

Measure Theory

Lectures by Claudio Landim

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1 Lecture 1	1	6 Lecture 6	25
2 Lecture 2	3	7 Lecture 7	28
3 Lecture 3	7	8 Lecture 8	29
4 Lecture 4	13	9 Lecture 9	30
5 Lecture 5	22	10 Lecture 10	31

Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. Taylor published by Cambridge University Press.

These notes were live-Texed, though I edited for typos and added diagrams requiring the TikZ package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

$$0. \lambda : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$$

$$1. \lambda((a, b]) = b - a$$

$$2. A \subseteq \mathbb{R}, A + x = \{x + y : y \in A\}, \forall A, A \subseteq \mathbb{R}, \forall x \in \mathbb{R} :$$

$$\lambda(A + x) = \lambda(A) \quad (1.1)$$

$$3. A = \bigcup_{j \geq 1} A_j, A_j \cap A_k = \emptyset :$$

$$\lambda(A) = \sum_k \lambda(A_k) \quad (1.2)$$

Definition 1.1. $x \sim y, x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R}, y - x \in \mathbb{Q}\}$.

$\Lambda = \mathbb{R}/\sim$, only one point represents the equivalence class of Ω , like α, β .

Ω is a class of equivalence class, if $\Omega \subseteq \mathbb{R}, \Omega \subseteq (0, 1)$

Claim 1.1. $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \emptyset \end{cases} \quad q, p \in \mathbb{Q}$

Proof. Assume that $\Omega + q \cap \Omega + q \neq \emptyset$ then, $x = \alpha + p = \beta + q, \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \emptyset]$. \square

Claim 1.2. $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \subseteq (-1, 2) \quad (1.3)$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda(E) \leq \lambda(F)$

Proof. $\because E \subseteq F \therefore F = E \cup (F \setminus E), E \cap (F \setminus E) = \emptyset$, then $\lambda(F) = \lambda(E) + \lambda((F \setminus E)) \Rightarrow \lambda(F) \geq \lambda(E)$. \square

Then,

$$\lambda \left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \right) \leq \lambda((-1, 2)) = 3 \quad (1.4)$$

and ,

$$\infty \cdot \lambda((\Omega + q)) = \infty \cdot \lambda(\Omega) \leq 3 \Rightarrow \lambda \left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \right) = 0 \quad (1.5)$$

Claim 1.4. $(0, 1) \subseteq \sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)$

Proof. \forall fixed $x \in (0, 1)$, $\exists \alpha \in [x] \cap \Omega$, $\alpha \in (0, 1)$, and we know that $\alpha - x = q \in \mathbb{Q}$, $- < q < 1 \Rightarrow x = \alpha + q$, $x \in \Omega + q$ \square

But, we get that:

$$1 = \lambda((0, 1)) \leq \lambda \left(\sum_{q \in \mathbb{Q}} \Omega + q \right) = 0 \quad (1.6)$$

it is impossible.

Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Definition 2.1. $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, \mathcal{S} is semi-algebra if:

1. $\Omega \in \mathcal{S}$
2. $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$
3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum_{i=1}^n E_j, \exists E_1, \dots, E_n \in \mathcal{S}, E_i, E_j (i \neq j)$ disjoint sets, n is finite number

Example 2.1. $\Omega = \mathbb{R}, \mathcal{S} = \{\mathbb{R}, \{(a, b), a < b, a, b \in \mathbb{R}\}, \{(-\infty, b], b \in \mathbb{R}\}, \{(a, \infty), a \in \mathbb{R}\}, \emptyset\}, (a, b]^c = (-\infty, a] \cup [b, +\infty)$

Example 2.2. $\Omega = \mathbb{R}^2$

$\mathcal{S} = \{\mathbb{R}^2, \{(a_1, b_1) \times (a_2, b_2), a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2], b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty), a_i \in \mathbb{R}\}, \emptyset\}$

Definition 2.2. $\mathcal{a} = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in \mathcal{a}$
2. $A, B \in \mathcal{a} \Rightarrow A \cap B \in \mathcal{a}$
3. $A \in \mathcal{a} \Rightarrow A^c \in \mathcal{a}$

Remark 2.1. \mathcal{a} algebra $\Rightarrow \mathcal{a}$ semi-algebra

Definition 2.3. σ -algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \in \mathcal{S}$
2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap_{j \geq 1} A_j \in \mathcal{S}$
3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

Remark 2.2. $\Omega, \mathcal{a}_\alpha \subseteq \mathcal{P}(\Omega), \mathcal{a}_\alpha$ algebra, $\alpha \in I \Rightarrow \mathcal{a} = \bigcap_{\alpha \in I} \mathcal{a}_\alpha$ is an algebra.

Proof. check the followings

1. $\Omega \in \mathcal{a}$
2. $A, B \in \mathcal{a} \Rightarrow A \cap B \in \mathcal{a}$
3. $A \in \mathcal{a} \Rightarrow A^c \in \mathcal{a}$

□

Remark 2.3. $\Omega, \mathcal{a}_\alpha \subseteq \mathcal{P}(\Omega), \alpha \in I, \mathcal{a}_\alpha, \sigma$ -algebra $\Rightarrow \mathcal{a} = \bigcap_{\alpha \in I} \mathcal{a}_\alpha$ is a σ -algebra

Proof. check the followings

1. $\Omega \in \mathcal{a}$
2. $A_j, j \geq 1 \in \mathcal{a} \Rightarrow \bigcap_{j \geq 1} A_j \in \mathcal{a}$

$$3. A \in a \Rightarrow A^c \in a$$

□

Definition 2.4 (minimal algebra generated by c). $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by c , and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \quad (2.1)$$

Remark 2.4. $a(c)$ exists, and $a = a(c) = \bigcap_{\alpha} a_{\alpha}$, $\forall \alpha$, $c \subseteq a_{\alpha}$, a_{α} is an algebra.

Definition 2.5 (minimal σ -algebra generated by c). $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a σ -algebra generated by c , and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is σ -algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \quad (2.2)$$

Remark 2.5. $a(c)$ exists, and $a = a(c) = \bigcap_{\alpha} a_{\alpha}$, $\forall \alpha$, $c \subseteq a_{\alpha}$, a_{α} is an σ -algebra.

Lemma 2.1. Ω, f semi-algebra $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ algebra generated by f then

$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leq j \leq n, A = \sum_{j=1}^n E_j \quad (2.3)$$

Proof.

1. \Leftarrow

$$A = \sum_{j=1}^n E_j, E_j \in f \in a(f)$$

By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. \Rightarrow

$$A \in a(f) \Rightarrow A = \sum_{j=1}^n E_j, E_j \in f$$

Then by remark 2.7, it will be proved easily.

□

Remark 2.6. $E, J \in a, E \cup F \in a, E \cup F = (E^c \cap F^c)^c$

Remark 2.7. $\mathcal{B} = \left\{ \sum_{j=1}^n F_j, F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. \mathcal{B} algebra
2. $\mathcal{B} \supseteq f$
3. $\mathcal{B} \supseteq a(f)$

Proof. We only prove that \mathcal{B} algebra, then check the following

1. $\Omega \in \mathcal{B}$

2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

$$\because A, B \in \mathcal{B}, \therefore A = \sum_{j=1}^n E_j, E_j \in f, B = \sum_{k=1}^m F_k, F_k \in f, \text{ then}$$

$$\begin{aligned} A \cap B &= \left(\sum_{j=1}^n E_j \right) \cap \left(\sum_{k=1}^m F_k \right) \\ &= \sum_{j=1}^n \sum_{k=1}^m \underbrace{(E_j \cap F_k)}_{\in f} \\ &\in \mathcal{B} \end{aligned} \tag{2.4}$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

$$A = \sum_{j=1}^n E_j, E_j \in f$$

By definition 2.1:

$$\begin{aligned} E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1}, F_{1,j} \in f \\ \dots &= \dots \\ E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i}, F_{i,j} \in f \end{aligned} \tag{2.5}$$

Then, we get that

$$\begin{aligned} A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1} \right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2} \right) \cap \dots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n} \right) \\ &= \sum_{k_1=1}^{l_1} \sum_{k_2=1}^{l_2} \dots \sum_{k_n=1}^{l_n} (F_{1,k_1} \cap F_{2,k_2} \cap \dots \cap F_{n,k_n}) \\ &\in \mathcal{B} \end{aligned} \tag{2.6}$$

□

Definition 2.6. $c \subseteq \mathcal{P}(\Omega), \emptyset \in c, \mu : c \rightarrow \mathbb{R}_+ \cup \{+\infty\}$. μ is additive if

1. $\mu(\emptyset) = 0$

2. $E_1, E_2, \dots, E_n \in c, E = \sum_{j=1}^n E_j \in c \Rightarrow \mu(E) = \sum_{j=1}^n \mu(E_j)$

Remark 2.8.

$$\exists A \in c, \mu(A) < \infty, A = A \cup \emptyset, \mu(A) = \mu(A) + \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0 \quad (2.7)$$

Remark 2.9. $c, \mu : c \rightarrow \mathbb{R}_+ \cup +\infty, E \subseteq F, F \setminus E \in c, E, F \in c$

$$F = E \cup (F \setminus E), \mu(F) = \mu(E) + \mu(F \setminus E) \quad (2.8)$$

1. $\mu(E) = +\infty, \mu(F) = +\infty$
2. $\mu(E) < +\infty, \mu(F \setminus E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leq \mu(F) \quad (2.9)$$

Example 2.3. Discrete measure: $\Omega, c \subseteq \mathcal{P}(\Omega), \{x_j, j \geq 1\}, x_j \in \Omega, \{p_j, j \geq 1\}, p_j \geq 0, A \in c$, define that

$$\mu(A) = \sum_{j \geq 1} p_j 1\{x_j \in A\} \quad (2.10)$$

then μ is additive

Definition 2.7. $c \in \mathcal{P}(\Omega), \emptyset \in c, \mu : c \rightarrow \mathbb{R}_+ \cup +\infty, \mu$ is σ -additive if

1. $\mu(\emptyset) = 0$
2. $E_j \in c, j \neq k, E_j \cap E_k = \emptyset, E = \sum_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum_{j \geq 1} \mu(E_j)$

Example 2.4. $\Omega = (0, 1), c = \{(a, b], 0 \leq a < b < 1\}, \mu : c \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a, b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \quad (2.11)$$

$(a, b] = \sum_{j=1}^n (a_j, b_j)$, we can get that μ is NOT σ -additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow \rightarrow 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geq 1} (x_{j+1}, x_j] = +\infty \quad (2.12)$$

it is impossible.

Lecture 3

Set Functions

Definition 3.1. $c \subseteq \mathcal{P}(\Omega)$, $\mu : c \rightarrow \mathbb{R}_+ \cup +\infty$:

1. $E \in c$, μ continuous from below at E , if $\forall (E_n)_{n \geq 1}$, $E_n \in c$, $E_n \uparrow E$ $\left(E_n \subseteq E_{n+1}, \bigcup_{n \geq 1} E_n = E \right)$:

$$\mu(E_n) \rightarrow \mu(E) \quad (3.1)$$

2. $E \in c$, μ continuous from above at E , if $\forall (E_n)_{n \geq 1}$, $E_n \in c$, $E_n \downarrow E$ $\left(E_{n+1} \subseteq E_n, \bigcap_{n \geq 1} E_n = E \right)$,
and $\exists n_0$, $\mu(E_{n_0}) < \infty$:

$$\mu(E_n) \rightarrow \mu(E) \quad (3.2)$$

Remark 3.1. For a sequence E_1, E_2, \dots of sets, we put

$$\limsup E_i = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} E_i \right), \liminf E_i = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} E_i \right) \quad (3.3)$$

and if $\{E_i\}$ is such that $\limsup E_i = \liminf E_i$ we say that the sequence converges to the set

$$E = \limsup E_i = \liminf E_i \quad (3.4)$$

Remark 3.2. μ need $\exists n_0$, $\mu(E_{n_0}) < \infty$, if not:

$$E_n = [n, +\infty), \mu(E_n) = +\infty, E_n \downarrow \emptyset, \lambda(\emptyset) = 0 \quad (3.5)$$

Lemma 3.1. $a \subseteq \mathcal{P}(\Omega)$, algebra; $\mu : a \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, additive;

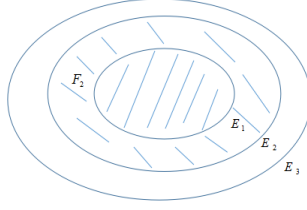
1. μ is σ -additive $\Rightarrow \mu$ continuous at E , $\forall E \in a$
2. μ is continuous from below $\Rightarrow \mu$ is σ -additive
3. μ is continuous from above at \emptyset & μ is finite $\Rightarrow \sigma$ -additive

Proof.

1.

- (i) μ is σ -additive $\Rightarrow \mu$ conti. from below at $E \in a$. $E \in a, E_n \uparrow E, E_n \in a$:

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \setminus E_1 \\ &\vdots \\ F_n &= E_n \setminus E_{n-1} \end{aligned} \quad (3.6)$$



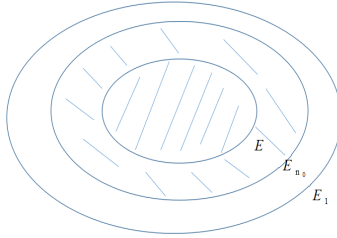
and we can get that

$$F_j \cap F_k = \emptyset, \quad \sum_{k=1}^n F_k = E_n, \quad \bigcup_{n \geq 1} E_n = \bigcup_{n \geq 1} F_n \quad (3.7)$$

so

$$\mu(E) = \sum_{k \geq 1} \mu(F_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(F_k) = \lim_{n \rightarrow \infty} \mu(E_n) \quad (3.8)$$

(ii) μ cont. from above $E \in a, E_n \in a, E_n \downarrow E, \mu(E_{n_0}) < \infty \Rightarrow \mu(E_n) \downarrow \mu(E)$



$$\begin{aligned} G_1 &= E_{n_0} \setminus E_{n_0+1} \\ G_2 &= E_{n_0} \setminus E_{n_0+2} \\ &\vdots = \vdots \\ G_k &= E_{n_0} \setminus E_{n_0+k} \end{aligned} \quad (3.9)$$

then $G_k \uparrow E_{n_0} \setminus E, G_k \in a \Rightarrow \mu(G_k) \uparrow \mu(E_{n_0} \setminus E)$, so

$$\begin{aligned} \mu(E_{n_0} \setminus E) &= \lim_{n \rightarrow \infty} \mu(E_{n_0} \setminus E_{n_0+k}) \\ \mu(E_{n_0} \setminus E) &= \mu(E_{n_0}) - \mu(E) \\ \mu(E_{n_0}) - \mu(E) &= \lim_{k \rightarrow \infty} (\mu(E_{n_0}) - \mu(E_{n_0+k})) \end{aligned} \quad (3.10)$$

2. μ cont. below, $E = \sum_{k \geq 1} E_k, E, E_k \in a$.

Obs.

$$\sum_{k=1}^n E_k \subseteq E \xrightarrow{\text{additive}} \begin{cases} \mu\left(\sum_{k=1}^n E_k\right) \leq \mu(E) \\ \sum_{k=1}^n \mu(E_k) \leq \mu(E) \end{cases} \quad (3.11)$$

then

$$\sum_{k \geq 1} \mu(E_k) \leq \mu(E) \quad (3.12)$$

$$F_n = \sum_{k=1}^n E_k \in a, F_n \uparrow E,$$

$$\sum_{k=1}^n \mu(E_k) = \mu(F_n) \uparrow \mu(E) \Rightarrow \sum_{k \geq 1} \mu(E_k) = \mu(E) \quad (3.13)$$

3. μ cont. at \emptyset , $\mu(\Omega) < \infty$, $E, E_k \in a$, $E = \sum_{k \geq 1} E_k$.

$$F_n = \sum_{k \geq m} E_k \in a \left(E \setminus \sum_{j=1}^{n-1} E_j \right) \quad (3.14)$$

$$F_n \downarrow \emptyset, \mu(F_1) < \infty, \mu(F_n) \rightarrow 0$$

$$\begin{aligned} \mu(E) &= \mu \left(\sum_{k=1}^n E_k \cup \sum_{k > n} E_k \right) \\ &= \underbrace{\mu \sum_{k=1}^n E_k}_{\rightarrow \sum_{k \geq 1} \mu(E_k)} + \underbrace{\mu \sum_{k > n} E_k}_{\rightarrow 0} \\ &\rightarrow \sum_{k \geq 1} \mu(E_k) \end{aligned} \quad (3.15)$$

□

Remark 3.3. Suppose E_α , $\alpha \in I$ is a class of subsets of X , and E_i is one set of the class, then

1. $\bigcap_{\alpha \in I} E_\alpha \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$
2. $X - \bigcup_{\alpha \in I} E_\alpha = \bigcap_{\alpha \in I} (X - E_\alpha)$
3. $X - \bigcap_{\alpha \in I} E_\alpha = \bigcup_{\alpha \in I} (X - E_\alpha)$

Proof.

1. This is immediate from the definition.
2. Suppose $x \in X - \bigcup_{\alpha \in I} E_\alpha$ then $x \in X$ and x is not in $\bigcup_{\alpha \in I} E_\alpha$, that is x is not in any E_α , $\alpha \in I$ so that $x \in X - E_\alpha$ for every $\alpha \in I$, and $x \in \bigcap_{\alpha \in I} (X - E_\alpha)$. Conversely if $x \in \bigcap_{\alpha \in I} (X - E_\alpha)$, then for every $\alpha \in I$, x is in X but not in E_α , so $x \in X$ but x is not in $\bigcup_{\alpha \in I} E_\alpha$, that is $x \in \bigcup_{\alpha \in I} (X - E_\alpha)$.
3. Similar to 2

Remark 3.3 (2) and (3) are also called as de Morgan's Law. \square

Example 3.1. $(0, 1), (a, b], 0 \leq a < b < 1$

$$\mu(a, b] = \begin{cases} b - a, & a > 0 \\ +\infty, & a = 0 \end{cases} \quad (3.16)$$

μ is additive but NOT σ -additive

Proof. $E_n \downarrow \emptyset, \mu(E_{n_0}) < \infty, E_n = (a_{n,1}, b_{n,1}] \cup \dots \cup (a_{n,k_n}, b_{n,k_n}], a_{n,j} < a_{n,j+1}.$

$$\begin{cases} a_{n,1} = 0, & \forall n \\ a_{n_0} > 0, & \text{some } n_0 \end{cases} \quad \square$$

Theorem 3.1 (Extension). $f \subseteq \mathcal{P}(\Omega)$ semi-algebra, $\mu : f \rightarrow \mathbb{R}_+ \cup \{\infty\}$ σ -additive, then $\exists \nu :$

$$\nu : a(f) \rightarrow \mathbb{R}_+ \cup \{\infty\} \quad (3.17)$$

such that:

1. ν σ -additive
2. $\nu(A) = \mu(A), \forall A \in f$
3. $\mu_1, \mu_2, a(f) \rightarrow \mathbb{R}_+ \cup \{\infty\},$ then $\mu_1(A) = \mu_2(A), \forall A \in f \Rightarrow \mu_1(E) = \mu_2(E), \forall E \in a(f)$

Proof. $A \in a(f) \Rightarrow A = \sum_{j=1}^n E_j, E_j \in f$ by Lemma 2.1.

$$\nu(A) \stackrel{add}{=} \sum_{j=1}^n \nu(E_j) \stackrel{ext}{=} \sum_{j=1}^n \mu(E_j) \quad (3.18)$$

we define that

$$\nu(A) = \sum_{j=1}^n \mu(E_j) \quad (3.19)$$

we want to show that $\nu(A) = \sum_{j=1}^n \mu(E_j)$ is well-defined:

1. ν is unique

$$\begin{aligned} A &= \sum_{j=1}^n E_j, E_j \in f \\ &= \sum_{k=1}^m F_k, F_k \in f \end{aligned} \quad (3.20)$$

then we will prove that

$$\begin{aligned} \nu(A) &= \sum_{j=1}^n \mu(E_j) \\ &= \sum_{k=1}^m \mu(F_k) \end{aligned} \quad (3.21)$$

$$\because E_j \subseteq A = \sum_{k=1}^m F_k \Rightarrow E_j = E_j \cap \left(\sum_{k=1}^m F_k \right) = \sum_{k=1}^m \underbrace{(E_j \cap F_k)}_{\in f} \quad (3.22)$$

$$\therefore \mu(E_j) = \mu \left(\sum_{k=1}^m (E_j \cap F_k) \right) \quad (3.23)$$

then

$$\nu(A) = \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^n \sum_{k=1}^m \mu(E_j \cap F_k) = \sum_{k=1}^m \mu(F_k) \quad (3.24)$$

2. ν is an additive, $\nu(A) = \sum_{j=1}^n \mu(E_j)$

Assume that

$$\begin{cases} A = \sum_{j=1}^n E_j, E_j \in f \\ B = \sum_{k=1}^m F_k, F_k \in f \end{cases}, A \cap B = \emptyset \quad (3.25)$$

We will show that

$$\nu(A \cup B) = \nu(A) + \nu(B) \quad (3.26)$$

$$\because A \cup B = \sum_{j=1}^n E_j + \sum_{k=1}^m F_k \quad (3.27)$$

therefore

$$\begin{aligned} \nu(A \cup B) &= \mu \left(\sum_{j=1}^n E_j + \sum_{k=1}^m F_k \right) \\ &= \sum_{j=1}^n \mu(E_j) + \sum_{k=1}^m \mu(F_k) \\ &= \nu(A) + \nu(B) \end{aligned} \quad (3.28)$$

3. $\nu(A) = \mu(A)$, $A \in f$ by Eq 3.19

4. ν is uniqueness, we want to show that:

Suppose that $\mu_1, \mu_2 : a(f) \rightarrow R_+ \cup \{+\infty\}, \forall A \in f, \mu_1, \mu_2$ additive, then

$$\mu_1(A) = \mu_2(A) \Rightarrow \mu_1(B) = \mu_2(B), \forall B \in a(f) \quad (3.29)$$

$$\because B \in a(f), \therefore B = \sum_{j=1}^n \mu_1(E_j), E_j \in f$$

$$\mu_1(B) = \sum_{j=1}^n \mu_1(E_j) = \sum_{j=1}^n \mu_2(E_j) = \mu_2(B) \quad (3.30)$$

Now we proof the extension of σ -additive, ie: $\mu - \sigma$ additive, f semi-algebra, $\nu - \sigma$ additive, $a(f)$ is a algebra generated by f . we want to show that

$$A = \sum_{j \geq 1} A_j, \quad A, A_j \in a(f) \Rightarrow \nu(A) = \sum_{j \geq 1} \nu(A_j) \quad (3.31)$$

by representation of an algebra:

$$A = \sum_{j=1}^m E_j, E_j \in f; \quad A_k = \sum_{l=1}^{m_k} E_{k,l}, E_{k,l} \in f \quad (3.32)$$

by Eq 3.19:

$$\nu(A) = \sum_{j=1}^m \nu(E_j), \quad \nu(A_k) = \sum_{l=1}^{m_k} \nu(E_{k,l}) \quad (3.33)$$

$$\because E_j = E_j \cap A = E_j \cap \left(\sum_{k \geq 1} A_k \right) = E_j \cap \left(\sum_{k \geq 1} \sum_{l=1}^{m_k} E_{k,l} \right) = \sum_{k \geq 1} \sum_{l=1}^{m_k} (E_j \cap E_{k,l}) \quad (3.34)$$

therefore

$$\begin{aligned} \nu(A) &= \sum_{j=1}^n \mu(E_j) \\ &= \sum_{j=1}^n \sum_{k \geq 1} \sum_{l=1}^{m_k} \mu(E_j \cap E_{k,l}) \\ &= \sum_{k \geq 1} \sum_{l=1}^{m_k} \underbrace{\mu(E_{k,l})}_{\subseteq A_k} \end{aligned} \quad (3.35)$$

Eq 3.35 holds because:

$$E_{k,l} = E_{k,l} \cap A = \sum_{j=1}^n (E_{k,l} \cap E_j) \quad (3.36)$$

and

$$\mu(E_{k,l}) = \sum_{j=1}^n \mu(E_{k,l} \cap E_j) \quad (3.37)$$

so we can get that

$$\nu(A) = \sum_{k \geq 1} \nu(A_k) \quad (3.38)$$

□

Lecture 4

Caratheodory Theorem

Intuition:

$$\begin{array}{lll}
 \sigma - add & \mu : f \rightarrow \mathbb{R}_+ \cup \{+\infty\} & f \subseteq \mathcal{P}(\Omega), f \text{ is semialgebra} \\
 \downarrow & \downarrow & \\
 \sigma - add & \nu : a(f) \rightarrow \mathbb{R}_+ \cup \{+\infty\} & a(f) \text{ algebra generated by } f \\
 \downarrow & \downarrow & \\
 \sigma - add & \pi : \mathcal{F}(a) \rightarrow \mathbb{R}_+ \cup \{+\infty\} & \mathcal{F}(a) \text{ is } \sigma - \text{algebra generated by algebra}
 \end{array} \tag{4.1}$$

The big picture of the proof to the Caratheodory Theorem:

1. Define the π^* outer measure:

$$\pi^* = \inf_{\{E_i\}} \sum_{i \geq 1} \nu(E_i) \tag{4.2}$$

2. \mathcal{M} σ -algebra, $\mathcal{M} \supseteq \mathcal{F}(a)$

- 3.

$$\pi^* : \mathcal{M} \rightarrow \mathbb{R}_+ \cup \{+\infty\} \tag{4.3}$$

is σ -additive, and

$$\pi^*|_a = \nu \tag{4.4}$$

4. (uniqueness) $\mu_1, \mu_2 : \mathcal{F}(a) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, Ω is σ -finite(μ_1), if $E_j \uparrow \Omega$, $\mu_1(E_j) < \infty, \forall j$, $E_j \in a$ and $\mu_1|_a = \mu_2|_a$ then implies that

$$\mu_1 = \mu_2 \tag{4.5}$$

$$\pi^* : \mathcal{P}(\Omega) \rightarrow \mathbb{R}_+ \cup \{+\infty\} \tag{4.6}$$

We will prove π^* is an outer measure.

And we will construct a family of subsets \mathcal{M}

$$\mathcal{M} \subseteq \mathcal{P}(\Omega) \tag{4.7}$$

we will also prove \mathcal{M} satisfies the following:

1. \mathcal{M} is a σ -algebra
2. $\mathcal{M} \supseteq a$
3. $\pi^*|_{\mathcal{M}}$ σ -additive
4. $\pi^*|_a = \nu$

Next, we will define π^* and \mathcal{M} .

Step 1

Definition 4.1 (π^*). $\pi^* : \mathcal{P}(\Omega) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, $A \in \Omega$, $\{E_i, i \geq 1\}, E_i \in \mathcal{A}, A \subseteq \bigcup E_i$, $\{E_i\}$ is a covering of A, then we define that

$$\pi^* = \inf_{\{E_i\}, A} \sum_{i \geq 1} \nu(E_i) \quad (4.8)$$

where $\nu : \mathcal{A} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, is σ -additive.

Definition 4.2 (Outer measure). $\mu : \mathcal{C} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, $\mathcal{C} \subseteq \mathcal{P}(\Omega)$, $\emptyset \in \mathcal{C}$, μ is a outer measure if

1. $\mu(\emptyset) = 0$
2. (monotone) $E \subseteq F, E, F \in \mathcal{C} \Rightarrow \mu(E) \leq \mu(F)$
3. (subadditive) $E, E_i \in \mathcal{C}, E \subseteq \bigcup_i E_i \Rightarrow \mu(E) \leq \sum_i \mu(E_i)$

Theorem 4.1. π^* in 4.1 is a outer measure.

Proof. We will check the conditions in Def 4.2.

1. check $\pi^*(\emptyset) = 0$

(a) $E_i = \emptyset, \emptyset \subseteq \bigcup_{i \geq 1} E_i$ then

$$\pi^*(\emptyset) = \inf_{\{E_i\}, \emptyset} \sum_{i \geq 1} \nu(E_i) \leq \sum_{i \geq 1} \nu(E_i) = 0 \quad (4.9)$$

(b) $E_i \in \mathcal{A}, \{E_i\}, \emptyset \subseteq \bigcup_{i \geq 1} E_i$, then

$$\sum_{i \geq 1} \nu(E_i) \geq 0 \Rightarrow \pi^*(\emptyset) \geq 0 \quad (4.10)$$

2. check $E \subseteq F, \pi^*(E) \leq \pi^*(F)$

Let's take any covering of $F: \{E_i\}, E_i \in \mathcal{A}, F \subseteq \bigcup_{i \geq 1} E_i$ is also a covering of E , then

$$\pi^*(E) = \inf_{\{E_i\}, E} \sum_{i \geq 1} \nu(E_i) \leq \pi^*(F) = \inf_{\{E_i\}, F} \sum_{i \geq 1} \nu(E_i) \quad (4.11)$$

3. check $E \subseteq \bigcup_{i \geq 1} E_i, \pi^*(E) \leq \sum_{i \geq 1} \pi^*(E_i)$

(a) $\pi^*(E_i) = \infty$ then

$$\pi^*(E) \leq \sum_{i \geq 1} \pi^*(E_i) \quad (4.12)$$

(b) $\pi^*(E_i) < \infty$, then

$$\pi^*(E_i) = \inf_{\{H_{ik}\}, E_i} \sum_{k \geq 1} \nu(H_{ik}) \quad (4.13)$$

then there $\exists \{H_{ik}\} \in a, E_i \subseteq \bigcup_{k \geq 1} H_{ik}$ such that

$$\pi^*(E_i) = \inf_{\{H_{ik}\}, E_i} \sum_{k \geq 1} \nu(H_{ik}) \leq \sum_{k \geq 1} \nu(H_{ik}) \leq \pi^*(E_i) + \frac{\varepsilon}{2^i} \quad (4.14)$$

$\{H_{ik}\}$ is a covering of E_i , then

$$\pi^*(E) \leq \sum_{i,k} \nu(H_{ik}) \leq \sum_{i \geq 1} \left(\pi^*(E_i) + \frac{\varepsilon}{2^i} \right) \leq \sum_{i \geq 1} \pi^*(E_i) + \varepsilon \quad (4.15)$$

so

$$\pi^*(E) \leq \sum_{i \geq 1} \pi^*(E_i) \quad (4.16)$$

□

Step 2

Definition 4.3 (Measurable set \mathcal{M}). A set called measurable set \mathcal{M} if $A \in \mathcal{M} \forall E \in \Omega$, we have that

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad (4.17)$$

Theorem 4.2. If \mathcal{M} defined as Def 4.3, then

1. $\mathcal{M} \supseteq a$
2. \mathcal{M} is a σ -algebra

Remark 4.1.

$$E \subseteq (E \cap A) \cup (E \cap A^c) \Rightarrow \pi^*(E) \leq \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad (4.18)$$

so we only to check \geq in Eq 4.17

Proof. π^* is an outer measurable by Thm 4.1, then by subadditive of outer measure. □

Now we proof Thm 4.2.

Proof.

1. $a \in \mathcal{M}$

Suppose that $A \in a, E \in \Omega$, we will show that

$$\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad (4.19)$$

assume that $\pi^*(E) < \infty$, given $\varepsilon, \exists \{E_i\}, E$, such that $E_i \in a, E \subseteq \bigcup_{i \geq 1} E_i$, then

$$\pi^*(E) \leq \sum_{i \geq 1} \pi^*(E_i) \leq \pi^*(E) + \varepsilon \quad (4.20)$$

$E_i \cap A \in \mathcal{a}, E \cap A \subseteq \bigcup_{i \geq 1} (E_i \cap A)$, so

$$\begin{aligned}\pi^*(E \cap A) &\leq \sum_{i \geq 1} \nu(E_i \cap A) \\ \pi^*(E \cap A^c) &\leq \sum_{i \geq 1} \nu(E_i \cap A^c)\end{aligned}\tag{4.21}$$

so

$$\pi^*(E \cap A) + \pi^*(E \cap A^c) \leq \sum_{i \geq 1} \nu(E_i \cap A) + \sum_{i \geq 1} \nu(E_i \cap A^c) \leq \sum_{i \geq 1} \nu(E_i) \leq \pi^*(E) + \varepsilon\tag{4.22}$$

2. \mathcal{M} is σ -algebra.

We need to show that

(a) $\Omega \in \mathcal{M}$

It is clearly that:

$$\pi^*(E) = \pi^*(E \cap \Omega) + \pi^*(E \cap \Omega^c)\tag{4.23}$$

(b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$

$$\therefore \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c)\tag{4.24}$$

(c) $A_i \in \mathcal{M} \Rightarrow \bigcup_{i \geq 1} A_i \in \mathcal{M}$

Finite union is closed: $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{M}$. Let's take $E \subseteq \Omega$. We will proof that

$$\pi^*(E) \geq \pi^*(E \cap (A \cup B)) + \pi^*(E \cap (A \cup B)^c)\tag{4.25}$$

$\therefore A \in \mathcal{M}$,

$$\therefore \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c)\tag{4.26}$$

$\therefore B \in \mathcal{M}$

$$\begin{aligned}\therefore \pi^*(E \setminus A) &= \pi^*(E \setminus A \cap B) + \pi^*(E \setminus A \cap B^c) \\ &= \pi^*(E \setminus A \cap B) + \pi^*(E \setminus (A \cup B))\end{aligned}\tag{4.27}$$

then

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \setminus A \cap B) + \pi^*(E \setminus (A \cup B))\tag{4.28}$$

We want to show

$$\pi^*(E \cap A) + \pi^*(E \setminus A \cap B) \geq \pi^*(E \cap (A \cup B))\tag{4.29}$$

By π^* is subadditive, we only to show that

$$E \cap (A \cup B) \subseteq (E \cap A) \cup (E \setminus A \cap B)\tag{4.30}$$

this is because

$$E \cap (A \cup B) = \underbrace{\{[E \cap (A \cup B)] \cap A\}}_{\subseteq E \cap A} \cup \underbrace{\{[E \cap (A \cup B)] \cap A^c\}}_{\subseteq (E \cap A^c) \cap B = (E \setminus A) \cap B}\tag{4.31}$$

Then Eq 4.25 holds. So \mathcal{M} is closed by finite(countable) union.

Now, we will show that countable infinite union is also closed. $A_i \in \mathcal{M}$, we want to show $A = \bigcup_{j \geq 1} A_j \in \mathcal{M}$, take $E \subseteq \Omega$,

$$\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad (4.32)$$

by Eq. 4.25, $\forall n$ we know that

$$\begin{aligned} \pi^*(E) &= \pi^*\left(E \cap \left(\bigcup_{j=1}^n A_j\right)\right) + \pi^*\left(E \cap \left(\bigcup_{j=1}^n A_j^c\right)\right) \\ &\geq \pi^*\left(E \cap \left(\bigcup_{j=1}^n A_j\right)\right) + \pi^*(E \setminus A) \end{aligned} \quad (4.33)$$

\geq holds in Eq 4.33 because $(E \setminus A) \subseteq \left(E \setminus \left(\bigcup_{j=1}^n A_j\right)\right)$.

Now, we define

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_1 \setminus A_2 \\ F_3 &= A_1 \setminus (A_2 \cup A_3) \\ &\vdots \\ F_n &= A_1 \setminus (A_2 \cup \dots \cup A_{n-1}) \\ &\vdots \end{aligned} \quad (4.34)$$

It is clear that

$$\bigcup_{i=1}^n A_i = \bigcup_{j=1}^n F_j \quad (4.35)$$

where $F_j \cap F_k = \emptyset, F_j \in \mathcal{M}$.

Then Eq 4.33 can be written as

$$\pi^*(E) \geq \pi^*\left(E \cap \sum_{j=1}^n F_j\right) + \pi^*(E \setminus A) \quad (4.36)$$

By Remark 4.2, we have

$$\begin{aligned} \pi^*(E) &\geq \pi^*\left(E \cap \left(\sum_{j=1}^n F_j\right)\right) + \pi^*(E \setminus A) \\ &= \sum_{j=1}^n \pi^*(E \cap F_j) + \pi^*(E \setminus A) \end{aligned} \quad (4.37)$$

\because n is any number in Remark 4.2, $\therefore \pi^* \left(E \cap \sum_{j=1}^{\infty} F_j \right) = \sum_{j=1}^{\infty} \pi^* (E \cap F_j)$, by π^* is subadditive

$$\begin{aligned}
\pi^* (E) &\geq \pi^* \left(E \cap \sum_j F_j \right) + \pi^* (E \setminus A) \\
&= \sum_{j \geq 1} \pi^* (E \cap F_j) + \pi^* (E \setminus A) \\
&\geq \pi^* \left(\bigcup_{j \geq 1} (E \cap F_j) \right) + \pi^* (E \setminus A) \\
&=\geq \pi^* \left(E \cap \left(\bigcup_{j \geq 1} F_j \right) \right) + \pi^* (E \setminus A) \\
&= \pi^* (E \cap A) + \pi^* (E \setminus A)
\end{aligned} \tag{4.38}$$

So \mathcal{M} is a σ -algebra.

□

Remark 4.2. $\forall n$, we have that

$$\pi^* \left(E \cap \sum_{j=1}^n F_j \right) = \sum_{j=1}^n \pi^* (E \cap F_j) \tag{4.39}$$

where F_j defined as Eq 4.34.

Proof. By induction

1. $n = 1$, Eq 4.39 holds

2. Support n holds then we will proof $n + 1$ holds. $F_k \in \mathcal{M}, F_{n+1} \in \mathcal{M}$, we have that

$$\begin{aligned}
\pi^* \left(E \cap \sum_{j=1}^{n+1} F_j \right) &= \pi^* \left(\left(E \cap \sum_{j=1}^{n+1} F_j \right) \cap F_{n+1} \right) + \pi^* \left(\left(E \cap \sum_{j=1}^{n+1} F_j \right) \cap F_{n+1}^c \right) \\
&= \pi^* (E \cap F_{n+1}) + \underbrace{\pi^* \left(E \cap \sum_{j=1}^n F_j \right)}_{\text{by assumption} = \sum_{j=1}^n \pi^* (E \cap F_j)} \\
&= \sum_{j=1}^{n+1} \pi^* (E \cap F_j)
\end{aligned} \tag{4.40}$$

□

By Thm 4.2 we have that $\mathcal{M} \supseteq \mathcal{F}(a)$.

Step 3

Theorem 4.3. $\pi^* : \mathcal{M} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ is σ -additive, then

$$\pi^*(A) = \nu(A), \quad \forall A \in a \quad (4.41)$$

Remark 4.3. Eq 4.41 is also

$$\pi^*|_a = \nu \quad (4.42)$$

Eq 4.2 holds because Thm 4.2, $a \in \mathcal{M}$.

Proof. (Thm 4.3)

$$1. \pi^*(A) = \nu(A), \quad \forall A \in a$$

(a) check $\pi^*(A) \leq \nu(A)$

Let's $\underbrace{A}_{E_1}, \underbrace{\emptyset}_{E_2}, \underbrace{\emptyset}_{E_3}, \dots$

$$\pi^*(A) = \inf_{\{E_i\}, A} \sum_i \nu(E_i) \leq \sum_i \nu(E_i) = \nu(A) \quad (4.43)$$

(b) check $\pi^*(A) \geq \nu(A)$

Let's take

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \setminus E_1 \\ F_3 &= E_3 \setminus (E_1 \cup E_2) \\ &\vdots \\ F_n &= E_n \setminus (E_1 \cup E_2 \cup \dots \cup E_{n-1}) \\ &\vdots \end{aligned} \quad (4.44)$$

$$F_j \in a, \bigcup_j F_j = \bigcup_j E_j, F_j \cap F_k = \emptyset, A \subseteq \bigcup_{j \geq 1} F_j, \text{ so } A = \sum_j F_j \cap A \in a.$$

Because ν is σ -additive we have that

$$\nu(A) = \sum_{j \geq 1} \nu(F_j \cap A) \quad (4.45)$$

$$\because F_j \subseteq E_j$$

$$\nu(A) = \sum_{j \geq 1} \nu(F_j \cap A) \leq \sum_{j \geq 1} \nu(E_j) \quad (4.46)$$

so

$$\nu(A) \leq \inf_{\{E_i\}, A} \sum_{j \geq 1} \nu(E_j) = \pi^*(A) \quad (4.47)$$

Then, we can get

$$\pi^*(A) = \nu(A), \quad \forall A \in a \quad (4.48)$$

2. $\pi^*|_{\mathcal{M}}$ is σ -additive

Suppose that $A_j \in \mathcal{M}, A_j \cap A_k = \emptyset$, we want to proof that

$$\pi^* \left(\sum_{j \geq 1} A_j \right) = \sum_{j \geq 1} \pi^* (A_j) \quad (4.49)$$

(a) check $\pi^* \left(\sum_{j \geq 1} A_j \right) \leq \sum_{j \geq 1} \pi^* (A_j)$ by π^* is an outer measure, π^* is subadditive

(b) check $\pi^* \left(\sum_{j \geq 1} A_j \right) \geq \sum_{j \geq 1} \pi^* (A_j)$

by π^* is an outer measure, π^* is monotone

$$\pi^* \left(\sum_{j \geq 1} A_j \right) \geq \pi^* \left(\sum_{j=1}^n A_j \right) \quad (4.50)$$

by Remark 4.2, we have that

$$\pi^* \left(\sum_{j=1}^n A_j \right) = \sum_{j=1}^n \pi^* (A_j), \quad \forall n \quad (4.51)$$

so

$$\pi^* \left(\sum_{j \geq 1} A_j \right) \geq \sum_{j \geq 1} \pi^* (A_j) \quad (4.52)$$

□

Step 4

Definition 4.4. Ω is σ -finite(μ_1) if $E_j \uparrow \Omega, \mu_1(E_j) < \infty, \forall j, E_j \in \mathcal{a}$.

Theorem 4.4 (Uniqueness). Suppose that $\mu_1, \mu_2 : \mathcal{F}(a) \rightarrow R_+ \cup \{+\infty\}, \Omega$ is σ -finite(μ_1), if $\mu_1|_a = \mu_2|_a$, then

$$\mu_1 = \mu_2, \quad \text{on } \mathcal{F}(a) \quad (4.53)$$

Definition 4.5. $\Omega, \mathcal{G} \subseteq \mathcal{P}(\Omega), \mathcal{G}$ is a monotone class if

1.

$$A_j \in \mathcal{G}, j \geq 1, A_j \subseteq A_{j+1} \Rightarrow A = \bigcup_{j \geq 1} A_j = \lim_{j \rightarrow \infty} A_j \in \mathcal{G} \quad (4.54)$$

2.

$$B_j \in \mathcal{G}, j \geq 1, B_j \supseteq B_{j+1} \Rightarrow B = \bigcap_{j \geq 1} B_j = \lim_{j \rightarrow \infty} B_j \in \mathcal{G} \quad (4.55)$$

Theorem 4.5. \mathcal{G}_α is a monotone class, $\alpha \in I$, then the followings hold

1. $\bigcap_{\alpha \in I} \mathcal{G}_\alpha$ is a monotone class

2. $c \subseteq \mathcal{P}(\Omega) \Rightarrow \mathcal{G}(c) = \bigcap_{\alpha \in I} \mathcal{G}_\alpha$, i.e. monotone classes generated by class c

Lemma 4.1. $a \subseteq \mathcal{P}(\Omega)$ is an algebra, $\mu(a)$ is monotone class generated by algebra a , $\mathcal{F}(a)$ is a σ -algebra generated by algebra a , then

$$\mu(a) = \mathcal{F}(a) \quad (4.56)$$

Proof. It will proof in the next lecture. \square

Proof. (Thm 4.4) $\mu_1, \mu_2 : \mathcal{F}(a) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, $\mu_1(A) = \mu_2(A)$, $\forall A \in a$, Ω σ -finite, $\Omega = \bigcup_{j \geq 1} E_j$, $E_j \in a$, $\mu_j(E_j) < \infty$, then $\mu_1 = \mu_2$ on $\mathcal{F}(a)$.

Fix E_n , we denote that

$$\mathcal{B}_n = \{E \in \mathcal{F}(a), \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\} \quad (4.57)$$

We claim that

1. $\mathcal{B}_n \supseteq a$
2. \mathcal{B}_n is a monotone class

We proof \mathcal{B}_n is a monotone class.

1. $\forall A_j \in \mathcal{B}_n, A_j \uparrow A = \bigcup_{j \geq 1} A_j$, then

$$\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n) \quad (4.58)$$

By Remark 3.1

$$\mu_1(A_j \cap E_n) \rightarrow \mu_1(A \cap E_n), \mu_2(A_j \cap E_n) \rightarrow \mu_2(A \cap E_n) \quad (4.59)$$

2. $\forall B_j \in \mathcal{B}_n, B_j \downarrow B = \bigcap_{j \geq 1} B_j$, then

$$\mu_1(B_j \cap E_n) = \mu_2(B_j \cap E_n) \quad (4.60)$$

By Remark 3.1

$$\mu_1(B_j \cap E_n) \rightarrow \mu_1(B \cap E_n), \mu_2(B_j \cap E_n) \rightarrow \mu_2(B \cap E_n) \quad (4.61)$$

So we can get that

$$\mathcal{B}_n \supseteq \mathcal{M}(a) \quad (4.62)$$

where $\mathcal{M}(a)$ is a monotone class generated by a . Then by Lemma 4.1

$$\mathcal{M}(a) = \mathcal{F}(a) \quad (4.63)$$

And by Eq 4.57,

$$\mathcal{B}_n(a) \subseteq \mathcal{F}(a) \quad (4.64)$$

so

$$\mathcal{B}_n(a) = \mathcal{F}(a) \quad (4.65)$$

Finally, $\mu_1(A) = \mu_2(A)$, $\forall A \in \mathcal{F}(a)$, by $\mathcal{B}_n = \mathcal{F}(a)$, then $A \in \mathcal{B}_n$. $B_j \uparrow \Omega$, apply Lemma 3.1 again, we have

$$\mu_1(A) = \mu_2(A) \quad (4.66)$$

\square

Lecture 5

Monotone Classes

Definition 5.1. Given Ω , define $\mathcal{M}(a) \subseteq \mathcal{P}(\Omega)$ is a monotone class is

1. $A_j \in \mathcal{M}, A_j \uparrow A \left(A_j \subseteq A_j, \bigcup_{j \geq 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$
2. $A_j \in \mathcal{M}, A_j \downarrow A \left(A_j \supseteq A_j, \bigcap_{j \geq 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$

Remark 5.1.

1. \mathcal{F} is σ -field(σ -algebra) $\Rightarrow \mathcal{F}$ is a monotone class
2. $\mathcal{M}_\alpha \subseteq \mathcal{P}(\Omega), (\alpha \in I)$ is monotone class, then $\mathcal{M} = \bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is a monotone class.

Notation 5.1. (Smallest monotone class contain c) $\mathcal{M}(c)$ is a monotone class generated by c if

$$c \subseteq \mathcal{M}(\Omega), \mathcal{M}(c) = \bigcap_{\alpha \in I} \mathcal{M}_\alpha \quad (5.1)$$

Definition 5.2. $E \subseteq \mathcal{M}(a)$, the set $\mathcal{G}(E)$ is defined as below

$$\mathcal{G}(E) = \{F \in \mathcal{M}(a), E \setminus F, E \cap F, F \setminus E \in \mathcal{M}(a)\} \quad (5.2)$$

Lemma 5.1.

1. If $E \in a \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$
2. If $E \in \mathcal{M}(a) \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$

Proof.

1. $E \in a$, we want to show that

$$(a) \mathcal{G}(E) \supseteq a$$

Take $H \in a \subseteq \mathcal{M}(a)$, then

$$\underbrace{E \setminus H}_{\in a}, \underbrace{E \cap H}_{\in a}, \underbrace{H \setminus E}_{\in a} \in \mathcal{G}(a) \quad (5.3)$$

so $H \in \mathcal{G}(E)$, then $a \subseteq \mathcal{G}(E)$

- (b) $\mathcal{G}(E)$ is a monotone class

Suppose that $H_k \uparrow H, H_k \in \mathcal{G}(E)$,

$$\because E \setminus H_k \in \mathcal{M}(a), E \setminus H_k \rightarrow E \setminus H, \therefore E \setminus H \in \mathcal{M}(a) \quad (5.4)$$

$$\because E \cap H_k \in \mathcal{M}(a), E \cap H_k \rightarrow E \cap H, \therefore E \cap H \in \mathcal{M}(a) \quad (5.5)$$

$$\because H_k \setminus E \in \mathcal{M}(a), H_k \setminus E \rightarrow H \setminus E, \therefore H \setminus E \in \mathcal{M}(a) \quad (5.6)$$

By Eq 5.6, $H \in \mathcal{M}(a)$, and by the definition 5.2, $H \in \mathcal{G}(E)$. So $\mathcal{G}(E)$ is a monotone class. We also get that $\mathcal{G}(E) \supseteq \mathcal{M}(a)$.

2. $E \in \mathcal{M}(a)$, we want to show that

(a) $\mathcal{G}(E)$ is a monotone class

$E \in \mathcal{M}(a)$, suppose $H_k \in \mathcal{G}(E), H_k \uparrow H$

$$\because E \setminus H_k \in \mathcal{M}(a), E \setminus H_k \downarrow E \setminus H \quad \therefore E \setminus H \in \mathcal{M}(a) \quad (5.7)$$

Similarity:

$$E \cap H \in \mathcal{M}(a) \quad (5.8)$$

$$H \setminus E \in \mathcal{M}(a) \quad (5.9)$$

then we can get $H \in \mathcal{G}(E)$, so $\mathcal{G}(E)$ is a monotone class.

(b) $\mathcal{G}(E) \supseteq a$

We need to show $H \in a \Rightarrow H \in \mathcal{G}(E)$.

By Lemma 5.1.1, we can get that

$$\mathcal{G}(H) \supseteq \mathcal{M}(a) \quad (5.10)$$

$\because E \in \mathcal{M}(a), \therefore E \in \mathcal{G}(H)$, by the Def 5.2, $H \setminus E, H \cap E, E \setminus H \in \mathcal{M}(a)$, so we can get $a \in \mathcal{G}(E)$

□

Theorem 5.1. a is a algebra, $a \subseteq \mathcal{P}(\Omega)$. $\mathcal{F}(a)$ is a σ -algebra generated by a , $\mathcal{M}(a)$ is a monotone class generated by a , then

$$\mathcal{F}(a) = \mathcal{M}(a) \quad (5.11)$$

Proof. By remark 5.1, $\mathcal{F}(a)$ is a monotone class, by Notation 5.1 $\mathcal{F}(a) \supseteq a$ and $\mathcal{F}(a) \supseteq \mathcal{M}(a)$.

So we have to show that

$$\mathcal{F}(a) \subseteq \mathcal{M}(a) \quad (5.12)$$

We will show that

1. $\mathcal{M}(a)$ is a algebra

(a) $\Omega \in \mathcal{M}(a)$ by $\Omega \subseteq a$

(b) $E \in \mathcal{M}(a) \Rightarrow E^c \in \mathcal{M}(a)$

By Lemma 5.1.1, let $E = \Omega$, then $\mathcal{M}(a) \subseteq \mathcal{G}(\Omega)$. $\because E \in \mathcal{M}(a)$, so $E \in \mathcal{G}(\Omega)$. By Definition 5.2, $\mathcal{G}(\Omega) = \{E \in \mathcal{M}(a), E^c, E, \emptyset \in \mathcal{M}(a)\}$

(c) $E, F \in \mathcal{M}(a) \Rightarrow E \cap F \in \mathcal{M}(a)$

By Lemma 5.1.2, $\mathcal{G}(E) \supseteq \mathcal{M}(a)$, so $F \in \mathcal{G}(E)$.

By Def 5.2 $F \in \mathcal{G}(E) = \{F \in \mathcal{M}(a), F \setminus E, F \cap E, E \setminus F \in \mathcal{M}(a)\}$, so $E \cap F \in \mathcal{M}(a)$

2. $\mathcal{M}(a)$ is a σ -algebra i.e. $A_j \in \mathcal{M}(a), j \geq 1 \Rightarrow \bigcup_{j \geq 1} A_j \in \mathcal{M}(a)$

By $\mathcal{M}(a)$ is a algebra, so $\bigcup_{j=1}^n A_j \in \mathcal{M}(a)$.

$\bigcup_{j=1}^n A_j \uparrow \bigcup_{j \geq 1} A_j$ and $\mathcal{M}(a)$ is a monotone class, so $\bigcup_{j \geq 1} A_j \in \mathcal{M}(a)$.

So $\mathcal{F}(a) \subseteq \mathcal{M}(a)$.

Above all,

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.13}$$

□

Lecture 6

The Lebesgue Measure I

Definition 6.1. $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$, we define \mathcal{S} as below:

$$\mathcal{S} = \{\emptyset, \mathbb{R}, (a, b], (a, \infty), (-\infty, b]\} \quad (6.1)$$

Remark 6.1. \mathcal{S} as above, then \mathcal{S} is a semialgebra

Proof. by Def 2.1. □

Definition 6.2. $\mu : \mathcal{S} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, additive, and

$$\mu(\emptyset) = 0, \mu((a, b]) = b - a, \mu((-\infty, b]) = +\infty, \mu(\mathbb{R}) = +\infty \quad (6.2)$$

Theorem 6.1. μ is additive on a semialgebra \mathcal{S} and defined as Def 6.2, then μ is σ -additive, i.e.

$$A = \sum_{j \geq 1} A_j \Rightarrow \mu(A) = \sum_{j \geq 1} \mu(A_j), \quad A, A_j \in \mathcal{S} \quad (6.3)$$

Remark 6.2. It is difficult to prove Thm 6.1 $(a, b] \cup (c, d]$ is not in the semialgebra \mathcal{S} . But, $\mathcal{S} \rightarrow \mathcal{a}(\mathcal{S})$ with respect to $\mu \rightarrow \nu$.

Proof.

1.

$$\because A = \sum_{j \geq 1} A_j \supseteq \sum_{j=1}^n A_j \quad (6.4)$$

By ν is additive $\Rightarrow \nu$ is monotone & subadditive,

$$\therefore \nu(A) \geq \nu\left(\sum_{j=1}^n A_j\right) = \sum_{j=1}^n \nu(A_j), \quad \forall n \quad (6.5)$$

so

$$\therefore \nu(A) \geq \sum_{j \geq 1} \nu(A_j) \quad (6.6)$$

2. (a) Assume that $A = (a, b], A_j = (a_j, b_j], A = \sum_{j \geq 1} A_j$, we want to show that

$$\nu(A) = b - a \leq \sum_{j \geq 1} (b_j - a_j) = \sum_{j \geq 1} \nu(A_j) \quad (6.7)$$

For any given $\epsilon > 0$, we have that

$$[a + \epsilon, b] \subseteq (a, b] = \sum_{j \geq 1} (a_j, b_j] \subseteq \bigcup_{j \geq 1} \left(a_j, b_j + \frac{\epsilon}{2^j}\right) \quad (6.8)$$

By a set K is compact i.e. K is closed and bounded \Rightarrow Any open cover for K has a finite subcover

$$[a + \varepsilon, b] \subseteq \bigcup_{k \geq 1} \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}} \right) \quad (6.9)$$

By ν is additive $\Rightarrow \nu$ is monotone & subadditive, we have

$$b - a - \varepsilon \leq \nu([a + \varepsilon, b]) = \nu \left(\bigcup_{k=1}^m \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}} \right) \right) \leq \sum_{k=1}^m \nu \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}} \right) \quad (6.10)$$

so we can get that

$$b - a - \varepsilon \leq \sum_{k=1}^m \left(b_{jk} - a_{jk} + \frac{\varepsilon}{2^{jk}} \right) \leq \sum_{j \geq 1} \left(b_j - a_j + \frac{\varepsilon}{2^j} \right) = \sum_{j \geq 1} (b - a) + \varepsilon \quad (6.11)$$

so Eq. 6.7 holds.

(b) General case $A \in \mathcal{S}$, $E_n = (-n, n] \uparrow \mathbb{R}$.

$$A \cap E_n = \sum_{j \geq 1} A_j \cap E_n.$$

By ν is additive on a semi-algebra

$$\nu(A \cap E_n) = \sum_{j \geq 1} \nu(A_j \cap E_n) \leq \sum_{j \geq 1} \nu(A_j) \quad (6.12)$$

By Remark 6.3, let $n \rightarrow \infty$, we have

$$\nu(A) = \lim_{n \rightarrow \infty} \nu(A \cap E_n) \leq \sum_{j \geq 1} \nu(A_j) \quad (6.13)$$

□

Remark 6.3. $E_n = (-n, n] \uparrow \mathbb{R}$, ν is additive on a semi-algebra then

$$\nu(A) = \lim_{n \rightarrow \infty} \nu(A \cap E_n) \quad (6.14)$$

Proof.

$$\because E_n \uparrow \mathbb{R}, \therefore A \cap E \uparrow, \therefore \lim_{n \rightarrow \infty} (A \cap E_n) = \bigcup_{n \geq 1} (A \cap E_n) = A \cap \left(\bigcup_{n \geq 1} E_n \right) = A \quad (6.15)$$

ν is additive,

$$\nu(A) = \nu \left(\bigcup_{n \geq 1} A \cap E_n \right) = \nu \left(\lim_{n \rightarrow \infty} A \cap E_n \right) \stackrel{\text{why}}{=} \lim_{n \rightarrow \infty} \nu(A \cap E_n) \quad (6.16)$$

why, because we will check via Def 6.1 except $A = (a, b]$

1. $A = \emptyset$

2. $A = \mathbb{R}$

3. $A = (a, \infty)$

(a) left hand of **why** in Eq. 6.16

$$\because A \cap E_n = (a, +\infty) \cap (-n, n) = \begin{cases} (a, n) & a \geq -n \\ (-n, n) & a < -n \end{cases} \quad (6.17)$$

$$\therefore \lim_{n \rightarrow \infty} (A \cap E_n) = (-\infty, +\infty) = \mathbb{R} \quad (6.18)$$

by Def 6.2

$$\mu \left(\lim_{n \rightarrow \infty} (A \cap E_n) \right) = \mu(\mathbb{R}) = +\infty \quad (6.19)$$

(b) right hand of **why** in Eq. 6.16

$$\because \nu(A \cap E_n) = \nu \left(\begin{cases} (a, n) & a \geq -n \\ (-n, n) & a < -n \end{cases} \right) = \begin{cases} n - a & a \geq -n \\ 2n & a < -n \end{cases} \quad (6.20)$$

$$\therefore \lim_{n \rightarrow \infty} \nu(A \cap E_n) = \lim_{n \rightarrow \infty} \begin{cases} n - a & a \geq -n \\ 2n & a < -n \end{cases} = +\infty \quad (6.21)$$

So Eq 6.16 holds.

4. $A = (-\infty, b]$

□

Lecture 7

The Lebesgue Measure II

Lecture 8

Complete Measures

Lecture 9

Approximation Theorems

Lecture 10

Integration: Measurable and Simple Functions