Chapter 1 Exercises

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Contents

1 Problem 1

Problem Evaluate numerically the integral

$$I = \int\limits_0^{\pi/2} \ln(\sin x)\, dx$$

and compare with the exact value $I = -\frac{\pi}{2}\ln 2$

Solution ...

Note that $\ln(\sin x) = \ln\left(x\frac{\sin x}{x}\right) = \ln x + \ln\frac{\sin x}{x}$
Therefore we just need to find $\int_0^{\pi/2} \ln x dx$ analytically and evaluate $\int_0^{\pi/2} \ln\frac{\sin x}{x} dx$ numerically.
$\int_0^{\pi/2} \ln x dx$ can be integrated by parts.
Let $u = \ln x, v^{'} = dx, u^{'} = \frac{1}{x}, v = x$. Therefore,

$$\int lnxdx = \int uv^{'}dx = uv - \int vu^{'}dx$$

$$uv - \int vu^{'}dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x$$

And so, $\int_0^{\pi/2} \ln x dx = x\ln x - x|_0^{\pi/2}$

$$\begin{aligned}\lim_{x\to 0} x\ln x &= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}\\ &= \lim_{x\to 0} -\frac{\frac{1}{x}}{\frac{1}{x^2}}\\ &= \lim_{x\to 0} -x\\ &= 0\end{aligned}$$

And, $=\int_0^{\pi/2} \ln x dx = \frac{pi}{2}(\ln x - 1) \approx -0.861$

```
from sympy import log, pi, integrate
from sympy.abc import x
print((pi/2 * (log(pi/2) - 1)).evalf())
print(integrate(log(x), (x, 0, pi/2)).evalf())
```

```
-0.861451872082119
-0.861451872082119
```

So we have a value for part of our problem. The other part is to evaluate the integral

$$\int_0^{\pi/2} \ln \frac{\sin x}{x} dx$$

We can do this with Simpson's rule. Recall that

$$S = \frac{H}{9} \sum_{i=0}^{n-1} f(x_0 + iH) + 4f\left(x_0 + \left(i + \frac{1}{2}\right) H\right) + f\left(x_0 + (i+1)\, H\right)$$

```python
from math import pi, sin, log
num_points = 10000
xmin = 0
xmax = pi/2
def f(x):
    if x == 0:
        return 0
    return log(sin(x)/x)
H = (xmax - xmin) / num_points
def s(x0, i, H, f):
    return H/6 * (
        f(x0 + i * H)
        + 4 * f(x0 + (i + 1/2) * H)
        + f(x0 + (i + 1) * H)
    )
print(sum(s(xmin, i, H, f) for i in range(num_points)))
```

```
-0.22734117306968255
```

```python
import matplotlib.pyplot as plt
import numpy as np
from math import pi, log, sin
xmin = 0
xmax = pi/2
def f(x):
    if x == 0:
        return 0
    return log(sin(x)/x)
```

```
X = np.linspace(xmin, xmax)
Y = [f(x) for x in X]
plt.plot(X, Y)
fname = "myfig.pdf"
plt.savefig(fname)
return fname
```

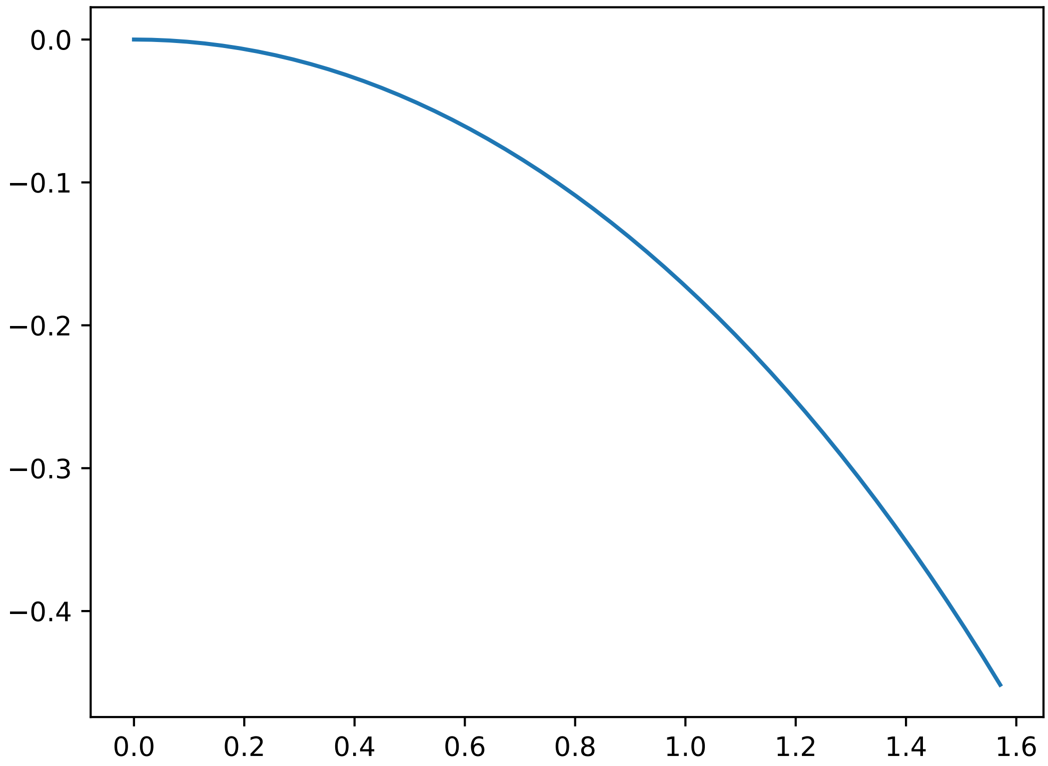

Figure 1

2 Problem 2

Problem Statement Evaluate numerically the integral $\int_1^2 \frac{dx}{x}$. Use Simpson's rule with an interval $h = 0.1$.

Solution ...

```
from math import log
```

```python
import numpy as np
xmin = 1
xmax = 2
h = 0.1
def f(x):
    return 1/x
def s(x, h, f):
    return h/3 * (f(x) + 4 * f(x + h) + f(x + 2 * h))
print(sum(s(x, h, f) for x in np.arange(xmin, xmax - 2 * h, 2 *
h)))
```

```
0.587789606907254
```

3 Problem 3

Problem Statement Integrate numerically the differential equation

$$\frac{dy}{dx} = y$$

with the initial condition $y = 1$ for $x = 0$. Use Adams' method with $h = 1$ in the range from $x = 0$ to $x = 1$.

Solution ...

Applying Adams' method, we first compute the Taylor's expansion of the solution in the vicinity of $x = 0$. The first derivative is

$$\frac{dy}{dx} = y$$

The second derivative is found by differentiating with respect to x:

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} = y$$

Differentiating again,

$$\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} = y$$

$$\frac{d^4y}{dx^4} = \frac{d^3y}{dx^3} = y$$

Using the values of these derivatives at $y = 1, x = 0$, we find the Taylor's expansion

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

```
def func(x, x0, y0, d0, d1, d2, d3, d4):
    return (
        d0(x0, y0)
        + d1(x0, y0) * (x - x0)
        + d2(x0, y0) * (x - x0)**2 / 2
        + d3(x0, y0) * (x - x0)**3 / 6
        + d4(x0, y0) * (x - x0)**4 / 24
    )
def get_deltas(l):
    return [lr - ll for lr, ll in zip(l[1:], l[:-1])]
def get_next_y(y, h, q0, q1, q2, q3, q4):
    return y + h * (q0 + 1/2 * q1 + 5/12 * q2 * 3/8 * q3 +
251/720 * q4)
epsilon = 1e-10
def get_values(x0, y0, xmax, h, d0, d1, d2, d3, d4):
    initial_x_vals = [x0 + i * h for i in range(5)]
    initial_y_vals = [func(x, x0, y0, d0, d1, d2, d3, d4) for x
in initial_x_vals]
    initial_qs = [d1(x, y) for x, y in zip(initial_x_vals,
initial_y_vals)]
    x = initial_x_vals[-1]
    qs = initial_qs
    y_vals = initial_y_vals
    x_vals = initial_x_vals
    while x < xmax - epsilon:
        x += h
        x_vals.append(x)
        dqs = get_deltas(qs)
        ddqs = get_deltas(dqs)
        dddqs = get_deltas(ddqs)
        ddddqs = get_deltas(dddqs)
        next_y = get_next_y(
            y_vals[-1],
            h,
            qs[-1],
            dqs[-1],
```

```
                ddqs[-1],
                dddqs[-1],
                ddddqs[-1]
            )
            qs.append(d1(x, next_y))
y_vals.append(next_y)
    return y_vals
```

now that we've defined a function for adams' method, we can reuse it.

```
def d0(x, y):
    return y
def d1(x, y):
    return y
def d2(x, y):
    return y
def d3(x, y):
    return y
def d4(x, y):
    return y
x0 = 0
y0 = 1
h = 0.1
xmax = 1
print(*[
    f"{y:.3f}" for y in get_values(x0, y0, xmax, h, d0, d1, d2,
d3, d4)
])
```

4 Problem 4

Problem Statement integrate numerically the differential equation

$$\frac{d^2y}{dx^2} - xy = 0$$

in the interval between $x = 0$ and $x = 1$ with the initial conditions $y = 1$ and $\frac{dy}{dx} = 0$ for $x = 0$

Solution introduce a new unknown variable u by the transformation

$$u = \frac{1}{y}\frac{dy}{dx}$$

or

$$u = \frac{d}{dx}(\ln y)$$

calculate and substitute $\frac{d^2y}{dx^2}$; show that the above differential equation becomes $\frac{du}{dx} + u^2 - x = 0$ and integrate using adams' method using $h = 0.1$.

$$\frac{du}{dx} = -\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = y\frac{du}{dx} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} - xy = y\frac{du}{dx} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - xy = 0$$

$$\frac{du}{dx} + \left(\frac{1}{y}\frac{dy}{dx}\right)^2 - x = 0$$

$$\frac{du}{dx} + u^2 - x = 0$$

we need to calculate derivatives of u for use in the taylor's series:

$$\begin{aligned}
\frac{du}{dx} &= x - u^2 \\
\frac{d^2u}{dx^2} &= 1 - 2u\frac{du}{dx} \\
&= 1 - 2xu + 2u^3 \\
\frac{d^3u}{dx^3} &= 2u + 2x\frac{du}{dx} + 6u^2\frac{du}{dx} \\
&= 2u + 2x^2 - 2xu^2 + 6xu^2 - 6u^4 \\
&= 2x^2 + 2u + 4xu^2 - 6u^4 \\
\frac{d^4}{dx^4} &= 4x + 2\frac{du}{dx} + 4u^2 + 8xu\frac{du}{dx} - 24u^3\frac{du}{dx} \\
&= 4x + 2x - 2u^2 + 4u^2 + 8x^2u - 8xu^3 - 24xu^3 + 24u^5 \\
&= 6x + 8x^2u + 2u^2 - 32xu^3 + 24u^5
\end{aligned}$$

```python
def d0(x, y):
    return y
def d1(x, y):
    return x - y**2
def d2(x, y):
    return 1 - 2 * x * y + 2 * y**3
def d3(x, y):
    return 2 * x**2 + 2 * y + 4 * x * y**2 - 6 * y**4
def d4(x, y):
    return 6 * x + 8 * x**2 * y + 2 * y**2 - 32 * x * y**3 + 24
* y**5
x0 = 0
y0 = 0
h = 0.1
xmax = 1
u_vals = get_values(x0, y0, xmax, h, d0, d1, d2, d3, d4)
print(*[
    f"{y:.3f}" for y in u_vals
], sep=",")
```

let's plot the vector field to get a idea if this is reasonable.

```python
import matplotlib.pyplot as plt
import numpy as np
x, y = np.meshgrid(np.linspace(0, 1, 25), np.linspace(0, 1, 25))
def slope(x, y):
    return x - y**2
u = np.ones_like(x)
v = slope(x, y)
plt.figure(figsize=(8, 8))
plt.quiver(x, y, u, v, scale=30)
plt.gcf().tight_layout()
plt.plot( np.linspace(0, 1, 11),
          [0.000,0.005,0.020,0.045,0.080,0.124,0.177,0.238,0.306,0.380,0.458]
)
plt.savefig("temp.pdf")
return "temp.pdf"
```

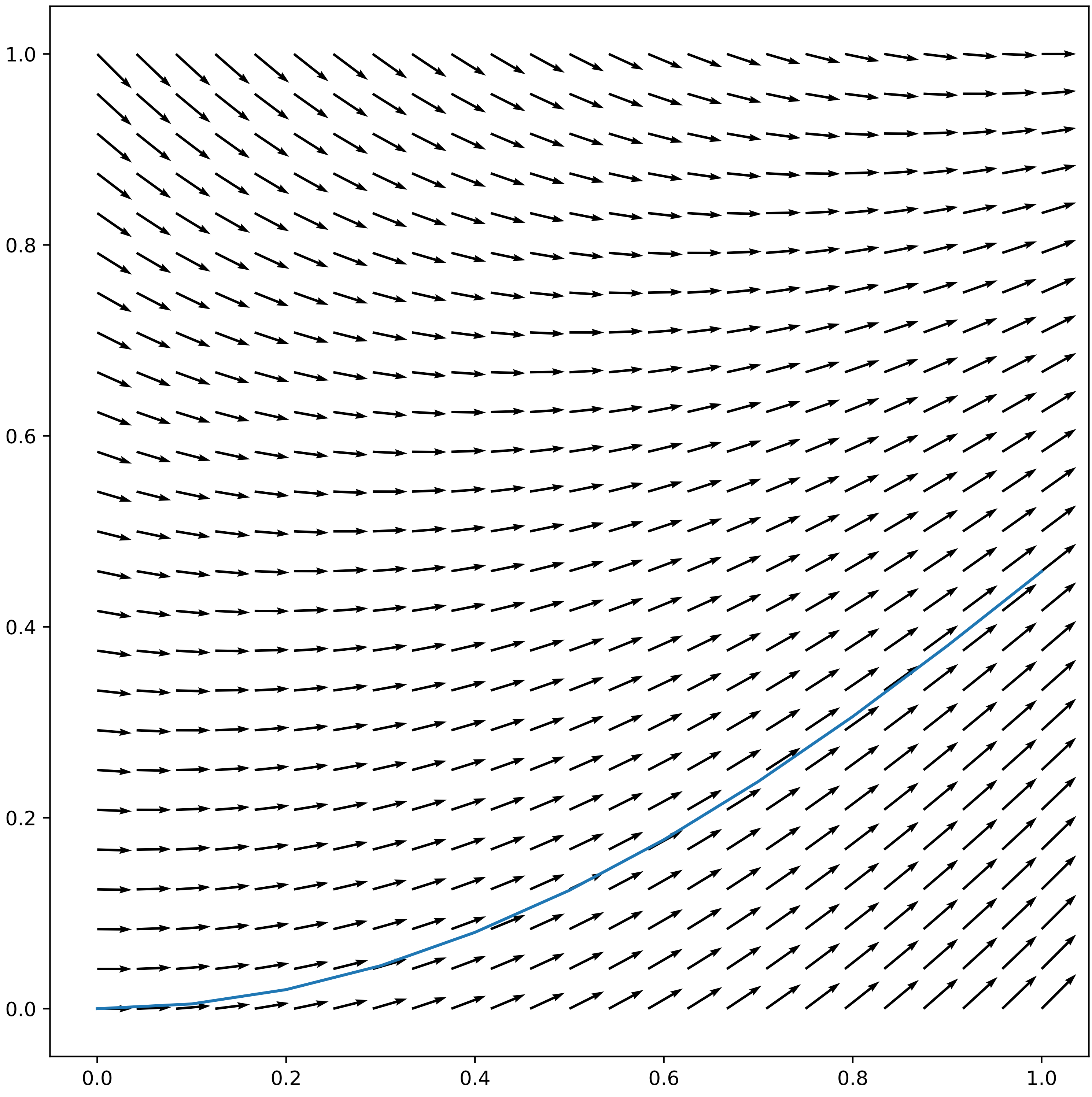

Figure 2

it remains to apply quadrature to the solution to convert back into a differential equation in terms of y

5 Problem 5

Problem statement find the general solution of

$$\frac{dy}{dx} = \frac{(1+x^2)\,y^3}{(y^2-1)\,x^3}$$

Solution ... this appears to be a separable equation of the form

$$\frac{dy}{dx} = \frac{\phi(x)}{\psi(y)}$$

equations of this form can be rewritten as

$$\psi(y)\frac{dy}{dx} = \phi(x)$$

which is equivalent to

$$\frac{d}{dx}\psi(y) = \phi(x)$$

where $\int \psi(y)dy = \psi(y)$ and so this can be rewritten as

$$\frac{d}{dx}\int_b^y \psi(y)dy = \phi(x)$$

and therefore

$$\int_b^y \psi(y)dy = \int_a^x \phi(x)dx$$

so, letting $\psi(y) = \frac{y^2-1}{y^3}$ and $\phi(x) = \frac{1+x^2}{x^3}$, we can solve:

$$\int_b^y \frac{y^2-1}{y^3}dy = \int_a^x \frac{1+x^2}{x^3}dx$$

both sides can be rewritten as the sum of multiple simpler integrals:

$$\int_b^y \frac{y^2-1}{y^3}dy = \int_b^y \frac{1}{y}dy - \int_b^y y^{-3}dy$$

$$= \ln y + \frac{1}{2y^2} - \ln b - \frac{1}{2b^2}$$

$$\int_a^x \frac{1+x^2}{x^3}dx = \int_a^x \frac{1}{x}dx + \int_a^x x^{-3}dx$$

$$= \ln x - \frac{1}{2x^2} - \ln a + \frac{1}{2a^2}$$

and as a result of this effort we have

$$\ln y + \frac{1}{2y^2} - \ln b - \frac{1}{2b^2} = \ln x - \frac{1}{2x^2} - \ln a + \frac{1}{2a^2}$$

which we can simplify, combining constants, to

$$\ln\frac{y}{x} + \frac{1}{2y^2} + \frac{1}{2x^2} = c$$

6 Problem 6

Problem statement find the general solution of

$$\frac{dy}{dx} = \frac{a^2+y^2}{2x\sqrt{ax-a^2}}$$

Solution this is a separable equation of the form $\frac{dy}{dx} = \frac{\phi(x)}{\psi(y)}$ where $\phi(x) = \frac{1}{2x\sqrt{ax-a^2}}$ and $\psi(y) = \frac{1}{a^2+y^2}$. the above equation can then be rewritten as

$$\psi(y)\frac{dy}{dx} = \phi(x)$$

letting $\psi(y) = \int \psi(y)dy$ we see that $\frac{d}{dx}\psi(y) = \psi(y)\frac{dy}{dx}$, so we can rewrite again as

$$\frac{d}{dx}\psi(y) = \phi(x)$$
$$\frac{d}{dx}\int \psi(y)dy = \phi(x)$$
$$\int \psi(y)dy = \int \phi(x)dx$$

substituting in our values for $\psi(y), \phi(x)$, we have

$$\int\limits_{y_0}^{y} \frac{1}{a^2+y^2}dy = \int\limits_{x_0}^{x} \frac{1}{2x\sqrt{ax-a^2}}dx$$

carrying out the integration on the left side, letting $u = \frac{y}{a}, du = \frac{dy}{a}$ we have

$$\begin{aligned}
\int \frac{1}{a^2+y^2}dy &= \int \frac{a}{a^2+a^2u^2}du \\
&= \frac{1}{a}\int \frac{1}{1+u^2}du \\
&= \frac{1}{a}\arctan u \\
&= \frac{1}{a}\arctan \frac{y}{a}
\end{aligned}$$

carrying out the integration on the right side, letting

$$\begin{aligned}
u &= \sqrt{ax-a^2} \\
du &= -\frac{a}{2\sqrt{ax-a^2}}dx \\
dx &= -\frac{2\sqrt{ax-a^2}}{a}du \\
&= -\frac{2u}{a}du \\
x &= \frac{u^2+a^2}{a}
\end{aligned}$$

we have

$$\int \frac{1}{2x\sqrt{ax-a^2}}dx = -\int \frac{1}{2\frac{u^2+a^2}{a}u}\frac{2u}{a}du$$
$$= \int \frac{1}{u^2+a^2}du$$

we can now perform a second substitution, letting $v = \frac{u}{a}, dv = \frac{du}{a}$ which gives $u = va, du = dva$

$$\int \frac{1}{u^2+a^2}du = a\int \frac{1}{a^2+a^2v^2}dv$$
$$= \frac{1}{a}\int \frac{1}{1+v^2}dv$$
$$= \frac{1}{a}\arctan v$$
$$= \frac{1}{a}\arctan \frac{u}{a}$$
$$= \frac{1}{a}\arctan \frac{\sqrt{ax-a^2}}{a}$$

inserting our results back into our original expression, we have

$$\frac{1}{a}\arctan \frac{y}{a} = \frac{1}{a}\arctan \frac{\sqrt{ax-a^2}}{a} + c$$
$$\arctan \frac{y}{a} = \arctan \frac{\sqrt{ax-a^2}}{a} + c$$

7 Problem 7

Problem statement find the family of curves which cut at right angles the parabolas given by

$$y^2 = 2p(x-a)$$

Solution ... we seek a family of curves such that the derivative of the curve is equal to the negative of the reciprocal of the derivative of the given parabola. the slope of the given parabola is found by

$$x = \frac{y^2}{2p} + a$$
$$\frac{dx}{dy} = \frac{y}{p}$$

the differential equation for our solution is given by

$$\frac{dx}{dy} = -\frac{p}{y}$$

we can integrate both sides of this equation with respect to y:

$$x = -p \int \frac{1}{y} dy$$
$$x = -p \ln y + c$$

8 Problem 8

Problem statement find the general solution of the differential equation

$$x^2 \frac{dy}{dx} = x^2 + y^2$$

Solution this equation can be rewritten in the form of a homogeneous equation:

$$\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2 = f\left(\frac{y}{x}\right)$$

letting $u = \frac{y}{x}; y = xu$ we have by differentiating

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

and therefore our previous equation becomes

$$\frac{du}{dx} = \frac{f(u) - u}{x}$$

this is now a separable equation in u and x. rewriting it with the usual abuse of notation, we have

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

and by integration

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x}$$

then the right hand side becomes $\log x + c$ and the left hand side depends on our particular $f(u)$. in this case, $f(u) = 1 + u^2$ and so the left hand side becomes

$$\int \frac{du}{u^2 - u + 1}$$

we can complete the square in the denominator:

$$u^2 - u + 1 = \left(u - \frac{1}{2}\right)^2 + \frac{3}{4}$$

and therefore, letting $v = \frac{2\sqrt{3}}{3}\left(u - \frac{1}{2}\right), dv = \frac{2\sqrt{3}}{3} du$, we have on the left hand side

$$\begin{aligned}
&\frac{\sqrt{3}}{2} \int \frac{dv}{3/4v^2 + 3/4} \\
&\qquad = \frac{2\sqrt{3}}{3} \int \frac{dv}{v^2 + 1} \\
&\qquad = \frac{2\sqrt{3}}{3} \arctan v \\
&\qquad = \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}}{3}\left(u - \frac{1}{2}\right)\right) \\
&\qquad = \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}}{3}\left(\frac{y}{x} - \frac{1}{2}\right)\right)
\end{aligned}$$

and so, with our right hand side from before, we have

$$\begin{aligned}
\log x + c &= \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\sqrt{3}}{3}\left(\frac{y}{x} - \frac{1}{2}\right)\right) \\
\frac{\sqrt{3}}{2} \log x + c &= \arctan\left(\frac{2\sqrt{3}}{3}\left(\frac{y}{x} - \frac{1}{2}\right)\right) \\
\tan\left(\frac{\sqrt{3}}{2} \log x + c\right) &= \frac{2\sqrt{3}}{3}\left(\frac{y}{x} - \frac{1}{2}\right) \\
\frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} \log x + c\right) + \frac{1}{2} &= \frac{y}{x} \\
x\left[\frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} \log x + c\right) + \frac{1}{2}\right] &= y
\end{aligned}$$

9 Problem 9

Problem Statement find the general solution of the differential equations:

$$\frac{d^4y}{dx^4} + y = 0$$

$$\frac{d^4y}{dx^4} - y = 0$$

Solution because these are both linear ordinary differential equations, we can solve by assuming a solution of the form $y = e^{ax}$. substituting this into the equations will yield a polynomial in a which we can solve for algebraically.

Solution 1

$$\frac{d^4y}{dx^4} + y = 0$$

let $y = e^{ax}$. therefore, we have

$$\begin{aligned}\frac{d^4}{dx^4}e^{ax}+e^{ax}&=0\\ a^4e^{ax}+e^{ax}&=0\\ a^4+1&=0\\ a^4&=-1\end{aligned}$$

so we're looking a such that $a^4=-1$. this is satisfied when $a^2=\pm i$ or when $a=\pm\sqrt{\frac{1}{2}}\pm i\sqrt{\frac{1}{2}}$.

$$\begin{aligned}&proof.\\ \left(\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}\right)^2&=\frac{1}{2}+2\frac{1}{2}i+\frac{1}{2}i^2=i\\ \left(\sqrt{\frac{1}{2}}-i\sqrt{\frac{1}{2}}\right)^2&=\frac{1}{2}-2\frac{1}{2}i+\frac{1}{2}i^2=-i\\ \left(-\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}\right)^2&=\frac{1}{2}-2\frac{1}{2}i+\frac{1}{2}i^2=-i\\ \left(-\sqrt{\frac{1}{2}}-i\sqrt{\frac{1}{2}}\right)^2&=\frac{1}{2}+2\frac{1}{2}i+\frac{1}{2}i^2=i\end{aligned}$$

so our final solution is

$$\begin{aligned}&c_1e^{\sqrt{1/2}(1+i)x}+c_2e^{\sqrt{1/2}(1-i)x}+c_3e^{\sqrt{1/2}(-1+i)x}+c_4e^{\sqrt{1/2}(-1-i)x}\\ &=c_1e^{\sqrt{1/2}x}\left(\cos\left(\sqrt{1/2}x\right)+i\sin\left(\sqrt{1/2}x\right)\right)\\ &\quad+c_2e^{\sqrt{1/2}x}\left(\cos\left(\sqrt{1/2}x\right)-i\sin\left(\sqrt{1/2}x\right)\right)\\ &\quad+c_3e^{-\sqrt{1/2}x}\left(\cos\left(\sqrt{1/2}x\right)+i\sin\left(\sqrt{1/2}x\right)\right)\\ &\quad+c_4e^{-\sqrt{1/2}x}\left(\cos\left(\sqrt{1/2}x\right)-i\sin\left(\sqrt{1/2}x\right)\right)\end{aligned}$$

or, combining arbitrary constants,

$$e^{\sqrt{1/2}x}\left(c_1\cos\left(\sqrt{1/2}x\right)+c_2\sin\left(\sqrt{1/2}x\right)\right)$$
$$+e^{-\sqrt{1/2}x}\left(c_3\cos\left(\sqrt{1/2}x\right)+c_4\sin\left(\sqrt{1/2}x\right)\right)$$

Solution 2

$$\frac{d^4y}{dx^4}-y=0$$

let $y=e^{ax}$. therefore, we have

$$\frac{d^4}{dx^4}e^{ax}-e^{ax}=0$$
$$a^4e^{ax}-e^{ax}=0$$
$$a^4-1=0$$
$$a^4=1$$

so we're looking a such that $a^4=-1$. this is satisfied when $a^2=\pm1$ or when $a=1, a=-1, a=i, a=-i$. this gives us

$$y=c_1e^x+c_2e^{-x}+c_3e^{ix}+c_4e^{-ix}$$

applying euler's identity to this gives us

$$y=c_1e^x+c_2e^{-x}+c_3\left(\cos x+i\sin x\right)+c_4\left(\cos x-i\sin x\right)$$

we can combine constants to get

$$y=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x$$

10 Problem 10

Problem statement find the general solution of the following differential equations:

$$\frac{dy}{dx}+y\cos x=\sin x\cos x$$

$$\frac{dy}{dx}=\frac{ny}{x+1}+e^x\left(x+1\right)^n$$

Solution 1 this is a linear differential equation of the first order of the form

$$\frac{dy}{dx} = \phi(x)y + \psi(x)$$

we can solve this by substituting $y = uv$. then,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

applying this to the general form of problem, we have

$$u\frac{dv}{dx} + v\frac{du}{dx} = \phi(x)y + \psi(x)$$

or in particular,

$$u\frac{dv}{dx} + v\frac{du}{dx} = -y\cos x + \sin x \cos x$$

let us choose v such that $u\frac{dv}{dx} = \phi(x)uv$ or $\frac{1}{v}\frac{dv}{dx} = \phi(x)$ which is a separable differential equation and is solved thus:

$$\begin{aligned} \frac{1}{v}\frac{dv}{dx} &= \phi(x) \\ \int \frac{1}{v}dv &= \int \phi(x)dx \\ \log v &= \int \phi(x)dx \\ v &= e^{\int \phi(x)dx} \end{aligned}$$

we can substitute this expression into our original general differential equation given our choice for v:

$$
\begin{aligned}
v\frac{du}{dx} &= \psi(x) \\
\frac{du}{dx} &= \frac{\psi(x)}{v} \\
\frac{du}{dx} &= e^{-\int \phi(x)dx}\psi(x) \\
u &= \int\limits_0^x e^{-\int\limits_0^x \phi(x)dx}\psi(x)dx \\
y &= e^{\int\limits_0^x \phi(x)dx}\int\limits_0^x e^{-\int\limits_0^x \phi(x)dx}\psi(x)dx \\
or \\
y &= v\int\limits_0^x \frac{\psi(x)}{v}dx
\end{aligned}
$$

from the original problem statement, we have $\phi(x) = -\cos x$. thus,

$$
\begin{aligned}
v &= e^{\int \phi(x)dx} \\
&= e^{\int -\cos x dx} \\
&= e^{-\sin x}
\end{aligned}
$$

substituting the value for v back into the problem statement we have

$$
y = e^{-\sin x}\int\limits_0^x e^{\sin x}\sin x\cos x dx
$$

this can be integrated by parts. letting

$$
\begin{aligned}
u &= \sin x \\
v^{'} &= e^{\sin x}\sin x \\
v &= e^{\sin x} \\
u^{'} &= \cos x \\
\int uv^{'} &= uv - \int u^{'}v \\
&= \sin(x)e^{\sin x} - \int \cos x e^{\sin x}dx
\end{aligned}
$$

this second part can be integrated by making a substitution, letting $u = \sin x, du = \cos x dx$ yields

$$\begin{aligned}\int \cos x e^{\sin x} dx &= \int e^u du \\ &= e^u \\ &= e^{\sin x}\end{aligned}$$

this gives us a result of

$$\sin{(x)} e^{\sin x} - e^{\sin x}$$

combining this result with our previous result we are left with

$$\begin{aligned}y &= e^{-\sin x}\left(\sin{(x)} e^{\sin x} - e^{\sin x}\right) \\ &= \sin x - 1\end{aligned}$$

checking our work, we have

$$\frac{dy}{dx} = \cos x$$

$$\begin{aligned}\frac{dy}{dx} + y\cos x &= \cos x + \sin x \cos x - \cos x \\ &= \sin x \cos x\end{aligned}$$

solution 2

$$\frac{dy}{dx} = \frac{ny}{x+1} + e^x (x+1)^n$$

this is an equation of the form

$$\frac{dy}{dx} = \phi(x)y + \psi(x)$$

let $u(x), v(x)$ such that $y = uv$, we then have

$$\frac{dy}{dx} = \frac{d}{dx}u(x)v(x) = u\frac{dv}{dx} + v\frac{du}{dx}$$

and therefore

$$u\frac{dv}{dx} + v\frac{du}{dx} = \phi(x)y + \psi(x)$$

we can arbitrarily choose that

$$v\frac{du}{dx} = \phi(x)uv$$

so that

$$\frac{1}{u}du = \phi(x)dx$$

and thus

$$\log u = \int \phi(x)dx$$

or

$$u = e^{\int \phi(x)dx}$$

given our value for u, we can return to our problem and find v:

$$\begin{aligned}
u\frac{dv}{dx} &= \psi(x) \\
\frac{dv}{dx} &= \frac{\psi(x)}{u} \\
\frac{dv}{dx} &= \psi(x)e^{-\int \phi(x)dx} \\
v &= \int \psi(x)e^{-\int \phi(x)dx}dx
\end{aligned}$$

back to our problem, we have

$$\begin{aligned}
\phi(x) &= \frac{n}{x+1} \\
\psi(x) &= e^x\,(x+1)^n \\
u &= e^{\int \frac{n}{x+1}}
\end{aligned}$$

we need to integrate $\int \frac{n}{x+1}dx$. this will be done with a u substitution letting $u = x+1, du = dx$. this gives

$$\int \frac{n}{x+1} = n\log u = n\log{(x+1)}$$

so

$$
\begin{aligned}
u &= e^{n \log(x+1)} \\
&= \left(e^{\log(x+1)}\right)^n \\
&= (x+1)^n \\
e^{-\int \phi(x)dx} &= e^{-n \log(x+1)} \\
&= (x+1)^{-n} \\
v &= \int \psi(x) e^{-\int \phi(x)dx} dx \\
&= \int e^x (x+1)^n (x+1)^{-n} dx \\
&= \int e^x \\
&= e^x \\
y &= e^x (x+1)^n
\end{aligned}
$$

we can check our solution:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(e^x (x+1)^n\right) \\
&= e^x (x+1)^n + n e^x (x+1)^{n-1} \\
\frac{ny}{x+1} + e^x (x+1)^n &= \frac{n e^x (x+1)^n}{x+1} + e^x (x+1)^n \\
&= n e^x (x+1)^{n-1} e^x (x+1)^n
\end{aligned}
$$

11 **TODO** Problem 11

Problem Statement a constant voltage e i applied to a resistance at the time $t = 0$. the heat capacity of the resistance is q, its heat transfer coefficient per unit time and unit surface is α, and its surface area is s. the resistance as a function of temperature is given by

$$r = r_0(1 + \beta\theta)$$

where θ is the temperature of the resistance above room temperature. find a relation between temperature θ and time.

Solution we find a differential equation for the temperature by expressing the fact that the heat produced per second by the current is partly stored in the resistance and partly lost by transfer through the surface. the equation of heat balance is $\frac{e^2}{1+\beta\theta} - \alpha s\theta = q\frac{d\theta}{dt}$. the final temperature θ_f must correspond to $\frac{d\theta}{dt} = 0$, hence it is given by $\frac{e^2}{r_0(1+\beta\theta_f)} = \alpha s\theta_f$. introducing the dimensionless variable $\eta = \frac{\theta}{\theta_f}$, we can rewrite the differential equation as $\frac{q}{\alpha s}\frac{d\eta}{dt} = \frac{1+\beta\theta_f}{1+\beta\theta_f\eta} - \eta$.
given the dimensionless equation, let's try substituting $u = \frac{1}{\eta}$.

$$\begin{aligned}\eta &= \frac{1}{u}\\ \frac{d\eta}{dt} &= \frac{d}{dt}\frac{1}{u}\\ &= -\frac{1}{u^2}\frac{du}{dt}\end{aligned}$$

let $\gamma = \frac{q}{\alpha s}$ and $\xi = \beta\theta_f$. then,

$$\begin{aligned}-\gamma\frac{1}{u^2}\frac{du}{dt} &= \frac{1+\xi}{1+\frac{\xi}{u}} - \frac{1}{u}\\ -\gamma\frac{du}{dt} &= u^2\frac{1+\xi}{1+\frac{\xi}{u}} - u\\ &= u^2\frac{1+\xi}{1+\frac{\xi}{u}} - u\end{aligned}$$

that doesn't work at all!
maybe we're overcomplicating this.

$$\frac{q}{\alpha s}\frac{d\eta}{dt} = \frac{1+\beta\theta_f}{1+\beta\theta_f\eta} - \eta$$

this looks like it could be separable! maybe i can just flip the sides around and integrate.

$$\frac{q}{\alpha s}\frac{1}{\frac{1+\beta\theta_f}{1+\beta\theta_f\eta} - \eta}d\eta = dt$$

this is an ugly fraction but so be it. here we go!

$$\frac{1+\beta\theta_f\eta}{1+\beta\theta_f-\eta-\beta\theta_f\eta^2}d\eta = \frac{\alpha s}{q}dt$$

now we just need to suffer through integrating both sides. we have to complete the square.

$$1+\beta\theta_f-\eta-\beta\theta_f\eta^2 = -\beta\theta_f\left(\eta^2+\frac{\eta}{\beta\theta_f}-\frac{1}{\beta\theta_f}-1\right)$$

$$\eta^2+\frac{\eta}{\beta\theta_f}-\frac{1}{\beta\theta_f}-1 = \left(\eta+\frac{1}{2\beta\theta_f}\right)^2-\frac{1}{\beta\theta_f}-1-\left(\frac{1}{2\beta\theta_f}\right)^2$$

This is nice. Let a be the stuff outside the parentheses and u be the stuff inside the parentheses. We then have an integral of the form $\frac{1+ku}{u^2-a^2}$, which can be split up into two parts: $\int\frac{1}{u^2-a^2}du$ and $\int\frac{u}{u^2-a^2}$. The first part evaluates to $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$ (from a table), and the second can be solved with a u substitution.
Let's figure out

$$\int\frac{u}{u^2-a^2}du$$

Let $v=u^2-a^2$, $dv=2udu$. Thus,

$$\begin{aligned}\int\frac{u}{u^2-a^2}du &= \frac{1}{2}\int\frac{1}{v}dv\\ &= \frac{1}{2}\log v\\ &= \frac{1}{2}\log\left(u^2-a^2\right)\end{aligned}$$

Returning back to our original problem, we need to manipulate the integrand into the form $\frac{1+ku}{u^2-a^2}$.

$$
\begin{aligned}
u &= \eta + \frac{1}{2\beta\theta_f} \\
du &= d\eta \\
\eta &= u - \frac{1}{2\beta\theta_f} \\
a &= 1 + \frac{1}{\beta\theta_f} + \left(\frac{1}{2\beta\theta_f}\right) \\
\frac{1+\beta\theta_f\eta}{1+\beta\theta_f-\eta-\beta\theta_f\eta^2} &= -\frac{1}{\beta\theta_f}\frac{1+\beta\theta_f\left(u-\frac{1}{2\beta\theta_f}\right)}{u^2-a^2} \\
&= -\frac{1}{\beta\theta_f}\frac{1+\beta\theta_f u-\frac{1}{2}}{u^2-a^2} \\
&= -\frac{1}{2\beta\theta_f}\frac{1}{u^2-a^2} - \frac{u}{u^2-a^2} \\
\int -\frac{1}{2\beta\theta_f}\frac{1}{u^2-a^2} - \frac{u}{u^2-a^2} du &= -\frac{1}{4\beta\theta_f}\log\left(u^2-a^2\right) - \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)
\end{aligned}
$$

12 **TODO** Problem 12

The pressure distribution $p(x)$ in the oil fil between the inclined surface S_a which moves with the velocity U and the fixed base S_2 is given by the equation

$$
\frac{dp}{dx} = \frac{12\mu}{h^3}\left(Q - \frac{Uh}{2}\right)
$$

13 **TODO** Problem 13

14 **TODO** Problem 14

15 **TODO** Problem 15

16 **TODO** Problem 16

17 Problem 17

Problem Statement Expand the function $y = \cosh^5 x$ as a series of hyperbolic cosines.

Solution Rewrite $\cosh x$ as $\frac{e^x+e^{-x}}{2}$. Therefore, we can write $\cosh^5 x$ as

$$
\begin{aligned}
&\frac{1}{32}\left(e^x + e^{-x}\right) \\
&= \frac{1}{32}\left(e^{5x} + 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} + 10e^{2x}e{-}3x + 5e^{x}e^{-4x} + e^{-5x}\right) \\
&= \frac{1}{32}\left(e^{5x} + 5e^{3x} + 10e^{x} + 10e^{-x} + 5e^{-3x} + e^{-5x}\right) \\
&= \frac{1}{32}\left(2\cosh 5x + 10\cosh 3x + 20\cosh x\right) \\
&= \frac{5}{8}\cosh x + + \frac{5}{16}\cosh 3x + \frac{1}{16}\cosh 5x
\end{aligned}
$$