Tensor Singular Value Decomposition

Via Tensor-tensor Product

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1.1 Based on FFT

Tensor singular value decomposition is more and more popular in data science. Let's talk about this hot topic right now.

Definition 1.1 ([1], [2])**.** *Let $\mathcal{X}$ be an $n_1 \times n_2 \times n_3$ tensor and $\mathcal{Y}$ be an $n_2 \times n_4 \times n_3$ tensor. Then the t-product, denote by $\mathcal{X} * \mathcal{Y}$, is the $n_1 \times n_4 \times n_3$ tensor given by*

$$\mathcal{X} * \mathcal{Y} = fold(BlockCirc(\mathcal{X}) \cdot unfold(\mathcal{Y})) \tag{1.1}$$

where

$$BlockCirc(\mathcal{X}) \doteq \begin{bmatrix} X^{(1)} & X^{(n_3)} & \cdots & X^{(2)} \\ X^{(2)} & X^{(1)} & \cdots & X^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(n_3)} & X^{(n_3-1)} & \cdots & X^{(1)} \end{bmatrix} \in \mathbb{R}^{(n_1 n_3) \times (n_2 n_3)},$$ *and where $X^{(i)}$ is the i-th frontal slice of $\mathcal{X}$, i.e. holding the 3-rd index fixed and varying first two,*

$$unfold(\mathcal{Y}) \doteq \begin{bmatrix} Y^{(1)} \\ Y^{(2)} \\ \vdots \\ Y^{(n_3)} \end{bmatrix} \in \mathbb{R}^{(n_2 n_3) \times n_4},$$

and the inverse operator fold takes $unfold(\mathcal{X})$ into a tensor: $fold(unfold(\mathcal{Y})) = \mathcal{Y}$.

Remark 1.1 (Quick Look)**.**

$$f\left(\underbrace{n_1 \times n_2}_{part\ 1} \times \underbrace{n_3}_{part\ 2}, \underbrace{n_2 \times n_4}_{part\ 1} \times \underbrace{n_3}_{part\ 2}\right) \in \mathbb{R}^{(n_1 \times n_4) \times n_3}. \tag{1.2}$$

where f is the t-product.

If $v = [v_0, v_1, v_2, v_3]^T$, then

$$circ\,(v) = \begin{bmatrix} v_0 & v_3 & v_2 & v_1 \\ v_1 & v_0 & v_3 & v_2 \\ v_2 & v_1 & v_0 & v_3 \\ v_3 & v_2 & v_1 & v_0 \end{bmatrix} \tag{1.3}$$

Remark 1.2. *$v \to fft(v)$, $\mathbb{R}^{n \times 1} \to \mathbb{R}^{n \times 1}$, so $fft\,(\cdot) \in \mathbb{R}^{n \times n}$.*

Proposition 1.1. *Circulant matrices can be diagonalized with the normalized discrete Fourier transform (DFT) matrix (Thm 4.8.2 in the book [3]).*

That is to say

$$F_n circ\,(v)\, F_n^* = Diag(\overline{v}) \tag{1.4}$$

where F_n is the $n \times n$ DFT matrix, and $\overline{v} = fft\,(v)$.

Proof. See Appendix. ☐

Proposition 1.2. *Suppose $\mathcal{X}$ is $n_1 \times n_2 \times n_3$ and F_{n_3} is the $n_3 \times n_3$ DFT matrix. Then*

$$\left(F_{n_3} \otimes I_{n_1}\right) BlockCirc\,(\mathcal{X}) \left(F_{n_3}^* \otimes I_{n_2}\right) = BlockDiag\left(\overline{\mathcal{X}}\right) \tag{1.5}$$

where $\otimes$ denotes the Kronecker product, $\overline{\mathcal{X}}$ as the result of DFT on $\mathcal{X}$ along the 3-rd dimension, and

$$BlockDiag\left(\overline{\mathcal{X}}\right) = \begin{bmatrix} \overline{X}^{(1)} & 0 & \cdots & 0 \\ 0 & \overline{X}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{X}^{(n_3)} \end{bmatrix} \in \mathbb{R}^{n_1 n_3 \times n_2 n_3} \tag{1.6}$$

where $\overline{X}^{(i)}$ is i-th frontal slice of $\overline{\mathcal{X}}$.

Remark 1.3.

$$F_{n_3} \otimes I_{n_1} = \begin{bmatrix} \underbrace{\begin{matrix} (F_{n_3})_{11} & 0 & \cdots & 0 \\ 0 & (F^*_{n_3})_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (F_{n_3})_{11} \end{matrix}}_{\in \mathbb{R}^{n_1 \times n_1}} & \cdots & \cdots & \underbrace{\begin{matrix} (F_{n_3})_{1n_3} & 0 & \cdots & 0 \\ 0 & (F_{n_3})_{1n_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (F_{n_3})_{1n_3} \end{matrix}}_{\in \mathbb{R}^{n_1 \times n_1}} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} (F_{n_3})_{n_31} & 0 & \cdots & 0 \\ 0 & (F_{n_3})_{n_31} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (F_{n_3})_{n_31} \end{matrix} & \cdots & \cdots & \begin{matrix} (F_{n_3})_{n_3n_3} & 0 & \cdots & 0 \\ 0 & (F_{n_3})_{n_3n_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (F_{n_3})_{n_3n_3} \end{matrix} \end{bmatrix} \in \mathbb{R}^{n_3 n_1 \times n_3 n_1} \tag{1.7}$$

and

$$F_{n_3}^* \otimes I_{n_2} =$$

$$\left[\begin{array}{ccccc}
\underbrace{\begin{matrix} \left(F_{n_3}^*\right)_{11} & 0 & \cdots & 0 \\ 0 & \left(F_{n_3}^*\right)_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(F_{n_3}^*\right)_{11} \end{matrix}}_{\in \mathbb{R}^{n_2 \times n_2}} & \cdots & \cdots & \underbrace{\begin{matrix} \left(F_{n_3}^*\right)_{1n_3} & 0 & \cdots & 0 \\ 0 & \left(F_{n_3}^*\right)_{1n_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(F_{n_3}^*\right)_{1n_3} \end{matrix}}_{\in \mathbb{R}^{n_2 \times n_2}} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\begin{matrix} \left(F_{n_3}^*\right)_{n_31} & 0 & \cdots & 0 \\ 0 & \left(F_{n_3}^*\right)_{n_31} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(F_{n_3}^*\right)_{n_31} \end{matrix} & \cdots & \cdots & \begin{matrix} \left(F_{n_3}^*\right)_{n_3n_3} & 0 & \cdots & 0 \\ 0 & \left(F_{n_3}^*\right)_{n_3n_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(F_{n_3}^*\right)_{n_3n_3} \end{matrix}
\end{array}\right] \tag{1.8}$$

$$\in \mathbb{R}^{n_3n_2 \times n_3n_2}$$

Remark 1.4. *Kronecker product $A \otimes B$ means that element-wise of the matrix A product the matrix B.*

$$\begin{aligned}
A \otimes B &= \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix} \\
&= \begin{bmatrix} \underbrace{\begin{matrix} A_{11}B_{11} & A_{11}B_{12} & \cdots & A_{11}B_{1l} \\ A_{11}B_{21} & A_{11}B_{22} & \cdots & A_{11}B_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ A_{11}B_{k1} & A_{11}B_{k2} & \cdots & A_{11}B_{kl} \end{matrix}}_{A_{11}B} & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}
\end{aligned} \tag{1.9}$$

Proof. See Appendix. □

Proposition 1.3 (The Relation Between T-product and Facewise Product)**.**

If we denote $\overline{X} = BlockDiag\left(\overline{\mathcal{X}}\right)$, then

$$\mathcal{X} = \mathcal{A} * \mathcal{B} \Leftrightarrow \overline{X} = \overline{A}\,\overline{B} \tag{1.10}$$

Before proof Proposition 1.3, we will give an example as below:

Example 1.1. [*bibid*] *Suppose* $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times 3}$ *and* $\mathcal{B} \in \mathbb{R}^{n_2 \times n_4 \times 3}$*. Then*

$$\begin{aligned} \mathcal{A} * \mathcal{B} &= fold(BlockCirc(\mathcal{A})unfold(\mathcal{C}) \\ &= fold\left(\begin{bmatrix} A^1 & A^3 & A^2 \\ A^2 & A^1 & A^3 \\ A^3 & A^2 & A^1 \end{bmatrix} \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix}\right) \end{aligned} \tag{1.11}$$

By Proposition 1.2

$$\left(F_3 \otimes I_{n_1}\right) BlockCirc\left(\mathcal{A}\right)\left(F_3^* \otimes I_{n_2}\right) = \begin{bmatrix} \left(\overline{A}\right)^1 & 0 & 0 \\ 0 & \left(\overline{A}\right)^2 & 0 \\ 0 & 0 & \left(\overline{A}\right)^2 \end{bmatrix} \tag{1.12}$$

where F_3 *is a* 3×3 *normalized Fourier transform matrix,* $\left(\overline{A}\right)^i$ *(*$i = 1, 2, 3$*) is the* i*-th frontal slice of* $\overline{\mathcal{A}}$*.*

Nextly,

$$\begin{aligned} & BlockCirc(\mathcal{A})unfold(\mathcal{B}) \\ &= \left(F_3^* \otimes I_{n_1}\right)\left(F_3 \otimes I_{n_1}\right) BlockCirc\left(\mathcal{A}\right)\left(F_3^* \otimes I_{n_2}\right)\left(F_3 \otimes I_{n_2}\right) unfold(\mathcal{B}) \\ &= \left(F_3^* \otimes I_{n_1}\right) \begin{bmatrix} \left(\overline{A}\right)^1 & 0 & 0 \\ 0 & \left(\overline{A}\right)^2 & 0 \\ 0 & 0 & \left(\overline{A}\right)^3 \end{bmatrix} \left(F_3 \otimes I_{n_2}\right) \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix} \\ &= \left(F_3^* \otimes I_{n_1}\right) \begin{bmatrix} \left(\overline{A}\right)^1 & 0 & 0 \\ 0 & \left(\overline{A}\right)^2 & 0 \\ 0 & 0 & \left(\overline{A}\right)^3 \end{bmatrix} \begin{bmatrix} \left(\overline{B}\right)^1 \\ \left(\overline{B}\right)^2 \\ \left(\overline{B}\right)^3 \end{bmatrix} \\ &= \left(F_3^* \otimes I_{n_1}\right) \begin{bmatrix} \left(\overline{A}\right)^1 \left(\overline{B}\right)^1 \\ \left(\overline{A}\right)^2 \left(\overline{B}\right)^2 \\ \left(\overline{A}\right)^3 \left(\overline{B}\right)^3 \end{bmatrix} \\ &= \begin{bmatrix} F_3^* \left(\overline{A}\right)^1 \left(\overline{B}\right)^1 \\ F_3^* \left(\overline{A}\right)^2 \left(\overline{B}\right)^2 \\ F_3^* \left(\overline{A}\right)^2 \left(\overline{B}\right)^3 \end{bmatrix} \end{aligned} \tag{1.13}$$

Proof.

By Def 1.1

$$unfold(\mathcal{X}) = BlockCirc(\mathcal{A}) \cdot unfold(\mathcal{B}) \tag{1.14}$$

by Rmk 1.2

$$\begin{aligned} unfold(\mathcal{X}) =& BlockCirc(\mathcal{A}) \cdot unfold(\mathcal{B}) \\ &= \left(F_{n_3}^{-1} \otimes I_{n_1}\right)\left(F_{n_3} \otimes I_{n_1}\right) BlockCirc\left(\mathcal{A}\right)\left(F_{n_3}^{-1} \otimes I_{n_2}\right)\left(\left(F_{n_3} \otimes I_{n_2}\right) unfold\left(\mathcal{B}\right)\right) \\ &= \left(F_{n_3}^{-1} \otimes I_{n_1}\right) \cdot \overline{A} \cdot unfold\left(\overline{\mathcal{B}}\right) \end{aligned} \tag{1.15}$$

Eq 1.15 multiplying both sides with $\left(F_{n_3} \otimes I_{n_1}\right) \Rightarrow unfold\left(\overline{\mathcal{X}}\right) = \overline{A} \cdot unfold\left(\overline{\mathcal{B}}\right) \Rightarrow \overline{X} = \overline{A}\,\overline{B}$. $\square$

Algorithm 1 Tensor-Tensor Product [4]

Input: $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}, \mathcal{B} \in \mathbb{R}^{n_2 \times n_4 \times n_3}$.

1. Compute $\overline{\mathcal{A}} = fft(\mathcal{A}, [\,], 3)$ and $\overline{\mathcal{B}} = fft(\mathcal{B}, [\,], 3)$
2. Compute each frontal slice $\overline{\mathcal{X}}$ by

$$\overline{X}^{(i)} = \begin{cases} \overline{A}^{(i)}\overline{B}^{i}, & i = 1, ..., \left[\frac{n_3+1}{2}\right], \\ conj\left(\overline{X}^{(n_3-i+2)}\right), & i = \left[\frac{n_3+1}{2}\right] + 1, ..., n_3. \end{cases}$$

3. Compute $\mathcal{X} = ifft(\overline{\mathcal{X}}, [\,], 3)$.

Output: $\mathcal{X} = \mathcal{A} * \mathcal{B} \in \mathbb{R}^{n_1 \times n_4 \times n_3}$.

Remark 1.5. *Proposition 1.3 suggests an efficient way based on FFT to computer t-product instead of using the definition of tensor-tensor product as Algorithm 1.*

Definition 1.2 (Conjugate Transpose)**.** *The conjugate transpose of a tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is the tensor $\mathcal{X}^T \in \mathbb{R}^{n_2 \times n_1 \times n_3}$ obtain by conjugate transposing each of the frontal slices and reversing the order of transposed frontal slice 2 through n_3, i.e.*

$$\left(X^T\right)^i = \begin{cases} \left(X^T\right)^1 = \left(X^1\right)^T, & i = 1 \\ \left(X^T\right)^i = \left(X^{n_3-i+2}\right)^T, & i = 2, 3, ..., n_3 \end{cases} \tag{1.16}$$

where $\left(X^T\right)^i$ is the i-th frontal slice of $\mathcal{X}^T$.

Remark 1.6. *If $\mathcal{X} = \left[X^1, X^2, X^3, \cdots, X^{n_3-2}, X^{n_3-1}, X^{n_3}\right]$ is ordered by frontal slices, then*

$$\begin{aligned} \mathcal{X} &= \left[X^1, X^2, X^3, \cdots, X^{n_3-2}, X^{n_3-1}, X^{n_3}\right] \\ \mathcal{X}^T &= \left[\left(X^1\right)^T, \left(X^{n_3}\right)^T, \left(X^{n_3-1}\right)^T, \cdots, \left(X^3\right)^T, \left(X^2\right)^T, \left(X^1\right)^T\right] \end{aligned} \tag{1.17}$$

$$BlockCirc\left(\mathcal{X}^T\right) = \begin{bmatrix} \left(X^1\right)^T & \left(X^2\right)^T & \cdots & \left(X^{n_3-1}\right)^T & \left(X^{n_3}\right)^T \\ \left(X^{n_3}\right)^T & \left(X^1\right)^T & \cdots & \left(X^{n_3-2}\right)^T & \left(X^{n_3-1}\right)^T \\ \left(X^{n_3-1}\right)^T & \left(X^{n_3}\right)^T & \cdots & \left(X^{n_3-3}\right)^T & \left(X^{n_3-2}\right)^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \left(X^2\right)^T & \left(X^3\right)^T & \cdots & \left(X^{n_3}\right)^T & \left(X^1\right)^T \end{bmatrix} \tag{1.18}$$

$$BlockCirc(\mathcal{X}) = \begin{bmatrix} X^1 & X^{n_3} & \cdots & X^3 & X^2 \\ X^2 & X^1 & \cdots & X^4 & X^3 \\ X^3 & X^2 & \cdots & X^5 & X^4 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X^{n_3} & X^{n_3-2} & \cdots & X^2 & X^1 \end{bmatrix} \tag{1.19}$$

then

$$BlockCirc\left(\mathcal{X}^T\right) = \left(BlockCirc(\mathcal{X})\right)^T \tag{1.20}$$

We can proof Eq 1.20 directly, but we give an example as below,

$$\mathcal{X} = \left(\underbrace{\begin{matrix}1&2&3\\4&5&6\\7&8&9\end{matrix}}_{X^1}, \underbrace{\begin{matrix}10&11&12\\13&14&15\\16&17&18\end{matrix}}_{X^2}, \underbrace{\begin{matrix}19&20&21\\22&23&24\\25&26&27\end{matrix}}_{X^3} \right) \tag{1.21}$$

then

$$BlockCirc(\mathcal{X}) = \begin{bmatrix} \underbrace{\begin{matrix}1&2&3\\4&5&6\\7&8&9\end{matrix}}_{X^1} & \underbrace{\begin{matrix}19&20&21\\22&23&24\\25&26&27\end{matrix}}_{X^3} & \underbrace{\begin{matrix}10&11&12\\13&14&15\\16&17&18\end{matrix}}_{X^2} \\ \underbrace{\begin{matrix}10&11&12\\13&14&15\\16&17&18\end{matrix}}_{X^2} & \underbrace{\begin{matrix}1&2&3\\4&5&6\\7&8&9\end{matrix}}_{X^1} & \underbrace{\begin{matrix}19&20&21\\22&23&24\\25&26&27\end{matrix}}_{X^3} \\ \underbrace{\begin{matrix}19&20&21\\22&23&24\\25&26&27\end{matrix}}_{X^3} & \underbrace{\begin{matrix}10&11&12\\13&14&15\\16&17&18\end{matrix}}_{X^2} & \underbrace{\begin{matrix}1&2&3\\4&5&6\\7&8&9\end{matrix}}_{X^1} \end{bmatrix} \tag{1.22}$$

$$\mathcal{X}^T = \left(\underbrace{\begin{matrix}1&2&3\\4&5&6\\7&8&9\end{matrix}}_{X^1=(X^T)^1}, \underbrace{\begin{matrix}19&20&21\\22&23&24\\25&26&27\end{matrix}}_{X^3=(X^T)^2}, \underbrace{\begin{matrix}10&11&12\\13&14&15\\16&17&18\end{matrix}}_{X^2=(X^T)^3} \right) \tag{1.23}$$

then

$$BlockCirc\left(\mathcal{X}^T\right) = \begin{pmatrix} \underbrace{\begin{matrix}1&4&7\\2&5&8\\3&6&9\end{matrix}}_{(X^1)^T=(X^T)^1} & \underbrace{\begin{matrix}10&13&16\\11&14&17\\12&15&18\end{matrix}}_{(X^2)^T=(X^T)^3} & \underbrace{\begin{matrix}19&22&25\\20&23&26\\21&24&27\end{matrix}}_{(X^3)^T=(X^T)^2} \\ \underbrace{\begin{matrix}19&22&25\\20&23&26\\21&24&27\end{matrix}}_{(X^3)^T=(X^T)^2} & \underbrace{\begin{matrix}1&4&7\\2&5&8\\3&6&9\end{matrix}}_{(X^1)^T=(X^T)^1} & \underbrace{\begin{matrix}10&13&16\\11&14&17\\12&15&18\end{matrix}}_{(X^2)^T=(X^T)^3} \\ \underbrace{\begin{matrix}10&13&16\\11&14&17\\12&15&18\end{matrix}}_{(X^2)^T=(X^T)^3} & \underbrace{\begin{matrix}19&22&25\\20&23&26\\21&24&27\end{matrix}}_{(X^3)^T=(X^T)^2} & \underbrace{\begin{matrix}1&4&7\\2&5&8\\3&6&9\end{matrix}}_{(X^1)^T=(X^T)^1} \end{pmatrix} = \left(BlockCirc\left(\mathcal{X}\right)\right)^T \tag{1.24}$$

Definition 1.3 (Identity Tensor)**.** *A $n \times n \times m$ identity tensor $\mathcal{I}$ is the tensor whose first frontal* *slice is the $n \times n$ identity matrix, and whose other frontal slices are all zeros, i.e.*

$$I^1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad I^i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{n\times n}, 2 \leqslant i \leqslant m \tag{1.25}$$

Remark 1.7.

$$\mathcal{X} * \mathcal{I} = fold\left(\begin{bmatrix} X^1 \in \mathbb{R}^{n_1\times n_2} & X^{n_3} & \cdots & X^2 \\ X^2 & X^1 & \cdots & X^3 \\ \vdots & \vdots & \ddots & \vdots \\ X^{n_3} & X^{n_3-1} & \cdots & X^1 \end{bmatrix}\begin{bmatrix} I \in \mathbb{R}^{n_1\times n_2} \\ 0 \in \mathbb{R}^{n_1\times n_2} \\ \vdots \\ 0 \in \mathbb{R}^{n_1\times n_2} \end{bmatrix}\right) = fold\left(\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^{n_3} \end{bmatrix}\right) = \mathcal{X} \tag{1.26}$$

and

$$\mathcal{I} * \mathcal{X} = fold\left(\begin{bmatrix} I \in \mathbb{R}^{n_1\times n_2} & 0 & \cdots & 0 \\ 0 \in \mathbb{R}^{n_1\times n_2} & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 \in \mathbb{R}^{n_1\times n_2} & 0 & \cdots & I \end{bmatrix}\begin{bmatrix} X^1 \in \mathbb{R}^{n_1\times n_2} \\ X^2 \in \mathbb{R}^{n_1\times n_2} \\ \vdots \\ X^{n_3} \in \mathbb{R}^{n_1\times n_2} \end{bmatrix}\right) = fold\left(\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^{n_3} \end{bmatrix}\right) = \mathcal{X} \tag{1.27}$$

Definition 1.4 (Orthogonal Tensor)**.** *A $n \times n \times m$ real-valued tensor $\mathcal{Q}$ is orthogonal if*

$$\mathcal{Q}^T * \mathcal{Q} = \mathcal{Q} * \mathcal{Q}^T = \mathcal{I}. \tag{1.28}$$

Definition 1.5 (f-diagonal)**.** *A tensor is f-diagonal if each frontal slice is diagonal. i.e expect $\mathcal{X}_{1,1,i}, \mathcal{X}_{2,2,i}, \ldots \mathcal{X}_{l,l,i}$ are all 0's, where $l = min\{n_1, n_2\}, 1 \leq i \leq n_3$*

Definition 1.6 (Inverse of Tensor)**.** *A $n \times n \times m$ tensor $\mathcal{A}$ has am inverse $\mathcal{B}$ if*

$$\mathcal{A} * \mathcal{B} = \mathcal{I} \quad \& \quad \mathcal{B} * \mathcal{A} = \mathcal{I} \tag{1.29}$$

where $\mathcal{I} \in \mathbb{C}^{n\times n\times m}$.

Proposition 1.4.

$$\mathcal{A} * (\mathcal{B} * \mathcal{C}) = (\mathcal{A} * \mathcal{B}) * \mathcal{C} \tag{1.30}$$

where $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$, $\mathcal{B} \in \mathbb{R}^{n_2\times n_4\times n_3}$ and $\mathcal{C} \in \mathbb{R}^{n_4\times n_5\times n_3}$

Proof. Firstly,

$$\underbrace{\mathcal{A} * \underbrace{(\mathcal{B} * \mathcal{C})}_{\mathcal{C}^{n_2\times n_5\times n_3}}}_{\mathbb{R}^{n_1\times n_5\times n_3}}, \quad \underbrace{\underbrace{(\mathcal{A} * \mathcal{B})}_{\mathbb{R}^{n_1\times n_4\times n_3}} * \mathcal{C}}_{\mathbb{R}^{n_1\times n_5\times n_3}} \tag{1.31}$$

so it's make sense.

Then

$$
\begin{aligned}
&\mathcal{A} * (\mathcal{B} * \mathcal{C}) \\
&\quad = \mathcal{A} * fold(\begin{bmatrix} B^1 & B^{n_3} & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^{n_3} & B^{n_3-1} & \cdots & B^2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix}) \\
&\quad = fold(\begin{bmatrix} A^1 & A^{n_3} & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^3 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3} & A^{n_3-1} & \cdots & A^1 \end{bmatrix} unfold\left(fold\left(\begin{bmatrix} B^1 & B^{n_3} & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^{n_3} & B^{n_3-1} & \cdots & B^2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix} \right)\right)) \\
&\quad = fold(\begin{bmatrix} A^1 & A^{n_3} & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^3 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3} & A^{n_3-1} & \cdots & A^1 \end{bmatrix} \left(\begin{bmatrix} B^1 & B^{n_3} & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^{n_3} & B^{n_3-1} & \cdots & B^2 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix} \right)) \\
&\quad = fold(\begin{bmatrix} A^1 & A^{n_3} & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^3 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3} & A^{n_3-1} & \cdots & A^1 \end{bmatrix} \begin{bmatrix} B^1 & B^{n_3} & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^{n_3} & B^{n_3-1} & \cdots & B^1 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix})
\end{aligned}
\tag{1.32}
$$

and

$$
\begin{aligned}
&(\mathcal{A} * \mathcal{B}) * \mathcal{C} \\
&= fold\left(\begin{bmatrix} A^1 & A^{n_3} & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^3 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3} & A^{n_3-1} & \cdots & A^1 \end{bmatrix} \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^{n_3} \end{bmatrix} \right) * \mathcal{C} \\
&= fold(\begin{bmatrix} A^1B^1 + A^{n_3}B^2 + \cdots A^2B^{n_3} & A^{n_3}B^1 + A^{n_3-1}B^2 + \cdots A^1B^{n_3} & \cdots & A^2B^1 + A^1B^2 + \cdots A^3B^{n_3} \\ A^2B^1 + A^1B^2 + \cdots A^3B^{n_3} & A^1B^1 + A^{n_3}B^2 + \cdots A^2B^{n_3} & \cdots & A^3B^1 + A^2B^2 + \cdots A^4B^{n_3} \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3}B^1 + A^{n_3-1}B^2 + \cdots A^1B^{n_3} & A^{n_3-1}B^1 + A^{n_3-2}B^2 + \cdots A^{n_3}B^{n_3} & \cdots & A^1B^1 + A^{n_3}B^2 + \cdots A^2B^{n_3} \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix}) \\
&= fold\left(\begin{bmatrix} A^1B^1 + A^{n_3}B^2 + \cdots + A^2B^{n_3} & A^1B^{n_3} + A^{n_3}B^1 + \cdots + A^2B^{n_3-1} & \cdots & A^1B^2 + A^2B^2 + \cdots + A^3B^1 \\ A^2B^1 + A^1B^2 + \cdots + A^3B^{n_3} & A^2B^{n_3} + A^1B^1 + \cdots + A^3B^{n_3-1} & \cdots & A^2B^2 + A^1B^3 + \cdots + A^3B^1 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3}B^1 + A^{n_3-1}B^2 + \cdots + A^1B^{n_3} & A^{n_3}B^{n_3} + A^{n_3-1}B^1 + \cdots A^1B^{n_3-1} & \cdots & A^{n_3}B^2 + A^{n_3-1}B^3 + \cdots + A^1B^1 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix} \right) \\
&= fold(\begin{bmatrix} A^1 & A^{n_3} & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^3 \\ \vdots & \vdots & \ddots & \vdots \\ A^{n_3} & A^{n_3-1} & \cdots & A^1 \end{bmatrix} \begin{bmatrix} B^1 & B^{n_3} & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^{n_3} & B^{n_3-1} & \cdots & B^1 \end{bmatrix} \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{n_3} \end{bmatrix})
\end{aligned}
\tag{1.33}
$$

$$
\therefore \mathcal{A} * (\mathcal{B} * \mathcal{C}) = (\mathcal{A} * \mathcal{B}) * \mathcal{C} \tag{1.34}
$$

Algorithm 2 Tensor SVD

Input: $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$.

$\mathcal{D} = fft(\mathcal{A}, [\,], 3)$

for = 1,...,n_3

[U,S,V] = SVD($\mathcal{D}(:,:,i)$);

$\mathcal{U}(:,:,i) = U, \mathcal{V}(:,:,i) = V, \mathcal{S}(:,:,i) = S.$

Output: $\mathcal{U} = ifft(\mathcal{U}, [\,], 3), \mathcal{V} = ifft(\mathcal{V}, [\,], 3), \mathcal{S} = ifft(\mathcal{S}, [\,], 3)$

□

Proposition 1.5.

$$(\mathcal{A} * \mathcal{B})^T = \mathcal{B}^T * \mathcal{A}^T \tag{1.35}$$

Proof.

$$\begin{aligned}
\mathcal{B}^T * \mathcal{A}^T &= fold\left(BlockCirc\left(B^T\right)\begin{bmatrix}\left(A^T\right)^1 \\ \left(A^T\right)^{n_3} \\ \vdots \\ \left(A^T\right)^3\end{bmatrix}\right) \\
&= fold\left(\begin{pmatrix}\left(B^T\right)^1 & \left(B^T\right)^2 & \cdots & \left(B^T\right)^{n_3} \\ \left(B^T\right)^{n_3} & \left(B^T\right)^1 & \cdots & \left(B^T\right)^{n_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ \left(B^T\right)^2 & \left(B^T\right)^3 & \cdots & \left(B^T\right)^1\end{pmatrix}\begin{bmatrix}\left(A^T\right)^1 \\ \left(A^T\right)^{n_3} \\ \vdots \\ \left(A^T\right)^3\end{bmatrix}\right)
\end{aligned} \tag{1.36}$$

$$content... \tag{1.37}$$

□

Proposition 1.6. *$\mathcal{Q}$ is an orthogonal tensor, then*

$$\|\mathcal{Q} * \mathcal{A}\|_F = \|\mathcal{A}\|_F \tag{1.38}$$

Theorem 1.1 (Tensor SVD)**.** *For any $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, the t-SVD of $\mathcal{X}$ is given by*

$$\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T \tag{1.39}$$

where $\mathcal{U} \in \mathbb{R}^{n_1 \times n_1 \times n_3}$ and $\mathcal{V} \in \mathbb{R}^{n_2 \times n_2 \times n_3}$ are orthogonal, $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ is f-diagonal.

Proof. By proposition 1.2, $BlockCirc(\mathcal{X})$ can be diagonalized as

$$\left(F_{n_3} \otimes I_{n_1}\right) BlockCirc\left(\mathcal{X}\right)\left(F_{n_3}^* \otimes I_{n_2}\right) = BlockDiag\left(\overline{\mathcal{X}}\right) = \begin{bmatrix} \left(\overline{X}\right)^1 & 0 & \cdots & 0 \\ 0 & \left(\overline{X}\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\overline{X}\right)^{n_3} \end{bmatrix} \tag{1.40}$$

Compute the SVD of each $\left(\overline{X}\right)^i$ as $\left(\overline{X}\right)^i = U^i S^i \left(V^T\right)^i$. Then

$$\begin{aligned} &\begin{bmatrix} \left(\overline{X}\right)^1 & 0 & \cdots & 0 \\ 0 & \left(\overline{X}\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\overline{X}\right)^{n_3} \end{bmatrix} \\ &= \begin{bmatrix} U^1 & 0 & \cdots & 0 \\ 0 & U^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & U^{n_3} \end{bmatrix} \begin{bmatrix} S^1 & 0 & \cdots & 0 \\ 0 & S^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & S^{n_3} \end{bmatrix} \begin{bmatrix} \left(V^T\right)^1 & 0 & \cdots & 0 \\ 0 & \left(V^T\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \left(V^T\right)^{n_3} \end{bmatrix} \end{aligned} \tag{1.41}$$

Then

$$\begin{aligned} &\left(F_{n_3}^* \otimes I_{n_1}\right) \begin{bmatrix} \underbrace{U^1}_{\in \mathbb{R}^{n_1 \times n_1}} & \underbrace{0}_{\in \mathbb{R}^{n_1 \times n_1}} & \cdots & 0 \\ 0 & U^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U^{n_3} \end{bmatrix} \left(F_{n_3} \otimes I_{n_1}\right) \\ &= \begin{bmatrix} \underbrace{\widetilde{U}^1}_{\in \mathbb{R}^{n_1 \times n_1}} & \widetilde{U}^{n_3} & \cdots & \widetilde{U}^2 \\ \widetilde{U}^2 & \widetilde{U}^1 & \cdots & \widetilde{U}^3 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{U}^{n_3} & \widetilde{U}^{(n_3-1)} & \cdots & \widetilde{U}^1 \end{bmatrix} \\ &= BlockCirc\left(\mathcal{U}\right) \end{aligned} \tag{1.42}$$

$$\begin{aligned} &\left(F_{n_3}^* \otimes I_{n_1}\right) \begin{bmatrix} \underbrace{S^1}_{\in \mathbb{R}^{n_1 \times n_2}} & \underbrace{0}_{\in \mathbb{R}^{n_1 \times n_2}} & \cdots & 0 \\ 0 & S^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S^{n_3} \end{bmatrix} \left(F_{n_3} \otimes I_{n_2}\right) \\ &= \begin{bmatrix} \underbrace{\widetilde{S}^1}_{\in \mathbb{R}^{n_1 \times n_2}} & \widetilde{S}^{n_3} & \cdots & \widetilde{S}^2 \\ \widetilde{S}^2 & \widetilde{S}^1 & \cdots & \widetilde{S}^3 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{S}^{n_3} & \widetilde{S}^{(n_3-1)} & \cdots & \widetilde{S}^1 \end{bmatrix} \\ &= BlockCirc\left(\mathcal{S}\right) \end{aligned} \tag{1.43}$$

and

$$
\begin{aligned}
&\left(F_{n_3}^* \otimes I_{n_2}\right)\begin{bmatrix} \underbrace{\left(V^T\right)^1}_{\in R^{n_2\times n_2}} & \underbrace{0}_{\in R^{n_2\times n_2}} & \cdots & 0 \\ 0 & \left(V^T\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(V^T\right)^{n_3} \end{bmatrix}\left(F_{n_3} \otimes I_{n_2}\right) \\
&= \begin{bmatrix} \underbrace{\left(\widetilde{V^T}\right)^1}_{\in R^{n_2\times n_2}} & \left(\widetilde{V^T}\right)^{n_3} & \cdots & \left(\widetilde{V^T}\right)^2 \\ \left(\widetilde{V^T}\right)^2 & \left(\widetilde{V^T}\right)^1 & \cdots & \left(\widetilde{V^T}\right)^3 \\ \vdots & \vdots & \ddots & \vdots \\ \left(\widetilde{V^T}\right)^{n_3} & \left(\widetilde{V^T}\right)^{(n_3-1)} & \cdots & \left(\widetilde{V^T}\right)^1 \end{bmatrix} \\
&= BlockCirc\left(\mathcal{V}^T\right)
\end{aligned} \tag{1.44}
$$

are block circulant matrices.

Then we define that

$$
unfold(\mathcal{U}) = \begin{bmatrix} \widetilde{U}^1 \\ \widetilde{U}^2 \\ \vdots \\ \widetilde{U}^{n_3} \end{bmatrix}, unfold(\mathcal{S}) = \begin{bmatrix} \widetilde{S}^1 \\ \widetilde{S}^2 \\ \vdots \\ \widetilde{S}^{n_3} \end{bmatrix} \;\; and \;\; unfold\left(\mathcal{V}\right) = \begin{bmatrix} \widetilde{V}^1 \\ \widetilde{V}^2 \\ \vdots \\ \widetilde{V}^{n_3} \end{bmatrix} \tag{1.45}
$$

then

$$
unfold\left(\mathcal{V}^T\right) = \begin{bmatrix} \left(\widetilde{V}^1\right)^T \\ \left(\widetilde{V}^{n_3}\right)^T \\ \vdots \\ \left(\widetilde{V}^2\right)^T \end{bmatrix} \tag{1.46}
$$

Now, we check that

$$
\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T \tag{1.47}
$$

and

$$
\mathcal{U} * \mathcal{U}^T = \mathcal{I}_{n_1\times n_1\times n_3}, \quad \mathcal{V} * \mathcal{V}^T = \mathcal{I}_{n_2\times n_2\times n_3} \tag{1.48}
$$

1.

$$
\begin{aligned}
& BlockCirc\,(\mathcal{X}) \\
&= \left(F_{n_3}^{*} \otimes I_{n_1}\right) \begin{bmatrix} \left(\overline{X}\right)^{1} & 0 & \cdots & 0 \\ 0 & \left(\overline{X}\right)^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\overline{X}\right)^{n_3} \end{bmatrix} \left(F_{n_3} \otimes I_{n_2}\right) \\
&= \left(F_{n_3}^{*} \otimes I_{n_1}\right) \begin{bmatrix} U^{1} & 0 & \cdots & 0 \\ 0 & U^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U^{n_3} \end{bmatrix} \left(F_{n_3} \otimes I_{n_1}\right) \left(F_{n_3}^{*} \otimes I_{n_1}\right) \\
& \begin{bmatrix} S^{1} & 0 & \cdots & 0 \\ 0 & S^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S^{n_3} \end{bmatrix} \left(F_{n_3} \otimes I_{n_2}\right) \left(F_{n_3}^{*} \otimes I_{n_2}\right) \begin{bmatrix} \left(V^{i}\right)^{T} & 0 & \cdots & 0 \\ 0 & \left(V^{2}\right)^{T} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(V^{n_3}\right)^{T} \end{bmatrix} \left(F_{n_3} \otimes I_{n_2}\right) \\
&= BlockCirc\,(\mathcal{U})\, BlockCirc\,(\mathcal{S})\, BlockCirc\left(\mathcal{V}^{T}\right)
\end{aligned}
\tag{1.49}
$$

By Eq 1.32 or Eq 1.33, we have that

$$
\mathcal{U} * \mathcal{S} * \mathcal{V}^{T} = fold\left(BolckCirc\,(\mathcal{U})\, BlockCirc\,(\mathcal{S})\, unfold\left(\mathcal{V}^{T}\right)\right) \tag{1.50}
$$

And

$$
\begin{aligned}
& BlockCirc\left(\mathcal{U} * \mathcal{S} * \mathcal{V}^{T}\right) \\
=& BlockCirc\left(fold\,(BolckCirc\,(\mathcal{U}))\, BlockCirc\,(\mathcal{S})\, unfold\left(\mathcal{V}^{T}\right)\right) \\
=& BlockCirc\,(\mathcal{U})\, BlockCirc\,(\mathcal{S})\, BlockCirc\left(\mathcal{V}^{T}\right) (by\ Remark\ 1.8) \\
=& BlockCirc(\mathcal{X})
\end{aligned}
\tag{1.51}
$$

By Eq 1.51

$$
\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^{T} \tag{1.52}
$$

2.

$$
\mathcal{U} * \mathcal{U}^{T} = fold\left(\begin{bmatrix} U^{1} & U^{n_3} & \cdots & U^{2} \\ U^{2} & U^{1} & \cdots & U^{3} \\ \vdots & \vdots & \ddots & \vdots \\ U^{n_3} & U^{n_3} & \cdots & U^{1} \end{bmatrix} \begin{bmatrix} \left(U^{1}\right)^{T} \\ \left(U^{n_3}\right)^{T} \\ \vdots \\ \left(U^{2}\right)^{T} \end{bmatrix} \right) = fold \begin{bmatrix} I_{n_1 \times n_1} \\ \underbrace{0}_{\in \mathbb{R}^{n_1 \times n_1}} \\ \vdots \\ 0 \end{bmatrix} \tag{1.53}
$$

where $U^{i} = \widetilde{U}^{i}$ in Eq 1.53.

$\square$

Remark 1.8.

$$\begin{bmatrix} \underbrace{A^1}_{\mathbb{R}^{k\times l}} & A^m & \cdots & A^2 \\ A^2 & A^1 & \cdots & A^2 \\ \vdots & \vdots & \ddots & \vdots \\ A^m & A^{m-1} & \cdots & A^1 \end{bmatrix} \begin{bmatrix} \underbrace{B^1}_{\mathbb{R}^{l\times n}} & B^m & \cdots & B^2 \\ B^2 & B^1 & \cdots & B^3 \\ \vdots & \vdots & \ddots & \vdots \\ B^m & B^{m-1} & \cdots & B^1 \end{bmatrix} = \begin{bmatrix} C^1 & C^m & \cdots & C^2 \\ C^2 & C^1 & \cdots & C^3 \\ \vdots & \vdots & \ddots & \vdots \\ C^m & C^{m-1} & \cdots & C^1 \end{bmatrix} \tag{1.54}$$

$$BlockCirc\ \left(fold\left(BlockCirl\mathcal{C}\right)unfold\left(\mathcal{D}\right)\right) = BlockCirc\left(\mathcal{C}\right)BlockCirc\left(\mathcal{D}\right) \tag{1.55}$$

Definition 1.7 (Multi Rank)**.** *The multi-rank of $\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times n_3}$ is a vector $\mathbb{R}^{n_3}$ with the i-th entry equal to the rank of the frontal slice of $\overline{\mathcal{X}}$ obtained by taking the Fourier transform along the 3r dimension of $\mathcal{X}$.*

Definition 1.8 (Tubal Rank)**.** *The tubal rank of $\mathcal{X}$, denoted by $rank_t\left(\mathcal{X}\right)$, is defined as the number of nonzero singular tubes of $\mathcal{S}$, i.e.,*

$$rank_t\left(\mathcal{X}\right) = \#\left\{i : \mathcal{S}\left(i,i,:\right) \neq 0 \in \mathbb{R}^{n_3\times 1}\right\} = \max\left\{r_1, \cdots, r_{n_3}\right\} \tag{1.56}$$

where $\mathcal{X} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T$ and $\left(\overline{X}\right)^i = U^i\Sigma^i\left(V^i\right)^T,\ r_i = Rank\left(\Sigma^i\right)$

Definition 1.9 (Tensor Nuclear Norm)**.** *The tensor nuclear norm of a tensor $\mathcal{X} \in \mathbb{R}^{n_1\times n_2\times n_3}$ is defined as*

$$\|\mathcal{X}\|_{TNN} = \frac{1}{n_3}\left\|\overline{X}\right\|_* \tag{1.57}$$

where

$$\overline{X} = BlockDiag\left(\overline{X}\right) = \begin{bmatrix} \overline{X}^1 & 0 & \cdots & 0 \\ 0 & \overline{X}^{(3)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{X}^{(n_3)} \end{bmatrix} \tag{1.58}$$

Definition 1.10 (Tensor Spectral Norm)**.** $\|\mathcal{X}\| \triangleq \left\|\overline{X}\right\|$

1.2 Acknowledge

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1.3 References

[1] M. Kilmer and C. Martin, "Factorization Strategies for Third-order Tensors", *Linear Algebra and its Applications*, 2011, Vol. 435(3), pp. 641–658.

[2] M. Kilmer, K. Braman, N. Hao and R. Hoover, "Third-order Tensors as 0perators on Matrices: A Theoretical and Computational Framework with Applications in Imaging", *SIAM Journal on Matrix Analysis and Applications*, 2013, Vol. 34(1), pp. 148–172.

[3] G. Golub and C. Van Loan, "Matrix Computations", *Johns Hopkins University Press*, 2013.

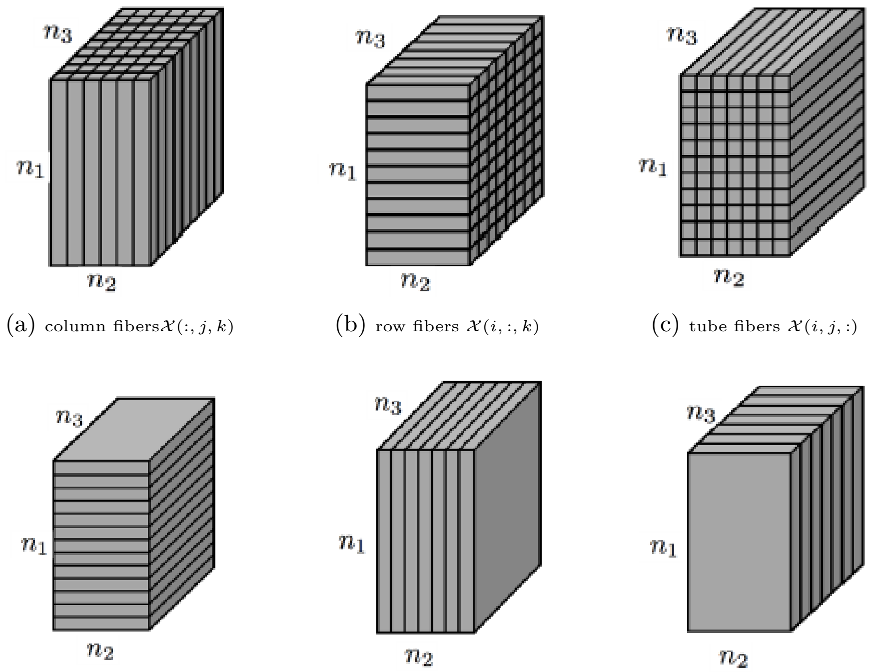
[4] C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin and S. Yan, "Tensor Robust Principal Component Analysis with a New Tensor Nuclear Norm", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2020, Vol. 42(4), pp. 925–938.

[5] E. Newman and M. Kilmer, "Nonnegative Tensor Patch Dictionary Approaches for Image Compression and Deblurring Applications", *SIAM Journal on Imaging Sciences*, 2020, Vol. 13(3), pp. 1084–1112.

[6] E. Kernfeld, M. Kilmer and S. Aeron, "Tensor–Tensor Products with Invertible Linear Transforms", *Linear Algebra and its Applications*, 2015, Vol. 485, pp. 545–570.

[7] G. Song, M. Ng and X. Zhang, "Robust Tensor Completion Using Transformed Tensor Singular Value Decomposition", *Numerical Linear Algebra with Application*, 2020, Vol. 27(3), pp. 1–27.

Appendix



(a) column fibers$\mathcal{X}(:, j, k)$ (b) row fibers $\mathcal{X}(i, :, k)$ (c) tube fibers $\mathcal{X}(i, j, :)$

(d) horizontal slices $\mathcal{X}(i, :, :)$ (e) lateral slices $\mathcal{X}(:, j, :)$ (f) frontal slices $\mathcal{X}(:, :, k)$

Figure 1: Fibers and slices of $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$

1.4 Proposition 1.1

A simple circulant matrix [3]:

$$C(z) = \begin{bmatrix} z_0 & z_4 & z_3 & z_2 & z_1 \\ z_1 & z_0 & z_4 & z_3 & z_2 \\ z_2 & z_1 & z_0 & z_4 & z_3 \\ z_3 & z_2 & z_1 & z_0 & z_4 \\ z_4 & z_3 & z_2 & z_1 & z_0 \end{bmatrix}, \; where \; z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \in \mathbb{C}^5. \tag{1.1}$$

Any circulant $C(z) \in \mathbb{C}^{n \times n}$ is a linear combination of $I_n, \mathcal{D}_n^1, ..., \mathcal{D}_n^{n-2}, \mathcal{D}_n^{n-1}$, where $\mathcal{D}_n$ is the downshift permutation. For example, if $n = 5$, then

$$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.2}$$

$$\mathcal{D}_5^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \mathcal{D}_5^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \mathcal{D}_5^3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \mathcal{D}_5^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.3}$$

Thus, the 5-by-5 circulant matrix displayed above is given by

$$C(z) = z_0 I_5 + z_1 \mathcal{D}_5^1 + z_2 \mathcal{D}_5^2 + z_3 \mathcal{D}_5^3 + z_4 \mathcal{D}_5^4 \tag{1.4}$$

Note that $\mathcal{D}_5^5 = I_5$.

More generally,

$$z = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} \Rightarrow C(z) = \sum_{k=0}^{n-1} z_k \mathcal{D}_n^k \tag{1.5}$$

where $\mathcal{D}_n^0 = I_n$, and $\mathcal{D}_n^k = \left(\mathcal{D}_n^1\right)^k$.

Note that if $V^{-1}\mathcal{D}_n^1 V = \Lambda$ is diagonal, then $\mathcal{D}_n^1 = V\Lambda V^{-1}$, and

$$\begin{aligned} V^{-1}C(z)V &= V^{-1}\left(\sum_{k=0}^{n-1} z_k \mathcal{D}_n^k\right)V = V^{-1}\left(\sum_{k=0}^{n-1} z_k \left(\mathcal{D}_n^1\right)^k\right)V \\ &= V^{-1}\left(\sum_{k=0}^{n-1} z_k \left(V\Lambda V^{-1}\right)^k\right)V = V^{-1}\left(\sum_{k=0}^{n-1} z_k \left(V\Lambda^k V^{-1}\right)\right)V \\ &= \sum_{k=0}^{n-1} z_k \left(V^{-1}V\Lambda^k V^{-1}V\right) = \sum_{k=0}^{n-1} z_k \left(\Lambda^k\right) \end{aligned} \tag{1.6}$$

Eq 1.6 shows that $V^{-1}C(z)V$ is diagonal.

It turns out that the DFT matrix diagonalizes the downshift permutation.

Theorem 1.2. *If $V = F_n$ then $V^{-1}\mathcal{D}_n V = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where*

$$\lambda_{j+1} = \overline{\omega}_n^j = \cos\left(\frac{2j\pi}{n}\right) + i\sin\left(\frac{2j\pi}{n}\right) = e^{\frac{2j\pi}{n}},\ 0 \leqslant j \leqslant n-1 \tag{1.7}$$

Proof.

$$\underbrace{\mathcal{D}_n F_n\left(:, j+1\right)}_{\mathcal{D}_n\alpha} = \mathcal{D}_n \begin{bmatrix} 1 \\ \omega_n^j \\ \omega_n^{2j} \\ \vdots \\ \omega_n^{(n-1)j} \end{bmatrix} = \begin{bmatrix} \omega_n^{(n-1)j} \\ 1 \\ \omega_n^j \\ \vdots \\ \omega_n^{(n-2)j} \end{bmatrix} = \underbrace{\overline{\omega}_n^j \begin{bmatrix} 1 \\ \omega_n^j \\ \omega_n^{2j} \\ \vdots \\ \omega_n^{(n-1)j} \end{bmatrix}}_{\lambda\alpha}, \ j = 0, ..., n-1. \tag{1.8}$$

□

Theorem 1.3. *Suppose $z \in \mathbb{C}^n$ and $C(z)$ are defined by Eq 1.6, If $V = F_n$ and $\lambda = \overline{F_n}z$, then*

$$V^{-1} C\left(z\right) V = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{1.9}$$

Proof. Denote

$$f = \begin{bmatrix} 1 \\ \overline{\omega_n} \\ \vdots \\ \overline{\omega_n^{n-1}} \end{bmatrix} \tag{1.10}$$

and note that the column of $\overline{F_n}$ are componentwise powers of this vector, i.e. $\overline{F_n}\left(:, k+1\right) = f_{.}^{\ k}$ where $\left[f_{.}^{\ k}\right]_j = f_j^k$. By the proof Thm 1.2, $\Lambda = diag\left(f\right)$. Then by Thm 1.2 again

$$\begin{aligned} V^{-1} C\left(z\right) V &= \sum_{k=0}^{n-1} z_k \Lambda^k = \sum_{k=0}^{n-1} z_k diag(f)^k \\ &= \sum_{k=0}^{n-1} z_k diag\left(f_{.}^{\ k}\right) = \sum_{k=0}^{n-1} diag\left(z_k f_{.}^{\ k}\right) \\ &= diag\left(\overline{F_n} z\right) \end{aligned} \tag{1.11}$$

□

1.5 Proposition 1.2

Definition 1.11. *Given a tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, we denote the permutation of the 1st and 3rd dimensions as $\mathcal{X}^P \in \mathbb{R}^{n_3 \times n_2 \times n_1}$. Furthermore,*

$$unfold\left(\mathcal{X}^P\right) = P \ unfold\left(\mathcal{X}\right) \tag{1.12}$$

where $P \in \mathbb{R}^{n_1 n_3 \times n_1 n_3}$ stride permutation matrix.

Remark 1.9. *In fact,*

$$
P\begin{bmatrix}
\underbrace{\begin{matrix} \mathcal{X}_{1,1,1} & \mathcal{X}_{1,2,1} & \cdots & \mathcal{X}_{1,n_2,1} \\ \mathcal{X}_{2,1,1} & \mathcal{X}_{2,2,1} & \cdots & \mathcal{X}_{2,n_2,1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{n_1,1,1} & \mathcal{X}_{n_1,1,1} & \cdots & \mathcal{X}_{n_1,n_2,1} \end{matrix}}_{X^1} \\
\underbrace{\begin{matrix} \mathcal{X}_{1,1,2} & \mathcal{X}_{1,2,2} & \cdots & \mathcal{X}_{1,n_2,2} \\ \mathcal{X}_{2,1,2} & \mathcal{X}_{2,2,2} & \cdots & \mathcal{X}_{2,n_2,2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{n_1,1,2} & \mathcal{X}_{n_1,1,2} & \cdots & \mathcal{X}_{n_1,n_2,2} \end{matrix}}_{X^2} \\
\vdots \\
\underbrace{\begin{matrix} \mathcal{X}_{1,1,n_3} & \mathcal{X}_{1,2,n_3} & \cdots & \mathcal{X}_{1,n_2,n_3} \\ \mathcal{X}_{2,1,n_3} & \mathcal{X}_{2,2,n_3} & \cdots & \mathcal{X}_{2,n_2,n_3} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{n_1,1,n_3} & \cdots & \cdots & \mathcal{X}_{n_1,n_2,n_3} \end{matrix}}_{X^{n_3}}
\end{bmatrix}
=
\begin{bmatrix}
\underbrace{\begin{matrix} \mathcal{X}_{1,1,1} & \mathcal{X}_{1,2,1} & \cdots & \mathcal{X}_{1,n_2,1} \\ \mathcal{X}_{1,1,2} & \mathcal{X}_{1,2,2} & \cdots & \mathcal{X}_{1,n_2,2} \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{\mathcal{X}_{1,1,n_3}}_{vec(\mathcal{X}_{1,1,:})} & \underbrace{\mathcal{X}_{1,2,n_3}}_{vec(\mathcal{X}_{1,2,:})} & \cdots & \underbrace{\mathcal{X}_{1,n_2,n_3}}_{vec(\mathcal{X}_{1,n_2,:})} \end{matrix}}_{(X^P)^1} \\
\underbrace{\begin{matrix} \mathcal{X}_{2,1,1} & \mathcal{X}_{2,2,1} & \cdots & \mathcal{X}_{2,n_2,1} \\ \mathcal{X}_{2,1,2} & \mathcal{X}_{2,2,2} & \cdots & \mathcal{X}_{2,n_2,2} \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{\mathcal{X}_{2,1,n_3}}_{vec(\mathcal{X}_{2,1,:})} & \underbrace{\mathcal{X}_{2,2,n_3}}_{vec(\mathcal{X}_{2,2,:})} & \cdots & \underbrace{\mathcal{X}_{2,n_2,n_3}}_{vec(\mathcal{X}_{2,n_2,:})} \end{matrix}}_{(X^P)^2} \\
\vdots \\
\underbrace{\begin{matrix} \mathcal{X}_{n_1,1,1} & \mathcal{X}_{n_1,2,1} & \cdots & \mathcal{X}_{n_1,n_2,1} \\ \mathcal{X}_{n_1,1,2} & \mathcal{X}_{n_1,2,2} & \cdots & \mathcal{X}_{n_1,n_2,2} \\ \vdots & \vdots & \vdots & \vdots \\ \underbrace{\mathcal{X}_{n_1,1,n_3}}_{vec(\mathcal{X}_{n_1,1,:})} & \underbrace{\mathcal{X}_{n_1,2,n_3}}_{vec(\mathcal{X}_{n_1,2,:})} & \cdots & \underbrace{\mathcal{X}_{n_1,n_2,n_3}}_{vec(\mathcal{X}_{n_1,n_2,:})} \end{matrix}}_{(X^P)^{n_1}}
\end{bmatrix}
\tag{1.13}
$$

(1.14)

i.e.

$$
\begin{aligned}
& P\, unfold\,(\mathcal{X}) \\
=& P\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^{n_3} \end{bmatrix} = \begin{bmatrix} vec\,(X_{11}) = \mathcal{X}_{1,1,:} \in \mathbb{R}^{n_3 \times 1} & vec\,(X_{12}) & \cdots & vec\,(X_{1n_2}) \\ vec\,(X_{21}) & vec\,(X_{22}) & \cdots & vec\,(X_{2n_2}) \\ \vdots & \vdots & \ddots & \vdots \\ vec\,(X_{n_1 1}) & \cdots & \cdots & vec\,(X_{n_1 n_2}) \end{bmatrix} \in \mathbb{R}^{n_1 n_3 \times n_2}
\end{aligned}
\tag{1.15}
$$

To put it simply, $unfold(\mathcal{X})$ via fixed n_3 firstly, but $unfold(\mathcal{X}^P)$ via fixed n_1 firstly.

Proposition 1.7. *If $X \in \mathbb{R}^{n_1 \times n_1}$ and $Y \in \mathbb{R}^{n_2 \times n_2}$, then*

$$P\,(X \otimes Y)\,P^T = Y \otimes X \tag{1.16}$$

where P is permutation matrix and $\otimes$ is the Kronecker product.

Proof. $\square$

Proof. (Proposition 1.2)
Given $P_1 \in \mathbb{R}^{n_1 n_3 \times n_1 n_3}$, $P_2 \in \mathbb{R}^{n_2 n_3 \times n_2 n_3}$ are permutation matrices respectively, then

$$P_1 P_1^T = P_1^T P_1 = I_{n_1 n_3} \in \mathbb{R}^{n_1 n_3 \times n_1 n_3} \text{ and } P_2 P_2^T = P_2^T P_2 = I_{n_2 n_3} \in \mathbb{R}^{n_2 n_3 \times n_2 n_3} \tag{1.17}$$

Then the left hand of Eq 1.5

$$\begin{aligned}
&\left(F_{n_3} \otimes I_{n_1}\right) BlockCirc\left(\mathcal{X}\right)\left(F_{n_3}^* \otimes I_{n_2}\right) \\
=&\left(P_1^T P_1\right)\left(F_{n_3} \otimes I_{n_1}\right)\left(P_1^T P_1\right) BlockCirc\left(\mathcal{X}\right)\left(P_2^T P_2\right)\left(F_{n_3}^* \otimes I_{n_2}\right)\left(P_2^T P_2\right) \\
=&P_1^T\left(P_1\left(F_{n_3} \otimes I_{n_1}\right) P_1^T\right)\left(P_1 BlockCirc\left(\mathcal{X}\right) P_2^T\right)\left(P_2\left(F_{n_3}^* \otimes I_{n_2}\right) P_2^T\right) P_2 \\
=&P_1^T\left(I_{n_1} \otimes F_{n_3}\right)\left(P_1 BlockCirc\left(\mathcal{X}\right) P_2^T\right)\left(I_{n_2} \otimes F_{n_3}^*\right) P_2 \\
=&P_1^T\left(I_{n_1} \otimes F_{n_3}\right)\begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n_2} \\ N_{21} & N_{22} & \cdots & N_{2n_2} \\ \cdots & \cdots & \ddots & \vdots \\ N_{n_1 1} & N_{n_1 2} & \cdots & N_{n_1 n_2} \end{bmatrix}\left(I_{n_2} \otimes F_{n_3}^*\right) P_2
\end{aligned} \tag{1.18}$$

where $N_{ij} = circ\left(vec\left(\mathcal{X}_{i,j,:}\right)\right)$.

We can rewrite Eq 1.18

$$P_1^T \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n_2} \\ D_{21} & D_{22} & \cdots & D_{2n_2} \\ \cdots & \cdots & \ddots & \vdots \\ D_{n_1 1} & D_{n_1 2} & \cdots & D_{n_1 n_2} \end{bmatrix} P_2 \tag{1.19}$$

where $D_{ij} = F_n N_{ij} F_n^* = diag\left(vec(\overline{X})_{i,j,:}\right)$.

Then Eq 1.19 can be written as

$$P_1^T \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n_2} \\ D_{21} & D_{22} & \cdots & D_{2n_2} \\ \cdots & \cdots & \ddots & \vdots \\ D_{n_1 1} & D_{n_1 2} & \cdots & D_{n_1 n_2} \end{bmatrix} P_2 = \begin{bmatrix} \left(\overline{X}\right)^1 & 0 & \cdots & 0 \\ 0 & \left(\overline{X}\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\overline{X}\right)^{n_3} \end{bmatrix} \tag{1.20}$$

□

Remark 1.10.

$$F_{n_3} \otimes I_{n_1} = \begin{bmatrix} (F_{n_3})_{11} I_{n_1} & (F_{n_3})_{12} I_{n_1} & \cdots & (F_{n_3})_{1n_3} I_{n_1} \\ (F_{n_3})_{21} I_{n_1} & (F_{n_3})_{22} I_{n_1} & \cdots & (F_{n_3})_{2n_3} I_{n_1} \\ \vdots & \vdots & \ddots & \vdots \\ (F_{n_3})_{n_3 1} I_{n_1} & (F_{n_3})_{n_3 2} I_{n_1} & \cdots & (F_{n_3})_{n_3 n_3} I_{n_1} \end{bmatrix} \in \mathbb{R}^{n_1 n_3 \times n_1 n_3} \tag{1.21}$$

where $(F_{n_3})_{ij} I_{n_1} \in \mathbb{R}^{n_1 \times n_1}$, $\forall\, 1 \leqslant i, j \leqslant n_3$.

$$blockcirc(\mathcal{X}) = \begin{bmatrix} X^{(1)} & X^{(n_3)} & \cdots & \mathcal{X}^{(2)} \\ X^{(2)} & X^{(2)} & \cdots & X^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(n_3)} & X^{(n_3-1)} & \cdots & {}^{(1)} \end{bmatrix} \in \mathbb{R}^{n_1 n_3 \times n_2 n_3} \tag{1.22}$$

$$F_{n_3} \cdot F_{n_3}^* = I_{n_3} \in \mathbb{R}^{n_3 \times n_3} \tag{1.23}$$

$$F_{n_3}^* \otimes I_{n_2} = \begin{bmatrix} \left(F_{n_3}^*\right)_{11} I_{n_2} & \left(F_{n_3}^*\right)_{12} I_{n_2} & \cdots & \left(F_{n_3}^*\right)_{1 n_3} I_{n_2} \\ \left(F_{n_3}^*\right)_{21} I_{n_2} & \left(F_{n_3}^*\right)_{22} I_{n_2} & \cdots & \left(F_{n_3}^*\right)_{2 n_3} I_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \left(F_{n_3}^*\right)_{n_3 1} I_{n_2} & \left(F_{n_3}^*\right)_{n_3 2} I_{n_2} & \cdots & \left(F_{n_3}^*\right)_{n_3 n_3} I_{n_2} \end{bmatrix} \in \mathbb{R}^{n_2 n_3 \times n_2 n_3} \tag{1.24}$$