Chapter 1 Exercises

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Contents

1 Problem 1

Problem Evaluate numerically the integral

$$I = \int\limits_0^{\pi/2} \ln(\sin x)\, dx$$

and compare with the exact value $I = -\frac{\pi}{2}\ln 2$

Solution ...

Note that $\ln(\sin x) = \ln\left(x\frac{\sin x}{x}\right) = \ln x + \ln\frac{\sin x}{x}$
Therefore we just need to find $\int_0^{\pi/2} \ln x dx$ analytically and evaluate $\int_0^{\pi/2} \ln\frac{\sin x}{x} dx$ numerically.
$\int_0^{\pi/2} \ln x dx$ can be integrated by parts.
Let $u = \ln x, v^{'} = dx, u^{'} = \frac{1}{x}, v = x.$ Therefore,

$$\int lnxdx = \int uv^{'}dx = uv - \int vu^{'}dx$$

$$uv - \int vu^{'}dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x$$

And so, $\int_0^{\pi/2} \ln x dx = \left. x\ln x - x\right|_0^{\pi/2}$

$$\begin{aligned}\lim_{x\to 0} x\ln x &= \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0} -\frac{\frac{1}{x}}{\frac{1}{x^2}} \\ &= \lim_{x\to 0} -x \\ &= 0\end{aligned}$$

And, $\int_0^{\pi/2} \ln x dx = \frac{pi}{2} (\ln x - 1) \approx -0.861$

```
from sympy import log, pi, integrate
from sympy.abc import x
print((pi/2 * (log(pi/2) - 1)).evalf())
print(integrate(log(x), (x, 0, pi/2)).evalf())
```

```
-0.861451872082119
-0.861451872082119
```

So we have a value for part of our problem. The other part is to evaluate the integral

$$\int_0^{\pi/2} \ln \frac{\sin x}{x} dx$$

We can do this with Simpson's rule. Recall that

$$S = \frac{H}{9} \sum_{i=0}^{n-1} f(x_0 + iH) + 4f\left(x_0 + \left(i + \frac{1}{2}\right) H\right) + f\left(x_0 + (i+1) H\right)$$

```
from math import pi, sin, log
num_points = 10000
xmin = 0
xmax = pi/2
def f(x):
    if x == 0:
        return 0
    return log(sin(x)/x)
H = (xmax - xmin) / num_points
def s(x0, i, H, f):
    return H/6 * (
        f(x0 + i * H)
        + 4 * f(x0 + (i + 1/2) * H)
        + f(x0 + (i + 1) * H)
    )
print(sum(s(xmin, i, H, f) for i in range(num_points)))
```

```
-0.22734117306968255
```