Deep Learning **Fall 2019**

Lecture 1: Convolutional Neural Networks

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Definition 1.1 (Integral operator)**.** *A transform T expressible with the kernel H and $t_1, t_2 \in \mathbb{R} \bigcup \{-\infty, \infty\}$ such that for any function f (for with Tf exists)*

$$(Tf)(u) = \int_{t_1}^{t_2} H(u,t) f(t) dt \tag{1.1}$$

is called an integral operator.

Example 1.1 (Fourier transform)**.**

$$(\mathcal{F}f)(u) \triangleq \int_{-\infty}^{\infty} e^{-2\pi i t u} f(t)\, dt \tag{1.2}$$

Definition 1.2 (Convolution)**.** *Given two functions f, h, their convolution is defined as*

$$(f * h)(u) \triangleq \int_{-\infty}^{\infty} h(u-t) f(t)\, dt = \int_{-\infty}^{\infty} f(u-t) h(t)\, dt \tag{1.3}$$

Remark 1.1.

1. *integral operator with kernel $H(u,t) = h(u-t)$*
2. *shift-invariant as $H(u-s, t-s) = h(u-t) = H(u,t) \quad (\forall s)$*

   Proof. content... □
3. *convolution operator is commutative*

   Proof. content... □
4. *existence depends on properties of f, h*
5. *typical use f = signal, h = fast decaying kernel function*

Definition 1.3 (Linear transform)**.** *T is linear, if for all functions f, g and the scalars α, β,*

$$T(\alpha f + \beta g) = \alpha T f + \beta T g \tag{1.4}$$

Definition 1.4 (Translation invariant transform)**.** *T is translation (or shift) invariant, if for any f and scalar τ,*

$$f_\tau(t) \triangleq f(t+\tau), \quad (Tf_\tau)(t) \triangleq (Tf)(t+\tau) \tag{1.5}$$

Remark 1.2. *content...*

Theorem 1.1. *Any linear, translation-invariant transformation T can be written as convolution with a suitable h.*

Proof. content... □

Signal processing with neural networks:

1. Transforms in deep networks: linear + simple non-linearity
2. Many signals (audio, image, etc.) obey translation invariance $\Rightarrow$ invariant feature maps: shift in input = shift in feature map

1 + 2 in above:

1. $\Rightarrow$ learn convolutions, not (full connectivity) weight matrices
2. $\Rightarrow$ convolutional layers for signal processing

For all practical purposes: signal are sampled, i.e. discrete.

Definition 1.5 (Discrete convolution (1-D))**.** *For $f, h : \mathbb{Z} \to \mathbb{R}$, we can define the discrete convolution via*

$$(f * h)[u] \triangleq \sum_{t=-\infty}^{\infty} f[t]\, h[u-t] \tag{1.6}$$

Remark 1.3.

1. *use of rectangular brackets to suggest "arrays"*
2. *2D case:*
$$\textit{content...} \tag{1.7}$$
3. *typical: h with finite support (window size)*

Example 1.2. *Small Gaussian kernel with support $[-2:2] \subset \mathbb{Z}$*

$$h[t] = \frac{1}{16} \begin{cases} 6 & t = 0 \\ 4 & |t| = 1 \\ 1 & |t| = 2 \\ 0 & \textit{otherwise} \end{cases} \tag{1.8}$$

Consequence: convolution sum can be truncated:

$$\begin{aligned} (f * h)[u] &= \sum_{t=u-2}^{u+2} f[t]\, h[u-t] = \sum_{t=-2}^{2} h[t]\, f[u-t] \\ &= \frac{6f[u] \;\; + 4f[u-1] + 4f[u+1] + f[u-2] + f[u+2]}{16} \end{aligned} \tag{1.9}$$

Remark 1.4. *content...*

Definition 1.6 (Discrete cross-correlation)**.** *Let $f, h : \mathbb{Z} \to \mathbb{R}$, then*

$$(h * f)[u] \triangleq \sum_{t=\infty}^{\infty} h[t]\, f[u+t] \tag{1.10}$$

Remark 1.5.

1. *Def. 1.6 also called a "sliding inner product", $u+t$ instead of $u-t$*
2. *note that cross-correlation and convolution are closely related:*

$$\begin{aligned}(h * f)[u] &= \sum_{t=\infty}^{\infty} h[t] f[u+t] \\ &= (h * f)[u] = \sum_{t=\infty}^{\infty} h[-t] f[u-t] \\ &= \left(\overline{h} * f\right)[u] \\ &= \left(f * \overline{h}\right)[u]\end{aligned} \tag{1.11}$$

   where $\overline{h}[t] \triangleq h[-t]$.

Only difference: kernel flipped over, but not non-commutative.

Convolution via matrices:

1. In practice: signal f and kernel h have finite support
2. Without loss of generality (w.l.o.g) $f[t] = 0$ for $t \notin [1:n]\,, h[t] = 0$ for $t \notin [1:m]$
3. We can think of f and h as vectors and define:

$$(f * h) = \underbrace{\begin{pmatrix} h_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1 & 0 & 0 & \cdots & 0 & 0 \\ h_3 & h_2 & h_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_m & h_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & h_m \end{pmatrix}}_{\triangleq H_n^h \in \mathbb{R}^{(n+m-1)\times n}} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} \tag{1.12}$$

Remark 1.6. *content...*

Definition 1.7 (Toeplitz matrix)**.** *A matrix $H \in \mathbb{R}^{k\times n}$ is a Toeplitz matrix, if there exists $n+k-1$ numbers c_l ($l \in [-(n-1):(k-1)] \subset \mathbb{Z}$) such that*

$$H_{ij} = c_{i-j} \tag{1.13}$$

Remark 1.7.

1. *in plain English, all NE-SE diagonals are constant*
2. *if $m \ll n$: additional sparseness (band matrix of width m)*
3. *H_n^h has only m degrees of freedom*
4. *locality (sparseness $m \ll n$) and weight sharing (kernel)*

Convolutions in higher dimensions: generalize concept of convolution to:

1. 2D: e.g. images, spectograms

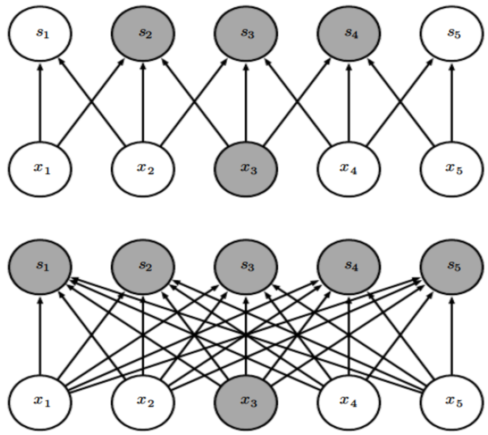


Figure 1.1: Sparse vs dense connectivity

2. 3D: e.g. color or multi-spectral images, voxel images, video
3. or even higher dimensions

Replace vector by:

1. matrices or fields (e.g. in discrete case)

$$(F * G)[i,j] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F[i-k, j-l] \cdot G[k,l] \tag{1.14}$$

2. tensors: for 3D and higher

Different options for border handling:

1. our definition: padding with zeros = same padding
2. only retain values from windows fully contained in support of signal f = valid padding

layout:

1. Convolved signal inherits topology of original signal
2. Hence: units in a convolutional layer are typically arranged on the same grid (1D, 2D, 3D,...)

Exploit structural sparseness in computing $\frac{\partial x_i^l}{\partial x_j^{l-1}}$:

1. receptive filed of $x_i^l : \mathcal{I}_i^l \triangleq \left\{j : W_{ij}^l \neq 0\right\}$, where W^l is the Toeplitz matrix of the convolution
2. obviously $\frac{\partial x_i^l}{\partial x_j^{l-1}} = 0$ for $j \notin \mathcal{I}_i^l$

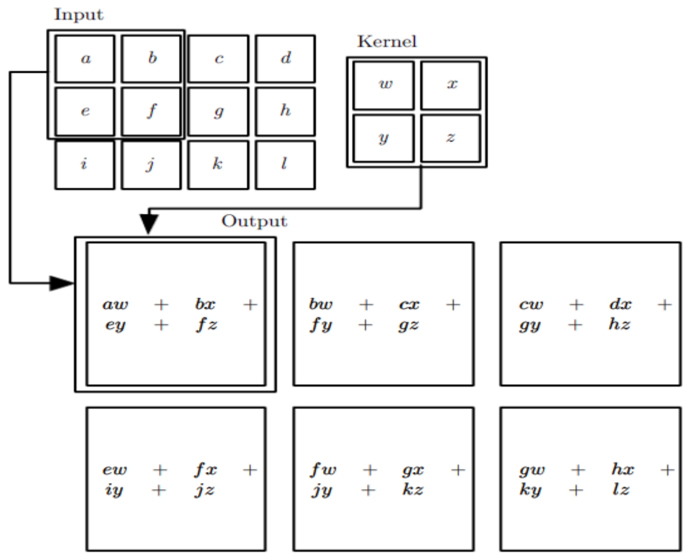


Figure 1.2

Weight sharing in computing $\frac{\partial \mathcal{R}}{\partial h_j^l}$, where h_j^l is a kernel weight

$$\frac{\partial \mathcal{R}}{\partial h_j^l} = \sum_i \frac{\partial \mathcal{R}}{\partial x_i^l} \frac{\partial x_i^l}{\partial h_j^l} \tag{1.15}$$

Weight is re-used for every unit within target layer $\Rightarrow$ additive combination of derivatives in chain rule. nesting of convolutions: receptive fields grow.

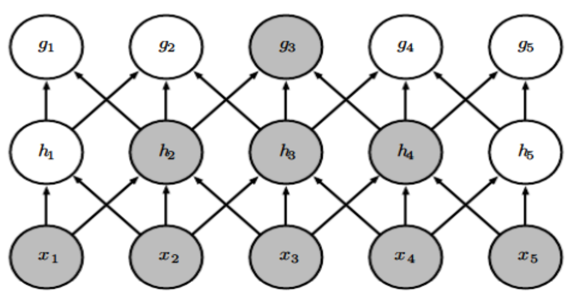


Figure 1.3

FFT (Fast Fourier Transform): compute convolutions fast(er).

1. Fourier transform of signal $f \to (\mathcal{F}f)$ and kernel $h \to (\mathcal{F}h)$
2. pointwise multiplication and inverse Fourier transform:

$$(f * h) = \mathcal{F}^{-1}\left((\mathcal{F}f) \cdot (\mathcal{F}h)\right) \tag{1.16}$$

3. FFT: signal of length n, can be done in $O\left(n \ \text{logn}\right)$

4. pays off, if many channels (amortizes computation of $\mathcal{F}f$)
5. small kernels ($m < \log n$): favor time / space domain

Remark 1.8. *content...*

Stages:

1. Non-linearites: detector stage. As always: scalar non-linearities (activation function)
2. Pooling stage: locally combine activities

Most frequently used pooling function:max pooling.

Definition 1.8 (Max Pooling)**.** *Define window size r (e.g. 3 or 3×3), then*

$$\begin{aligned} 1D: \quad & x_i^{\max} = \max\left\{x_{i+k} : 0 \leqslant k < r\right\}, \\ 2D: \quad & x_{ij}^{\max} = \max\left\{x_{i+k,j+l} : 0 \leqslant k, l < r\right\} \end{aligned} \tag{1.17}$$

Remark 1.9.

1. *maximum over a small patch of units*
2. *other functions are possible: average, soft-maximization*

Max-pooling: invariance

1. set of invertible transformations $\mathcal{T}$: group w.r.t composition
2. $\mathcal{T}-$invariance through maximization $f_{\mathcal{T}}(x) \triangleq \max\limits_{\tau \in \mathcal{T}} f(\tau x)$

Proposition 1.1. *$f_{\mathcal{T}}$ is invariant under $\tau \in \mathcal{T}$.*

Proof.

$$f_{\mathcal{T}}(\tau x) = \max_{\rho \in \mathcal{T}} f(\rho(\tau x)) = \max_{\rho \in \mathcal{T}} (f(\rho \circ \tau) x) = \max_{\sigma \in \mathcal{T}} f(\sigma x) \tag{1.18}$$

as $\forall \sigma, \sigma = \rho \circ \tau$ with $\rho = \sigma \circ \tau^{-1}$. □

sub-sampling(also known as (aka) strides):

1. often, it is desirable to reduce the size of feature maps
2. sub-sampling: reduce temporal/spatial resolution. Often: combined with (max-)pooling (aka. stride)
3. example: max-pool, filter 2×2, stride 2×2
4. disadvantage: loss of information

Learn multiple convolution kernel (or filters) = multiple channels:

1. typically: all channels use same window size

2. channels form additional dimension for next layer (e.g. 2D signal $\times$ channels = 3D tensor)
3. number of channels: design parameter

http://cs231n.github.io/assets/conv-demo/index.html

Note that kernels (across channels) form a linear map:

$$h : \mathbb{R}^{r^2 \times d} \to \mathbb{R}^k \tag{1.19}$$

where $r \times r$ is the window size and d is the depth.

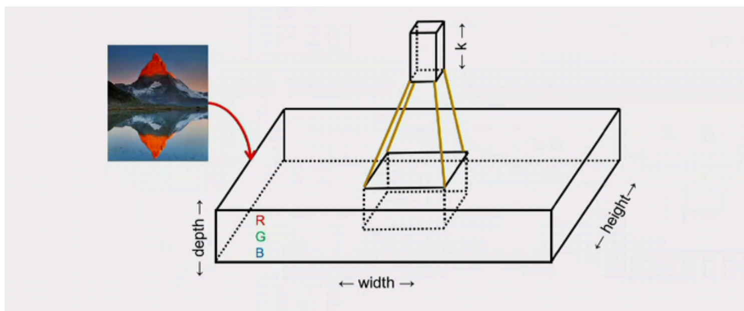


Figure 1.4: convolutional layers for vision

Convolutional networks: multiple, stacked feature maps

$$\underbrace{y[r]}_{r-th\ channel}[s,t] = \sum_u \sum_{\Delta s, \Delta t} \underbrace{w[r,u][\Delta s, \Delta t]}_{parameters} \underbrace{x[u]}_{u-th\ channel}[s+\Delta s, t+\Delta t] \tag{1.20}$$

1. x, y tensor, 3-rd order
2. number of parameters:
$$\underbrace{\#r \cdot \#u}_{fully\ connected} \cdot \underbrace{\#\Delta s \cdot \#\Delta t}_{window\ size} \tag{1.21}$$
3. pointwise non-linearities (e.g. ReLU)
4. interleaved with: pooling (e.g. max, average)
5. optionally: downsampling (use of strides)

Convolutional pyramid:
Typical use of convolution in vision: sequence of convolutions that

1. reduce spatial dimensions (sub-sampling)
2. increase number of channels

$\Rightarrow$ smaller, but more feature maps.

LeNet5 [1,2]

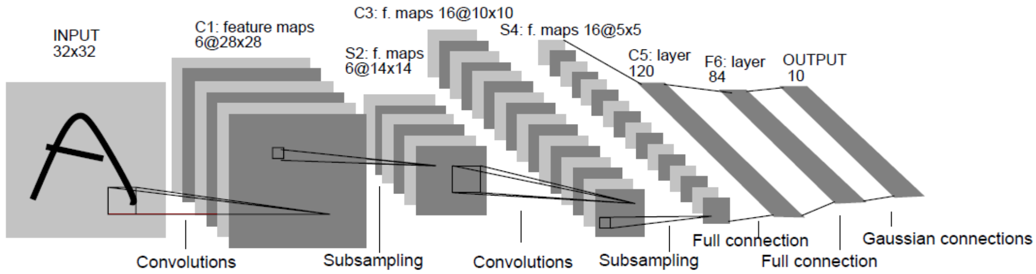


Figure 1.5: Architecture of LeNet-5, a convolutional neural network, here for digits recognition. Each plan is a feature map, i.e. a set of units whose weights are constrained to be identical. [2]

1. C1/S2: 6 channels, 5×5 kernels, 2×2 sub (4704 units)
2. C3/S4: 16 channels,6×6 kernels, 2×2 sub (1600 units)
3. C5: 120 channels, F6: fully-connected
4. output: Gaussian noise model (squared loss)

AlexNet[3]

1. Pyramidal architecture: reduce spatial resolution, increase channels with depth
2. Challenge: many channels (width) + large windows + depth
3. Number of parameters
   (a) 384 to 384 channels with 3×3 windows: > 1.3 M
   (b) $13 \times 13 \times 384$ tensor to 4096, fully connected: > 265 M

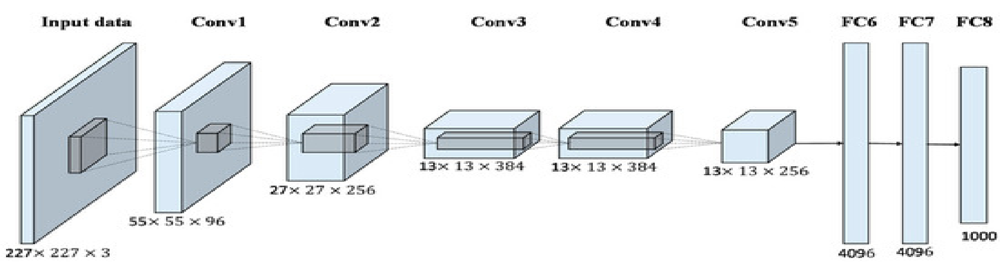


Figure 1.6: AlexNet architecture

Deep ConvNets: key challenges

1. avoid blow-up of model size (e.g. # parameters)
2. preserve computational efficiency of learning (e.g. gradients)
3. allow for large depth (as it is known to be a plus)
4. allow for sufficient width (as it is known to be a plus, too)

Very deep convolutional networks: VGG [4]

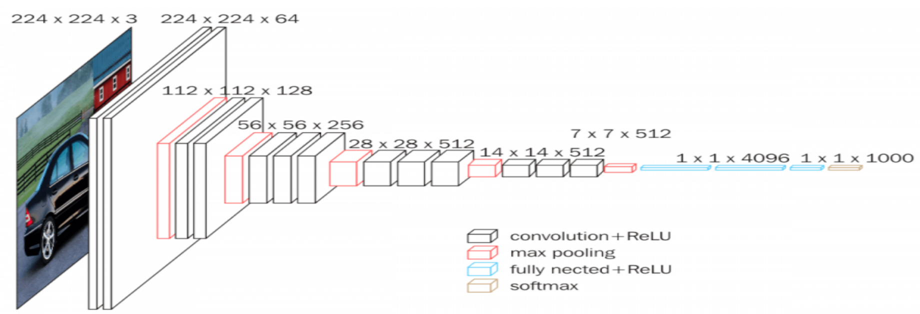


Figure 1.7: VGG 16

1. use very small receptive fields (maximally 3×3)
2. avoid downsampling/pooling
3. stacking small receptive fields: more depth, fewer parameters
4. example: $3 \cdot (3 \times 3) = 27 < 49 (7 \times 7)$

Many channels needed for high accuracy, typically $k \sim 200 - 1000$ (e.g. AlexNet: 2×192).

Observation (motivated by Arora et al, 2013 [5]): when convolving, dimension reduction across channels may be acceptable.

Dimension reduction: m channels of a $1 \times 1 \times k$ convolution $m \leq k$:

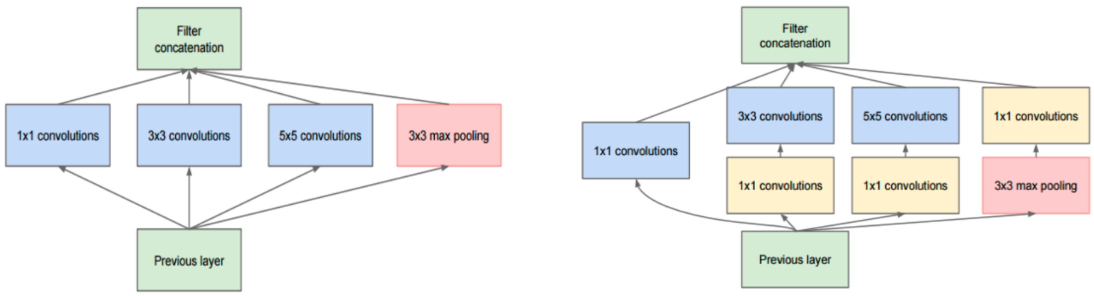
$$x_{ij}^{+} = \sigma\left(W x_{ij}\right), \quad W \in \mathbb{R}^{m \times k} \tag{1.22}$$

1. 1×1 convolution = no convolution
2. inception module (Szegedy et al. [6])
3. network within a network (Lin et al, [7])
4. i.e. W is shared for all (i, j) (translation invariance)

Inception module: mixing

Instead of fixed window size convolution: mix 1×1 with 3×3 and 5×5, max-polling. Use 1×1 convolutions for dimension reduction before convolving with large kernels.

Google inception networks [6]



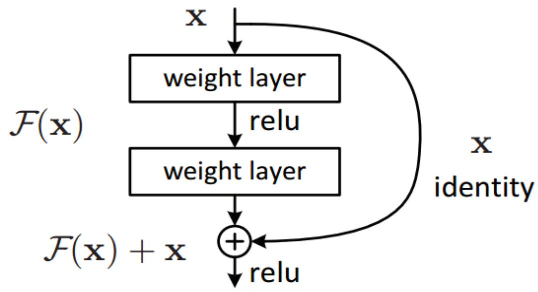
(a) naive version (b)

Figure 1.8: Inception module [8]

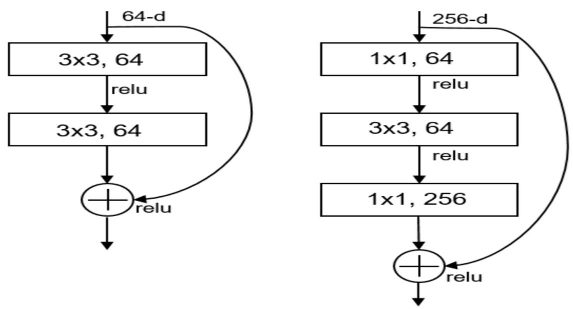
Very deep network: many inception modules (green boxes: concatenation points). Additional trick: connect softmax layer (and loss) at intermediate stages (yellow boxes) $\Rightarrow$ gradient shortcuts.

Figure 1.9: Google inception networks

Residual networks: ResNets [9]



(a)



(b)

Figure 1.10: Residual Networks module [9]

1. learn changes to the identity map (aka. shortcut connections)
2. use small filters (VGG), use dimension reduction (inception)
3. reach depth of 100 + layers (+ increase accuracy + trainable)

Next topic is convolutional sequence models and recurrent networks

Reading List

[1] Y. LeCun, B. Boser, J. Denker, D. Henderson, R. Howard, W. Hubbard and L. Jackel, "Back-propagation applied to handwritten zip code recognition," *Neural Computation* , 1989, Vol. 1(4), pp. 541–551.

[2] Y. Lecun, L. Bottou, Y. Bengio and P. Haffner, "Gradient-based learning applied to document recognition," *Proceedings of the IEEE* , 1998, Vol. 86(11), pp. 2278–2324.

[3] A. Krizhevsky, I. Sutskever and G. Hinton, "ImageNet classification with deep convolutional neural networks," *NIPS 2012, Neural Information Processing Systems* , 2012, Vol. 60, pp. 84–90.

[4] K. Simonyan and A. Zisserman, "Very deep convolutional networks for large-scale image recognition," *International Conference on Learning Representations* , 2015.

[5] M. Arora and H. Kaur, “Performance analysis of communication system with convolutional coding over fading channel,” *International Journal of Scientific & Engineering Research* , 2013, Vol. 4(5), pp. 1116–1120.

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[8] C. Vasconcelos, B. Vasconcelos, “Network in convolutional neural network committees for melanoma classification with classical and expert knowledge based image transforms data augmentation,” *arXiv:1702.07025*, 2017.

[9] K. He, X. Zhang, S. Ren and J. Sun, “Deep Residual Learning for Image Recognition,” *2016 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)* , 2016, pp. 770–778.