An Infinity Norm Proof

Definition 1. *The L^p norm is formally defined as:*

$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p} \tag{1.1}$$

The L^p norm has several cases that supposedly arise often in linear algebra, numerical analysis, and machine learning. The L_∞ norm:

$$\|x\|_\infty = \max_i |x_i| \tag{1.2}$$

Theorem 1. *We have that*

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty \tag{1.3}$$

Proof. We can do that

1. $p \geq 1$

$$\begin{aligned} \|x\|_\infty^p &= \max_i |x_i|^p \\ \|x\|_p^p &= \sum_i |x_i|^p \end{aligned} \tag{1.4}$$

$$\therefore \quad \|x\|_\infty \leqslant \|x\|_p \tag{1.5}$$

2. $p \geq 1$

$$\|x\|_p = \|x\|_\infty \cdot \frac{\left(\sum_i |x_i|^p \right)^{1/p}}{\|x\|_\infty} = \|x\|_\infty \cdot \left(\sum_i \left(\frac{|x_i|}{\|x\|_\infty} \right)^p \right)^{1/p} \leqslant \|x\|_\infty \cdot n^{1/p} \tag{1.6}$$

3. finally, by Eq. 1.5 and 1.6, we can get

$$\|x\|_\infty \leqslant \|x\|_p \leqslant \|x\|_\infty n^{1/p} \tag{1.7}$$

so, taking a limit as $p \to \infty$, we have

$$\|x\|_\infty \leqslant \lim_{p \to \infty} \|x\|_p \leqslant \|x\|_\infty \lim_{p \to \infty} n^{1/p} = \|x\|_\infty \tag{1.8}$$

$$\therefore \|x\|_\infty = \lim_{p \to \infty} \|x\|_p. \tag{1.9}$$

□

October 25, 2019