Measure Theory

Lectures by Claudio Landim

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Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. J. Taylor published by Cambridge University Press.

There are many other very good books on the subject. Here is a partial list:

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

0. $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a,b]) = b - a$
2. $A \subseteq \mathbb{R},\ A + x = \{x + y :\ y \in A\},\ \ \forall A,\ A \subseteq \mathbb{R},\ \forall x \in \mathbb{R} :$
$$\lambda(A + x) = \lambda(A) \tag{1.1}$$
3. $A = \bigcup_{j\geqslant 1} A_j,\ \ A_j \cap A_k = \varnothing :$
$$\lambda(A) = \sum_k \lambda(A_k) \tag{1.2}$$

Definition 1.1. $x \sim y,\ x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R},\ y - x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_{\sim}$, only one point represents the equivalence class of Ω , like α, β.

Ω is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0,1)$

Claim 1.1. $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \varnothing \end{cases} \quad q, p \in \mathbb{Q}$

Proof. Assume that $\Omega + q \cap \Omega + q \neq \varnothing$ then, $x = \alpha + p = \beta + q,\ \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \varnothing]$. □

Claim 1.2. $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q) \subseteq (-1, 2) \tag{1.3}$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda(E) \leqslant \lambda(F)$

Proof. $\because E \subseteq F \therefore F = E \cup (F \backslash E),\ E \cap (F \backslash E) = \varnothing$, then $\lambda(F) = \lambda(E) + \lambda((F \backslash E)) \Rightarrow \lambda(F) \geqslant \lambda(E)$. □

Then,

$$\lambda\left(\sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)\right) \leqslant \lambda((-1, 2)) = 3 \tag{1.4}$$

On the other hand,

$$\lambda((\Omega + q)) = \lambda(\Omega) = 0 \Rightarrow \lambda\left(\sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)\right) = 0 \tag{1.5}$$

Claim 1.4. $(0,1) \subseteq \sum\limits_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)$

Proof. $\forall$ fixed $x \in (0,1)\,, \exists \alpha \in [x] \cap \Omega,\ \alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q},\ - < q < 1 \Rightarrow x = \alpha + q,\ x \in \Omega + q$ $\square$

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda\left(\sum_{q\in\mathbb{Q}} \Omega + q\right) = 0 \tag{1.6}$$

it is impossible.

Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Lecture 3

Set Functions

Lecture 4

Caratheodory Theorem

Lecture 5

Monotone Classes

Lecture 6

The Lebesgue Measure I

Lecture 7

The Lebesgue Measure II

Lecture 8

Complete Measures

Lecture 9

Approximation Theorems

Lecture 10

Integration: Measurable and Simple Functions

Lecture 11

Measurable Functions

Lecture 12

Definition of The Integral

Lecture 13

Integral of Simple Functions

Lecture 14

Properties of The Integral I

Lecture 15

Properties of The Integral II

Lecture 16

Theorems on The Convergence of Integrals

Lecture 17

Product Measures

Lecture 18

Measure On a Countable Product of Spaces

Lecture 19

Fubini's Theorem

Lecture 20

Hahn-Jordan Theorem

Lecture 21

Radon-Nikodym Theorem

Lecture 22

Almost Sure and Almost Uniform

Lecture 23

Convergence in Measure

Lecture 24

Höelder and Minkowski inequalities

Lecture 25

L_p Spaces

Lecture 26

From Convergence in Measure to Convergence in L_p

Lecture 27

Bounded Linear Operators in L_p

Lecture 28

Vitali's Covering Lemma

Lecture 29

Differentiability of Functions of Bounded Variations

Lecture 30

Absolutely Continuous Functions

Lecture 31

Decomposition of Distribution

Lecture 32

Cantor Ternary Set and Function