Measure Theory

Lectures by Claudio Landim

Notes by Yao Zhang

Instituto de Matemática Pura e Aplicada, Spring 2018

Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. Taylor published by Cambridge University Press.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the TikZ package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to `jaafar_zhang@163.com`.

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Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

0. $\lambda: \ \mathcal{P}\left(\mathbb{R}\right) \to \mathbb{R}_+ \cup \{+\infty\}$

1. $\lambda\left((a,b]\right) = b - a$

2. $A \subseteq \mathbb{R}, \ A + x = \{x + y: \ y \in A\}, \ \ \forall A, \ A \subseteq \mathbb{R}, \ \forall x \in \mathbb{R}:$

$$\lambda\left(A + x\right) = \lambda\left(A\right) \tag{1.1}$$

3. $A = \bigcup_{j \geqslant 1} A_j, \ \ A_j \cap A_k = \varnothing:$

$$\lambda\left(A\right) = \sum_k \lambda\left(A_k\right) \tag{1.2}$$

Definition 1.1. $x \sim y, \ x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}. \ [x] = \{y \in \mathbb{R}, \ y - x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_\sim$, only one point represents the equivalence class of Ω , like α, β.

Ω is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0,1)$

Claim 1.1. $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \varnothing \end{cases} \quad q, p \in \mathbb{Q}$

Proof. Assume that $\Omega + q \cap \Omega + q \neq \varnothing$ then, $x = \alpha + p = \beta + q, \ \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \varnothing]$. $\qquad\square$

Claim 1.2. $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q) \subseteq (-1, 2) \tag{1.3}$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda\left(E\right) \leqslant \lambda\left(F\right)$

Proof. $\because E \subseteq F \therefore F = E \cup (F \backslash E), \ E \cap (F \backslash E) = \varnothing$, then $\lambda\left(F\right) = \lambda\left(E\right) + \lambda\left((F \backslash E)\right) \Rightarrow \lambda\left(F\right) \geqslant \lambda\left(E\right)$. $\qquad\square$

Then,

$$\lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) \leqslant \lambda\left((-1, 2)\right) = 3 \tag{1.4}$$

and ,

$$\infty \cdot \lambda\left((\Omega + q)\right) = \infty \cdot \lambda\left(\Omega\right) \leq 3 \Rightarrow \lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)\right) = 0 \tag{1.5}$$

Claim 1.4. $(0,1) \subseteq \sum\limits_{\substack{q \in \mathbb{Q} \\ -1 < q < 1}} (\Omega + q)$

Proof. $\forall$ fixed $x \in (0,1)$, $\exists \alpha \in [x] \cap \Omega$, $\alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q}$, $- < q < 1 \Rightarrow$
$x = \alpha + q$, $x \in \Omega + q$ □

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda\left(\sum_{q \in \mathbb{Q}} \Omega + q\right) = 0 \tag{1.6}$$

it is impossible.

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Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Definition 2.1. $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is semi-algebra if:

1. $\Omega \subseteq \mathcal{S}$

2. $A, B \in \mathcal{S} \Rightarrow A \bigcap B \in \mathcal{S}$

3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum_{i=1}^{n} E_j$, $\exists E_1, \cdots, E_n \in \mathcal{S}$, E_i, E_j $(i \neq j)$ disjoint sets, n is finite number

Example 2.1. $\Omega = \mathbb{R}$, $\mathcal{S} = \{\mathbb{R}, \{(a,b), a < b, a, b \in \mathbb{R}\}, \{(-\infty, b], b \in \mathbb{R}\}, \{(a, \infty), a \in \mathbb{R}\}, \varnothing\}$,
$(a,b]^c = (-\infty, a] \cup [b, +\infty)$

Example 2.2. $\Omega = \mathbb{R}^2$

$\mathcal{S} = \{\mathbb{R}^2, \{(a_1, b_1) \times (a_2, b_2), a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2], b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty), a_i \in \mathbb{R}\}, \varnothing\}$

Definition 2.2. $a = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in a$

2. $A, B \in a \Rightarrow A \bigcap B \in a$

3. $A \in a \Rightarrow A^c \in a$

Remark 2.1. a algebra $\Rightarrow a$ semi-algebra

Definition 2.3. σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \subseteq \mathcal{S}$

2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap_{j \geqslant 1} A_j \in \mathcal{S}$

3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

Remark 2.2. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)$, a_α algebra, $\alpha \in I \Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is an algebra.

Proof. check the followings

1. $\Omega \in a$

2. $A, B \in a \Rightarrow A \bigcap B \in a$

3. $A \in a \Rightarrow A^c \in a$

$\square$

Remark 2.3. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega), \alpha \in I, a_\alpha, \sigma$-algebra $\Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is a σ-algebra

Proof. check the followings

1. $\Omega \in a$

2. $A_j, j \geq 1 \in a \Rightarrow \bigcap_{j \geqslant 1} A_j \in a$

3. $A \in a \Rightarrow A^c \in a$

$\square$

Definition 2.4 (minimal algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$

2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.1}$$

Remark 2.4. $a(c)$ exits, and $a = a(c) = \bigcap_\alpha a_\alpha$, $\forall \alpha$, $c \subseteq a_\alpha$, a_α is an algebra.

Definition 2.5 (minimal σ-algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a σ-algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$

2. $\forall \mathcal{B}$ is σ-algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:
$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.2}$$

Remark 2.5. $a(c)$ exits, and $a = a(c) = \bigcap_\alpha a_\alpha$, $\forall \alpha$, $c \subseteq a_\alpha$, a_α is an σ-algebra.

Lemma 2.1. Ω, f *semi-algebra* $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ *algebra generated by* f *then*

$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leqslant j \leqslant n, \ A = \sum_{j=1}^{n} E_j \tag{2.3}$$

Proof.

1. $\Leftarrow$

   $A = \sum\limits_{j=1}^{n} E_j, \ E_j \in f \in a(f)$

   By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. $\Rightarrow$

   $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j, E_j \in f$

   Then by remark 2.7, it will be proved easily.

$\square$

Remark 2.6. $E, J \in a, E \bigcup F \in a, E \bigcup F = (E^c \bigcap F^c)^c$

Remark 2.7. $\mathcal{B} = \left\{ \sum\limits_{j=1}^{n} F_j, \ F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. $\mathcal{B}$ algebra

2. $\mathcal{B} \supseteq f$

3. $\mathcal{B} \supseteq a(f)$

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Proof. We only prove that $\mathcal{B}$ algebra, then check the following

1. $\Omega \in \mathcal{B}$

2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

$\because A, B \in \mathcal{B}, \therefore A = \sum_{j=1}^{n} E_j, \ E_j \in f, \ B = \sum_{k=1}^{m} F_k, \ F_k \in f$, then

$$
\begin{aligned}
A \cap B &= \left(\sum_{j=1}^{n} E_j \right) \cap \left(\sum_{k=1}^{m} F_k \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \\
&\in \mathcal{B}
\end{aligned}
\tag{2.4}
$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

$A = \sum_{j=1}^{n} E_j, \ E_j \in f$

By definition 2.1:

$$
\begin{aligned}
E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1}, \ F_{1,j} \in f \\
\cdots &= \cdots \\
E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i}, \ F_{i,j} \in f
\end{aligned}
\tag{2.5}
$$

Then, we get that

$$
\begin{aligned}
A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1} \right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2} \right) \cap \cdots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n} \right) \\
&= \sum_{k_1=1}^{l_1} \sum_{k_2=1}^{l_2} \cdots \sum_{k_n=1}^{l_n} (F_{1,k_1} \cap F_{2,k_2} \cap F_{n,k_n}) \\
&\in \mathcal{B}
\end{aligned}
\tag{2.6}
$$

$\square$

Definition 2.6. $c \subseteq \mathcal{P}(\Omega), \varnothing \in c, \mu : \ c \to \mathbb{R}_+ \cup \{+\infty\}$. μ is additive if

1. $\mu(\varnothing) = 0$

2. $E_1, E_2, ..., E_n \in c, \ E = \sum_{j=1}^{n} E_j \in c \Rightarrow \mu(E) = \sum_{j=1}^{n} \mu(E_k)$

Remark 2.8.

$$\exists A \in c,\ \mu(A) < \infty,\ A = A \cup \varnothing,\ \mu(A) = \mu(A) + \mu(\varnothing) \Rightarrow \mu(\varnothing) = 0 \tag{2.7}$$

Remark 2.9. $c,\ \mu : c \to \mathbb{R}_+ \bigcup +\infty,\ E \subseteq F,\ F\backslash E \in c,\ E, F \in c$

$$F = E \cup (F\backslash E),\ \mu(F) = \mu(E) + (F\backslash E) \tag{2.8}$$

1. $\mu(E) = +\infty,\ \mu(F) = +\infty$
2. $\mu(E) < +\infty,\ \mu(F\backslash E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leqslant \mu(F) \tag{2.9}$$

Example 2.3. Discrete measure: $\Omega,\ c \subseteq \mathcal{P}(\Omega),\ \{x_j,\ j \geqslant 1\},\ x_j \in \Omega,\ \{p_j,\ j \geqslant 1\},\ p_j, \geqslant 0,\ A \in c$, define that

$$\mu(A) = \sum_{j \geqslant 1} p_j 1\{x_j \in A\} \tag{2.10}$$

then μ is additive

Definition 2.7. $c \in \mathcal{P}(\Omega),\ \varnothing \in c,\ \mu :\ c \to \mathbb{R}_+ \bigcup +\infty,\ \mu$ is σ-additive if

1. $\mu(\varnothing) = 0$
2. $E_j \in c,\ j \neq k, E_j \bigcap E_k = \varnothing,\ \ E = \sum_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum_{j \geq 1} \mu(E_j)$

Example 2.4. $\Omega = (0, 1),\ c = \{(a, b],\ 0 \leqslant a < b < 1\},\ \mu :\ c \to \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a, b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \tag{2.11}$$

$(a, b] = \sum_{j=1}^{n} (a_j, b_j)$, we can get that μ is NOT σ-additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow \to 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geqslant 1} (x_{j+1}, x_j] = +\infty \tag{2.12}$$

it is impossible.

Lecture 3

Set Functions

Definition 3.1. $c \subseteq \mathcal{P}(\Omega)$, $\mu : c \to \mathbb{R}_+ \bigcup +\infty$:

1. $E \in c$, μ continuous from below at E, if $\forall (E_n)_{n \geqslant 1}$, $E_n \in c$, $E_n \uparrow E$ $\left(E_n \subseteq E_{n+1}, \bigcup\limits_{n \geqslant 1} E_n = E \right)$:

$$\mu(E_n) \to \mu(E) \tag{3.1}$$

2. $E \in c$, μ continuous from above at E, if $\forall (E_n)_{n \geqslant 1}$, $E_n \in c$, $E_n \downarrow E$ $\left(E_{+1} \subseteq E_n, \bigcap\limits_{n \geqslant 1} E_n = E \right)$, and $\exists n_0$, $\mu(E_{n_0}) < \infty$:

$$\mu(E_n) \to \mu(E) \tag{3.2}$$

Remark 3.1. For a sequence $E_1, E_2, \ldots$ of sets, we put

$$\limsup E_i = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} E_i \right), \liminf E_i = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} E_i \right) \tag{3.3}$$

and if $\{E_i\}$ is such that $\limsup E = \liminf E_i$ we say that the sequence converges to the set

$$E = \limsup E = \liminf E_i \tag{3.4}$$

Remark 3.2. 2 need $\exists n_0$, $\mu(E_{n_0}) < \infty$, if not:

$$E_n = [n, +\infty), \ \mu(E_n) = +\infty, \ E_n \downarrow \varnothing, \ \lambda(\varnothing) = 0 \tag{3.5}$$

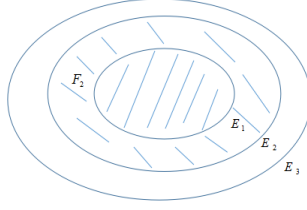**Lemma 3.1.** $a \subseteq \mathcal{P}(\Omega)$, *algebra;* $\mu : \ a \to \mathbb{R}_+ \cup \{+\infty\}$, *additive;*

1. *μ is σ-additive $\Rightarrow \mu$ continuous at E, $\forall E \in a$*

2. *μ is continuous from below $\Rightarrow \mu$ is σ-additive*

3. *μ is continuous from above at $\varnothing$ & μ is finite $\Rightarrow \sigma$-additive*

Proof.

1.

(i) μ is σ-additive $\Rightarrow \mu$ conti. from below at $E \in a$. $E \in a, E_n \uparrow E, \ E_n \in a$:

$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ \vdots &= \vdots \\ F_n &= E_n \backslash E_{n-1} \end{aligned} \tag{3.6}$$

and we can get that

$$F_j \cap F_k = \varnothing, \quad \sum_{k=1}^{n} F_k = E_n, \quad \bigcup_{n \geqslant 1} E_n = \bigcup_{n \geqslant 1} F_n \tag{3.7}$$

so

$$\mu(E) = \sum_{k \geqslant 1} \mu(F_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n \to \infty} \mu(E_n) \tag{3.8}$$

(ii) μ cont. from above $E \in a, E_n \in a, E_n \downarrow E, \mu(E_{n_0}) < \infty \Rightarrow \mu(E_n) \downarrow \mu(E)$
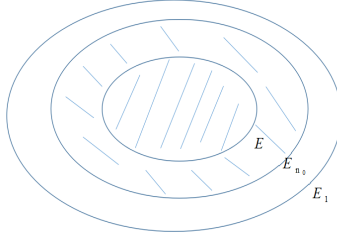
$$
\begin{aligned}
G_1 &= E_{n_0} \backslash E_{n_0+1} \\
G_2 &= E_{n_0} \backslash E_{n_0+2} \\
\vdots &= \vdots \\
G_k &= E_{n_0} \backslash E_{n_0+k}
\end{aligned}
\tag{3.9}
$$

then $G_k \uparrow E_{n_0} \backslash E, G_k \in a \Rightarrow \mu(G_k) \uparrow \mu(E_{n_0} \backslash E)$, so

$$
\begin{aligned}
\mu(E_{n_0} \backslash E) &= \lim_{n \to \infty} \mu(E_{n_0} \backslash E_{n_0+k}) \\
\mu(E_{n_0} \backslash E) &= \mu(E_{n_0}) - \mu(E) \\
\mu(E_{n_0}) - \mu(E) &= \lim_{k \to \infty} (\mu(E_{n_0}) - \mu(E_{n_0+k}))
\end{aligned}
\tag{3.10}
$$

2. μ cont. below, $E = \sum_{k \geqslant 1} E_k, \ E, E_k \in a$.

   Obs.

$$\sum_{k=1}^{n} E_k \subseteq E \stackrel{additive}{\Rightarrow} \begin{cases} \mu\left(\sum_{k=1}^{n} E_k\right) \leqslant \mu(E) \\ \sum_{k=1}^{n} \mu(E_k) \leqslant \mu(E) \end{cases} \tag{3.11}$$

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then
$$\sum_{k \geqslant 1} \mu(E_k) \leqslant \mu(E) \tag{3.12}$$

$F_n = \sum\limits_{k=1}^{n} E_k \in a, \ F_n \uparrow E,$

$$\sum_{k=1}^{n} \mu(E_k) = \mu(F_n) \uparrow \mu(E) \Rightarrow \sum_{k \geqslant 1} \mu(E_k) = \mu(E) \tag{3.13}$$

3. μ cont. at $\varnothing$, $\mu(\Omega) < \infty$, $E, E_k \in a, E = \sum\limits_{k \geqslant 1} E_k$.

$$F_n = \sum_{k \geqslant m} E_k \in a \ \left(E \backslash \sum_{j=1}^{n-1} E_j \right) \tag{3.14}$$

$F_n \downarrow \varnothing, \mu(F_1) < \infty, \ \mu(F_n) \to 0$

$$\begin{aligned}
\mu(E) &= \mu\left(\sum_{k=1}^{n} E_k \cup \sum_{k>n} E_k \right) \\
&= \underbrace{\mu \sum_{k=1}^{n} E_k}_{\to \sum\limits_{k \geqslant 1} \mu(E_n)} + \underbrace{\mu \sum_{k>n} E_k}_{\to 0} \\
&\to \sum_{k \geqslant 1} \mu(E_n)
\end{aligned} \tag{3.15}$$

$\square$

Remark 3.3. Suppose E_α, $\alpha \in I$ is a class of subsets of X, and E_i is one set of the class, then

1. $\bigcap\limits_{\alpha \in I} E_\alpha \subseteq E_i \subseteq \bigcup\limits_{\alpha \in I} E_\alpha$

2. $X - \bigcup\limits_{\alpha \in I} E_\alpha = \bigcap\limits_{\alpha \in I} (X - E_\alpha)$

3. $X - \bigcap\limits_{\alpha \in I} E_\alpha = \bigcup\limits_{\alpha \in I} (X - E_\alpha)$

Proof.

1. This is immediate from the definition.

2. Suppose $x \in X - \bigcup\limits_{\alpha \in I} E_\alpha$ then $x \in X$ and x is not in $\bigcup\limits_{\alpha \in I} E_\alpha$, that is x is not in any E_α, $\alpha \in I$ so that $x \in X - E_\alpha$ for every $\alpha \in I$, and $x \in \bigcap\limits_{\alpha \in I} (X - E_\alpha)$. Conversely if $x \in \bigcap\limits_{\alpha \in I} (X - E_\alpha)$, then for every $\alpha \in I$, x is in X but not in E_α, so $x \in X$ but x is not in $\bigcup\limits_{\alpha \in I} E_\alpha$, that is $x \in \bigcup\limits_{\alpha \in I} (X - E_\alpha)$.

3. Similar to 2

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Remark 3.3 (2) and (3) are also called as de Morgan's Law. □

Example 3.1. $(0,1), (a,b], 0 \leqslant a < b < 1$

$$\mu(a,b] = \begin{cases} b-a, & a > 0 \\ +\infty, & a = 0 \end{cases} \tag{3.16}$$

μ is additive but NOT σ-additive

Proof. $E_n \downarrow \varnothing, \mu(E_{n_0}) < \infty, E_n = (a_{n,1}, b_{n,1}] \cup \cdots \cup (a_{n,k_n}, b_{n,k_n}], a_{n,j} < a_{n,j+1}.$

$$\begin{cases} a_{n,1} = 0, & \forall n \\ a_{n_0} > 0, & some \ n_0 \end{cases} \qquad\qquad □$$

Theorem 3.1 (Extension)**.** $f \subseteq \mathcal{P}(\Omega)$ *semi-algebra,* $\mu : f \to \mathbb{R}_+ \cup \{\infty\}$ σ-*additive, then* $\exists \nu$:

$$\nu : a(f) \to \mathbb{R}_+ \cup \{\infty\} \tag{3.17}$$

such that:

1. ν σ-*additive*

2. $\nu(A) = \mu(A), \ \forall A \in f$

3. $\mu_1, \mu_2, a(f) \to \mathbb{R}_+ \bigcup \{+\infty\}, \ then \ \mu_1(A) = \mu_2(A), \forall A \in s \Rightarrow \mu_1(E) = \mu_2(E), \forall E \in a(f)$

Proof. $A \in a(f) \Rightarrow A = \sum_{j=1}^{n} E_j, \ E_j \in f$ by Lemma 2.1.

$$\nu(A) \overset{add}{=} \sum_{j=1}^{n} \nu(E_j) \overset{ext}{=} \sum_{j=1}^{n} \mu(E_j) \tag{3.18}$$

we define that

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) \tag{3.19}$$

we want to show that $\nu(A) = \sum_{j=1}^{n} \mu(E_j)$ is well-defined:

1. ν is unique

$$\begin{aligned} A &= \sum_{j=1}^{n} E_j, \ E_j \in f \\ &= \sum_{k=1}^{m} F_k, \ F_k \in f \end{aligned} \tag{3.20}$$

then we will prove that

$$\begin{aligned} \nu(A) &= \sum_{j=1}^{n} \mu(E_j) \\ &= \sum_{k=1}^{m} \mu(F_k) \end{aligned} \tag{3.21}$$

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$$\because E_j \subseteq A = \sum_{k=1}^{m} F_k \Rightarrow E_j = E_j \cap \left(\sum_{k=1}^{m} F_k \right) = \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \tag{3.22}$$

$$\therefore \mu(E_j) = \mu \left(\sum_{k=1}^{m} (E_j \cap F_k) \right) \tag{3.23}$$

then

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(E_j \cap F_k) = \sum_{k=1}^{m} \mu(F_k) \tag{3.24}$$

2. ν is an additive, $\nu(A) = \sum_{j=1}^{n} \mu(E_j)$

Assume that

$$\begin{cases} A = \sum_{j=1}^{n} E_j, \ E_j \in f \\ B = \sum_{k=1}^{m} F_k, \ F_k \in f \end{cases}, A \cap B = \varnothing \tag{3.25}$$

We will show that

$$\nu(A \cup B) = \nu(A) + \nu(B) \tag{3.26}$$

$$\because A \cup B = \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \tag{3.27}$$

therefore

$$\begin{aligned} \nu(A \cup B) &= \mu \left(\sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \right) \\ &= \sum_{j=1}^{n} \mu(E_j) + \sum_{k=1}^{m} \mu(F_k) \\ &= \nu(A) + \nu(B) \end{aligned} \tag{3.28}$$

3. $\nu(A) = \mu(A), \ A \in f$ by Eq 3.19

4. ν is uniqueness, we want to show that:

Suppose that $\mu_1, \mu_2 : \ a(f) \to R_+ \cup \{+\infty\}, \forall A \in f, \mu_1, \mu_2 \ additive$, then

$$\mu_1(A) = \mu_2(A) \Rightarrow \mu_1(B) = \mu_2(B), \ \forall B \in a(f) \tag{3.29}$$

$$\because B \in a(f), \therefore B = \sum_{j=1}^{n} \mu_1(E_j), \ E_j \in f$$

$$\mu_1(B) = \sum_{j=1}^{n} \mu_1(E_j) = \sum_{j=1}^{n} \mu_2(E_j) = \mu_2(B) \tag{3.30}$$

Now we proof the extension of σ-additive, ie: $\mu - \sigma$ additive, f semi-algebra, $\nu - \sigma$ additive, $a(f)$ is a algebra generated by f. we want to show that

$$A = \sum_{j \geqslant 1} A_j, \ A, A_j \in a\left(f\right) \Rightarrow \nu\left(A\right) = \sum_{j \geqslant 1} \nu\left(A_j\right) \tag{3.31}$$

by representation of an algebra:

$$A = \sum_{j=1}^{m} E_j, E_j \in f; \quad A_k = \sum_{l=1}^{m_k} E_{k,l}, E_{k,l} \in f \tag{3.32}$$

by Eq 3.19:

$$\nu\left(A\right) = \sum_{j=1}^{m} \nu\left(E_j\right), \quad \nu\left(A_k\right) = \sum_{l=1}^{m_k} \nu\left(E_{k,l}\right) \tag{3.33}$$

$$\because E_j = E_j \cap A = E_j \cap \left(\sum_{k \geqslant 1} A_k\right) = E_j \cap \left(\sum_{k \geqslant 1} \sum_{l=1}^{m_k} E_{k,l}\right) = \sum_{k \geqslant 1} \sum_{l=1}^{m_k} \left(E_j \cap E_{k,l}\right) \tag{3.34}$$

therefore

$$\begin{aligned}
\nu\left(A\right) &= \sum_{j=1}^{n} \mu\left(E_j\right) \\
&= \sum_{j=1}^{n} \sum_{k \geqslant 1} \sum_{l=1}^{m_k} \mu\left(E_j \cap E_{k,l}\right) \\
&= \sum_{k \geqslant 1} \underbrace{\sum_{l=1}^{m_k} \mu\left(E_{k,l}\right)}_{\subseteq A_k}
\end{aligned} \tag{3.35}$$

Eq 3.35 holds because:

$$E_{k,l} = E_{k,l} \cap A = \sum_{j=1}^{n} \left(E_{k,l} \cap E_j\right) \tag{3.36}$$

and

$$\mu\left(E_{k,l}\right) = \sum_{j=1}^{n} \mu\left(E_{k,l} \cap E_j\right) \tag{3.37}$$

so we can get that

$$\nu\left(A\right) = \sum_{k \geqslant 1} \nu\left(A_k\right) \tag{3.38}$$

$$\square$$

Lecture 4

Caratheodory Theorem

Intuition:

$$
\begin{array}{lll}
\sigma-add & \mu: f \to \mathbb{R}_+ \cup \{+\infty\} & f \subseteq \mathcal{P}(\Omega),\, f \text{ is semialgebra} \\
\quad\downarrow & \quad\downarrow & \\
\sigma-add & \nu: a(f) \to \mathbb{R}_+ \cup \{+\infty\} & a(f) \text{ algebra generated by } f \qquad (4.1) \\
\quad\downarrow & \quad\downarrow & \\
\sigma-add & \pi: \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\} & \mathcal{F}(a) \text{ is } \sigma-\text{algebra generated by algebra}
\end{array}
$$

The big picture of the proof to the Caratheodory Theorem:

1. Define the π^* outer measure:
$$
\pi^* = \inf_{\{E_i\}} \sum_{i \geqslant 1} \nu(E_i) \tag{4.2}
$$

2. $\mathcal{M}$ σ-algebra, $\mathcal{M} \supseteq \mathcal{F}(a)$

3.
$$
\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.3}
$$

   is σ-additive, and
$$
\pi^*|_a = \nu \tag{4.4}
$$

4. (uniqueness) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \bigcup\{+\infty\}$, Ω is σ-finite(μ_1), if $E_j \uparrow \Omega$, $\mu_1(E_j) < \infty, \forall j,\ E_j \in a$ and $\mu_1|_a = \mu_2|_a$ then implies that
$$
\mu_1 = \mu_2 \tag{4.5}
$$

$$
\pi^* : \ \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.6}
$$

We will prove π^* is an outer measure.

And we will construct a family of subsets $\mathcal{M}$

$$
\mathcal{M} \subseteq \mathcal{P}(\Omega) \tag{4.7}
$$

we will also prove $\mathcal{M}$ satisfies the following:

1. $\mathcal{M}$ is a $\sigma-$algebra

2. $\mathcal{M} \supseteq a$

3. $\pi^*|_{\mathcal{M}}$ $\sigma-$additive

4. $\pi^*|_a = \nu$

Next, we will define π^* and $\mathcal{M}$.

Step 1

Definition 4.1 (π^*). $\pi^*:\ \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$, $A \in \Omega$, $\{E_i, i \geqslant 1\}$, $E_i \in a, A \subseteq \bigcup E_i$, $\{E_i\}$ is a covering of A, then we define that

$$\pi^* = \inf_{\{E_i\},A} \sum_{i \geqslant 1} \nu(E_i) \tag{4.8}$$

where $\nu:\ a(f) \to \mathbb{R}_+ \cup \{+\infty\}$, *is σ-additive.*

Definition 4.2 (Outer measure). $\mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}, c \subseteq P(\Omega), \varnothing \in c, \mu$ is a outer measure if

1. $\mu(\varnothing) = 0$

2. (monotone) $E \subseteq F,\ E, F \in c\ \Rightarrow \mu(E) \leqslant \mu(F)$

3. (subadditive) $E, E_i \in c,\ E \subseteq \bigcup\limits_i E_i \Rightarrow \mu(E) \leqslant \sum\limits_i \mu(E_i)$

Theorem 4.1. *π^* in 4.1 is a outer measure.*

Proof. We will check the conditions in Def 4.2.

1. check $\pi^*(\varnothing) = 0$

   (a) $E_i = \varnothing, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$ then

   $$\pi^*(\varnothing) = \inf_{\{E_i\},\varnothing} \sum_{i \geqslant 1} \nu(E_i) \leqslant \sum_{i \geqslant 1} \nu(E_i) = 0 \tag{4.9}$$

   (b) $E_i \in a, \{E_i\}, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$, then

   $$\sum_{i \geqslant 1} \nu(E_i) \geqslant 0 \Rightarrow \pi^*(\varnothing) \geqslant 0 \tag{4.10}$$

2. check $E \subseteq F,\ \pi^*(E) \leqslant \pi^*(F)$

   Let's take any covering of F:$\{E_i\}, E_i \in a, F \subseteq \bigcup\limits_{i \geqslant 1} E_i$ is also a covering of E, then

   $$\pi^*(E) = \inf_{\{E_i\},E} \sum_{i \geqslant 1} \nu(E_i) \leqslant \pi^*(F) = \inf_{\{E_i\},F} \sum_{i \geqslant 1} \nu(E_i) \tag{4.11}$$

3. check $E \subseteq \bigcup\limits_{i \geqslant 1} E_i,\ \pi^*(E) \leqslant \sum\limits_{i \geqslant 1} \pi^*(E_i)$

   (a) $\pi^*(E_i) = \infty$ then
   $$\pi^*(E) \leqslant \sum_{i \geqslant 1} \pi^*(E_i) \tag{4.12}$$

   (b) $\pi^*(E_i) < \infty$, then
   $$\pi^*(E_i) = \inf_{\{H_{ik}\},\ E_i} \sum_{k \geqslant 1} \nu(H_{ik}) \tag{4.13}$$

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then there $\exists \{H_{ik}\} \in a, E_i \subseteq \bigcup\limits_{k \geqslant 1} H_{ik}$ such that

$$\pi^* (E_i) = \inf_{\{H_{ik}\},\, E_i} \sum_{k \geqslant 1} \nu (H_{ik}) \leqslant \sum_{k \geqslant 1} \nu (H_{ik}) \leqslant \pi^* (E_i) + \frac{\varepsilon}{2^i} \tag{4.14}$$

$\{H_{ik}\}$ is a covering of E, then

$$\pi^* (E) \leqslant \sum_{i,k} \nu (H_{ik}) \leqslant \sum_{i \geqslant 1} \left(\pi^* (E_i) + \frac{\varepsilon}{2^i} \right) \leqslant \sum_{i \geqslant 1} \pi^* (E_i) + \varepsilon \tag{4.15}$$

so

$$\pi^* (E) \leqslant \sum_{i \geqslant 1} \pi^* (E_i) \tag{4.16}$$

$\square$

Step 2

Definition 4.3 (Measurable set $\mathcal{M}$)**.** A set called measurable set $\mathcal{M}$ if $A \in \mathcal{M}\ \forall E \in \Omega$, we have that

$$\pi^* (E) = \pi^* \left(E \bigcap A \right) + \pi^* \left(E \bigcap A^c \right) \tag{4.17}$$

Theorem 4.2. *If $\mathcal{M}$ definited as Def 4.3, then*

1. *$\mathcal{M} \supseteq a$*

2. *$\mathcal{M}$ is a $\sigma-algebra$*

Remark 4.1.

$$E \subseteq (E \cap A) \cup (E \cap A^c) \Rightarrow \pi^* (E) \leqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.18}$$

so we only to check $\geq$ in Eq 4.17

Proof. π^* is an outer measurable by Thm 4.1, then by subadditive of outer measure. $\square$

Now we proof Thm 4.2.

Proof.

1. $a \in \mathcal{M}$

   Suppose that $A \in a, E \in \Omega$, we will show that

   $$\pi^* (E) \geqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.19}$$

   assume that $\pi^* (E) < \infty$, given $\varepsilon, \exists \{E_i\}, E,\ such\ that\ E_i \in a, E \subseteq \bigcup\limits_{i \geqslant 1} E_i$, then

   $$\pi^* (E) \leqslant \sum_{i \geqslant 1} \nu (E_i) \leqslant \pi^* (E) + \varepsilon \tag{4.20}$$

$E_i \cap A \in a, E \cap A \subseteq \bigcup_{i \geqslant 1} (E_i \bigcap A)$, so

$$\pi^* (E \cap A) \leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right)$$
$$\pi^* (E \cap A^c) \leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right)$$
$$(4.21)$$

so

$$\pi^* (E \cap A) + \pi^* (E \cap A^c) \leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right) + \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right) \leq \sum_{i \geqslant 1} \nu (E_i) \leqslant \pi^* (E) + \varepsilon \tag{4.22}$$

2. $\mathcal{M}$ is σ-algebra.

   We need to show that

   (a) $\Omega \in \mathcal{M}$

   It is clearly that:
   $$\pi^* (E) = \pi^* (E \cap \Omega) + \pi^* (E \cap \Omega^c) \tag{4.23}$$

   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$
   $$\because \pi^* (E) = \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.24}$$

   (c) $A_i \in \mathcal{M} \Rightarrow \bigcup_{i \geqslant 1} A_i \subseteq \mathcal{M}$

   Finite union is closed: $A, B \in \mathcal{F} \Rightarrow A \bigcup B \in M$. Let's take $E \subseteq \Omega$. We will proof that
   $$\pi^* (E) \geqslant \pi^* \left(E \cap \left(A \bigcup B \right) \right) + \pi^* \left(E \cap \left(A \bigcup B \right)^c \right) \tag{4.25}$$

   $\because A \in \mathcal{M}$,
   $$\therefore \quad \pi^* (E) = \pi^* \left(E \bigcap A \right) + \pi^* \left(E \bigcap A^C \right) \tag{4.26}$$

   $\because B \in \mathcal{M}$
   $$\therefore \quad \pi^* (E \backslash A) = \pi^* (E \backslash A \cap B) + \pi^* (E \backslash A \cap B^c)$$
   $$= \pi^* (E \backslash A \cap B) + \pi^* \left(E \backslash \left(A \bigcup B \right) \right) \tag{4.27}$$

   then
   $$\pi^* (E) = \pi^* (E \cap A) + \pi^* (E \backslash A \cap B) + \pi^* (E \backslash (A \cup B)) \tag{4.28}$$

   We want to show
   $$\pi^* (E \cap A) + \pi^* (E \backslash A \cap B) \geqslant \pi^* (E \cap (A \cup B)) \tag{4.29}$$

   By π^* is subadditive, we only to show that
   $$E \cap (A \cup B) \subseteq (E \cap A) \cup (E \backslash A \cap B) \tag{4.30}$$

   this is because
   $$E \cap (A \cup B) = \underbrace{\{[E \cap (A \cup B)] \cap A\}}_{\subseteq E \cap A} \bigcup \underbrace{\{[E \cap (A \cup B)] \cap A^c\}}_{\subseteq (E \cap A^c) \cap B = (E \backslash A) \bigcap B} \tag{4.31}$$

Then Eq 4.25 holds. So $\mathcal{M}$ is closed by finite(countable) union.

Now, we will show that countable infinite union is also closed. $A_i \in \mathcal{M}$, we want to show $A = \bigcup_{j \geqslant 1} A_j \in \mathcal{M}$, take $E \subseteq \Omega$,

$$\pi^* (E) \geqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.32}$$

by Eq. 4.25, $\forall\, n$ we know that

$$
\begin{aligned}
\pi^* (E) &= \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j^c \right) \right) \\
&\geq \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* (E \backslash A)
\end{aligned}
\tag{4.33}
$$

$\geq$ holds in Eq 4.33 because $(E \backslash A) \subseteq \left(E \backslash \left(\bigcup_{j=1}^{n} A_j \right) \right)$.

Now, we define

$$
\begin{aligned}
F_1 &= A_1 \\
F_2 &= A_1 \backslash A_2 \\
F_3 &= A_1 \backslash (A_2 \cup A_3) \\
&\;\;\vdots \\
F_n &= A_1 \backslash (A_2 \cup \cdots \cup A_{n-1}) \\
&\;\;\vdots
\end{aligned}
\tag{4.34}
$$

It is clear that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} F_j \tag{4.35}$$

where $F_j \cap F_k = \varnothing, F_j \in \mathcal{M}$.

Then Eq 4.33 can be written as

$$\pi^* (E) \geqslant \pi^* \left(E \cap \sum_{j=1}^{n} F_j \right) + \pi^* (E \backslash A) \tag{4.36}$$

By Remark 4.2, we have

$$
\begin{aligned}
\pi^* (E) &\geqslant \pi^* \left(E \cap \left(\sum_{j=1}^{n} F_j \right) \right) + \pi^* (E \backslash A) \\
&= \sum_{j=1}^{n} \pi^* (E \cap F_j) + \pi^* (E \backslash A)
\end{aligned}
\tag{4.37}
$$

$\because$ n is any number in Remark 4.2 , $\therefore$ $\pi^* \left(E \cap \sum_{j=1}^{\infty} F_j \right) = \sum_{j=1}^{\infty} \pi^* (E \cap F_j)$, by π^* is subadditive

$$\pi^* (E) \geqslant \pi^* \left(E \cap \sum_{j} F_j \right) + \pi^* (E \backslash A)$$

$$= \sum_{j \geqslant 1} \pi^* (E \cap F_j) + \pi^* (E \backslash A)$$

$$\geqslant \pi^* \left(\bigcup_{j \geqslant 1} (E \cap F_j) \right) + \pi^* (E \backslash A) \tag{4.38}$$

$$=\geqslant \pi^* \left(E \cap \left(\bigcup_{j \geqslant 1} F_j \right) \right) + \pi^* (E \backslash A)$$

$$= \pi^* (E \cap A) + \pi^* (E \backslash A)$$

So $\mathcal{M}$ is a $\sigma-$algebra.

$\square$

Remark 4.2. $\forall n$, we have that

$$\pi^* \left(E \cap \sum_{j=1}^{n} F_j \right) = \sum_{j=1}^{n} \pi^* (E \cap F_j) \tag{4.39}$$

where F_j defined as Eq 4.34.

Proof. By induction

1. $n = 1$, Eq 4.39 holds

2. Support n holds then we will proof $n + 1$ holds. $F_k \in \mathcal{M}, F_{n+1} \in \mathcal{M}$, we have that

$$\pi^* \left(E \cap \sum_{j=1}^{n+1} F_j \right) = \pi^* \left(\left(E \cap \sum_{j=1}^{n+1} F_j \right) \cap F_{n+1} \right) + \pi^* \left(\left(E \cap \sum_{j=1}^{n+1} F_j \right) \cap F_{n+1}^c \right)$$

$$= \pi^* (E \cap F_{n+1}) + \underbrace{\pi^* \left(E \cap \sum_{j=1}^{n} F_j \right)}_{\text{by assumption } = \sum_{j=1}^{n} \pi^*(E \cap F_j)} \tag{4.40}$$

$$= \sum_{j=1}^{n+1} \pi^* (E \cap F_j)$$

$\square$

By Thm 4.2 we have that $\mathcal{M} \supseteq \mathcal{F}(a)$.

Step 3

Theorem 4.3. $\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$ *is $\sigma-$ additive, then*

$$\pi^* (A) = \nu (A), \quad \forall A \in a \tag{4.41}$$

Remark 4.3. Eq 4.41 is also

$$\pi^*|_a = v \tag{4.42}$$

Eq 4.2 holds because Thm 4.2, $a \in \mathcal{M}$.

Proof. (Thm 4.3)

1. $\pi^* (A) = \nu (A), \ \forall A \in a$

   (a) check $\pi^* (A) \leqslant \nu (A)$

   Let's $\underbrace{A}_{E_1}, \underbrace{\varnothing}_{E_2}, \underbrace{\varnothing}_{E_3}, \underbrace{\cdots}_{E_j}$

   $$\pi^* (A) = \inf_{\{E_i\},A} \sum_i \nu (E_i) \leqslant \sum_i \nu (E_i) = \nu (A) \tag{4.43}$$

   (b) check $\pi^* (A) \geqslant \nu (A)$

   Let's take

   $$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ F_3 &= E_3 \backslash (E_1 \cup E_2) \\ &\vdots \\ F_n &= E_n \backslash (E_1 \cup E_2 \cup \cdots \cup E_{n-1}) \\ &\vdots \end{aligned} \tag{4.44}$$

   $F_j \in a, \bigcup_j F_j = \bigcup_j E_j, F_j \cap F_k = \varnothing, A \subseteq \bigcup_{j \geqslant 1} F_j$, so $A = \sum_j F_j \cap A \in a$.

   Because ν is $\sigma-$additive we have that

   $$\nu (A) = \sum_{j \geqslant 1} \nu (F_j \cap A) \tag{4.45}$$

   $\because F_j \subseteq E_j$

   $$\nu (A) = \sum_{j \geqslant 1} \nu (F_j \cap A) \leqslant \sum_{j \geqslant 1} \nu (E_j) \tag{4.46}$$

   so

   $$\nu (A) \leqslant \inf_{\{E_i\},A} \sum_{j \geqslant 1} \nu (E_j) = \pi^* (A) \tag{4.47}$$

   Then, we can get

   $$\pi^* (A) = \nu (A), \ \forall A \in a \tag{4.48}$$

2. $\pi^*|_{\mathcal{M}}$ is $\sigma-$additive

Suppose that $A_j \in \mathcal{M}, A_j \cap A_k = \varnothing$, we want to proof that

$$\pi^* \left(\sum A_j \right) = \sum_{j \geqslant 1} \pi^* (A_j) \tag{4.49}$$

(a) check $\pi^* \left(\sum A_j \right) \leqslant \sum_{j \geqslant 1} \pi^* (A_j)$ by π^* is an outer measure, π^* is subadditive

(b) check $\pi^* \left(\sum A_j \right) \geqslant \sum_{j \geqslant 1} \pi^* (A_j)$

   by π^* is an outer measure, π^* is monotone

$$\pi^* \left(\sum_{j \geqslant 1} A_j \right) \geqslant \pi^* \left(\sum_{j=1}^{n} A_j \right) \tag{4.50}$$

   by Remark 4.2, we have that

$$\pi^* \left(\sum_{j=1}^{n} A_j \right) = \sum_{j=1}^{n} \pi^* (A_j), \quad \forall n \tag{4.51}$$

so

$$\pi^* \left(\sum_{j \geqslant 1} A_j \right) \geqslant \sum_{j \geqslant 1} \pi^* (A_j) \tag{4.52}$$

$\square$

Step 4

Definition 4.4. Ω is σ-finite(μ_1) if $E_j \uparrow \Omega, \mu_1 (E_j) < \infty, \ \forall j, \ E_j \in a$.

Theorem 4.4 (Uniqueness). *Suppose that* $\mu_1, \mu_2 : \mathcal{F} (a) \to R_+ \cup \{+\infty\}, \Omega$ *is* σ-finite(μ_1), *if* $\mu_1|_a = \mu_2|_a$, *then*

$$\mu_1 = \mu_2, \quad on \quad \mathcal{F}(a) \tag{4.53}$$

Definition 4.5. $\Omega, \mathcal{G} \subseteq \mathcal{P} (\Omega), \mathcal{G}$ is a monotone class if

1.
$$A_j \in \mathcal{G}, j \geqslant 1, A_j \subseteq A_{j+1} \Rightarrow A = \bigcup_{j \geqslant 1} A_j = \lim_{j \to \infty} A_j \in \mathcal{G} \tag{4.54}$$

2.
$$B_j \in \mathcal{G}, j \geqslant 1, B_j \supseteq B_{j+1} \Rightarrow B = \bigcap_{j \geqslant 1} B_j = \lim_{j \to \infty} B_j \in \mathcal{G} \tag{4.55}$$

Theorem 4.5. $\mathcal{G}_\alpha$ *is a monotone class,* $\alpha \in I$, *then the followings hold*

1. $\bigcap_{\alpha \in I} \mathcal{G}_\alpha$ *is a monotone class*

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2. $c \subseteq \mathcal{P}(\Omega) \Rightarrow \mathcal{G}(c) = \bigcap\limits_{\alpha \in I} \mathcal{G}_\alpha$, *i.e. monotone classes generated by class c*

Lemma 4.1. $a \subseteq \mathcal{P}(\Omega)$ *is an algebra, $\mu(a)$ is monotone class generated by algebra a, $\mathcal{F}(a)$ is a $\sigma-$algebra generated by algebra a, then*

$$\mu(a) = \mathcal{F}(a) \tag{4.56}$$

Proof. It will proof in the next lecture. $\qquad \square$

Proof. (Thm 4.4) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\}, \mu_1(A) = \mu_2(A), \forall A \in a, \Omega$ σ-finite, $\Omega = \bigcup\limits_{j \geqslant 1} E_j, E_j \in a, \mu_j(E_j) < \infty$, then $\mu_1 = \mu_2$ on $\mathcal{F}(a)$.

Fix E_n, we denote that

$$\mathcal{B}_n = \{E \in \mathcal{F}(a), \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\} \tag{4.57}$$

We claim that

1. $\mathcal{B}_n \supseteq a$

2. $\mathcal{B}_n$ is a monotone class

We proof $\mathcal{B}_n$ is a monotone class.

1. $\forall A_j \in \mathcal{B}_n, A_j \uparrow A = \bigcup\limits_{j \geqslant 1} A_j$, then

$$\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n) \tag{4.58}$$

   By Remark 3.1

$$\mu_1(A_j \cap E_n) \to \mu_1(A \cap E_n), \mu_2(A_j \cap E_n) \to \mu_2(A \cap E_n) \tag{4.59}$$

2. $\forall B_j \in \mathcal{B}_n, B_j \downarrow B = \bigcap\limits_{j \geqslant 1} B_j$, then

$$\mu_1(B_j \cap E_n) = \mu_2(B_j \cap E_n) \tag{4.60}$$

   By Remark 3.1

$$\mu_1(B_j \cap E_n) \to \mu_1(B \cap E_n), \mu_2(B_j \cap E_n) \to \mu_2(B \cap E_n) \tag{4.61}$$

So we can get that

$$\mathcal{B}_n \supseteq \mathcal{M}(a) \tag{4.62}$$

where $\mathcal{M}(a)$ is a monotone class generated by a. Then by Lemma 4.1

$$\mathcal{M}(a) = \mathcal{F}(a) \tag{4.63}$$

And by Eq 4.57,

$$\mathcal{B}_n(a) \subseteq \mathcal{F}(a) \tag{4.64}$$

so

$$\mathcal{B}_n(a) = \mathcal{F}(a) \tag{4.65}$$

Finally, $\mu_1(A) = \mu_2(A), \forall A \in \mathcal{F}(a)$, by $\mathcal{B}_n = \mathcal{F}(a)$, then $A \in \mathcal{B}_n$. $B_j \uparrow \Omega$, apply Lemma 3.1 again, we have

$$\mu_1(A) = \mu_2(A) \tag{4.66}$$

$\qquad \square$

Lecture 5

Monotone Classes

Definition 5.1. Given Ω, define $\mathcal{M}(a) \subseteq \mathcal{P}(\Omega)$ is a monotone class is

1. $A_j \in \mathcal{M}, A_j \uparrow A \left(A_j \subseteq A_j, \bigcup_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$

2. $A_j \in \mathcal{M}, A_j \downarrow A \left(A_j \supseteq A_j, \bigcap_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$

Remark 5.1.

1. $\mathcal{F}$ is σ-filed(σ-algebra) $\Rightarrow \mathcal{F}$ is a monotone class

2. $\mathcal{M}_\alpha \subseteq P(\Omega)$, $(\alpha \in I)$ is monotone class, then $\mathcal{M} = \bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is a monotone class.

Notation 5.1. (Smallest monotone class contain c) $\mathcal{M}(c)$ is a monotone class generated by c if

$$c \subseteq \mathcal{M}(\Omega), \mathcal{M}(c) = \bigcap_{\alpha \in I} \mathcal{M}_\alpha \tag{5.1}$$

Definition 5.2. $E \subseteq \mathcal{M}(a)$, the set $\mathcal{G}(E)$ is defined as below

$$\mathcal{G}(E) = \{F \in \mathcal{M}(a), E \backslash F, E \cap F, F \backslash E \in \mathcal{M}(a)\} \tag{5.2}$$

Lemma 5.1.

1. If $E \in a \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$

2. If $E \in \mathcal{M}(a) \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$

Proof.

1. $E \in a$, we want to show that

   (a) $\mathcal{G}(E) \supseteq a$

   Take $H \in a \subseteq \mathcal{M}(a)$, then

   $$\underbrace{E \backslash H}_{\in a}, \underbrace{E \cap H}_{\in a}, \underbrace{H \backslash E}_{\in a} \in \mathcal{G}(a) \tag{5.3}$$

   so $H \in \mathcal{G}(E)$, then $a \subseteq \mathcal{G}(E)$

   (b) $\mathcal{G}(E)$ is a monotone class

   Suppose that $H_k \uparrow H$, $H_k \in \mathcal{G}(E)$,

   $$\because E \backslash H_k \in \mathcal{M}(a), \ E \backslash H_k \to E \backslash H, \therefore E \backslash H \in \mathcal{M}(a) \tag{5.4}$$

   $$\because E \cap H_k \in \mathcal{M}(a), \ E \cap H_k \to E \cap H, \therefore E \cap H \in \mathcal{M}(a) \tag{5.5}$$

$$\because H_k \backslash E \in \mathcal{M}(a), \ H_k \backslash E \to H \backslash E, \therefore H \backslash E \in \mathcal{M}(a) \tag{5.6}$$

By Eq 5.6, $H \in \mathcal{M}(a)$, and by the definition 5.2, $H \in \mathcal{G}(E)$. So $\mathcal{G}(E)$ is a monotone class. We also get that $\mathcal{G}(E) \supseteq \mathcal{M}(a)$.

2. $E \in \mathcal{M}(a)$, we want to show that

    (a) $\mathcal{G}(E)$ is a monotone class

    $E \in \mathcal{M}(a)$, suppose $H_k \in \mathcal{G}(E), H_k \uparrow H$

    $$\because E \backslash H_k \in \mathcal{M}(a), E \backslash H_k \downarrow E \backslash H \quad \therefore E \backslash H \in \mathcal{M}(a) \tag{5.7}$$

    Similarity:
    $$E \cap H \in \mathcal{M}(a) \tag{5.8}$$
    $$H \backslash E \in \mathcal{M}(a) \tag{5.9}$$

    then we can get $H \in \mathcal{G}(E)$, so $\mathcal{G}(E)$ is a monotone class.

    (b) $\mathcal{G}(E) \supseteq a$

    We need to show $H \in a \Rightarrow H \in \mathcal{G}(E)$.

    By Lemma 5.1.1, we can get that

    $$\mathcal{G}(H) \supseteq \mathcal{M}(a) \tag{5.10}$$

    $\because E \in \mathcal{M}(a), \therefore E \in \mathcal{G}(H)$, by the Def 5.2, $H \backslash E, H \cap E, E \backslash H \in \mathcal{M}(a)$, so we can get $a \in \mathcal{G}(E)$

    $\square$

Theorem 5.1. *a is a algebra, $a \subseteq \mathcal{P}(\Omega)$. $\mathcal{F}(a)$ is a σ-algebra generated by a, $\mathcal{M}(a)$ is a monotone class generated by a, then*

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.11}$$

Proof. By remark 5.1, $\mathcal{F}(a)$ is a monotone class, by Notation 5.1 $\mathcal{F}(a) \supseteq a$ and $\mathcal{F}(a) \supseteq \mathcal{M}(a)$.

So we have to show that

$$\mathcal{F}(a) \subseteq \mathcal{M}(a) \tag{5.12}$$

We will show that

1. $\mathcal{M}(a)$ is a algebra

    (a) $\Omega \in \mathcal{M}(a)$ by $\Omega \subseteq a$

    (b) $E \in \mathcal{M}(a) \Rightarrow E^c \in \mathcal{M}(a)$

    By Lemma 5.1.1, let $E = \Omega$, then $\mathcal{M}(a) \subseteq \mathcal{G}(\Omega)$. $\because E \in \mathcal{M}(a)$, so $E \in \mathcal{G}(\Omega)$. By Definition 5.2, $\mathcal{G}(\Omega) = \{E \in \mathcal{M}(a), E^c, E, \varnothing \in \mathcal{M}(a)\}$

    (c) $E, F \in \mathcal{M}(a) \Rightarrow E \cap F \in \mathcal{M}(a)$

    By Lemma 5.1.2, $\mathcal{G}(E) \supseteq \mathcal{M}(a)$, so $F \in \mathcal{G}(E)$.

    By Def 5.2 $F \in \mathcal{G}(E) = \{F \in \mathcal{M}(a), F \backslash E, F \cap E, E \backslash F \in \mathcal{M}(a)\}$, so $E \bigcap F \in \mathcal{M}(a)$

2. $\mathcal{M}(a)$ is a σ-algebra i.e. $A_j \in \mathcal{M}(a)$, $j \geqslant 1 \Rightarrow \bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$

By $\mathcal{M}(a)$ is a algebra, so $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{M}(a)$.

$\bigcup\limits_{j=1}^{n} A_j \uparrow \bigcup\limits_{j \geqslant 1} A_j$ and $\mathcal{M}(a)$ is a monotone class, so $\bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$.

So $\mathcal{F}(a) \subseteq \mathcal{M}(a)$.

Above all,

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.13}$$

$\square$

Lecture 6

The Lebesgue Measure I

Definition 6.1. $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$, we define $\mathcal{S}$ as below:

$$\mathcal{S} = \{\varnothing, \mathbb{R}, (a, b], (a, \infty), (-\infty, b]\} \tag{6.1}$$

Remark 6.1. $\mathcal{S}$ as above, then $\mathcal{S}$ is a semialgebra

Proof. by Def 2.1. $\qquad\square$

Definition 6.2. $\mu : \mathcal{S} \to \mathbb{R}_+ \bigcup \{+\infty\}$, additive, and

$$\mu(\varnothing) = 0, \mu((a, b]) = b - a, \mu((-\infty, b]) = +\infty, \mu(\mathbb{R}) = +\infty \tag{6.2}$$

Theorem 6.1. *μ is additive on a semialgebra $\mathcal{S}$ and defined as Def 6.2, then μ is $\sigma-$additive, i.e.*

$$A = \sum_{j \geqslant 1} A_j \Rightarrow \mu(A) = \sum_{j \geqslant 1} \mu(A_j), \qquad A, A_j \in \mathcal{S} \tag{6.3}$$

Remark 6.2. It is difficult to prove Thm 6.1 $(a, b] \cup (c, d]$ is not in the semialgebra $\mathcal{S}$. But, $\mathcal{S} \to a(\mathcal{S})$ with respect to $\mu \to \nu$.

Proof.

1.

$$\because A = \sum_{j \geqslant 1} A_j \supseteq \sum_{j=1}^{n} A_j \tag{6.4}$$

By ν is additive $\Rightarrow \nu$ is monotone & subadditive,

$$\therefore \nu(A) \geqslant \nu\left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \nu(A_j), \quad \forall n \tag{6.5}$$

so

$$\therefore \nu(A) \geqslant \sum_{j \geqslant 1} \nu(A_j) \tag{6.6}$$

2. (a) Assume that $A = (a, b], A_j = (a_j, b_j], A = \sum_{j \geqslant 1} A_j$, we want to show that

$$\nu(A) = b - a \leqslant \sum_{j \geqslant 1} (b_j - a_j) = \sum_{j \geqslant 1} \nu(A_j) \tag{6.7}$$

For any given $\epsilon > 0$, we have that

$$[a + \varepsilon, b] \subseteq (a, b] = \sum_{j \geqslant 1} (a_j, b_j] \subseteq \bigcup_{j \geqslant 1} \left(a_j, b_j + \frac{\varepsilon}{2^j}\right) \tag{6.8}$$

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By a set K is compact i.e. K is closed and bounded $\Rightarrow$ Any open cover for K has a finite subcover

$$[a + \varepsilon, b] \subseteq \bigcup_{k \geqslant 1} \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right) \tag{6.9}$$

By ν is additive $\Rightarrow \nu$ is monotone & subadditive, we have

$$b - a - \varepsilon \leqslant \nu\left([a + \varepsilon, b]\right) = \nu \left(\bigcup_{k=1}^{m} \left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right)\right) \leqslant \sum_{k=1}^{m} \nu\left(a_{jk}, b_{jk} + \frac{\varepsilon}{2^{jk}}\right) \tag{6.10}$$

so we can get that

$$b - a - \varepsilon \leqslant \sum_{k=1}^{m} \left(b_{jk} - a_{jk} + \frac{\varepsilon}{2^{jk}}\right) \leqslant \sum_{j \geqslant 1} \left(b_j - a_j + \frac{\varepsilon}{2^j}\right) = \sum_{j \geqslant 1} (b - a) + \varepsilon \tag{6.11}$$

so Eq. 6.7 holds.

(b) General case $A \in \mathcal{S}$, $E_n = (-n, n] \uparrow \mathbb{R}$.

$A \cap E_n = \sum_{j \geqslant 1} A_j \cap E_n$.

By ν is additive on a semi-algebra

$$\nu\left(A \cap E_n\right) = \sum_{j \geqslant 1} \nu\left(A_j \cap E_n\right) \leqslant \sum_{j \geqslant 1} \nu\left(A_j\right) \tag{6.12}$$

By Remark 6.3, let $n \to \infty$, we have

$$\nu\left(A\right) = \lim_{n \to \infty} \nu\left(A \cap E_n\right) \leqslant \sum_{j \geqslant 1} \nu\left(A_j\right) \tag{6.13}$$

$\square$

Remark 6.3. $E_n = (-n, n] \uparrow \mathbb{R}$, ν is additive on a semi-algebra then

$$\nu\left(A\right) = \lim_{n \to \infty} \nu\left(A \cap E_n\right) \tag{6.14}$$

Proof.

$$\because E_n \uparrow \mathbb{R}, \therefore A \cap E \uparrow, \therefore \lim_{n \to \infty} (A \cap E_n) = \bigcup_{n \geqslant 1} (A \cap E_n) = A \cap \left(\bigcup_{n \geqslant 1} E_n\right) = A \tag{6.15}$$

ν is additive,

$$\nu\left(A\right) = \nu\left(\bigcup_{n \geqslant 1} A \cap E_n\right) = \nu\left(\lim_{n \to \infty} A \cap E_n\right) \overset{why}{=} \lim_{n \to \infty} \nu\left(A \cap E_n\right) \tag{6.16}$$

why , because we will check via Def 6.1 except $A = (a, b]$

1. $A = \varnothing$

2. $A = \mathbb{R}$

3. $A = (a, \infty)$

   (a) left hand of why in Eq. 6.16

   $$\because A \cap E_n = (a, +\infty) \cap (-n, n) = \begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \tag{6.17}$$

   $$\therefore \lim_{n \to \infty} (A \cap E_n) = (-\infty, +\infty) = \mathbb{R} \tag{6.18}$$

   by Def 6.2

   $$\mu \left(\lim_{n \to \infty} (A \cap E_n) \right) = \mu(\mathbb{R}) = +\infty \tag{6.19}$$

   (b) right hand of why in Eq. 6.16

   $$\because \nu(A \cap E_n) = \nu \left(\begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \right) = \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} \tag{6.20}$$

   $$\therefore \lim_{n \to \infty} \nu(A \cap E_n) = \lim_{n \to \infty} \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} = +\infty \tag{6.21}$$

   So Eq 6.16 holds.

4. $A = (-\infty, b]$

   $\square$

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Lecture 7

The Lebesgue Measure II

$\mathcal{S} = \{\varnothing, \mathbb{R}, (a, b], (a, \infty), (-\infty, b]\}, \mu : a(\mathcal{S}) \to \mathbb{R}_+ \cup \{+\infty\},$

$$\mu((a, b]) = b - a \tag{7.1}$$

Theorem 7.1. μ is σ-additive on $a(\mathcal{S})$

Remark 7.1. $E_k \in (-N, N]$, μ is finite and μ is continuous from below at $\varnothing$ (i.e. $E_k \in a, E_k \downarrow \varnothing \Rightarrow \mu(E_k) \to 0$), by Lemma 3.1 can imply Thm 7.1 hold.

Proof. Now we want to show that $E_k \downarrow \varnothing, E_k \in a, E_k \in (-N, N]$, then

$$\mu(E_k) \to 0 \tag{7.2}$$

If not, $\exists \delta > 0, \exists E_k \downarrow \varnothing, E_k \in a, E_k \in (-N, N]$, such that

$$\mu(E_k) \geqslant 2\delta > 0 \tag{7.3}$$

If $\exists$ a compact set $\{G_k\}$, s.t. $G_k \supseteq G_{k+1}, G_k \subseteq E_k$, but

$$\varnothing \neq \bigcap_{k \geqslant 1} G_k \subseteq \bigcap_{k \geqslant 1} E_k = \varnothing \tag{7.4}$$

Then, we will find a sequence of compact sets $\{G_k\}$ by induction.

Our goal is : $E_k \subseteq (-N, N], \mu(E_n) \geqslant 2\delta, (F_k)_{1 \leqslant k \leqslant M} G_k = \overline{F_k}$. F_k satisfy the flowing three conditions:

1. $\overline{F_k} \subseteq E_k, \quad 1 \leqslant k \leqslant n - 1$
2. $F_{k+1} \subseteq F_k, \quad 1 \leqslant k \leqslant n - 1$
3. $\mu(E_n \backslash F_n) \leqslant \frac{\delta}{2} + \frac{\delta}{4} + \cdots + \frac{\delta}{2^n} = \delta$

Now,

1. by $E_1 \in a$, then E_1 can be written as

$$E_1 = \sum_{j=1}^{n_1} (a_{1,j}, b_{1,j}] \tag{7.5}$$

   define F_1 as

$$F_1 = \sum_{j=1}^{n_1} (a_{1,j} + \varepsilon_1, b_{1,j}] \in a \tag{7.6}$$

$\mu(E_1 \backslash F_1) = m_1 \varepsilon_1.$

We will pick a small enough ϵ to meet $\mu(E_1 \backslash F_1) \leqslant \frac{\delta}{2}$, i.e. $m_1 \varepsilon_1 \leqslant \frac{\delta}{2}$, and $b_{1,j} - a_{1,j} \geqslant \varepsilon_1$, i.e. $\min_j \{b_{1,j} - a_{1,j}\} \geqslant \varepsilon_1$, so we choose $0 < \varepsilon_1 \leqslant \min \left\{ \frac{\delta}{2m_1}, \min_{1 \leqslant j \leqslant m_1} \{b_{1,j} - a_{1,j}\} \right\}.$

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2. We will show $\mu\left(E_2 \cap F_1\right)$ have a lower positive bound , i e. $E_2 \cap F_1 \neq \varnothing$

$$2\delta \leqslant \mu\left(E_2\right) = \mu\left(E_2 \cap F_1\right) + \underbrace{\mu\left(E_2 \backslash F_1\right)}_{\leqslant \mu\left(E_1 \backslash F_1\right) \leqslant \frac{\delta}{2}} \Rightarrow \mu\left(E_2 \cap F_1\right) \geqslant 2\delta - \frac{\delta}{2} > 0 \qquad (7.7)$$

by $E_2 \cap F_1 \neq \varnothing, E_2 \cap F_1 \in a$, then $E_2 \cap F_1$ can be written as

$$E_2 \cap F_1 = \sum_{j=1}^{m_2}\left(a_{2,j}, b_{2,j}\right] \qquad (7.8)$$

Define F_2 :

$$F_2 = \sum_{j=1}^{m_2}\left(a_{2,j} + \varepsilon_2, b_{2,j}\right] \qquad (7.9)$$

choose a small enough ϵ_2 satisfies that

$$F_2 \subseteq \overline{F_2} \subseteq E_2 \cap F_1 \qquad (7.10)$$

then $F_2 \subseteq F_1, \overline{F_2} \subseteq E_2$, and $F_2 \subseteq F_1 \Rightarrow \overline{F_2} \subseteq \overline{F_1}$, then we get that

$$F_2 \subseteq \overline{F_2} \subseteq E_2$$
$$F_2 \subseteq F_1 \qquad (7.11)$$
$$\mu\left(E_2 \backslash F_2\right) \leqslant \frac{\delta}{2} + \frac{\delta}{4}$$

3. assume the F_n satisfies the three conditions as our goal above

$$2\delta \leqslant \mu\left(E_{n+1}\right) = \mu\left(E_{n+1} \cap F_n\right) + \underbrace{\mu\left(E_{n+1} \backslash F_n\right)}_{\mu\left(E_n \backslash F\right) \leqslant \delta} \Rightarrow \mu\left(E_{n+1} \cap F_n\right) \geqslant \delta > 0 \qquad (7.12)$$

by $E_{n+1} \cap F_n \neq \varnothing$ and $E_{n+1} \cap F_n \in a$ then

$$E_{n+1} \cap F_n = \sum_{j=1}^{k_{n+1}}\left(a_{n+1,j}, b_{n+1,j}\right] \qquad (7.13)$$

then we define F_{n+1} as

$$F_{n+1} = \sum_{j=1}^{k_{n+1}}\left(a_{n+1,j} + \varepsilon_{n+1}, b_{n+1,j}\right] \qquad (7.14)$$

choose a small enough ϵ_{n+1} satisfies that

$$F_{n+1} \subseteq \overline{F_{n+1}} \subseteq E_{n+1} \cap F_n \qquad (7.15)$$

then $F_{n+1} \subseteq E_{n+1}, F_{n+1} \subseteq F_n$, and $\overline{F_{n+1}} \subseteq \overline{F_n}$, let $\varepsilon_{n+1} = \frac{\delta}{k_{n+1} \cdot 2^{n+1}}$, then $\mu\left(\left(E_{n+1} \cap F_n\right) \backslash F_{n+1}\right) \leqslant \frac{\delta}{2^{n+1}}$.

Then

$$\mu\left(E_{n+1}\backslash F_{n+1}\right) = \mu\left((E_{n+1}\cap F_n)\backslash F_{n+1}\right) + \underbrace{\mu\left((E_{n+1}\backslash F_n)\backslash F_{n+1}\right)}_{\underbrace{\leqslant \mu\left(E_{n+1}\backslash F_n\right)}_{\leqslant \mu(E_n\backslash F_n)\leqslant \frac{\delta}{2}+\cdots+\frac{\delta}{2^n}}} \tag{7.16}$$

$$\leq \frac{\delta}{2^{n+1}} + \frac{\delta}{2} + \frac{\delta}{4} + \cdots + \frac{\delta}{2^n} = \delta\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$$

define $G_k = \overline{F_k}$, then $G_{k+1} = \overline{F_{k+1}} \subseteq \overline{F_k} = G_k$ G_k : satisfies that

(a) $G_{k+1} \subseteq G_k$

(b) G_k compact

(c) $G_k \neq \varnothing$

Why $G_k \neq \varnothing$ because:

$$2\delta \leqslant \mu\left(E_k\right) = \mu\left(E_k\backslash F_k\right) + \mu\left(E_k \cap F_k\right) \leqslant \delta + \mu\left(F_k\right) \Rightarrow \mu\left(F_k\right) \geq \delta \tag{7.17}$$

Then $F_k \neq \varnothing \Rightarrow G_k = \overline{F_k} \neq \varnothing$.

But

$$\varnothing \neq \bigcap_{k\geqslant 1} G_k \subseteq \bigcap_{k\geqslant 1} E_k = \varnothing \tag{7.18}$$

Above all, $E_k \in (-N, N]$, μ is finite and μ is continuous from below at $\varnothing$, then Lebesgue μ is σ-additive on $a(\mathcal{S})$. $\qquad\square$

Lecture 8

Complete Measures

Definition 8.1. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is σ-algebra, $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \infty$ is additive. $(\mu, \mathcal{F})$ is complete if : $A \in \mathcal{F}$ such that $\mu(A) = 0$, $\forall E \subseteq A$ then $E \in \mathcal{F}$.

Remark 8.1. In Def 8.1, by monotone $\mu(E) = 0$.

Next, our goal is: $\overline{\mathcal{F}} \supseteq \mathcal{F}$, and $\overline{\mu} : \overline{\mathcal{F}} \to \mathbb{R}_+ \cup \{+\infty\}$: $\begin{cases} \overline{\mu}|_{\mathcal{F}} = \mu, \\ (\overline{\mu}, \overline{\mathcal{F}}) \text{ is complete} \end{cases}$

Definition 8.2. $\overline{\mathcal{F}} = \{A \cup N, \text{ where } A \in \mathcal{F} \text{ and } N \subseteq E \in \mathcal{F}, \text{ such that } \mu(E) = 0\}$

Claim 8.1. $\overline{\mathcal{F}}$ is a σ-algebra.

Proof. We will check :

1. $\Omega \in \overline{\mathcal{F}}$, $\because \Omega = \Omega \cup \varnothing, \varnothing \subseteq \varnothing \in \mathcal{F}$

2. $A \in \overline{\mathcal{F}} \Rightarrow A^c \in \overline{\mathcal{F}}$

   $\because A \subseteq \overline{\mathcal{F}}, A = E \cup N$ where $E \in \mathcal{F}$, $N \subseteq H \in \mathcal{F}$ such that $\mu(H) = 0$

$$
\begin{aligned}
A^c &= (E \cup N)^c \\
&= \underbrace{[(E \cup N)^c \cap H]}_{\subseteq H} \cup \underbrace{[(E \cup N)^c \cap H^c]}_{\underset{\subseteq E^c \cap H^c \in \mathcal{F}}{E^c \cap N^c \cap H^c}}
\end{aligned}
\tag{8.1}
$$

   by Def 8.2, $A^c \in \overline{\mathcal{F}}$.

3. $A_j = E_j \cup H_j$ where $E_j \in \mathcal{F}, H_j \subseteq W_j$ where $w_j \in \mathcal{F}, \mu(W_j) = 0$ then $\bigcup_{j \geqslant 1} A_j \in \overline{\mathcal{F}}$

$$
\begin{aligned}
\because \bigcup_{j \geqslant 1} A_j &= \bigcup_{j \geqslant 1} (E_j \cup H_j) \\
&= \underbrace{\bigcup_{j \geqslant 1} E_j}_{\mathcal{F}} \cup \underbrace{\bigcup_{j \geqslant 1} H_j}_{\subseteq \bigcup_{j \geqslant 1} W_j \triangleq W}
\end{aligned}
\tag{8.2}
$$

   and $\mu(W) = \mu\left(\bigcup_{j \geqslant 1} W_j\right) \leqslant \sum_{j \geqslant 1} \mu(W_j) = 0$

   $\square$

We want to define $\overline{\mu}$ on $\overline{\mathcal{F}}$:

$$
\because \quad \underbrace{\overline{\mu}(A \cup N)}_{\geqslant \overline{\mu}(A) = \mu(A)} \leqslant \overline{\mu}(A \cup E) \leqslant \underbrace{\overline{\mu}(A) + \overline{\mu}(E)}_{= \mu(A) + \mu(E) = \mu(A)}
\tag{8.3}
$$

So we give the following definition.

Definition 8.3. $\overline{\mu}(A \cup N) = \mu(A)$

Proof. By the Def 8.3

1. check $\overline{\mu}$ is well defined

   Assume that $A \cup N = B \cup M$, *where* $A, B \in \mathcal{F}, N \subseteq E \in \mathcal{F}$ *where* $\mu(E) = 0$, $M \subseteq F \in \mathcal{F}$ *where* $\mu(F) = 0$. We need to show that $\mu(A) = \mu(B)$.

   $$\because A \subseteq A \cup N = B \cup M \subseteq B \cup M \tag{8.4}$$

   by μ is $\sigma-$additive, then μ is monotone,

   $$\mu(A) \leqslant \mu(B \cup F) \leqslant \mu(B) + \mu(F) = \mu(B) \tag{8.5}$$

   similarly, $\mu(B) \leqslant \mu(A)$.

2. check $\overline{\mu}|_{\mathcal{F}} = \mu$

   by $A \in \mathcal{F}$, $A = A \bigcup \varnothing$ then $\overline{\mu}(A \cup \varnothing) = \mu(A)$

3. check $\overline{\mu}$ is $\sigma-$additive i.e. $A_j \in \overline{\mathcal{F}}$, $A = \sum_{j \geqslant 1} A_j \Rightarrow \overline{\mu}(A) = \sum_{j \geqslant 1} \mu(A_j)$

   $$\because A_j \in \overline{\mathcal{F}}, \therefore A_j = E_j \cup N_j \text{ where } E_j \in \mathcal{F}, N_j \subseteq H_j \subseteq \mathcal{F} \text{ where } \mu(H_j) = 0 \tag{8.6}$$

   $$\therefore A = \sum_{j \geqslant 1} A_j = \sum_{j \geqslant 1} E_j \cup \sum_{j \geqslant 1} N_j$$

   $$\therefore \overline{\mu}(A) = \mu\left(\sum_{j \geqslant 1} E_j\right) = \sum_{j \geqslant 1} \mu(E_j) = \sum_{j \geqslant 1} \overline{\mu}(A_j) \tag{8.7}$$

4. check $(\overline{\mu}, \overline{\mathcal{F}})$ is complete, i.e. $\overline{\mathcal{F}}$ *is* $\overline{\mu}$-complete.

   Assume that $A \subseteq E \in \overline{\mathcal{F}}$ where $\overline{\mu}(E) = 0$. We have to show that $A \in \overline{\mathcal{F}}$.

   $\because E \in \overline{\mathcal{F}} \therefore E = B \cup N$ *where* $B \in \mathcal{F}$, $N \subseteq H \in \mathcal{F}$ *where* $\mu(H) = 0$

   $A = \varnothing \cup A$, $\varnothing \in F, A \subseteq E \subseteq B \cup N \subseteq \underbrace{B}_{\in \mathcal{F}} \cup \underbrace{H}_{\in \mathcal{F}} \in \mathcal{F}$, so $\mu(B \cup N) \leqslant \mu(B) + \mu(N) = 0$ by

   $\overline{\mu}(E) = \mu(B) = 0, \mu(A) \leqslant \mu(B) \Rightarrow \mu(A) = 0$, so $A \in \overline{\mathcal{F}}$

5. check $\overline{\mu}$ is unique. $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \{+\infty\}$,

   And, extension $\overline{\mathcal{F}_\mu} = \{E \cup N, \text{ where } E \in \mathcal{F}, N \subseteq H \in \mathcal{F}, \text{ where } \mu(H) = 0\}, \overline{\mu} : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$.

   Assume that $\nu : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$, and $\nu(A) = \overline{\mu}(A), \forall A \in \mathcal{F}$. Then we want show that $\nu(B) = \overline{\mu}(B), \forall B \in \overline{\mathcal{F}_\mu}$.

   Let $B \in \overline{\mathcal{F}_\mu}, B = E \cup N$ *where* $E \in \mathcal{F}$, $N \subseteq H \in \mathcal{F}$, *where* $\mu(H) = 0, \nu(H) = \overline{\mu}(H) = \mu(H) = 0$.

   fix B, $\overline{\mu}(B) = \mu(E) \underbrace{=}_{by\ E\ \in \mathcal{F}} v(E) \leqslant \nu(B)$

   $\nu(B) = \nu(E \cup N) \leqslant \nu(E \cup H) \leqslant \nu(E) + \nu(H) = \nu(E) = \overline{\mu}(B)$, then

   $$\nu(B) = \overline{\mu}(B) \tag{8.8}$$

$\square$

$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$.

Claim 8.2. $\mathcal{M}$ is π^*-complete.

Proof. π^*-complete, i.e. $A \subseteq B, B \subseteq \mathcal{M}, \pi^*(B) = 0 \Rightarrow A \in \mathcal{M}$

We have to show $\forall E \subseteq \Omega, \ \pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c)$

    1. $\because E \cap A \subseteq A \subseteq B \ \therefore \pi^*(E \cap A) \leqslant \pi^*(B) = 0$

    2. $\pi^*(E \cap A^c) \leqslant \pi^*(E)$

So, $A \in \mathcal{M}$ $\hfill \square$

Lecture 9

Approximation Theorems

Lecture 10

Integration: Measurable and Simple Functions