EE 120: Signals and Systems
Department of EECS
UNIVERSITY OF CALIFORNIA BERKELEY

Problem Set 5
Issued: 20 October 2018
Due: 27 October 2018, 11:59 pm

FIRST Name ____________________ LAST Name ____________________

Discussion Section Time: __________ SID (All Digits): ______________

Circumstances Favorable and Unfavorable to Original Ideas

It will be fairly clear to the reader that the really fundamental and seminal idea is to a large extent a lucky and unpredictable accident. There was no absolute necessity for Euclid to develop the axiomatic theory of geometry, nor for Gibbs to insist so strongly on the notion of probability in thermodynamics. These innovations might easily have occurred somewhat earlier or considerably later, and are no more satisfactorily subject to betting about them, say, than about the particular house in the village which would next be struck by lightning.

For all that, though lightning is a sporadic phenomenon even for good betting, we do have a general idea of what circumstances are favorable for lightning and what are unfavorable. We do not build a house on top of a high and isolated hill without being particularly attentive to our lightning rods. So too in matters of invention, occasional and sporadic as the phenomenon is, we may look to certain circumstances to favor it, just as we may look to other circumstances to cut down the risk of lightning.

There are certain procedures which are undoubtedly favorable for invention and discovery. One of the most potent tools in reanimating a science is mathematics. To some extent, a mathematical treatment of a science consists in writing down its data and its questions in a numerical or a quantitative form, but it is perhaps better to consider that here number and quantity are secondary to a logically precise language.

If a certain question is to be asked in biology, and if we are to ask it in biological language, then we ourselves and whoever reads our work are likely to be strongly conditioned to think of what we have done as the answer to a biological question. However, if we express our ideas in a mathematical form, we are using what is much more likely to be a colorless and indifferent language.

Just because of that, we are far more likely to recognize the same question even if it is asked in a totally different field. This greater scope is far from of trivial significance.[a]

[a]Excerpts from *Invention: The Care and Feeding of Ideas*, by Norbert Wiener, The MIT Press, 1994, pp. 25–26, ISBN: 0-262-73111-8.

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- On the solution document that you turn in for grading, you must write the names of your collaborators below your own; each teammate must submit for our evaluation a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- Unless we explicitly state otherwise, you will receive full credit *only if* you explain your work succinctly, but clearly and convincingly.
- Typically, we evaluate your solutions for only a subset of the assigned problems. A priori, you do not know which subset we will grade. It is to your advantage to make a bona fide effort at tackling *every* assigned problem.
- If you are asked to provide a "sketch," it refers to a *hand-drawn* sketch, well-labeled to indicate all the salient features—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." We do NOT grade such problems. Nevertheless, you are responsible for learning the subject matter within their scope.

Overview

This problem set is designed to review concepts related to the continuous-time Fourier transform (CTFT), signal energy, and orthogonality of functions (signals).

List of Your Collaborators

- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________
- Name: ______________________________ SID: ________________________

HW5.1 Determine X, the continuous-time Fourier transform (CTFT) of each of the following signals x.

(a) **One-sided decaying exponential:** $x(t) = e^{-at}\,u(t), \forall t$, where $a > 0$. Provide a well-labeled sketch of $|X(\omega)|, \forall \omega$. Describe how the bandwidth of x varies in relation to the decay rate a of the one-sided exponential? Consider as the bandwidth frequency ω_B the threshold where $|X(\omega_B)| = \dfrac{1}{\sqrt{2}} \max_{\omega} |X(\omega)|$. This bandwidth frequency (or cutoff frequency) is chosen by convention to correspond to the "3 dB (decibel)" drop in magnitude. Notice that $20 \log_{10}(1/\sqrt{2}) \approx -3$ dB. This has historical roots in circuit theory and design.

(b) **Two-sided decaying exponential:** $x(t) = e^{-a|t|}, \forall t$, where $a > 0$.

(c) **Signum:** Let the signal x be characterized by $x(t) = \mathrm{sgn}(t), \forall t$, where

$$\mathrm{sgn}(t) = \begin{cases} +1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

is, according to one popular definition, the *signum function.*

Hint: Write the signum function in terms of the unit step function and a constant. Then use the differentiation property of the CTFT to arrive at the answer.

(d) **Unit Step:** $x(t) = u(t), \ \forall t$.

Hint: Write the unit step $u(t)$ in terms of the signum function $\mathrm{sgn}(t)$ and a constant. Then use your result from the previous part to determine the CTFT.

(e) **Infinite Impulse Train:** $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \forall t$, where $0 < T$. Hint: This signal is periodic; it has a Fourier series (FS) expansion. What is the transform of each term in the FS expansion of x? How does the shape of the CTFT of x change as T varies between 0 to $+\infty$?

(f) **Rectangular pulse:**

$$x(t) = \begin{cases} b & |t| \leq a \\ 0 & \text{elsewhere.} \end{cases}$$

The parameters a and b have appropriately-chosen positive values.

HW5.2 (CTFT Properties) Consider a real-valued continuous-time signal x whose contrinuout-time Fourier transform (CTFT) is X.

1. **Time-Frequency Spread Reciprocity (Time-Frequency Scaling Property)**

   Let $\widehat{x}$ be such that

   $$\forall t \in \mathbb{R}, \quad \widehat{x}(t) = x(at)$$

   for some real number a.

   Show that $\widehat{X}$, the CTFT of $\widehat{x}$, is

   $$\forall \omega \in \mathbb{R}, \quad \widehat{X}(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

   Note that if $|a| > 1$, the signal is contracted in the time domain but dilated in the frequency domain, and if $|a| < 1$, the signal is dilated in the time domain, but contracted in the frequency domain.

   What can you say about $\widehat{X}$ if $\widehat{x}$ is the time-reversed version of x, i.e., if $\forall t \in \mathbb{R}, \widehat{x}(t) = x(-t)$?

2. **Differentiation in Time** Let $\widehat{x}$ be such that

$$\forall t \in \mathbb{R}, \quad \widehat{x}(t) = \dot{x}(t) \triangleq \frac{dx(t)}{dt}.$$

Show that $\widehat{X}$, the CTFT of $\widehat{x}$, is

$$\forall \omega \in \mathbb{R}, \quad \widehat{X}(\omega) = i\omega X(\omega).$$

Let $\widetilde{x}$ be such that

$$\forall t \in \mathbb{R}, \quad \widetilde{x}(t) = \frac{d^n x(t)}{dt^n}$$

Determine a simple expression for $\widetilde{X}(\omega), \forall \omega \in \mathbb{R}$, where $\widetilde{X}$ is the CTFT of $\widetilde{x}$. Your expression must be in terms of $X(\omega)$.

3. **Differentiation in Frequency** Let $\widehat{x}$ be such that

$$\forall t \in \mathbb{R}, \quad \widehat{x}(t) = tx(t).$$

Show that $\widehat{X}$, the CTFT of $\widehat{x}$, is

$$\forall \omega \in \mathbb{R}, \quad \widehat{X}(\omega) = i\,\frac{dX(\omega)}{d\omega}.$$

Rewriting in the form

$$-i\,t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{dX(\omega)}{d\omega},$$

we recognize the frequency differentiation property as simply the dual of the time differentiation property.

Let $\widetilde{x}$ be such that

$$\forall t \in \mathbb{R}, \quad \widetilde{x}(t) = t^n x(t).$$

Determine a simple expression for $\widetilde{X}(\omega), \forall \omega \in \mathbb{R}$, where $\widetilde{X}$ is the CTFT of $\widetilde{x}$. Your expression must be in terms of $X(\omega)$.

Express, in terms of $X(\omega)$, the n^{th} moment x, defined by

$$\int_{-\infty}^{+\infty} t^n x(t)\,dt\,, \quad n \in \mathbb{Z}_+.$$

4. **Integration in Time** Let $\widehat{x}$ be such that

$$\forall t \in \mathbb{R}, \quad \widehat{x}(t) = \int_{-\infty}^{t} x(\tau) d\tau.$$

Show that $\widehat{X}$, the CTFT of $\widehat{x}$, is

$$\forall \omega \in \mathbb{R}, \quad \widehat{X}(\omega) = \frac{X(\omega)}{i\,\omega} + \pi X(0)\delta(\omega).$$

<u>Hint</u>: Show, and take advantage of the fact, that $\widehat{x}$ can be thought of as the convolution of x with the unit-step function u, i.e., $\widehat{x} = x * u$.

5. **Duality** Consider a signal $x : \mathbb{R} \to \mathbb{C}$ whose CTFT is $X : \mathbb{R} \to \mathbb{C}$. More compactly, let

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega),$$

where $\mathcal{F}$ denotes the Fourier transform.

Prove that

$$X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega).$$

6. **Parseval-Plancherel-Rayleigh Identity**[1]

In this part, $y : \mathbb{R} \to \mathbb{C}$ is a continuous-time signal whose CTFT is $Y : \mathbb{R} \to \mathbb{C}$. Let

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \qquad \text{and} \qquad \langle X, Y \rangle = \int_{-\infty}^{\infty} X(\omega)\, Y^*(\omega)\, d\omega \ .$$

(i) Prove the identity $\langle x, y \rangle = \dfrac{1}{2\pi} \langle X, Y \rangle$. That is, prove

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, Y^*(\omega)\, d\omega \ .$$

<u>Hint</u>: Write $x(t)$ using the CTFT synthesis equation. Thereafter, exchange the order of the integrals, and proceed from there.

(ii) Let $y = x$ to show that

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}_{\langle x, x \rangle} = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}_{\langle X, X \rangle} \ .$$

[1] Originally, Parseval proved the analogue of this identity in the context of the Fourier series representation of a periodic function. However, the generalization to the Fourier transform still bears his name.

HW5.3 In this problem, you'll enhance your knowledge of the properties of the CTFT.

(a) **Effect of Differentiation in Convolution:** Suppose a continuous-time signal h is obtained by convolving two continuous-time signals f and g, respectively, i.e., $h = f * g$.

Use the *convolution property* and the *time-derivative property* of the CTFT to establish the following identity:

$$\frac{dh(t)}{dt} = \left(\frac{df}{dt} * g\right)(t) = \left(f * \frac{dg}{dt}\right)(t)$$

In short,

$$\dot{h}(t) = (\dot{f} * g)(t) = (f * \dot{g})(t)\,.$$

(b) **Triangular Pulse and Convolution:** Determine an expression for, and provide a well-labeled sketch of, the CTFT of the following *triangular pulse* x:

$$\forall t \in \mathbb{R}, \quad x(t) = \begin{cases} b^2 \left(1 - \dfrac{|t|}{2a}\right) & |t| < 2a \\ 0 & |t| \geq 2a. \end{cases}$$

Hint: The signal x is the convolution of two rectangular pulses. Determine what those rectangular pulses are and then use one or more properties of the CTFT to determine $X(\omega)$.

HW5.4 (Orthogonality-Preserving Property of the CTFT) In this problem, we set out to prove that the continuous-time Fourier transform (CTFT) preserves mutual orthogonality of signals, and that the inverse of the CTFT preserves mutual orthogonality of signal spectra.

Consider a set $\{\phi_k\}, k \in \mathbb{Z}$, of mutually-orthogonal functions

$$\phi_k : \mathbb{R} \to \mathbb{C}$$

each of whose elements ϕ_k has finite energy E_ϕ, i.e.,

$$\langle \phi_k, \phi_l \rangle \triangleq \int_{-\infty}^{\infty} \phi_k(t)\, \phi_l^*(t)\, dt = E_\phi\, \delta(k-l),$$

where δ is the Kronecker delta function and * denotes complex conjugation. Let Φ_k be the CTFT (spectrum) of ϕ_k, i.e., for $k \in \mathbb{Z}$,

$$\Phi_k : \mathbb{R} \to \mathbb{C}$$
$$\forall \omega \in \mathbb{R}, \qquad \Phi_k(\omega) = \int_{-\infty}^{\infty} \phi_k(t)\, e^{-i\omega t}\, dt\ .$$

Show that

$$\langle \Phi_k, \Phi_l \rangle \triangleq \int_{-\infty}^{\infty} \Phi_k(\omega)\, \Phi_l^*(\omega)\, d\omega = 2\pi\, E_\phi\, \delta(k-l).$$

Hint: Use the CTFT Analysis Equation

$$\Phi_l(\omega) = \int_{-\infty}^{\infty} \phi_l(t)\, e^{-i\omega t}\, dt$$

and insert the expression for $\Phi_l^*(\omega)$ in the inner product $\langle \Phi_k, \Phi_l \rangle$ to arrive at the appropriate conclusion.

Please write your solutions in the blank space at the top of the next page.
Work written here will not be graded.

We have shown that the continuous-time Fourier transforms of mutually-orthogonal signals are themselves mutually-orthogonal functions (of the frequency variable ω). Using a similar method, we can show that the converse is also true (you need not show it here, however)—namely, that signals whose Fourier transforms are mutually orthogonal are themselves mutually orthogonal. Therefore, we can state the following theorem:

> **Orthogonality-Preserving Property of the CTFT:** A set of signals $\{\phi_k\}$, $k \in \mathbb{Z}$, is orthogonal according to
>
> $$\langle \phi_k, \phi_l \rangle \triangleq \int_{-\infty}^{\infty} \phi_k(t)\, \phi_l^*(t)\, dt = E_\phi\, \delta(k-l)$$
>
> if, and only if, the set $\{\Phi_k\}$, $k \in \mathbb{Z}$, of the signals' respective continuous-time Fourier transforms (spectra) satisfies the following orthogonality property:
>
> $$\langle \Phi_k, \Phi_l \rangle \triangleq \int_{-\infty}^{\infty} \Phi_k(\omega)\, \Phi_l^*(\omega)\, d\omega = 2\pi\, E_\phi\, \delta(k-l).$$

HW5.5 (Orthogonal Functions) We can use the orthogonality-preserving property of the continuous-time Fourier transform to establish the mutual orthogonality of some well-known signals.

(a) In this part, you'll prove that appropriately-shifted sinc functions form a mutually orthogonal set. In particular, consider the set $\{\phi_k\}$, $k \in \mathbb{Z}$, of sinc functions, i.e.,

$$\begin{aligned} &\phi_k : \mathbb{R} \to \mathbb{R} \\ \forall t \in \mathbb{R}, \qquad &\phi_k(t) = \mathsf{sinc}(t-k) \triangleq \frac{\sin(\pi(t-k))}{\pi(t-k)} \,. \end{aligned}$$

Using the signal $\phi_0(t) = \mathsf{sinc}(t)$ and its spectrum $\Phi_0(\omega)$ as benchmarks, show that $\Phi_k \perp \Phi_l$, $\forall k \neq l$, and invoke the orthogonality-preserving property of the CTFT to prove that the shifted sinc functions ϕ_k are mutually orthogonal.

Note: To appreciate the wisdom of invoking the orthogonality-preserving property of the continuous-time Fourier transform, you may want to try proving the mutual orthogonality of the shifted sinc functions without it (i.e., in the time domain). You will quickly encounter complicated integrals evaluating which yields little, if any, meaningful insight into our knowledge of signals and systems.

Discussion: An inference that we can draw from this exercise is that shifted sinc functions can serve as an orthogonal basis set for representing other functions. Indeed, linear combinations of shifted sinc functions are used to reconstruct bandlimited signals from their appropriately-spaced samples. This topic is further explored in the discussion of the *sampling theorem* wherein shifted sinc functions are shown to interpolate between samples of a bandlimited signal to reconstruct portions of the signal discarded in the sampling process.

Please write your solutions in the blank space at the top of the next page.
Work written here will not be graded.

HW5.5(a) (Continued)

(b) (The Shannon Wavelet) Consider the function ψ whose CTFT is:

$$\Psi(\omega) = \begin{cases} 1 & \text{if } |\omega| \in [\pi, 2\pi] \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Determine an expression for $\psi(t)$, the time-domain representation of the Shannon wavelet ψ.

You may find the following trigonometric identity useful:

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta.$$

(ii) Determine the value of $||\psi||^2 \triangleq \langle \psi, \psi \rangle$.

(iii) Consider the set of Shannon wavelets $\{\psi_k\}$, $k \in \mathbb{Z}$, where $\psi_k(t) = \psi(t-k)$, $\forall t$. Show that the wavelets are mutually orthogonal, i.e.,

$$\langle \psi_k, \psi_l \rangle = A\,\delta(k-l),\ \exists A > 0.$$