\infty}s_k e^{jk\Omega_s t}$$

s_k is the Fourier series coefficients,

$$s_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\Omega_s t}dt=\frac{1}{T}$$

Therefore,

$$S(j\Omega)=\mathcal{F}\left\{\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\Omega_s t}\right\}$$

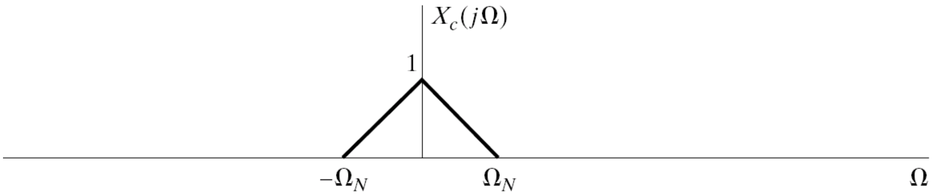
$$=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(j\Omega-jk\Omega_s)$$

By using $x_s(t)=\sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$, we get

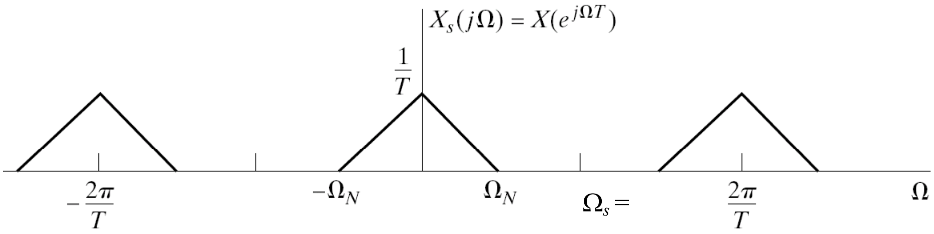
$$X_s(j\Omega)=\sum_{k=-\infty}^{\infty}x_c(nT)e^{-jn\Omega T}=\sum_{k=-\infty}^{\infty}x[n]e^{-jn\Omega T} \quad (1)$$

Since $X(e^{jw})=\sum_{k=-\infty}^{\infty}x[n]e^{-jwn}$, (2)

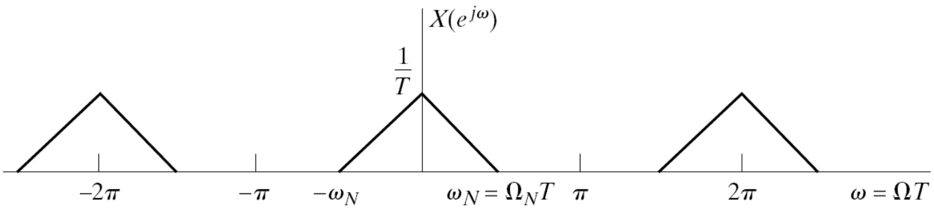
Comparing (1) and (2), we see that $X(e^{jw})=X_s(j\Omega)\big|_{\Omega=\frac{w}{T}}=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j\frac{1}{T}(w-2\pi k)\right)$



(a)



(b)



(c)

** The sampled signal spectrum is the sum of shifted copies of the original. **

Note:

Ω is the analog frequency

w is the digital frequency

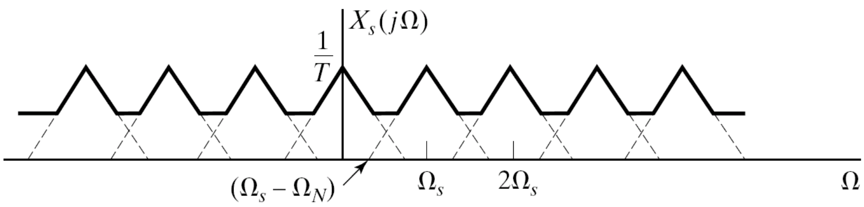
$w = \Omega T$, where T is the sampling period

$T = 1/f_s$, where f_s is the sampling frequency in Hz

$\Omega_s = 2\pi f_s = 2\pi / T$ is the sampling frequency in radius.

- Aliasing

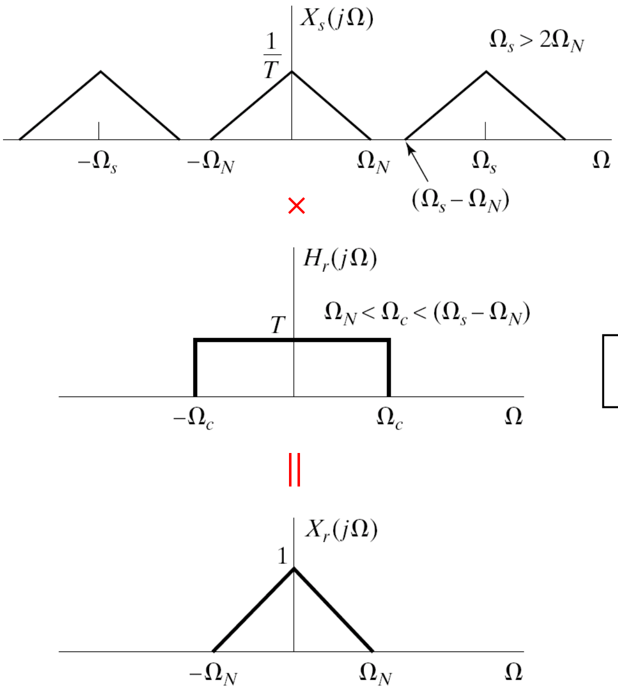
  If $\Omega_s < 2\Omega_N$, the copies of $X_c(j\Omega)$ overlap, which results in distortion.



- Nyquist sampling theorem

  To avoid aliasing, the sampling frequency must be at least the double of the highest frequency of the analog signal, i.e., $\Omega_s \geq 2\Omega_N$

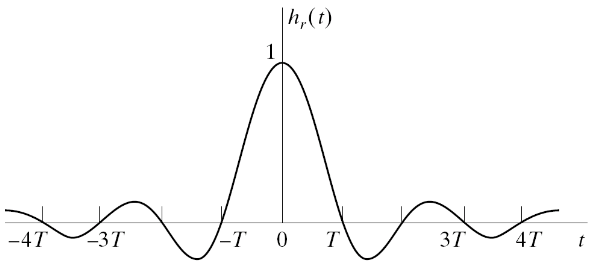
- Reconstruction of an analog signal from its samples



(e)

$\boxed{\text{Usually, } \Omega_c \text{ is set to be } \Omega_s/2 = \pi/T}$

$$h_r(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T e^{j\Omega t} d\Omega = \frac{\sin(\pi t/T)}{\pi t/T}$$

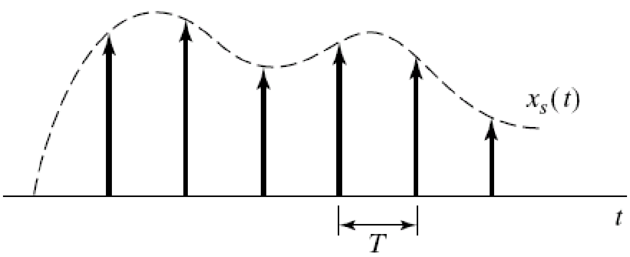


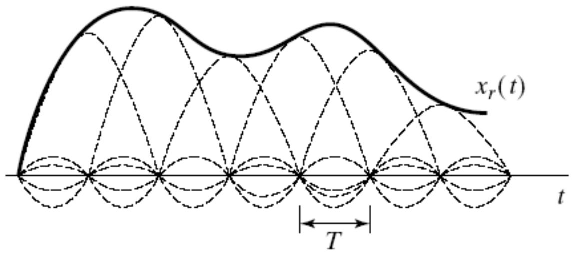
Since $X_r(j\Omega) = X_s(j\Omega)H_r(j\Omega)$,

$$x_r(t) = x_s(t) * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

The reconstruction is an interpolation.





5.2 Change the Sampling Rate Using Discrete-time Processing

$\boxed{\downarrow M}$

- Sampling rate reduction by an integer factor (Decimation)

Sampling rate compressor: If $x_d[n] = x[Mn]$, where $x[n] = x_c(nT)$ is obtained by sampling $x_c(t)$ with period T, what is the relationship between $X_d(e^{jw})$ and $X(e^{jw})$?

Taking a simple example that $M = 2$,

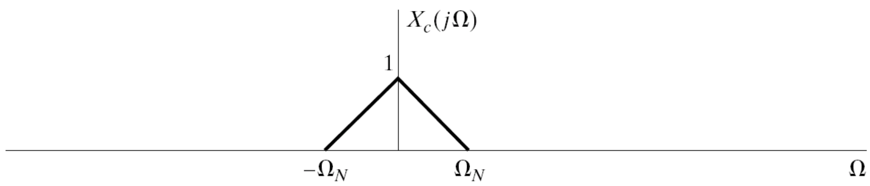
$$X_d(e^{jw}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} x[2n]e^{-jwn} = \sum_{m\in even} x[m]e^{-jwm/2}$$
$$= \sum_{m=-\infty}^{\infty} \frac{1}{2}\{x[m] + (-1)^m x[m]\}e^{-jwm/2} = \frac{1}{2}X(e^{jw/2}) + \frac{1}{2}X(e^{j(w/2-\pi)}).$$

Note: $(-1)^m = e^{-j\pi m}$.

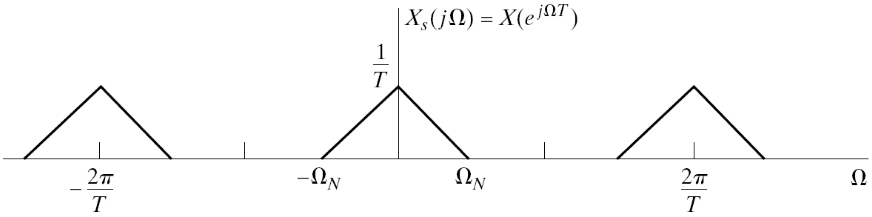
Now consider $x_d[n] = x[Mn]$,

$$X_d(e^{jw}) = \sum_{n=-\infty}^{\infty} x[Mn]e^{-jwn} = \sum_{m=0,\pm M,\pm 2M\ldots} x[m]e^{-jwm/M} = \sum_{m=-\infty}^{\infty} \frac{1}{M}\left\{\sum_{k=0}^{M-1} e^{j2\pi km/M} x[m]\right\}e^{-jwm/M}$$
$$= \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(w/M-2\pi k/M)}).$$

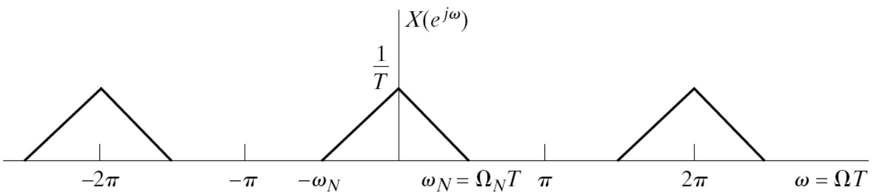
$$\sum_{k=0}^{M-1} e^{j2\pi km/M} = \begin{cases} M & ,m = Mr \\ \dfrac{1-e^{j2\pi m}}{1-e^{j2\pi m/M}} = 0, & m \neq Mr \end{cases}$$



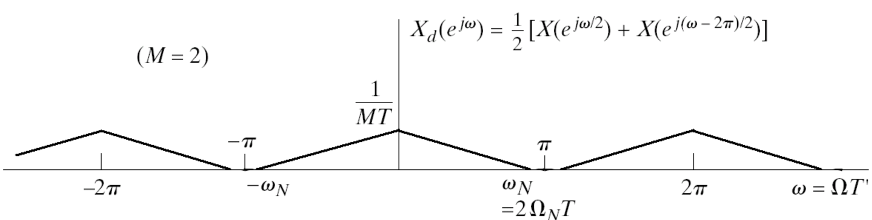
(a)



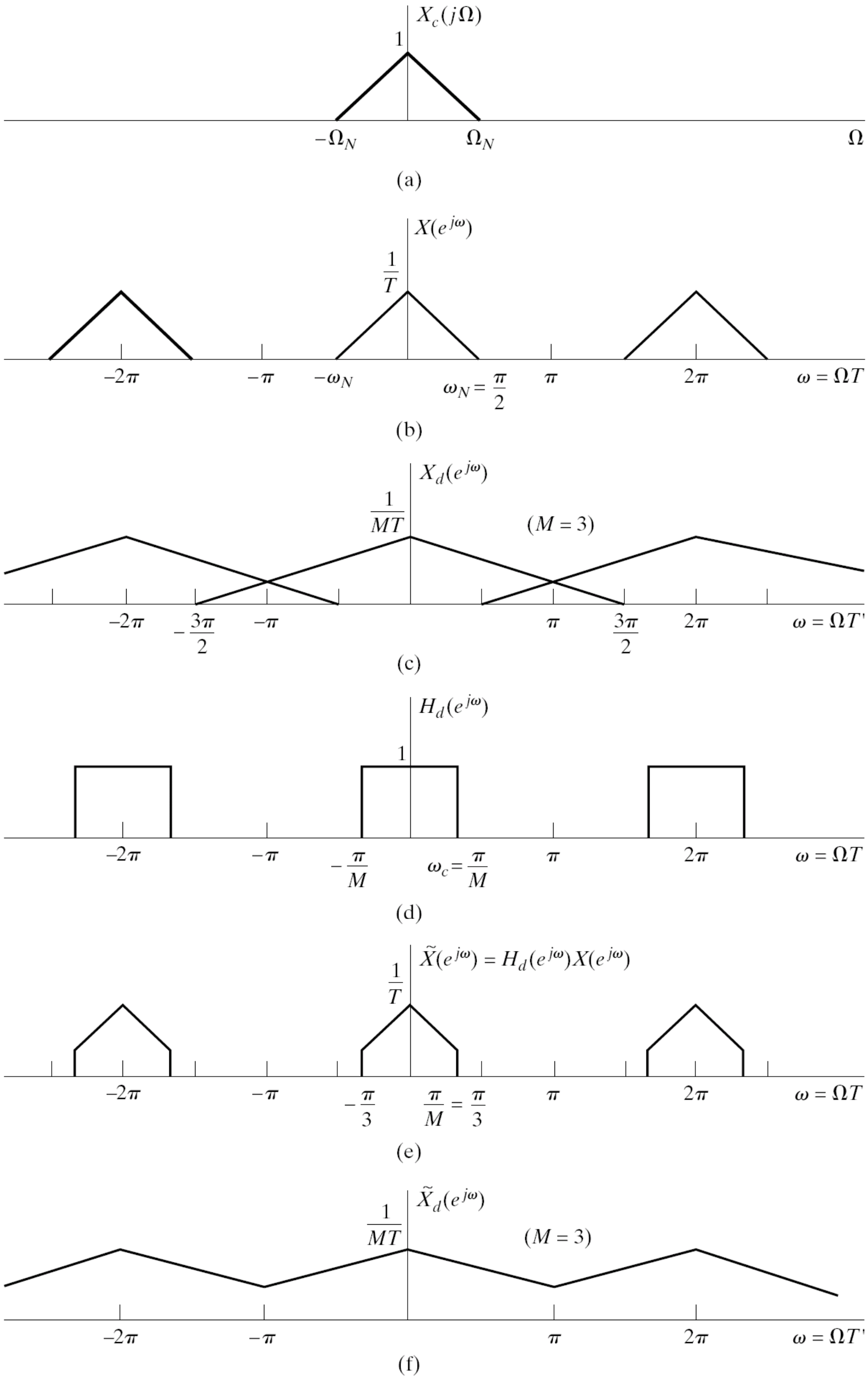
(b)



(c)

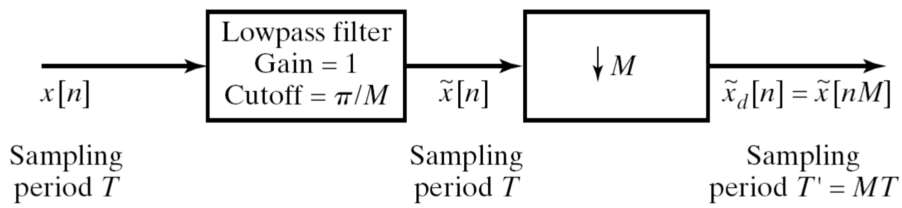


(d)



(Alaising occurs due to the decimation)

Procedure of downsampling:

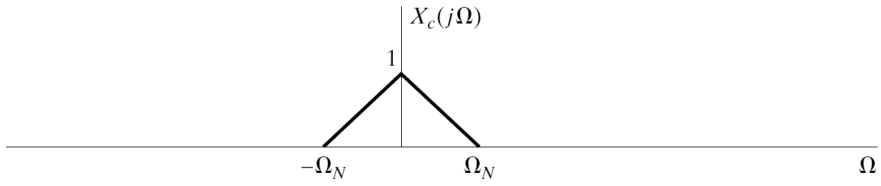


- Sampling rate increase by an integer factor (Interpolation)

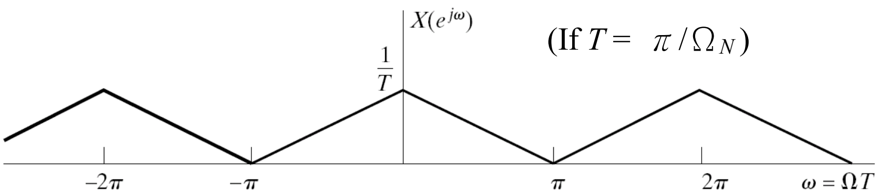
Sampling rate expander:

If $x_e[n] = \begin{cases} x[n/L], n = 0, \pm L, \pm 2L, \dots \\ 0 \quad , \quad \text{otherwise} \end{cases}$ $\qquad \boxed{\uparrow L}$

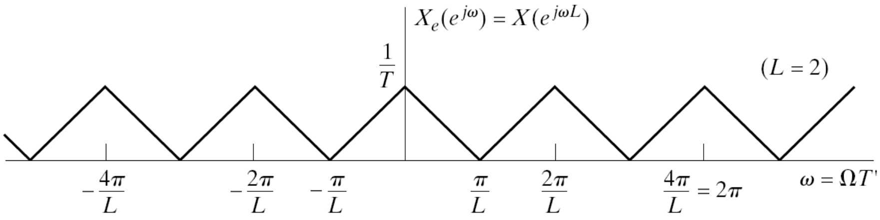
$$X_e(e^{jw}) = \sum_{n=0,\pm L,\pm 2L\dots}^{\infty} x[n/L]e^{-jwn} = \sum_{m=-\infty}^{\infty} x[m]e^{-jwmL} = X(e^{jwL}).$$



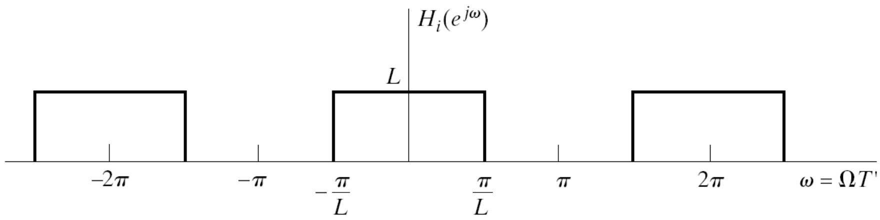
(a)



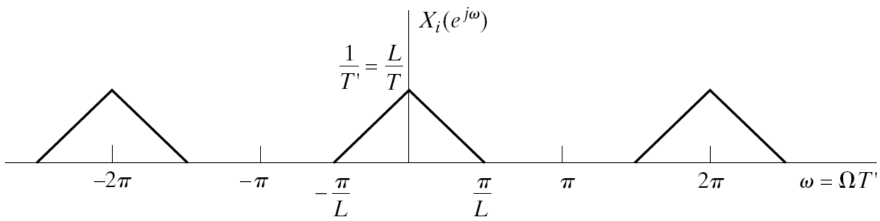
(b)



(c)

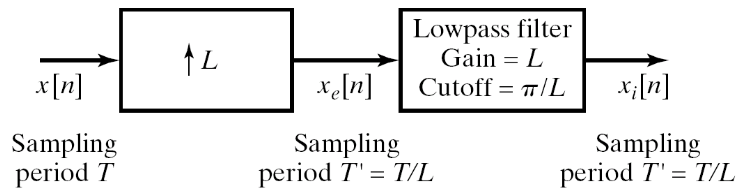


(d)



(e)

To obtain a signal $x_i[n] = x_c(nT')$, where $T' = T/L$, we need $H_e(e^{jw}) = \begin{cases} L, |w| < \pi/L \\ 0, \text{otherwise} \end{cases}$



$$X_i(e^{jw}) = X_e(e^{jw})H_i(e^{jw}).$$

$$x_i[n] = x_e[n] * h_i[n] = x_e[n] * \frac{\sin(\pi n/L)}{\pi n/L} = \sum_{m=0,\pm L,\pm 2L\ldots} x[m/L]\cdot\frac{\sin\left(\pi(n-m)/L\right)}{\pi(n-m)/L}$$
$$= \sum_{k=-\infty}^{\infty} x[k]\cdot\frac{\sin\left(\pi(n-kL)/L\right)}{\pi(n-kL)/L}$$

In practice, an ideal lowpass filter cannot be realized, because of its infinite length. A linear interpolation is commonly used instead. Specifically,

$$h_{lin}[n] = \begin{cases} 1-|n|/L, n \le L \\ 0 \quad , \quad \text{otherwise} \end{cases} \qquad H_{lin}(e^{jw}) = \frac{1}{L}\left[\frac{\sin(wL/2)}{\sin(w/2)}\right]^2$$

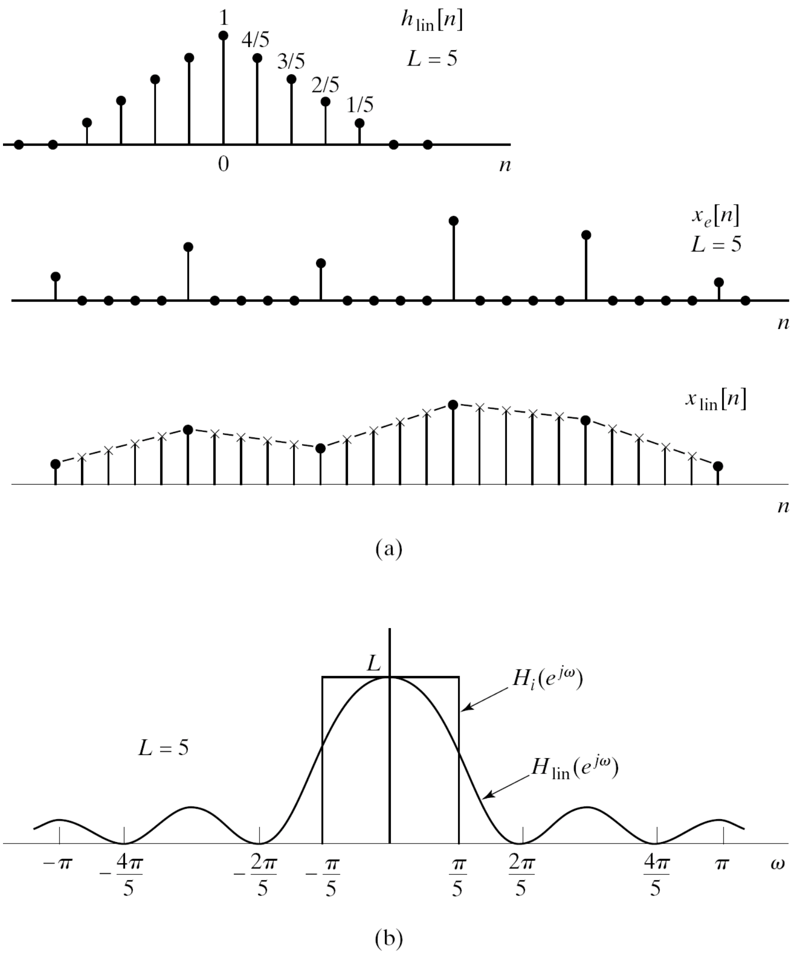
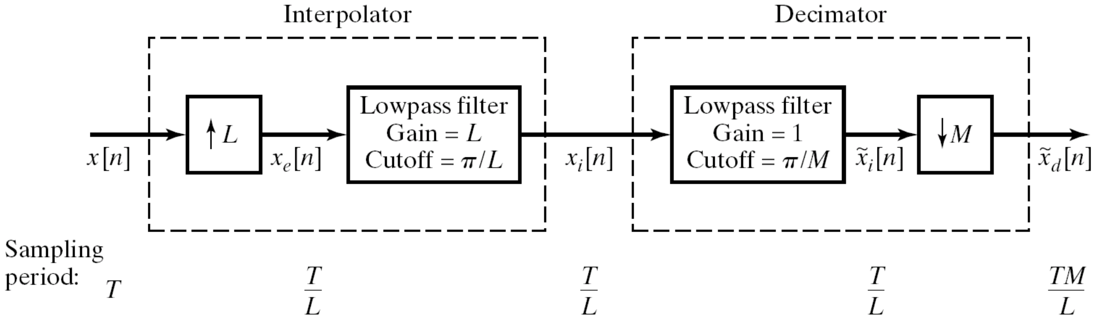


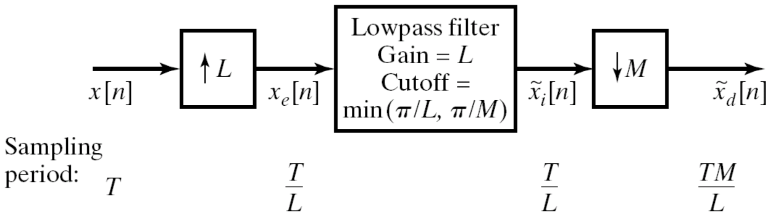
Figure 4.27 (a) Illustration of linear interpolation by filtering. (b) Frequency response of linear interpolator compared with ideal lowpass interpolation filter.

- Sampling rate change by a non-integer factor k

If k is rational, i.e., $k = M/L$, the change of sampling rate can be done by an interpolation followed by a decimation.



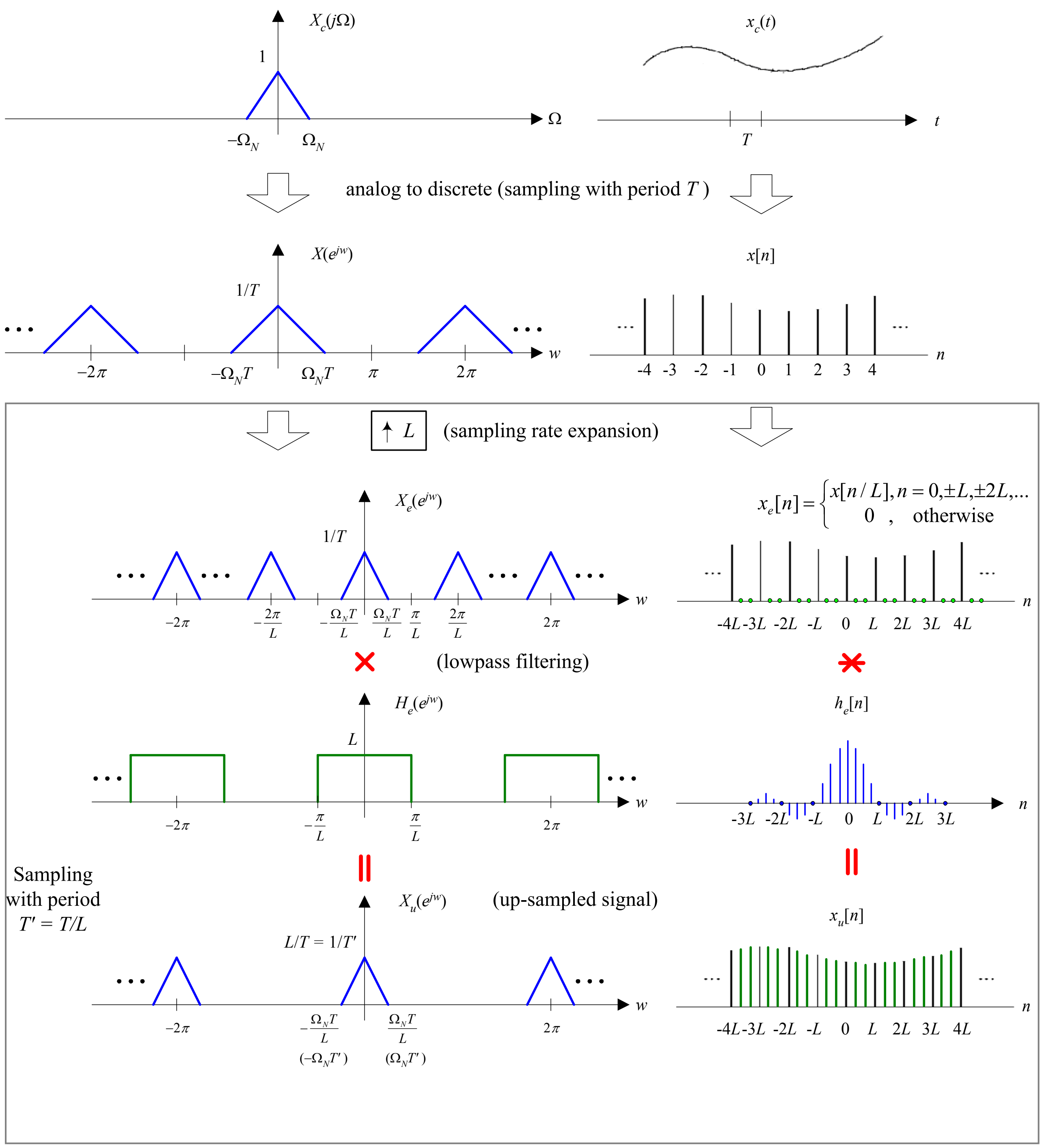
(a)



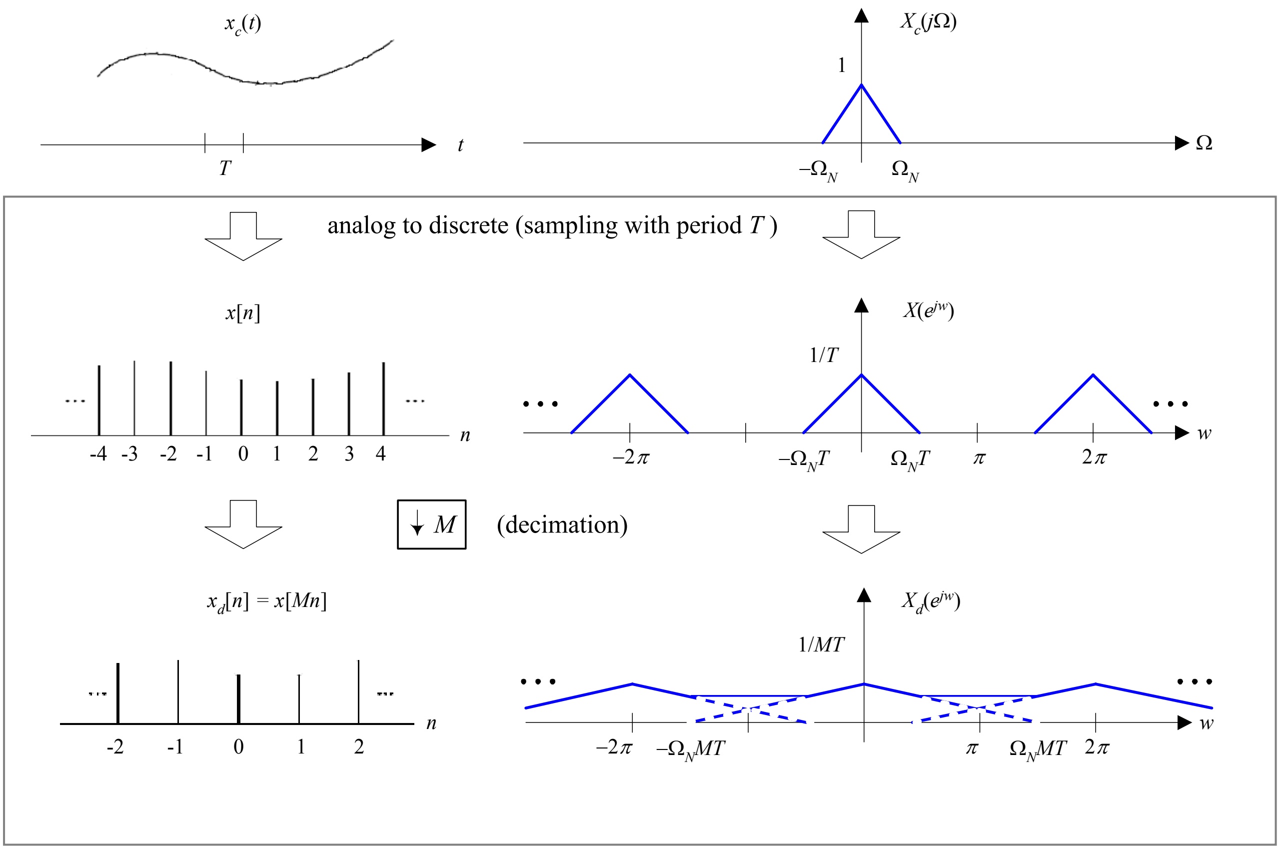
(b)

If k is not rational, the change of sampling rate must be done by converting the discrete-time signals into the continuous signals and then sampling the continuous signals again.

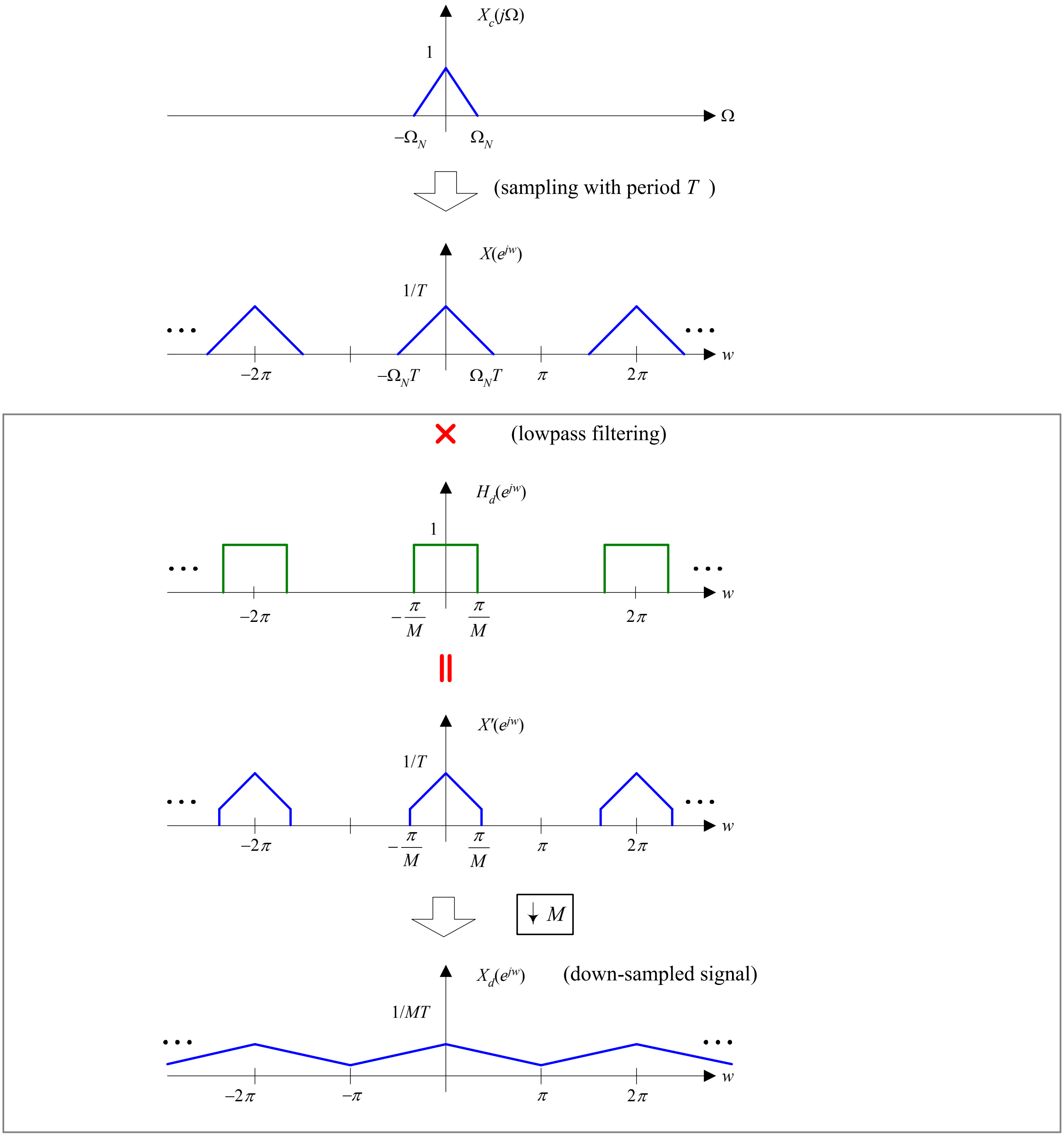
Review



Upsampling


x_c(t)
T
t
X_c(jΩ)
1
Ω
−Ω_N
Ω_N
analog to discrete (sampling with period T)
x[n]
-4 -3 -2 -1 0 1 2 3 4
n
X(e^jw)
1/T
−2π
−Ω_N T
Ω_N T
π
2π
w
↓ M
(decimation)
x_d[n] = x[Mn]
-2 -1 0 1 2
X_d(e^jw)
1/MT
−Ω_N MT
Ω_N MT

Decimation



Downsampling

Remark

1. Express $Y(e^{jw})$ in terms of $X(e^{jw})$, if
   (i) $y[n] = x[2n]$, $n = 0, \pm1, \pm2, \ldots$

   (ii) $y[n] = \begin{cases} x[n], n = 0, \pm 2, \pm 4, \ldots \\ 0, \quad \text{otherwise} \end{cases}$

   (iii) $y[n] = \begin{cases} x[n/2], \; n = 0, \pm 2, \pm 4, \ldots \\ 0, \; \text{otherwise} \end{cases}$

(i)

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty} x[2n]e^{-jwn} = \sum_{m \in even} x[m]e^{-jwm/2}$$
$$= \sum_{m=-\infty}^{\infty} \frac{1}{2}\{x[m] + (-1)^m x[m]\}e^{-jwm/2} = \frac{1}{2}X(e^{jw/2}) + \frac{1}{2}X(e^{j(w/2-\pi)})$$

(ii)

$$Y(e^{jw}) = \sum_{n \in even} x[n]e^{-jwn} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\{x[m] + (-1)^m x[m]\}e^{-jwm}$$
$$= \frac{1}{2}X(e^{jw}) + \frac{1}{2}X(e^{j(w-\pi)})$$

(iii)

$$Y(e^{jw}) = \sum_{n \in even} x[n/2]e^{-jwn} = \sum_{m=-\infty}^{\infty} x[m]e^{-j2wm} = X(e^{j2w})$$

2. Express $Y[k]$ in terms of $X[k]$, if $X[k]$ is the N-pt DFT of $x[n]$, $0 \le n \le N-1$, and

(i) $y[n] = x[2n]$, $0 \le n \le N/2 - 1$, find $(N/2)$-pt DFT of $y[n]$

(ii) $y[n] = \begin{cases} x[n], & n = 0, 2, 4, \ldots, (N-2) \\ 0, & \text{otherwise} \end{cases}$, find N-pt DFT of $y[n]$

(iii) $y[n] = \begin{cases} x[n/2], & n = 0, 2, 4, \ldots, (2N-2) \\ 0, & \text{otherwise} \end{cases}$, find $2N$-pt DFT of $y[n]$

(iv) $y[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le 2N-1 \end{cases}$, find $2N$-pt DFT of $y[n]$

(i)

$$Y[k] = \sum_{n=0}^{N/2-1} x[2n] e^{-j\frac{2\pi}{N/2}kn} = \sum_{m=0, m\in even}^{N-2} x[m] e^{-j\frac{2\pi}{N/2}k\frac{m}{2}}$$

$$= \sum_{m=0}^{N-1} \frac{1}{2}\{x[m] + (-1)^m x[m]\} e^{-j\frac{2\pi}{N}km} = \frac{1}{2}X[k] + \frac{1}{2}X[k+\frac{N}{2}] \ ,0 \le k \le N/2-1$$

(ii)

$$Y[k] = \sum_{n=0, n\in even}^{N-2} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \frac{1}{2}\{x[n] + (-1)^n x[n]\} e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{2}X[k] + \frac{1}{2}X[((k+\frac{N}{2}))_N] \ ,0 \le k \le N-1$$

(iii)

$$Y[k] = \sum_{n=0}^{2N-2} y[n] e^{-j\frac{2\pi}{2N}kn} = \sum_{n=0, n\in even}^{2N-2} x[n/2] e^{-j\frac{2\pi}{2N}kn}$$

$$= \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}km} = \begin{cases} X[k] & ,0 \le k \le N-1 \\ X[k-N] & ,N \le k \le 2N-1 \end{cases}$$

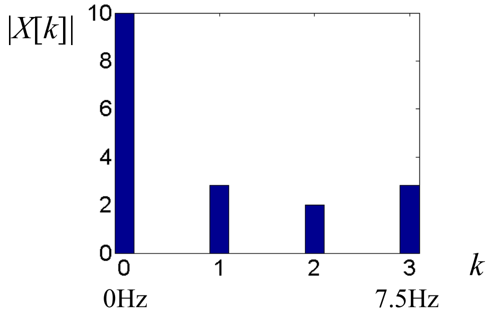
(iv)

$$Y[k] = \sum_{n=0}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}kn} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\frac{k}{2}n} = X[k/2],\ k \in \text{even}.$$

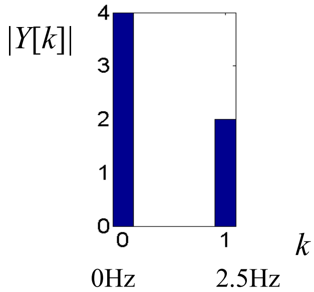
However, it is not easy to express $Y[k]$ in terms of $X[k]$ for $k \in$ odd

If $\mathbf{x}$ = [1 2 3 4], then $\mathbf{X} = \mathrm{DFT}_4(\mathbf{x})$ = [10   -2+2j -2   -2-2j]

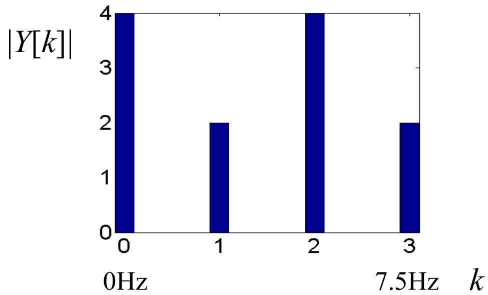
> Suppose $x[n]$ is obtained by sampling $x(t)$ with sampling frequency of 10 Hz.



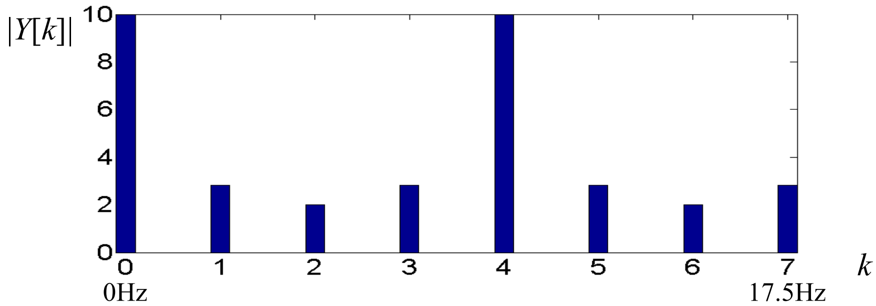
If $\mathbf{y}$ = [1 3], then $\mathbf{Y} = \mathrm{DFT}_2(\mathbf{y})$ = [4 -2]



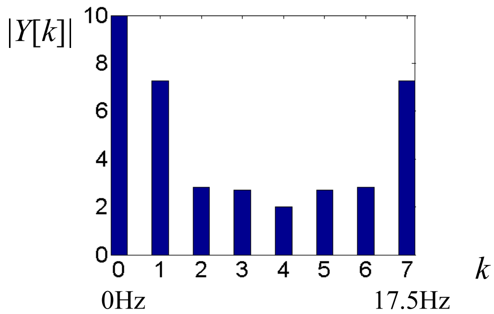
If $\mathbf{y}$ = [1 0 3 0], then $\mathbf{Y} = \mathrm{DFT}_2(\mathbf{y})$ = [4 -2 4 -2]



If $\mathbf{y}$ = [1 0 2 0 3 0 4 0], then $\mathbf{Y} = \mathrm{DFT}_8(\mathbf{y})$ = [10   -2+2j -2   -2-2j 10   -2+2j -2 -2-2j]



If $\mathbf{y}$ = [1 2 3 4 0 0 0 0], then $\mathbf{Y} = \mathrm{DFT}_8(\mathbf{y})$ = [10   -0.4142-7.24j -2+2j 2.4142-1.2426j -2
2.4142+1.2426j -2-2j -0.4142+7.24j]



Chapter 6 Structures for Discrete-Time Systems

6.1 IIR and FIR systems

- Infinite Impulse Response (IIR)

  - If the impulse response function of a system is non-zero over an infinite length of time, the system is called Infinite Impulse Response (IIR).

  - Consider a linear constant-coefficient difference equation

  $$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

  If there is a coefficient $a_k \neq 0$, for $k \neq 0$, the system is IIR. For example, $y[n] - y[n\text{-}1] = x[n] + x[n\text{-}1]$ is an IIR system.

  - Consider a system function

  $$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}.$$

  If there is at least one nonzero pole not cancelled by zero, the system is IIR.

- Finite Impulse Response (FIR)

  - If the impulse response function of a system is non-zero over a finite length of time, the system is called Finite Impulse Response (FIR).

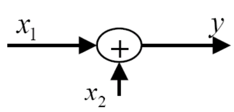
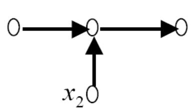
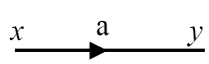
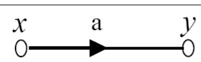
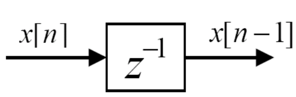
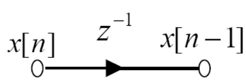
  - An FIR system has the following difference equation and system function

  $$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

  $$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} = b_0 \prod_{k=1}^{M} (1 - c_k z^{-1}).$$

6.2 Block diagram and signal flow graph

- Three elements in LTI discrete-time systems:

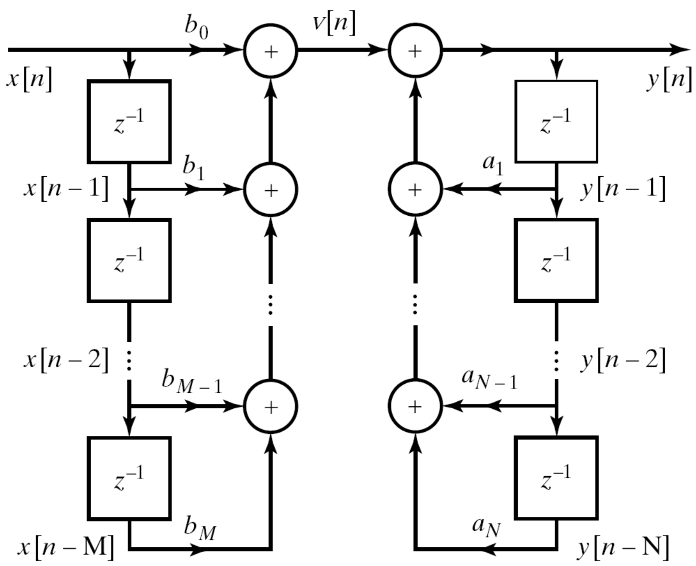
	Block diagram	Signal flow graph
Adder		
Scalar (Multiplication by a constant)		
Unit delay		

6.3 Basic structures for IIR systems

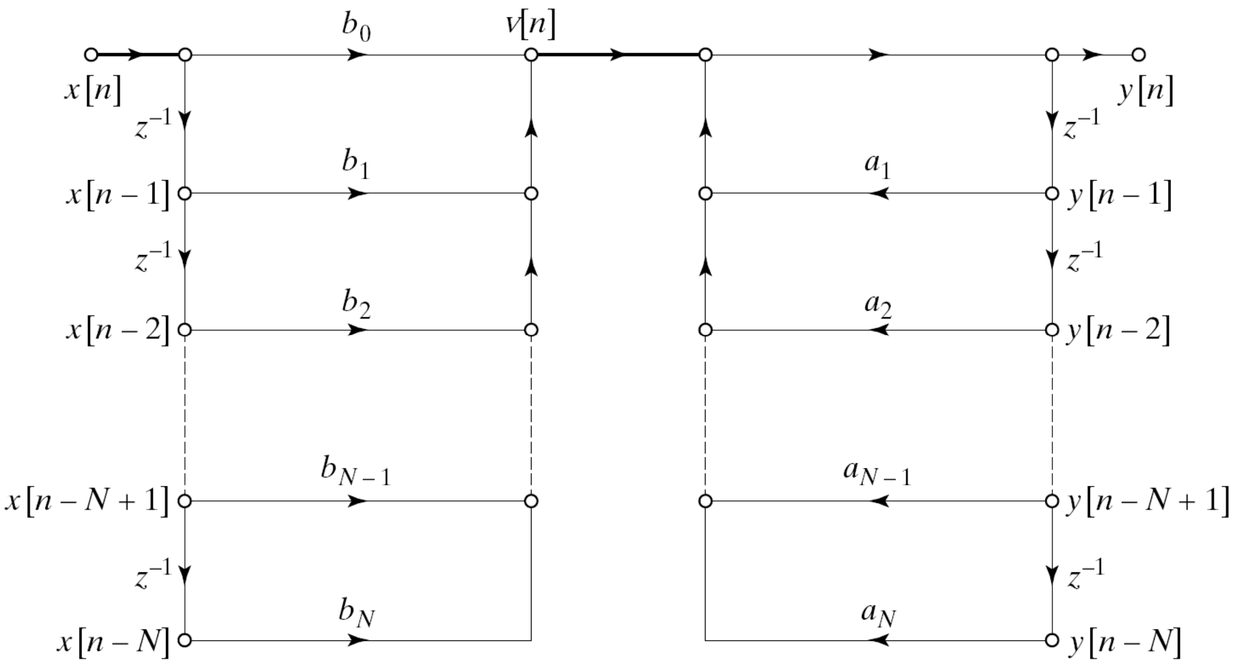
- Types of representation
  - Direct form I
  - Direct form II (Canonic form)
  - Cascade form
  - Parallel form
- Direct form I

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}, \quad y[n]=\sum_{k=1}^{N} a_k y[n-k]+\sum_{k=0}^{M} b_k x[n-k]$$

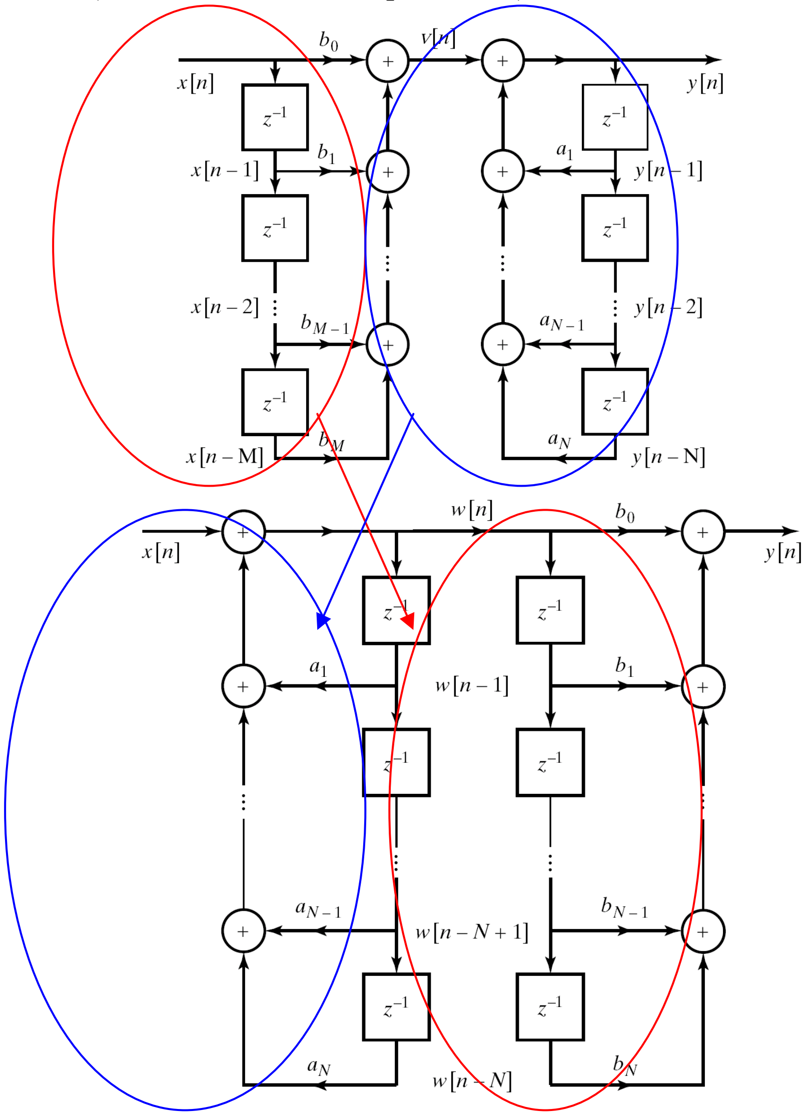
Block diagram

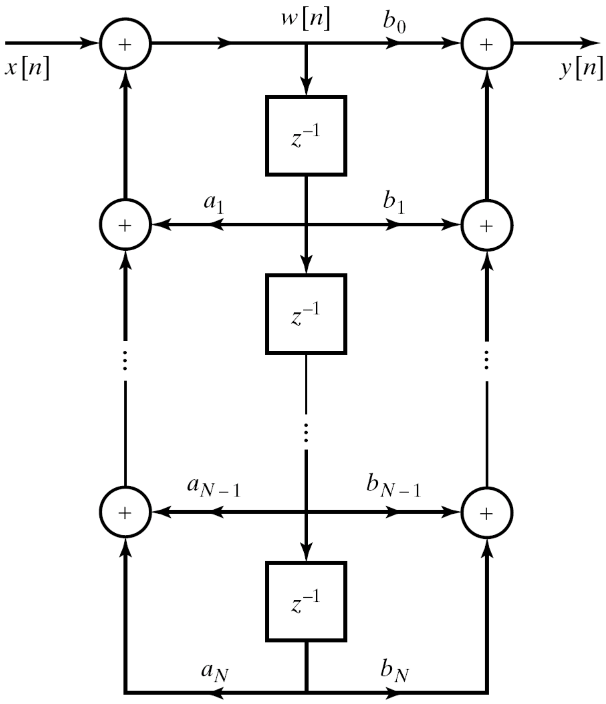


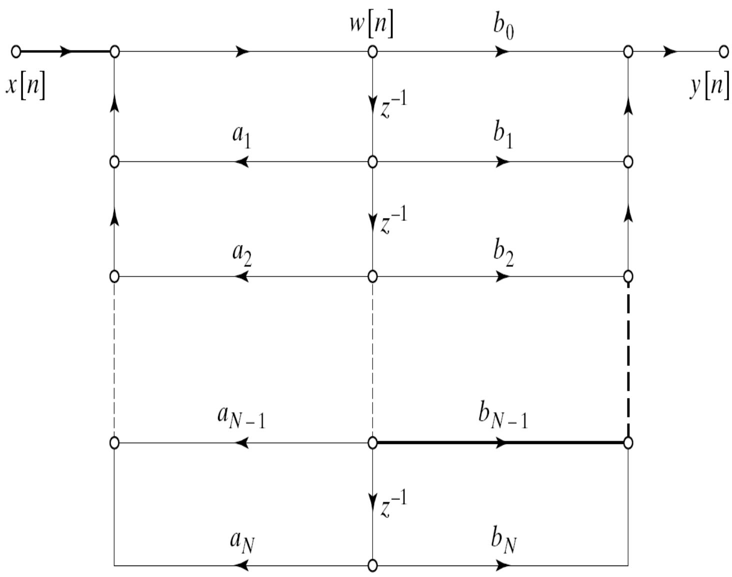
Signal flow graph



- Direct form II (Canonic form: the simplest form)

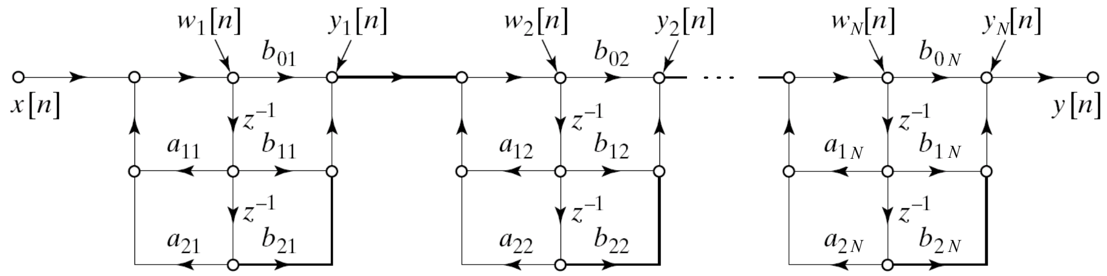






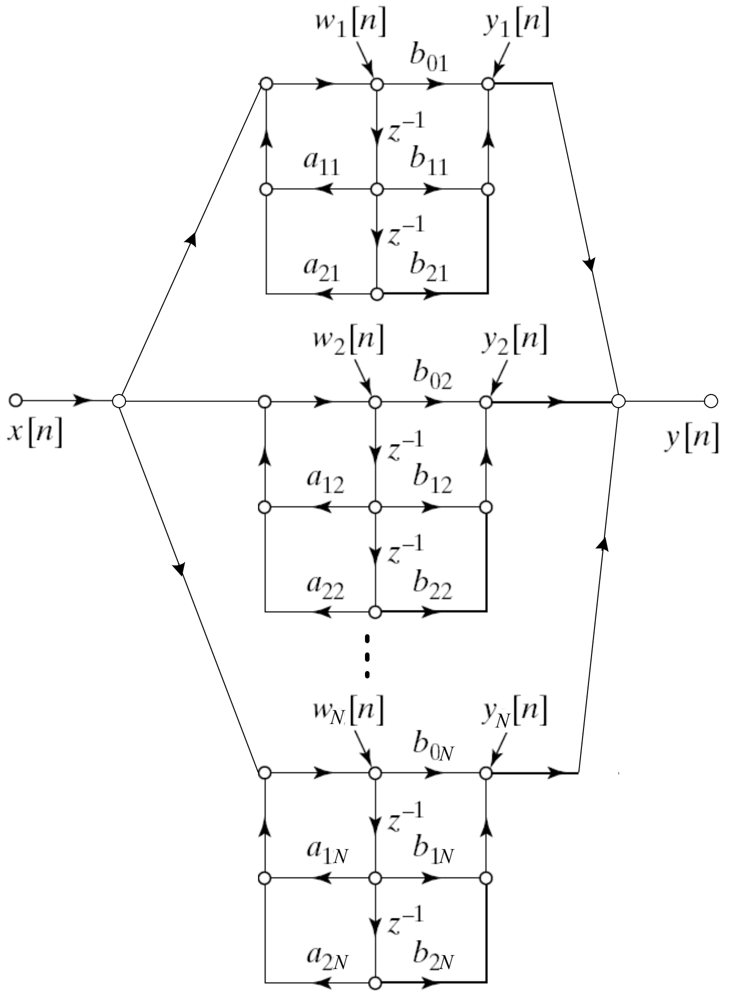
- Cascade form: serial connection of 1st and 2nd order factors

$$H(z) = \prod_{k=1}^{N} H_k(z), \quad H_k(z) = \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$



- Parallel form: parallel connection of 1st and 2nd order factors

$$H(z) = \sum_{k=1}^{N} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$



Example:

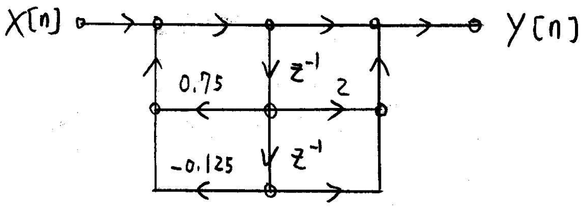
$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} \qquad (1)$$

$$=8+\frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}} \qquad (2)$$

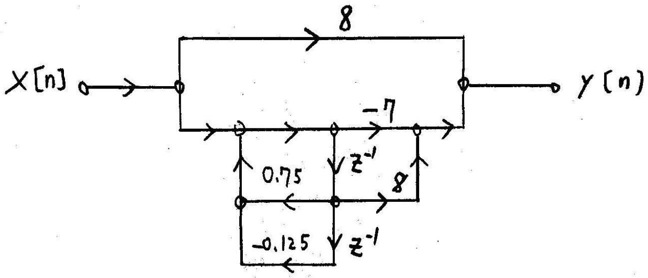
$$=8+\frac{18}{1-0.5z^{-1}}-\frac{25}{1-0.25z^{-1}} \qquad (3)$$

$$=\left(\frac{1+z^{-1}}{1-0.5z^{-1}}\right)\left(\frac{1+z^{-1}}{1-0.25z^{-1}}\right) \qquad (4)$$

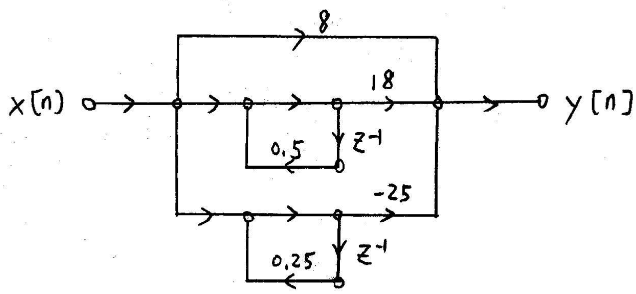
By (1)



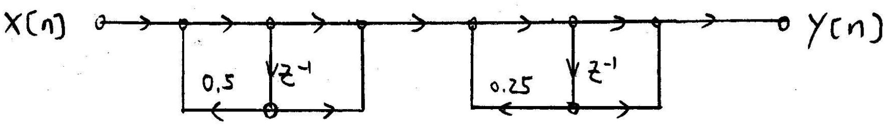
By (2)



By (3)



By (4)

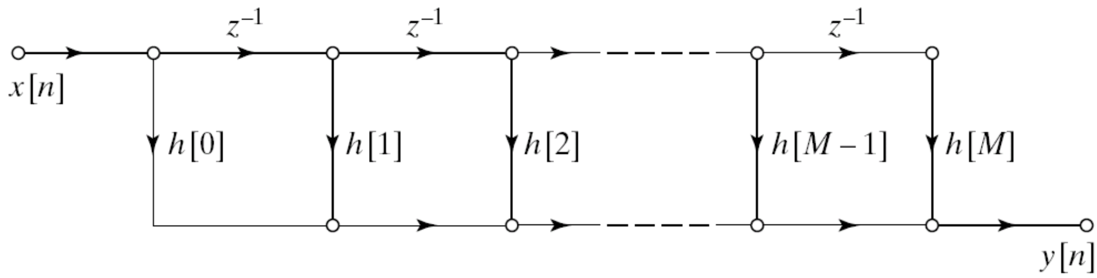


6.4 Basic structures for FIR systems

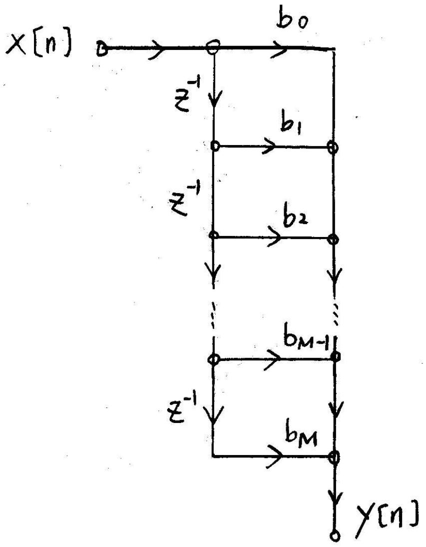
- Types of representation
  - Direct form (Parallel form)
  - Cascade form
  - Symmetric form
  - Lattice form

- Direct form I

$$H(z)=\sum_{k=0}^{M} b_k z^{-k}$$

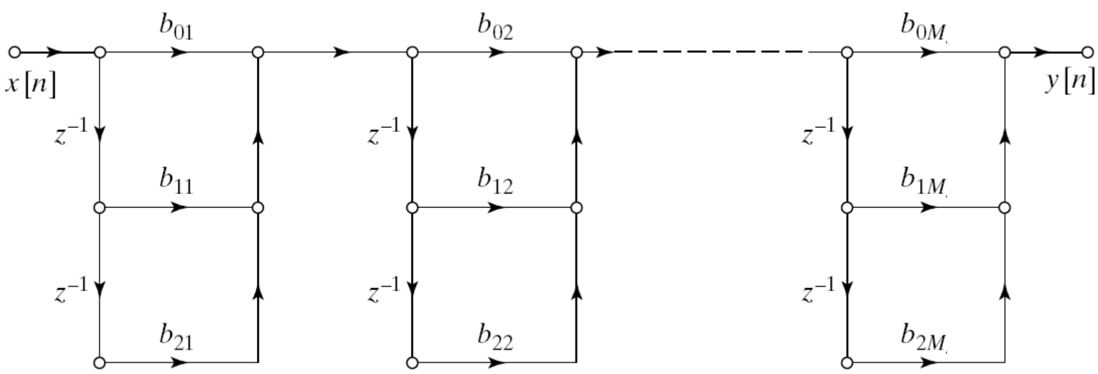


or



- Cascade form

$$H(z)=\prod_{k=0}^{M}(b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2})$$

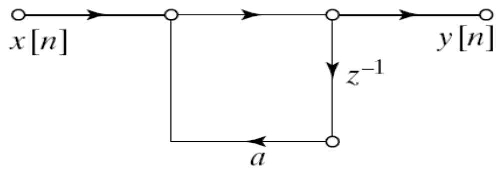


- Transpose:

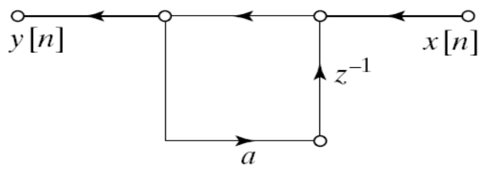
- Reversing the directions of all branches in the network and reversing the roles of the input and output
- The system is unchanged after the transposition

Example 1:

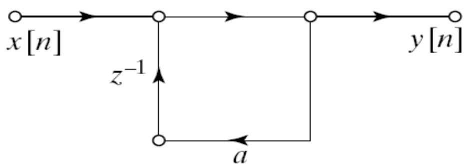
$y[n] = a\, y[n-1] + x[n]$



After transposition

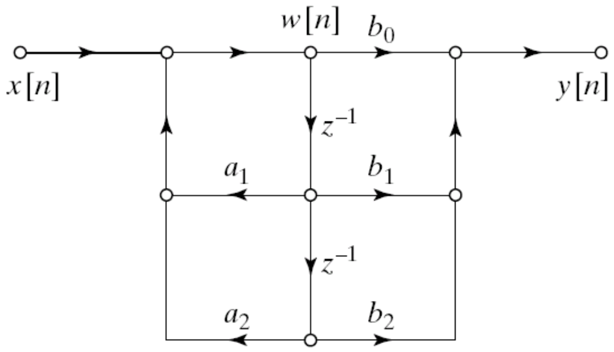


Redrawing with input on left becomes

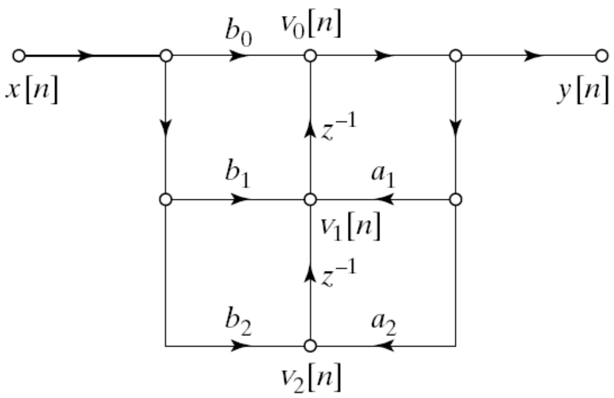


It is easy to write the difference equation: $y[n] = a\, y[n-1] + x[n]$

Example 2:



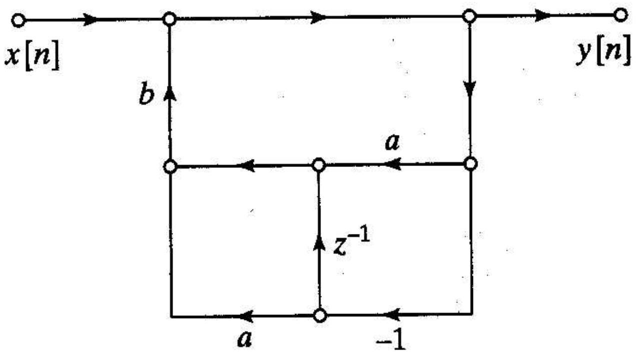
After transposition and redrawing with input on left becomes



It is easy to write the difference equation:

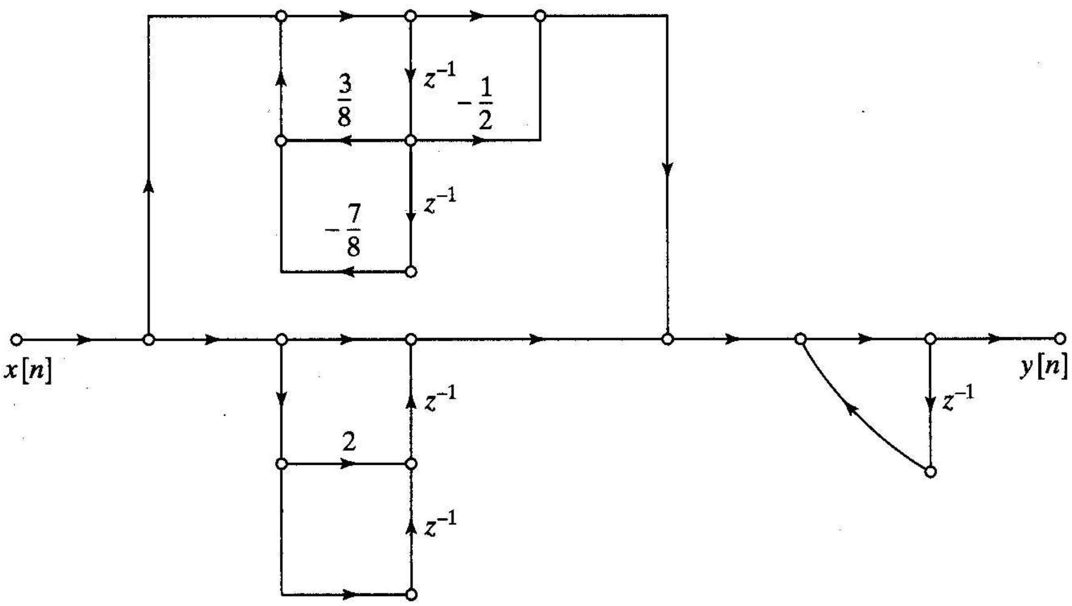
$y[n] = a_1\, y[n-1] + a_2\, y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2\, x[n-2]$

Exercise 1: write the difference equation of the following system.



Answer: $y[n] = -b\,y[n-1] + x[n]$

Exercise 2: find the following system function.

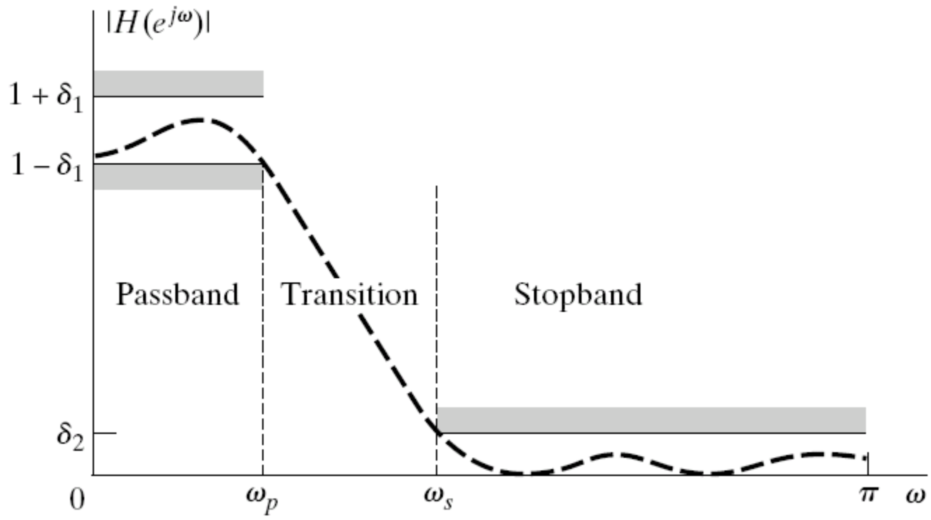


Answer: $$H(z)=\left(\frac{1-\frac{1}{2}z^{-1}}{1-\frac{3}{8}z^{-1}+\frac{7}{8}z^{-2}}+(1+2z^{-1}+z^{-2})\right)\cdot\frac{1}{1-z^{-1}}$$

Chapter 7 Filter Design Techniques

7.1 Filter design basics

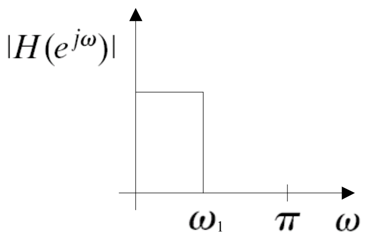
- Three stages in filter design
  - Specification: specifications in frequency domain, usually magnitude only.



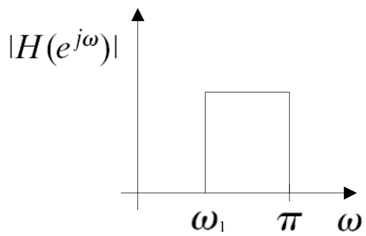
  - Design: approximation of the given specification via a causal system
    That is, finding $H(z)$.
  - Realization: implementation of the architectures and circuits (IC)

- Filter types:

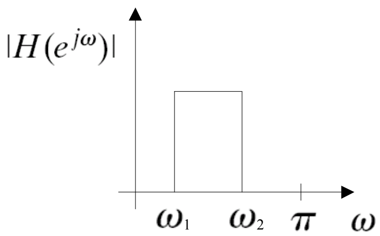
Lowpass



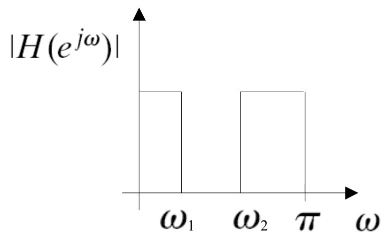
Highpass



Bandpass



Bandstop



7.2 Design of Digital IIR Filters from Analog Filters

Design of digital filter is usually done by transforming an analog filter to digital filter instead of direct design.

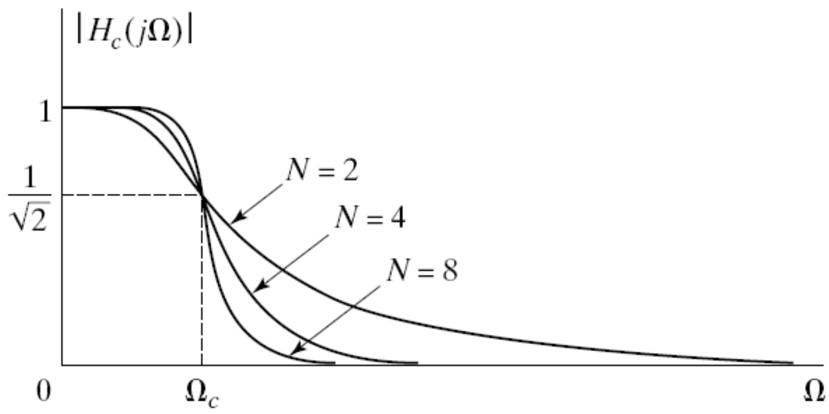
- Why based on analog filters?
  - Analog filter design methods have been well developed.
  - Analog filters often have simple closed-from design formulas, but direct digital filter design methods often have no closed-form formulas.

- There are two types of transformations
  - Transformation from analog to digital
  - Transformation from one type filter (usually lowpass) to another type

- Typical analog filters
  (1) Butterworth lowpass filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}},$$

where N is filter order, and Ω_c is the 3-dB frequency, i.e., magnitude is $1/\sqrt{2}$.



Poles of the filter:

Let $s = j\Omega$. Then, $H_c(s)H_c(-s) = \dfrac{1}{1+(s/j\Omega_c)^{2N}}$

By letting $H_c(s)H_c(-s) = \infty$, we have $1+(s/j\Omega_c)^{2N} = 0$.

Thus, $(s/j\Omega_c)^{2N} = -1 = e^{\pm j\pi} = e^{\pm j\pi + 2k\pi}$, where k is an integer.

The roots of $H_c(s)H_c(-s) = \infty$ are

$$s_k = j\Omega_c e^{j(2k\pi\pm\pi)/2N} = \Omega_c e^{j\pi/2} e^{j(2k\pi\pm\pi)/2N} = \Omega_c e^{(j\pi/2N)(2k+N\pm1)},$$

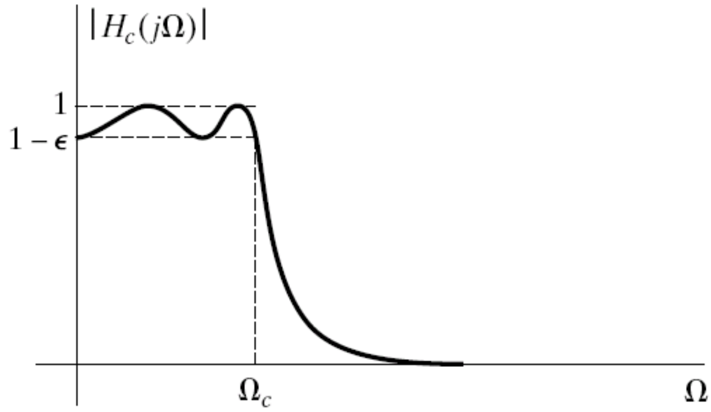
where $k = 0,1,\ldots,(2N\text{-}1)$ (there are $2N$ roots)

If $H_c(s)$ is stable, the ploes of $H_c(s)$ must lie in the left side of s-plane, i.e., the real part of each pole of $H_c(s)$ is negative.

Hence, $H_c(s) = A \prod_{\forall k, \mathrm{Re}(s_k)<0} \dfrac{1}{s - s_k}$, where A is a constant making $|H_c(0)| = 1$.

(2) Chebyshev lowpass filter

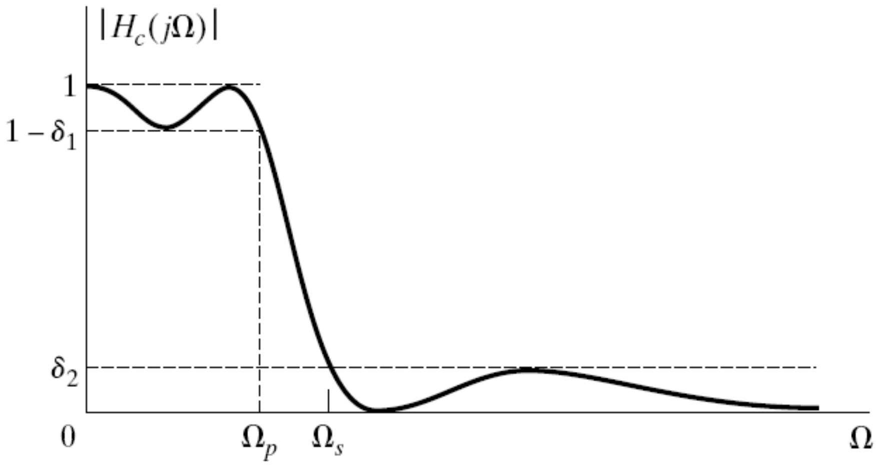
$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)}, \text{ where } V_N(x) = \cos(N\cos^{-1} x)$$



Note: Chebyshev lowpass filter has a sharper transition band than Butterworth filter does, but it has ripples in the passband.

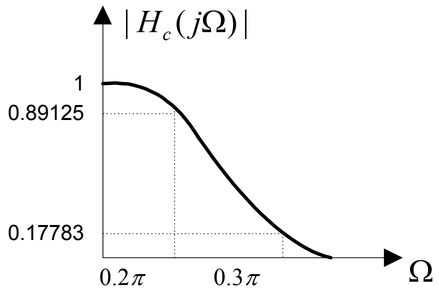
(3) Elliptic lowpass filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^2(\Omega/\Omega_p)}, \text{ where } U_N() \text{ is a Jacobian elliptic function}$$



Note: Elliptic lowpass filter has the smallest (Ω_s - Ω_p), compared to those of the Butterworth lowpass filter and Chebyshev lowpass filter.

Example: Design an analog Butterworth lowpass filter with the following specification: -1dB at 0.2π rad/sec, and -15dB at 0.3π rad/sec



(-1dB = 20 log 0.89125)
(-15dB = 20 log 0.17783)

The task is to find Ω_c and N that fit the specification.

Since $|H_c(j\Omega)|^2 = \dfrac{1}{1+(j\Omega/j\Omega_c)^{2N}}$, we have

$$\begin{cases} 1+\left(\dfrac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.89125}\right)^2 \\ 1+\left(\dfrac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.17783}\right)^2 \end{cases} \Rightarrow \begin{cases} \left(\dfrac{0.2\pi}{\Omega_c}\right)^{2N} = 0.2589 & (1) \\ \left(\dfrac{0.3\pi}{\Omega_c}\right)^{2N} = 30.622 & (2) \end{cases}$$

Dividing (1) by (2), we can solve $N = 5.8858$.
Taking an integer $N = 6$, we obtain $\Omega_c = 0.7032$ rad/sec.

Thus, the roots of $H_c(s)H_c(s) = \infty$ are

$$s_k = \Omega_c e^{(j\pi/2N)(2k+N-1)} = 0.7032\, e^{j\frac{\pi}{12}(2k+5)},\ k = 0,1,\ldots 11.$$

The poles of $H_c(s)$ are those s_k with negative real part:

-0.182 $\pm j$(0.679), -0.497 $\pm j$(0.497), -0.679 $\pm j$(0.182)

Thus, $H_c(s)$ =

$$A\frac{1}{(s+0.182)^2+(0.679)^2}\frac{1}{(s+0.497)^2+(0.497)^2}\frac{1}{(s+0.679)^2+(0.182)^2},$$

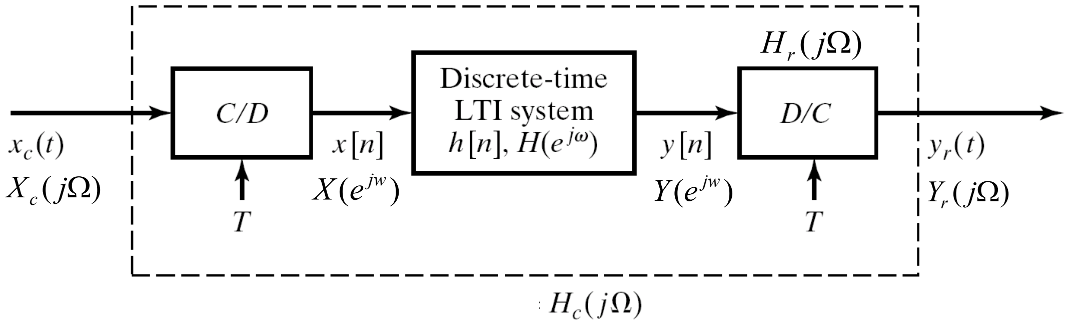
where A can be computed by letting $|H_c(0)| = 1$.

We have

$$H_c(s) = \frac{0.12093}{(s^2+0.364s+0.495)(s^2+0.995s+0.495)(s^2+1.359s+0.495)}$$

- Transformation methods (transforming an analog filter to digital filter)
  - Impulse invariance
  - Bilinear transform

- Impulse invariance



We want to find the relationship between $H_c(j\Omega)$ and $H(e^{jw})$, so that $H(e^{jw})$ can be designed by a transformation of $H_c(j\Omega)$.

$$X(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{w}{T} - j\frac{2\pi}{T}k\right)$$

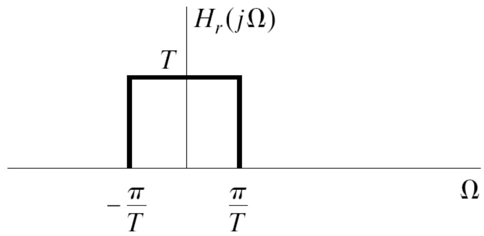
$$Y(e^{jw}) = H(e^{jw})\, X(e^{jw})$$

$$Y_r(j\Omega) = H_r(j\Omega)\, Y(e^{j\Omega T}) = H_r(j\Omega)\, H(e^{j\Omega T})\cdot\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{w}{T} - j\frac{2\pi}{T}k\right)$$

$$= \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| \le \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \quad \text{...(a)}$$

Recall that

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$



In the continuous-time domain, $Y_r(j\Omega) = H_c(j\Omega)\, X_c(j\Omega)$ ....(b)

Comparing Eqs. (a) and (b), we have

$H_c(j\Omega) = H(e^{j\Omega T})$, $|\Omega| \le \frac{\pi}{T}$. Or equivalently, $H(e^{jw}) = H_c(j\frac{w}{T})$, $|w| \le \pi$.

According to the sampling theory, if a discrete-time signal $d[n]$ is sampled from an analog signal $d_c(t)$, with sampling period T, then $d[n] = d_c(nT)$, and

$D(e^{jw}) = \frac{1}{T} D_c(j\frac{w}{T})$, $|w| \le \pi$.

Since $H(e^{jw}) = H_c(j\frac{w}{T})$, $|w| \le \pi$, we have $h[n] = T\, h_c(nT)$. This is the so-called <u>*impulse invariance*</u>

- Analog-digital filter conversion via the impulse invariance

  Let $H_c(s)$ be the frequency response of a well-designed filter.

  Assume $H_c(s)$ is of the form:

  $$H_c(s) = A \prod_{\forall k, \mathrm{Re}(s_k)<0} \frac{1}{s - s_k} = \sum_{k=1}^{K} \frac{B_k}{s - s_k}$$. (by a partial fraction expansion)

  Taking the inverse Laplace transform, we have the impulse response of $H_c(s)$ as

  $$h_c(t) = \sum_{k=1}^{K} B_k e^{s_k t} u(t).$$

  By using the impulse invariance, a digital filter corresponding to $h_c(t)$ is

  $$h[n] = Th_c(nT) = \sum_{k=1}^{K} B_k T e^{s_k nT} u[n] = \sum_{k=1}^{K} B_k T (e^{s_k T})^n u[n]$$

  By taking the z-transform, we have

  $$H(z) = \sum_{k=1}^{K} \frac{B_k T}{1 - e^{s_k T} z^{-1}}.$$

  And, we note that

  $$H_c(s) = \frac{1}{s-a} \xleftarrow{} \underline{\text{Transformation via the impulse invariance}} \rightarrow H(z) = \frac{T}{1 - e^{aT} z^{-1}}$$

Note: impulse invariance is only valid for the band-limited system. This is because if analog filter $H_c(j\Omega)$ is not band-limited, there must be aliasing in the resulting digital filter $H(e^{jw})$. However, in practice, most analog filters are not band-limited.

For example, $H_c(s) = \dfrac{A}{(s-a)(s-b)(s-c)} = \dfrac{D}{s-a} + \dfrac{E}{s-b} + \dfrac{F}{s-c}$

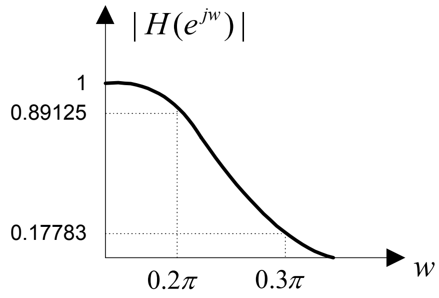
$D(s-b)(s-c) + E(s-a)(s-c) + F(s-a)(s-b) = A$

Let $s = a$, we have $D(a-b)(a-c) = A, \Rightarrow D = A/(a-b)(a-c)$

Let $s = b$, we have $E(b-a)(b-c) = A, \Rightarrow E = A/(b-a)(b-c)$

Let $s = c$, we have $F(c-a)(c-b) = A, \Rightarrow F = A/(c-a)(c-b)$

Example: Using a Butterworth lowpass filter to design a digital lowpass filter with the following specification:



$0.89125 \le |H(e^{jw})| \le 1, \quad 0 \le |w| \le 0.2\pi$

$|H(e^{jw})| \le 0.17783, \quad 0.3\pi \le |w| \le \pi$

Step 1: converting the specification to the analog domain by impulse invariance:

$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le 0.2\pi / T$

$|H_c(j\Omega)| \le 0.17783, \quad 0.3\pi / T \le |\Omega| \le \pi / T$

Step 2: designing a Butterworth filter by determining a proper Ω_c and N that fit the above specification.

By $\begin{cases} 1+\left(\dfrac{0.2\pi}{T\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.89125}\right)^2 \\ 1+\left(\dfrac{0.3\pi}{T\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.17783}\right)^2 \end{cases}$, we can solve $N = 5.8858$.

Taking $N = 6$, we obtain $T\Omega_c = 0.7032$.

Thus, the roots of $H_c(s)H_c(s) = \infty$ are

$$s_k = \Omega_c e^{(j\pi/2N)(2k+N-1)} = \left(\frac{0.7032}{T}\right) e^{j\frac{\pi}{12}(2k+5)}, \; k = 0,1,\ldots 11.$$

The poles of $H_c(s)$ are those s_k with negative real part:

$[-0.182 \pm j(0.679)] / T, \quad [-0.497 \pm j(0.497)] / T, \quad [-0.679 \pm j(0.182)] / T$

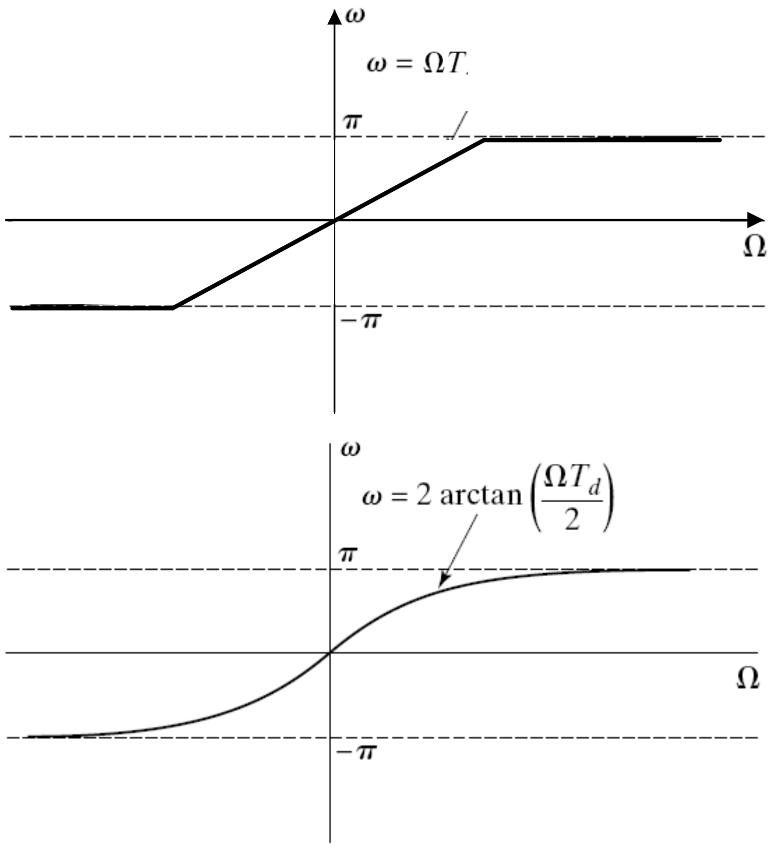
Step 3: the frequency response of the digital filter can be represented by

$$H(z) = \frac{TA_1}{1-e^{0.182+j(0.679)}z^{-1}} + \frac{TA_2}{1-e^{0.182-j(0.679)}z^{-1}} + \frac{TA_3}{1-e^{0.479+j(0.479)}z^{-1}} + \frac{TA_4}{1-e^{0.479-j(0.479)}z^{-1}} + \frac{TA_5}{1-e^{0.679+j(0.182)}z^{-1}} + \frac{TA_6}{1-e^{0.679-j(0.182)}z^{-1}}$$

where A_1, A_2, A_3, A_4, A_5, and A_6 are subject to $|H(e^{j0})| = 1$

- Bilinear transform

The transformation is developed to better handle the problem arising from the fact that analog filters are not bandlimited. Its basic idea is to replace the relationship $w = \Omega T$ by a non-linear mapping $w = 2\tan^{-1}(\Omega T/2)$.



In the impulse invariance, $z \equiv e^{jw} = e^{j\Omega T} = e^{sT}$

In the bilinear transform, $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$. This yields that

$$j\Omega = \frac{2}{T}\left(\frac{1-e^{-jw}}{1+e^{-jw}}\right) = \frac{2}{T}\left[\frac{e^{-jw/2}(e^{jw/2}-e^{-jw/2})}{e^{-jw/2}(e^{jw/2}+e^{-jw/2})}\right] = \frac{2}{T}\left[\frac{j\sin(w/2)}{\cos(w/2)}\right] = j\frac{2}{T}\tan(w/2).$$

Hence, $\Omega = \frac{2}{T}\tan(w/2)$ and $w = 2\tan^{-1}(\Omega T/2)$

Example: On the basis of bilinear transform, design a digital filter with the following specification using a Butterworth lowpass filter.

$0.18925 \le |H(e^{jw})| \le 1, \qquad 0 \le |w| \le 0.2\pi$

$|H(e^{jw})| \le 0.17783, \qquad 0.3\pi \le |w| \le \pi$

Step 1: converting the specification to the analog domain by bilinear transform

$$0.18925 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le \frac{2}{T}\tan(0.2\pi/2)$$

$$|H_c(j\Omega)| \le 0.17783, \qquad \frac{2}{T}\tan(0.3\pi/2) \le |\Omega| \le \infty$$

Step 2: designing a Butterworth filter by determining a proper Ω_c and N that fit the above specification.

$$\text{By } \begin{cases} 1+\left(\dfrac{2\tan(0.1\pi)}{T\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.89125}\right)^2 & (1) \\ 1+\left(\dfrac{2\tan(0.15\pi)}{T\Omega_c}\right)^{2N} = \left(\dfrac{1}{0.17783}\right)^2 & (2) \end{cases}$$

Dividing (1) by (2), we have

$$N = \frac{\log\left\{\left[\left(\dfrac{1}{0.17783}\right)^2 - 1\right] \Big/ \left[\left(\dfrac{1}{0.89125}\right)^2 - 1\right]\right\}}{2\log\left[\dfrac{\tan(0.15\pi)}{\tan(0.1\pi)}\right]} = 5.305$$

Taking $N = 6$, we obtain $T\Omega_c = 0.76622$.

Thus, the roots of $H_c(s)H_c(s) = \infty$ are

$$s_k = \Omega_c e^{(j\pi/2N)(2k+N-1)} = \left(\frac{0.76622}{T}\right) e^{j\frac{\pi}{12}(2k+5)}, \quad k = 0,1,\ldots 11.$$

The poles of $H_c(s)$ are those s_k with negative real part:

$[-0.1998 \pm j(0.7397)] / T$, $[-0.5418 \pm j(0.5418)] / T$, $[-0.7401 \pm j(0.1984)] / T$

$$H_c(s) = A \prod_{\forall k, \mathrm{Re}(s_k)<0} \frac{1}{s - s_k}$$

Step 3: the frequency response of the digital filter can be represented by

$$H(z) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

7.3 Transforming one-type filter to another

Typically, we first design a "frequency-normalized prototype" lowpass filter and then use an algebraic transformation to derive the desired filters from the prototype lowpass filter. The transformation can be represented by

$$H(z) = H_{lp}(\tilde{z})\Big|_{\tilde{z}^{-1}=G(z^{-1})},$$

and $G(z^{-1})$ is of an allpass-like form

$$G(z^{-1}) = \pm\prod_{k=1}^{K}\left(\frac{z^{-1}-\alpha_k}{1-\alpha_k z^{-1}}\right).$$

- Lowpass to lowpass transformation

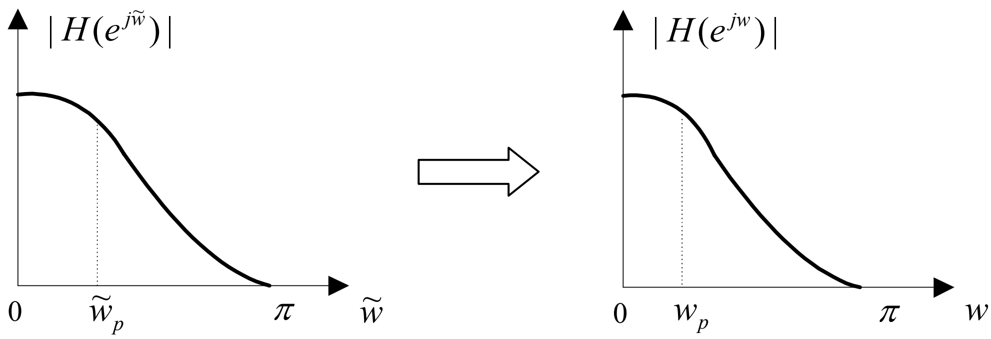
  This transformation is used to change the passband and stopband frequencies while maintain the magnitude of the filter.

  For this purpose, we can choose $\tilde{z}^{-1} = G(z^{-1}) = \dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$

  Then, $\tilde{z}^{-1} = e^{-j\tilde{w}} = \dfrac{e^{-jw}-\alpha}{1-\alpha e^{-jw}}$

  $\Rightarrow\ e^{-j\tilde{w}} - \alpha e^{-j(w+\tilde{w})} = e^{-jw} - \alpha$

  $\Rightarrow\ \alpha = \dfrac{e^{-j\tilde{w}} - e^{-jw}}{e^{-j(\tilde{w}+w)} - 1} = \dfrac{e^{-j(\tilde{w}+w)/2}\left[e^{-j(\tilde{w}-w)/2} - e^{j(\tilde{w}-w)/2}\right]}{e^{-j(\tilde{w}+w)/2}\left[e^{-j(\tilde{w}+w)/2} - e^{j(\tilde{w}+w)/2}\right]} = \dfrac{\sin[(\tilde{w}-w)/2]}{\sin[(\tilde{w}+w)/2]}$



  <Design approach>

  If we have a digital lowpass filter $H_1(\tilde{z})$ with cutoff frequency $\tilde{w}_p$, how to design another filter $H_2(z)$ with cutoff frequency w_p?

  Step 1: Compute $\alpha = \dfrac{\sin[(\tilde{w}_p - w_p)/2]}{\sin[(\tilde{w}_p + w_p)/2]}$

  Step 2: Apply $H_2(z) = H_1(\tilde{z})\Big|_{\tilde{z}^{-1}=\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}}$

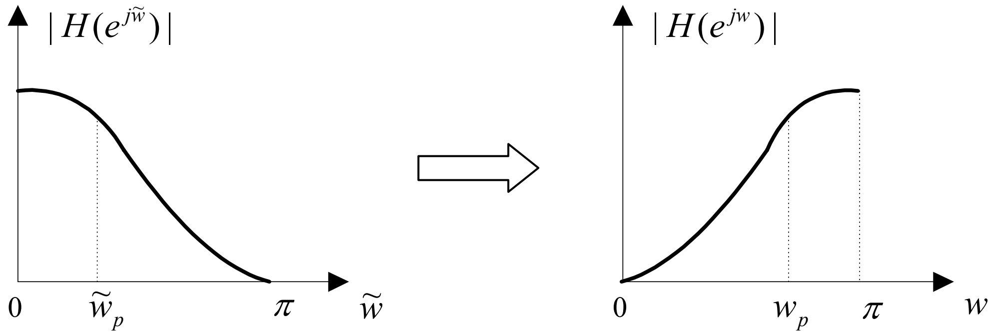
- Typical Transformation

Filter Type	Transformation	Associated Design Formulas
Lowpass	$\widetilde{Z}^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\dfrac{\theta_p - \omega_p}{2}\right)}{\sin\left(\dfrac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$\widetilde{Z}^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\dfrac{\theta_p + \omega_p}{2}\right)}{\cos\left(\dfrac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$\widetilde{Z}^{-1} = -\dfrac{z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + \dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$\widetilde{Z}^{-1} = \dfrac{z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + \dfrac{1-k}{1+k}}{\dfrac{1-k}{1+k} z^{-2} - \dfrac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency

- Lowpass to highpass transformation

We can choose $\tilde{z}^{-1} = G(z^{-1}) = -\dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$ or $-\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$

Then, $\alpha = -\dfrac{\cos[(\tilde{w} + w)/2]}{\cos[(\tilde{w} - w)/2]}$



<Design approach>

If we have a digital lowpass filter $H_1(\tilde{z})$ with cutoff frequency $\tilde{w}_p$, how to design a highpass filter $H_2(z)$ with cutoff frequency w_p?

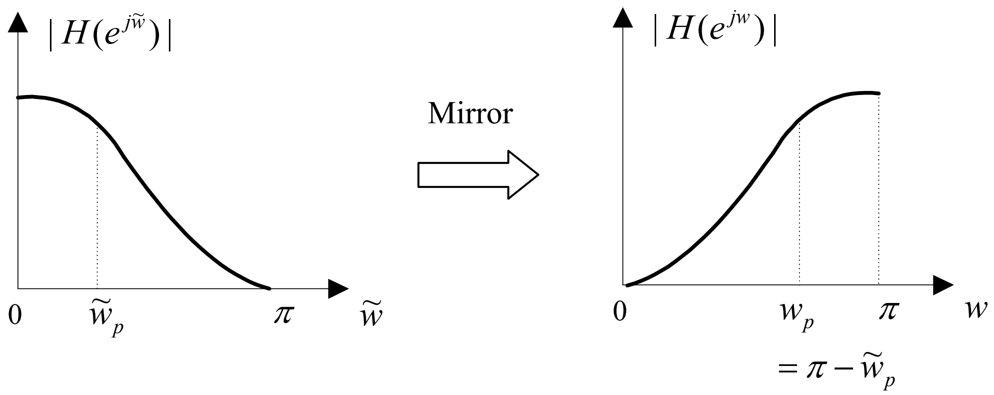
Step 1: Compute $\alpha = -\dfrac{\cos[(\tilde{w}_p + w_p)/2]}{\cos[(\tilde{w}_p - w_p)/2]}$

Step 2: Apply $H_2(z) = H_1(\tilde{z})\Big|_{\tilde{z}^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}}$

A simple way for lowpass-to-highpass conversion is to let $\alpha = 0$. This yields $\tilde{z}^{-1} = -z^{-1}$.

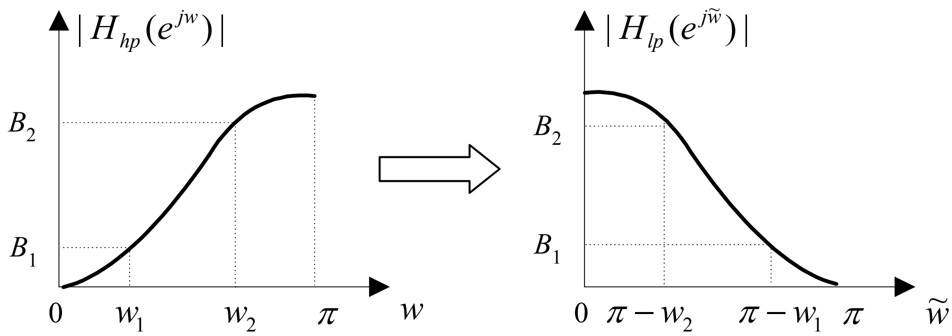
Then, $e^{-j\tilde{w}_p} = -e^{-jw_p} = e^{j\pi} e^{-jw_p} = e^{-j(w_p - \pi)}$, $\Rightarrow$ $\tilde{w}_p = w_p - \pi \equiv \pi - w_p$

$\Rightarrow$ $w_p = \pi - \tilde{w}_p$



- Steps for designing a digital highpass filter using an analogy lowpass filter

Step 1: Converting the specification of the desired digital highpass filter to the specification of a digital lowpass filter by the transform $\tilde{z}^{-1} = -z^{-1}$.



Step 2: Converting the specification of the digital lowpass filter to the specification of an analog lowpass filter using bilinear transform.

$$\Omega = \frac{2}{T}\tan(w/2)$$

Step 3: Determine the order and cutoff frequency of the analog lowpass filter, and find the system function $H_c(s)$.

Step 4: Find $H_{lp}(\tilde{z}) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-\tilde{z}^{-1}}{1+\tilde{z}^{-1}}\right)}$

Step 5: Find $H_{hp}(z) = H_{lp}(\tilde{z})\Big|_{\tilde{z}^{-1}=-z^{-1}}$

7.4 Design of FIR filters by windowing

- Rectangular window

  The simplest way to obtaining a causal FIR filter is to truncate an IIR filter $h_{IIR}[n]$ as $h[n] = \begin{cases} h_{IIR}[n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$, or

  $h[n] = h_{IIR}[n]\, \mathrm{w}[n]$,

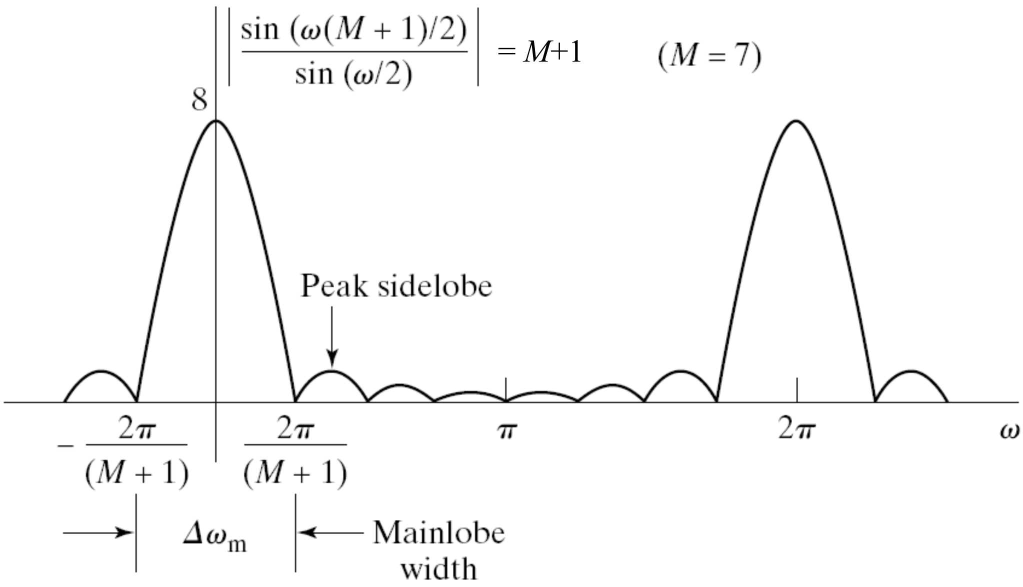
  where $\mathrm{w}[n] = \begin{cases} 1, 0 \le n \le M \\ 0, \text{otherwise} \end{cases}$ is a rectangular window.

  The frequency response of the rectangular window is

  $$\mathrm{W}(e^{jw}) = \sum_{n=0}^{M} e^{-jwn} = \frac{1-e^{-jw(M+1)}}{1-e^{-jw}} = e^{-jwM/2}\frac{\sin[w(M+1)/2]}{\sin(w/2)}.$$

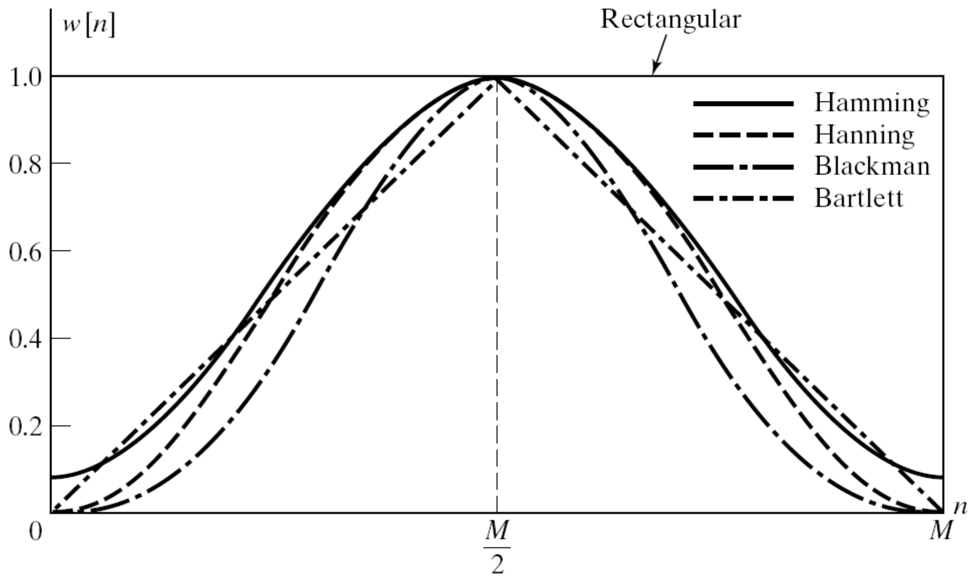
  Letting $\dfrac{\sin[w(M+1)/2]}{\sin(w/2)} = 0$, we have $w(M+1) = 2\pi \Rightarrow \ w = \dfrac{2\pi}{M+1}$

  When $w = 0$, $|\mathrm{W}(e^{j0})| = \left.\dfrac{\sin[w(M+1)/2]}{\sin(w/2)}\right|_{w=0} = M+1$



As M increases, the mainlobe width decreases, and the mainlobe peak increases; however, the undesired sidelobe amplitude increases correspondingly.

- Other windows – tradeoff between sidelobe peak and mainlobe width



Bartlett (triangular) Window:

$$w[n] = \begin{cases} \dfrac{2n}{M}, & 0 \le n \le \dfrac{M}{2} \\ 2 - \dfrac{2n}{M}, & \dfrac{M}{2} < n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hanning Window:

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\dfrac{2n\pi}{M}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hamming Window:

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2n\pi}{M}\right),, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Blackman Window:

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\dfrac{2n\pi}{M}\right) + 0.08\cos\left(\dfrac{2n\pi}{M}\right), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

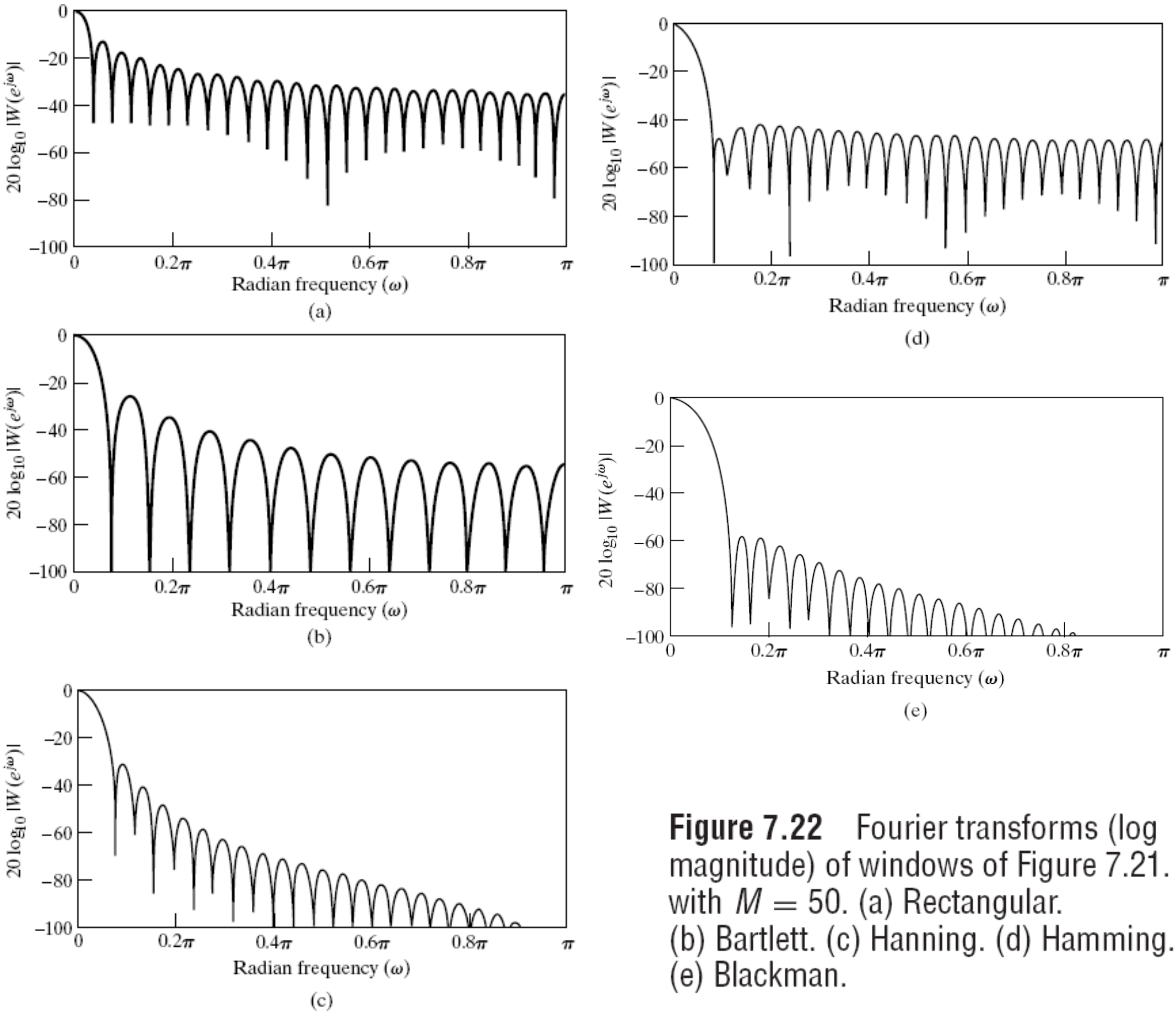


Figure 7.22 Fourier transforms (log magnitude) of windows of Figure 7.21. with $M = 50$. (a) Rectangular. (b) Bartlett. (c) Hanning. (d) Hamming. (e) Blackman.

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$