Measure Theory

Lectures by Claudio Landim

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Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. Taylor published by Cambridge University Press.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the TikZ package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

0. $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{+} \cup \{+\infty\}$
1. $\lambda((a, b]) = b - a$
2. $A \subseteq \mathbb{R},\ A + x = \{x + y :\ y \in A\},\ \ \forall A,\ A \subseteq \mathbb{R},\ \forall x \in \mathbb{R} :$

$$\lambda(A + x) = \lambda(A) \tag{1.1}$$

3. $A = \bigcup\limits_{j \geqslant 1} A_j,\ \ A_j \cap A_k = \varnothing :$

$$\lambda(A) = \sum_{k} \lambda(A_k) \tag{1.2}$$

Definition 1.1. $x \sim y,\ x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R},\ y - x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_{\sim}$, only one point represents the equivalence class of Ω , like α, β.

Ω is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0, 1)$

Claim 1.1. $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \varnothing \end{cases} \quad q, p \in \mathbb{Q}$

Proof. Assume that $\Omega + q \cap \Omega + q \neq \varnothing$ then, $x = \alpha + p = \beta + q,\ \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \varnothing]$. □

Claim 1.2. $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \subseteq (-1, 2) \tag{1.3}$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda(E) \leqslant \lambda(F)$

Proof. $\because E \subseteq F \therefore F = E \cup (F \backslash E),\ E \cap (F \backslash E) = \varnothing$, then $\lambda(F) = \lambda(E) + \lambda((F \backslash E)) \Rightarrow \lambda(F) \geqslant \lambda(E)$. □

Then,

$$\lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) \leqslant \lambda((-1, 2)) = 3 \tag{1.4}$$

and ,

$$\infty \cdot \lambda((\Omega + q)) = \infty \cdot \lambda(\Omega) \leq 3 \Rightarrow \lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) = 0 \tag{1.5}$$

Claim 1.4. $(0,1) \subseteq \sum\limits_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)$

Proof. $\forall$ fixed $x \in (0,1)$, $\exists \alpha \in [x] \cap \Omega,\ \alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q},\ - < q < 1 \Rightarrow x = \alpha + q,\ x \in \Omega + q$ $\square$

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda\left(\sum_{q\in\mathbb{Q}} \Omega + q\right) = 0 \tag{1.6}$$

it is impossible.

Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Definition 2.1. $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is semi-algebra if:

1. $\Omega \subseteq \mathcal{S}$
2. $A, B \in \mathcal{S} \Rightarrow A \bigcap B \in \mathcal{S}$
3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum_{i=1}^{n} E_j, \ \ \exists E_1, \cdots, E_n \in \mathcal{S}, \ E_i, E_j \, (i \neq j)$ disjoint sets, n is finite number

Example 2.1. $\Omega = \mathbb{R}, \ \ \mathcal{S} = \{\mathbb{R}, \{(a,b), a < b, a, b \in \mathbb{R}\}, \{(-\infty, b], b \in \mathbb{R}\}, \{(a, \infty), a \in \mathbb{R}\}, \varnothing\}$, $(a,b]^c = (-\infty, a] \cup [b, +\infty)$

Example 2.2. $\Omega = \mathbb{R}^2$

$\mathcal{S} = \ \left\{\mathbb{R}^2, \{(a_1, b_1) \times (a_2, b_2), \ a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2], b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty), a_i \in \mathbb{R}\}, \varnothing\right\}$

Definition 2.2. $a = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

Remark 2.1. a algebra $\Rightarrow$ a semi-algebra

Definition 2.3. σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \subseteq \mathcal{S}$
2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap_{j \geqslant 1} A_j \in \mathcal{S}$
3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

Remark 2.2. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)$, a_α algebra, $\alpha \in I \Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is an algebra.

Proof. check the followings

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

□

Remark 2.3. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega), \alpha \in I, a_\alpha$, σ-algebra $\Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is a σ-algebra

Proof. check the followings

1. $\Omega \in a$
2. $A_j, j \geq 1 \in a \Rightarrow \bigcap_{j \geqslant 1} A_j \in a$

3. $A \in a \Rightarrow A^c \in a$

$\square$

Definition 2.4 (minimal algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.1}$$

Remark 2.4. $a(c)$ exits, and $a = a(c) = \bigcap_{\alpha} a_\alpha,\ \forall \alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an algebra.

Definition 2.5 (minimal σ-algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a σ-algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is σ-algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.2}$$

Remark 2.5. $a(c)$ exits, and $a = a(c) = \bigcap_{\alpha} a_\alpha,\ \forall \alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an σ-algebra.

Lemma 2.1. *Ω, f semi-algebra $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ algebra generated by f then*

$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leqslant j \leqslant n,\ A = \sum_{j=1}^{n} E_j \tag{2.3}$$

Proof.

1. $\Leftarrow$

   $A = \sum_{j=1}^{n} E_j,\ E_j \in f \in a(f)$

   By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. $\Rightarrow$

   $A \in a(f) \Rightarrow A = \sum_{j=1}^{n} E_j, E_j \in f$

   Then by remark 2.7, it will be proved easily.

$\square$

Remark 2.6. $E, J \in a, E \bigcup F \in a, E \bigcup F = (E^c \bigcap F^c)^c$

Remark 2.7. $\mathcal{B} = \left\{ \sum_{j=1}^{n} F_j,\ F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. $\mathcal{B}$ algebra
2. $\mathcal{B} \supseteq f$
3. $\mathcal{B} \supseteq a(f)$

Proof. We only prove that $\mathcal{B}$ algebra, then check the following

1. $\Omega \in \mathcal{B}$
2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

   $\because A, B \in \mathcal{B}, \therefore A = \sum_{j=1}^{n} E_j,\ E_j \in f,\ B = \sum_{k=1}^{m} F_k,\ F_k \in f$, then

$$\begin{aligned} A \cap B &= \left(\sum_{j=1}^{n} E_j \right) \cap \left(\sum_{k=1}^{m} F_k \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \\ &\in \mathcal{B} \end{aligned} \tag{2.4}$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

   $A = \sum_{j=1}^{n} E_j,\ E_j \in f$

   By definition 2.1:

$$\begin{aligned} E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1},\ F_{1,j} \in f \\ \cdots &= \cdots \\ E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i},\ F_{i,j} \in f \end{aligned} \tag{2.5}$$

   Then, we get that

$$\begin{aligned} A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1} \right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2} \right) \cap \cdots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n} \right) \\ &= \sum_{k_1=1}^{l_1} \sum_{k_2=1}^{l_2} \cdots \sum_{k_n=1}^{l_n} (F_{1,k_1} \cap F_{2,k_2} \cap F_{n,k_n}) \\ &\in \mathcal{B} \end{aligned} \tag{2.6}$$

□

Definition 2.6. $c \subseteq \mathcal{P}(\Omega), \varnothing \in c,\ \mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}$. μ is additive if

1. $\mu(\varnothing) = 0$
2. $E_1, E_2, ..., E_n \in c,\ E = \sum_{j=1}^{n} E_j \in c \Rightarrow \mu(E) = \sum_{j=1}^{n} \mu(E_k)$

Remark 2.8.

$$\exists A \in c,\ \mu(A) < \infty,\ A = A \cup \varnothing,\ \mu(A) = \mu(A) + \mu(\varnothing) \Rightarrow \mu(\varnothing) = 0 \tag{2.7}$$

Remark 2.9. $c,\ \mu : c \to \mathbb{R}_+ \bigcup +\infty,\ E \subseteq F,\ F \backslash E \in c,\ E, F \in c$

$$F = E \cup (F \backslash E),\ \mu(F) = \mu(E) + (F \backslash E) \tag{2.8}$$

1. $\mu(E) = +\infty$, $\mu(F) = +\infty$
2. $\mu(E) < +\infty$, $\mu(F \backslash E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leqslant \mu(F) \tag{2.9}$$

Example 2.3. Discrete measure: Ω, $c \subseteq \mathcal{P}(\Omega)$, $\{x_j,\ j \geqslant 1\},\ x_j \in \Omega,\ \{p_j,\ j \geqslant 1\},\ p_j, \geqslant 0,\ A \in c$, define that

$$\mu(A) = \sum_{j \geqslant 1} p_j 1 \{x_j \in A\} \tag{2.10}$$

then μ is additive

Definition 2.7. $c \in \mathcal{P}(\Omega), \varnothing \in c$, $\mu:\ c \to \mathbb{R}_+ \bigcup +\infty$, μ is σ-additive if

1. $\mu(\varnothing) = 0$
2. $E_j \in c,\ j \neq k, E_j \bigcap E_k = \varnothing,\quad E = \sum\limits_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j \geq 1} \mu(E_j)$

Example 2.4. $\Omega = (0, 1),\ c = \{(a, b],\ 0 \leqslant a < b < 1\},\ \mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a, b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \tag{2.11}$$

$(a, b] = \sum\limits_{j=1}^{n} (a_j, b_j)$, we can get that μ is NOT σ-additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow\rightarrow 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geqslant 1} (x_{j+1}, x_j] = +\infty \tag{2.12}$$

it is impossible.

Lecture 3

Set Functions

Definition 3.1. $c \subseteq \mathcal{P}(\Omega)$, $\mu : c \to \mathbb{R}_+ \bigcup +\infty$:

1. $E \in c$, μ continuous from below at E, if $\forall (E_n)_{n\geqslant 1}$, $E_n \in c$, $E_n \uparrow E$ $\left(E_n \subseteq E_{n+1}, \bigcup_{n\geqslant 1} E_n = E\right)$:
$$\mu(E_n) \to \mu(E) \tag{3.1}$$

2. $E \in c$, μ continuous from above at E, if $\forall (E_n)_{n\geqslant 1}$, $E_n \in c$, $E_n \downarrow E$ $\left(E_{+1} \subseteq E_n, \bigcap_{n\geqslant 1} E_n = E\right)$, and $\exists n_0,\ \mu(E_{n_0}) < \infty$:
$$\mu(E_n) \to \mu(E) \tag{3.2}$$

Remark 3.1. For a sequence $E_1, E_2, ...$ of sets, we put

$$\limsup E_i = \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right), \liminf E_i = \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \tag{3.3}$$

and if $\{E_i\}$ is such that $\limsup E = \liminf E_i$ we say that the sequence converges to the set

$$E = \limsup E = \liminf E_i \tag{3.4}$$

Remark 3.2. 2 need $\exists n_0,\ \mu(E_{n_0}) < \infty$, if not:

$$E_n = [n, +\infty),\ \mu(E_n) = +\infty,\ E_n \downarrow \varnothing,\ \lambda(\varnothing) = 0 \tag{3.5}$$

Lemma 3.1. *$a \subseteq \mathcal{P}(\Omega)$, algebra; $\mu :\ a \to \mathbb{R}_+ \cup \{+\infty\}$, additive;*

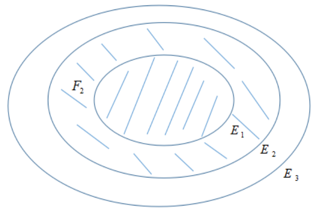
1. *μ is σ-additive $\Rightarrow$ μ continuous at E, $\forall E \in a$*
2. *μ is continuous from below $\Rightarrow$ μ is σ-additive*
3. *μ is continuous from above at $\varnothing$& μ is finite $\Rightarrow$ σ-additive*

Proof.

1.

 (i) μ is σ-additive $\Rightarrow$ μ conti. from below at $E \in a$. $E \in a, E_n \uparrow E,\ E_n \in a$:

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ \vdots &= \vdots \\ F_n &= E_n \backslash E_{n-1} \end{aligned} \tag{3.6}$$



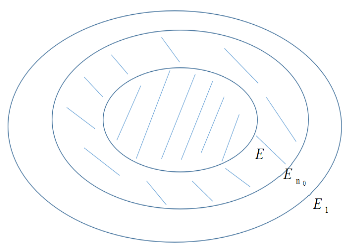
and we can get that

$$F_j \cap F_k = \varnothing, \quad \sum_{k=1}^{n} F_k = E_n, \quad \bigcup_{n\geqslant 1} E_n = \bigcup_{n\geqslant 1} F_n \tag{3.7}$$

so

$$\mu(E) = \sum_{k\geqslant 1} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu(E_n) \tag{3.8}$$

(ii) μ cont. from above $E \in a, E_n \in a, E_n \downarrow E, \mu(E_{n_0}) < \infty \Rightarrow \mu(E_n) \downarrow \mu(E)$



$$\begin{aligned} G_1 &= E_{n_0} \backslash E_{n_0+1} \\ G_2 &= E_{n_0} \backslash E_{n_0+2} \\ \vdots &= \vdots \\ G_k &= E_{n_0} \backslash E_{n_0+k} \end{aligned} \tag{3.9}$$

then $G_k \uparrow E_{n_0} \backslash E, G_k \in a \Rightarrow \mu(G_k) \uparrow \mu(E_{n_0} \backslash E)$, so

$$\begin{aligned} \mu(E_{n_0} \backslash E) &= \lim_{n\to\infty} \mu(E_{n_0} \backslash E_{n_0+k}) \\ \mu(E_{n_0} \backslash E) &= \mu(E_{n_0}) - \mu(E) \\ \mu(E_{n_0}) - \mu(E) &= \lim_{k\to\infty} (\mu(E_{n_0}) - \mu(E_{n_0+k})) \end{aligned} \tag{3.10}$$

2. μ cont. below, $E = \sum_{k\geqslant 1} E_k,\ E, E_k \in a$.

   Obs.

$$\sum_{k=1}^{n} E_k \subseteq E \overset{additive}{\Rightarrow} \begin{cases} \mu\left(\sum_{k=1}^{n} E_k\right) \leqslant \mu(E) \\ \sum_{k=1}^{n} \mu(E_k) \leqslant \mu(E) \end{cases} \tag{3.11}$$

then

$$\sum_{k\geqslant 1}\mu\left(E_k\right)\leqslant\mu\left(E\right) \tag{3.12}$$

$F_n=\sum\limits_{k=1}^{n}E_k\in a,\ F_n\uparrow E,$

$$\sum_{k=1}^{n}\mu\left(E_k\right)=\mu\left(F_n\right)\uparrow\mu\left(E\right)\Rightarrow\sum_{k\geqslant 1}\mu\left(E_k\right)=\mu\left(E\right) \tag{3.13}$$

3. μ cont. at $\varnothing$, $\mu\left(\Omega\right)<\infty$, $E,E_k\in a, E=\sum\limits_{k\geqslant 1}E_k$.

$$F_n=\sum_{k\geqslant m}E_k\in a\ \left(E\backslash\sum_{j=1}^{n-1}E_j\right) \tag{3.14}$$

$F_n\downarrow\varnothing,\mu\left(F_1\right)<\infty,\ \mu\left(F_n\right)\to 0$

$$\begin{aligned}\mu\left(E\right)&=\mu\left(\sum_{k=1}^{n}E_k\cup\sum_{k>n}E_k\right)\\&=\underbrace{\mu\sum_{k=1}^{n}E_k}_{\to\sum\limits_{k\geqslant 1}\mu(E_n)}+\underbrace{\mu\sum_{k>n}E_k}_{\to 0}\\&\to\sum_{k\geqslant 1}\mu\left(E_n\right)\end{aligned} \tag{3.15}$$

□

Remark 3.3. Suppose E_α, $\alpha\in I$ is a class of subsets of X, and E_i is one set of the class, then

1. $\bigcap\limits_{\alpha\in I}E_\alpha\subseteq E_i\subseteq\bigcup\limits_{\alpha\in I}E_\alpha$
2. $X-\bigcup\limits_{\alpha\in I}E_\alpha=\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$
3. $X-\bigcap\limits_{\alpha\in I}E_\alpha=\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$

Proof.

1. This is immediate from the definition.
2. Suppose $x\in X-\bigcup\limits_{\alpha\in I}E_\alpha$ then $x\in X$ and x is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is x is not in any E_α, $\alpha\in I$ so that $x\in X-E_\alpha$ for every $\alpha\in I$, and $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$. Conversely if $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$, then for every $\alpha\in I$, x is in X but not in E_α, so $x\in X$ but x is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is $x\in\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$.
3. Similar to 2

Remark 3.3 (2) and (3) are also called as de Morgan's Law. □

Example 3.1. $(0,1), (a,b], 0 \leqslant a < b < 1$

$$\mu(a,b] = \begin{cases} b-a, & a>0 \\ +\infty, & a=0 \end{cases} \tag{3.16}$$

μ is additive but NOT σ-additive

Proof. $E_n \downarrow \varnothing,\ \mu(E_{n_0}) < \infty,\ E_n = (a_{n,1}, b_{n,1}] \cup \cdots \cup (a_{n,k_n}, b_{n,k_n}], a_{n,j} < a_{n,j+1}$.

$\begin{cases} a_{n,1} = 0, & \forall n \\ a_{n_0} > 0, & \textit{some } n_0 \end{cases}$ □

Theorem 3.1 (Extension). *$f \subseteq \mathcal{P}(\Omega)$ semi-algebra, $\mu : f \to \mathbb{R}_+ \cup \{\infty\}$ σ-additive, then $\exists \nu$:*

$$\nu : a(f) \to \mathbb{R}_+ \cup \{\infty\} \tag{3.17}$$

such that:

1. *ν σ-additive*
2. *$\nu(A) = \mu(A), \ \forall A \in f$*
3. *$\mu_1, \mu_2, a(f) \to \mathbb{R}_+ \bigcup \{+\infty\}$, then $\mu_1(A) = \mu_2(A), \forall A \in s \Rightarrow \mu_1(E) = \mu_2(E), \forall E \in a(f)$*

Proof. $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f$ by Lemma 2.1.

$$\nu(A) \overset{add}{=} \sum_{j=1}^{n} \nu(E_j) \overset{ext}{=} \sum_{j=1}^{n} \mu(E_j) \tag{3.18}$$

we define that

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) \tag{3.19}$$

we want to show that $\nu(A) = \sum\limits_{j=1}^{n} \mu(E_j)$ is well-defined:

1. ν is unique

$$\begin{aligned} A &= \sum_{j=1}^{n} E_j,\ E_j \in f \\ &= \sum_{k=1}^{m} F_k,\ F_k \in f \end{aligned} \tag{3.20}$$

then we will prove that

$$\begin{aligned} \nu(A) &= \sum_{j=1}^{n} \mu(E_j) \\ &= \sum_{k=1}^{m} \mu(F_k) \end{aligned} \tag{3.21}$$

$$\because E_j \subseteq A = \sum_{k=1}^{m} F_k \Rightarrow E_j = E_j \cap \left(\sum_{k=1}^{m} F_k\right) = \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \tag{3.22}$$

$$\therefore \mu(E_j) = \mu\left(\sum_{k=1}^{m} (E_j \cap F_k)\right) \tag{3.23}$$

then

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{n}\sum_{k=1}^{m} \mu(E_j \cap F_k) = \sum_{k=1}^{m} \mu(F_k) \tag{3.24}$$

2. ν is an additive, $\nu(A) = \sum\limits_{j=1}^{n} \mu(E_j)$

   Assume that

$$\begin{cases} A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f \\ B = \sum\limits_{k=1}^{m} F_k,\ F_k \in f \end{cases}, A \cap B = \varnothing \tag{3.25}$$

   We will show that

$$\nu(A \cup B) = \nu(A) + \nu(B) \tag{3.26}$$

$$\because A \cup B = \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \tag{3.27}$$

   therefore

$$\begin{aligned} \nu(A \cup B) &= \mu\left(\sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k\right) \\ &= \sum_{j=1}^{n} \mu(E_j) + \sum_{k=1}^{m} \mu(F_k) \\ &= \nu(A) + \nu(B) \end{aligned} \tag{3.28}$$

3. $\nu(A) = \mu(A),\ A \in f$ by Eq 3.19

4. ν is uniqueness, we want to show that:

   Suppose that $\mu_1, \mu_2 :\ a(f) \to R_+ \cup \{+\infty\}, \forall A\ \in f, \mu_1, \mu_2$ *additive*, then

$$\mu_1(A) = \mu_2(A) \Rightarrow \mu_1(B) = \mu_2(B),\ \forall B \in a(f) \tag{3.29}$$

   $\because B \in a(f), \therefore B = \sum\limits_{j=1}^{n} \mu_1(E_j),\ E_j \in f$

$$\mu_1(B) = \sum_{j=1}^{n} \mu_1(E_j) = \sum_{j=1}^{n} \mu_2(E_j) = \mu_2(B) \tag{3.30}$$

Now we proof the extension of σ-additive, ie: $\mu-\sigma$ additive, f semi-algebra, $\nu-\sigma$ additive, $a(f)$ is a algebra generated by f. we want to show that

$$A=\sum_{j\geqslant 1} A_j,\ A, A_j \in a\left(f\right) \Rightarrow \nu\left(A\right)=\sum_{j\geqslant 1} \nu\left(A_j\right) \tag{3.31}$$

by representation of an algebra:

$$A=\sum_{j=1}^{m} E_j, E_j \in f; \quad A_k=\sum_{l=1}^{m_k} E_{k,l}, E_{k,l} \in f \tag{3.32}$$

by Eq 3.19:

$$\nu\left(A\right)=\sum_{j=1}^{m} \nu\left(E_j\right), \quad \nu\left(A_k\right)=\sum_{l=1}^{m_k} \nu\left(E_{k,l}\right) \tag{3.33}$$

$$\because E_j=E_j \cap A=E_j \cap\left(\sum_{k\geqslant 1} A_k\right)=E_j \cap\left(\sum_{k\geqslant 1} \sum_{l=1}^{m_k} E_{k,l}\right)=\sum_{k\geqslant 1} \sum_{l=1}^{m_k}\left(E_j \cap E_{k,l}\right) \tag{3.34}$$

therefore

$$\begin{aligned} \nu\left(A\right) &= \sum_{j=1}^{n} \mu\left(E_j\right) \\ &= \sum_{j=1}^{n} \sum_{k\geqslant 1} \sum_{l=1}^{m_k} \mu\left(E_j \cap E_{k,l}\right) \\ &= \sum_{k\geqslant 1} \underbrace{\sum_{l=1}^{m_k} \mu\left(E_{k,l}\right)}_{\subseteq A_k} \end{aligned} \tag{3.35}$$

Eq 3.35 holds because:

$$E_{k,l}=E_{k,l} \cap A=\sum_{j=1}^{n}\left(E_{k,l} \cap E_j\right) \tag{3.36}$$

and

$$\mu\left(E_{k,l}\right)=\sum_{j=1}^{n} \mu\left(E_{k,l} \cap E_j\right) \tag{3.37}$$

so we can get that

$$\nu\left(A\right)=\sum_{k\geqslant 1} \nu\left(A_k\right) \tag{3.38}$$

□

Lecture 4

Caratheodory Theorem

Intuition:

$$\begin{array}{llr} \sigma - add & \mu : f \to \mathbb{R}_+ \cup \{+\infty\} & f \subseteq \mathcal{P}(\Omega), f \ is \ semialgebra \\ \quad\downarrow & \qquad\qquad\downarrow & \\ \sigma - add & \nu : a(f) \to \mathbb{R}_+ \cup \{+\infty\} & a(f) \ algebra \ generated \ by \ f \\ \quad\downarrow & \qquad\qquad\downarrow & \\ \sigma - add & \pi : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\} & \mathcal{F}(a) \ is \ \sigma - algebra \ generated \ by \ algebra \end{array} \tag{4.1}$$

The big picture of the proof to the Caratheodory Theorem:

1. Define the π^* outer measure:
$$\pi^* = \inf_{\{E_i\}} \sum_{i \geqslant 1} \nu(E_i) \tag{4.2}$$
2. $\mathcal{M}$ σ-algebra, $\mathcal{M} \supseteq \mathcal{F}(a)$
3. $$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.3}$$
is σ-additive, and
$$\pi^*|_a = \nu \tag{4.4}$$
4. (uniqueness) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \bigcup \{+\infty\}$, Ω is σ-finite(μ_1), if $E_j \uparrow \Omega$, $\mu_1(E_j) < \infty, \forall j,\ E_j \in a$ and $\mu_1|_a = \mu_2|_a$ then implies that
$$\mu_1 = \mu_2 \tag{4.5}$$

$$\pi^* :\ \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.6}$$

We will prove π^* is an outer measure.

And we will construct a family of subsets $\mathcal{M}$

$$\mathcal{M} \subseteq \mathcal{P}(\Omega) \tag{4.7}$$

we will also prove $\mathcal{M}$ satisfies the following:

1. $\mathcal{M}$ is a $\sigma-$algebra
2. $\mathcal{M} \supseteq a$
3. $\pi^*|_{\mathcal{M}}$ $\sigma-$additive
4. $\pi^*|_a = \nu$

Next, we will define π^* and $\mathcal{M}$.

Step 1

Definition 4.1 (π^*)**.** $\pi^* : \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$, $A \in \Omega$, $\{E_i, i \geqslant 1\}, E_i \in a, A \subseteq \bigcup E_i$, $\{E_i\}$ is a covering of A, then we define that

$$\pi^* = \inf_{\{E_i\},A} \sum_{i \geqslant 1} \nu(E_i) \tag{4.8}$$

where $\nu : \ a(f) \to \mathbb{R}_+ \cup \{+\infty\}, \ is$ σ-additive.

Definition 4.2 (Outer measure)**.** $\mu : \ c \to \mathbb{R}_+ \cup \{+\infty\}, c \subseteq P(\Omega), \varnothing \in c$, μ is a outer measure if

1. $\mu(\varnothing) = 0$
2. (monotone) $E \subseteq F, \ E, F \in c \ \Rightarrow \mu(E) \leqslant \mu(F)$
3. (subadditive) $E, E_i \in c, \ E \subseteq \bigcup\limits_i E_i \Rightarrow \mu(E) \leqslant \sum\limits_i \mu(E_i)$

Theorem 4.1. *π^* in 4.1 is a outer measure.*

Proof. We will check the conditions in Def 4.2.

1. check $\pi^*(\varnothing) = 0$

   (a) $E_i = \varnothing, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$ then

   $$\pi^*(\varnothing) = \inf_{\{E_i\},\varnothing} \sum_{i \geqslant 1} \nu(E_i) \leqslant \sum_{i \geqslant 1} \nu(E_i) = 0 \tag{4.9}$$

   (b) $E_i \in a, \{E_i\}, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$, then

   $$\sum_{i \geqslant 1} \nu(E_i) \geqslant 0 \Rightarrow \pi^*(\varnothing) \geqslant 0 \tag{4.10}$$

2. check $E \subseteq F, \ \pi^*(E) \leqslant \pi^*(F)$

   Let's take any covering of F:$\{E_i\}, E_i \in a, F \subseteq \bigcup\limits_{i \geqslant 1} E_i$ is also a covering of E, then

   $$\pi^*(E) = \inf_{\{E_i\},E} \sum_{i \geqslant 1} \nu(E_i) \leqslant \pi^*(F) = \inf_{\{E_i\},F} \sum_{i \geqslant 1} \nu(E_i) \tag{4.11}$$

3. check $E \subseteq \bigcup\limits_{i \geqslant 1} E_i, \ \ \pi^*(E) \leqslant \sum\limits_{i \geqslant 1} \pi^*(E_i)$

   (a) $\pi^*(E_i) = \infty$ then

   $$\pi^*(E) \leqslant \sum_{i \geqslant 1} \pi^*(E_i) \tag{4.12}$$

   (b) $\pi^*(E_i) < \infty$, then

   $$\pi^*(E_i) = \inf_{\{H_{ik}\}, \ E_i} \sum_{k \geqslant 1} \nu(H_{ik}) \tag{4.13}$$

then there $\exists \{H_{ik}\} \in a, E_i \subseteq \bigcup\limits_{k\geqslant 1} H_{ik}$ such that

$$\pi^*(E_i) = \inf_{\{H_{ik}\},\, E_i} \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \pi^*(E_i) + \frac{\varepsilon}{2^i} \tag{4.14}$$

$\{H_{ik}\}$ is a covering of E, then

$$\pi^*(E) \leqslant \sum_{i,k} \nu(H_{ik}) \leqslant \sum_{i\geqslant 1} \left(\pi^*(E_i) + \frac{\varepsilon}{2^i}\right) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) + \varepsilon \tag{4.15}$$

so

$$\pi^*(E) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) \tag{4.16}$$

□

Step 2

Definition 4.3 (Measurable set $\mathcal{M}$)**.** A set called measurable set $\mathcal{M}$ if $A \in \mathcal{M}$ $\forall E \in \Omega$, we have that

$$\pi^*(E) = \pi^*\left(E \bigcap A\right) + \pi^*\left(E \bigcap A^c\right) \tag{4.17}$$

Theorem 4.2. *If $\mathcal{M}$ definited as Def 4.3, then*

1. $\mathcal{M} \supseteq a$
2. $\mathcal{M}$ *is a* $\sigma-$*algebra*

Remark 4.1.

$$E \subseteq (E \cap A) \cup (E \cap A^c) \Rightarrow \pi^*(E) \leqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.18}$$

so we only to check $\geq$ in Eq 4.17

Proof. π^* is an outer measurable by Thm 4.1, then by subadditive of outer measure. □

Now we proof Thm 4.2.

Proof.

1. $a \in \mathcal{M}$

   Suppose that $A \in a$, $E \in \Omega$, we will show that

   $$\pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.19}$$

   assume that $\pi^*(E) < \infty$, given $\varepsilon, \exists \{E_i\}, E,$ *such that* $E_i \in a, E \subseteq \bigcup\limits_{i\geqslant 1} E_i$, then

   $$\pi^*(E) \leqslant \sum_{i\geqslant 1} \nu(E_i) \leqslant \pi^*(E) + \varepsilon \tag{4.20}$$

$E_i \cap A \in a, E \cap A \subseteq \bigcup_{i \geqslant 1} (E_i \bigcap A)$, so

$$\begin{aligned} \pi^* (E \cap A) &\leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right) \\ \pi^* (E \cap A^c) &\leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right) \end{aligned} \tag{4.21}$$

so

$$\pi^* (E \cap A) + \pi^* (E \cap A^c) \leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right) + \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right) \leq \sum_{i \geqslant 1} \nu (E_i) \leqslant \pi^* (E) + \varepsilon \tag{4.22}$$

2. $\mathcal{M}$ is σ-algebra.

   We need to show that

   (a) $\Omega \in \mathcal{M}$

   It is clearly that:

$$\pi^* (E) = \pi^* (E \cap \Omega) + \pi^* (E \cap \Omega^c) \tag{4.23}$$

   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$

$$\because \pi^* (E) = \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.24}$$

   (c) $A_i \in \mathcal{M} \Rightarrow \bigcup_{i \geqslant 1} A_i \subseteq \mathcal{M}$

   Finite union is closed: $A, B \in \mathcal{F} \Rightarrow A \bigcup B \in M$. Let's take $E \subseteq \Omega$. We will proof that

$$\pi^* (E) \geqslant \pi^* \left(E \cap \left(A \bigcup B \right) \right) + \pi^* \left(E \cap \left(A \bigcup B \right)^c \right) \tag{4.25}$$

   $\because A \in \mathcal{M}$,

$$\therefore \quad \pi^* (E) = \pi^* \left(E \bigcap A \right) + \pi^* \left(E \bigcap A^C \right) \tag{4.26}$$

   $\because B \in \mathcal{M}$

$$\begin{aligned} \therefore \quad \pi^* (E \backslash A) &= \pi^* (E \backslash A \cap B) + \pi^* (E \backslash A \cap B^c) \\ &= \pi^* (E \backslash A \cap B) + \pi^* \left(E \backslash \left(A \bigcup B \right) \right) \end{aligned} \tag{4.27}$$

   then

$$\pi^* (E) = \pi^* (E \cap A) + \pi^* (E \backslash A \cap B) + \pi^* (E \backslash (A \cup B)) \tag{4.28}$$

   We want to show

$$\pi^* (E \cap A) + \pi^* (E \backslash A \cap B) \geqslant \pi^* (E \cap (A \cup B)) \tag{4.29}$$

   By π^* is subadditive, we only to show that

$$E \cap (A \cup B) \subseteq (E \cap A) \cup (E \backslash A \cap B) \tag{4.30}$$

   this is because

$$E \cap (A \cup B) = \underbrace{\{[E \cap (A \cup B)] \cap A\}}_{\subseteq E \cap A} \bigcup \underbrace{\{[E \cap (A \cup B)] \cap A^c\}}_{\subseteq (E \cap A^c) \cap B = (E \backslash A) \bigcap B} \tag{4.31}$$

Then Eq 4.25 holds. So $\mathcal{M}$ is closed by finite(countable) union.

Now, we will show that countable infinite union is also closed. $A_i \in \mathcal{M}$, we want to show $A = \bigcup_{j \geqslant 1} A_j \in \mathcal{M}$, take $E \subseteq \Omega$,

$$\pi^* (E) \geqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.32}$$

by Eq. 4.25, $\forall\ n$ we know that

$$\begin{aligned} \pi^* (E) &= \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j^c \right) \right) \\ &\geq \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* (E \backslash A) \end{aligned} \tag{4.33}$$

$\geq$ holds in Eq 4.33 because $(E \backslash A) \subseteq \left(E \backslash \left(\bigcup_{j=1}^{n} A_j \right) \right)$.

Now, we define

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_1 \backslash A_2 \\ F_3 &= A_1 \backslash (A_2 \cup A_3) \\ &\vdots \\ F_n &= A_1 \backslash (A_2 \cup \cdots \cup A_{n-1}) \\ &\vdots \end{aligned} \tag{4.34}$$

It is clear that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} F_j \tag{4.35}$$

where $F_j \cap F_k = \varnothing, F_j \in \mathcal{M}$.

Then Eq 4.33 can be written as

$$\pi^* (E) \geqslant \pi^* \left(E \cap \sum_{j=1}^{n} F_j \right) + \pi^* (E \backslash A) \tag{4.36}$$

By Remark 4.2, we have

$$\begin{aligned} \pi^* (E) &\geqslant \pi^* \left(E \cap \left(\sum_{j=1}^{n} F_j \right) \right) + \pi^* (E \backslash A) \\ &= \sum_{j=1}^{n} \pi^* (E \cap F_j) + \pi^* (E \backslash A) \end{aligned} \tag{4.37}$$

$\because$ n is any number in Remark 4.2 , $\therefore$ $\pi^*\left(E\cap\sum\limits_{j=1}^{\infty}F_j\right)=\sum\limits_{j=1}^{\infty}\pi^*\left(E\cap F_j\right)$, by π^* is subadditive

$$\begin{aligned}
\pi^*(E) &\geqslant \pi^*\left(E\cap\sum_j F_j\right)+\pi^*(E\backslash A)\\
&=\sum_{j\geqslant 1}\pi^*(E\cap F_j)+\pi^*(E\backslash A)\\
&\geqslant \pi^*\left(\bigcup_{j\geqslant 1}(E\cap F_j)\right)+\pi^*(E\backslash A)\\
&=\geqslant \pi^*\left(E\cap\left(\bigcup_{j\geqslant 1}F_j\right)\right)+\pi^*(E\backslash A)\\
&=\pi^*(E\cap A)+\pi^*(E\backslash A)
\end{aligned}\tag{4.38}$$

So $\mathcal{M}$ is a $\sigma-$algebra.

$\square$

Remark 4.2. $\forall n$, we have that

$$\pi^*\left(E\cap\sum_{j=1}^{n}F_j\right)=\sum_{j=1}^{n}\pi^*(E\cap F_j)\tag{4.39}$$

where F_j defined as Eq 4.34.

Proof. By induction

1. $n=1$, Eq 4.39 holds
2. Support n holds then we will proof $n+1$ holds. $F_k\in\mathcal{M}, F_{n+1}\in\mathcal{M}$, we have that

$$\begin{aligned}
\pi^*\left(E\cap\sum_{j=1}^{n+1}F_j\right)&=\pi^*\left(\left(E\cap\sum_{j=1}^{n+1}F_j\right)\cap F_{n+1}\right)+\pi^*\left(\left(E\cap\sum_{j=1}^{n+1}F_j\right)\cap F_{n+1}^c\right)\\
&=\pi^*(E\cap F_{n+1})+\underbrace{\pi^*\left(E\cap\sum_{j=1}^{n}F_j\right)}_{by\ assumption\ =\sum_{j=1}^{n}\pi^*(E\cap F_j)}\\
&=\sum_{j=1}^{n+1}\pi^*(E\cap F_j)
\end{aligned}\tag{4.40}$$

$\square$

By Thm 4.2 we have that $\mathcal{M}\supseteq\mathcal{F}(a)$.

Step 3

Theorem 4.3. *$\pi^*: \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$ is $\sigma-$ additive, then*

$$\pi^*(A) = \nu(A), \quad \forall A \in a \tag{4.41}$$

Remark 4.3. Eq 4.41 is also

$$\pi^*|_a = v \tag{4.42}$$

Eq 4.2 holds because Thm 4.2, $a \in \mathcal{M}$.

Proof. (Thm 4.3)

1. $\pi^*(A) = \nu(A), \; \forall A \in a$
   (a) check $\pi^*(A) \leqslant \nu(A)$

   Let's $\underbrace{A}_{E_1}, \underbrace{\varnothing}_{E_2}, \underbrace{\varnothing}_{E_3}, \underbrace{\cdots}_{E_j}$

$$\pi^*(A) = \inf_{\{E_i\},A} \sum_i \nu(E_i) \leqslant \sum_i \nu(E_i) = \nu(A) \tag{4.43}$$

   (b) check $\pi^*(A) \geqslant \nu(A)$

   Let's take

$$\begin{aligned}
F_1 &= E_1 \\
F_2 &= E_2 \backslash E_1 \\
F_3 &= E_3 \backslash (E_1 \cup E_2) \\
&\vdots \\
F_n &= E_n \backslash (E_1 \cup E_2 \cup \cdots \cup E_{n-1}) \\
&\vdots
\end{aligned} \tag{4.44}$$

   $F_j \in a, \bigcup_j F_j = \bigcup_j E_j, F_j \cap F_k = \varnothing, A \subseteq \bigcup_{j \geqslant 1} F_j$, so $A = \sum_j F_j \cap A \in a$.

   Because ν is $\sigma-$additive we have that

$$\nu(A) = \sum_{j \geqslant 1} \nu(F_j \cap A) \tag{4.45}$$

   $\because F_j \subseteq E_j$

$$\nu(A) = \sum_{j \geqslant 1} \nu(F_j \cap A) \leqslant \sum_{j \geqslant 1} \nu(E_j) \tag{4.46}$$

   so

$$\nu(A) \leqslant \inf_{\{E_i\},A} \sum_{j \geqslant 1} \nu(E_j) = \pi^*(A) \tag{4.47}$$

   Then, we can get

$$\pi^*(A) = \nu(A), \; \forall A \in a \tag{4.48}$$

2. $\pi^*|_{\mathcal{M}}$ is $\sigma-$additive

   Suppose that $A_j \in \mathcal{M}, A_j \cap A_k = \varnothing$, we want to proof that

$$\pi^*\left(\sum A_j\right) = \sum_{j\geqslant 1} \pi^*\left(A_j\right) \tag{4.49}$$

   (a) check $\pi^*\left(\sum A_j\right) \leqslant \sum\limits_{j\geqslant 1} \pi^*\left(A_j\right)$ by π^* is an outer measure, π^* is subadditive

   (b) check $\pi^*\left(\sum A_j\right) \geqslant \sum\limits_{j\geqslant 1} \pi^*\left(A_j\right)$

   by π^* is an outer measure, π^* is monotone

$$\pi^*\left(\sum_{j\geqslant 1} A_j\right) \geqslant \pi^*\left(\sum_{j=1}^{n} A_j\right) \tag{4.50}$$

   by Remark 4.2, we have that

$$\pi^*\left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \pi^*\left(A_j\right), \quad \forall n \tag{4.51}$$

   so

$$\pi^*\left(\sum_{j\geqslant 1} A_j\right) \geqslant \sum_{j\geqslant 1} \pi^*\left(A_j\right) \tag{4.52}$$

□

Step 4

Definition 4.4. Ω is σ-finite(μ_1) if $E_j \uparrow \Omega, \mu_1\left(E_j\right) < \infty, \ \forall j, \ E_j \in a$.

Theorem 4.4 (Uniqueness)**.** *Suppose that $\mu_1, \mu_2 : \mathcal{F}\left(a\right) \to R_+ \cup \{+\infty\}, \Omega$ is σ-finite(μ_1), if $\mu_1|_a = \mu_2|_a$, then*

$$\mu_1 = \mu_2, \quad \textit{on} \ \ \mathcal{F}(a) \tag{4.53}$$

Definition 4.5. Ω, $\mathcal{G} \subseteq \mathcal{P}\left(\Omega\right), \mathcal{G}$ is a monotone class if

1.
$$A_j \in \mathcal{G}, j \geqslant 1, A_j \subseteq A_{j+1} \Rightarrow A = \bigcup_{j\geqslant 1} A_j = \lim_{j\to\infty} A_j \in \mathcal{G} \tag{4.54}$$

2.
$$B_j \in \mathcal{G}, j \geqslant 1, B_j \supseteq B_{j+1} \Rightarrow B = \bigcap_{j\geqslant 1} B_j = \lim_{j\to\infty} B_j \in \mathcal{G} \tag{4.55}$$

Theorem 4.5. *$G\mathcal{G}_\alpha$ is a monotone class, $\alpha \in I$, then the followings hold*

1. $\bigcap\limits_{\alpha\in I} \mathcal{G}_\alpha$ *is a monotone class*

2. $c \subseteq \mathcal{P}(\Omega) \Rightarrow \mathcal{G}(c) = \bigcap_{\alpha \in I} \mathcal{G}_\alpha$, *i.e. monotone classes generated by class c*

Lemma 4.1. *$a \subseteq \mathcal{P}(\Omega)$ is an algebra, $\mu(a)$ is monotone class generated by algebra a, $\mathcal{F}(a)$ is a $\sigma-$algebra generated by algebra a, then*

$$\mu(a) = \mathcal{F}(a) \tag{4.56}$$

Proof. It will proof in the next lecture. □

Proof. (Thm 4.4) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\}$, $\mu_1(A) = \mu_2(A), \forall A \in a, \Omega$ σ-finite, $\Omega = \bigcup_{j \geqslant 1} E_j, E_j \in a, \mu_j(E_j) < \infty$, then $\mu_1 = \mu_2$ on $\mathcal{F}(a)$.

Fix E_n, we denote that

$$\mathcal{B}_n = \{E \in F(a), \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\} \tag{4.57}$$

We claim that

1. $\mathcal{B}_n \supseteq a$
2. $\mathcal{B}_n$ is a monotone class

We proof $\mathcal{B}_n$ is a monotone class.

1. $\forall A_j \in \mathcal{B}_n, A_j \uparrow A = \bigcup_{j \geqslant 1} A_j$, then

$$\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n) \tag{4.58}$$

   By Remark 3.1

$$\mu_1(A_j \cap E_n) \to \mu_1(A \cap E_n), \mu_2(A_j \cap E_n) \to \mu_2(A \cap E_n) \tag{4.59}$$

2. $\forall B_j \in \mathcal{B}_n, B_j \downarrow B = \bigcap_{j \geqslant 1} B_j$, then

$$\mu_1(B_j \cap E_n) = \mu_2(B_j \cap E_n) \tag{4.60}$$

   By Remark 3.1

$$\mu_1(B_j \cap E_n) \to \mu_1(B \cap E_n), \mu_2(B_j \cap E_n) \to \mu_2(B \cap E_n) \tag{4.61}$$

So we can get that

$$\mathcal{B}_n \supseteq \mathcal{M}(a) \tag{4.62}$$

where $\mathcal{M}(a)$ is a monotone class generated by a. Then by Lemma 4.1

$$\mathcal{M}(a) = \mathcal{F}(a) \tag{4.63}$$

And by Eq 4.57,

$$\mathcal{B}_n(a) \subseteq \mathcal{F}(a) \tag{4.64}$$

so

$$\mathcal{B}_n(a) = \mathcal{F}(a) \tag{4.65}$$

Finally, $\mu_1(A) = \mu_2(A), \forall A \in \mathcal{F}(a)$, by $\mathcal{B}_n = \mathcal{F}(a)$, then $A \in \mathcal{B}_n$. $B_j \uparrow \Omega$, apply Lemma 3.1 again, we have

$$\mu_1(A) = \mu_2(A) \tag{4.66}$$

□

Lecture 5

Monotone Classes

Lecture 6

The Lebesgue Measure I