Measure Theory

Lectures by Claudio Landim

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Introduction

These lectures are mainly based on the books Introduction to measure and integration by S. Taylor published by Cambridge University Press.

These notes were live-TeXed, though I edited for typos and added diagrams requiring the Ti*k*Z package separately. I used the editor TeXstudio.

I am responsible for all faults in this document, mathematical or otherwise; any merits of the material here should be credited to the lecturer, not to me.

Please email any corrections or suggestions to jaafar_zhang@163.com.

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Lecture 1

Introduction: a Non-measurable Set

λ satisfies the flowing:

0. $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a, b]) = b - a$
2. $A \subseteq \mathbb{R},\ A + x = \{x + y :\ y \in A\},\ \ \forall A,\ A \subseteq \mathbb{R},\ \forall x \in \mathbb{R} :$
$$\lambda(A + x) = \lambda(A) \tag{1.1}$$
3. $A = \bigcup_{j \geqslant 1} A_j,\ \ A_j \cap A_k = \varnothing :$
$$\lambda(A) = \sum_k \lambda(A_k) \tag{1.2}$$

Definition 1.1. $x \sim y,\ x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $[x] = \{y \in \mathbb{R},\ y - x \in \mathbb{Q}\}$.
$\Lambda = \mathbb{R}|_{\sim}$, only one point represents the equivalence class of Ω , like α, β.

Ω is a class of equivalence class, if $\Omega \subseteq R, \Omega \subseteq (0, 1)$

Claim 1.1. $\begin{cases} \Omega + q = \Omega + q \\ \Omega + q \cap \Omega + q = \varnothing \end{cases} \quad q, p \in \mathbb{Q}$

Proof. Assume that $\Omega + q \cap \Omega + q \neq \varnothing$ then, $x = \alpha + p = \beta + q,\ \alpha, \beta \in \Omega \Rightarrow \alpha - \beta = q - p \in \mathbb{Q} \Rightarrow \alpha = \beta \Rightarrow [q \neq p, p, q \in \mathbb{Q} \Rightarrow (\Omega + q) \cap (\Omega + p) = \varnothing]$. □

Claim 1.2. $\Omega + q \subseteq (-1, 2)$, if $-1 < q < 1$.

then we can get

$$\sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} (\Omega + q) \subseteq (-1, 2) \tag{1.3}$$

Claim 1.3. $E \subseteq F \Rightarrow \lambda(E) \leqslant \lambda(F)$

Proof. $\because E \subseteq F \therefore F = E \cup (F \backslash E),\ E \cap (F \backslash E) = \varnothing$, then $\lambda(F) = \lambda(E) + \lambda((F \backslash E)) \Rightarrow \lambda(F) \geqslant \lambda(E)$. □

Then,

$$\lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} (\Omega + q)\right) \leqslant \lambda((-1, 2)) = 3 \tag{1.4}$$

and ,

$$\infty \cdot \lambda((\Omega + q)) = \infty \cdot \lambda(\Omega) \leq 3 \Rightarrow \lambda\left(\sum_{\substack{q \in \mathbb{Q} \\ -1<q<1}} (\Omega + q)\right) = 0 \tag{1.5}$$

Claim 1.4. $(0,1) \subseteq \sum\limits_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)$

Proof. $\forall$ fixed $x \in (0,1)\,, \exists \alpha \in [x] \cap \Omega,\ \alpha \in (0,1)$, and we know that $\alpha - x = q \in \mathbb{Q},\ - < q < 1 \Rightarrow x = \alpha + q,\ x \in \Omega + q$ $\square$

But, we get that:

$$1 = \lambda\left((0,1)\right) \leqslant \lambda \left(\sum_{q\in\mathbb{Q}} \Omega + q \right) = 0 \tag{1.6}$$

it is impossible.

Lecture 2

Classes of Subsets (Semi-algebras, Algebras and Sigma-algebras) and Set Functions

Definition 2.1. $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is semi-algebra if:

1. $\Omega \subseteq \mathcal{S}$
2. $A, B \in \mathcal{S} \Rightarrow A \bigcap B \in \mathcal{S}$
3. $\forall A \in \mathcal{S} \Rightarrow A^c = \sum_{i=1}^{n} E_j$, $\exists E_1, \cdots, E_n \in \mathcal{S}$, $E_i, E_j\,(i \neq j)$ disjoint sets, n is finite number

Example 2.1. $\Omega = \mathbb{R}$, $\mathcal{S} = \{\mathbb{R}, \{(a,b), a < b, a, b \in \mathbb{R}\}, \{(-\infty, b], b \in \mathbb{R}\}, \{(a, \infty), a \in \mathbb{R}\}, \varnothing\}$, $(a,b]^c = (-\infty, a] \cup [b, +\infty)$

Example 2.2. $\Omega = \mathbb{R}^2$

$\mathcal{S} = \{\mathbb{R}^2, \{(a_1, b_1) \times (a_2, b_2),\ a_i < b_i, a_i, b_i \in \mathbb{R}, \{(-\infty, b_1] \times (-\infty, b_2], b_i \in \mathbb{R}\}, \{(a_1, \infty) \times (a_2, \infty), a_i \in \mathbb{R}\}, \varnothing\}$

Definition 2.2. $a = \mathcal{P}(\Omega)$ is an algebra:

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

Remark 2.1. a algebra $\Rightarrow$ a semi-algebra

Definition 2.3. σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$:

1. $\Omega \subseteq \mathcal{S}$
2. $A_j \in \mathcal{S}, j \leq 1 \Rightarrow \bigcap_{j \geqslant 1} A_j \in \mathcal{S}$
3. $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$

Remark 2.2. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega)$, a_α algebra, $\alpha \in I \Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is an algebra.

Proof. check the followings

1. $\Omega \in a$
2. $A, B \in a \Rightarrow A \bigcap B \in a$
3. $A \in a \Rightarrow A^c \in a$

$\square$

Remark 2.3. $\Omega, a_\alpha \subseteq \mathcal{P}(\Omega), \alpha \in I, a_\alpha$, σ-algebra $\Rightarrow a = \bigcap_{\alpha \in I} a_\alpha$ is a σ-algebra

Proof. check the followings

1. $\Omega \in a$
2. $A_j, j \geq 1 \in a \Rightarrow \bigcap_{j \geqslant 1} A_j \in a$

3. $A \in a \Rightarrow A^c \in a$

□

Definition 2.4 (minimal algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is an algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.1}$$

Remark 2.4. $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall \alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an algebra.

Definition 2.5 (minimal σ-algebra generated by c)**.** $\Omega, c \subseteq \mathcal{P}(\Omega)$, $a(c)$ is a σ-algebra generated by c, and $a = a(c)$:

1. $c \subseteq a$
2. $\forall \mathcal{B}$ is σ-algebra, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$c \subseteq \mathcal{B} \Rightarrow a \subseteq \mathcal{B} \tag{2.2}$$

Remark 2.5. $a(c)$ exits, and $a = a(c) = \bigcap\limits_{\alpha} a_\alpha,\ \forall \alpha,\ c \subseteq a_\alpha,\ a_\alpha$ is an σ-algebra.

Lemma 2.1. *Ω, f semi-algebra $f \subseteq \mathcal{P}(\Omega)$, $a(f)$ algebra generated by f then*

$$A \in a(f) \Leftrightarrow \exists E_j \in f, 1 \leqslant j \leqslant n,\ A = \sum_{j=1}^{n} E_j \tag{2.3}$$

Proof.

1. $\Leftarrow$

   $A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f \in a(f)$

   By definition 2.1 and remark 2.6 $\Rightarrow A \in a(f)$

2. $\Rightarrow$

   $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j, E_j \in f$

   Then by remark 2.7, it will be proved easily.

□

Remark 2.6. $E, J \in a, E \bigcup F \in a, E \bigcup F = (E^c \bigcap F^c)^c$

Remark 2.7. $\mathcal{B} = \left\{ \sum\limits_{j=1}^{n} F_j,\ F_j \in f \right\}$, $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ then

1. $\mathcal{B}$ algebra
2. $\mathcal{B} \supseteq f$
3. $\mathcal{B} \supseteq a(f)$

Proof. We only prove that $\mathcal{B}$ algebra, then check the following

1. $\Omega \in \mathcal{B}$
2. $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$

   $\because A, B \in \mathcal{B}, \therefore A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f,\ B = \sum\limits_{k=1}^{m} F_k,\ F_k \in f$, then

$$\begin{aligned} A \cap B &= \left(\sum_{j=1}^{n} E_j \right) \cap \left(\sum_{k=1}^{m} F_k \right) \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \\ &\in \mathcal{B} \end{aligned} \tag{2.4}$$

3. $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$

   $A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f$

   By definition 2.1:

$$\begin{aligned} E_1^c &= \sum_{k_1=1}^{l_1} F_{1,k_1},\ F_{1,j} \in f \\ \cdots &= \cdots \\ E_i^c &= \sum_{k_i=1}^{l_i} F_{i,k_i},\ F_{i,j} \in f \end{aligned} \tag{2.5}$$

   Then, we get that

$$\begin{aligned} A^c &= \left(\sum_{k_1=1}^{l_1} F_{1,k_1} \right) \cap \left(\sum_{k_2=1}^{l_2} F_{2,k_2} \right) \cap \cdots \cap \left(\sum_{k_n=1}^{l_n} F_{n,k_n} \right) \\ &= \sum_{k_1=1}^{l_1} \sum_{k_2=1}^{l_2} \cdots \sum_{k_n=1}^{l_n} (F_{1,k_1} \cap F_{2,k_2} \cap F_{n,k_n}) \\ &\in \mathcal{B} \end{aligned} \tag{2.6}$$

□

Definition 2.6. $c \subseteq \mathcal{P}(\Omega), \varnothing \in c,\ \mu:\ c \to \mathbb{R}_+ \cup \{+\infty\}$. μ is additive if

1. $\mu(\varnothing) = 0$
2. $E_1, E_2, ..., E_n \in c,\ E = \sum\limits_{j=1}^{n} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j=1}^{n} \mu(E_k)$

Remark 2.8.

$$\exists A \in c,\ \mu(A) < \infty,\ A = A \cup \varnothing,\ \mu(A) = \mu(A) + \mu(\varnothing) \Rightarrow \mu(\varnothing) = 0 \tag{2.7}$$

Remark 2.9. $c,\ \mu : c \to \mathbb{R}_+ \bigcup +\infty,\ E \subseteq F,\ F \backslash E \in c,\ E, F \in c$

$$F = E \cup (F \backslash E),\ \mu(F) = \mu(E) + (F \backslash E) \tag{2.8}$$

1. $\mu(E) = +\infty$, $\mu(F) = +\infty$
2. $\mu(E) < +\infty$, $\mu(F \backslash E) = \mu(F) - \mu(E)$

so,

$$\mu(E) \leqslant \mu(F) \tag{2.9}$$

Example 2.3. Discrete measure: Ω, $c \subseteq \mathcal{P}(\Omega)$, $\{x_j,\ j \geqslant 1\},\ x_j \in \Omega,\ \{p_j,\ j \geqslant 1\},\ p_j, \geqslant 0,\ A \in c$, define that

$$\mu(A) = \sum_{j \geqslant 1} p_j 1\{x_j \in A\} \tag{2.10}$$

then μ is additive

Definition 2.7. $c \in \mathcal{P}(\Omega), \varnothing \in c$, $\mu :\ c \to \mathbb{R}_+ \bigcup +\infty$, μ is σ-additive if

1. $\mu(\varnothing) = 0$
2. $E_j \in c,\ j \neq k, E_j \bigcap E_k = \varnothing,\ \ E = \sum\limits_{j \geq 1} E_j \in c \Rightarrow \mu(E) = \sum\limits_{j \geq 1} \mu(E_j)$

Example 2.4. $\Omega = (0,1),\ c = \{(a,b],\ 0 \leqslant a < b < 1\},\ \mu :\ c \to \mathbb{R}_+ \cup \{+\infty\}$, define that

$$\mu(a,b] = \begin{cases} +\infty & a = 0 \\ b - a & a > 0 \end{cases} \tag{2.11}$$

$(a,b] = \sum\limits_{j=1}^{n} (a_j, b_j)$, we can get that μ is NOT σ-additive.

If $x_1 = \frac{1}{2}, x_j > x_{j+1}, x_j \downarrow\to 0$, then

$$\frac{1}{2} = \left(0, \frac{1}{2}\right] = \sum_{j \geqslant 1} (x_{j+1}, x_j] = +\infty \tag{2.12}$$

it is impossible.

Lecture 3

Set Functions

Definition 3.1. $c \subseteq \mathcal{P}(\Omega)$, $\mu : c \to \mathbb{R}_+ \bigcup +\infty$:

1. $E \in c$, μ continuous from below at E, if $\forall (E_n)_{n \geqslant 1},\ E_n \in c,\ E_n \uparrow E\ \left(E_n \subseteq E_{n+1},\ \bigcup\limits_{n\geqslant 1} E_n = E\right)$:
$$\mu(E_n) \to \mu(E) \tag{3.1}$$

2. $E \in c$, μ continuous from above at E, if $\forall (E_n)_{n \geqslant 1},\ E_n \in c,\ E_n \downarrow E\ \left(E_{+1} \subseteq E_n,\ \bigcap\limits_{n\geqslant 1} E_n = E\right)$, and $\exists n_0,\ \mu(E_{n_0}) < \infty$:
$$\mu(E_n) \to \mu(E) \tag{3.2}$$

Remark 3.1. For a sequence $E_1, E_2, ...$ of sets, we put
$$\limsup E_i = \bigcap_{n=1}^{\infty}\left(\bigcup_{i=n}^{\infty} E_i\right), \liminf E_i = \bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} E_i\right) \tag{3.3}$$
and if $\{E_i\}$ is such that $\limsup E = \liminf E_i$ we say that the sequence converges to the set
$$E = \limsup E = \liminf E_i \tag{3.4}$$

Remark 3.2. 2 need $\exists n_0,\ \mu(E_{n_0}) < \infty$, if not:
$$E_n = [n, +\infty),\ \mu(E_n) = +\infty,\ E_n \downarrow \varnothing,\ \lambda(\varnothing) = 0 \tag{3.5}$$

Lemma 3.1. *$a \subseteq \mathcal{P}(\Omega)$, algebra; $\mu :\ a \to \mathbb{R}_+ \cup \{+\infty\}$, additive;*

1. μ is σ-additive $\Rightarrow$ μ continuous at E, $\forall E \in a$

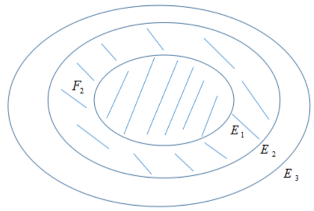
2. μ is continuous from below $\Rightarrow$ μ is σ-additive

3. μ is continuous from above at $\varnothing$& μ is finite $\Rightarrow$ σ-additive

Proof.

1.

(i) μ is σ-additive $\Rightarrow$ μ conti. from below at $E \in a$. $E \in a, E_n \uparrow E,\ E_n \in a$:
$$\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 \backslash E_1 \\ \vdots &= \vdots \\ F_n &= E_n \backslash E_{n-1} \end{aligned} \tag{3.6}$$



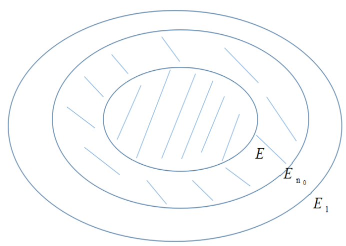
and we can get that

$$F_j \cap F_k = \varnothing, \quad \sum_{k=1}^{n} F_k = E_n, \quad \bigcup_{n \geqslant 1} E_n = \bigcup_{n \geqslant 1} F_n \tag{3.7}$$

so

$$\mu(E) = \sum_{k \geqslant 1} \mu(F_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n \to \infty} \mu(E_n) \tag{3.8}$$

(ii) μ cont. from above $E \in a, E_n \in a, E_n \downarrow E, \mu(E_{n_0}) < \infty \Rightarrow \mu(E_n) \downarrow \mu(E)$



$$\begin{aligned} G_1 &= E_{n_0} \backslash E_{n_0+1} \\ G_2 &= E_{n_0} \backslash E_{n_0+2} \\ \vdots &= \vdots \\ G_k &= E_{n_0} \backslash E_{n_0+k} \end{aligned} \tag{3.9}$$

then $G_k \uparrow E_{n_0} \backslash E, G_k \in a \Rightarrow \mu(G_k) \uparrow \mu(E_{n_0} \backslash E)$, so

$$\begin{aligned} \mu(E_{n_0} \backslash E) &= \lim_{n \to \infty} \mu(E_{n_0} \backslash E_{n_0+k}) \\ \mu(E_{n_0} \backslash E) &= \mu(E_{n_0}) - \mu(E) \\ \mu(E_{n_0}) - \mu(E) &= \lim_{k \to \infty} (\mu(E_{n_0}) - \mu(E_{n_0+k})) \end{aligned} \tag{3.10}$$

2. μ cont. below, $E = \sum_{k \geqslant 1} E_k, \ E, E_k \in a$.

   Obs.

$$\sum_{k=1}^{n} E_k \subseteq E \overset{additive}{\Rightarrow} \begin{cases} \mu\left(\sum_{k=1}^{n} E_k\right) \leqslant \mu(E) \\ \sum_{k=1}^{n} \mu(E_k) \leqslant \mu(E) \end{cases} \tag{3.11}$$

then

$$\sum_{k\geqslant 1}\mu\left(E_k\right)\leqslant\mu\left(E\right) \tag{3.12}$$

$F_n=\sum\limits_{k=1}^{n}E_k\in a,\ F_n\uparrow E,$

$$\sum_{k=1}^{n}\mu\left(E_k\right)=\mu\left(F_n\right)\uparrow\mu\left(E\right)\Rightarrow\sum_{k\geqslant 1}\mu\left(E_k\right)=\mu\left(E\right) \tag{3.13}$$

3. μ cont. at $\varnothing$, $\mu\left(\Omega\right)<\infty$, $E,E_k\in a,E=\sum\limits_{k\geqslant 1}E_k$.

$$F_n=\sum_{k\geqslant m}E_k\in a\ \left(E\backslash\sum_{j=1}^{n-1}E_j\right) \tag{3.14}$$

$F_n\downarrow\varnothing,\mu\left(F_1\right)<\infty,\ \mu\left(F_n\right)\to 0$

$$\begin{aligned}\mu\left(E\right)&=\mu\left(\sum_{k=1}^{n}E_k\cup\sum_{k>n}E_k\right)\\&=\underbrace{\mu\sum_{k=1}^{n}E_k}_{\to\sum\limits_{k\geqslant 1}\mu(E_n)}+\underbrace{\mu\sum_{k>n}E_k}_{\to 0}\\&\to\sum_{k\geqslant 1}\mu\left(E_n\right)\end{aligned} \tag{3.15}$$

□

Remark 3.3. Suppose E_α, $\alpha\in I$ is a class of subsets of X, and E_i is one set of the class, then

1. $\bigcap\limits_{\alpha\in I}E_\alpha\subseteq E_i\subseteq\bigcup\limits_{\alpha\in I}E_\alpha$
2. $X-\bigcup\limits_{\alpha\in I}E_\alpha=\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$
3. $X-\bigcap\limits_{\alpha\in I}E_\alpha=\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$

Proof.

1. This is immediate from the definition.
2. Suppose $x\in X-\bigcup\limits_{\alpha\in I}E_\alpha$ then $x\in X$ and x is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is x is not in any E_α, $\alpha\in I$ so that $x\in X-E_\alpha$ for every $\alpha\in I$, and $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$. Conversely if $x\in\bigcap\limits_{\alpha\in I}\left(X-E_\alpha\right)$, then for every $\alpha\in I$, x is in X but not in E_α, so $x\in X$ but x is not in $\bigcup\limits_{\alpha\in I}E_\alpha$, that is $x\in\bigcup\limits_{\alpha\in I}\left(X-E_\alpha\right)$.
3. Similar to 2

Remark 3.3 (2) and (3) are also called as de Morgan's Law. □

Example 3.1. $(0,1), (a,b], 0 \leqslant a < b < 1$

$$\mu(a,b] = \begin{cases} b-a, & a>0 \\ +\infty, & a=0 \end{cases} \tag{3.16}$$

μ is additive but NOT σ-additive

Proof. $E_n \downarrow \varnothing,\ \mu(E_{n_0}) < \infty,\ E_n = (a_{n,1}, b_{n,1}] \cup \cdots \cup (a_{n,k_n}, b_{n,k_n}], a_{n,j} < a_{n,j+1}$.

$\begin{cases} a_{n,1} = 0, & \forall n \\ a_{n_0} > 0, & \textit{some } n_0 \end{cases}$ □

Theorem 3.1 (Extension)**.** *$f \subseteq \mathcal{P}(\Omega)$ semi-algebra, $\mu : f \to \mathbb{R}_+ \cup \{\infty\}$ σ-additive, then $\exists \nu$:*

$$\nu : a(f) \to \mathbb{R}_+ \cup \{\infty\} \tag{3.17}$$

such that:

1. *ν σ-additive*
2. *$\nu(A) = \mu(A), \ \forall A \in f$*
3. *$\mu_1, \mu_2, a(f) \to \mathbb{R}_+ \bigcup \{+\infty\}$, then $\mu_1(A) = \mu_2(A), \forall A \in s \Rightarrow \mu_1(E) = \mu_2(E), \forall E \in a(f)$*

Proof. $A \in a(f) \Rightarrow A = \sum\limits_{j=1}^{n} E_j,\ E_j \in f$ by Lemma 2.1.

$$\nu(A) \overset{add}{=} \sum_{j=1}^{n} \nu(E_j) \overset{ext}{=} \sum_{j=1}^{n} \mu(E_j) \tag{3.18}$$

we define that

$$\nu(A) = \sum_{j=1}^{n} \mu(E_j) \tag{3.19}$$

we want to show that $\nu(A) = \sum\limits_{j=1}^{n} \mu(E_j)$ is well-defined:

1. ν is unique

$$\begin{aligned} A &= \sum_{j=1}^{n} E_j,\ E_j \in f \\ &= \sum_{k=1}^{m} F_k,\ F_k \in f \end{aligned} \tag{3.20}$$

then we will prove that

$$\begin{aligned} \nu(A) &= \sum_{j=1}^{n} \mu(E_j) \\ &= \sum_{k=1}^{m} \mu(F_k) \end{aligned} \tag{3.21}$$

$$\because E_j \subseteq A = \sum_{k=1}^{m} F_k \Rightarrow E_j = E_j \cap \left(\sum_{k=1}^{m} F_k \right) = \sum_{k=1}^{m} \underbrace{(E_j \cap F_k)}_{\in f} \tag{3.22}$$

$$\therefore \mu (E_j) = \mu \left(\sum_{k=1}^{m} (E_j \cap F_k) \right) \tag{3.23}$$

then

$$\nu (A) = \sum_{j=1}^{n} \mu (E_j) = \sum_{j=1}^{n} \sum_{k=1}^{m} \mu (E_j \cap F_k) = \sum_{k=1}^{m} \mu (F_k) \tag{3.24}$$

2. ν is an additive, $\nu (A) = \sum\limits_{j=1}^{n} \mu (E_j)$

   Assume that

$$\begin{cases} A = \sum\limits_{j=1}^{n} E_j, \; E_j \in f \\ B = \sum\limits_{k=1}^{m} F_k, \; F_k \in f \end{cases}, A \cap B = \varnothing \tag{3.25}$$

   We will show that

$$\nu (A \cup B) = \nu (A) + \nu (B) \tag{3.26}$$

$$\because A \cup B = \sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \tag{3.27}$$

   therefore

$$\begin{aligned} \nu (A \cup B) &= \mu \left(\sum_{j=1}^{n} E_j + \sum_{k=1}^{m} F_k \right) \\ &= \sum_{j=1}^{n} \mu (E_j) + \sum_{k=1}^{m} \mu (F_k) \\ &= \nu (A) + \nu (B) \end{aligned} \tag{3.28}$$

3. $\nu (A) = \mu (A), \; A \in f$ by Eq 3.19

4. ν is uniqueness, we want to show that:

   Suppose that $\mu_1, \mu_2 : \; a(f) \to R_+ \cup \{+\infty\}, \forall A \; \in f, \mu_1, \mu_2$ *additive*, then

$$\mu_1 (A) = \mu_2 (A) \Rightarrow \mu_1 (B) = \mu_2 (B), \; \forall B \in a(f) \tag{3.29}$$

   $\because B \in a(f), \therefore B = \sum\limits_{j=1}^{n} \mu_1 (E_j), \; E_j \in f$

$$\mu_1 (B) = \sum_{j=1}^{n} \mu_1 (E_j) = \sum_{j=1}^{n} \mu_2 (E_j) = \mu_2 (B) \tag{3.30}$$

Now we proof the extension of σ-additive, ie: $\mu - \sigma$ additive, f semi-algebra, $\nu - \sigma$ additive, $a(f)$ is a algebra generated by f. we want to show that

$$A = \sum_{j\geqslant 1} A_j,\ A, A_j \in a\left(f\right) \Rightarrow \nu\left(A\right) = \sum_{j\geqslant 1} \nu\left(A_j\right) \tag{3.31}$$

by representation of an algebra:

$$A = \sum_{j=1}^{m} E_j, E_j \in f; \quad A_k = \sum_{l=1}^{m_k} E_{k,l}, E_{k,l} \in f \tag{3.32}$$

by Eq 3.19:

$$\nu\left(A\right) = \sum_{j=1}^{m} \nu\left(E_j\right), \quad \nu\left(A_k\right) = \sum_{l=1}^{m_k} \nu\left(E_{k,l}\right) \tag{3.33}$$

$$\because E_j = E_j \cap A = E_j \cap \left(\sum_{k\geqslant 1} A_k\right) = E_j \cap \left(\sum_{k\geqslant 1}\sum_{l=1}^{m_k} E_{k,l}\right) = \sum_{k\geqslant 1}\sum_{l=1}^{m_k} \left(E_j \cap E_{k,l}\right) \tag{3.34}$$

therefore

$$\begin{aligned} \nu\left(A\right) &= \sum_{j=1}^{n} \mu\left(E_j\right) \\ &= \sum_{j=1}^{n}\sum_{k\geqslant 1}\sum_{l=1}^{m_k} \mu\left(E_j \cap E_{k,l}\right) \\ &= \sum_{k\geqslant 1}\underbrace{\sum_{l=1}^{m_k} \mu\left(E_{k,l}\right)}_{\subseteq A_k} \end{aligned} \tag{3.35}$$

Eq 3.35 holds because:

$$E_{k,l} = E_{k,l} \cap A = \sum_{j=1}^{n} \left(E_{k,l} \cap E_j\right) \tag{3.36}$$

and

$$\mu\left(E_{k,l}\right) = \sum_{j=1}^{n} \mu\left(E_{k,l} \cap E_j\right) \tag{3.37}$$

so we can get that

$$\nu\left(A\right) = \sum_{k\geqslant 1} \nu\left(A_k\right) \tag{3.38}$$

□

Lecture 4

Caratheodory Theorem

Theorem 4.1 (Caratheodory Theorem)**.**

$$\begin{array}{llr} \sigma - add & \mu : f \to \mathbb{R}_+ \cup \{+\infty\} & f \subseteq \mathcal{P}(\Omega), f \ is \ semialgebra \\ \quad\downarrow & \qquad\qquad\downarrow & \\ \sigma - add & \nu : a(f) \to \mathbb{R}_+ \cup \{+\infty\} & a(f) \ algebra \ generated \ by \ f \\ \quad\downarrow & \qquad\qquad\downarrow & \\ \sigma - add & \pi : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\} & \mathcal{F}(a) \ is \ \sigma - algebra \ generated \ by \ algebra \ a \end{array} \tag{4.1}$$

The big picture of the proof:

1. Define the π^* outer measure:
$$\pi^* = \inf_{\{E_i\}} \sum_{i \geqslant 1} \nu(E_i) \tag{4.2}$$

2. $\mathcal{M}$ σ-algebra, $\mathcal{M} \supseteq \mathcal{F}(a)$

3. $$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.3}$$
is σ-additive, and
$$\pi^*|_a = \nu \tag{4.4}$$

4. (uniqueness) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \bigcup \{+\infty\}$, Ω is σ-finite(μ_1), if $E_j \uparrow \Omega$, $\mu_1(E_j) < \infty, \forall j, \ E_j \in a$ and $\mu_1|_a = \mu_2|_a$ then implies that
$$\mu_1 = \mu_2 \tag{4.5}$$

Finally, we define $\pi(E) = \pi^*(E), \ \forall E \in \mathcal{F}(a) \subseteq \mathcal{M}$.

Now, let
$$\pi^* : \ \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\} \tag{4.6}$$

We will prove π^* is an outer measure.

And we will construct a family of subsets $\mathcal{M}$

$$\mathcal{M} \subseteq \mathcal{P}(\Omega) \tag{4.7}$$

we will also prove $\mathcal{M}$ satisfies the following:

1. $\mathcal{M}$ is a $\sigma-$algebra
2. $\mathcal{M} \supseteq a$
3. $\pi^*|_{\mathcal{M}}$ $\sigma-$additive
4. $\pi^*|_a = \nu$

Next, we will define π^* and $\mathcal{M}$.

Step 1

Definition 4.1 (π^*)**.** $\pi^*: \mathcal{P}(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$, $A \in \Omega$, $\{E_i, i \geqslant 1\}, E_i \in a, A \subseteq \bigcup E_i$, $\{E_i\}$ is a covering of A, then we define that

$$\pi^* = \inf_{\{E_i\}, A} \sum_{i \geqslant 1} \nu(E_i) \tag{4.8}$$

where $\nu: \ a(f) \to \mathbb{R}_+ \cup \{+\infty\}, \ is$ σ-additive.

Definition 4.2 (Outer measure)**.** $\mu: \ c \to \mathbb{R}_+ \cup \{+\infty\}, c \subseteq P(\Omega), \varnothing \in c$, μ is a outer measure if

1. $\mu(\varnothing) = 0$
2. (monotone) $E \subseteq F, \ E, F \in c \ \Rightarrow \mu(E) \leqslant \mu(F)$
3. (subadditive) $E, E_i \in c, \ E \subseteq \bigcup\limits_i E_i \Rightarrow \mu(E) \leqslant \sum\limits_i \mu(E_i)$

Theorem 4.2. *π^* in 4.1 is a outer measure.*

Proof. We will check the conditions in Def 4.2.

1. check $\pi^*(\varnothing) = 0$
   (a) $E_i = \varnothing, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$ then
   $$\pi^*(\varnothing) = \inf_{\{E_i\}, \varnothing} \sum_{i \geqslant 1} \nu(E_i) \leqslant \sum_{i \geqslant 1} \nu(E_i) = 0 \tag{4.9}$$
   (b) $E_i \in a, \{E_i\}, \varnothing \subseteq \bigcup\limits_{i \geqslant 1} E_i$, then
   $$\sum_{i \geqslant 1} \nu(E_i) \geqslant 0 \Rightarrow \pi^*(\varnothing) \geqslant 0 \tag{4.10}$$
2. check $E \subseteq F$, $\pi^*(E) \leqslant \pi^*(F)$

   Let's take any covering of F:$\{E_i\}, E_i \in a, F \subseteq \bigcup\limits_{i \geqslant 1} E_i$ is also a covering of E, then
   $$\pi^*(E) = \inf_{\{E_i\}, E} \sum_{i \geqslant 1} \nu(E_i) \leqslant \pi^*(F) = \inf_{\{E_i\}, F} \sum_{i \geqslant 1} \nu(E_i) \tag{4.11}$$
3. check $E \subseteq \bigcup\limits_{i \geqslant 1} E_i, \ \ \pi^*(E) \leqslant \sum\limits_{i \geqslant 1} \pi^*(E_i)$
   (a) $\pi^*(E_i) = \infty$ then
   $$\pi^*(E) \leqslant \sum_{i \geqslant 1} \pi^*(E_i) \tag{4.12}$$
   (b) $\pi^*(E_i) < \infty$, then
   $$\pi^*(E_i) = \inf_{\{H_{ik}\}, \ E_i} \sum_{k \geqslant 1} \nu(H_{ik}) \tag{4.13}$$

then there $\exists \{H_{ik}\} \in a, E_i \subseteq \bigcup\limits_{k\geqslant 1} H_{ik}$ such that

$$\pi^*(E_i) = \inf_{\{H_{ik}\},\ E_i} \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \sum_{k\geqslant 1} \nu(H_{ik}) \leqslant \pi^*(E_i) + \frac{\varepsilon}{2^i} \tag{4.14}$$

$\{H_{ik}\}$ is a covering of E, then

$$\pi^*(E) \leqslant \sum_{i,k} \nu(H_{ik}) \leqslant \sum_{i\geqslant 1} \left(\pi^*(E_i) + \frac{\varepsilon}{2^i}\right) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) + \varepsilon \tag{4.15}$$

so

$$\pi^*(E) \leqslant \sum_{i\geqslant 1} \pi^*(E_i) \tag{4.16}$$

□

Step 2

Definition 4.3 (Measurable set $\mathcal{M}$)**.** A set called measurable set $\mathcal{M}$ if $A \in \mathcal{M}$ $\forall E \in \Omega$, we have that

$$\pi^*(E) = \pi^*\left(E \bigcap A\right) + \pi^*\left(E \bigcap A^c\right) \tag{4.17}$$

Theorem 4.3. *If $\mathcal{M}$ definited as Def 4.3, then*

1. $\mathcal{M} \supseteq a$
2. $\mathcal{M}$ *is a* $\sigma-$*algebra*

Remark 4.1.

$$E \subseteq (E \cap A) \cup (E \cap A^c) \Rightarrow \pi^*(E) \leqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.18}$$

so we only to check $\geq$ in Eq 4.17

Proof. π^* is an outer measurable by Thm 4.1, then by subadditive of outer measure. □

Now we proof Thm 4.3.

Proof.

1. $a \in \mathcal{M}$

   Suppose that $A \in a$, $E \in \Omega$, we will show that

   $$\pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c) \tag{4.19}$$

   assume that $\pi^*(E) < \infty$, given $\varepsilon, \exists \{E_i\}, E,\ such\ \ that\ E_i \in a, E \subseteq \bigcup\limits_{i\geqslant 1} E_i$, then

   $$\pi^*(E) \leqslant \sum_{i\geqslant 1} \nu(E_i) \leqslant \pi^*(E) + \varepsilon \tag{4.20}$$

$E_i \cap A \in a, E \cap A \subseteq \bigcup\limits_{i \geqslant 1} (E_i \bigcap A)$, so

$$\begin{aligned} \pi^* (E \cap A) &\leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right) \\ \pi^* (E \cap A^c) &\leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right) \end{aligned} \tag{4.21}$$

so

$$\pi^* (E \cap A) + \pi^* (E \cap A^c) \leqslant \sum_{i \geqslant 1} \nu \left(E_i \bigcap A \right) + \sum_{i \geqslant 1} \nu \left(E_i \bigcap A^c \right) \leq \sum_{i \geqslant 1} \nu (E_i) \leqslant \pi^* (E) + \varepsilon \tag{4.22}$$

2. $\mathcal{M}$ is σ-algebra.

   We need to show that

   (a) $\Omega \in \mathcal{M}$

   It is clearly that:

$$\pi^* (E) = \pi^* (E \cap \Omega) + \pi^* (E \cap \Omega^c) \tag{4.23}$$

   (b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$

$$\because \pi^* (E) = \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.24}$$

   (c) $A_i \in \mathcal{M} \Rightarrow \bigcup\limits_{i \geqslant 1} A_i \subseteq \mathcal{M}$

   Finite union is closed: $A, B \in \mathcal{F} \Rightarrow A \bigcup B \in M$. Let's take $E \subseteq \Omega$. We will proof that

$$\pi^* (E) \geqslant \pi^* \left(E \cap \left(A \bigcup B \right) \right) + \pi^* \left(E \cap \left(A \bigcup B \right)^c \right) \tag{4.25}$$

   $\because A \in \mathcal{M}$,

$$\therefore \quad \pi^* (E) = \pi^* \left(E \bigcap A \right) + \pi^* \left(E \bigcap A^C \right) \tag{4.26}$$

   $\because B \in \mathcal{M}$

$$\begin{aligned} \therefore \quad \pi^* (E \backslash A) &= \pi^* (E \backslash A \cap B) + \pi^* (E \backslash A \cap B^c) \\ &= \pi^* (E \backslash A \cap B) + \pi^* \left(E \backslash \left(A \bigcup B \right) \right) \end{aligned} \tag{4.27}$$

   then

$$\pi^* (E) = \pi^* (E \cap A) + \pi^* (E \backslash A \cap B) + \pi^* (E \backslash (A \cup B)) \tag{4.28}$$

   We want to show

$$\pi^* (E \cap A) + \pi^* (E \backslash A \cap B) \geqslant \pi^* (E \cap (A \cup B)) \tag{4.29}$$

   By π^* is subadditive, we only to show that

$$E \cap (A \cup B) \subseteq (E \cap A) \cup (E \backslash A \cap B) \tag{4.30}$$

   this is because

$$E \cap (A \cup B) = \underbrace{\{[E \cap (A \cup B)] \cap A\}}_{\subseteq E \cap A} \bigcup \underbrace{\{[E \cap (A \cup B)] \cap A^c\}}_{\subseteq (E \cap A^c) \cap B = (E \backslash A) \bigcap B} \tag{4.31}$$

Then Eq 4.25 holds. So $\mathcal{M}$ is closed by finite(countable) union.

Now, we will show that countable infinite union is also closed. $A_i \in \mathcal{M}$, we want to show $A = \bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}$, take $E \subseteq \Omega$,

$$\pi^* (E) \geqslant \pi^* (E \cap A) + \pi^* (E \cap A^c) \tag{4.32}$$

by Eq. 4.25, $\forall\ n$ we know that

$$\begin{aligned} \pi^* (E) &= \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j^c \right) \right) \\ &\geq \pi^* \left(E \cap \left(\bigcup_{j=1}^{n} A_j \right) \right) + \pi^* (E \backslash A) \end{aligned} \tag{4.33}$$

$\geq$ holds in Eq 4.33 because $(E \backslash A) \subseteq \left(E \backslash \left(\bigcup\limits_{j=1}^{n} A_j \right) \right)$.

Now, we define

$$\begin{aligned} F_1 &= A_1 \\ F_2 &= A_1 \backslash A_2 \\ F_3 &= A_1 \backslash (A_2 \cup A_3) \\ &\vdots \\ F_n &= A_1 \backslash (A_2 \cup \cdots \cup A_{n-1}) \\ &\vdots \end{aligned} \tag{4.34}$$

It is clear that

$$\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{n} F_j \tag{4.35}$$

where $F_j \cap F_k = \varnothing, F_j \in \mathcal{M}$.

Then Eq 4.33 can be written as

$$\pi^* (E) \geqslant \pi^* \left(E \cap \sum_{j=1}^{n} F_j \right) + \pi^* (E \backslash A) \tag{4.36}$$

By Remark 4.2, we have

$$\begin{aligned} \pi^* (E) &\geqslant \pi^* \left(E \cap \left(\sum_{j=1}^{n} F_j \right) \right) + \pi^* (E \backslash A) \\ &= \sum_{j=1}^{n} \pi^* (E \cap F_j) + \pi^* (E \backslash A) \end{aligned} \tag{4.37}$$

$\because$ n is any number in Remark 4.2 , $\therefore$ $\pi^*\left(E \cap \sum\limits_{j=1}^{\infty} F_j\right) = \sum\limits_{j=1}^{\infty} \pi^*(E \cap F_j)$, by π^* is subadditive

$$
\begin{aligned}
\pi^*(E) &\geqslant \pi^*\left(E \cap \sum_j F_j\right) + \pi^*(E \backslash A) \\
&= \sum_{j \geqslant 1} \pi^*(E \cap F_j) + \pi^*(E \backslash A) \\
&\geqslant \pi^*\left(\bigcup_{j \geqslant 1} (E \cap F_j)\right) + \pi^*(E \backslash A) \\
&=\geqslant \pi^*\left(E \cap \left(\bigcup_{j \geqslant 1} F_j\right)\right) + \pi^*(E \backslash A) \\
&= \pi^*(E \cap A) + \pi^*(E \backslash A)
\end{aligned} \tag{4.38}
$$

So $\mathcal{M}$ is a $\sigma-$algebra.

$\square$

Remark 4.2. $\forall n$, we have that

$$
\pi^*\left(E \cap \sum_{j=1}^{n} F_j\right) = \sum_{j=1}^{n} \pi^*(E \cap F_j) \tag{4.39}
$$

where F_j defined as Eq 4.34.

Proof. By induction

1. $n = 1$, Eq 4.39 holds
2. Support n holds then we will proof $n+1$ holds. $F_k \in \mathcal{M}, F_{n+1} \in \mathcal{M}$, we have that

$$
\begin{aligned}
\pi^*\left(E \cap \sum_{j=1}^{n+1} F_j\right) &= \pi^*\left(\left(E \cap \sum_{j=1}^{n+1} F_j\right) \cap F_{n+1}\right) + \pi^*\left(\left(E \cap \sum_{j=1}^{n+1} F_j\right) \cap F_{n+1}^c\right) \\
&= \pi^*(E \cap F_{n+1}) + \underbrace{\pi^*\left(E \cap \sum_{j=1}^{n} F_j\right)}_{by\ \ assumption\ \ = \sum_{j=1}^{n} \pi^*(E \cap F_j)} \\
&= \sum_{j=1}^{n+1} \pi^*(E \cap F_j)
\end{aligned} \tag{4.40}
$$

$\square$

By Thm 4.3 we have that $\mathcal{M} \supseteq \mathcal{F}(a)$.

Step 3

Theorem 4.4. *$\pi^*:\mathcal{M}\to\mathbb{R}_+\cup\{+\infty\}$ is $\sigma-$ additive, then*

$$\pi^*(A)=\nu(A),\ \ \forall A\in a \tag{4.41}$$

Remark 4.3. Eq 4.41 is also

$$\pi^*|_a=v \tag{4.42}$$

Eq 4.2 holds because Thm 4.3, $a\in\mathcal{M}$.

Proof. (Thm 4.4)

1. $\pi^*(A)=\nu(A),\ \forall A\in a$

   (a) check $\pi^*(A)\leqslant\nu(A)$

   Let's $\underbrace{A}_{E_1},\underbrace{\varnothing}_{E_2},\underbrace{\varnothing}_{E_3},\underbrace{\cdots}_{E_j}$

$$\pi^*(A)=\inf_{\{E_i\},A}\sum_i\nu(E_i)\leqslant\sum_i\nu(E_i)=\nu(A) \tag{4.43}$$

   (b) check $\pi^*(A)\geqslant\nu(A)$

   Let's take

$$\begin{aligned}F_1&=E_1\\F_2&=E_2\backslash E_1\\F_3&=E_3\backslash(E_1\cup E_2)\\&\vdots\\F_n&=E_n\backslash(E_1\cup E_2\cup\cdots\cup E_{n-1})\\&\vdots\end{aligned} \tag{4.44}$$

   $F_j\in a,\bigcup_j F_j=\bigcup_j E_j, F_j\cap F_k=\varnothing, A\subseteq\bigcup_{j\geqslant1}F_j$, so $A=\sum_j F_j\cap A\in a$.

   Because ν is $\sigma-$additive we have that

$$\nu(A)=\sum_{j\geqslant1}\nu(F_j\cap A) \tag{4.45}$$

   $\because F_j\subseteq E_j$

$$\nu(A)=\sum_{j\geqslant1}\nu(F_j\cap A)\leqslant\sum_{j\geqslant1}\nu(E_j) \tag{4.46}$$

   so

$$\nu(A)\leqslant\inf_{\{E_i\},A}\sum_{j\geqslant1}\nu(E_j)=\pi^*(A) \tag{4.47}$$

   Then, we can get

$$\pi^*(A)=\nu(A),\ \forall A\in a \tag{4.48}$$

2. $\pi^*|_{\mathcal{M}}$ is $\sigma-$additive

   Suppose that $A_j \in \mathcal{M}, A_j \cap A_k = \varnothing$, we want to proof that

$$\pi^*\left(\sum A_j\right) = \sum_{j\geqslant 1} \pi^*(A_j) \tag{4.49}$$

   (a) check $\pi^*(\sum A_j) \leqslant \sum\limits_{j\geqslant 1} \pi^*(A_j)$ by π^* is an outer measure, π^* is subadditive

   (b) check $\pi^*(\sum A_j) \geqslant \sum\limits_{j\geqslant 1} \pi^*(A_j)$

   by π^* is an outer measure, π^* is monotone

$$\pi^*\left(\sum_{j\geqslant 1} A_j\right) \geqslant \pi^*\left(\sum_{j=1}^{n} A_j\right) \tag{4.50}$$

   by Remark 4.2, we have that

$$\pi^*\left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \pi^*(A_j), \quad \forall n \tag{4.51}$$

   so

$$\pi^*\left(\sum_{j\geqslant 1} A_j\right) \geqslant \sum_{j\geqslant 1} \pi^*(A_j) \tag{4.52}$$

□

Step 4

Definition 4.4. Ω is σ-finite(μ_1) if $E_j \uparrow \Omega, \mu_1(E_j) < \infty, \ \forall j, \ E_j \in a$.

Theorem 4.5 (Uniqueness)**.** *Suppose that $\mu_1, \mu_2 : \mathcal{F}(a) \to R_+ \cup \{+\infty\}, \Omega$ is σ-finite(μ_1), if $\mu_1|_a = \mu_2|_a$, then*

$$\mu_1 = \mu_2, \quad on \ \ \mathcal{F}(a) \tag{4.53}$$

Definition 4.5. Ω, $\mathcal{G} \subseteq \mathcal{P}(\Omega), \mathcal{G}$ is a monotone class if

1.
$$A_j \in \mathcal{G}, j \geqslant 1, A_j \subseteq A_{j+1} \Rightarrow A = \bigcup_{j\geqslant 1} A_j = \lim_{j\to\infty} A_j \in \mathcal{G} \tag{4.54}$$

2.
$$B_j \in \mathcal{G}, j \geqslant 1, B_j \supseteq B_{j+1} \Rightarrow B = \bigcap_{j\geqslant 1} B_j = \lim_{j\to\infty} B_j \in \mathcal{G} \tag{4.55}$$

Theorem 4.6. *$\mathcal{G}_\alpha$ is a monotone class, $\alpha \in I$, then the followings hold*

1. $\bigcap\limits_{\alpha\in I} \mathcal{G}_\alpha$ is a monotone class

2. $c \subseteq \mathcal{P}(\Omega) \Rightarrow \mathcal{G}(c) = \bigcap_{\alpha \in I} \mathcal{G}_\alpha$, *i.e. monotone classes generated by class* c

Lemma 4.1. *$a \subseteq \mathcal{P}(\Omega)$ is an algebra, $\mu(a)$ is monotone class generated by algebra a, $\mathcal{F}(a)$ is a $\sigma-$algebra generated by algebra a, then*

$$\mu(a) = \mathcal{F}(a) \tag{4.56}$$

Proof. It will proof in the next lecture. □

Proof. (Thm 4.5) $\mu_1, \mu_2 : \mathcal{F}(a) \to \mathbb{R}_+ \cup \{+\infty\}$, $\mu_1(A) = \mu_2(A), \forall A \in a$, Ω σ-finite, $\Omega = \bigcup_{j \geqslant 1} E_j, E_j \in a, \mu_j(E_j) < \infty$, then $\mu_1 = \mu_2$ on $\mathcal{F}(a)$.

Fix E_n, we denote that

$$\mathcal{B}_n = \{E \in \mathcal{F}(a), \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\} \tag{4.57}$$

We claim that

1. $\mathcal{B}_n \supseteq a$
2. $\mathcal{B}_n$ is a monotone class

We proof $\mathcal{B}_n$ is a monotone class.

1. $\forall A_j \in \mathcal{B}_n, A_j \uparrow A = \bigcup_{j \geqslant 1} A_j$, then

$$\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n) \tag{4.58}$$

By Remark 3.1

$$\mu_1(A_j \cap E_n) \to \mu_1(A \cap E_n), \mu_2(A_j \cap E_n) \to \mu_2(A \cap E_n) \tag{4.59}$$

2. $\forall B_j \in \mathcal{B}_n, B_j \downarrow B = \bigcap_{j \geqslant 1} B_j$, then

$$\mu_1(B_j \cap E_n) = \mu_2(B_j \cap E_n) \tag{4.60}$$

By Remark 3.1

$$\mu_1(B_j \cap E_n) \to \mu_1(B \cap E_n), \mu_2(B_j \cap E_n) \to \mu_2(B \cap E_n) \tag{4.61}$$

So we can get that

$$\mathcal{B}_n \supseteq \mathcal{M}(a) \tag{4.62}$$

where $\mathcal{M}(a)$ is a monotone class generated by a. Then by Lemma 4.1

$$\mathcal{M}(a) = \mathcal{F}(a) \tag{4.63}$$

And by Eq 4.57,

$$\mathcal{B}_n(a) \subseteq \mathcal{F}(a) \tag{4.64}$$

so

$$\mathcal{B}_n(a) = \mathcal{F}(a) \tag{4.65}$$

Finally, $\mu_1(A) = \mu_2(A), \forall A \in \mathcal{F}(a)$, by $\mathcal{B}_n = \mathcal{F}(a)$, then $A \in \mathcal{B}_n$. $B_j \uparrow \Omega$, apply Lemma 3.1 again, we have

$$\mu_1(A) = \mu_2(A) \tag{4.66}$$

□

Lecture 5

Monotone Classes

Definition 5.1. Given Ω, define $\mathcal{M}(a) \subseteq \mathcal{P}(\Omega)$ is a monotone class is

1. $A_j \in \mathcal{M}, A_j \uparrow A \left(A_j \subseteq A_j, \bigcup_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$
2. $A_j \in \mathcal{M}, A_j \downarrow A \left(A_j \supseteq A_j, \bigcap_{j \geqslant 1} A_j = A \right) \Rightarrow A \in \mathcal{M}$

Remark 5.1.

1. $\mathcal{F}$ is σ-filed(σ-algebra) $\Rightarrow \mathcal{F}$ is a monotone class
2. $\mathcal{M}_\alpha \subseteq P(\Omega)$, $(\alpha \in I)$ is monotone class, then $\mathcal{M} = \bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is a monotone class.

Notation 5.1. (Smallest monotone class contain c) $\mathcal{M}(c)$ is a monotone class generated by c if

$$c \subseteq \mathcal{M}(\Omega), \mathcal{M}(c) = \bigcap_{\alpha \in I} \mathcal{M}_\alpha \tag{5.1}$$

Definition 5.2. $E \subseteq \mathcal{M}(a)$, the set $\mathcal{G}(E)$ is defined as below

$$\mathcal{G}(E) = \{F \in \mathcal{M}(a), E \backslash F, E \cap F, F \backslash E \in \mathcal{M}(a)\} \tag{5.2}$$

Lemma 5.1.

1. *If $E \in a \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$*
2. *If $E \in \mathcal{M}(a) \Rightarrow \mathcal{G}(E) \supseteq \mathcal{M}(a)$*

Proof.

1. $E \in a$, we want to show that
   (a) $\mathcal{G}(E) \supseteq a$

   Take $H \in a \subseteq \mathcal{M}(a)$, then

$$\underbrace{E \backslash H}_{\in a}, \underbrace{E \cap H}_{\in a}, \underbrace{H \backslash E}_{\in a} \in \mathcal{G}(a) \tag{5.3}$$

   so $H \in \mathcal{G}(E)$, then $a \subseteq \mathcal{G}(E)$

   (b) $\mathcal{G}(E)$ is a monotone class

   Suppose that $H_k \uparrow H$, $H_k \in \mathcal{G}(E)$,

$$\because E \backslash H_k \in \mathcal{M}(a), \ E \backslash H_k \to E \backslash H, \therefore E \backslash H \in \mathcal{M}(a) \tag{5.4}$$

$$\because E \cap H_k \in \mathcal{M}(a), \ E \cap H_k \to E \cap H, \therefore E \cap H \in \mathcal{M}(a) \tag{5.5}$$

$$\because H_k \backslash E \in \mathcal{M}\left(a\right), \; H_k \backslash E \to H \backslash E, \therefore H \backslash E \in \mathcal{M}\left(a\right) \tag{5.6}$$

By Eq 5.6, $H \in \mathcal{M}(a)$, and by the definition 5.2, $H \in \mathcal{G}(E)$. So $\mathcal{G}(E)$ is a monotone class. We also get that $\mathcal{G}(E) \supseteq \mathcal{M}(a)$.

2. $E \in \mathcal{M}(a)$, we want to show that

   (a) $\mathcal{G}(E)$ is a monotone class

   $E \in \mathcal{M}(a)$, suppose $H_k \in \mathcal{G}\left(E\right), H_k \uparrow H$

   $$\because E \backslash H_k \in \mathcal{M}\left(a\right), E \backslash H_k \downarrow E \backslash H \quad \therefore E \backslash H \in \mathcal{M}\left(a\right) \tag{5.7}$$

   Similarity:

   $$E \cap H \in \mathcal{M}\left(a\right) \tag{5.8}$$

   $$H \backslash E \in \mathcal{M}\left(a\right) \tag{5.9}$$

   then we can get $H \in \mathcal{G}\left(E\right)$, so $\mathcal{G}\left(E\right)$ is a monotone class.

   (b) $\mathcal{G}(E) \supseteq a$

   We need to show $H \in a \Rightarrow H \in \mathcal{G}(E)$.

   By Lemma 5.1.1, we can get that

   $$\mathcal{G}(H) \supseteq \mathcal{M}(a) \tag{5.10}$$

   $\because E \in \mathcal{M}(a), \therefore E \in \mathcal{G}(H)$, by the Def 5.2, $H \backslash E, H \cap E, E \backslash H \in \mathcal{M}\left(a\right)$, so we can get $a \in \mathcal{G}\left(E\right)$

□

Theorem 5.1. *a is a algebra, $a \subseteq \mathcal{P}(\Omega)$. $\mathcal{F}(a)$ is a σ-algebra generated by a, $\mathcal{M}(a)$ is a monotone class generated by a, then*

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.11}$$

Proof. By remark 5.1, $\mathcal{F}(a)$ is a monotone class, by Notation 5.1 $\mathcal{F}(a) \supseteq a$ and $\mathcal{F}(a) \supseteq \mathcal{M}(a)$.

So we have to show that

$$\mathcal{F}(a) \subseteq \mathcal{M}(a) \tag{5.12}$$

We will show that

1. $\mathcal{M}(a)$ is a algebra

   (a) $\Omega \in \mathcal{M}\left(a\right)$ by $\Omega \subseteq a$

   (b) $E \in \mathcal{M}(a) \Rightarrow E^c \in \mathcal{M}(a)$

   By Lemma 5.1.1, let $E = \Omega$, then $\mathcal{M}(a) \subseteq \mathcal{G}(\Omega)$. $\because E \in \mathcal{M}(a)$, so $E \in \mathcal{G}(\Omega)$. By Definition 5.2, $\mathcal{G}\left(\Omega\right) = \{E \in \mathcal{M}\left(a\right), E^c, E, \varnothing \in \mathcal{M}\left(a\right)\}$

   (c) $E, F \in \mathcal{M}\left(a\right) \Rightarrow E \cap F \in \mathcal{M}\left(a\right)$

   By Lemma 5.1.2, $\mathcal{G}\left(E\right) \supseteq \mathcal{M}\left(a\right)$, so $F \in \mathcal{G}(E)$.

   By Def 5.2 $F \in \mathcal{G}\left(E\right) = \{F \in \mathcal{M}\left(a\right), F \backslash E, F \cap E, E \backslash F \in \mathcal{M}\left(a\right)\}$, so $E \bigcap F \in \mathcal{M}(a)$

2. $\mathcal{M}(a)$ is a σ-algebra i.e. $A_j \in \mathcal{M}(a),\ j \geqslant 1 \ \Rightarrow \bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$

   By $\mathcal{M}(a)$ is a algebra, so $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{M}(a)$.

   $\bigcup\limits_{j=1}^{n} A_j \uparrow \bigcup\limits_{j \geqslant 1} A_j$ and $\mathcal{M}(a)$is a monotone class, so $\bigcup\limits_{j \geqslant 1} A_j \in \mathcal{M}(a)$.

So $\mathcal{F}(a) \subseteq \mathcal{M}(a)$.

Above all,

$$\mathcal{F}(a) = \mathcal{M}(a) \tag{5.13}$$

$\square$

Lecture 6

The Lebesgue Measure I

Definition 6.1. $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$, we define $\mathcal{S}$ as below:

$$\mathcal{S} = \{\varnothing, \mathbb{R}, (a,b], (a,\infty), (-\infty,b]\} \tag{6.1}$$

Remark 6.1. $\mathcal{S}$ as above, then $\mathcal{S}$ is a semialgebra

Proof. by Def 2.1. □

Definition 6.2. $\mu : \mathcal{S} \to \mathbb{R}_+ \bigcup \{+\infty\}$, additive, and

$$\mu(\varnothing) = 0, \mu((a,b]) = b - a, \mu((-\infty,b]) = +\infty, \mu(\mathbb{R}) = +\infty \tag{6.2}$$

Theorem 6.1. *μ is additive on a semialgebra $\mathcal{S}$ and defined as Def 6.2, then μ is $\sigma-$additive, i.e.*

$$A = \sum_{j\geqslant 1} A_j \Rightarrow \mu(A) = \sum_{j\geqslant 1} \mu(A_j), \qquad A, A_j \in \mathcal{S} \tag{6.3}$$

Remark 6.2. It is difficult to prove Thm 6.1 $(a,b] \cup (c,d]$ is not in the semialgebra $\mathcal{S}$. But, $\mathcal{S} \to a(\mathcal{S})$ with respect to $\mu \to \nu$.

Proof.

1.
$$\because A = \sum_{j\geqslant 1} A_j \supseteq \sum_{j=1}^{n} A_j \tag{6.4}$$

   By ν is additive $\Rightarrow$ ν is monotone & subadditive,

$$\therefore \nu(A) \geqslant \nu\left(\sum_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \nu(A_j), \quad \forall n \tag{6.5}$$

   so

$$\therefore \nu(A) \geqslant \sum_{j\geqslant 1} \nu(A_j) \tag{6.6}$$

2. (a) Assume that $A = (a,b], A_j = (a_j, b_j], A = \sum_{j\geqslant 1} A_j$, we want to show that

$$\nu(A) = b - a \leqslant \sum_{j\geqslant 1} (b_j - a_j) = \sum_{j\geqslant 1} \nu(A_j) \tag{6.7}$$

   For any given $\epsilon > 0$, we have that

$$[a+\varepsilon, b] \subseteq (a,b] = \sum_{j\geqslant 1} (a_j, b_j] \subseteq \bigcup_{j\geqslant 1} \left(a_j, b_j + \frac{\varepsilon}{2^j}\right) \tag{6.8}$$

By a set K is compact i.e. K is closed and bounded $\Rightarrow$ Any open cover for K has a finite subcover

$$[a+\varepsilon, b] \subseteq \bigcup_{k\geqslant 1}\left(a_{jk}, b_{jk}+\frac{\varepsilon}{2^{jk}}\right) \tag{6.9}$$

By ν is additive $\Rightarrow$ ν is monotone & subadditive, we have

$$b-a-\varepsilon \leqslant \nu\left([a+\varepsilon, b]\right) = \nu\left(\bigcup_{k=1}^{m}\left(a_{jk}, b_{jk}+\frac{\varepsilon}{2^{jk}}\right)\right) \leqslant \sum_{k=1}^{m}\nu\left(a_{jk}, b_{jk}+\frac{\varepsilon}{2^{jk}}\right) \tag{6.10}$$

so we can get that

$$b-a-\varepsilon \leqslant \sum_{k=1}^{m}\left(b_{jk}-a_{jk}+\frac{\varepsilon}{2^{jk}}\right) \leqslant \sum_{j\geqslant 1}\left(b_j-a_j+\frac{\varepsilon}{2^{j}}\right) = \sum_{j\geqslant 1}(b-a)+\varepsilon \tag{6.11}$$

so Eq. 6.7 holds.

(b) General case $A \in \mathcal{S}$, $E_n = (-n, n] \uparrow \mathbb{R}$.

$A \cap E_n = \sum_{j\geqslant 1} A_j \cap E_n$.

By ν is additive on a semi-algebra

$$\nu\left(A \cap E_n\right) = \sum_{j\geqslant 1}\nu\left(A_j \cap E_n\right) \leqslant \sum_{j\geqslant 1}\nu\left(A_j\right) \tag{6.12}$$

By Remark 6.3, let $n \to \infty$, we have

$$\nu\left(A\right) = \lim_{n\to\infty}\nu\left(A \cap E_n\right) \leqslant \sum_{j\geqslant 1}\nu\left(A_j\right) \tag{6.13}$$

□

Remark 6.3. $E_n = (-n, n] \uparrow \mathbb{R}$, ν is additive on a semi-algebra then

$$\nu\left(A\right) = \lim_{n\to\infty}\ \nu\left(A \cap E_n\right) \tag{6.14}$$

Proof.

$$\because E_n \uparrow \mathbb{R}, \therefore A \cap E \uparrow, \therefore \lim_{n\to\infty}\left(A \cap E_n\right) = \bigcup_{n\geqslant 1}\left(A \cap E_n\right) = A \cap \left(\bigcup_{n\geqslant 1} E_n\right) = A \tag{6.15}$$

ν is additive,

$$\nu\left(A\right) = \nu\left(\bigcup_{n\geqslant 1} A \cap E_n\right) = \nu\left(\lim_{n\to\infty} A \cap E_n\right) \overset{why}{=} \lim_{n\to\infty}\nu\left(A \cap E_n\right) \tag{6.16}$$

why , because we will check via Def 6.1 except $A = (a, b]$

1. $A = \varnothing$

2. $A = \mathbb{R}$
3. $A = (a, \infty)$
   (a) left hand of why in Eq. 6.16

$$\because A \cap E_n = (a, +\infty) \cap (-n, n) = \begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \tag{6.17}$$

$$\therefore \lim_{n \to \infty} (A \cap E_n) = (-\infty, +\infty) = \mathbb{R} \tag{6.18}$$

   by Def 6.2

$$\mu \left(\lim_{n \to \infty} (A \cap E_n) \right) = \mu (\mathbb{R}) = +\infty \tag{6.19}$$

   (b) right hand of why in Eq. 6.16

$$\because \nu (A \cap E_n) = \nu \left(\begin{cases} (a, n) & a \geqslant -n \\ (-n, n) & a < -n \end{cases} \right) = \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} \tag{6.20}$$

$$\therefore \lim_{n \to \infty} \nu (A \cap E_n) = \lim_{n \to \infty} \begin{cases} n - a & a \geqslant -n \\ 2n & a < -n \end{cases} = +\infty \tag{6.21}$$

   So Eq 6.16 holds.
4. $A = (-\infty, b]$

$\square$

Lecture 7

The Lebesgue Measure II

$\mathcal{S}=\{\varnothing,\mathbb{R},(a,b],(a,\infty),(-\infty,b]\}$, $\mu: a(\mathcal{S})\to\mathbb{R}_+\cup\{+\infty\}$,

$$\mu((a,b])=b-a \tag{7.1}$$

Theorem 7.1. *μ is σ-additive on $a(\mathcal{S})$*

Remark 7.1. $E_k\in(-N,N]$, μ is finite and μ is continuous from below at $\varnothing$ (i.e. $E_k\in a, E_k\downarrow\varnothing\Rightarrow\mu(E_k)\to 0$), by Lemma 3.1 can imply Thm 7.1 hold.

Proof. Now we want to show that $E_k\downarrow\varnothing, E_k\in a, E_k\in(-N,N]$, then

$$\mu(E_k)\to 0 \tag{7.2}$$

If not, $\exists\delta>0$, $\exists E_k\downarrow\varnothing, E_k\in a, E_k\in(-N,N]$, such that

$$\mu(E_k)\geqslant 2\delta>0 \tag{7.3}$$

If $\exists$ a compact set $\{G_k\}$, *s.t.* $G_k\supseteq G_{k+1}, G_k\subseteq E_k$, but

$$\varnothing\neq\bigcap_{k\geqslant 1}G_k\subseteq\bigcap_{k\geqslant 1}E_k=\varnothing \tag{7.4}$$

Then, we will find a sequence of compact sets $\{G_k\}$ by induction.

Our goal is : $E_k\subseteq(-N,N]$, $\mu(E_n)\geqslant 2\delta$, $(F_k)_{1\leqslant k\leqslant M}G_k=\overline{F_k}$. F_k satisfy the flowing three conditions:

1. $\overline{F_k}\subseteq E_k,\quad 1\leqslant k\leqslant n-1$
2. $F_{k+1}\subseteq F_k,\quad 1\leqslant k\leqslant n-1$
3. $\mu(E_n\backslash F_n)\leqslant\frac{\delta}{2}+\frac{\delta}{4}+\cdots+\frac{\delta}{2^n}=\delta$

Now,

1. by $E_1\in a$, then E_1 can be written as

$$E_1=\sum_{j=1}^{n_1}(a_{1,j},b_{1,j}] \tag{7.5}$$

define F_1 as

$$F_1=\sum_{j=1}^{n_1}(a_{1,j}+\varepsilon_1,b_{1,j}]\in a \tag{7.6}$$

$\mu(E_1\backslash F_1)=m_1\varepsilon_1$.

We will pick a small enough ϵ to meet $\mu(E_1\backslash F_1)\leqslant\frac{\delta}{2}$, *i.e.* $m_1\varepsilon_1\leqslant\frac{\delta}{2}$, and $b_{1,j}-a_{1,j}\geqslant\varepsilon_1$, *i.e.* $\min_j\{b_{1,j}-a_{1,j}\}\geqslant\varepsilon_1$, so we choose $0<\varepsilon_1\leqslant\min\left\{\frac{\delta}{2m_1},\min_{1\leqslant j\leqslant m_1}\{b_{1,j}-a_{1,j}\}\right\}$.

2. We will show $\mu(E_2 \cap F_1)$ have a lower positive bound , i e. $E_2 \cap F_1 \neq \varnothing$

$$2\delta \leqslant \mu(E_2) = \mu(E_2 \cap F_1) + \underbrace{\mu(E_2 \backslash F_1)}_{\leqslant \mu(E_1 \backslash F_1) \leqslant \frac{\delta}{2}} \Rightarrow \mu(E_2 \cap F_1) \geqslant 2\delta - \frac{\delta}{2} > 0 \tag{7.7}$$

by $E_2 \cap F_1 \neq \varnothing, E_2 \cap F_1 \in a$, then $E_2 \cap F_1$ can be written as

$$E_2 \cap F_1 = \sum_{j=1}^{m_2} (a_{2,j}, b_{2,j}] \tag{7.8}$$

Define F_2 :

$$F_2 = \sum_{j=1}^{m_2} (a_{2,j} + \varepsilon_2, b_{2,j}] \tag{7.9}$$

choose a small enough ϵ_2 satisfies that

$$F_2 \subseteq \overline{F_2} \subseteq E_2 \cap F_1 \tag{7.10}$$

then $F_2 \subseteq F_1, \overline{F_2} \subseteq E_2$, and $F_2 \subseteq F_1 \Rightarrow \overline{F_2} \subseteq \overline{F_1}$, then we get that

$$\begin{aligned} F_2 \subseteq \overline{F_2} \subseteq E_2 \\ F_2 \subseteq F_1 \\ \mu(E_2 \backslash F_2) \leqslant \frac{\delta}{2} + \frac{\delta}{4} \end{aligned} \tag{7.11}$$

3. assume the F_n satisfies the three conditions as our goal above

$$2\delta \leqslant \mu(E_{n+1}) = \mu(E_{n+1} \cap F_n) + \underbrace{\mu(E_{n+1} \backslash F_n)}_{\mu(E_n \backslash F) \leqslant \delta} \Rightarrow \mu(E_{n+1} \cap F_n) \geqslant \delta > 0 \tag{7.12}$$

by $E_{n+1} \cap F_n \neq \varnothing$ and $E_{n+1} \cap F_n \in a$ then

$$E_{n+1} \cap F_n = \sum_{j=1}^{k_{n+1}} (a_{n+1,j}, b_{n+1,j}] \tag{7.13}$$

then we define F_{n+1} as

$$F_{n+1} = \sum_{j=1}^{k_{n+1}} (a_{n+1,j} + \varepsilon_{n+1}, b_{n+1,j}] \tag{7.14}$$

choose a small enough ϵ_{n+1} satisfies that

$$F_{n+1} \subseteq \overline{F_{n+1}} \subseteq E_{n+1} \cap F_n \tag{7.15}$$

then $F_{n+1} \subseteq E_{n+1}, F_{n+1} \subseteq F_n$, and $\overline{F_{n+1}} \subseteq \overline{F_n}$, let $\varepsilon_{n+1} = \frac{\delta}{k_{n+1} \cdot 2^{n+1}}$, then $\mu((E_{n+1} \cap F_n) \backslash F_{n+1}) \leqslant \frac{\delta}{2^{n+1}}$.

Then

$$\mu\left(E_{n+1}\backslash F_{n+1}\right)=\mu\left(\left(E_{n+1}\cap F_{n}\right)\backslash F_{n+1}\right)+\underbrace{\underbrace{\mu\left(\left(E_{n+1}\backslash F_{n}\right)\backslash F_{n+1}\right)}_{\leqslant\mu\left(E_{n+1}\backslash F_{n}\right)}}_{\leqslant\mu\left(E_{n}\backslash F_{n}\right)\leqslant\frac{\delta}{2}+\cdots+\frac{\delta}{2^{n}}}$$
$$\leq\frac{\delta}{2^{n+1}}+\frac{\delta}{2}+\frac{\delta}{4}+\cdots+\frac{\delta}{2^{n}}=\delta\left(1-\left(\frac{1}{2}\right)^{n+1}\right) \tag{7.16}$$

define $G_k=\overline{F_k}$, then $G_{k+1}=\overline{F_{k+1}}\subseteq\overline{F_k}=G_k$ G_k : satisfies that

(a) $G_{k+1}\subseteq G_k$

(b) G_k compact

(c) $G_k\neq\varnothing$

Why $G_k\neq\varnothing$ because:

$$2\delta\leqslant\mu\left(E_k\right)=\mu\left(E_k\backslash F_k\right)+\mu\left(E_k\cap F_k\right)\leqslant\delta+\mu\left(F_k\right)\Rightarrow\mu\left(F_k\right)\geq\delta \tag{7.17}$$

Then $F_k\neq\varnothing\Rightarrow G_k=\overline{F_k}\neq\varnothing$.

But

$$\varnothing\neq\bigcap_{k\geqslant1}G_k\subseteq\bigcap_{k\geqslant1}E_k=\varnothing \tag{7.18}$$

Above all, $E_k\in(-N,N]$, μ is finite and μ is continuous from below at $\varnothing$, then Lebesgue μ is σ-additive on $a(\mathcal{S})$. $\square$

Lecture 8

Complete Measures

Definition 8.1. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is σ-algebra, $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \infty$ is additive. $(\mu, \mathcal{F})$ is complete if : $A \in \mathcal{F}$ such that $\mu(A) = 0$, $\forall E \subseteq A$ then $E \in \mathcal{F}$.

Remark 8.1. In Def 8.1, by monotone $\mu(E) = 0$.

Next, our goal is: $\overline{\mathcal{F}} \supseteq \mathcal{F}$, and $\overline{\mu} : \overline{\mathcal{F}} \to \mathbb{R}_+ \cup \{+\infty\}$: $\begin{cases} \overline{\mu}|_{\mathcal{F}} = \mu, \\ (\overline{\mu}, \overline{\mathcal{F}}) \ \textit{is complete} \end{cases}$

Definition 8.2. $\overline{\mathcal{F}} = \{A \cup N, \ \textit{where } A \in \mathcal{F} \textit{ and } N \subseteq E \in \mathcal{F}, \ \textit{such that } \mu(E) = 0\}$

Claim 8.1. $\overline{\mathcal{F}}$ is a σ-algebra.

Proof. We will check :

1. $\Omega \in \overline{\mathcal{F}}$, $\because \Omega = \Omega \cup \varnothing, \varnothing \subseteq \varnothing \in \mathcal{F}$
2. $A \in \overline{\mathcal{F}} \Rightarrow A^c \in \overline{\mathcal{F}}$

   $\because A \subseteq \overline{\mathcal{F}}, A = E \cup N \textit{ where } E \in \mathcal{F}, \ N \subseteq H \in \mathcal{F} \textit{ such that } \mu(H) = 0$

   $$\begin{aligned} A^c &= (E \cup N)^c \\ &= \underbrace{[(E \cup N)^c \cap H]}_{\subseteq H} \cup \underbrace{[(E \cup N)^c \cap H^c]}_{\underbrace{E^c \cap N^c \cap H^c}_{\subseteq E^c \cap H^c \in \mathcal{F}}} \end{aligned} \tag{8.1}$$

   by Def 8.2, $A^c \in \overline{\mathcal{F}}$.
3. $A_j = E_j \cup H_j \ \textit{ where } \ E_j \in \mathcal{F}, H_j \subseteq W_j \textit{ where } w_j \in \mathcal{F}, \mu(W_j) = 0$ then $\bigcup\limits_{j \geqslant 1} A_j \in \overline{\mathcal{F}}$

   $$\begin{aligned} \because \bigcup_{j \geqslant 1} A_j &= \bigcup_{j \geqslant 1} (E_j \cup H_j) \\ &= \underbrace{\bigcup_{j \geqslant 1} E_j}_{\mathcal{F}} \cup \underbrace{\bigcup_{j \geqslant 1} H_j}_{\subseteq \bigcup\limits_{j \geqslant 1} W_j \triangleq W} \end{aligned} \tag{8.2}$$

   and $\mu(W) = \mu\left(\bigcup\limits_{j \geqslant 1} W_j\right) \leqslant \sum\limits_{j \geqslant 1} \mu(W_j) = 0$

$\square$

We want to define $\overline{\mu}$ *on* $\overline{\mathcal{F}}$:

$$\because \quad \underbrace{\overline{\mu}(A \cup N)}_{\geqslant \overline{\mu}(A) = \mu(A)} \leqslant \overline{\mu}(A \cup E) \leqslant \underbrace{\overline{\mu}(A) + \overline{\mu}(E)}_{= \mu(A) + \mu(E) = \mu(A)} \tag{8.3}$$

So we give the following definition.

Definition 8.3. $\overline{\mu}(A \cup N) = \mu(A)$

Proof. By the Def 8.3

1. check $\overline{\mu}$ is well defined

   Assume that $A \cup N = B \cup M,\ where\ A, B \in \mathcal{F}, N \subseteq E \in \mathcal{F}\ where\ \mu(E) = 0,\ M \subseteq F \in \mathcal{F}\ where\ \mu(F) = 0$. We need to show that $\mu(A) = \mu(B)$.

   $$\because A \subseteq A \cup N = B \cup M \subseteq B \cup M \tag{8.4}$$

   by μ is $\sigma-$additive, then μ is monotone,

   $$\mu(A) \leqslant \mu(B \cup F) \leqslant \mu(B) + \mu(F) = \mu(B) \tag{8.5}$$

   similarly, $\mu(B) \leqslant \mu(A)$.

2. check $\overline{\mu}|_{\mathcal{F}} = \mu$

   by $A \in \mathcal{F}$, $A = A \bigcup \varnothing$ then $\overline{\mu}(A \cup \varnothing) = \mu(A)$

3. check $\overline{\mu}$ is $\sigma-$additive i.e. $A_j \in \overline{\mathcal{F}},\ A = \sum\limits_{j \geqslant 1} A_j \Rightarrow \overline{\mu}(A) = \sum\limits_{j \geqslant 1} \mu(A_j)$

   $$\because A_j \in \overline{\mathcal{F}}, \therefore A_j = E_j \cup N_j\ where\ E_j \in \mathcal{F},\ N_j \subseteq H_j \subseteq \mathcal{F}\ where\ \mu(H_j) = 0 \tag{8.6}$$

   $\therefore A = \sum\limits_{j \geqslant 1} A_j = \sum\limits_{j \geqslant 1} E_j \cup \sum\limits_{j \geqslant 1} N_j$

   $$\therefore \overline{\mu}(A) = \mu\left(\sum_{j \geqslant 1} E_j\right) = \sum_{j \geqslant 1} \mu(E_j) = \sum_{j \geqslant 1} \overline{\mu}(A_j) \tag{8.7}$$

4. check $\left(\overline{\mu}, \overline{\mathcal{F}}\right)$ is complete, i.e. $\overline{\mathcal{F}}\ is\ \overline{\mu}$-complete.

   Assume that $A \subseteq E \in \overline{\mathcal{F}}$ where $\overline{\mu}(E) = 0$. We have to show that $A \in \overline{\mathcal{F}}$.

   $\because E \in \overline{\mathcal{F}} \therefore E = B \cup N\ where\ B \in \mathcal{F},\ N \subseteq H \in \mathcal{F}\ where\ \mu(H) = 0$

   $A = \varnothing \cup A,\ \varnothing \in F, A \subseteq E \subseteq B \cup N \subseteq \underbrace{B}_{\in \mathcal{F}} \cup \underbrace{H}_{\in \mathcal{F}} \in \mathcal{F}$, so $\mu(B \cup N) \leqslant \mu(B) + \mu(N) = 0$ by $\overline{\mu}(E) = \mu(B) = 0, \mu(A) \leqslant \mu(B) \Rightarrow \mu(A) = 0$, so $A \in \overline{\mathcal{F}}$

5. check $\overline{\mu}$ is unique. $\mu : \mathcal{F} \to \mathbb{R}_+ \bigcup \{+\infty\}$,

   And, extension $\overline{\mathcal{F}_\mu} = \{E \cup N,\ where\ E \in \mathcal{F}, N \subseteq H \in \mathcal{F},\ where\ \mu(H) = 0\}$, $\overline{\mu} : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$.

   Assume that $\nu : \overline{\mathcal{F}_\mu} \to \mathbb{R}_+ \cup \{+\infty\}$, and $\nu(A) = \overline{\mu}(A), \forall A \in \mathcal{F}$. Then we want show that $\nu(B) = \overline{\mu}(B), \forall B \in \overline{\mathcal{F}_\mu}$.

   Let $B \in \overline{\mathcal{F}_\mu}, B = E \cup N\ where\ E \in \mathcal{F},\ N \subseteq H \in \mathcal{F},\ where\ \mu(H) = 0, \nu(H) = \overline{\mu}(H) = \mu(H) = 0$.

   fix B, $\overline{\mu}(B) = \mu(E) \underbrace{=}_{by\ E \in \mathcal{F}} v(E) \leqslant \nu(B)$

   $\nu(B) = \nu(E \cup N) \leqslant \nu(E \cup H) \leqslant \nu(E) + \nu(H) = \nu(E) = \overline{\mu}(B)$, then

   $$\nu(B) = \overline{\mu}(B) \tag{8.8}$$

□

$\pi^* : \mathcal{M} \to \mathbb{R}_+ \cup \{+\infty\}$.

Claim 8.2. $\mathcal{M}$ is π^*-complete.

Proof. π^*-complete, i.e. $A \subseteq B, B \subseteq \mathcal{M}, \pi^*(B) = 0 \Rightarrow A \in \mathcal{M}$

We have to show $\forall E \subseteq \Omega$, $\pi^*(E) \geqslant \pi^*(E \cap A) + \pi^*(E \cap A^c)$

1. $\because E \cap A \subseteq A \subseteq B \;\therefore \pi^*(E \cap A) \leqslant \pi^*(B) = 0$
2. $\pi^*(E \cap A^c) \leqslant \pi^*(E)$

So, $A \in \mathcal{M}$ □

Lecture 9

Approximation Theorems

Goal: $\pi^*(A) < \infty, A \in \mathcal{M}, F \in \mathcal{F},$ *where* $\mathcal{F}$ *is* $\sigma - algebra, A \subseteq F, \pi^*(A) = \pi^*(F)$.

Theorem 9.1. *$a \subseteq \mathcal{P}(\Omega)$, where a is an algebra, $\mathcal{F}$ is a $\sigma-$algebra generated by a, $\mathcal{F}(a) = \mathcal{F}$, we have $\mu : \mathcal{F} \to \overline{\mathbb{R}}_+$, where μ is a measure, and $\mu|_a = v$, $A \subseteq \mathcal{F}, \mu(A) < \infty, \forall \epsilon > 0$, there*

$$\exists\ E \in a,\ s.t.\ \ \mu(E \backslash A) + \mu(A \backslash E) < \varepsilon \tag{9.1}$$

Proof. $A \in \mathcal{F}, \mu(A) < \infty$, by Thm 4.1, then

$$\mu(A) = \pi^*(A) = \inf_{\{A_j\} \supseteq A, A_{j \in a}} \sum \nu(A_i) \tag{9.2}$$

but μ here is π in Thm 4.1.

$\forall \epsilon, \exists \{A_i\} \quad A_i \in a,\ A \subseteq \cup A_i, \quad s.t.$

$$\pi^*(A) \leqslant \sum_{j \geqslant 1} \nu(A_i) \leqslant \pi^*(A) + \varepsilon \tag{9.3}$$

so

$$\exists\ m_0, \quad s.t. \sum_{i \geqslant m_0} \nu(A_i) \leqslant \varepsilon \tag{9.4}$$

Let $E = \bigcup\limits_{i=1}^{m_0} A_i \in a$, then we need to proof the following:

$$\pi^*(E \backslash A) \leqslant \varepsilon, \quad \pi^*(A \backslash E) \leqslant \varepsilon \tag{9.5}$$

By Thm 4.2, $\pi^*(A)$ is an out-measure, $\pi^*(A)$ is monotone and by Tmm 4.4, $\pi^*(A)$ is σ-additive.

$$\begin{aligned}
\therefore \pi^*(E \backslash A) &= \pi^*\left(\bigcup_{i=1}^{n_0} A_i \backslash A\right) \\
&\leqslant \pi^*\left(\bigcup_{i \geqslant 1} A_i \backslash A\right) \\
&= \pi^*\left(\bigcup_{i \geqslant 1} A_i\right) - \pi^*(A) \quad \textit{by } \pi^*(A) = \mu(A) < \infty \\
&\leqslant \sum_{i \geqslant 1} \pi^*(A_i) - \pi^*(A) \\
&= \sum_{i \geqslant 1} \nu(A_i) - \pi^*(A)\ \textit{ by } \pi^*|_{\mathcal{F}} = \mu,\ \mu|_a = v,\ A_i \in a \therefore \pi^*(A_i) = \nu(A_i) \\
&\leq \varepsilon
\end{aligned} \tag{9.6}$$

On the other hand,

$$\pi^*(A\backslash E) = \pi^*\left(A\backslash\bigcup_{i=1}^{n_0} A_i\right) \leqslant \pi^*\left(\bigcup_{i\geqslant1} A_i\backslash\bigcup_{j=1}^{n_0} A_j\right) \leqslant \pi^*\left(\bigcup_{j\geqslant n_0+1}^{n_0} A_j\right) \leqslant \sum_{j\geqslant m_0}\left(\bigcup_{j\geqslant n_0+1}^{n_0} A_j\right) \leqslant \varepsilon \quad (9.7)$$

$\square$

Remark 9.1. Ω is $\sigma-$finite(μ) (i.e. $\Omega = \bigcup_{i\geqslant1} E_i$ *where* $E_i \in a, \mu(E_i) < \infty$), $\overline{\mu} : \overline{\mathcal{F}} \to \mathbb{R}_+ \cup \{+\infty\}$, $A \in \overline{\mathcal{F}}, \forall\varepsilon > 0, \exists E \in a$, such that

$$\overline{\mu}(E\backslash A) + \overline{\mu}(A\backslash E) < \varepsilon. \quad (9.8)$$

Ω is topological space (open, closed sets), $\mathcal{B}$ is Borel σ-algebra set (the smallest σ set which contains all open, closed sets in Ω).

Definition 9.1 (Regular Measure). $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{\infty\}$ where $\mathcal{B} \subseteq \mathcal{F}$, is a measure. Then μ is a regular measure if: $\forall A \in \mathcal{F}, \forall\epsilon > 0$, there $\exists F \subseteq A \subseteq G$, where $F \in \mathcal{B}$ closed, $G \in \mathcal{B}$ open, such that:

$$\mu(G\backslash F) \leqslant \varepsilon \quad (9.9)$$

Remark 9.2. $\mu < \infty$ is not necessary.

Remark 9.3. $\mu(G\backslash A) \leqslant \varepsilon$ and $\mu(A\backslash F) \leqslant \varepsilon$.

Remark 9.4. $\mathcal{B} \subseteq \mathcal{F}$, μ *is regular* $\Rightarrow \mathcal{F} \subseteq \overline{\mathcal{B}_\mu}$

Proof. $A \in \mathcal{F}, n \geq 1$, *by* μ is regular, then $\exists F_n, G_n \in \mathcal{B}, F_n \subseteq \mathcal{B}$, such that $\mu(F_n\backslash G_n) \leqslant \frac{1}{n}$.

Let's define $F = \bigcup_{n\geqslant1} F_n \in \mathcal{B}$, $G = \bigcap_{n\geqslant1} G_n \in \mathcal{B}$, then $F \subseteq F_n \subseteq A \subseteq G_n \subseteq G$, *i.e.* $F \subseteq A \subseteq G$. By

$$G_n\backslash\left(\bigcup_{k\geqslant1} F_k\right) = G_n \cap \left(\bigcup_{k\geqslant1} F_k\right)^c = G_n \cap \left(\bigcap_{k\geqslant1} F_k^c\right) = \bigcap_{k\geqslant1}(G_n \cap F_k^c) = \bigcap_{k\geqslant1}(G_n\backslash F_k) \subseteq G_n\backslash F_n \quad (9.10)$$

then

$$\mu(G\backslash F) \leqslant \mu\left(G_n\backslash\left(\bigcup_{k\geqslant1} F_k\right)\right) \leqslant \mu(G_n\backslash F_n) \leqslant \frac{1}{n} \to 0 \quad (9.11)$$

Finally,

$$A = \underbrace{F}_{\in\mathcal{B}} \cup \underbrace{(A\backslash F)}_{\subseteq G\backslash F\in\mathcal{B}} \in \mathcal{B} \Rightarrow A \in \overline{\mathcal{B}} \quad (9.12)$$

$\square$

Theorem 9.2. *$\mathcal{L}$ is a σ-algebra generated by $a(\mathcal{S})$, where $\mathcal{S}$ is a set which defined as in Lecture 7, i.e. $\mathcal{S} = \{\varnothing, \mathbb{R}, (a,b], (a,\infty), (-\infty,b]\}$. $\mu : \mathcal{L} \to \mathbb{R}_+ \cup \{\infty\}$, is Lebesgue measure, then μ is regular measure. (if $A \in \mathcal{L}$, there $\exists F$ closed, G open, $F \subseteq A \subseteq G$ such that $\mu(G\backslash F) \leqslant \varepsilon$).*

Proof.

1. goal: $A \in \mathcal{L}, \varepsilon > 0$, there exists G open, such that $A \subseteq G,\ \mu(G \backslash A) \leqslant \varepsilon$.

   Denote $E_n = [-n, n]$, $A_n = A \cap E_n$, then $\mu(A_n) < \infty$. By the construction of Caratheodory Thm 4.1, there $\exists \{B_{n,k}\}_{k\geqslant 1}, B_{n,k} \in a, A_n \subseteq \bigcup\limits_{k\geqslant 1} B_{n,k}$, such that

   $$\mu(A_n) \leqslant \sum_{k\geqslant 1} \mu(B_{n,k}) \leqslant \mu(A_n) + \frac{\varepsilon}{2^n} \tag{9.13}$$

   By $B_{n,k} \in a,\ \therefore B_{n,k} = \sum\limits_{j=1}^{l_{n,k}} I_{n,k,j} \subseteq G_{n,k}$, where $I_{n,k,j} = (a_{n,k,j}, b_{n,k,j}]$.

   Then we denote $c_{n,k,j} = b_{n,k,j} + \underbrace{\delta_{n,k,j}}_{>0}, J_{n,k,j} = (a_{n,k,j}, c_{n,k,j})$, then $B_{n,k} \subseteq G_{n,k} = \bigcup\limits_{j=1}^{l_{n,k}} J_{n,k,j}$, then

   $$\mu(G_{n,k}) \leqslant \sum_{j=1}^{l_{n,k}} \mu(I_{n,k,j}) + \delta_{n,k,j} = \underbrace{\sum_{j=1}^{l_{n,k}} \mu(I_{n,k,j})}_{\mu(B_{n,k})} + \underbrace{\sum_{j=1}^{l_{n,k}} \delta_{n,k,j}}_{\leqslant \frac{\varepsilon}{2^n 2^k}} \tag{9.14}$$

   $\because B_{n,k} \subseteq G_{n,k},$ *and* $G_{n,k}$ *open set* $\therefore \mu(G_{n,k}) \leqslant \mu(B_{n,k}) + \frac{\varepsilon}{2^n 2^k}$. $\because A_n \subseteq \bigcup\limits_{k\geqslant 1} B_{n,k}, B_{n,k} \subseteq G_{n,k} \therefore A_n \subseteq \bigcup\limits_{k\geqslant 1} G_{n,k} = G_n$.

   On the other hand,

   $$\mu(G_n) \leqslant \sum_{k\geqslant 1} \mu(G_{n,k}) \leqslant \sum_{k\geqslant 1} \mu(B_{n,k}) + \frac{\varepsilon}{2^n} \leqslant \mu(A_n) + \frac{2\varepsilon}{2^n} \tag{9.15}$$

   $\because A_n \subseteq G_n$ *open, and* $\mu(G_n) \leqslant \mu(A_n) + \frac{2\varepsilon}{2^n}$.

   Then define $G = \bigcup\limits_{n\geqslant 1} G_n,$ *open and* $A = \bigcup\limits_{n\geqslant 1} A_n,\ A \subseteq G$.

   $$\begin{aligned} \because \bigcup_{n\geqslant 1} G_n \backslash \bigcup_{k\geqslant 1} A_k &= \bigcup_{n\geqslant 1} G_n \cap \left(\bigcup_{k\geqslant 1} A_k\right)^c = \bigcup_{n\geqslant 1} G_n \cap \left(\bigcap_{k\geqslant 1} A_k^c\right) \\ &= \bigcap_{k\geqslant 1} \left(\bigcup_{n\geqslant 1} G_n \bigcap A_k^c\right) \subseteq \left(\bigcup_{n\geqslant 1} G_n \bigcap A_n^c\right) = \bigcup_{n\geqslant 1} G_n \backslash A_n \end{aligned} \tag{9.16}$$

$$
\begin{aligned}
\therefore \mu(G \backslash A) &= \mu\left(\bigcup_{n \geqslant 1} G_n \backslash \bigcup_{k \geqslant 1} A_k\right) \\
&\leqslant \mu\left(\bigcup_{n \geqslant 1} G_n \backslash A_n\right) \quad by \;\; Eq.\ 9.16 \\
&\leqslant \sum_{n \geqslant 1} \mu(G_n \backslash A_n) \\
&= \sum_{n \geqslant 1} \left[\mu(G_n) - \mu(A_n)\right] \quad by\ \mu(A_n) < \infty \\
&\leq 2\varepsilon
\end{aligned}
\tag{9.17}
$$

2. goal: $A \in \mathcal{L}, \varepsilon > 0$, there exists F closed , such that $F \subseteq A,\ \mu(A \backslash F) \leqslant \varepsilon$.

   By above 1, $\exists H,\ A^c \subseteq H,\ H\ open\ set,\ \mu(H \backslash A^c) \leqslant \varepsilon,\ then\ F = H^c \subseteq A,\ F\ closed$.

   Finally,

$$
\mu(A \backslash F) = \mu(A \cap F^c) = \mu(A \cap H) = \mu(H \cap (A^c)^c) = \mu(H \backslash A^c) \leqslant \varepsilon. \tag{9.18}
$$

□

Remark 9.5. $\mathcal{F}_\sigma$: countable union closed sets, $\mathcal{G}_\sigma$: countable injection open sets. $\forall A \in \mathcal{L}$ there $\exists R \in \mathcal{F}_\sigma$ and $S \in \mathcal{G}_\sigma$, such that

$$
R \subseteq A \subseteq S, \quad \mu(S \backslash R) = 0. \tag{9.19}
$$

Lecture 10

Integration: Measurable and Simple Functions