Probability

Lectures by Steven Miller

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Introduction

While probability began with a study of games, it has grown to become a discipline with numerous applications throughout mathematics and the sciences. Drawing on gaming examples for motivation, this course will present axiomatic and mathematical aspects of probability. Included will be discussions of random variables, expectation, independence, laws of large numbers, and the Central Limit Theorem. Many interesting and important applications will also be presented, including some from coding theory, number theory and nuclear physics.

These lectures are mainly based on the books The Probability Lifesaver by Steven Miller published by Princeton University Press.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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I would like to especially thank Professor Miller who puts this course in website.

Lecture 1

Introduction

Lecture 2

Combinatorics, Birthday Problem, Pentium Bug, QWERTY, Programming, Babylonian Mathematics

Definition 2.1. Factorials:

$$n! = n \times (n-1) \times \cdots \times 2 \times 1 \tag{2.1}$$

Then $0! = 1,\ \left(\frac{1}{2}\right)! = \sqrt{\pi}$.

Definition 2.2.

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \frac{n!\ \times (n-k)!}{k!} \tag{2.2}$$

Then $\binom{8}{3} = \frac{8\times 7\times 6}{3!} \times \frac{5!}{5!},\ (-1)! = ?,\ \binom{5}{3} = \frac{5!}{3!\times 2!} = 10,\ \binom{3}{5} = \frac{3!}{5!\times(-2)!} = 0,\ (-2)! = \infty$

p_n = probability no 2 in n share a birthday:

$$p_n = \frac{(365-0)\times(365-1)\times\cdots(365-(n-1))}{365^n} \tag{2.3}$$

then Eq. 2.3

$$p_n = \left(1 - \frac{0}{365}\right) \times \left(1 - \frac{1}{365}\right) \times \cdots \times \left(1 - \frac{n-1}{365}\right),\ n \leqslant 365 \tag{2.4}$$

then Eq. 2.4 can be written as

$$p_n = \prod_{k=0}^{n-1} \left(1 - \frac{k}{365}\right) \tag{2.5}$$

then

$$p_n = \prod_{k=0}^{n-1} \left(1 - \frac{k}{D}\right) \Rightarrow \log\ p_n = \sum_{k=0}^{n-1} \log\left(1 - \frac{k}{D}\right) \tag{2.6}$$

Remark 2.1.

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^k}{k} \cdots \tag{2.7}$$

Then

$$\log\ p_n \approx \sum_{k=0}^{n} -\frac{k}{D} \Rightarrow -D \log\ p_n \approx \sum_{k=0}^{n} k = \frac{n\,(n-1)}{2} \tag{2.8}$$

If we let $p_n = \frac{1}{2}$, then

$$D\ \log 2 \approx \frac{n\,(n-1)}{2} \approx \frac{\left(n - \frac{1}{2}\right)^2}{2} \Rightarrow n \approx \sqrt{2D\ \log 2} + \frac{1}{2} \Rightarrow n \approx \sqrt{D}\sqrt{\log 4} + \frac{1}{2} \tag{2.9}$$

$1 \leqslant a, b, c, d, e, f, g, h, i \leqslant 9$

$$\frac{a}{10b+c} + \frac{d}{10e+f} + \frac{g}{1h+i} = 1 \tag{2.10}$$

A computer should try $9! \approx 330,000$ times.

$$\begin{aligned} \frac{1}{998\,999} &= ? \\ \frac{1}{9998\,9999} &= ? \\ \cdots &= ? \end{aligned} \tag{2.11}$$

$$\binom{10}{2} + \binom{10}{1} \tag{2.12}$$

$$xy = \frac{(x+y)^2 - (x-y)^2}{4} \tag{2.13}$$