Measure Theory

Lectures by Claudio Landim

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Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. J. Taylor published by Cambridge University Press.

There are many other very good books on the subject. Here is a partial list:

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

Acknowledgments

Thank you to all of my friends who will send me suggestions and corrections. My notes will be much improved due to your help.

I would like to especially thank the IMPA and Professor Landim who put their courses in website.

Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

0. $\lambda:\ \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a,b]) = b-a$
2. $A \subseteq \mathbb{R},\ A+x = \{x+y:\ y \in A\},\ \ \forall A,\ A \subseteq \mathbb{R},\ \forall x \in \mathbb{R}:$
$$\lambda(A+x) = \lambda(A) \tag{1.1}$$
3. $A = \bigcup\limits_{j\geqslant 1} A_j,\ \ A_j \cap A_k = \varnothing:$
$$\lambda(A) = \sum_k \lambda(A_k) \tag{1.2}$$

Definition 1.1. $x \sim y,\ x,y \in \mathbb{R}$ if $y-x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R},\ y-x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_{\sim}$, only one point represents the equivalence class of Ω , like α, β.

Ω is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0,1)$

Claim 1.1. $\begin{cases} \Omega+q = \Omega+q \\ \Omega+q \cap \Omega+q = \varnothing \end{cases} \quad q,p \in \mathbb{Q}$

Proof. Assume that $\Omega+q \cap \Omega+q \neq \varnothing$ then, $x = \alpha+p = \beta+q,\ \alpha,\beta \in \Omega \Rightarrow \alpha-\beta = q-p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p,q \in \mathbb{Q} \Rightarrow (\Omega+q) \cap (\Omega+p) = \varnothing]$. □

Claim 1.2. $\Omega+q \subseteq (-1,2)$, if $-1<q<1$.

then we can get

$$\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}} (\Omega+q) \subseteq (-1,2) \tag{1.3}$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda(E) \leqslant \lambda(F)$

Proof. $\because E \subseteq F \therefore F = E \cup (F\backslash E),\ E \cap (F\backslash E) = \varnothing$, then $\lambda(F) = \lambda(E) + \lambda((F\backslash E)) \Rightarrow \lambda(F) \geqslant \lambda(E)$. □

Then,
$$\lambda\left(\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}} (\Omega+q)\right) \leqslant \lambda((-1,2)) = 3 \tag{1.4}$$

On the other hand,

$$\lambda((\Omega+q)) = \lambda(\Omega) = 0 \Rightarrow \lambda\left(\sum_{\substack{q\in\mathbb{Q}\\ -1<q<1}} (\Omega+q)\right) = 0 \tag{1.5}$$

Claim 1.4. $(0,1) \subseteq \sum\limits_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)$

Proof. $\forall$ fixed $x \in (0,1)\,, \exists \alpha \in [x] \cap \Omega,\ \alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q},\ - < q < 1 \Rightarrow x = \alpha + q,\ x \in \Omega + q$ $\square$

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda\left(\sum_{q\in\mathbb{Q}} \Omega + q\right) = 0 \tag{1.6}$$

it is impossible.

Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Definition 2.1. $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is semi-algebra if:

1. $\Omega \subseteq \mathcal{S}$
2. $A, B \in \mathcal{S} \Rightarrow A \bigcap B \in \mathcal{S}$
3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum\limits_{i=1}^{n} E_j,\ \ \exists E_1, \cdots, E_n \in \mathcal{S},\ E_i, E_j\,(i \neq j)$ disjoint sets, n is finite number

Example 2.1. $\Omega = \mathbb{R},\ \ \mathcal{S} = \{\mathbb{R}, \{(a,b)\,, a < b, a, b \in \mathbb{R}\}, \{(-\infty, b]\,, b \in \mathbb{R}\}, \{(a, \infty)\,, a \in \mathbb{R}\}, \varnothing\}$, $(a,b]^c = (-\infty, a] \cup [b, +\infty)$

Example 2.2. $\Omega = \mathbb{R}^2$

$\mathcal{S} = \ \left\{\mathbb{R}^2\,, \{(a_1, b_1) \times (a_2, b_2)\,,\ a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2]\,, b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty)\,, a_i \in \mathbb{R}\}, \varnothing\right\}$

Definition 2.2. $a = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

Remark 2.1. a algebra $\Rightarrow$ a semi-algebra

Definition 2.3. σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \subseteq \mathcal{S}$
2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap\limits_{j \geqslant 1} A_j \in \mathcal{S}$
3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

Remark 2.2. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)$, a_α algebra, $\alpha \in I \Rightarrow a = \bigcap\limits_{\alpha \in I} a_\alpha$ is an algebra.

Proof. check the followings

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

$\square$

Remark 2.3. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)\,, \alpha \in I, a_\alpha$, σ-algebra $\Rightarrow a = \bigcap\limits_{\alpha \in I} a_\alpha$ is a σ-algebra

Proof. check the followings

1. $\Omega \in a$
2. $A_j, j \geq 1 \in a \Rightarrow \bigcap\limits_{j \geqslant 1} A_j \in a$

3. $A \in a \Rightarrow A^c \in a$

$\square$

Definition 2.4 (minimal algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.1}$$

Remark 2.4. $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall\alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an algebra.

Definition 2.5 (minimal σ-algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a σ-algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is σ-algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.2}$$

Remark 2.5. $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall\alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an σ-algebra.

Lemma 2.1. *Ω, f semi-algebra $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ algebra generated by f then*

$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leqslant j \leqslant n,\ A = \sum_{j=1}^{n} E_j \tag{2.3}$$

Proof.

1. $\Leftarrow$

   $A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f \in a(f)$

   By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. $\Rightarrow$

   $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j, E_j \in f$

   Then by remark 2.7, it will be proved easily.

$\square$

Remark 2.6. $E, J \in a, E \bigcup F \in a, E \bigcup F = (E^c \bigcap F^c)^c$

Remark 2.7. $\mathcal{B} = \left\{ \sum\limits_{j=1}^{n} F_j,\ F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. $\mathcal{B}$ algebra
2. $\mathbb{B} \supseteq f$

at last $\mathbb{B} \supseteq a(f)$

Proof. We only prove that $\mathcal{B}$ algebra, then check the following

1. $\Omega \in \mathcal{B}$
2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

   $\because A, B \in \mathcal{B}, \therefore A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f,\ B = \sum\limits_{k=1}^{m} F_k,\ F_k \in f$, then

$$\begin{aligned} A \cap B &= \left(\sum_{j=1}^{n} E_j\right) \cap \left(\sum_{k=1}^{m} F_k\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \\ &\in \mathcal{B} \end{aligned} \tag{2.4}$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

   $A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f$

   By definition 2.1:

$$\begin{aligned} E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1},\ F_{1,j} \in f \\ \cdots &= \cdots \\ E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i},\ F_{i,j} \in f \end{aligned} \tag{2.5}$$

   Then, we get that

$$\begin{aligned} A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1}\right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2}\right) \cap \cdots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n}\right) \\ &= \sum_{k_1=1}^{l_1}\sum_{k_2=1}^{l_2}\cdots\sum_{k_n=1}^{l_n} \left(F_{1,k_1} \cap F_{2,k_2} \cap F_{n,k_n}\right) \\ &\in \mathcal{B} \end{aligned} \tag{2.6}$$

$\square$

Definition 2.6. $c \subseteq \mathcal{P}(\Omega), \varnothing \in c,\ \mu :\ c \to \mathbb{R}_+ \cup \{+\infty\}$. μ is additive if

1. $\mu(\varnothing) = 0$
2. $E_1, E_2, ..., E_n \in c,\ E = \sum\limits_{j=1}^{n} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j=1}^{n} \mu(E_k)$

Remark 2.8.

$$\exists A \in c,\ \mu(A) < \infty,\ A = A \cup \varnothing,\ \mu(A) = \mu(A) + \mu(\varnothing) \Rightarrow \mu(\varnothing) = 0 \tag{2.7}$$

Remark 2.9. $c,\ \mu : c \to \mathbb{R}_+ \bigcup +\infty,\ E \subseteq F,\ F \backslash E \in c,\ E, F \in c$

$$F = E \cup (F \backslash E),\ \mu(F) = \mu(E) + (F \backslash E) \tag{2.8}$$

1. $\mu(E) = +\infty$, $\mu(F) = +\infty$
2. $\mu(E) < +\infty$, $\mu(F \backslash E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leqslant \mu(F) \tag{2.9}$$

Example 2.3. Discrete measure: Ω, $c \subseteq \mathcal{P}(\Omega)$, $\{x_j,\ j \geqslant 1\},\ x_j \in \Omega,\ \{p_j,\ j \geqslant 1\},\ p_j, \geqslant 0,\ A \in c$, define that

$$\mu(A) = \sum_{j \geqslant 1} p_j 1 \{x_j \in A\} \tag{2.10}$$

then μ is additive

Definition 2.7. $c \in \mathcal{P}(\Omega), \varnothing \in c$, $\mu :\ c \to \mathbb{R}_+ \bigcup +\infty$, μ is σ-additive if

1. $\mu(\varnothing) = 0$
2. $E_j \in c,\ j \neq k, E_j \bigcap E_k = \varnothing,\quad E = \sum\limits_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j \geq 1} \mu(E_j)$

Example 2.4. $\Omega = (0,1),\ c = \{(a,b],\ 0 \leqslant a < b < 1\},\ \mu :\ c \to \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a,b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \tag{2.11}$$

$(a,b] = \sum\limits_{j=1}^{n} (a_j, b_j)$, we can get that μ is NOT σ-additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow\!\to 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geqslant 1} (x_{j+1}, x_j] = +\infty \tag{2.12}$$

it is impossible.