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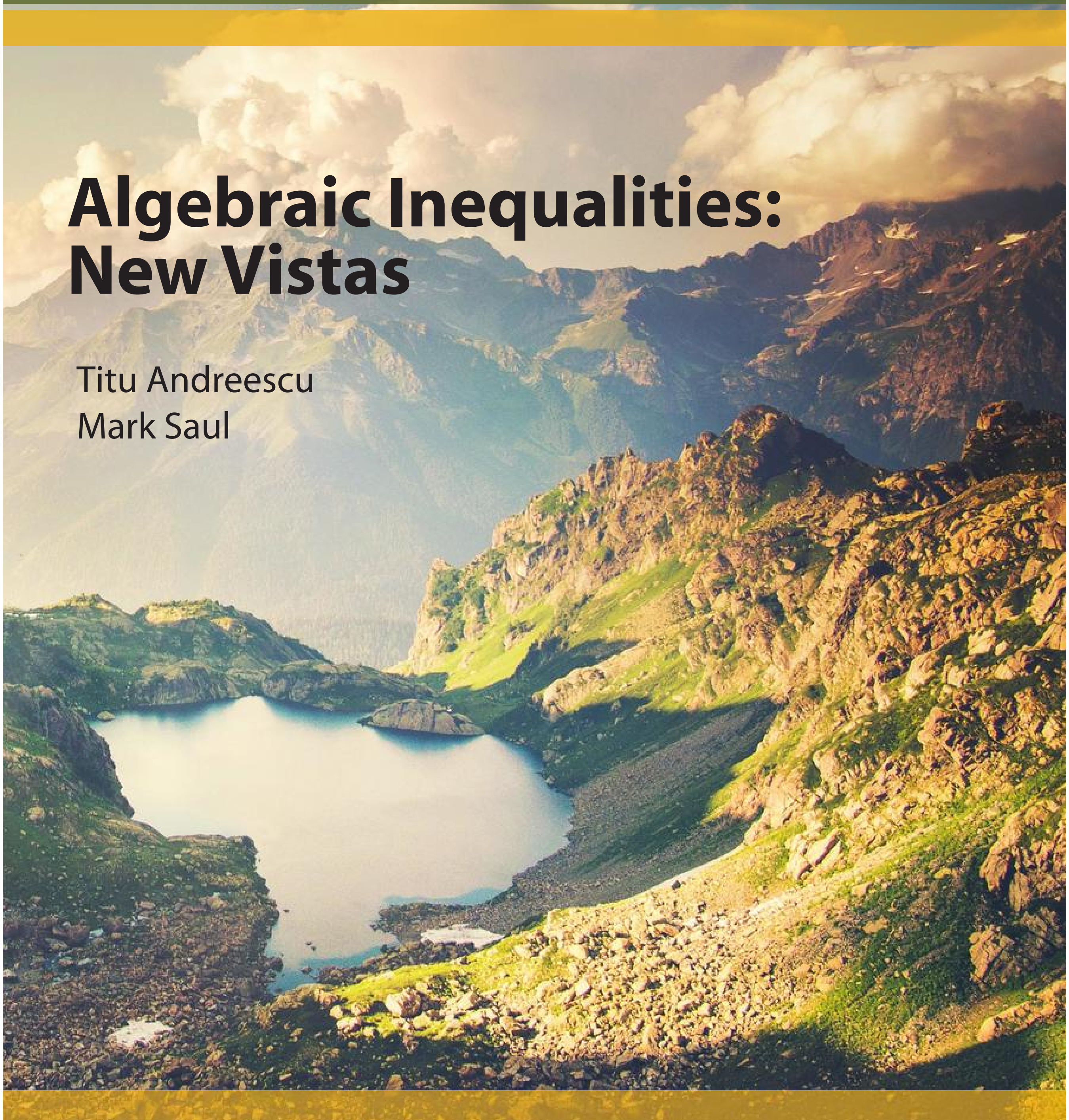

Algebraic Inequalities: New Vistas

Titu Andreescu

Mark Saul

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MSRI
Mathematical Sciences Research Institute
Berkeley, California

AMS
American Mathematical Society
Providence, Rhode Island

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Titu Andreescu
Mark Saul

Introduction

This book is about algebraic inequalities. We have chosen this topic because we can start with almost no background, and build meaningful sets of problems.

At the outset, nothing is required of the reader beyond an elementary knowledge of the rational numbers. Later, we require some algebra, and later still more algebra, up to and including the solution of quadratic equations. For all but a very few problems, no more than intermediate algebra is required as background.

The exposition is not exactly linear. We have taken the opportunity to link our central topic with others in mathematics. Sometimes these topics relate to much more than algebraic inequalities, but arise naturally as we discuss our topic. An example is the long digression on algebraic symmetry in the middle of the book. The topic creeps into the early chapters, and is addressed directly in the middle chapters.

There are also 'secret' pathways through the book. Each chapter has a subtext, a theme which prepares the student for learning other mathematical topics, concepts, or habits of mind. The early chapters on the AM/GM inequality, for example, show how very simple observations can be leveraged to yield useful and interesting results. The later chapters give examples of how one can generalize a mathematical statement. The chapter on the Cauchy-Schwarz inequality provides an introduction to vectors as mathematical objects. And there are many other secret pathways that we hope the reader will discover—and follow.

There is another level to this book. If you are reading this, you probably enjoy solving math problems. It is a joy to find a good solution to a problem, and perhaps the greatest joy comes from the most difficult problem. But if you have not solved difficult math problems, how do you learn this skill?

In regular math classes, or regular textbooks, you learn to read mathematics. You learn about the objects of mathematics and acquire tools for working with them. The mathematical tasks set in most textbooks offer exercise, but not challenge.

On the other hand, contest problems, or problems posed in journals, are often very difficult. Many people are successful in school mathematics, but don't have the problem solving skills to approach even a relatively easy problem that is not part of the structure they have learned in school.

How do you learn to solve problems? How do you get there from here?

This book attempts an answer. Each chapter begins with problems requiring only a little thought, but still—we hope—evoking thoughts that the solver had not yet had. Each chapter then leads the reader through more and more difficult problems. The final problems are olympiad level and sometimes require advanced knowledge.

We have linked the problems, so that each contains a hint, or a preparatory result, for the next. We have given hints—fewer as the difficulty of the problem progresses. We have also written solutions which peer forward to later problems and back to earlier ones, in the hopes that that the reader learns something valuable from each solution.

Thus it is important to read our solutions, even if you have your own. And even if your own is better.

We think of our work as marking a trail for the student, a trail which starts out flat and smooth, leads through tougher and tougher terrain, and ends with a scramble to the summit. But you need not get all the way up to enjoy the trip. We have striven to provide vistas of learning at each turn in the path.

The hike may be strenuous, but the vistas are worth it. We hope you enjoy the climb!

Titu Andreescu
Mark Saul

Chapter 0

Some Introductory Problems

Sometimes we take for granted the most important things. For example, it is 'obvious', but very important, that real numbers can be compared to each other. Of two real numbers, one must be greater than the other—unless they are equal. This is a basic property both of the real numbers and of the inequality relation on them. The arithmetic problems below are intended to bring out a few more important, easy, but subtle properties of arithmetic inequalities. Please do not use a calculator for these problems.

0.1. A man was collecting from an audience for a certain charity. He said, "Look at the money in your pocket. You can certainly donate $\frac{1}{10}$ of that money to us. But if you can't afford to give $\frac{1}{10}$, maybe you can afford to give $\frac{1}{9}$ or $\frac{1}{8}$." What comment do you have on this scene?

0.2. Which is larger: (A) $\dfrac{9999}{10000}$ or (B) $\dfrac{10000}{10001}$?

0.3. Which is larger: (A) $\dfrac{90046}{90049}$ or (B) $\dfrac{90047}{90050}$?

0.4. Which is larger: (A) $\dfrac{1}{2+\frac{3}{7}}$ or (B) $\dfrac{1}{2+\frac{4}{7}}$?

0.5. Which is larger: (A) $\dfrac{1}{2+\dfrac{3}{6-\frac{4}{11}}}$ or (B) $\dfrac{1}{2+\dfrac{3}{6-\frac{5}{11}}}$?

0.6. A remark attributed to Joseph Stalin: "That tendency is not just a negative quantity. It is a negative quantity squared." Comment?

0.7. If a, b, and c are positive real numbers, which is larger:
(A) the average (arithmetic mean) of a, b, and c,
or
(B) the average (arithmetic mean) of a^2, b^2, and c^2?
Hint: Can you always answer this question?

0.8. If c and d are positive numbers, which is larger:
(A) $7c + 9d$ or (B) $9(c + d)$?

Two questions with tricks in them:

0.9. If e and f are real numbers, and $e^2 - 1 > f^2 - 1$, then which is larger: (A) e or (B) f?

0.10. If g and h are real numbers, and $g^4 - 1 > h^4 - 1$, then which is larger: (A) g^2 or (B) h^2?

How Do Inequalities Behave?

Most students think that inequalities are just like equations: "Whatever you do to one side, you can do to the other". This is not quite true, even for equations. But it is "less true" for inequalities.

Let us make some more precise statements about how inequalities differ from equations.

Statement 1. For real numbers a, b, if $a > b$ then $-a < -b$.

Check that this statement is true, whether a and b are positive, negative, or zero.

Statement 2. For positive real numbers a, b, if $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

What can you say if either of the numbers a, b is negative?

Statement 3. If a, b, c, d are positive real numbers and $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$.

This property of inequalities is not really very different from the corresponding property for equations. But it is sometimes overlooked, so we list it separately here. Can you prove statement 3 from statements 1 and 2?

There is one more important statement about inequalities:

Statement 4. (Transitivity Property of Inequality) For real numbers a, b, c, if $a > b$ and $b > c$, then $a > c$.

Note that if we replace the "$>$" sign with "$=$", the statement is still true. So this is not exactly a difference between equations and inequalities. But, as we will see, the transitive property becomes much more vivid in working with inequalities than in solving equations.

We will not give a formal treatment of the algebra of inequalities here. An axiomatic description of an ordered field can be found in most books on abstract algebra. We simply want to point out how the algebra of inequalities differs from what we may be used to in working with equations.

Solutions

0.1. A man was collecting from an audience for a certain charity. He said, "Look at the money in your pocket. You can certainly donate $\frac{1}{10}$ of that money to us. But if you can't afford to give $\frac{1}{10}$, maybe you can afford to give $\frac{1}{9}$ or $\frac{1}{8}$." What comment do you have on this scene?

Solution. The man thinks that because $10 > 9 > 8$, then $\frac{1}{10}$, $\frac{1}{9}$, $\frac{1}{8}$ are in the same order.

But in fact they are in the opposite order. This is what statement 2 above asserts.

0.2. Which is larger: (A) $\frac{9999}{10000}$ or (B) $\frac{10000}{10001}$?

Solution. An easy way to think of this problem is to ask which fraction is closer to 1. We have:

$$\frac{9999}{10000} = 1 - \frac{1}{10000},$$
$$\frac{10000}{10001} = 1 - \frac{1}{10001},$$

and we know that $\frac{1}{10000} > \frac{1}{10001}$. So, in the first case, you must subtract a larger fraction from 1 than in the second case, to get the fraction on the left.

That is, both fractions are less than 1, but the first is farther from 1 than the second, so the first is smaller.

0.3. Which is larger: (A) $\frac{90046}{90049}$ or (B) $\frac{90047}{90050}$?

Solution. We can proceed here in the same way as in Problem 0.2:

$$\frac{90046}{90049} = 1 - \frac{3}{90049},$$
$$\frac{90047}{90050} = 1 - \frac{3}{90050},$$

and $\frac{3}{90049} > \frac{3}{90050}$, so we see that fraction (B) is larger than fraction (A).

0.4. Which is larger: (A) $\frac{1}{2+\frac{3}{7}}$ or (B) $\frac{1}{2+\frac{4}{7}}$?

Solution 1. We must work from the "bottom up".

Fraction (A) is equal to $\frac{1}{(\frac{17}{7})} = \frac{7}{17}$.

Fraction (B) is equal to $\frac{1}{(\frac{18}{7})} = \frac{7}{18}$, and, as in Problem 0.1, the first fraction is larger than the second. Thus fraction (A) is larger than fraction (B).

Solution 2. Just look to see that the denominator in (B) (that is, $2 + \frac{4}{7}$) is larger than denominator in (A) (that is, $2 + \frac{3}{7}$), so the first fraction is larger than the second.

0.5. Which is larger: (A) $\frac{1}{2+\frac{3}{6-\frac{4}{11}}}$ or (B) $\frac{1}{2+\frac{3}{6-\frac{5}{11}}}$?

Solution 1. Again, work from the bottom up:

$$\frac{1}{2+\cfrac{3}{6-\frac{4}{11}}}=\frac{1}{2+\cfrac{3}{\left(\frac{62}{11}\right)}}=\frac{1}{2+\frac{33}{62}}=\frac{1}{\left(\frac{157}{62}\right)}=\frac{62}{157},$$

$$\frac{1}{2+\cfrac{3}{6-\frac{5}{11}}}=\frac{1}{2+\cfrac{3}{\left(\frac{61}{11}\right)}}=\frac{1}{2+\frac{33}{61}}=\frac{1}{\left(\frac{157}{61}\right)}=\frac{61}{157}.$$

Hence fraction (A) is greater than fraction (B).

Solution 2. This can be done (pretty easily!) by "just looking" at denominators on the way up the two fractions. But it's hard to write down the thought process. Try it yourself.

0.6. A remark attributed to Joseph Stalin: "That tendency is not just a negative quantity. It is a negative quantity squared." Comment?

Solution. Well, this is not exactly a "solution". But note that when you square a negative quantity, you certainly get a positive quantity. Stalin was trying to say that the new quantity is worse (more negative) than the original negative quantity, and got mixed up.

It's not clear that this is an authentic quote. The sources we have found so far are all written by mathematicians.

Can you prove, from our statements (1) and (2), that if $a < 0$, then $a^2 > 0$?

0.7. If a, b, and c are positive real numbers, which is larger:
(A) the average (arithmetic mean) of a, b, and c,
or
(B) the average (arithmetic mean) of a^2, b^2, and c^2?
Hint: Can you always answer this question?

Solution. The average of three bigger numbers is certainly larger than the average of three smaller numbers. So it looks like the average of a^2, b^2, c^2 should be larger than the average of a, b, c.

But in fact squaring does not always make a number larger! Not even a positive number: if $0 < x < 1$, then $0 < x^2 < x < 1$. So you can't tell which of these two averages is the larger.

Can you prove the statement we just made (about x), from our statements (1) and (2)?

0.8. If c and d are positive numbers, which is larger:
(A) $7c + 9d$ or (B) $9(c + d)$?

Solution. We have $9(c + d) = 9c + 9d = 7c + 9d + 2c$. So we have to add something positive to expression (A) to get to expression (B). Hence (B) must be the larger.

Is the same thing true if c and d are negative numbers?

0.9. If e and f are real numbers, and $e^2 - 1 > f^2 - 1$, then which is larger:
(A) e or (B) f?

Solution. If $e^2 - 1 > f^2 - 1$, then we know $e^2 > f^2$. But does that mean that $e > f$?
Certainly not. Find some examples for yourself.

0.10. If g and h are real numbers, and $g^4 - 1 > h^4 - 1$, then which is larger: (A) g^2 or (B) h^2?

Solution. If $g^4 - 1 > h^4 - 1$, then we know $g^4 > h^4$. But does that mean that $g^2 > h^2$?
Yes, it does! It does because all the quantities we are comparing are squares ($g^4 = (g^2)^2$ counts as a square), so we don't have those annoying negative numbers to worry about.

Chapter 1

Squares Are Never Negative

Theorem 1.1. *For any real number N, we have $N^2 \geq 0$, with equality if and only if $N = 0$.*

This seems simple. Math students know that the square of a real number cannot be negative. There's hardly anything here to prove.

But look at what we can do with this result.

Example 1.1. Show that for any two real numbers a, b, we have $a^2 + b^2 \geq 2ab$.

Solution. We know that $(a - b)^2 \geq 0$ (because a square is never negative). So $a^2 - 2ab + b^2 \geq 0$, and $a^2 + b^2 \geq 2ab$. Isn't this now "obvious"? Of course! "Obvious" really just means "I figured it out".

But how could we think of starting with $(a - b)^2$? Well, we could have worked backwards. We want $a^2 + b^2 \geq 2ab$, which is equivalent to $a^2 - 2ab + b^2 \geq 0$. The expression on the left looks familiar, and factoring it reduces the original statement to an "obvious" one: $(a - b)^2 \geq 0$. Then we write down the proof in the more logical order, as in the first sentence of this solution.

When can $a^2 + b^2 = 2ab$? Again, we can follow the logic "backwards". We are asking when $(a - b)^2 = 0$, and this can only occur when $a = b$.

In general, whenever we have an inequality, we want to know when equality holds. This will become important in later problems.

Problems

1.1. Show that for any two positive numbers c, d, we have $c + d \geq 2\sqrt{cd}$.

1.2. For any real numbers a and b, prove that

$$12(a^2 - ab + b^2) \geq 6(a^2 + b^2) \geq 4(a^2 + ab + b^2) \geq 3(a + b)^2 \geq 12ab.$$

1.3. For $a \geq 0$, prove that $a + 1 \geq 2\sqrt{a}$.

1.4.
- **a.** For any three real numbers a, b, c, show that $a^2 + 2b^2 + c^2 \geq 2ab + 2bc$.
- **b.** For any three real numbers a, b, c, show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.
- **c.** State and prove an analogous result for four real numbers a, b, c, d.

1.5. If a is a positive number, find the minimum possible value of the expression $a - 2\sqrt{a}$.

1.6. For any real number b, find the minimum possible value of $b^2 - 6b$.
1.7. For any real number c, find the minimum possible value of $c^2 - 8c + 7$.

♦

Example 1.2. For any two positive numbers x and y, show that

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

Solution. Clearing fractions, we have $x^2 + y^2 \geq 2xy$, which is true from Example 1.1.

But why must we stipulate that x and y be positive? Well, the statement is certainly wrong if we don't. Let $x = 1$, $y = -1$. Then $\frac{x}{y} + \frac{y}{x} = -2$, which is certainly less than 2. Where did the proof go wrong?

Recall (from Chapter 0) that inequalities do not behave just like equalities. If $P > Q$, then $kP > kQ$ if k is positive, but $kP < kQ$ if k is negative. When we "cleared fractions" above, we multiplied both sides of the inequality by xy. So we had to be sure that this quantity is positive. The condition that x and y both be positive assures us of this.

In this case, we can go a bit further: we can say that if $xy > 0$ (that is, if x and y have the same sign), then $\frac{x}{y} + \frac{y}{x} \geq 2$.

♦

1.8. For any positive number a, find the minimum value of the expression

$$a + \frac{1}{a}.$$

1.9. For any positive number c, show that $c + \frac{2}{c} \geq 2\sqrt{2}$.

1.10. For any positive number d, what is the minimum value of the expression

$$d + \frac{3}{d}?$$

1.11. If $a > b > 0$, show that

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \cdot \frac{(a-b)^2}{b}.$$

Later in this volume, we will see that this result puts bounds on the difference between the arithmetic and geometric mean of two numbers.

1.12. If a, b, c, d are four real numbers such that $ab \geq 0$ and $cd \geq 0$, show that

$$ad + bc \geq 2\sqrt{abcd}.$$

Can the inequality be made "sharper"? That is, can the number 2 on the right-hand side be replaced by any larger number?

1.13. For $a, b, c, d \geq 0$, show that $\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$.

1.14. If a, b, c, d are all positive, and $\frac{a}{b} \leq \frac{c}{d}$, show that $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$.

1.15. Suppose we have n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$. If m and M are the smallest and largest of these fractions (respectively), show that

$$m \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq M.$$

This problem generalizes Problem 1.14.

1.16. Prove that for all real numbers x,

$$4(x^3 + x + 1) \leq (x^2 + 1)(x^2 + 5).$$

♦

The next few problems involve the expressions $(a + b)^3$, $a^3 + b^3$, and $a^3 - b^3$. Recall that

$$\begin{aligned} (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ a^3 + b^3 &= (a + b)(a^2 - ab + b^2), \\ a^3 - b^3 &= (a - b)(a^2 + ab + b^2). \end{aligned}$$

♦

1.17. For $a, b \geq 0$, show that $a^2 - ab + b^2 \geq ab$.

1.18. Make sure you know how to factor $a^3 + b^3$. Then show that for $a, b \geq 0$,

$$a^3 + b^3 \geq ab(a + b).$$

1.19. Show that for non-negative numbers a, b, we have

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a + b}{2}\right)^3.$$

1.20. For any non-negative number a, prove that $1 - \frac{a}{2} \leq \frac{1}{1 + a^2}$.

Hint: If you run out of ideas, try factoring.

1.21. Let a and b be positive real numbers. Prove that

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| \leq 3\left|\frac{a - b}{a + b}\right|.$$

1.22. Compute the product $(x + y + z)(x^2 + y^2 + x^2 - xy - yz - zx)$.

1.23. Show that the rightmost factor in the product of Problem 1.22 can be written as $\frac{1}{2}\left((x - y)^2 + (y - z)^2 + (z - x)^2\right)$.

1.24. If x, y, z are real numbers, show that the expression

$$x^2 + y^2 + x^2 - xy - yz - zx$$

is zero if and only if $x = y = z$.

1.25. In a triangle whose sides have lengths a, b, c, with $a < b < c$, show that

$$a^3 + b^3 + c^3 + 3abc > 2c^3.$$

1.26. (This problem requires some knowledge of complex numbers.) If we have complex numbers v, w, z such that $v^2+w^2+z^2-vw-wz-zv=0$, show that either $v=w=z$ or the points represented by v, w, z in the complex plane form an equilateral triangle.

Solutions

1.1. Show that for any two positive numbers c, d, we have $c+d \geq 2\sqrt{cd}$.

Solution. This falls to a typical algebraic "trick". In Example 1.1, let $a=\sqrt{c}$, and let $b=\sqrt{d}$, and the statement falls out.
Equality holds if and only if $c=d$.
Why did we have to say that c and d were positive numbers here?
This innocent-looking statement will soon become a powerful tool.
(If you're impatient to find out how, look for *Arithmetic-Geometric Mean Inequality* later in this book.)

1.2. For any real numbers a and b, prove that

$$12(a^2-ab+b^2) \geq 6(a^2+b^2) \geq 4(a^2+ab+b^2) \geq 3(a+b)^2 \geq 12ab.$$

Solution. Each link in this chain opens to routine algebraic manipulation. We offer the first two as examples:

(i) We need to show that

$$12(a^2-ab+b^2) \geq 6(a^2+b^2),$$

which we can write as:

$$12(a^2-ab+b^2)-6(a^2+b^2) \geq 0.$$

We have:

$$\begin{aligned} 12a^2-12ab+12b^2-6a^2-6b^2 &= 6a^2-12ab+6b^2 \\ &= 6(a^2-2ab+b^2) \\ &= 6(a-b)^2, \end{aligned}$$

which is certainly greater than or equal to 0. Equality holds just when $a=b$.

(ii) We have:

$$\begin{aligned} 6(a^2+b^2) &\geq 4(a^2+ab+b^2) \\ 6a^2+6b^2 &\geq 4a^2+4ab+4b^2 \\ 2a^2-4ab+2b^2 &\geq 0 \\ 2(a-b)^2 &\geq 0, \end{aligned}$$

which is certainly true, with equality exactly when $a=b$.

1.3. For $a \geq 0$, prove that $a+1 \geq 2\sqrt{a}$.

Solution. We can write the given inequality as

$$a-2\sqrt{a}+1 \geq 0.$$

The form of the expression on the left is reminiscent of a perfect square, especially given the title of this set of problems. And in fact this inequality is equivalent to

$$(\sqrt{a}-1)^2 \geq 0,$$

which is just the theme we are varying here. Equality holds if and only if $a = 1$.

1.4. **a.** For any three real numbers a, b, c, show that $a^2 + 2b^2 + c^2 \geq 2ab + 2bc$.

Solution. Following the theme of this chapter, we can try to show that this is the consequence of the fact that a square is never negative. So let's write it as a statement that something is never negative. The required inequality is equivalent to

$$a^2 + 2b^2 + c^2 - 2ab - 2bc \geq 0.$$

And now we can create squares, by factoring. Notice that b^2 is treated differently from a^2 or c^2. That's what makes the problem 'ugly'. So there must be a reason for this.

Indeed, we can make the problem prettier by writing it as:

$$a^2 + b^2 + b^2 + c^2 - 2ab - 2bc \geq 0,$$

or

$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 \geq 0,$$

or

$$(a-b)^2 + (b-c)^2 \geq 0,$$

which is certainly true. Of course, we must check to see that the reasoning is reversible. Happily, it is.

Did you see the squares coming?

b. For any three real numbers a, b, c, show that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Solution. Comparing this problem with Problem 1.4a might give us an idea. Rewrite the required inequality as:

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 0.$$

Now what? We want something like $(a-b)^2 = a^2 - 2ab + b^2$, etc. We are lacking certain squares, but even more conspicuously, we are lacking double copies of ab, bc, ca. So let's supply them, by writing the inequality as:

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \geq 0,$$

which is still equivalent. And let's try to 'cannibalize' the expression for squares:

$$a^2 + a^2 - 2ab + b^2 + b^2 + 2c^2 - 2bc - 2ca \geq 0.$$

We have an 'extra' copy of a^2, but let's keep going:

$$a^2 + (a-b)^2 + b^2 - 2bc + c^2 + c^2 - 2ca \geq 0,$$

$$a^2 + (a-b)^2 + (b-c)^2 + c^2 - 2ca \geq 0,$$

and we're home. The 'extra' pieces make up yet another square:

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0.$$

Many students try to start this problem by computing $(a-b-c)^2$, or some such trinomial square. The problem is that the result we want is *symmetric* in a, b, and c, but the trinomial we are squaring is not. This concept will be explored in some depth later on in this volume.

c. State and prove an analogous result for four real numbers a, b, c, d.
Solution. Having worked on the solution to Problem 1.4b, we can start from the end, and see what inequality results. The natural way to do this is to write

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2 \geq 0;$$

$$a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2cd + d^2 + d^2 - 2da + a^2 \geq 0,$$

and eventually

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da.$$

Can you generalize for n real numbers, $a_1, a_2, \ldots, a_n$? As an exercise in subscripts, try writing down the result, without necessarily deriving. The derivation will not be anything new.
Problems like this one are of course not well-defined. Perhaps you may think of another generalization, even more clever than this one.

1.5. If a is a positive number, find the minimum possible value of the expression $a - 2\sqrt{a}$.

Solution. This is just another way of asking the question of Problem 1.3. From that problem, we have $a - 2\sqrt{a} \geq -1$, with equality when $a = 1$. Hence the minimum possible value of the expression is -1.
Inequalities are often used to solve problems involving maxima and minima. Note that without having solved Problem 1.3, we would have to think much harder about how to solve this problem. But see the note to the next problem.

1.6. For any real number b, find the minimum possible value of $b^2 - 6b$.

Solution. There are many ways to solve this problem. We present a proof using the theme of this chapter.
We want to relate the given expression to the square of a real number, and in fact, we have the "makings" of a trinomial square. If we just add 9 to this expression, we have:

$$b^2 - 6b + 9 = (b-3)^2 \geq 0.$$

So $b^2 - 6b \geq -9$, with equality when $b = 3$.
That is, the minimum value of the given expression is -9, and is achieved when $b = 3$.
This method may seem artificial, but in fact, it is another method that generalizes powerfully.

1.7. For any real number c, find the minimum possible value of $c^2 - 8c + 7$.

Solution. We can again look for a perfect square, hiding behind the expression we are given:

$$c^2 - 8c + 7 = c^2 - 8c + 16 - 9 = (c - 4)^2 - 9 \geq 0 - 9 = -9,$$

with equality just when $c = 4$.
How did we decide to express 7 as $16 - 9$? If your hindsight doesn't show you this, look up the method of *completing the square* in any intermediate algebra text.

1.8. For any positive number a, find the minimum value of the expression

$$a + \frac{1}{a}.$$

Solution 1. Taking a hint from Example 1.2, we let $x = a$, $y = 1$ to get

$$\frac{a}{1} + \frac{1}{a} \geq 2,$$

with equality when $a = 1$.
What is the corresponding result if a is a negative number?

Solution 2. We can proceed directly as in Problem 1.1, then use the fact that $(A - B)^2 \geq 0$ for any real numbers A, B.
Here we let $A = \sqrt{a}$, $B = \dfrac{1}{\sqrt{a}}$, and expand. We get that

$$a + \frac{1}{a} - 2 \geq 0,$$

which leads to the same result as in the first solution.

1.9. For any positive number c, show that $c + \dfrac{2}{c} \geq 2\sqrt{2}$.

Solution. We cannot use the result of Example 1.2 directly. But we can "imitate" the proof that got us its result. That is, clearing fractions, we have $c^2 + 2 \geq 2c\sqrt{2}$. Letting $a = c$, $b = \sqrt{2}$ in Example 1.1, we have our result.
The condition for equality in Example 1.1 implies that equality holds here when $c = \sqrt{2}$. Check to make sure that this is correct.
Later we will see another way to think of this problem.

1.10. For any positive number d, what is the minimal value of the expression

$$d + \frac{3}{d}?$$

Solution. Looking at the solution to Problem 1.9, we might guess that the minimum value is $2\sqrt{3}$.
But pretend we didn't guess this. Let the minimum value be some

number k. Then we have $d + \frac{3}{d} \geq k$, or $d^2 + 3 \geq kd$. If we match this with Example 1.1, we can try letting $a = d$, $b = \sqrt{3}$. Then Example 1.1 tells us that $d^2 + 3 \geq 2d\sqrt{3}$. Since $d > 0$, we can write this as

$$d + \frac{3}{d} \geq 2\sqrt{3}.$$

So we can feel more secure that the minimum value of the given expression is $2\sqrt{3}$. But is this true? Is this minimum actually achieved? The answer is yes, from the equality condition of Example 1.1. If $d = \sqrt{3}$, the minimum we have "predicted" is achieved.

See how important it is to determine the case for equality?

1.11. If $a > b > 0$, show that

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8} \cdot \frac{(a-b)^2}{b}.$$

Later in this volume, we will see that this result puts bounds on the difference between the arithmetic and geometric mean of two numbers.

Solution. We give a proof for the inequality on the left. The one on the right is obtained analogously.

We want:

$$\frac{1}{8} \cdot \frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab}.$$

Direct computation will be a mess, unless we have some insight into where we are going. So look at the right-hand side. Does it look familiar? Well, twice this expression is $a + b - 2\sqrt{ab} = \left(\sqrt{a} - \sqrt{b}\right)^2$, so we can rewrite our inequality as:

$$\begin{aligned}
\frac{1}{4} \cdot \frac{(a-b)^2}{a} &\leq \left(\sqrt{a} - \sqrt{b}\right)^2, \\
\frac{1}{4a}\left(\sqrt{a} + \sqrt{b}\right)^2\left(\sqrt{a} - \sqrt{b}\right)^2 &\leq \left(\sqrt{a} - \sqrt{b}\right)^2, \\
\frac{1}{4a}\left(\sqrt{a} + \sqrt{b}\right)^2 &\leq 1, \\
\left(\sqrt{a} + \sqrt{b}\right)^2 &\leq 4a,
\end{aligned}$$

and since everything is positive, we can write this as $\sqrt{a} + \sqrt{b} \leq 2\sqrt{a}$, or $\sqrt{b} \leq \sqrt{a}$, which is certainly true for $0 \leq b \leq a$.

Must a and b be non-negative?

1.12. If a, b, c, d are four real numbers such that $ab \geq 0$ and $cd \geq 0$, show that

$$ad + bc \geq 2\sqrt{abcd}.$$

Can the inequality be made "sharper"? That is, can the number 2 on the right-hand side be replaced by any larger number?

Solution. The result follows from the fact that

$$\left(\sqrt{ad}-\sqrt{bc}\right)^2 \ge 0.$$

Equality holds when $ad = bc$, and the fact that equality holds for these values shows that the number 2 cannot be replaced by a smaller number. The inequality is as "sharp" as can be.

1.13. For $a, b, c, d \ge 0$, show that $\sqrt{(a+c)(b+d)} \ge \sqrt{ab} + \sqrt{cd}$.

Solution. Since all variables are non-negative, the required result is equivalent to:

$$(a+c)(b+d) \ge ab + cd + 2\sqrt{abcd},$$

or

$$ab + cd + ad + bc \ge ab + cd + 2\sqrt{abcd},$$

or

$$ad + bc \ge 2\sqrt{abcd},$$

which follows from the fact that

$$\left(\sqrt{ad}-\sqrt{bc}\right)^2 \ge 0.$$

1.14. If a, b, c, d are all positive, and $\frac{a}{b} \le \frac{c}{d}$, show that $\frac{a}{b} \le \frac{a+c}{b+d} \le \frac{c}{d}$.

Solution. Since a, b, c, d are positive, we know that $\frac{a}{b} \le \frac{c}{d}$ implies that $ad \le bc$, so that

$$ab + ad \le ab + bc,$$

or

$$a(b+d) \le b(a+c),$$

or

$$\frac{a}{b} \le \frac{a+c}{b+d}.$$

The other half of the inequality is proved analogously. Notice that we have written the proof starting with what we know (that $\frac{a}{b} \le \frac{c}{d}$) and we have shown that this implies the desired inequality. If you were to see our "scratchwork", you would know that we transformed both the initial and the final inequalities, then worked with the second to get the first.

But the logic must flow the other way, which is how we have written the solution.

The question of what happens when some of the variables are negative is a complicated one which can be analyzed case-by-case.

It is not hard to see that equality holds if and only if $\frac{a}{b} = \frac{c}{d}$.

This inequality forms the basis of a very interesting object called the *Farey sequence*, which readers may enjoy investigating.

1.15. Suppose we have n fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$. If m and M are the smallest and largest of these fractions (respectively), show that

$$m \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq M.$$

This problem generalizes Problem 1.14.

Solution. (Watch the subscripts!) By the very definitions of m and M, we have, for every value of i,

$$m \leq \frac{a_i}{b_i} \leq M.$$

(Note that this represents n inequalities, one each for $i = 1, 2, \ldots, n$.) Hence we also have $mb_i \leq a_i \leq Mb_i$, for each i. Adding these n inequalities, we have:

$$mb_1 + mb_2 + \ldots + mb_n \leq a_1 + a_2 + \ldots + a_n \leq Mb_1 + Mb_2 + \ldots + Mb_n,$$
$$m(b_1 + b_2 + \ldots + b_n) \leq a_1 + a_2 + \ldots + a_n \leq M(b_1 + b_2 + \ldots + b_n),$$
$$m \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq M.$$

The reader who is confused by subscripts is urged to write out the inequalities for $i = 1, 2$, and 3. The reader who is used to subscripts is urged to rewrite this proof using "sigma" (Σ) notation.

1.16. Prove that for all real numbers x,

$$4(x^3 + x + 1) \leq (x^2 + 1)(x^2 + 5).$$

Solution. Multiplying out and placing everything to one side of the inequality, we have (magically!):

$$x^4 - 4x^3 + 6x^2 - 4x + 1 \geq 0.$$

The key is noting the appearance of the binomial coefficients $1, 4, 6, 4, 1$ in the polynomial. Using this hint, we have:

$$(x - 1)^4 \geq 0,$$

which is certainly true.
Note that a fourth power is the square of a square, so it must also be non-negative. This is another hint that we should compare the given polynomial with the binomial expansion noted above.

1.17. For $a, b \geq 0$, show that $a^2 - ab + b^2 \geq ab$.

Solution. We have $(a - b)^2 = a^2 - 2ab + b^2 \geq 0$, hence the result.

1.18. Make sure you know how to factor $a^3 + b^3$. Then show that for $a, b \geq 0$,

$$a^3 + b^3 \geq ab(a + b).$$

Solution. Multiply the inequality of Problem 1.17 by $a + b$.

1.19. Show that for non-negative numbers a, b, we have

$$\frac{a^3 + b^3}{2} \geq \left(\frac{a + b}{2}\right)^3.$$

Solution. Multiplying the result of Problem 1.18 by 3, and adding $a^3 + b^3$, we have

$$4a^3 + 4b^3 \geq (a+b)^3,$$

which is equivalent to the desired result.

1.20. For any non-negative number a, prove that $1 - \dfrac{a}{2} \leq \dfrac{1}{1+a^2}$.

Hint: If you run out of ideas, try factoring.

Solution. (a) Since $1 + a^2$ cannot be negative, we can multiply both sides of the inequality by it, and also by 2, to get:

$$(1+a^2)(2-a) = 2 - a + 2a^2 - a^3 \leq 2,$$

or

$$a^3 - 2a^2 + a \geq 0.$$

But this is immediate, since we have

$$a^3 - 2a^2 + a = a(a-1)^2 \geq 0.$$

1.21. Let a and b be positive real numbers. Prove that

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| \leq 3\left|\frac{a-b}{a+b}\right|.$$

Solution. We need to show that:

$$\left|\frac{a^3 - b^3}{a^3 + b^3}\right| = \left|\frac{a-b}{a+b}\right| \cdot \left|\frac{a^2 + ab + b^2}{a^2 - ab + b^3}\right| \leq 3\left|\frac{a-b}{a+b}\right|$$

or

$$\left|\frac{a^2 + ab + b^2}{a^2 - ab + b^2}\right| \leq 3.$$

This last follows from the inequality established between the first and third expression in Problem 1.2.

Remember that a formal proof would mean reading up in this chain of computations.

1.22. Compute the product $(x+y+z)(x^2+y^2+x^2-xy-yz-zx)$.

Solution. The computation is straightforward, and the answer is amazing:

$$x^3 + y^3 + z^3 - 3xyz.$$

Most people do this by multiplying the second factor by x, then by y, then by z, and adding the results. If you proceed this way, you may notice that the three partial products are very similar. They involve the three variables in the same way, just with one replacing another. This phenomenon is called *algebraic symmetry*, and we will have much more to say about it in later chapters. For now, notice that it helps you organize and check this complicated computation.

If you didn't notice the symmetry, it might be useful to go back and redo the computation using this property of the factors.

1.23. Show that the rightmost factor in the product of Problem 1.22 can be written as $\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)$.

Solution. Again, the computation is straightforward. And again, symmetry helps organize the computation.

1.24. If x, y, z are real numbers, show that the expression

$$x^2+y^2+x^2-xy-yz-zx$$

is zero if and only if $x=y=z$.

Solution. We use the results of Problem 1.23. We can write:

$$x^2+y^2+x^2-xy-yz-zx=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right),$$

The long factor in the right-hand side is the sum of three squares. Since a square is never negative, it can be zero if and only if each addend is zero. This means that $x=y=z$.

1.25. In a triangle whose sides have lengths a, b, c, with $a<b<c$, show that

$$a^3+b^3+c^3+3abc>2c^3.$$

Solution. From Problems 1.22 and 1.23 we know that we can write

$$z^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left((x-y)^2+(y-z)^2+(z-x)^2\right).$$

Setting $x=a, y=b, z=-c$ in this identity, we get:

$$a^3+b^3-c^3+3abc=\frac{1}{2}(a+b-c)\left((a-b)^2+(b+c)^2+(-c-a)^2\right).$$

It is not hard to see that the right-hand side of this identity cannot be negative. Indeed, the second factor is a sum of squares, and so is non-negative. And the triangle inequality ensures us that $a+b-c>0$, so the whole product is non-negative. But this means that $a^3+b^3-c^3+3abc\geq 0$. Adding $2c^3$ to both sides gives us the required result.

1.26. (This problem requires some knowledge of complex numbers.) If we have complex numbers v, w, z such that $v^2+w^2+z^2-vw-wz-zv=0$, show that either $v=w=z$ or the points represented by v, w, z in the complex plane form an equilateral triangle.

Solution. This problem requires somewhat advanced knowledge. If you don't know about completing the square or the geometry of complex numbers, perhaps it will whet your appetite.

We multiply the given equation by 2, and use the result of Problem 1.23 to get the equivalent condition

$$(v-w)^2+(w-z)^2+(z-v)^2=0. \tag{1.1}$$

Now (here's the trick!) we look at just the first two terms, and complete the square:

$$\begin{aligned}(v-w)^2+(w-z)^2 &= (v-w)^2+2(v-w)(w-z)\\ &\quad +(w-z)^2-2(v-w)(w-z)\\ &= [(v-w)+(w-z)]^2-2(v-w)(w-z)\\ &= (v-z)^2-2(v-w)(w-z).\end{aligned}$$

Substituting in equation (1.1), we find $2(z-v)^2-2(v-w)(w-z)=0$, or $(z-v)^2=(v-w)(w-z)$ (for three complex numbers satisfying the given condition).

Taking absolute values of these complex numbers, we find

$$|z-v|^2=|v-w||w-z|.$$

In the same way, we can show that $|v-w|^2=|w-z||z-v|$. Now we look at the three (non-negative) numbers $|v-w|, |w-z|, |z-v|$. We can show that if even one of them is zero, then they are all zero. For example, suppose $|v-w|=0$, so that $v=w$. Then

$$|v-w|^2=0=|w-z||z-v|=|v-z||z-v|=|z-v|^2,$$

so $z=v$ as well, and all three are equal. An analogous (i.e., *symmetric*) argument shows that if any other of the absolute values of the differences are 0, then they are all 0.

But if none of the absolute values of the differences are 0, we can show that they must be equal. For example, we can show that $|z-v|=|v-w|$:

$$|z-v|^2=|v-w||w-z| \quad \text{and} \quad |v-w|^2=|w-z||z-v|,$$

thus

$$\frac{|z-v|^2}{|v-w|^2}=\frac{|v-w|}{|z-v|}.$$

(By forming these fractions, we use the fact that $|z-v|$ is not zero.) Clearing fractions, we have $|z-v|^3=|v-w|^3$, or (since the cubes are real numbers) $|z-v|=|v-w|$. A similar argument holds for any other pair of these absolute values.

Now in the geometry of complex numbers, $|z-w|$ is just the length of the line segment between the points representing the complex numbers z and w. So a geometric interpretation of what we just proved is that the points representing v, w, z form an equilateral triangle.

Chapter 2

The Arithmetic-Geometric Mean Inequality, Part I

The theme of this chapter is a classic inequality called the *Arithmetic Mean-Geometric Mean Inequality* (which we will shorten to "AM-GM inequality"). This first sequence of problems concerns a version of the inequality for two variables. We will then generalize it to more than two variables.

Theorem 2.1. *The arithmetic mean of any two non-negative real numbers is greater than or equal to their geometric mean. The two means are equal if and only if the two numbers are equal.*

In other words, if $a, b \geq 0$, then

$$\frac{a+b}{2} \geq \sqrt{ab},$$

and

$$\frac{a+b}{2} = \sqrt{ab}$$

if and only if $a = b$.

Proof. The square of a real number cannot be negative. Therefore

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

But this means that

$$\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab} \geq 0,$$

or

$$a + b \geq 2\sqrt{ab},$$

which gives us the required result. Equality holds only when $\left(\sqrt{a} - \sqrt{b}\right)^2 = 0$ which occurs when $a = b$.

Often the AM-GM inequality is used to compare a product to a sum, or to transform one into the other. Watch how this theme unfolds.

Problems

2.1. Figure 2.1 shows a semicircle with center O. Its diameter has been divided at point X into two segments of lengths $AX = a$ and $XB = b$. Which is larger, OP or XY?

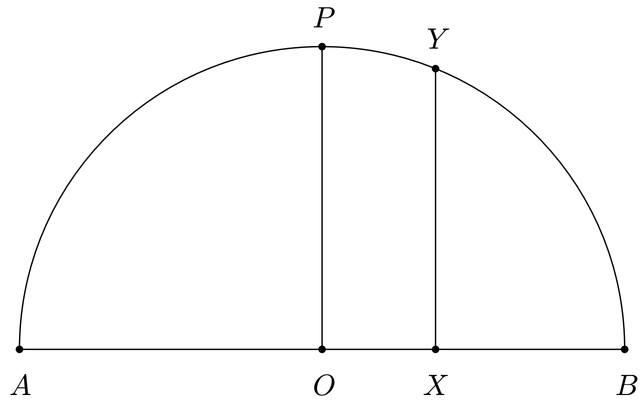

Figure 2.1

2.2. In trapezoid $ABCD$, segment MN connects the midpoints of legs AD and BC. Segment XY divides the trapezoid into two smaller trapezoids similar to each other. Figure 2.2 shows XY closer to the smaller base than to the larger base, and therefore smaller than MN. Is this correct?

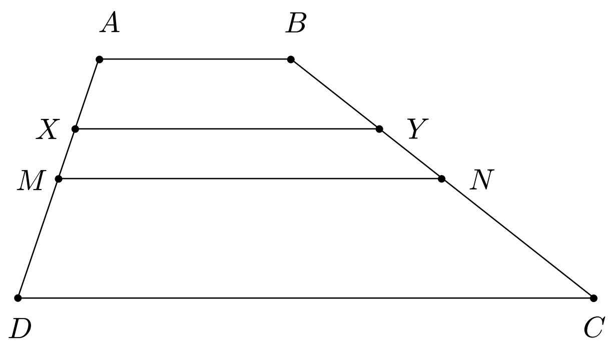

Figure 2.2

2.3. A rectangle has perimeter 20. What is its largest possible area?

2.4. A rectangle has area 100. What is its smallest possible perimeter?

2.5. Generalize the solutions to Problems 2.3 and 2.4 to show that
(a) if the sum of two positive numbers is constant, then their product is maximal when they are equal, and
(b) if the product of two positive numbers is constant, then their sum is minimal when they are equal.

2.6. During the 12 days of Christmas (in the old song), you receive not just 1 partridge in a pear tree, but 12: the gift of 1 partridge is repeated for each of 12 days. And you receive not just 2 turtle doves, but 22 turtle doves: a pair on each of the 12 days of Christmas, except the first. Finally, on the twelfth day, you receive 12 drummers drumming, but this gift is not repeated. Which gift do you receive the most of?

2.7. If x is a positive real number, find the smallest possible value of the expression

$$x + \frac{1}{x}.$$

2.8. If x is a positive real number, show that $2\sqrt{x} - x \leq 1$.

2.9. If x is a real number, find the largest possible value of the expression

$$(x + 4)(6 - x).$$

2.10. If $0 \leq x \leq \dfrac{\pi}{2}$, find the smallest possible value of $\tan x + \cot x$.

2.11. For any real number x, find the largest possible value of $(\sin^2 x)(\cos^2 x)$.

2.12. If x is a real number, find the minimum value of the expression $2^x + 2^{-x}$.

2.13. If x, y, and z are non-negative real numbers, show that

$$x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \leq xy + yz + xz.$$

2.14. If a, b, c, d are positive real numbers, show that

$$\sqrt{(a + c)(b + d)} \geq \sqrt{ab} + \sqrt{cd}.$$

2.15. Point D is chosen in the interior of angle $\angle ABC$. A variable line passes through D, intersecting ray BA at M and ray BC at N. Find the position of line MN that gives the smallest possible area for triangle MBN.

(For a hint to this rather difficult problem, glance at the diagram in the solution without reading the details.)

2.16. We start with n positive numbers $x_1, x_2, \ldots, x_n$, whose product is 1. Show that if we add 1 to each number, the product of the new numbers must be greater than or equal to 2^n.

2.17. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that

$$\sqrt{a_1 a_2} + \sqrt{a_2 a_3} + \ldots + \sqrt{a_{n-1} a_n} + \sqrt{a_n a_1} \leq a_1 + a_2 + \ldots + a_n.$$

2.18. Note that for $a \geq 0$, we have $\sqrt[4]{a} = \sqrt{\sqrt{a}}$. Show that if $a, b, c, d \geq 0$, then

$$\frac{a + b + c + d}{4} \geq \sqrt[4]{abcd},$$

and determine when equality occurs.

2.19. Show that if $a, b, c \geq 0$, then $\dfrac{a + b + c}{3} \geq \sqrt[3]{abc}$, and determine when equality occurs.

(Warning: You may find the proof for three variables more difficult than the corresponding proof for four variables.)

Solutions

2.1. Figure 2.1 shows a semicircle with center O. Its diameter has been divided at point X into two segments of lengths $AX = a$ and $XB = b$. Which is larger, OP or XY?

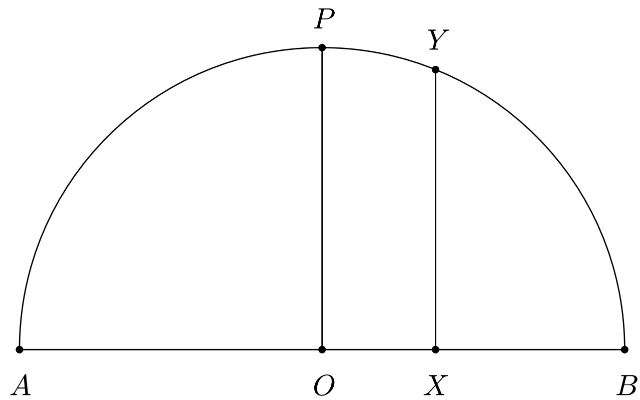

Figure 2.1

Solution. Segment XY is half of a certain chord, and OP is half of a diameter. Since a diameter of a circle is its longest chord, $OP > XY$. Analytically, this follows from (or can be considered a proof of) the main theorem, since $OP = \dfrac{a+b}{2}$ and $XY = \sqrt{ab}$.
If you're not sure why $XY = \sqrt{ab}$, note that AB is a diameter, so the angle subtended by AB at the circumference of the circle is a right angle. That is, $\angle AYB = 90°$. Also, $\angle AXY = \angle BXY = 90°$. So triangles AXY and YXB are similar. Hence

$$\frac{AX}{XY} = \frac{YX}{XB} \Rightarrow AX \cdot XB = (XY)^2 \Rightarrow XY = \sqrt{ab}.$$

2.2. In trapezoid $ABCD$, segment MN connects the midpoints of legs AD and BC. Segment XY divides the trapezoid into two smaller trapezoids similar to each other. Figure 2.2 shows XY closer to the smaller base than to the larger base, and therefore smaller than MN. Is this correct?

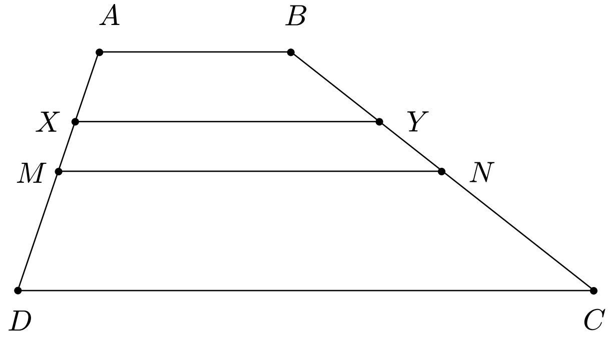

Figure 2.2

Solution. From a well-known theorem in geometry, we know that

$$MN = \frac{1}{2}(AB + CD),$$

the arithmetic mean of the bases. It is less well known that XY is the geometric mean of the bases. But it's not hard to prove.
Trapezoids $ABYX$ and $XYCD$ are similar, so their bases are in proportion: $AB : XY = XY : CD$, which means that $XY = \sqrt{AB \cdot CD}$. So our theorem tells us that $XY < MN$ and is thus closer to the smaller base.
Did you somehow think that any line parallel to the bases of a trapezoid divides into two similar trapezoids? The argument above shows that this is *not* true.

2.3. A rectangle has perimeter 20. What is its largest possible area?

Solution. If the length and width of the rectangle are denoted by a and b respectively, then we have $a + b = 10$, and we must find the maximum of ab. But the AM-GM inequality says

$$2\sqrt{ab} \leq a + b = 10,$$

so that

$$ab \leq \left(\frac{a+b}{2}\right)^2 = 25.$$

A quick check will show that if $a = b = 5$, then the maximum is achieved. In this case, the rectangle is a square.

2.4. A rectangle has area 100. What is its smallest possible perimeter?

Solution. If the length and width of the rectangle are denoted by a and b respectively, then we have $ab = 100$, and we must find the minimum of $2a + 2b$, or equivalently, the minimum of $a + b$. Again, the AM-GM inequality says $a + b \geq 2\sqrt{ab} = 20$, with equality if and only if $a = b = 10$. The shape of this rectangle of minimal perimeter is (again) a square, and its perimeter is 40.

2.5. Generalize the solutions to Problems 2.3 and 2.4 to show that
(a) if the sum of two positive numbers is constant, then their product is maximal when they are equal, and
(b) if the product of two positive numbers is constant, then their sum is minimal when they are equal.

Solution. The generalizations are immediate.

(a) If $a + b$ is constant, then $\left(\frac{a+b}{2}\right)^2$ is also constant, and is an upper bound for ab. The two expressions are equal if and only if $a = b$.

(b) If ab is constant, then $2\sqrt{ab}$ is also constant, and this is a lower bound for $a + b$, achieved also when $a = b$.

2.6. During the 12 days of Christmas (in the old song), you receive not just 1 partridge in a pear tree, but 12: the gift of 1 partridge is repeated for each of 12 days. And you receive not just 2 turtle doves, but 22 turtle doves: a pair on each of the 12 days of Christmas, except the first. Finally, on the twelfth day, you receive 12 drummers drumming, but this gift is not repeated. Which gift do you receive the most of?

Solution. On day n you receive n of a certain gift. And you receive that gift for a total of m days, where $m = 13 - n$. (Try it for $n = 4$, if you're not sure where the number 13 came from.) So the number of gifts of a given type that you receive is mn, where $m + n = 13$. So the maximal value for mn occurs when $m = n = 6.5$. However, m and n, from the song, must be integers. So the gift you receive the most of can be determined by finding when the values of m and n are as close to 6.5 as possible; that is, when $n = 6$ or $n = 7$. (These numbers of gifts are the same.)
Consulting the song, we find that these gifts are "geese a-laying" and "swans a-swimming".

2.7. If x is a positive real number, find the smallest possible value of the expression
$$x + \frac{1}{x}.$$

Solution. Since the product of x and $\frac{1}{x}$ is constant (it is 1), their sum is minimal when they are equal, which occurs when $x = \frac{1}{x} = 1$. Because x is positive, the smallest possible value of the expression is 2.

2.8. If x is a positive real number, show that $2\sqrt{x} - x \leq 1$.

Solution. This looks different from the previous problem. But we can make it look the same if we rewrite it so that it compares a sum (rather than a difference) to a product, which is what the AM-GM inequality does for us. Here, we can write $1 + x \geq 2\sqrt{x}$. Then, letting $a = 1$ and $b = x$ in the AM-GM inequality, we have our result.

We could also have utilized what we learned in the previous chapter to get:
$$1 - (2\sqrt{x} - x) = x + 1 - 2\sqrt{x} = (\sqrt{x} - 1)^2 \geq 0,$$
which also proves the inequality.

2.9. If x is a real number, find the largest possible value of the expression
$$(x + 4)(6 - x).$$

Solution. One could, of course, multiply this out, get a quadratic function in x, and use some standard techniques for finding the maximum. However, we can also note that $(x+4)+(6-x) = 10$, a constant, so the product of the two numbers is maximal when they are equal. This occurs when $x = 1$, and the largest possible value of the expression is 5.
Note that we have not (yet) violated the condition of the AM-GM inequality that requires both numbers to be positive. However, one might ask if we could get a still larger product if either term were negative, a

situation not covered by the AM-GM inequality. But of course, in this case, the product is negative, and our maximum value is larger. The reader is invited to explore the situation for expressions of the form $(x-a)(b-x)$ for various values of a and b.

2.10. If $0 \le x \le \dfrac{\pi}{2}$, find the smallest possible value of $\tan x + \cot x$.

Solution. On the domain indicated, and for any real number x for which $\tan x$ and $\cot x$ are defined, we have $(\tan x)(\cot x) = 1$. Thus their sum is minimal when $\tan x = \cot x$, which occurs when $\tan x = 1$. The required minimum value is 2.

2.11. For any real number x, find the largest possible value of $(\sin^2 x)(\cos^2 x)$.

Solution. The sum $\sin^2 x + \cos^2 x$ is constant (it equals 1). So the largest value of the given product occurs when they are equal; for example, when $x = \dfrac{\pi}{4}$. This largest value is $\dfrac{1}{4}$. Note that this implies, for $0 < x < \dfrac{\pi}{2}$, that the largest value of $\sin x \cos x$ is $\dfrac{1}{2}$. This result leads to another solution when we note that $\sin 2x = 2\sin x \cos x$, so that $\sin x \cos x = \dfrac{1}{2}\sin 2x$, whose maximal value is $\dfrac{1}{2}$.

2.12. If x is a real number, find the minimum value of the expression $2^x + 2^{-x}$.

Solution. The product $(2^x)(2^{-x})$ is constant (it is 1), so the expression is minimal when $2^x = 2^{-x}$, which occurs when $x = 0$. The minimal value is 2.

If we consider the related expression $\dfrac{e^x + e^{-x}}{2}$ (where the number e is the base of the natural logarithm), then we are studying the function $y = \cosh x$ (the hyperbolic cosine of x), which is of importance in engineering and theoretical work. Its minimal value over the real numbers is 1.

2.13. If x, y, and z are non-negative real numbers, show that

$$x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \le xy + yz + xz.$$

Solution. The square roots on the left side of the given inequality are an open invitation to apply the AM-GM inequality. We have

$$\begin{aligned} x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} &\le x\left(\frac{y+z}{2}\right) + y\left(\frac{x+z}{2}\right) + z\left(\frac{x+y}{2}\right) \\ &= xy + yz + zx. \end{aligned}$$

If only all our estimates would fall out so neatly! The two expressions are certainly equal when $x = y = z$. But are there any other possibilities for equality?

2.14. If a, b, c, d are positive real numbers, show that

$$\sqrt{(a+c)(b+d)} \ge \sqrt{ab} + \sqrt{cd}.$$

Solution. You can try writing the square roots as sums immediately, but it probably won't work. In this case, it is easier to square both sides and work with the equivalent inequality

$$(a+c)(b+d) \geq ab + cd + 2\sqrt{abcd}.$$

This simplifies to $ad + bc \geq 2\sqrt{abcd}$ and now we can use the AM-GM inequality directly.

2.15. Point D is chosen in the interior of angle $\angle ABC$. A variable line passes through D, intersecting ray BA at M and ray BC at N. Find the position of line MN that gives the smallest possible area for triangle MBN.

(For a hint to this rather difficult problem, glance at the diagram in the solution without reading the details.)

Solution. As shown in Figure 2.3, we draw line DE parallel to BC (with E on ray BA), and line DF parallel to AB (with F on ray BC). Then we draw segment EF. Let S_1 denote the area of triangle MED, S_2 the area of triangle DFN, and S the area of triangle DEF. Then, because diagonal EF bisects the area of parallelogram $EDFB$, the area of triangle EFB is also S. So the area to minimize is $S_1 + S_2 + 2S$.

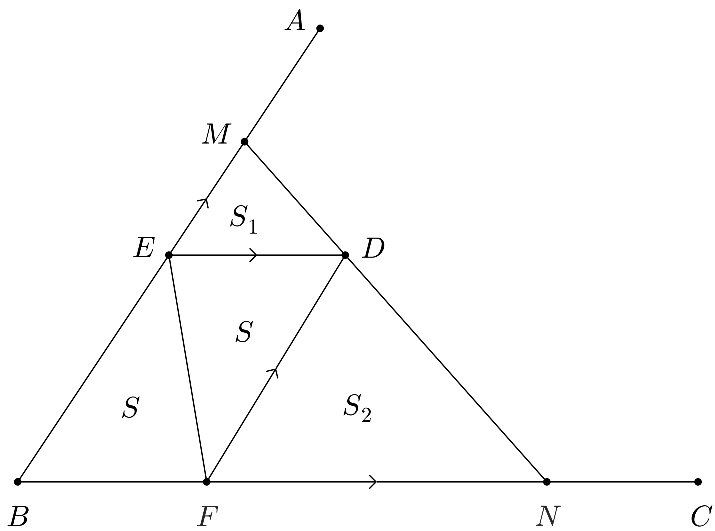

Figure 2.3

Notice that the positions of lines DE and DF, and thus the value of S, do not depend on the position of line MN, so we only need to minimize the sum $S_1 + S_2$. How shall we do this? The placement of this problem in a chapter on the AM-GM inequality provides a hint. We should look to see if the product S_1S_2 is constant. So let us inquire into relationships among the triangles we've drawn.

Triangles MED and EBF have equal altitudes from points D and F, since DF is parallel to AB, so the ratio of their areas is just the ratio of their bases, or

$$\frac{S_1}{S} = \frac{ME}{EB}.$$

Similarly, the ratio of the areas of triangles DFN and EFB is equal to the ratio of their bases, or

$$\frac{S_2}{S} = \frac{FN}{BF}.$$

Finally, since triangles MED and MBN are similar, the ratio

$$\frac{ME}{EB} = \frac{MD}{DN}.$$

Likewise, since triangles DFN and MBN are similar, we have

$$\frac{FN}{BF} = \frac{DN}{MD}.$$

(This is also true because parallel lines intercept proportional segments on any transversal.) Therefore

$$\frac{S_1}{S} \cdot \frac{S_2}{S} = \frac{ME}{EB} \cdot \frac{FN}{BF} = \frac{MD}{DN} \cdot \frac{DN}{MD} = 1.$$

Thus $S_1 S_2 = S^2$, a constant, and $S_1 + S_2$ is minimal when $S_1 = S_2 = S$. This happens when MN is parallel to EF.

2.16. We start with n positive numbers $x_1, x_2, \ldots, x_n$, whose product is 1. Show that if we add 1 to each number, the product of the new numbers must be greater than or equal to 2^n.

Solution. We know $x_1, x_2, \ldots, x_n = 1$, and we want to show that

$$(1 + x_1)(1 + x_2) \ldots (1 + x_n) \geq 2^n.$$

We can use the AM-GM inequality to transform each sum on the left to a product. We have

$$1 + x_1 \geq 2\sqrt{1 \cdot x_1} = 2\sqrt{x_1},$$

and similarly for the other factors. Multiplying, we find that

$$(1 + x_1)(1 + x_2) \ldots (1 + x_n) \geq (2\sqrt{x_1})\,(2\sqrt{x_2}) \ldots (2\sqrt{x_n}) = 2^n \cdot 1,$$

which is the result we need.

2.17. For n non-negative numbers $a_1, a_2, \ldots, a_n$, show that

$$\sqrt{a_1 a_2} + \sqrt{a_2 a_3} + \ldots + \sqrt{a_{n-1} a_n} + \sqrt{a_n a_1} \leq a_1 + a_2 + \ldots + a_n.$$

Solution. We have

$$\begin{aligned}
\sqrt{a_1 a_2} &\leq \frac{a_1 + a_2}{2}, \\
\sqrt{a_2 a_3} &\leq \frac{a_2 + a_3}{2}, \\
&\vdots \\
\sqrt{a_{n-1} a_n} &\leq \frac{a_{n-1} + a_n}{2}, \\
\sqrt{a_n a_1} &\leq \frac{a_n + a_1}{2}.
\end{aligned}$$

Adding, and noting that there are two copies of each addend a_i in the right-hand column, we have the required result.

2.18. Note that for $a \geq 0$, we have $\sqrt[4]{a} = \sqrt{\sqrt{a}}$. Show that if $a, b, c, d \geq 0$, then

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd},$$

and determine when equality occurs.

Solution. We have

$$\begin{aligned}\frac{a+b+c+d}{4} &= \frac{(a+b)+(c+d)}{4} \\ &= \frac{1}{2} \cdot \frac{a+b}{2} + \frac{1}{2} \cdot \frac{c+d}{2} \\ &\geq \frac{1}{2}\sqrt{ab} + \frac{1}{2}\sqrt{cd} \\ &= \frac{\sqrt{ab}+\sqrt{cd}}{2} \\ &\geq \sqrt{\sqrt{abcd}} \\ &= \sqrt[4]{abcd}.\end{aligned}$$

Equality occurs when $a = b = c = d$.

2.19. Show that if $a, b, c \geq 0$, then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$, and determine when equality occurs.

(Warning: You may find the proof for three variables more difficult than the corresponding proof for four variables.)

Solution I. We can apply the result of 2.18 by reducing four variables to three. When is the average (arithmetic mean) of four variables equal to the arithmetic mean of three of them? When the missing (fourth) variable is equal to the average of the other three! So we apply the result of 2.18 to the four numbers a, b, c, $\dfrac{a+b+c}{3}$. We have, as expected:

$$\begin{aligned}\frac{a+b+c+\dfrac{a+b+c}{3}}{4} &= \frac{\dfrac{3a}{3}+\dfrac{3b}{3}+\dfrac{3c}{3}+\dfrac{a+b+c}{3}}{4} \\ &= \frac{4a+4b+4c}{12} = \frac{a+b+c}{3}.\end{aligned}$$

And we know this is greater than or equal to $\sqrt[4]{abc\left(\dfrac{a+b+c}{3}\right)}$.

We must now reduce the fourth root to a cube root. We can do this by recognizing that the expression $\dfrac{a+b+c}{3}$ is on both sides of our inequality. That is, letting $\beta = \dfrac{a+b+c}{3}$, we have

$$\beta \geq \sqrt[4]{abc\beta}, \quad \text{or} \quad \beta^4 \geq abc\beta, \quad \text{or} \quad \beta^3 \geq abc,$$

which is equivalent to what we wanted to prove. Whew!

Solution II. This solution is nicer, but more difficult to think of. We use the results from Problems 1.22 and 1.23.

From those problems, we know that

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\left((x-y)^2 + (y-z)^2 + (z-x)^2\right).$$

If $x, y, z \geq 0$, then the right-hand side of this identity is certainly non-negative. But this means that $x^3 + y^3 + z^3 - 3xyz \geq 0$.

Now we can substitute $x = \sqrt[3]{a}, y = \sqrt[3]{b}, z = \sqrt[3]{c}$, to get

$$a + b + c - 3\sqrt[3]{abc} \geq 0,$$

which gives us the inequality we want. Equality holds when the three squares $(x-y)^2, (y-z)^2, (z-x)^2$ are all zero, which implies that $a = b = c$.

But without Problems 1.22, 1.23, how would you have thought of this?

Chapter 3

The Arithmetic-Geometric Mean Inequality, Part II

We begin this chapter with an interlude: a story-proof of the AM-GM inequality for any set of non-negative numbers. We then look at problems using the AM-GM inequality for three variables. Many of these generalize to any number of variables in ways that offer no difficulty, once the reader has worked the case $n = 3$. Finally, we offer a few advanced problems using the AM-GM inequality.

Interlude: Cauchy's Great-Granddaughter

We have looked at the AM-GM inequality in two variables: that is, if $a, b \geq 0$, then $\sqrt{ab} \leq \dfrac{a+b}{2}$. We have also looked at the corresponding inequalities for three numbers, and four numbers:

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3},$$

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}.$$

It is in fact true generally that for any n:

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

An early proof of this statement is credited to Augustin-Louis Cauchy, and is one of the most beautiful in the literature. But we search in vain for Cauchy's motivation in doing this. In the best mathematical tradition, Cauchy was given to hiding his scratch work and showing only his results.

And, in fact, while some mathematicians (Poincaré, Pólya) did allow us into their mental workshop, Cauchy did not. Cauchy was not exactly a people person. He was deeply conservative in his habits and beliefs. (He followed the French King Charles X into exile after the revolution of 1830.) So even if we could conjure him up, it is not likely he would reveal his mind to us.

Instead, let us conjure up his great-great-great-granddaughter, Augusta-Louise Cauchy[1].

Her cumbersome first name is an issue. She never liked it. Her parents call her Gussie-Lou, but she bridles at that. Her grandparents wanted "Moon Unit", not realizing how it dated them. And "Augie" sounds like a cartoon character. We'll settle for "Augusta-Louise".

Augusta-Louise is a junior in an American high school. But we should let her talk:

"It's so embarrassing. My math teachers always tell everyone about my last name. And they pronounce it wrong. It's co-SHEE, not CO-shee. The French teacher, Mme. de Trop, gets it right, but I don't take French. Japanese.

"And it's a good thing I'm OK in math, or I'd never hear the end of it. 'CO-shee this. CO-shee that. Why don't you live up to your name?' Thank goodness I'm spared those silly comments.

"So anyway, I have this extra-credit problem, to prove the AM-GM inequality. And the teacher said that CO-shee already proved it. OK. So I have to live up to my name. I can do this.

"The teacher said to use induction. In fact, he said 'It's a simple induction'. Right. As if induction is simple. OK. Let's look.

"We know that $\sqrt{ab} \leq \dfrac{a+b}{2}$. So that's the base of the induction. How do we get from there to $\sqrt[3]{abc}$? Well, the square root of the square root. . . .

"No. That won't work. The square root of the square root is the fourth root."

(Augusta-Louise is thinking: $\sqrt{\sqrt{x}} = \sqrt[4]{x}$, or $\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{4}}$.)

"Well, it sort of helps. Because we can show that the case for $n = 2$ implies the case for $n = 4$. I'll write that down, so at least I'll get partial credit. Let's see.

"We know $\sqrt{ab} \leq \dfrac{a+b}{2}$, for any a and b. We want the same thing for four letters a, b, c, d. But I'm going to get confused between the 'old' a, b and the 'new' a, b. So let's write what we want with a_1, a_2, a_3, a_4. I know it's going to have to be subscripts at some point.

"We want $\sqrt[4]{a_1a_2a_3a_4} \leq \dfrac{a_1 + a_2 + a_3 + a_4}{4}$.

"So we let $a = \sqrt{a_1a_2}$, $b = \sqrt{a_3a_4}$. Luckily, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. So I can write:

$$\sqrt[4]{a_1a_2a_3a_4} = \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} \leq \frac{\sqrt{a_1a_2} + \sqrt{a_3a_4}}{2}.$$

"Now what? The left side is OK, but the right side? Looks funny.

"Oh. Oh. Wait. We know about $\sqrt{a_1a_2}$. It's $\leq \dfrac{a_1 + a_2}{2}$. That's the induction hypothesis.

[1]There is no such person. Cauchy had two daughters, so any living descendants are likely to have a different family name.

"Can you use the induction hypothesis twice in the same proof? I guess you can. If we assume it's true, it can't suddenly become false. So I can write:

$$\begin{aligned}
\sqrt[4]{a_1a_2a_3a_4} &= \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} \\
&\leq \frac{\sqrt{a_1a_2}+\sqrt{a_3a_4}}{2} \\
&\leq \frac{\dfrac{a_1+a_2}{2}+\dfrac{a_1+a_2}{2}}{2} \\
&= \frac{a_1+a_2+a_3+a_4}{4}.
\end{aligned}$$

"And it's neat how the algebra works.

"The same thing will get us from $n = 4$ to $n = 6$. NO! It will get us from $n = 4$ to $n = 8$. I don't know how to get $n = 6$. But it will also get us from $n = 8$ to $n = 16$. It really is just an easy induction. The algebra is the same. I'll need subscripts, which I hate. But OK. At least I proved it for powers of 2, for $n = 2^k$."

So Augusta-Louise (we can call her that, because she's not listening) wrote it all out, with subscripts. She had to be careful, but the same algebra, and the same logic, that brought her from 2 to 4 brought her from 2^k to 2^{k+1}. She showed her math teacher, who praised her. But Augusta-Louise wondered out loud how to get $n = 3$. The math teacher wouldn't tell her. "Go look in a textbook", he said.

So she did. And she saw the phrase 'backwards induction'.

"How can induction work backwards?" Augusta-Louise wondered. "Well, it can. Sort of. You can prove that if it's true for $n+1$ then it's true for n. Or, you could say the same thing by proving, 'If it's true for n, then it's true for $n-1$.' What good does that do?"

Augusta-Louise sat and thought about it for a while.

Then: "OMG! That would work! If I can prove that n implies $n-1$, then from 16 we can get to 15, then to 14, and so on. I mean, let's say it's right. From $n = 2^k$ we can get to $n = 2^k - 1$, then to $n = 2^k - 2$, and all the way down to 2^{k-1}. Then it also proves it from 2^{k-1} down to 2^{k-2}, and so on, all the way down to $n = 3$ and $n = 5$ and $n = 6$, which I was missing. That. Is. So. Cool.

"OK. Let's do it. I'll practice going from 6 to 5, then write it in general. We have to prove that:

$$\text{if } \sqrt[6]{a_1a_2a_3a_4a_5a_6} \leq \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6},$$

for any values of $a_1, a_2, \ldots, a_6$,

$$\text{then } \sqrt[5]{b_1b_2b_3b_4b_5} \leq \frac{b_1+b_2+b_3+b_4+b_5}{5}.$$

"OK. This is gonna be hard. How do we take the sixth root of six things, and make them into the fifth root of five things? It's gonna be hard.

"Wait. Wait. Last time I made the left sides the same. Maybe I should make the right sides the same this time. Let's see. The right side is just an average, like the average of five or six tests."

That analogy came easily to Augusta-Louise, as it does to most students.

"I know what to do. I can do what Mme. de Trop does. I helped her once. She has this computer program for averaging six tests. But if a student is absent, and took only five tests, she doesn't give a makeup. She asked me, 'What's a fair grade to enter for the sixth test?' And that was easy. You just enter the average of the five tests as the score for the sixth test. Everyone knows that."

Well, most people (except math teachers) don't know that, but usually understand it once a student like Augusta-Louise points it out to them.

"So: I can let $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$, $a_5 = b_5$ and try

$$a_6 = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5}.$$

With any luck, the algebra will work out."

(Of course it works out, or you wouldn't be reading this.)

"Let's see. I'm going to start writing the inequalities the 'other way', with the arithmetic mean on the left and a '$\geq$' sign in between. I find it's easier for me to think that way, and there's no real difference. We have:

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} = \frac{b_1 + b_2 + b_3 + b_4 + b_5 + \dfrac{b_1 + b_2 + b_3 + b_4 + b_5}{5}}{6}.$$

Daunting. But let's plug on. In terms of the b's, the left-hand side is:

$$\begin{aligned} &\frac{b_1 + b_2 + b_3 + b_4 + b_5 + \dfrac{b_1 + b_2 + b_3 + b_4 + b_5}{5}}{6} \\ &= \frac{5b_1 + 5b_2 + 5b_3 + 5b_4 + 5b_5 + b_1 + b_2 + b_3 + b_4 + b_5}{30} \\ &= \frac{6b_1 + 6b_2 + 6b_3 + 6b_4 + 6b_5}{30} = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5}, \end{aligned}$$

which is just what I told Mme. de Trop.

"OK. Now for the right-hand side. It doesn't look great. We know, from the a's, that

$$\begin{aligned} \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} &= \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5} \\ &\geq \sqrt[6]{a_1 a_2 a_3 a_4 a_5 a_6} \\ &= \sqrt[6]{b_1 b_2 b_3 b_4 b_5 \left(\frac{b_1 + b_2 + b_3 + b_4 + b_5}{5}\right)}. \end{aligned}$$

"But what do we do now? How do we turn a sixth root into a fifth root?

"Wait. The same thing appears on the left and the right. We can divide by it, in a funny way. Let's let

$$\beta = \frac{b_1 + b_2 + b_3 + b_4 + b_5}{5},$$

the arithmetic mean of the b's. Then all we really have is:

$$\beta \geq \sqrt[6]{b_1 b_2 b_3 b_4 b_5 \beta},$$

or

$$\beta^6 \geq b_1 b_2 b_3 b_4 b_5 \beta,$$

or

$$\beta^5 \geq b_1 b_2 b_3 b_4 b_5,$$

and

$$\beta \geq \sqrt[5]{b_1 b_2 b_3 b_4 b_5}.$$

"And that's what I need to prove. Yesss! Extra credit!"

Augusta-Louise smiled proudly at herself for a minute. She wrote it up in general, for any n. Then a look came across her face. "Why did that last step work? Was I just lucky?"

Maybe a reader can give a satisfactory answer to that. It's the one step in the proof that Augusta-Louise can't tell us about, any more than her great-great-great-grandfather can.

Problems

3.1. For positive numbers a, b, c, x, y, z show that

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}.$$

3.2. Show that if a, b, c are positive numbers such that $a + b + c = 1$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

3.3. More generally, for positive numbers a, b, c, show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}.$$

3.4. If $a_1, a_2, \ldots, a_n$ are n positive numbers, show that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2.$$

3.5. If $a_1, a_2, \ldots, a_n$ are positive numbers, show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} \geq n.$$

3.6. Let a, b, c, d be positive real numbers such that $a+b+c+d=1$. Prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\left(1+\frac{1}{d}\right) \geq 625.$$

Can you extend this result to n variables?

Solutions

3.1. For positive numbers a, b, c, x, y, z show that

$$\sqrt[3]{(a+x)(b+y)(c+z)} \geq \sqrt[3]{abc} + \sqrt[3]{xyz}.$$

Solution. We can cube both sides, to get the equivalent inequality

$$(a+x)(b+y)(c+z) \geq \left(\sqrt[3]{abc} + \sqrt[3]{xyz}\right)^3.$$

Then we transform the two sides of the inequality separately. We have:

$$\begin{aligned}(a+x)(b+y)(c+z) &= (a+x)(bc+bz+yc+yz) \\ &= abc+abz+acy+ayz+bcx+bxz+cxy+xyz \\ &= abc+xyz+abz+acy+ayz+bcx+bxz+cxy.\end{aligned}$$

On the right-hand side we have

$$\left(\sqrt[3]{abc} + \sqrt[3]{xyz}\right)^3 = abc+xyz+3\left(\sqrt[3]{a^2b^2c^2xyz} + \sqrt[3]{abcx^2y^2z^2}\right).$$

Comparing the two results, we see that the inequality

$$abz+acy+ayz+bcx+bxz+cxy \geq 3\left(\sqrt[3]{a^2b^2c^2xyz} + \sqrt[3]{abcx^2y^2z^2}\right)$$

is again equivalent to the one we want to prove. This (finally!) resembles a previous result, that of Problem 2.19. We can separate the six terms on the left into two sets of three so that their products are just what is underneath each radical, and from Problem 2.19, we have:

$$\frac{abz+acy+bcx}{3} \geq \sqrt[3]{a^2b^2c^2xyz},$$

$$\frac{ayz+bxz+cxy}{3} \geq \sqrt[3]{abcx^2y^2z^2}.$$

Adding these two inequalities, and following the argument backwards, we obtain the required result.

Notes:

(a) Can you really cube both sides of an inequality to get an equivalent inequality? Yes, because $y = x^3$ is a continuous function which is always increasing. In any such situation, you can take the functional value of both sides of inequality to get an equivalent inequality.

(b) Do you see why this principle does not allow us always to square both sides of an inequality? We don't always get an equivalent inequality, because the argument will not run backwards.

(c) Go back and look at the argument to see that the six variables concerned all entered into the original inequality symmetrically, and that we treated them symmetrically at each step.

3.2. Show that if a, b, c are positive numbers such that $a+b+c=1$, then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9.$$

Solution. Using the AM-GM inequality for three variables, we can write:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

Since $a+b+c=1$, this can be written as

$$\frac{1}{\sqrt[3]{abc}} \geq 3.$$

Using this inequality, and applying the AM-GM inequality a second time, we have

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \frac{1}{\sqrt[3]{abc}} \geq 3.$$

The result follows.

3.3. More generally, for positive numbers a, b, c, show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c}.$$

Solution. From the AM-GM inequality, we have:

$$a+b+c \geq 3\sqrt[3]{abc}$$

and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}} = \frac{3}{\sqrt[3]{abc}}.$$

Multiplying these two inequalities, we get the desired result. Equality holds occurring only when $a=b=c$.

Notes: This is a tricky problem. The hint is that it is placed here, right after the AM-GM inequality for three variables. The sum of three addends gives us another clue. The solution is essentially motivated by comparing each side to $\sqrt[3]{abc}$.

Were the variables used symmetrically in this (very tricky) solution?

3.4. If $a_1, a_2, \ldots, a_n$ are n positive numbers, show that

$$(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \geq n^2.$$

Solution. From the general AM-GM inequality, we have:

$$a_1+a_2+\ldots+a_n \geq n\sqrt[n]{a_1a_2\ldots a_n}$$

and

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geq \frac{n}{\sqrt[n]{a_1 a_2 \ldots a_n}}.$$

We get the required result by multiplying these two inequalities, with equality just when $a_1 = a_2 = \ldots = a_n$.

3.5. If $a_1, a_2, \ldots, a_n$ are positive numbers, show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} \geq n.$$

Solution. By the generalized AM-GM inequality, we have:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} \geq n \sqrt[n]{\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \ldots \frac{a_n}{a_1}} = n \sqrt[n]{1} = n.$$

The case for equality is not the usual. The two expressions are equal when

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_n}{a_1} = 1.$$

3.6. Let a, b, c, d be positive real numbers such that $a+b+c+d = 1$. Prove that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right)\left(1 + \frac{1}{d}\right) \geq 625.$$

Can you extend this result to n variables?

Solution. Since this is a chapter on the AM-GM inequality, our first instinct might be to write $1 + \frac{1}{a} \geq 2\sqrt{\frac{1}{a}}$, and rewrite the other terms similarly. This turns out not to work too well, as we might suspect. First of all, this set of problems is about the AM-GM inequality for more than two variables. But more mathematically, equality holds when $a = 1$, which leaves no 'room' for the other variables: The four of them sum to 1.

But looking at the case for equality will give us a hint. Why does the inequality we desire have the number 625? Well, this is 5^4, and if all the factors on the left were equal, they would all have to be equal to 5. And in fact if $a = b = c = d = \frac{1}{4}$, the inequality does turn into an equality.

We can continue to think of the case for equality, as splitting $1 + \frac{1}{a}$ into five pieces, all of which are equal if $a = b = c = d$. Since $a + b + c + d = 1$, we can write:

$$1 + \frac{1}{a} = 1 + \frac{a+b+c+d}{a} = 1 + 1 + \frac{b}{a} + \frac{c}{a} + \frac{d}{a} \geq 5\sqrt[5]{\frac{bcd}{a^3}},$$

by the AM-GM inequality for several variables. Similarly,

$$1 + \frac{1}{b} \geq 5\sqrt[5]{\frac{cda}{b^3}},$$

$$1+\frac{1}{c}\geq 5\sqrt[5]{\frac{dab}{c^3}},$$
$$1+\frac{1}{d}\geq 5\sqrt[5]{\frac{abc}{d^3}}.$$

Multiplying these four inequalities yields the conclusion.

And the result can easily be extended: If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1+a_2+\cdots+a_n=1$, then

$$\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\ldots\left(1+\frac{1}{a_n}\right)\geq (n+1)^n.$$

Chapter 4

The Harmonic Mean

Introduction

Here are a few typical "word problems" from algebra textbooks.

Example 4.1. Dora walked to school, a distance of 1 mile, at a rate of 5 miles per hour. She got a lift home in a car which went 15 miles per hour. What was her average rate for the round trip?

Solution. Most people, without thinking much, would say that the average rate is $\frac{5+15}{2} = 10$ MPH. What is wrong with this?

Well, let us look at how much time she took going and coming. Going to school she took $\frac{1}{5}$ hour $= 12$ minutes walking. (In algebra, $T = \frac{D}{R} = \frac{1}{5}$ hours, but it's easier than that if you think arithmetically.) Coming home she took $\frac{1}{15}$ hour $= 4$ minutes. So altogether she took $12 + 4 = 16$ minutes $= \frac{4}{15}$ hours.

A reasonable way to think of the "average rate" is to say that if she somehow managed to go and come at this same average rate r (say, by bicycle), then the round trip at rate r should take her the same time as the trip coming and going at two different rates. (Stop and think if you agree with this idea of what "average" means.)

Time is distance divided by rate. So if she travels at this average rate r, she will be going a distance of 2 miles, and $\frac{2}{r} = \frac{4}{15}$, so $r = \frac{15}{2}$ or 7.5 MPH. (You can check that this works: if she goes 1 mile to school and 1 mile back at this rate, she will spend 16 minutes $= \frac{4}{15}$ hours traveling.)

Note that this is less than the "naïve average" of 10 MPH. This makes sense: Dora is spending much more time walking than riding, and somehow the "average" we take should reflect this.

The point is that when we talk about "averaging" rates, we are not talking about the same operation as "averaging" test grades. When we average two test grades, we are asking what single grade on both tests would measure the same achievement as two different grades. This kind of average is the arithmetic mean of the two grades.

When we average two rates (over the same distance) we are asking which single rate would make up, in time, for the two different rates. This kind of average is called the *harmonic mean* of the two rates. (The term "average" is not really a mathematical term. It means any concept of a "typical" or "usual" value. How we actually compute this "typical" value depends on how we use the value.)

Let us look at this harmonic mean in general. Suppose we have two trips of the same distance d miles (in the example above, the fact that we had a round trip guaranteed that the two distances were equal). Suppose one trip is made at the rate f (for fast) MPH and another at the rate s (for slow) MPH. Let us compute the time taken in various legs of the journey.

For the fast leg, the time is $\frac{d}{f}$. For the slow leg, the time is $\frac{d}{s}$. And if the full journey (a distance $2d$) is taken at an average rate r, the time is $\frac{2d}{r}$. So we must have:

$$\frac{d}{f} + \frac{d}{s} = \frac{2d}{r}.$$

But look! The d's cancel out. (The average rate should be the same whether we go 1 mile or 100 miles.) And we get:

$$\frac{2}{r} = \frac{1}{f} + \frac{1}{s}.$$

This is the arithmetic definition of the harmonic mean of two numbers. We can rewrite it in several ways. One way is:

$$\frac{1}{r} = \left(\frac{1}{2}\right)\left(\frac{1}{f} + \frac{1}{s}\right).$$

That is, the reciprocals (multiplicative inverses) of the rates are averaged in the usual way: The reciprocal of r is half the sum of the reciprocals of f and s. This make algebraic sense, because when we compute the time of a particular trip, the rates appear in the denominator, not in the numerator. So we sort of have to "average the denominators", which requires this unusual arithmetic.

Another way to write the harmonic mean is:

$$r = \frac{2}{\frac{1}{f} + \frac{1}{s}}. \tag{4.1}$$

Yet another way to write this expression, which is not nearly so informative, is:

$$r = \frac{2fs}{f+s}.$$

This is sometimes given as the formal definition of the harmonic mean.

You can memorize the formula for the harmonic mean, and make average rate problems into plug-ins. But you will probably get more confused than if you think them through more logically.

It's more than just $D = RT$

The harmonic mean shows up in solving many types of verbal problems, and not just in $D = RT$ problems. For example, let us look at a problem about filling a swimming pool.

Example 4.2. One pipe can fill a swimming pool in 2 hours. Another can fill it in 3 hours. If both pipes are turned on, how long will it take to fill the pool?

Solution. Let us again look at a "naïve" solution. Well, the easiest, and most wrong solution, is to say that the first pipe takes 2 hours and the second pipe takes 3 hours, so together it will take them 5 hours. This is silly. How can it take them longer working together than either took alone? A better — but still naïve and wrong — solution is to say that if they work together they do 5 hours "worth" of work, so each one has to do $5/2 = 2.5$ hours of work, and it will take them that long to fill the tank. Note that this is the "testing average" (the mathematical term is the *arithmetic mean*) of the two rates.

But in fact the faster pipe does more work. Thus the two pipes shouldn't be counted equally in the "average". What is going on? We can see this a little more clearly if we think of this as a problem about rates. The first pipe fills the pool in two hours, so in one hour it fills $\frac{1}{2}$ the pool. That is, it fills pools at a rate of $\frac{1}{2}$ pool per hour. The second pipe fills the pool in three hours, so it fills pools at the rate of $\frac{1}{3}$ pool per hour. We want to find an average rate at which a pipe could fill the pool which matches these two rates.

Now we can add: together, the two pipes fill $\frac{1}{2}+\frac{1}{3}$ of the pool, or $\frac{5}{6}$ of the pool, in one hour. What rate does this correspond to? That's asking how many times does $\frac{5}{6}$ fit into 1 (pool), and that's just division. The answer is $\frac{1}{\frac{5}{6}}$ or $\frac{6}{5}$ of an hour, and this is the time it takes for both pipes to fill the pool together.

Note that this is half the harmonic mean of 2 and 3.

We can see what is going on if we compare this problem to $D = RT$ problems. In this one, we can think of the first pipe as filling the pool, from the bottom, at a rate of $\frac{1}{2}$ pool per hour. If you like, think of the water level traveling at this rate up the side of the pool.

Now for a little imagination. We can think of the second pipe as "filling" the pool from the top down. So the water coming out of this pipe magically floats at the top of the pool, and gets deeper as the pool fills. Using this model, the water level is traveling downwards, at a rate of $\frac{1}{3}$ pool per hour (since it would take 3 hours to travel all the way down).

So we have a $D = RT$ problem in which the two travelers are going towards each other. Let's make this more specific.

Example 4.3. Tom and Jerry live a mile apart. They start walking towards each other's house, at constant rates, along the same path. In t hours, Tom walks 1 mile, and in j hours, Jerry walks 1 mile. How long will it be before they meet?

Solution. As you might suspect, it's the same problem. Tom walks $\frac{1}{t}$ miles, that is, $\frac{1}{t}$ of the path, in one hour, while Jerry walks $\frac{1}{j}$ miles in one hour. So together, in one hour, they walk $\frac{1}{t} + \frac{1}{j} = \frac{t+j}{tj}$ miles of the path, or at that rate in miles per hour. How many hours does it take them to walk the path? That's just asking how many times $\frac{t+j}{tj}$ goes into 1 mile, which is $\frac{1}{\frac{t+j}{tj}} = \frac{tj}{t+j}$ hours.

Sound familiar? The algebra is just the same as when we were filling the pool. The answer is half the harmonic mean of the two rates. Half, because they are not traversing the path twice. Similarly, in filling the pool, the result is half the harmonic mean of the two rates because we are not filling then emptying the pool.

Example 4.4. Suppose you spend \$6 on pink pills costing a cents per dozen, and \$6 on blue pills costing b cents per dozen. What was the average price per dozen of the pills you've bought?

Solution. Sigh. It's going to be the harmonic mean, of course. But let's see why.

The average price will be the total amount spent divided by the total number of pills we bought. We spent \$12 altogether, but on how many pills? Well, we bought the pink pills in dozens, each one costing a cents. How many dozens did we buy? That's asking how many times a goes into 600: $\frac{600}{a}$. Similarly, the number of dozens of blue pills is $\frac{600}{b}$, and the total number of pills is the sum of these, or $\frac{600a + 600b}{ab}$. We must divide this into \$12 = 1200 cents to get our average, which is $\frac{1200ab}{600a + 600b} = \frac{2ab}{a+b}$ cents per dozen, the harmonic mean of a and b.

The reader is invited to check that if each dozen pills had cost this much, we would have spent the same amount of money.

Notes and Summary

For two positive numbers a and b, we call $\frac{2ab}{a+b}$ the *harmonic mean* of a and b.

Note that if h is the harmonic mean of a and b, we have $\dfrac{2}{h} = \dfrac{1}{a} + \dfrac{1}{b}$.

Note further that if $\dfrac{1}{k} = \dfrac{1}{a} + \dfrac{1}{b}$, then k is *half* the harmonic mean of a and b.

Problems

4.1. Show that the geometric mean of a and b is also the geometric mean of $\dfrac{a+b}{2}$ and $\dfrac{2ab}{a+b}$.

4.2. Show that if AM, GM, HM are respectively the arithmetic, geometric, and harmonic means of two positive numbers, then $HM \leq GM \leq AM$.

4.3. We have so far begged an important question. Is the harmonic mean of a and b always a mean? That is, is it always between a and b? Show that if $0 \leq a < b$, then $a < \dfrac{2ab}{a+b} < b$.

4.4. **a.** Look back at Figure 2.1, and connect O to Y as shown in Figure 4.1 below. Let point Z be the foot of the perpendicular from X to OY. Using the same segment lengths as given in Problem 2.1, show that YZ is the harmonic mean of a and b. Deduce geometrically that the harmonic mean of a and b is less than or equal to their arithmetic mean.

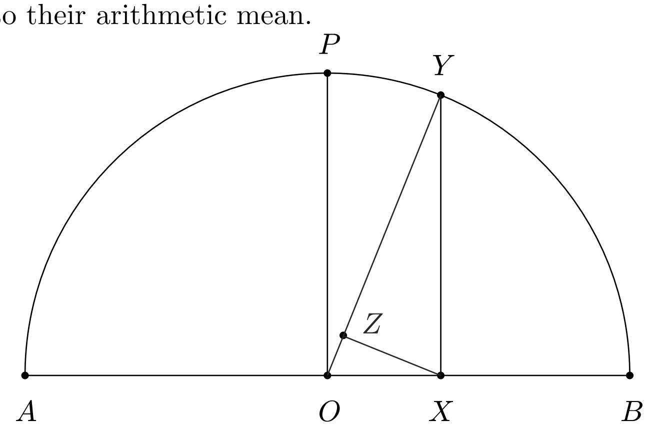

Figure 4.1

b. (This problem requires the 'triangle inequality', a geometric inequality which we haven't discussed in this volume.) Let h_a, h_b, h_c be the lengths of the altitudes of a triangle (with sides a, b, c). Prove that $2h_a$ is greater than the harmonic mean of h_b and h_c.

The Harmonic Mean of Several Quantities

Just as we can take the arithmetic or geometric mean of several quantities, we can also take the harmonic mean of several quantities. Recall that we can define the harmonic mean h of a and b by the equation

$$h = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

(i.e., equation (4.1)). Likewise, the harmonic mean h of a, b, and c is defined by:

$$h = \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

And in general, for n positive numbers $a_1, a_2, \ldots, a_n$, the harmonic mean is defined by:

$$h = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}.$$

And now we have some work.

Problems

4.5. Suppose we have two sets of n positive numbers each:

$$S_1 = \{a_1, a_2, a_3, \ldots, a_n\} \text{ and } S_2 = \{a_1, x, a_3, \ldots, a_n\}.$$

If $a_2 > x$, show that the harmonic mean of the numbers in S_1 is greater than the harmonic mean of those in S_2.

That is, replacing a number in a set by a smaller number decreases the harmonic mean of the set.

4.6. Show that the harmonic mean of n numbers is between the largest and the smallest of them.

4.7. If AM, GM, HM stand respectively for the arithmetic, geometric, and harmonic means of n positive numbers, show that $AM \geq GM \geq HM$. (We have already shown the first inequality in Chapter 3. Show the second.)

The Harmonic Mean in Geometry

For this next set of problems, the term "find" means "estimate the location of". Unless otherwise indicated, it does not mean "construct by straightedge and compass" (or other tools).

Problems

4.8. Given two points A and B on a line, find a point W on line segment AB such that $AW = WB$.

4.9. Given two points A and B on a line, find a point X on line segment AB such that $BX = 2XA$.

4.10. Given two points A and B on a line, find a point Y on line segment AB such that $AY = 2YB$.

4.11. Given two points A and B on a line, find another point Y' on line AB (distinct from the point Y in Problem 4.10) such that $AY' = 2Y'B$.

4.12. Are there any other points Z, anywhere in the plane, other than Y' (from Problem 4.11) such that $AZ = 2BZ$?

4.13. (Extra credit) Given two points A and B on a line, construct, with straightedge and compass, the points Y and Y' referred to in Problems 4.10 and 4.11.

♦

In Problem 4.10, we started with a point Y between A and B such that $AY = 2YB$. We then found another point Y' on line AB (but outside segment AB) such that $AY' = 2Y'B$. We say that Y and Y' are *harmonic conjugates* with respect to segment AB. In general, if we have a line segment AB, and a given ratio r, there are two points Y, Y' on line AB such that $AY : YB = AY' : Y'B = r$. The term 'harmonic conjugates' is used for the pair Y, Y', no matter what the ratio r may be.

♦

4.14. If X and Y are harmonic conjugates with respect to A and B, show that A and B are also harmonic conjugates with respect to X and Y.

4.15. Generalize Problem 4.13. Given points A and B, and a point X between them, construct, with straightedge and compass, the point Y that is the harmonic conjugate of X. What if point X is given outside segment AB?

4.16. Let X and Y be harmonic conjugates with respect to A and B. Let $AX = a$, $AB = h$, and $AY = b$. Show that h is the harmonic mean of a and b.

4.17. For two points A and B, where is the harmonic conjugate of the midpoint of AB?

4.18. If you ride your bike up a hill at 15 MPH, at what rate must you ride down the hill, so that your average speed for the two trips is 30 MPH? If you have trouble with this problem and Problem 4.17, you are in good company.
Albert Einstein is on record for finding the solution counterintuitive. But can you make a geometric diagram which shows what is happening?

Solutions

4.1. Show that the geometric mean of a and b is also the geometric mean of $\dfrac{a+b}{2}$ and $\dfrac{2ab}{a+b}$.

Solution. It's just algebra, and it's easy if we put it in the following form:

$$\left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab,$$

or that the arithmetic mean times the harmonic mean is the square of the geometric mean. The truth of this equation is easy to see from the form of the fractions.

4.2. Show that if AM, GM, HM are respectively the arithmetic, geometric, and harmonic means of two positive numbers, then $HM \le GM \le AM$.

Solution. This is again immediate: we know that the geometric mean of two numbers is between the larger and smaller of them, and this is all that the inequality says. The means are equal, of course, when the numbers are equal.

We have shown that the geometric mean of two positive real numbers lies between their harmonic mean and their arithmetic mean. It would be interesting to know if the geometric mean is closer to the harmonic mean or to the arithmetic mean. We show that it is closer to the harmonic mean by proving that

$$\sqrt{ab} - \frac{2ab}{a+b} \le \frac{a+b}{2} - \sqrt{ab}.$$

Indeed, this is equivalent to

$$2\sqrt{ab} \le \frac{2ab}{a+b} + \frac{a+b}{2},$$

which is just $2\sqrt{xy} \le x + y$ for $x = \dfrac{2ab}{a+b}$ and $y = \dfrac{a+b}{2}$.

4.3. We have so far begged an important question. Is the harmonic mean of a and b always a mean? That is, is it always between a and b? Show that if $0 \le a < b$, then $a < \dfrac{2ab}{a+b} < b$.

Solution. We have:

$$\begin{aligned} a &< b \\ a^2 &< ab, \text{ (since } b > 0) \\ a^2 + ab &< 2ab, \\ a(a+b) &< 2ab, \\ a &< \frac{2ab}{a+b}, \end{aligned}$$

which is what we want. A similar argument will show that $\dfrac{2ab}{a+b} < b$. This shows that the harmonic mean is truly a mean.

As you solve this problem, you may find yourself writing this argument "backwards" from our solution above. Remember that the logic of inequalities is not always “reversible”. We must be sure we can start with something we know and end up with our result.

4.4. **a.** Look back at Figure 2.1, and connect O to Y as shown in Figure 4.1 below. Point Z be the foot of the perpendicular from X to OY. Using the same segment lengths as given in Problem 2.1, show that YZ is the harmonic mean of a and b. Deduce geometrically that the harmonic mean of a and b is less than or equal to their arithmetic mean.

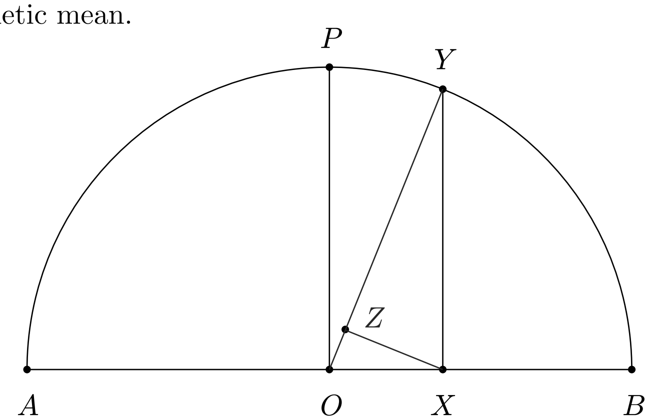

Figure 4.1

Solution. Triangles XYZ and OYX are similar, so $\dfrac{XY}{YZ} = \dfrac{OY}{XY}$. It follows that

$$YZ = \frac{XY^2}{OY} = \frac{\left(\sqrt{ab}\right)^2}{(a+b)/2} = \frac{2ab}{a+b},$$

which is just the harmonic mean of a and b.

Now since YZ is a leg of right triangle XYZ, while XY is its hypotenuse, we have $YZ \le XY$, or $\dfrac{2ab}{a+b} \le \dfrac{a+b}{2}$.

Equality holds if and only if triangle XYZ degenerates into a straight line. Then points X and O coincide, which means that $a = b$.

b. (This problem requires the ‘triangle inequality’, a geometric inequality which we haven’t discussed in this volume.) Let h_a, h_b, h_c be the lengths of the altitudes of a triangle (with sides a, b, c). Prove that $2h_a$ is greater than the harmonic mean of h_b and h_c.

Solution. The key to this problem is the classic *triangle inequality*, and the connection between the altitudes of a triangle and its area. The triangle inequality says that the sum of any two sides of a triangle must be greater than the third side. So we know that $b + c > a$.

Now if K is the area of the triangle, then

$$K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c,$$

so we can rewrite the triangle inequality as:

$$\frac{2K}{h_b} + \frac{2K}{h_c} > \frac{2K}{h_a}.$$

Then $\dfrac{1}{h_b} + \dfrac{1}{h_c} > \dfrac{1}{h_a}$, which means that

$$\frac{h_b + h_c}{h_b h_c} > \frac{1}{h_a}.$$

Hence

$$2h_a > \frac{2h_b h_c}{h_b + h_c},$$

as desired.

4.5. Suppose we have two sets of n positive numbers each:

$$S_1 = \{a_1, a_2, a_3, \ldots, a_n\} \text{ and } S_2 = \{a_1, x, a_3, \ldots, a_n\}.$$

If $a_2 > x$, show that the harmonic mean of the numbers in S_1 is greater than the harmonic mean of those in S_2.

That is, replacing a number in a set by a smaller number decreases the harmonic mean of the set.

Solution. We write the solution for two sets of four numbers. The generalization is immediate.

We have:

$$a_2 > x;$$
$$\frac{1}{a_2} < \frac{1}{x};$$
$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} < \frac{1}{a_1} + \frac{1}{x} + \frac{1}{a_3} + \frac{1}{a_4};$$
$$\frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4}} > \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{x} + \dfrac{1}{a_3} + \dfrac{1}{a_4}}.$$

This last inequality is the statement that the harmonic mean of S_1 is greater than or equal to that of S_2.

Note the various reversals of the direction of this inequality as we manipulate it.

4.6. Show that the harmonic mean of n numbers is between the largest and the smallest of them.

Solution. (This solution was suggested by Daniel Vitek.)

We use the notation $HM(S)$ to denote the harmonic mean of a set S of numbers.

Let $S_1 = \{a_1, a_2, a_3, \ldots, a_n\}$, where $a_1 \le a_2 \le a_3 \le \ldots \le a_n$. We will show that $HM(S_1) \ge a_1$.

Let $S_2 = \{a_1, a_1, a_3, \ldots, a_n\}$, where we have replaced a_2 by a_1 (which is smaller). Then, by Problem 4.3, we have $HM(S_1) \ge HM(S_2)$.

Let $S_3 = \{a_1, a_1, a_1, \ldots, a_n\}$, where we have replaced a_3 by a_1 (which is smaller). Then, by Problem 4.3, we have $HM(S_2) \geq HM(S_3)$, so $HM(S_1) \geq HM(S_3)$ as well.

Continuing in this way, we finally arrive at S_n, which consists of n copies of a_1. Then

$$HM(S_1) \geq HM(S_n) = \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_1} + \ldots + \dfrac{1}{a_1}} = \frac{n}{\left(\dfrac{n}{a_1}\right)} = a_1.$$

In the same way, we can prove that $HM(S_1) \leq a_n$.

4.7. If AM, GM, HM stand respectively for the arithmetic, geometric, and harmonic means of n positive numbers, show that $AM \geq GM \geq HM$. (We have already shown the first inequality in Chapter 3. Show the second.)

Solution. The inequality to be proven can be written as

$$\frac{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_1}}{n} \geq \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}},$$

which is just a special case of the AM-GM inequality for n quantities.

4.8. Given two points A and B on a line, find a point W on line segment AB such that $AW = WB$.

Solution. Point W is just the midpoint of line segment AB. There are other points W such that $AW = WB$, but they are not on segment AB.

4.9. Given two points A and B on a line, find a point X on line segment AB such that $BX = 2XA$.

Solution. We just trisect line segment AB, and choose as X the trisection point closer to A.

4.10. Given two points A and B on a line, find a point Y on line segment AB such that $AY = 2YB$.

Solution. Point Y is the other trisection point of line segment AB, the one closer to B.

4.11. Given two points A and B on a line, find another point Y' on line AB (distinct from the point Y in Problem 4.10) such that $AY' = 2Y'B$.

Solution. Take the point Y' such that $Y'B = BA$. Then $AY' = AB + BY' = 2Y'B$, as required. Note that the second trisection point of segment AB does not work.

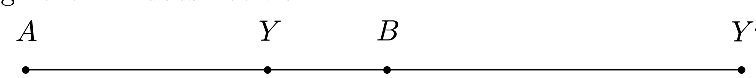

Figure 4.2

4.12. Are there any other points Z, anywhere in the plane, other than Y' (from Problem 4.11) such that $AZ = 2BZ$?

Solution. Yes. There are lots of them, not on line AB. It is a fascinating question, which we will not pursue here, where they all lie.

For a peek at this topic, look up *circles of Apollonius* in any advanced geometry book.

4.13. (Extra credit) Given two points A and B on a line, construct, with straightedge and compass, the points Y and Y' referred to in Problems 4.10 and 4.11.

Solution. A set of parallel lines cuts off proportional segments along any transversal.

Thus we have the following construction. Through point A we draw any line at all, different from AB. Along this line we want to create a set of four points in the ratios we require along line AB.

We can do this by marking off six equal segments, of any length, along this line, starting at point A: $AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6$. Then $AP_2 : P_2P_3 = 2 : 1$ and $AP_6 : P_6P_3 = 2 : 1$ as well.

Next we join P_3 to point B, and draw a line through P_2 parallel to P_3B. This line will intersect AB at point Y inside segment AB such that $AY : YB = 2 : 1$.

Finally, we draw a line parallel to P_3B through point P_6. This line will intersect AB at point Y' outside segment AB such that $AY' : Y'B = 2 : 1$.

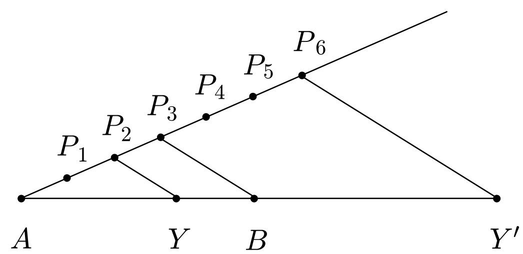

Figure 4.3

Given two points A and B on a line, if $AX : XB = AY : YB$, then points X and Y are called *harmonic conjugates* with respect to A and B. The points $\{A, X, B, Y\}$ are called a *harmonic range* on the line.

4.14. If X and Y are harmonic conjugates with respect to A and B, show that A and B are also harmonic conjugates with respect to X and Y.

Solution. We have $AX : XB = AY : YB$. We want $XA : YA = XB : YB$. These two proportions have the same cross products, so each implies the other.

We sometimes say that the second proportion is obtained from the first by "alternation of the means".

4.15. Generalize Problem 4.13. Given points A and B, and a point X between them, construct, with straightedge and compass, the point Y that is the harmonic conjugate of X. What if point X is given outside segment AB?

Solution. We can use a construction similar to that of Problem 4.13. We draw any line through B, and choose some point B' on this

line. We connect B to B', then draw a parallel to BB' through X, intersecting AB' at point X'. We now have $AX' : X'B' = AX : XB$. If we can construct Y' on AB' so that $AY' : Y'B' = AX' : X'B'$, we will have the required proportions along line AB', and can use parallel lines to locate point Y on AB with the required ratio.
Let us do this by drawing a third line through A, reversing the roles of X' and B' for new points along this third line. That is, we mark off a segment $AX'' = AX'$ along it. If we then mark off segment $B''X'' = B'X'$ along AB'' so that X'' is outside segment AB'' we have $AX'' : X''B'' = AX' : X'B' = AX : XB$. Drawing a line $X''Y'$ parallel to $B'B''$, with Y' on line AB', we have $AY' : Y'B' = AX'' : X''B'' = AX' : X'B' = AX : XB$.

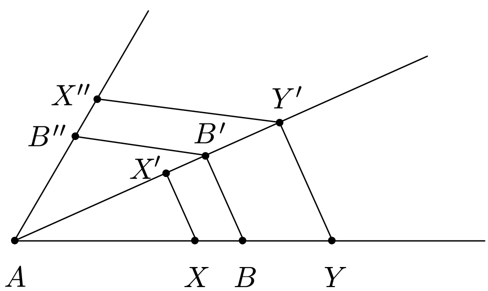

Figure 4.4

In turn, if we draw a line through Y' parallel to BB', it will intersect AB at point Y such that $AY : YB = AY' : Y'B' = AX : XB$, and Y is the harmonic conjugate of X with respect to segment AB.
If point X is outside segment AB, the construction proceeds analogously.

4.16. Let X and Y be harmonic conjugates with respect to A and B. Let $AX = a$, $AB = h$, and $AY = b$. Show that h is the harmonic mean of a and b.

Solution. We have

$$AX = a,\ XB = h - a,\ AY = b,\ YB = AY - AB = b - h,$$

and we have:

$$\begin{aligned} AX : XB &= AY : YB, \\ a : (h-a) &= b : (b-h), \\ ab - ah &= bh - ba, \\ 2ab &= h(a+b), \\ h &= \frac{2ab}{a+b}, \end{aligned}$$

which is the harmonic mean of a and b.

4.17. For two points A and B, where is the harmonic conjugate of the midpoint of AB?

Solution. This is an interesting question. Let M be the midpoint of AB. We need a point P such that $AM : MB = 1 = AP : PB$. But that means that $AP = PB$. This cannot happen, since $BP = AP - AB$, and AB is not zero. (We will not bother to talk about the situation when $AB = 0$. Then A and B will coincide, and there is no midpoint to talk about. Or, $AP = PB$ for any point on line AB.)
But notice that as point P recedes from segment AB, in either direction, the ratio $AP : PB$ approaches 1. We sometimes say that the harmonic conjugate of the midpoint M of AB is the *point at infinity* along line AB. This concept is made very clear in projective geometry. For now, it is just a figure of speech. But also see Problem 4.18.

4.18. If you ride your bike up a hill at 15 MPH, at what rate must you ride down the hill, so that your average speed for the two trips is 30 MPH? If you have trouble with this problem and Problem 4.17, you are in good company.
Albert Einstein is on record for finding the solution counterintuitive. But can you make a geometric diagram which shows what is happening?

Solution and discussion (for this problem and Problem 4.17) Let's look at the motion problem first. The harmonic mean h of rates a and b is $\dfrac{2ab}{a+b}$. Here, we have $a = 15$, $h = 30$, and we are asked for b. So we need:

$$\begin{aligned} \frac{30b}{15+b} &= 30, \\ \frac{b}{15+b} &= 1, \\ \frac{15+b}{b} &= 1, \\ 15+b &= b, \\ 15 &= 0. \text{ (?!?!)} \end{aligned}$$

So what's going on? It seems that there is no number b that works. In fact, if we do this more generally, and require the average of two rates to be double one of them, the algebra will again show no solution.
We can illustrate this situation geometrically. We know that if four points A, M, B, P form a harmonic range along line AB, then AB is the harmonic mean of AM and AP. We think of line AB as a number line, and let A have coordinate 0, M have coordinate 15, and B have coordinate 30. Then we want a point P so that the harmonic mean of AM and AP is AB. That will be asking for the harmonic conjugate of point M along line AB, and we know that this point does not exist.
Or, we know that it is the point at infinity. As Einstein wrote to a colleague, "Not until calculating did I notice that there is no time left for the way down!"

Chapter 5

Symmetry in Algebra, Part I

Symmetry is a fundamental mathematical concept. The study of symmetry, which is called *group theory*, has been a productive area of mathematical research for two centuries, and its treasury of uses and results shows no sign of being depleted.

In geometry, the symmetry in certain figures strikes the eye immediately, and the difficulty lies in harnessing it to achieve certain results. The same concept, in algebra, is more subtle. Algebraic symmetry appeals to the mind, not the eye, and reveals itself only slowly, as one works through a series of problems.

Example 5.1. Solve the following system of equations:

$$\begin{cases} x + 5y = 9 \\ 5x + y = 15. \end{cases}$$

Solution. Following the usual textbook solution, one would multiply one of the equations by 5, then subtract. This will of course get us the answer, and the method generalizes to any pair of simultaneous linear equations, and to simultaneous equations with more than two variables.

Or, we could solve one equation for x and substitute into the other equation. This method also generalizes for any pairs of simultaneous linear equations (although it gets difficult when we involve more variables).

But here's a more subtle way to solve this system, a way that generalizes in another direction: We have $x + 5y = 9$ and $5x + y = 15$. Adding, we find that $6x + 6y = 24$, so $x + y = 4$. Then we subtract this from the first equation to get $4y = 5$, and from the second to get $4x = 11$, and the solution is easy:

$$x = \frac{11}{4},\ y = \frac{5}{4}.$$

Why does this method work? Neither equation is symmetric in x and y on its own, but as a system, there is symmetry: the two variables play the same roles in the system. In other words, if we looked only at the left sides of the equations, and interchanged x and y, we would not know the difference. And in fact it is the *form* of the left sides of the equations, not the particular *numbers* on the right, that dictates the algebraic procedures we use to solve them.

The following problems can be thought of as generalizations of this first simple example. In general, if we perceive algebraic symmetry in a system of equations, it is helpful to act on them so as to preserve this symmetry.

Problems

5.1. Solve simultaneously $\begin{cases} x + 2y + z = 14 \\ 2x + y + z = 12 \\ x + y + 2z = 18 \end{cases}$

5.2. Solve simultaneously $\begin{cases} x + y = 7 \\ y + z = -2 \\ z + x = 9 \end{cases}$

5.3. Solve simultaneously $\begin{cases} xy = 6 \\ yz = 15 \\ zx = 10 \end{cases}$

5.4. Solve simultaneously $\begin{cases} (x+1)(y+1) = 24 \\ (y+1)(z+1) = 30 \\ (z+1)(x+1) = 20 \end{cases}$

5.5. Solve simultaneously $\begin{cases} xy - x - y = 11 \\ yz - y - z = 14 \\ zx - z - x = 19 \end{cases}$

5.6. Solve simultaneously $\begin{cases} x(x+y+z) = 4 \\ y(x+y+z) = 6 \\ z(x+y+z) = 54 \end{cases}$

5.7. Solve simultaneously $\begin{cases} x + [y] + \{z\} = 1.1 \\ \{x\} + y + [z] = 2.2 \\ [x] + \{y\} + z = 3.3 \end{cases}$

In this problem, the notation $[x]$ means "the greatest integer not exceeding x", and $\{x\}$ means "the fractional part of x", that is, $\{x\} = x - [x]$. So, for example, $[5.2] = 5$ and $\{5.2\} = 0.2$, while $[7] = 7$ and $\{7\} = 0$.

5.8. If a is a fixed positive real number, solve simultaneously:

$$\begin{cases} x^2 - xy = a \\ y^2 - xy = a(a-1). \end{cases}$$

5.9. Solve the following system of n equations in n unknowns (where n is some integer greater than 2):

$$\begin{cases} x_2 + x_3 + x_4 + \ldots + x_n = 1 \\ x_1 + x_3 + x_4 + \ldots + x_n = 2 \\ x_1 + x_2 + x_4 + \ldots + x_n = 3 \\ \ldots \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} = n. \end{cases}$$

5.10. A triangle has sides of lengths 13, 14, and 15. Its inscribed circle divides each side into two segments, making six segments in all. Find the length of each segment.

5.11. The three altitudes of acute triangle ABC (with sides a, b, and c) determine six segments along the triangle's sides (see Figure 5.1). If we let $x = \cos A$, $y = \cos B$, $z = \cos C$, then the trigonometry of the right triangle lets us represent the six segments as shown in Figure 5.1.

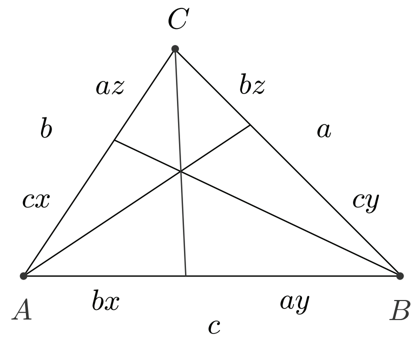

Figure 5.1

Solve this system of simultaneous equations for x, y, and z in terms of a, b, and c.

Symmetry in Inequalities

Algebraic symmetry often plays a role in working with inequalities. In fact, it is ubiquitous. Here's a simple example:

Example 5.2. We have been working with the arithmetic and geometric means. But we never proved that they actually are means. That is, do the AM and GM of two positive numbers a, b actually lie between the larger and smaller of a and b?

Solution. Let us assume that $a < b$. Then $a+a < a+b$, and $\frac{a+a}{2} < \frac{a+b}{2}$. So the AM is larger than the smaller of a, b.

Similarly, we have $\frac{a+b}{2} < \frac{b+b}{2} = b$. So the AM is smaller than the larger of a, b. That is, the AM of a, b lies between the two values.

But what if $b < a$? We could write the proof over again under this assumption. But it's pretty clear that if we exchange a and b, the algebra will look the same. That is, we can assume $a < b$, without loss of generality, from the symmetry of the situation.

You may want to check that the proof is the same if we assume that $b < a$. You will quickly find the computation tedious and boring, and the outcome obvious because of the algebraic symmetry.

Problems

5.12. Show that the geometric mean of the positive numbers a, b lies between their two values.

5.13. Generalize Example 5.2 and Problem 5.12 for n positive numbers $a_1, a_2, \ldots, a_n$.

5.14. Prove that for any real numbers a, b, c,

$$7\sqrt{a^2+b^2+c^2} \le \sqrt{3}\max\{|-2a+3b+6c|, |6a-2b+3c|, |3a+6b-2c|\}.$$

5.15. Let a, b, c, x, y, z be real numbers such that

$$4x+y=b+4c,\ 4y+z=c+4a,\ 4z+x=a+4b.$$

Prove that

$$x^2+y^2+z^2 \ge ab+bc+ca.$$

5.16. Prove that for any real numbers a, b, c,

$$a^2+b^2+c^2-ab-bc-ca \ge \frac{3}{4}(a-b)^2.$$

5.17. (Dominik Teiml, https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2015-05/mr_4_2015_solutions_2.pdf, accessed June 2016) Find the maximum possible value of k for which

$$\frac{a^2+b^2+c^2}{3} - \left(\frac{a+b+c}{3}\right)^2 \ge k \cdot \max\{(a-b)^2, (b-c)^2, (c-a)^2\},$$

for all real numbers a, b, c.

5.18. In triangle ABC, $\frac{\pi}{7} < A \le B \le C < \frac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4} - \sin\frac{7B}{4} + \sin\frac{7C}{4} > \cos\frac{7A}{4} - \cos\frac{7B}{4} + \cos\frac{7B}{4}.$$

5.19. In triangle ABC, $\max\{A, B, C\} < 120°$. Prove that

$$\sin A - \sin B + \sin C < \sqrt{3}(\cos A - \cos B + \cos C).$$

5.20. In any triangle ABC, show that

$$\cos\frac{A}{2} + \cos\frac{B}{2} > \cos\frac{C}{2}.$$

Solutions

5.1. Solve simultaneously $\begin{cases} x+2y+z=14 \\ 2x+y+z=12 \\ x+y+2z=18 \end{cases}$

Solution. Adding the three given equations, we get

$$4x+4y+4z=44,$$

or

$$x+y+z=11.$$

If we subtract this equation from each of the given equations in turn, we find very quickly that $y = 3$, $x = 1$, and $z = 7$. The reader who has tried substitution will appreciate how much easier our solution is.

5.2. Solve simultaneously $\begin{cases} x + y = 7 \\ y + z = -2 \\ z + x = 9 \end{cases}$

Solution. Adding the three equations, we find

$$2x + 2y + 2z = 14,$$

or

$$x + y + z = 7.$$

Subtracting each of the given equations in turn from this one, we find $z = 0$, $x = 9$, and $y = -2$.

5.3. Solve simultaneously $\begin{cases} xy = 6 \\ yz = 15 \\ zx = 10 \end{cases}$

Solution. Taking a hint from Problem 5.2, we multiply the three equations together to find

$$x^2y^2z^2 = 6 \cdot 15 \cdot 10,$$

so $xyz = \pm 30$. Then we divide this equation by each of the given equations to find

$$(x, y, z) = \pm(2, 3, 5).$$

5.4. Solve simultaneously $\begin{cases} (x + 1)(y + 1) = 24 \\ (y + 1)(z + 1) = 30 \\ (z + 1)(x + 1) = 20 \end{cases}$

Solution. Let's be quick about this. Let $p = x + 1$, $q = y + 1$, and $r = z + 1$. Then we have $pq = 24$, $qr = 30$, and $rp = 20$, and we have the same kind of equation as in Problem 5.3. We find that $(p, q, r) = (4, 6, 5)$ or $(-4, -6, -5)$. The corresponding values for (x, y, z) are $(3, 5, 4)$ and $(-5, -7, -6)$.

5.5. Solve simultaneously $\begin{cases} xy - x - y = 11 \\ yz - y - z = 14 \\ zx - z - x = 19 \end{cases}$

Solution. We can make this problem resemble Problem 5.4 by adding 1 to both sides of each equation. For example, the first equation becomes

$$xy - x - y + 1 = 12,$$

or

$$(x - 1)(y - 1) = 12.$$

We then let $p = x - 1$, $q = y - 1$, $r = z - 1$ and proceed as before. We find that $(x, y, z) = (5, 4, 6)$ or $(-3, -2, -4)$.

5.6. Solve simultaneously $\begin{cases} x(x+y+z) = 4 \\ y(x+y+z) = 6 \\ z(x+y+z) = 54 \end{cases}$

Solution. Adding all three equations, and factoring the left side, we find that

$$(x+y+z)^2 = 64,$$

so

$$x+y+z = \pm 8.$$

Then we divide each equation by this relation, to find

$$(x,y,z) = \left(\frac{1}{2}, \frac{3}{4}, \frac{27}{4}\right) \text{ or } \left(-\frac{1}{2}, -\frac{3}{4}, -\frac{27}{4}\right).$$

5.7. Solve simultaneously $\begin{cases} x + [y] + \{z\} = 1.1 \\ \{x\} + y + [z] = 2.2 \\ [x] + \{y\} + z = 3.3 \end{cases}$

In this problem, the notation $[x]$ means "the greatest integer not exceeding x", and $\{x\}$ means "the fractional part of x", that is, $\{x\} = x - [x]$. So, for example, $[5.2] = 5$ and $\{5.2\} = 0.2$, while $[7] = 7$ and $\{7\} = 0$.

Solution. Adding the given equations, we get $2x + 2y + 2z = 6.6$, or

$$x + y + z = 3.3. \tag{5.1}$$

The key to the rest of the solution is that for any a, we have $a = [a] + \{a\}$. Subtracting the first of the given equations from (5.1) gives us

$$\{y\} + [z] = 2.2,$$

which means that $[z] = 2$ and $\{y\} = 0.2$. Subtracting the third of the given equations from (5.1), we find that

$$\{x\} + [y] = 0,$$

which shows that $\{x\} = 0$ and $[y] = 0$ as well. Hence x is an integer. From these two steps we can conclude that $y = 0.2$. Subtracting the second of the given equations from (5.1), we find that

$$[x] + \{z\} = 1.1,$$

so $[x] = 1$ and $\{z\} = 0.1$, and from the previous relations, we can infer that $x = 1$ and $z = 2.1$.

5.8. If a is a fixed positive real number, solve simultaneously:

$$\begin{cases} x^2 - xy = a \\ y^2 - xy = a(a-1). \end{cases}$$

Solution. Adding the equations, we find

$$x^2 - 2xy + y^2 = (x-y)^2 = a^2,$$

so $x - y = \pm a$. We can now write the given equations as

$$x(x-y) = a; \quad -y(x-y) = a(a-1),$$

and divide each in turn by $x-y$. (Note that in this step we are using the fact that $a \neq 0$.) We find that $(x, y) = (1, 1-a)$ or $(x, y) = (-1, a-1)$.

5.9. Solve the following system of n equations in n unknowns (where n is some integer greater than 2):

$$\begin{cases} x_1 + x_2 + x_3 + \ldots + x_n = 1 \\ x_1 + x_3 + x_4 + \ldots + x_n = 2 \\ x_1 + x_2 + x_4 + \ldots + x_n = 3 \\ \ldots \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} = n. \end{cases}$$

Solution. Rather than adding all the equations together, we can take advantage of their symmetric relation to the sum $x_1+x_2+\ldots+x_n$. We denote this sum by S, and forget for a moment that we know that its numerical value is 1. Then we can write the system as:

$$\begin{cases} S - x_2 = 2 \\ S - x_3 = 3 \\ \ldots \\ S - x_{n-1} = n-1 \\ S - x_n = n \end{cases}$$

Then we recall that in fact $S = 1$, and we see immediately that

$$x_2 = -1, \; x_3 = -2, \; x_4 = -3, \ldots, x_n = -n+1.$$

Next, from a well-known formula,

$$x_2 + x_3 + \ldots + x_n = -[1 + 2 + 3 + \ldots + (n-1)] = -\frac{n(n-1)}{2},$$

and finally,

$$x_1 = 1 - (x_2 + x_3 + \ldots + x_n) = 1 + \frac{n(n-1)}{2}.$$

5.10. A triangle has sides of lengths 13, 14, and 15. Its inscribed circle divides each side into two segments, making six segments in all. Find the length of each segment.

Solution. Since tangents to a circle from a point outside are equal, the segments of the sides of the triangle are equal in pairs. Let their lengths (Figure 5.2) be x, y, z. We then have

$$\begin{cases} x + y = 13 \\ y + z = 14 \\ z + x = 15. \end{cases}$$

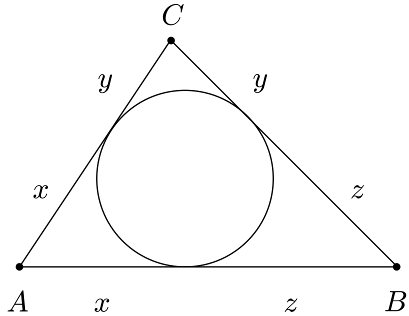

Figure 5.2

We can solve these using the method of Problem 5.2. We find that $x = 7$, $y = 6$, and $z = 8$.

5.11. The three altitudes of acute triangle ABC (with sides a, b, and c) determine six segments along the triangle's sides (see Figure 5.1). If we let $x = \cos A$, $y = \cos B$, $z = \cos C$, then the trigonometry of the right triangle lets us represent the six segments as shown in Figure 5.1.

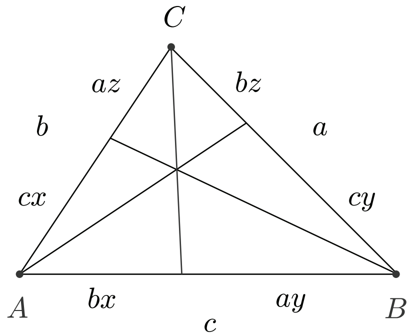

Figure 5.1

Solve this system of simultaneous equations for x, y, and z in terms of a, b, and c.

Solution. We cannot use the method of Problem 5.2 (or Problem 5.3), because the coefficients of x, y, and z do not follow the same patterns as in those problems. But suppose we divide the equations by ab, ac, and bc, respectively. We obtain:

$$\frac{y}{b} + \frac{x}{a} = \frac{c}{ab},$$

$$\frac{x}{a} + \frac{z}{c} = \frac{b}{ac},$$

$$\frac{z}{c} + \frac{y}{b} = \frac{a}{bc},$$

and now we can let

$$A = \frac{x}{a},\ B = \frac{y}{b},\ C = \frac{z}{c},$$

and apply the method of Problem 5.2. We find that

$$A = \left(\frac{c^2+b^2+a^2}{2abc}\right) - \frac{a}{bc} = \frac{c^2+b^2-a^2}{2abc},$$

with corresponding expressions for B and C. Then we can easily find that:

$$x = \frac{b^2+c^2-a^2}{2bc},$$

$$y = \frac{a^2+c^2-b^2}{2ac},$$

$$z = \frac{a^2+b^2-c^2}{2ab}.$$

But if you remember the Law of Cosines in trigonometry, you already knew this! Indeed, this problem gives an algebraic derivation of the Law of Cosines, for an acute-angled triangle. Can you extend the derivation to cover the case of an obtuse triangle?

5.12. Show that the geometric mean of the positive numbers a, b lies between their two values.

Solution. From symmetry, we lose no generality by assuming that $a < b$. Then $\sqrt{a} < \sqrt{b}$, and multiplying both sides by the positive number $\sqrt{a}$, we have $\sqrt{a^2} = a < \sqrt{ab}$.
Similarly, we can show that $\sqrt{ab} < \sqrt{b^2} = b$.
Notice that we can write $\sqrt{a^2} = a$ because $a > 0$. If we didn't know this, we would have to write $\sqrt{a^2} = |a|$.

5.13. Generalize Example 5.2 and Problem 5.12 for n positive numbers $a_1, a_2, \ldots, a_n$.

Solution. The expression for the geometric mean of n numbers is symmetric in each of the numbers. So without loss of generality, we can assume that $a_1 \le a_2 \le \ldots \le a_n$. In particular,

$$a_1 = \sqrt[n]{a^n} \le \sqrt[n]{a_1 a_2 \ldots a_n} \le \sqrt[n]{a_n^n} = a_n.$$

5.14. Prove that for any real numbers a, b, c,

$$7\sqrt{a^2+b^2+c^2} \le \sqrt{3}\max\{|-2a+3b+6c|, |6a-2b+3c|, |3a+6b-2c|\}.$$

Solution. We have

$$3\max\{(-2a+3b+6c)^2, (6a-2b+3c)^2, (3a+6b-2c)^2\}$$

$$\ge (-2a+3b+6c)^2 + (6a-2b+3c)^2 + (3a+6b-2c)^2 = 49(a^2+b^2+c^2),$$

hence the conclusion.

5.15. Let a, b, c, x, y, z be real numbers such that

$$4x+y = b+4c,\ 4y+z = c+4a,\ 4z+x = a+4b.$$

Prove that

$$x^2+y^2+z^2 \ge ab+bc+ca.$$

Solution. This is a strange problem. The given relationships are linear, but the result is quadratic. So we can anticipate that at some point we will need to multiply linear expressions. And since the result involves squares, we can anticipate squaring.

Sometimes the best way to proceed is the most obvious. Here, we can solve for x, y, z and square. We find

$$\begin{aligned} 13x &= -3a + 4b + 12c, \\ 13y &= 12a - 3b + 4c, \\ 13z &= 4a + 12b - 3c. \end{aligned}$$

Then

$$\begin{aligned} (13x)^2 + (13y)^2 + (13z)^2 = (&-3a + 4b + 12c)^2 + (12a - 3b + 4c)^2 \\ &+ (4a + 12b - 3c)^2, \end{aligned}$$

implying $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$.

Since we know (from Problem 1.4) that $a^2 + b^2 + c^2 \geq ab + bc + ca$, the result now follows.

5.16. Prove that for any real numbers a, b, c,

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{3}{4}(a - b)^2.$$

Solution. The left side of the inequality is symmetric in a, b, and c. But the right side isn't. What can we do?
Well, look at the expressions $(a - b)^2$, $(b - c)^2$, $(c - a)^2$. One of them is larger than the other two. Without loss of generality (to the left-hand side), let us assume that the largest is $(a - b)^2$.
Then we can rewrite the inequality in a more symmetric form:

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{3}{4}\max\{(a - b)^2, (b - c)^2, (c - a)^2\},$$

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq \frac{3}{2}(a - b)^2.$$

Setting $a - b = u$ and $b - c = v$, we obtain the equivalent inequality

$$u^2 + v^2 + (-u - v)^2 \geq \frac{3}{2}u^2,$$

which reduces to the inequality $(u + 2v)^2 \geq 0$, which is certainly true.

5.17. (Dominik Teiml, https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2015-05/mr_4_2015_solutions_2.pdf, accessed June 2016) Find the maximum possible value of k for which

$$\frac{a^2 + b^2 + c^2}{3} - \left(\frac{a + b + c}{3}\right)^2 \geq k \cdot \max\{(a - b)^2, (b - c)^2, (c - a)^2\},$$

for all real numbers a, b, c.

Solution. Both sides of the inequality are symmetric in a, b, c, so we may assume, without loss of generality, that $a \geq b \geq c$. Let $u = a - c$, $v = a - b$, and note that $u \geq v \geq 0$. We have:

$$\begin{aligned}\frac{a^2+b^2+c^2}{3} - \left(\frac{a+b+c}{3}\right)^2 &= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{9} \\ &= \frac{v^2+(u-v)^2+u^2}{9} = \frac{3u^2+(u-2v)^2}{18} \\ &\geq \frac{u^2}{6} = \frac{1}{6}\max\{(a-b)^2,(b-c)^2,(c-a)^2\}.\end{aligned}$$

So the maximal value for k is $\frac{1}{6}$, with equality just when $u = 2v$; that is, if a, b, c are in arithmetic progression.

5.18. In triangle ABC, $\frac{\pi}{7} < A \leq B \leq C < \frac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4} - \sin\frac{7B}{4} + \sin\frac{7C}{4} > \cos\frac{7A}{4} - \cos\frac{7B}{4} + \cos\frac{7B}{4}.$$

Solution. The conditions of the problem ensure that

$$0 < \frac{7A}{4} - 45^\circ < 180^\circ,\ 0 < \frac{7B}{4} - 45^\circ < 180^\circ,\ 0 < \frac{7C}{4} - 45^\circ < 180^\circ.$$

In addition, we have

$$\frac{7A}{4} - 45^\circ + \frac{7B}{4} - 45^\circ + \frac{7C}{4} - 45^\circ = 7\cdot 45^\circ - 3\cdot 45^\circ = 180^\circ.$$

So there exists a triangle $A'B'C'$ with angles

$$A' = \frac{7A}{4} - 45^\circ,\ B' = \frac{7B}{4} - 45^\circ,\ C' = \frac{7C}{4} - 45^\circ.$$

Then

$$\sin\left(\frac{7A}{4} - 45^\circ\right) + \sin\left(\frac{7C}{4} - 45^\circ\right) > \sin\left(\frac{7B}{4} - 45^\circ\right),$$

implying

$$\begin{aligned}\frac{\sqrt{2}}{2}\sin\frac{7A}{4} - \frac{\sqrt{2}}{2}\cos\frac{7A}{4} + \frac{\sqrt{2}}{2}\sin\frac{7C}{4} - \frac{\sqrt{2}}{2}\cos\frac{7C}{4}& \\ > \frac{\sqrt{2}}{2}\sin\frac{7B}{4} - \frac{\sqrt{2}}{2}\cos\frac{7B}{4}&,\end{aligned}$$

and the conclusion follows.

5.19. In triangle ABC, $\max\{A, B, C\} < 120^\circ$. Prove that

$$\sin A - \sin B + \sin C < \sqrt{3}(\cos A - \cos B + \cos C).$$

Solution. We don't have symmetry. And how do we use the condition that the angles A, B, C are less than 120°?

We can use the technique of Problem 5.18. Consider the triangle $A'B'C'$ with $A' = 120° - A$, $B' = 120° - B$ and $C' = 120° - C$. Indeed, there is such a triangle $A'B'C'$, since $A', B', C' > 0$ and

$$A' + B' + C' = 3 \cdot 120° - 180° = 180°.$$

Let $a' = B'C'$, $b' = C'A'$, $c' = A'B'$ and let R' be the circumradius of triangle $A'B'C'$. From the triangle inequality, we have $b' < a' + c'$ and so, using the Extended Law of Sines, $2R'' \sin B' < 2R'' \sin A' + 2R'' \sin C'$. It follows that

$$\sin(120° - B) < \sin(120° - A) + \sin(120° - C),$$

implying

$$\frac{\sqrt{3}}{2}\cos B - \frac{1}{2}\sin B < \frac{\sqrt{3}}{2}\cos A - \frac{1}{2}\sin A + \frac{\sqrt{3}}{2}\cos C - \frac{1}{2}\sin C.$$

Hence the conclusion.

5.20. In any triangle ABC, show that

$$\cos\frac{A}{2} + \cos\frac{B}{2} > \cos\frac{C}{2}.$$

Solution. We use the ideas of the previous problem. We want to construct a triangle with angles

$$A' = 90° - \frac{A}{2},\ B' = 90° - \frac{B}{2},\ C' = 90° - \frac{C}{2}.$$

Because $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°$, there is in fact such a triangle. Following the reasoning of the previous solution, we find that

$$\sin\left(90° - \frac{A}{2}\right) + \sin\left(90° - \frac{B}{2}\right) > \sin\left(90° - \frac{C}{2}\right),$$

which is equivalent to the result we want.

Chapter 6

Symmetry in Algebra, Part II

The topic of algebraic symmetry is central to a study of many aspects of algebra. For this reason, we continue our discussion of it. We will come back to our central theme of inequalities in a later chapter.

We start with a simple quadratic equation. We know how to solve quadratic equations by factoring.

Example 6.1. Solve $x^2 - 5x + 6 = 0$.

Solution.

$$\begin{aligned}
&x^2 - 5x + 6 = 0 \\
&(x-3)(x-2) = 0 \\
&x = 3 \text{ or } x = 2.
\end{aligned}$$

So if $x - 2$ is a factor of the original polynomial, then $x = 2$ is a root of the associated polynomial equation. In fact, this will always work. There is nothing special about the number 2 or the factor $x - 2$:

It is not hard to see that the following statement is true in general:

Theorem. *If $x - a$ is a factor of the quadratic polynomial $P(x)$, then*

$$P(a) = 0. \tag{6.1}$$

Is the converse of this statement true? Let us look at another example:

Example 6.2. Solve $6x^2 - x - 1 = 0$.

Solution.

$$\begin{aligned}
&6x^2 - x - 1 = 0 \\
&(2x-1)(3x+1) = 0 \\
&2x - 1 = 0 \text{ or } 3x + 1 = 0 \\
&x = \frac{1}{2} \text{ or } x = -\frac{1}{3}.
\end{aligned}$$

We see that $x = \dfrac{1}{2}$ is a root of the equation we started with, but $x - \dfrac{1}{2}$ is not a factor. Or is it? If we had a bit more fondness for fractions, we could have rewritten the original equation as:

$$x^2 - \frac{x}{6} - \frac{1}{6} = 0$$
$$\left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = 0$$
$$x = \frac{1}{2} \text{ or } x = -\frac{1}{3}.$$

Why didn't we do this in the first place? Because factoring takes some guesswork, and it's easier for us to guess about integers than about rational numbers. But this is our failing, and not that of the equation we are solving. Considering the second solution of our equation, we can see that it is true that if $x = \frac{1}{2}$ is a root, then $x - \frac{1}{2}$ is a factor.

Indeed, the whole story is made simpler if we consider only quadratic equations whose lead coefficient (the coefficient of x^2) is 1. We will do so for the remainder of this chapter, and now the converse of statement (6.1) is in fact true. We can state the very interesting *Factor Theorem* for quadratic polynomials:

Theorem. *If the lead coefficient of the quadratic polynomial $P(x)$ is 1, then $x - a$ is a factor of $P(x)$ if and only if $P(a) = 0$.*

It is important to note that this statement is true for clumsy irrational roots as well as for neat integer roots or rational roots.

In working the next set of problems, it is important to understand the solutions to Problems 6.1 and 6.10. Please read our solutions, even after you've constructed your own.

Problems

6.1. Check that the roots of the quadratic equation $x^2 - 3x - 5 = 0$ are

$$\frac{3 + \sqrt{29}}{2} \text{ and } \frac{3 - \sqrt{29}}{2}.$$

Then show that the polynomial $x^2 - 3x - 5$ does indeed factor as

$$\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right).$$

6.2. If α and β are the roots of $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + 2\alpha\beta + \beta^2$.

6.3. If α and β are the roots of $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + \beta^2$.

6.4. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\alpha^2 + \beta^2$ (in terms of p and q).

6.5. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ in terms of p and q (you may assume that $\alpha, \beta \neq 0$).

♦

Example 6.3. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\alpha^3 + \beta^3$ (in terms of p and q).

Solution 1. By direct calculation (or applying the binomial theorem, if you are familiar with it), we have:

$$(\alpha + \beta)^3 = \alpha^2 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta),$$

so

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = p^3 - 3pq.$$

Solution 2. Since α and β are both solutions to the equation $x^2 - px + q = 0$. they must also satisfy $x^3 - px^2 + qx = 0$. So we have

$$\alpha^3 - p\alpha^2 + q\alpha = 0,$$
$$\beta^3 - p\beta^2 + q\beta = 0.$$

Adding, we have

$$\alpha^3 + \beta^3 - p(\alpha^2 + \beta^2) + q(\alpha + \beta) = 0,$$

or (from the result of Problem 6.4),

$$\alpha^3 + \beta^3 - p(p^2 - 2q) + pq = 0.$$

Hence

$$\alpha^3 + \beta^3 = p^3 - 2pq - pq = p^3 - 3pq,$$

as before.

Example 6.4. Use the result of Example 6.3 to find the value of $\alpha^4 + \beta^4$ (in terms of p and q).

Solution. Let S_4 be the expression in p and q that represents $\alpha^4 + \beta^4$. Let S_1, S_2, S_3 represent the corresponding expressions for $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$, respectively. We already know that:

$$S_1 = \alpha + \beta = p$$
$$S_2 = \alpha^2 + \beta^2 = p^2 - 2q$$
$$S_3 = \alpha^3 + \beta^3 = p^3 - 3pq.$$

We use the idea of solution 2 from Example 6.3. The numbers α and β must satisfy

$$\alpha^4 - p\alpha^3 + q\alpha^2 = 0$$
$$\beta^4 - p\beta^3 + q\beta^2 = 0.$$

Adding, we have

$$S_4 - pS_3 + qs_2 = 0 \quad \text{or} \quad S_4 = pS_3 - qS_2.$$

Substituting the values we already know, we have

$$S_4 = p(p^3 - 3pq) - q(p^2 - 2q) = p^4 - 3p^2q - p^2q + 2q^2 = p^4 - 4p^2q + 2q^2.$$

Note. Using this "bootstrapping" method (whose formal name is the *Principle of Mathematical Induction*), we can find the sum of any integral powers of α and β in terms of p and q.

Suppose we know, for instance, the value of S_{999} (the sum of the 999th powers of the roots) for $x^2 - px + q = 0$, in terms of p and q. And suppose we know S_{998} as well, in the same sense. We can then find the corresponding value of S_{1000}. We write:

$$\alpha^{1000} - p\alpha^{999} + q\alpha^{998} = 0,$$
$$\beta^{1000} - p\beta^{999} + q\beta^{998} = 0.$$

Adding, we find that $S_{1000} - pS_{999} + qS_{998} = 0$, so $S_{1000} = pS_{999} - qS_{998}$.

This relationship was noticed by Isaac Newton, who is credited with an explicit formula resulting from this insight. We won't develop the whole formula, but the reader is invited to explore it. In general, we can show (by mathematical induction, as we just indicated) that

$$S_n = pS_{n-1} - qS_{n-2}, \tag{6.2}$$

where $S_n = \alpha^n + \beta^n$.

♦

6.6. What value should we assign to S_0? For S_1? Show that equation (6.2) remains valid for $n = 2$.

6.7. Does equation (6.2) hold for $n = 1$? For $n = 0$? For $n = \frac{3}{2}$? Note that a proof by mathematical induction does not hold for these values of n.

6.8. Does equation (6.2) hold for $n = -1$? $n = -2$?

6.9. (Trick question!) If α and β are the roots of $x^2 - px + q = 0$ (with $\alpha \neq \beta$), find (in terms of p and q) the value of

$$\frac{\alpha + 2\beta}{\alpha - \beta} + \frac{2\alpha + \beta}{\beta - \alpha}.$$

♦

What has all this to do with symmetry? You can check that all the expressions whose values we are asked to compute in the problems above are symmetric in α and β. It turns out that each of the problems above is a particular example of a very general statement:

Any symmetric rational function of α and β can be represented in terms of the two functions $\alpha + \beta$ and $\alpha\beta$.

Equivalently: *Any symmetric rational function of α and β can be represented in terms of the coefficients of a quadratic equation with lead coefficient 1 and roots α and β.*

(A rational function is just a quotient of two polynomials.)

A proof of this statement is beyond the scope of this collection of problems, but can be found in many texts discussing symmetric rational functions.

This discussion contains the seeds of some profound mathematical results. Let us look at a cubic equation, say

$$x^3 + 3x^2 - x - 3 = 0.$$

If we knew the factors of the polynomial on the left, we could solve the equation by setting each factor equal to zero.

And the converse is also true: if we knew the roots of the equation, we could factor the associated polynomial. In fact, it is not difficult to check that 1, -1, and -3 are the roots of this particular equation, and that the polynomial $x^3 + 3x^2 - x - 3$ can be factored as $(x-1)(x+1)(x+3)$.

♦

6.10. Check that the assertions made above are correct.

For Problems 6.11 through 6.15, if α, β, and γ are the roots of the equation

$$x^3 - px^2 + qx - r = 0, \tag{6.3}$$

express in terms of p, q, and r the value of:

6.11. $\alpha^2 + \beta^2 + \gamma^2$.

6.12. $\alpha^3 + \beta^3 + \gamma^3$.

6.13. $\alpha^4 + \beta^4 + \gamma^4$.

6.14. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

6.15. $\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

6.16. Suppose α, β, and γ are roots of equation (6.3), and suppose S_n is an expression in p, q, and r equal to the sum $\alpha^n + \beta^n + \gamma^n$, where n is a natural number. Find an expression for S_n in terms of S_{n-1}, S_{n-2}, and S_{n-3}. This will generalize the "bootstrapping" operation described in the solution to Example 6.4.

Solve the following systems of equations:

6.17. $\begin{cases} \alpha + \beta &= 5 \\ \alpha\beta &= 6 \end{cases}$

6.18. $\begin{cases} \alpha + \beta &= 8 \\ \alpha\beta &= -7 \end{cases}$

6.19. $\begin{cases} \alpha + \beta + \gamma &= -9 \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 19 \\ \alpha\beta\gamma &= -11 \end{cases}$

6.20. $\begin{cases} \alpha + \beta + \gamma &= 5 \\ \alpha^2 + \beta^2 + \gamma^2 &= 29 \\ \alpha\beta\gamma &= -24 \end{cases}$

6.21. Solve the following system of equations:

$$\begin{cases} a + b + c &= k \\ a^2 + b^2 + c^2 &= k^2 \\ a^3 + b^3 + c^3 &= k^3 \end{cases}$$

(Moscow Mathematical Olympiad, 1937)

6.22. Let a, b, c, d, e be integers such that both

$$a + b + c + d + e \text{ and } a^2 + b^2 + c^2 + d^2 + e^2$$

are divisible by some odd integer n.

Prove that $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by n.

6.23. If ω is a root of the equation $x^2 + x + 1 = 0$, find the numerical value of

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5.$$

6.24. If ρ is a root of the equation $x^3 - 3x - 1 = 0$, show that

$$\rho^5 + \rho^4 + \rho^3 = 4\rho^2 + 13\rho + 4.$$

6.25. Let r_1 and r_2 be the roots of the equation $x^2 + 2x + 3 = 0$. Compute

$$\frac{r_1^2 + 4r_1 + 5}{r_1^2 + 5r_1 + 4} + \frac{r_2^2 + 4r_2 + 5}{r_2^2 + 5r_2 + 4}.$$

6.26. Let a, b, c be the zeros of the polynomial $x^3 + tx + u$.

Evaluate $a^3b + b^3c + c^3a$ in terms of t and u.

6.27. If p and q are complex numbers such that two of the roots a, b, c of the cubic equation $x^3 + 3px^2 + 3qx + 3pq = 0$ are equal, evaluate $a^2b + b^2c + c^2a$.

Solutions

6.1. Check that the roots of the quadratic equation $x^2 - 3x - 5 = 0$ are

$$\frac{3 + \sqrt{29}}{2} \text{ and } \frac{3 - \sqrt{29}}{2}.$$

Then show that the polynomial $x^2 - 3x - 5$ does indeed factor as

$$\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right).$$

Solution. (This is a lot of computation. You won't have to do this more than once or twice to get the idea.) We can rephrase Problem 6.1 slightly differently. Let us set

$$\alpha = \frac{3 + \sqrt{29}}{2}, \ \beta = \frac{3 - \sqrt{29}}{2}.$$

Then, noting that a quadratic polynomial can have only two linear factors, the problem actually states that

$$(x - \alpha)(x - \beta) = x^2 - 3x - 5.$$

If we multiply out the left-hand side, we get

$$x^2 - (\alpha + \beta)x + \alpha\beta.$$

Since this must equal $x^2 - 3x - 5$, it is not difficult to see that we must have

$$\alpha + \beta = 3$$
$$\alpha\beta = -5.$$

Again, there is nothing special about the awkward numbers α and β. The factor theorem for quadratic polynomials that we stated above has the following consequence:

The numbers α and β are roots of the polynomial equation $x^2 - px + q = 0$ if and only if $\alpha + \beta = p$ and $\alpha\beta = q$.

6.2. If α and β are the roots of $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + 2\alpha\beta + \beta^2$.

Solution. We have

$$\alpha^2 + 2\alpha\beta + \beta^2 = (\alpha + \beta)^2 = 3^2 = 9.$$

6.3. If α and β are the roots of $x^2 - 3x - 5 = 0$, find the numerical value of $\alpha^2 + \beta^2$.

Solution. Method I: We have

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 3^2 - 2(-5) = 19.$$

Method II: Since α and β are roots of the given equation, we know that

$$\alpha^2 - 3\alpha - 5 = 0$$
$$\beta^2 - 3\beta - 5 = 0$$

Adding we find that

$$\alpha^2 + \beta^2 - 3(\alpha + \beta) - 10 = 0,$$

or

$$\alpha^2 + \beta^2 - 3 \cdot 3 - 10 = 0,$$

(since $\alpha + \beta = 3$), or

$$\alpha^2 + \beta^2 = 9 + 10 = 19.$$

6.4. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\alpha^2 + \beta^2$ (in terms of p and q).

Solution. Using either Method I or Method II from Problem 6.3, we find that

$$\alpha^2 + \beta^2 = p^2 - 2q.$$

6.5. If α and β are the roots of $x^2 - px + q = 0$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$ in terms of p and q (you may assume that $\alpha, \beta \neq 0$).

Solution. We have

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{p}{q}.$$

6.6. What value should we assign to S_0? For S_1? Show that equation (6.1) remains valid for $n = 2$.

Solution. If $n = 0$, then we can write $S_0 = \alpha^0 + \beta^0 = 2$, no matter what equation we are talking about. If $n = 1$, then we can write

$$S_1 = \alpha^1 + \beta^1 = p.$$

Let us look at equation (6.1) for the case $n = 2$.
It asserts that $S_2 = pS_1 - qS_0$, or $\alpha^2 + \beta^2 = p \cdot p - q \cdot 2$, which we already know is true from Problem 6.4.

6.7. Does equation (6.1) hold for $n = 1$? For $n = 0$? For $n = \frac{3}{2}$? Note that a proof by mathematical induction does not hold for these values of n.

Solution. For $n = 1$ we have $S_1 = pS_0 - qS_{-1}$, or

$$\alpha + \beta = 2p - q\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{6.4}$$

We recall that $q = \alpha\beta$ and $p = \alpha + \beta$, so that we can rewrite (6.4) as:

$$\alpha + \beta = 2p - (\beta + \alpha) = 2(\alpha + \beta) - (\beta + \alpha),$$

which we can easily see is true.

For $n = 0$, we have $S_0 = pS_{-1} - qS_{-2}$, or

$$2 = (\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \alpha\beta\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right), \tag{6.5}$$

and we have a computation on our hands. We have:

$$(\alpha + \beta)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{\alpha + \beta}{\alpha} + \frac{\alpha + \beta}{\beta} = 1 + \frac{\beta}{\alpha} + 1 + \frac{\alpha}{\beta},$$

$$\alpha\beta\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = \frac{\beta}{\alpha} + \frac{\alpha}{\beta}.$$

Combining these two, we find that the right-hand side of (6.5) is indeed equal to 2. Equation (6.2) is true for $n = 0$.

Finally, for $n = \frac{3}{2}$, we have $S_{\frac{3}{2}} = pS_{\frac{1}{2}} - qS_{-\frac{1}{2}}$, or

$$\alpha^{\frac{3}{2}} + \beta^{\frac{3}{2}} = p\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right) - q\left(\frac{1}{\alpha^{\frac{1}{2}}} + \frac{1}{\beta^{\frac{1}{2}}}\right).$$

The first term on the right is equal to

$$(\alpha + \beta)\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right) = \alpha^{\frac{3}{2}} + \beta^{\frac{3}{2}} + \alpha^{\frac{1}{2}}\beta + \beta^{\frac{1}{2}}\alpha.$$

The second term is equal to

$$\alpha\beta\left(\frac{1}{\alpha^{\frac{1}{2}}} + \frac{1}{\beta^{\frac{1}{2}}}\right) = \alpha^{\frac{1}{2}}\beta + \beta^{\frac{1}{2}}\alpha.$$

Subtracting the second term from the first verifies that equation (6.2) is in fact true for $n = \frac{3}{2}$.

6.8. Does equation (6.1) hold for $n = -1$? $n = -2$?

Solution. We could go back to the definition of S_n and calculate. But there is a slightly easier way.

First note that in general:

$$S_{-n} = \frac{S_n}{q^n}.$$

(This is easy algebra.)

The problem asks if $S_{-1} = pS_{-2} - qS_{-3}$, which we now see is equivalent to

$$\frac{S_1}{q} = p\frac{S_2}{q^2} - q\frac{S_3}{q^3}.$$

Multiplying by q^2, we see that this in turn is equivalent to

$$qS_1 = pS_2 - S_3.$$

This, in turn, is equivalent to

$$S_3 = pS_2 - qS_1,$$

which is just a special case of equation (6.2).

The case for S_{-3} is exactly the same, and the situation generalized for negative integers.

6.9. (Trick question!) If α and β are the roots of $x^2 - px + q = 0$ (with $\alpha \neq \beta$), find (in terms of p and q) the value of

$$\frac{\alpha + 2\beta}{\alpha - \beta} + \frac{2\alpha + \beta}{\beta - \alpha}.$$

Solution. Adding the two fractions, we find that their value is simply -1, and is not dependent either on α or β or any equation they might satisfy (!).

6.10. Check that the assertions made above are correct.

Solution. In short, the Factor Theorem for Quadratic Polynomials can be extended to polynomial equations of higher degree:

Factor Theorem. *The polynomial $P(x)$, with lead coefficient 1, has a factor $x - a$ if and only if $P(a) = 0$.*

A proof of the Factor Theorem – really just a restatement of observations we've already made – can be found in standard textbooks.

We can apply this general statement to a cubic equation, just to get used to what it says. Suppose we start with a cubic polynomial $P(x)$ whose lead coefficient is 1. Then $P(x)$ factors as $(x - \alpha)(x - \beta)(x - \gamma)$ if and only if α, β and γ are roots of the equation $P(x) = 0$.
If $P(x) = x^3 - px^2 + qx - r$, then we can write:

$$\begin{aligned} x^3 - px^2 + qx - r &= (x - \alpha)(x - \beta)(x - \gamma) \\ &= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma. \end{aligned}$$

It follows that

$$\begin{aligned} \alpha + \beta + \gamma &= p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= q \\ \alpha\beta\gamma &= r. \end{aligned}$$

For Problems 6.11 through 6.15, if α, β, and γ are the roots of the equation

$$x^3 - px^2 + qx - r = 0, \tag{6.6}$$

express in terms of p, q, and r the value of:

6.11. $\alpha^2 + \beta^2 + \gamma^2$.

Solution. Following Problem 6.3, Method II, we find that

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma),$$

or

$$p^2 = \alpha^2 + \beta^2 + \gamma^2 + 2q.$$

It follows that

$$\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q.$$

Compare this result with the result of Problem 6.4.
How does it generalize?

6.12. $\alpha^3 + \beta^3 + \gamma^3$.

Solution. We again follow Problem 6.3, Method II:

$$\begin{aligned} \alpha^3 - p\alpha^2 + q\alpha - r &= 0 \\ \beta^3 - p\beta^2 + q\beta - r &= 0 \\ \gamma^3 - p\gamma^2 + q\gamma - r &= 0. \end{aligned}$$

Adding, we have

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 &= p(\alpha^2 + \beta^2 + \gamma^2) - q(\alpha + \beta + \gamma) + 3r \\ &= p(p^2 - 2q) - pq + 3r \\ &= p^3 - 3pq + 3r. \end{aligned}$$

6.13. $\alpha^4 + \beta^4 + \gamma^4$.

Solution. We can "bootstrap" this computation, as we did in Problem 6.3, Method II, to find the sums of higher powers of α, β and γ. In fact, we have been beaten to this idea by none other than Sir Isaac Newton, who is credited with discovering a general formula for the sums of powers of roots of a polynomial equation.

Suppose α, β, γ are the roots of

$$x^3 - px^2 + qx - r = 0.$$

Then $x^4 - px^3 + qx^2 - rx = 0$ as well, so

$$\alpha^4 - p\alpha^3 + q\alpha^2 - r\alpha = 0$$
$$\beta^4 - p\beta^3 + q\beta^2 - r\beta = 0$$
$$\gamma^4 - p\gamma^3 + q\gamma^2 - r\gamma = 0.$$

Adding, we obtain

$$\alpha^4 + \beta^4 + \gamma^4 - p(\alpha^3 + \beta^3 + \gamma^3) + q(\alpha^2 + \beta^2 + \gamma^2) - r(\alpha + \beta + \gamma) = 0,$$

or

$$\alpha^4 + \beta^4 + \gamma^4 - p(p^3 - 3pq + 3r) + q(p^2 - 2q) - rp = 0.$$

This gives us the required expression:

$$\alpha^4 + \beta^4 + \gamma^4 = p^4 - 4p^2q + 4pr + 2q^2.$$

6.14. $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

Solution. The given expression equals

$$\frac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}.$$

The denominator of this fraction is clearly r^2. The numerator looks like it is related to q^2. And indeed, we have

$$q^2 = (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + 2\alpha\beta\gamma(\alpha + \beta + \gamma).$$

Using this information, we quickly find that the required expression equals

$$\frac{q^2 - 2pr}{r^2}.$$

6.15. $\dfrac{\alpha\beta}{\gamma} + \dfrac{\alpha\gamma}{\beta} + \dfrac{\beta\gamma}{\alpha}$. (Assume that $\alpha, \beta, \gamma \neq 0$.)

Solution. The given expression equals

$$\frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha\beta\gamma}.$$

Using the computation from Problem 6.14, we find that this equals

$$\frac{q^2 - 2pr}{r}.$$

6.16. Suppose α, β, and γ are roots of equation (6.3), and suppose S_n is an expression in p, q, and r equal to the sum $\alpha^n + \beta^n + \gamma^n$, where n is a natural number. Find an expression for S_n in terms of S_{n-1}, S_{n-2}, and S_{n-3}. This will generalize the "bootstrapping" operation described in the solution to Example 6.4.

Solution. As in Example 6.4, we know that α, β, γ are solutions to the equation

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} = 0,$$

so we have:

$$\alpha^n - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} = 0$$
$$\beta^n - p\beta^{n-1} + q\beta^{n-2} - r\beta^{n-3} = 0$$
$$\gamma^n - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} = 0.$$

Adding gives us

$$S_n - pS_{n-1} + qS_{n-2} - rS_{n-3} = 0,$$

so

$$S_n = pS_{n-1} - qS_{n-2} + rS_{n-3},$$

which gives the required expression.

As indicated in earlier problems, this idea can be generalized significantly, to apply to any value of n (not just natural numbers) and to equations of any degree (not just quadratic and cubic equations).

Solve the following systems of equations:

6.17. $\begin{cases} \alpha + \beta &= 5 \\ \alpha\beta &= 6 \end{cases}$

Solution. This system of equations is not difficult to solve, for example, by substitution. But we can make the solution easier still by forming a quadratic equation with roots α and β. This equation is $x^2 - 5x + 6 = 0$, and it can be solved easily by factoring. Its roots are 2 and 3. Hence $\alpha = 2$ and $\beta = 3$, or $\alpha = 3$ and $\beta = 2$. Note that the solution relies on the fact that the equations are symmetric in α and β.

6.18. $\begin{cases} \alpha + \beta &= 8 \\ \alpha\beta &= -7 \end{cases}$

Solution. We can proceed as in Problem 6.17. The equation satisfied by α and β is $x^2 - 8x - 7 = 0$, and this time it doesn't factor over the integers, so we use the quadratic formula to find that $\alpha = 4 + \sqrt{23}$, $\beta = 4 - \sqrt{23}$, or vice versa.

6.19. $\begin{cases} \alpha + \beta + \gamma &= -9 \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 19 \\ \alpha\beta\gamma &= -11 \end{cases}$

Solution. Let the numbers α, β, γ be solutions to the equation

$$x^3 + 9x^2 + 19x + 11 = 0.$$

By inspection, $x = -1$ is a solution to this equation, so $x + 1$ is a factor of the polynomial. Division reveals that the other factor is $x^2 + 8x + 11$. Setting this factor equal to zero, we find that the other two roots are $-4 \pm \sqrt{5}$. But which value is α? The symmetry tells us that α could be any of the numbers 1, $-4 + \sqrt{5}$, $-4 - \sqrt{5}$, and β and γ are the other two. There are six solutions in all.

6.20. $\begin{cases} \alpha + \beta + \gamma &= 5 \\ \alpha^2 + \beta^2 + \gamma^2 &= 29 \\ \alpha\beta\gamma &= -24 \end{cases}$

Solution. We would like to write down a cubic equation

$$x^3 - px^2 + qx - r = 0$$

whose roots are α, β, γ.
We have easily $p = \alpha + \beta + \gamma = 5$ and $r = \alpha\beta\gamma = -24$. But how do we get $q = \alpha\beta + \alpha\gamma + \beta\gamma$? We can use the results of Problem 6.11. From that result, we have:

$$\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q = 29,$$

from which it follows that $q = -2$.

Now we know that the required equation is

$$x^3 - 5x^2 - 2x + 24 = 0.$$

As in Problem 6.19, we can find one root from inspection, then the others by division. We find that the values of α, β, γ are $\{-2, 3, 4\}$ (in any order).

6.21. Solve the following system of equations:

$$\begin{cases} a + b + c &= k \\ a^2 + b^2 + c^2 &= k^2 \\ a^3 + b^3 + c^3 &= k^3 \end{cases}$$

(Moscow Mathematical Olympiad, 1937)

Solution. Suppose a, b, and c are the roots of the cubic equation

$$x^3 - px^2 + qx - r = 0.$$

Then we have immediately $p = a + b + c = k$. Let us find relationships among p, q, and r.
We can get a relationship involving q and r using the technique of Problem 6.3, Method II (which is truly a method: we used it in Problems 6.6, 6.11, 6.12, and 6.13 as well):

$$a^3 - pa^2 + qa - r = 0$$
$$b^3 - pb^2 + qb - r = 0$$
$$c^3 - pc^2 + qc - r = 0.$$

Adding, we have $k^3 - pk^2 + qk - 3r = 0$. But $p = k$, so this becomes simply $qk - 3r = 0$.
We can get another such relationship using the technique of Problem 6.3, Method I (which is also used in Problems 6.11, 6.12, and 6.13):

$$p^2 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac) = k^2 + 2q.$$

So $k^2 + 2q = p^2 = k^2$ and $q = 0$. But we know $qk - 3r = 0$, so r is also 0, and the equation collapses to $x^3 - kx^2 = 0$, whose roots are 0, 0, k in some order. There are three solutions: $(0, 0, k)$, $(0, k, 0)$, and $(k, 0, 0)$.

6.22. Let a, b, c, d, e be integers such that both

$$a + b + c + d + e \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 + e^2$$

are divisible by some odd integer n.
Prove that $a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$ is also divisible by n.

Solution. Let

$$P(x) = x^5 - px^4 + qx^3 - rx^2 + sx - t$$

be the polynomial with lead coefficient 1 and roots a, b, c, d, e. We proceed as in Problem 6.3, Method II. Writing algebraically the fact that these roots satisfy $P(x) = 0$ and adding, we have

$$\begin{aligned} a^5 + b^5 + c^5 + d^5 + c^5 - p(a^4 + b^4 + c^4 + d^4 + e^4) & \\ + q(a^3 + b^3 + c^3 + d^3 + e^3) & \\ - r(a^2 + b^2 + c^2 + d^2 + e^2) & \\ + s(a + b + c + d + e) & \\ - 5t = 0. & \end{aligned}$$

Examining this equation, we will show that all the terms except the sum of the first and last must be multiples of n. Since the sum of all the terms is zero, this will show that the sum of the first and last is also a multiple of n.

We are given that p is a multiple of n (it is the sum of the five given numbers). For the same reason, the term containing s above is a multiple of n. And in the term containing r, the second factor is, by hypothesis, a multiple of n. So it remains to show that the term containing q is a multiple of n.

But this is easy: we will show that q itself is a multiple of n. Indeed, an argument analogous to that in Problems 6.4 and 6.11 shows that $a^2 + b^2 + c^2 + d^2 + e^2 = p^2 - 2q$. Since the left side of the equation and p are, by hypothesis, a multiple of n, so is $2q$. But n is odd, so q itself must be a multiple of n.

Thus all the terms on the left side of our equation, except the first and last, are multiples of n. So the sum of the first and last must also be a multiple of n, and this is the required result.

Note: We have used, several times, the following principle: *In an equation involving integers, if several of the terms are all multiples of some number n, then the sum of the remaining terms is also a multiple of n.* This principle is very useful, especially when all but one term of an equation is a multiple of n.

6.23. If ω is a root of the equation $x^2 + x + 1 = 0$, find the numerical value of

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5.$$

Solution. We know how to solve quadratic equations explicitly. So we could in fact get the value(s) of ω, which are complex numbers, and compute directly the value of the given expression. But that is too much work. Let us look at a less direct but easier way to solve the problem.

Since ω satisfies the given equation, we know that $\omega^2+\omega+1=0$, so the first three terms of the given expression sum to 0. We can extend this observation to deal with the remaining terms:

$$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5 = 1+\omega+\omega^2+\omega^3(1+\omega+\omega^2) = 0.$$

Note first that there are actually two possible values of ω, because there are two roots of the quadratic equation. The given expression sums to 0 for either of these values.

Note also that $(x-1)(x^2+x+1) = x^3-1$, so ω is also a solution to the equation $x^3=1$; that is, ω is a "cube root of unity". More generally, the equation $x^n-1=0$, for positive integers n, has n different complex roots, which play very important roles in many topics in algebra and number theory.

6.24. If ρ is a root of the equation $x^3-3x-1=0$, show that

$$\rho^5+\rho^4+\rho^3 = 4\rho^2+13\rho+4.$$

Solution. We have:

$$\begin{aligned}\rho^3 &= 3\rho+1\\ \rho^4 &= 3\rho^2+\rho\\ \rho^5 &= 3\rho^3+\rho^2\\ &= 3(3\rho+1)+\rho^2\\ &= \rho^2+9\rho+3.\end{aligned}$$

Hence

$$\rho^5+\rho^4+\rho^3 = (\rho^2+9\rho+3)+(3\rho^2+\rho)+(3\rho+1) = 4\rho^2+13\rho+4.$$

Using the same method, we can reduce any polynomial in ρ to a quadratic polynomial in ρ. More generally, if ρ is a root of a polynomial equation of degree n, then a polynomial in ρ can be reduced to another polynomial in ρ, of degree less than n.

6.25. Let r_1 and r_2 be the roots of the equation $x^2+2x+3=0$. Compute

$$\frac{r_1^2+4r_1+5}{r_1^2+5r_1+4}+\frac{r_2^2+4r_2+5}{r_2^2+5r_2+4}.$$

Solution. Since r_1 and r_2 are the roots of a quadratic equation, we can express r_1^2 and r_2^2 in terms of r_1 and r_2:

$$r_1^2 = -2r_1-3 = 0 \text{ and } r_2^2 = -2r_2-3.$$

So the expression whose value we want to compute can be written as:

$$\frac{2r_1+2}{3r_1+1}+\frac{2r_2+2}{3r_2+1} = \frac{12r_1r_2+8(r_1+r_2)+4}{9r_1r_2+3(r_1+r_2)+1}.$$

Since $r_1+r_2=-2$ and $r_1r_2=3$, the value of the expression is

$$\frac{36-16+4}{27-6+1} = \frac{12}{11}.$$

6.26. Let a, b, c be the zeros of the polynomial $x^3 + tx + u$.
Evaluate $a^3b + b^3c + c^3a$ in terms of t and u.

Solution. From the relations between the zeros and coefficients, we have $a + b + c = 0$, $ab + bc + ca = t$ and $abc = -u$. Replacing, for instance, a by $-(b + c)$ in the expression that we need to evaluate will break the cyclic symmetry. We will preserve this symmetry and write the condition that a, b, c are zeros of the given polynomial:

$$\begin{aligned} a^3 &= -(ta + u), \\ b^3 &= -(tb + u), \\ c^3 &= -(tc + u). \end{aligned}$$

Hence

$$\begin{aligned} a^3b + b^3c + c^3a &= -(tab + ub) - (tbc + uc) - (tca + ua) \\ &= -t(ab + bc + ca) - u(a + b + c) = -t^2. \end{aligned}$$

6.27. If p and q are complex numbers such that two of the roots a, b, c of the cubic equation $x^3 + 3px^2 + 3qx + 3pq = 0$ are equal, evaluate $a^2b + b^2c + c^2a$.

Solution. The desired expression is not exactly symmetric in a, b, c. So, for example, if we assumed that the two equal roots are a and b, we would need to evaluate $a^3 + a^2c + ac^2$, which is a mess.
But the expression is *cyclically symmetric*. That is, if we replace a with b, b with c, and c with a, we get the same expression back.
So let us keep the cyclic symmetry.
The key is to note that $(a - b)(b - c)(c - a) = 0$, no matter which of the two roots are equal. Then we have:

$$\begin{aligned} 0 &= (a - b)(b - c)(c - a) \\ &= abc - a^2b - b^2c - c^2a + ab^2 + bc^2 + ca^2 - abc \\ &= ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2a), \end{aligned}$$

so that

$$a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2.$$

Then

$$\begin{aligned} 2(a^2b + b^2c + c^2a) &= (a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) \\ &= (a + b + c)(ab + bc + ca) - 3abc \\ &= (-3p)(3q) - 3(-3pq) = 0, \end{aligned}$$

which implies that

$$a^2b + b^2c + c^2a = 0.$$

Chapter 7

Symmetry in Algebra, Part III

Let's go back to basics. Suppose we wanted to factor $x^3 - 5x^2 + 5x - 1$. We can note that if $x = 1$, the value of the given polynomial is 0. It follows from the factor theorem that $(x - 1)$ is a factor of the polynomial, and we can obtain the other factor by division. Indeed,

$$x^3 - 5x^2 + 5x - 1 = (x - 1)(x^2 - 4x + 1).$$

Remember the Factor Theorem?

If the value of a polynomial is 0 when $x = k$, we sometimes say that the polynomial *vanishes* when $x = k$.

Factor Theorem. *For any number a, $(x - a)$ is a factor of the polynomial $P(x)$ if and only if $P(a) = 0$.*

Problems

7.1. Factor $150x^2 - 77x - 73$.
7.2. Factor $x^3 + 15x^2 + 15x + 1$.
7.3. Factor $x^3 - 1$.
7.4. Factor $x^3 - 7x^2 + 7x - 1$.
7.5. Factor $x^3 - 137x^2 + 137x - 1$.
7.6. Factor $x^3 - ax^2 + ax - 1$.
7.7. Factor $x^3 + ax^2 + ax + 1$.
7.8. Factor $x^3 - ax^2 + 2ax - 2a^2$.
Hint. What happens if $x = a$?
7.9. Factor $x^4 - 6x^3y + 4xy^3 + y^4$.
7.10. Factor $ab(a - b) + bc(b - c) + ca(c - a)$.
7.11. Factor $(a - b)^3 + (b - c)^3 + (c - a)^3$.
7.12. Factor $(a + b + c)^3 - (a^3 + b^3 + c^3)$.
Hint. What happens if $a = -b$?
7.13. For any three real numbers a, b, c, x, with a, b, c all distinct, prove that

$$a^2 \frac{(x - b)(x - c)}{(a - b)(a - c)} + b^2 \frac{(x - c)(x - a)}{(b - c)(b - a)} + c^2 \frac{(x - a)(x - b)}{(c - a)(c - b)} = x^2.$$

Hint. Consider the problem as an equation in x. What is its degree? How many roots can you find by inspection? What kind of equation has more roots than its degree?

7.14. For all real numbers a, b, c, x with a, b, c all distinct, prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(x-b)} = 1.$$

7.15. Let m and n be two odd integers. Show that

$$\frac{1}{a^m + b^m + c^m} = \frac{1}{a^m} + \frac{1}{b^m} + \frac{1}{c^m},$$

if and only if

$$\frac{1}{a^n + b^n + c^n} = \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}.$$

Hint. One approach is to construct a cubic equation for which a^m, b^m, c^m are the roots. Then guess at one of the roots of the equation.

7.16. Factor $x^3 + y^3 + z^3 - 3xyz$.

Hint 1. Try letting $y + z = -x$.

Hint 2. Alternatively, and if you know something about determinants, note that the given polynomial is equal to the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

7.17. Let the symbol $\overline{abc}$ denote the decimal numeral with a in the hundreds place, b in the tens place, and c in the units place. Prove that if the numbers $\overline{abc}$, $\overline{bca}$, $\overline{cab}$ are all divisible by some integer n, then $a^3 + b^3 + c^3 - 3abc$ is also divisible by n. (Note: The solution we give depends on properties of determinants, and is related to the second solution to Problem 7.16.)

Solutions

7.1. Factor $150x^2 - 77x - 73$.

Solution. If we let $x = 1$, the value of $150x^2 - 77x - 73$ is 0. Thus $x-1$ is a factor. Using division of polynomials, or otherwise, we quickly find out that the other factor is $150x + 73$.

7.2. Factor $x^3 + 15x^2 + 15x + 1$.

Solution. The given polynomial vanishes when $x = -1$. Thus one factor is $x = 1$, and the other factor turns out to be $x^2 + 14x + 1$.

7.3. Factor $x^3 - 1$.

Solution. The given polynomial vanishes when $x = 1$. This leads to the factorization

$$x^3 - 1 = (x-1)(x^2 + x + 1).$$

Some readers may have encountered this factorization already. Both x^3-1 and x^3+1 can be factored, and it may be difficult to remember how each factored form looks. But if we recall the factor theorem, it is easy to see that $x-1$ must be a factor of x^3-1 and that $x+1$ must be a factor of x^3-1.

7.4. Factor x^3-7x^2+7x-1.

Solution. Answer: $(x-1)(x^2-6x+1)$.

7.5. Factor $x^3-137x^2+137x-1$.

Solution. Answer: $(x-1)(x^2-136x+1)$.

What's going on? Problems 7.4 and 7.5 are not very interesting. What's interesting is the pattern that they indicate.

7.6. Factor x^3-ax^2+ax-1.

Solution. Once more it is clear that one factor of this polynomial is $x-1$. We can obtain the other factor easily, for example by division: it is $x^2+(1-a)x+1$. Thus we have the complicated looking, but really not so difficult, identity:

$$x^3-ax^2+ax-1=(x-1)(x^2+(1-a)x+1),$$

which can be checked by multiplication. The reader is invited to look back at Problems 7.4 and 7.5 to see that the answers are indeed of this form.

7.7. Factor x^3+ax^2+ax+1.

Solution. The polynomial vanishes when $x=-1$. This allows us to find the factorization

$$(x+1)(x^2+(a-1)x+1).$$

7.8. Factor $x^3-ax^2+2ax-2a^2$.

Solution. Since the polynomial vanishes when $x=a$, one factor is $x-a$. Thus we have the factorization $(x-a)(x^2+2a)$.

7.9. Factor $x^4-6x^3y+4xy^3+y^4$.

Solution. The polynomial vanishes when $x=y$, so we get the factorization

$$(x-y)(x^3-5x^2y-5xy^2-y^3).$$

7.10. Factor $ab(a-b)+bc(b-c)+ca(c-a)$.

Solution. Let us first consider this expression as a polynomial in a, and think of b and c as "constants". The polynomial vanishes when $a=b$ and when $a=c$, so it has factors $(a-b)$ and $(a-c)$. Now let us consider the expression as a polynomial in b. We already know that it vanishes when $b=a$, but it also vanishes when $b=c$. Thus it has another factor of $(b-c)$. Therefore we can write:

$$ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(a-c)(b-c)M,$$

where M is some polynomial in a, b, and c. Let us think again of these two expressions (whose identity is being asserted) as polynomials in a. Then the left-hand polynomial is quadratic in a, so the right side must also be quadratic in a, and M cannot contain any positive powers of a. But the same is true for b and c, so M must be a constant. We can find the value of the constant, for instance, by plugging in numerical values for a, b, and c. We quickly find that $M = 1$.

7.11. Factor $(a-b)^3 + (b-c)^3 + (c-a)^3$.

Solution. We know that $(a-b)^3 = -(b-a)^3$, and the expression vanishes when $a = b$, and also when $a = c$ and $b = c$. So once again, we can write

$$(a-b)^3 + (b-c)^3 + (c-a)^3 = (a-b)(b-c)(c-a)M,$$

where M is a polynomial in a, b, and c. It is somewhat surprising, but still true, that the original expression is quadratic (and not cubic) in a, and so is the expression

$$(a-b)(b-c)(a-c).$$

And of course (by symmetry) the same holds for b and c. It follows once more that M is a constant, and some judicious plugging in of numbers (try $a = 3$, $b = 2$, $c = 1$) will show that $M = 3$.

7.12. Factor $(a+b+c)^3 - (a^3 + b^3 + c^3)$.

Solution. The expression vanishes when $a = -b$, when $a = -c$, and when $b = -c$. Thus it has factors $(a+b)(b+c)(a+c)$. An argument similar to those used in the previous solutions lets us conclude that

$$(a+b+c)^3 - (a^3 + b^3 + c^3) = 3(a+b)(a+c)(b+c).$$

7.13. For all real numbers a, b, c, x with a, b, c all distinct, prove that

$$a^2 \frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2 \frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2 \frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

Solution. Following the hint, we note that the equation is quadratic in x. Furthermore, it is true when $x = a$, $x = b$, and $x = c$. If any two of these values are equal, the left side of the equation has no sense. Thus we are looking at a quadratic equation satisfied by three different numbers, which must therefore be an identity.

7.14. For all real numbers a, b, c, x with a, b, c all distinct, prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(x-b)} = 1.$$

Solution. As in Problem 7.13, we can think of this as an equation in x, and once again, it is quadratic. It is satisfied when $x = a$, $x = b$, or $x = c$, and exactly the same reasoning applies as in Problem 7.13, leading to the desired conclusion.

7.15. Let m and n be two odd integers. Show that

$$\frac{1}{a^m + b^m + c^m} = \frac{1}{a^m} + \frac{1}{b^m} + \frac{1}{c^m},$$

if and only if

$$\frac{1}{a^n + b^n + c^n} = \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}.$$

Solution. We will show that both assertions in the problem (for exponent m and for exponent n) are equivalent to the statement that there are two "opposite" numbers among a, b, and c (that is, that $a = -b$, or $b = -c$, or $c = -a$).

Certainly if the set $\{a, b, c\}$ contains two opposite numbers, then for any odd exponent k, we have

$$\frac{1}{a^k + b^k + c^k} = \frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k}.$$

Let us prove the converse. Following the given hints, we suppose the cubic equation

$$x^3 - px^2 + qx - r = 0$$

has roots a^k, b^k, c^k. Then, since

$$\frac{1}{a^k + b^k + c^k} = \frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k},$$

we know that $r = pq$, so that the equation has the form

$$x^3 - px^2 + qx - pq = 0.$$

The left side vanishes when $x = p$, and so factors into $(x - p)(x^2 + q)$. Thus p is itself one root of the equation. This means that $a^k + b^k + c^k$ has one of the values a^k, b^k, or c^k. If $a^k + b^k + c^k = a^k$, then $b^k + c^k = 0$, so $b^k = -c^k$, and (since k is odd), $b = -c$. A similar conclusion follows if $a^k + b^k + c^k = b^k$, or if it equals c^k.

If this solution is difficult to read, try formulating the problem with $m = 1$ first, then looking at the general situation.

7.16. Factor $x^3 + y^3 + z^3 - 3xyz$.

Solution. Method I, using hint 1: When $x = -(y + z)$, a simple computation shows that the polynomial vanishes. So $x + y + z$ is a factor, and the other factor can be obtained by long division.

Method II, using hint 2: A computation will show that

$$x^3 + y^3 + x^3 - 3xyz$$

is indeed equal to

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

Then, computing with determinants, we find

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \begin{vmatrix} x+y+z & y & z \\ x+y+z & x & y \\ x+y+z & z & x \end{vmatrix} = (x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix}$$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

7.17. Let the symbol $\overline{abc}$ denote the decimal numeral with a in the hundreds place, b in the tens place and c in the units place. Prove that if the numbers $\overline{abc}$, $\overline{bca}$, $\overline{cab}$ are all divisible by some integer n, then the value of $a^3+b^3+c^3-3abc$ is also divisible by n.

Solution. As in Problem 7.16, we write

$$a^3+b^3+c^3-3abc = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Multiplying the first column by 100, the second by 10, and adding these to the third column, we find that

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 100a+10b+c & b & c \\ 100c+10a+b & a & b \\ 100b+10c+a & c & a \end{vmatrix},$$

so

$$a^3+b^3+c^3-3abc = \begin{vmatrix} \overline{abc} & b & c \\ \overline{cab} & a & b \\ \overline{bca} & c & a \end{vmatrix}.$$

And since the numbers $\overline{abc}$, $\overline{cab}$, $\overline{bca}$ are all divisible by n, so is

$$a^3+b^3+c^3-3abc.$$

Chapter 8

The Rearrangement Inequality

Suppose you have two sacks of coins. One is full of nickels, and the other is full of dimes. You're allowed to take a coins from one sack, and b coins from the other sack, where $a > b$. And suppose you want the most money you can get. (Not unlikely!)

Clearly you want to take a dimes and b nickels: you want the largest number of the most valuable coin. And clearly this would not depend on the actual value of the coins, but only on the fact that the coins in each sack are identical. If one sack is full of coins worth x cents, and the other is full of coins worth y cents, with $x > y$, then you will want to take a coins worth x cents, and b coins worth y cents.

That is, if $a > b$ and $x > y$, then $ax + by > ay + bx$.

Example 8.1. Prove this inequality.

Hint. Find a way to factor, then show that something is never negative.

Solution. There is nothing to factor in the form given. But the hint suggests that we show that something is ≥ 0, so let's get a zero into the picture:

$$\begin{aligned}
&ax + by > ay + bx\\
&ax + by - ay - bx > 0\\
&ax - ay + bx - by > 0\\
&a(x - y) + b(x - y) > 0\\
&(a - b)(x - y) > 0,
\end{aligned}$$

which is certainly true, since $a > b$ and $x > y$.

Example 8.2. What is the case for equality?

Solution. The case for equality is when $(a - b)(x - y) = 0$, which means that either $a - b = 0$ or $x - y = 0$. Many find this unexpected, so check some examples.

Let's philosophize a bit. Isn't it obvious to anyone sufficiently greedy that you should take the larger number of coins from the sack with the greater value? Why do we need to prove it? What have we learned from the proof?

For one thing, we have found the case for equality, which is not so obvious to most people.

For another thing the proof is not so easy. It's not as simple as just multiplying some inequalities and adding. So we got to show how clever we are.

But we already know how clever we are. Why indulge in a proof of something which we know, in our hearts and our pocketbooks, must be true?

A deeper answer lies in the meaning of mathematics. When we do mathematics, we don't really care whether a statement describes a real situation or is a lie. We don't care who made us choose the coins, or what we are going to spend them on. We don't characterize anyone, or anything, as "greedy". We simply decide whether one statement follows from another.

That is, doing mathematics does not involve observation, or value judgment, or description of reality. Doing mathematics means linking statements, creating a fabric of truths, all interwoven. We want to start with the most simple truths, those that are least likely to fail us in reality. But after this, we don't worry about reality. Our concern is whether new statements follow from the ones we've already accepted.

So in our case, all our senses, all our feelings, all our greed, tells us to take the larger number of the more valuable coins. But as mathematicians, we must show how this statement follows from others. That is what the proof shows.

The General Rearrangement Inequality

Now suppose you have more than two sacks of coins, say seven sacks. Each sack is filled with coins of the same type, and coins from different sacks have different values. You want to take 3 coins from one sack, and 2 coins from a second sack, and 10 coins from a third sack, and so on; that is, the number of coins per sack is fixed. Of course you want to maximize the value of the coins you will be taking. How do you do this?

Well, the answer is just as "obvious": you must take the largest number of coins from the most valuable sack, the second largest number of coins from the second most valuable sack, ... and the smallest number of coins from the least valuable sack.

And of course the same is true for any number of sacks, and any collection of "weights" (numbers of coins to be taken from each sack). If we have weights $x_1 < x_2 < x_3 < \ldots < x_n$ and values $a_1 < a_2 < a_3 < \ldots < a_n$, then we get the largest value by taking the sum $x_1a_1 + x_2a_2 + x_3a_3 + \ldots + x_na_n$.

How do we prove this? So that we don't just "look smart"? One idea for a proof is relatively simple. It is based on the case $n = 2$ which we've already proved. Suppose we have a sum of coins we have chosen, and we have taken a smaller number of coins from a sack with a larger value. That is, suppose within our sum we have $a_ix_i + a_jx_j$, for some sacks numbered i and j, and while $x_i < x_j$, we also have $a_i > a_j$.

If we've done that, then we can increase our sum merely by interchanging a_i and a_j. For the case $n = 2$ says just this: if $x_i < x_j$ (note that this is what we had assumed) and $a_j < a_i$, then $a_j x_i + a_i x_j > a_i x_i + a_j x_j$. That is, we can increase our sum as long as we can rearrange even two weights that are not in increasing order. Another way to say this is that we can increase our sum if even one of the weights is out of order.

And yet another way to say this is to say that the largest possible sum is the one where the weights and the values are all in order.

Notice that this is an indirect proof. We assumed that some arrangement that was not in order was maximal, and showed that in fact it could not be maximal. This contradiction shows us that the maximal arrangement is the one in which both the weights and the values are in order.

Permutations

There is something else odd about this proof. We haven't quite proved an inequality. Where is the "greater than" or "less than" sign? Some readers may find this formulation sufficient. We give below a more formal treatment of the same proof, using the notion of a *permutation*. Permutations are among the most important objects in algebra. Even if the reader is convinced by the informal proof above, it may be useful to understand the formality below.

Simply put, a permutation is a rearrangement of a list of objects. So, for example, if we have an ordered list of seven numbers, say $(3, 1, 2, 5, 6, 3, 6)$ numbers (often called an "ordered 7-tuple"), then the following are permutations of that list:

$$\begin{gathered}(3, 1, 2, 6, 5, 3, 6)\\(2, 3, 1, 5, 6, 6, 3)\\(1, 2, 3, 3, 5, 6, 6),\end{gathered}$$

and so on.

We also need some notation for a general permutation. We will use "hat" notation:

If the list of weights is $(x_1, x_2, \ldots, x_7)$, then a permutation of this list will be written as $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_7$, where $\widehat{x}_1$ is one of the weights (which may or may not be x_1 itself), $\widehat{x}_2$ is another of the weights, and so on.

We can think of the "hats" as disguises: each x_i is under one of the hats, but we cannot tell (i.e., we must not assume that we know) which x_i is masquerading as $\widehat{x}_j$.

More formally still, a permutation is nothing more than a function, whose domain is the ordered list $(x_1, x_2, \ldots, x_n)$, and $\widehat{x}_i$ is simply the image of x_i under this function. But we won't need this much formality for our discussion. All we need is a way to write down a rearrangement of the x_i's.

Now we can state the result formally, for the case $n = 7$. Suppose our weights are $x_1 < x_2 < x_3 < \ldots < x_7$, and our values (of the coins in the sacks) are $a_1 < a_2 < a_3 < \ldots < a_7$. Suppose that $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_7$ is a permutation of the set of weights. Then

$$x_1a_1 + x_2a_2 + \ldots + x_na_n \geq \widehat{x}_1a_1 + \widehat{x}_2a_2 + \ldots + \widehat{x}_7a_7.$$

Notice that we don't need to consider permutations of the values a_i (of the coins). It is enough to consider permutations of the weights. Notice also that the original list of the weights, in increasing order, is also a permutation of the weights. So we are saying that of all permutations of the weights, the maximal value of the collection of coins is achieved if the weights are in increasing order.

Since we are still mathematicians, let us write the proof using this new notation. It's not different: it's just a new way of writing the same thought.

Suppose there were some permutations that gave a larger value than the one we claim is largest. That is, suppose we have $\widehat{x}_1a_1 + \widehat{x}_2a_2 + \ldots + \widehat{x}_7a_7$ bigger than the corresponding value in which the x_i's are arranged in order. Then, for some subscripts $p < q$, we must have $\widehat{x}_q < \widehat{x}_p$. That is, one pair of x_i's must be "out of order". We will show that this permutation in fact cannot give the maximal value: if we "unscramble" these two x_i's, we will increase the value.

We can do this if we apply the case $n = 2$ to the sum $\widehat{x}_pa_p + \widehat{x}_qa_q$. We know that $p < q$, so $a_p < a_q$ (that is, the a_i's are already in order). We know from the case $n = 2$ that if $\widehat{x}_q > \widehat{x}_p$, then $\widehat{x}_pa_p + \widehat{x}_qa_q > \widehat{x}_qa_p + \widehat{x}_pa_q$. This means that if we leave the other terms in place, and just switch around these two, we will have increased the value of $\widehat{x}_1a_1 + \widehat{x}_2a_2 + \ldots + \widehat{x}_7a_7$.

That is, the permutation of the x_i's that we thought gave us the maximal choice in fact does not: it can be increased by "untwisting" one of the pairs that is not in order.

We have written the proof for 7 elements, but we have not used any property of the number 7. In fact, the same argument works for any number of elements. This observation concludes the proof.

An important note: Nowhere have we used the fact that the weights are *integers*. For our story about the coins, they must be integers. We don't want to break our teeth by taking pieces of coins. But the "weights" can be any positive real numbers, and the proof is exactly the same.

Now that we have written our result as an inequality, we can ask about the case for equality. The answer is pretty easy. The sum in question can be made larger so long as even one of the $\widehat{x}_i$'s is out of order. Therefore the sums resulting from the two permutations in the statement of the inequality are equal if none of them are out of order; that is, when $\widehat{x}_i = x_i$ for every subscript i.

A Bit More Philosophy

We humans have a habit of finding meaning in things. We are always finding a camel in a cloud, an omen in an incident, a moral in a story. As soon as our attention is drawn to something, we tend to relate it to the rest of our lives.

Numbers, on the other hand, are devoid of meaning. We think of them as counts, or lengths, or maybe ratios. But they themselves have no idea what they are counting or measuring or comparing. The number 3 is the same whether it be 3 apples, 3 cartons of apples, 3 minutes, or even 3 fives. It is we who give numbers their meaning, the relationship between them and reality.

Let us look back at the rearrangement inequality from this rather poetic viewpoint. We have two sequences of numbers, $x_1 < x_2 < \ldots < x_n$ and $a_1 < a_2 < \ldots < a_n$. We thought of the x_i's as weights, and the a_i's as value, so that we could tell our story about nickels and dimes. And from our story we got a statement about numbers (the rearrangement inequality itself).

So our assignment of different meanings to x_i and a_i led us to an important relationship. But in doing this, we have lost the symmetry of the situation. The two sequences enter into the inequality in a perfectly symmetric manner. If we interchanged their labels, the statement would remain the same.

So here is another way to think of the rearrangement inequality, a way that preserves the symmetry of the situation. We take two sequences $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$, without considering any meaning to the numbers. We make a square array out of the products $a_i b_j$:

$$
\begin{array}{c|ccccc}
 & b_1 & b_2 & b_3 & \ldots & b_n \\
\hline
a_1 & a_1b_1 & a_1b_2 & a_1b_3 & \ldots & a_1b_n \\
a_2 & a_2b_1 & a_2b_2 & a_2b_3 & \ldots & a_2b_n \\
a_3 & a_3b_1 & a_3b_2 & a_3b_3 & \ldots & a_3b_n \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
a_n & a_nb_1 & a_nb_2 & a_nb_3 & \ldots & a_nb_n
\end{array}
$$

The rearrangement inequality tells us that the sum along the diagonal is larger than any other sum which includes one just one product from each row and just one product from each column. For example:

$$
\begin{array}{c|cccccc}
 & b_1 & b_2 & b_3 & \dots & \dots & b_n \\
\hline
a_1 & \boxed{a_1b_1} & a_1b_2 & a_1b_3 & \dots & \dots & a_1b_n \\
a_2 & a_2b_1 & \boxed{a_2b_2} & a_2b_3 & a_2b_4 & \dots & a_2b_n \\
a_3 & a_3b_1 & a_3b_2 & \boxed{a_3b_3} & \dots & \dots & a_3b_n \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
a_n & a_nb_1 & a_nb_2 & a_nb_3 & \dots & \dots & \boxed{a_nb_n}
\end{array}
$$

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \geq a_1b_3 + a_2b_4 + a_3b_1 + \ldots + a_nb_n.$$

You can check to see that this is exactly what the rearrangement inequality states.

Problems

A critical step in solving many of the problems below will be identifying which sequence of numbers are the "weights", and which are the "coins".

8.1. How many permutations of a list of 7 elements are there? Of n elements?

8.2. If $a_1 < a_2 < \ldots < a_n$, prove that the sum

$$x_1a_1 + x_2a_2 + \cdots + x_na_n$$

is *minimal* if $x_1 > x_2 > \cdots > x_n$.

8.3. (Based on a 2012AMC10A contest question)

As you may know, you can't keep a "running average" the way you keep a "running total". That is, if we expect five pieces of data a, b, c, d, e, coming to us one at a time, and need their sum, we can compute $a + b$, then add this sum to c, then keep adding numbers as they come in.

But you can't do this with the arithmetic mean. If we take the mean of a and b, then the mean of that number with c, and so on, we will not end up with the arithmetic mean of all five numbers. Worse, the result we get will depend on the order in which we receive the numbers.

Prove that the smallest "running average" results when the numbers a, b, c, d, e come in increasing order, while the largest "running average" results when the numbers come in decreasing order.

8.4. For $a, b \geq 0$, show that $a^2 + b^2 \geq 2ab$. (Yes, we know we've already proved this. Show that we could do so using the rearrangement inequality.)

8.5. Let a, b, c be positive real numbers, with $a \le b \le c$. Prove that
$$ab + bc + ca \le a^2 + b^2 + c^2.$$

(This was Problem 1.4b. We revisit it here to show that it is just a special case of the very powerful rearrangement inequality.)

8.6. For n positive numbers $a_1 \le a_2 \le \ldots \le a_n$, show that
$$a_1^2 + a_2^2 + \ldots + a_n^2 \ge a_1a_2 + a_2a_3 + \ldots + a_na_1.$$

8.7. For three non-negative numbers a, b, c show that:

(i) $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \ge \dfrac{c}{b+c} + \dfrac{a}{a+c} + \dfrac{b}{a+b}$;

(ii) $\dfrac{a+b}{b+c} + \dfrac{b+c}{a+c} + \dfrac{a+c}{a+b} \ge 3$.

8.8. Let a, b, c be positive real numbers. Prove that
$$\frac{a+b+c}{abc} \le \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

8.9. Let a, b, c be positive real numbers. Prove that:
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \ge \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

8.10. For positive numbers $x \le y \le z$, show that
$$x^3 + y^3 + z^3 \ge x^2y + y^2z + z^2x.$$

8.11. Suppose a, b, c are positive numbers with $abc = 1$. Prove that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a + b + c.$$

8.12. Let a, b, c be three positive numbers. Prove that
$$\frac{a^2 + bc}{b+c} + \frac{b^2 + ac}{c+a} + \frac{c^2 + ab}{a+b} \ge a + b + c.$$

Chebyshev's Inequality

We can get an important inequality from the table shown above by considering one particular set of products.

Chebyshev's Inequality. *Suppose we have two non-decreasing sequences of real numbers, of equal finite length:* $a_1 \le a_2 \le \ldots \le a_n$ *and* $b_1 \le b_2 \le \ldots \le b_n$. *Then*
$$\frac{a_1b_1 + a_2b_2 + \ldots + a_nb_n}{n} \ge \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \cdot \left(\frac{b_1 + b_2 + \ldots + b_n}{n}\right).$$

Proof. The left numerator is just the diagonal of our "multiplication table" on page 95. We have already noted that it is the largest sum of "its kind": it is larger than any sum of products, one chosen from each row and column in our table. This is a hint towards a proof. Clearing fractions makes this hint even stronger. That is, we will prove (equivalently) that:
$$n(a_1b_1 + a_2b_2 + \ldots + a_nb_n) \ge (a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n).$$

And now we can seek n copies of the diagonal, each larger than some sum (to be determined). Adding these up, we should get the right-hand side. But the sums dominated by each copy of the left-hand side will be sums of products, and the right side is a product of sums. It will be easier to find what we seek if we multiply out the right-hand side:

$$\begin{aligned}(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) &= a_1(b_1 + b_2 + \ldots + b_n)\\ &+ a_2(b_1 + b_2 + \ldots + b_n)\\ &+ \ldots\\ &+ a_n(b_1 + b_2 + \ldots + b_n)\\ &= a_1b_1 + a_1b_2 + \ldots + a_1b_n\\ &+ a_2b_1 + a_2b_2 + \ldots + a_2b_n\\ &+ \ldots\\ &+ a_nb_1 + a_nb_2 + \ldots + a_nb_n.\end{aligned}$$

Each row of the last sum consists of one product selected from each row and column of our "multiplication table" from page 95.

So each row in the last sum is less than one copy of the diagonal sum. Adding, we get Chebyshev's inequality.

We have written this proof "backwards", the way we thought about it. But in mathematics, this is considered scratch work. The formal argument flows the other way: we write down a number of inequalities, then add them up and put it in the form that we quoted in the statement of the theorem.

Exercise. Write this proof formally, starting with the rearrangement inequality applied to various parts of our table and ending with the statement of Chebyshev's inequality.

Problems

8.13. Let a, b, c be three positive numbers. Prove that

$$3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2).$$

8.14. Suppose $x_1, x_2, \ldots, x_n$ is a sequence of real numbers. Prove that

$$\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^2 \leq \frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}.$$

8.15. (Mircea Becheanu, Mathematical Reflections, Problem J138, issue 5, 2009)

Let a, b, c be positive numbers. Prove that

$$\frac{a^3}{b^2 + c^2} + \frac{b^3}{a^2 + c^2} + \frac{c^3}{a^2 + b^2} \geq \frac{a + b + c}{2}.$$

8.16. If $a \geq b > 1$, show that $\dfrac{a-2}{a+1} \geq \dfrac{b-2}{b+1}$ and that $\dfrac{a+2}{a-1} \leq \dfrac{b+2}{b-1}$.

8.17. (Po-Ru Loh) Suppose a, b, c are three numbers greater than 1 which satisfy

$$\frac{1}{a^2-1}+\frac{1}{b^2-1}+\frac{1}{c^2-1}=1.$$

Show that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\leq 1.$$

8.18. Suppose x, y, z are positive numbers. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)}$$
$$\geq\left(\frac{x^3+y^3+z^3}{3}\right)\left(\frac{3+x+y+z}{(1+x)(1+y)(1+z)}\right).$$

8.19. (IMO Shortlist 1998) Suppose x, y, z are positive numbers such that $xyz=1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

(Note that we have added a condition on x, y, z from Problem 8.18.)

Solutions

8.1. How many permutations of a list of 7 elements are there? Of n elements?

Solution. For 7 elements, the number is $1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7=5040$, by a standard argument that can be found in any text on combinatorics. For n elements, the number is $1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)\cdot n$, which we denote by $n!$. Briefly, the standard argument begins by noting that there are n choices for the element in the first position, then $n-1$ choices for the second position, $n-2$ choices for the third position, and so on.

8.2. If $a_1 < a_2 < \ldots < a_n$, prove that the sum

$$x_1a_1+x_2a_2+\cdots+x_na_n$$

is *minimal* if $x_1 > x_2 > \cdots > x_n$.

Solution. Multiply everything by -1 and the required inequality becomes the one we've already proved.

8.3. (Based on a 2012AMC10A contest question)

As you may know, you can't keep a "running average" the way you keep a "running total". That is, if we expect five pieces of data a, b, c, d, e, coming to us one at a time, and need their sum, we can compute $a+b$, then add this sum to c, then keep adding numbers as they come in.

But you can't do this with the arithmetic mean. If we take the mean of a and b, then the mean of that number with c, and so on, we will not end up with the arithmetic mean of all five numbers. Worse, the result we get will depend on the order in which we receive the numbers.

Prove that the smallest "running average" results when the numbers a, b, c, d, e come in increasing order, while the largest "running average" results when the numbers come in decreasing order.

Solution. The "running average" of five numbers is just:

$$\frac{\dfrac{\dfrac{\dfrac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{1}{16}(a+b+2c+4d+8e).$$

The result follows from the rearrangement inequality with "coins" a, b, c, d, e and weights as shown by the algebra.

8.4. For $a, b \geq 0$, show that $a^2 + b^2 \geq 2ab$.

Solution. From symmetry, we lose no generality if we assume that $a \leq b$. Then the rearrangement inequality implies that

$$a \cdot a + b \cdot b \geq a \cdot b + b \cdot a,$$

which is just what we wanted to prove.

8.5. Let a, b, c be positive real numbers, with $a \leq b \leq c$. Prove that

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Solution. As the reader may suspect, this is a simple application of the rearrangement inequality, applied to the sequence a, b, c (twice!). The left-hand side has the products in the "wrong" order to be maximal, and the right-hand side has the products in the maximal order.

8.6. For n positive numbers $a_1 \leq a_2 \leq \ldots \leq a_n$, show that

$$a_1^2 + a_2^2 + \ldots + a_n^2 \geq a_1a_2 + a_2a_3 + \ldots + a_na_1.$$

Solution. The statement follows directly from the rearrangement inequality applied to the sequence $a_1 \leq a_2 \leq \ldots \leq a_n$ twice.
Note that this problem generalizes Problem 8.5. But do you see why we had to add here the condition $a_1 \leq a_2 \leq \ldots \leq a_n$?

8.7. For three non-negative numbers a, b, c show that:

(i) $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \geq \dfrac{c}{b+c} + \dfrac{a}{a+c} + \dfrac{b}{a+b}$;

(ii) $\dfrac{a+b}{b+c} + \dfrac{b+c}{a+c} + \dfrac{a+c}{a+b} \geq 3$.

Solution. (i) The left expression suggests that we choose as one sequence a, b, c and the other $\dfrac{1}{b+c}, \dfrac{1}{a+c}, \dfrac{1}{a+b}$. But are they in the right order?
Well, from symmetry, we can assume that $a \leq b \leq c$. Then we see that $a+b \leq a+c \leq b+c$, because going from each sum to the next, we are increasing one of the addends. So

$$\frac{1}{b+c} \leq \frac{1}{a+c} \leq \frac{1}{a+b},$$

and statement (i) follows from the rearrangement inequality.

(ii) This statement follows from statement (i) by adding the expression

$$\frac{b}{b+c}+\frac{c}{a+c}+\frac{a}{a+b}$$

to both sides (!).

8.8. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+c}{abc} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Solution. The problem is symmetric in a, b, and c, so without loss of generality, we can assume $a \leq b \leq c$. Then we can "tear apart" the left-hand side to get:

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

which we can write as:

$$\left(\frac{1}{b}\right)\left(\frac{1}{a}\right)+\left(\frac{1}{c}\right)\left(\frac{1}{b}\right)+\left(\frac{1}{a}\right)\left(\frac{1}{c}\right) \leq \left(\frac{1}{a}\right)\left(\frac{1}{a}\right)+\left(\frac{1}{b}\right)\left(\frac{1}{b}\right)+\left(\frac{1}{c}\right)\left(\frac{1}{c}\right).$$

And we are done! This is just the rearrangement inequality applied to two copies of the sequence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. (The sequence is used as the "coins" and also the "weights".) The right-hand side has the products in maximal order, and the left-hand side has the products in some other order.

8.9. Let a, b, c be positive real numbers. Prove that:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

Solution. It is not easy, in this situation, to distinguish the "coins" and the "weights". In fact, the right-hand side of the proposed inequality does not have any products at all. We must struggle to interpret it as the sum of three products. How can we do this?

Well, for example, we can rewrite $\frac{b}{a}$ as $b\left(\frac{1}{a}\right)$, and the other fractions similarly. This would require one of our sequences to be $\{a, b, c\}$ and the other to be $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$. But looking for this same sequence (either in the "correct" arrangement or in some other rearrangement), leaves an ugly set of terms for the other sequence, which certainly doesn't match the left-hand side.

And in fact, no "obvious" choices of sequences work. But the fact that the variables enter into the inequality in ratios gives us a clue. We can let

$$x=\frac{a}{b},\ y=\frac{b}{c},\ z=\frac{c}{a}.$$

Without loss of generality, we can assume that $x \le y \le z$.
Then we can apply the rearrangement inequality to the sequence x, y, z. But we've already done this in Problem 8.5. We have:

$$xy + yz + zx \le x^2 + y^2 + z^2,$$

which is magically equivalent to the inequality we want, if we write it in terms of a, b, and c.
We glossed over one little issue. How do we know that we can assume that $x \le y \le z$, when they are defined in terms of other variables? Well, their values in terms of a, b, c are just three ratios, no matter what a, b, and c are. That is, they are three numbers. That observation, and the symmetry of the equation, shows that we can order x, y, z as we like – without loss of generality.

8.10. For positive numbers $x \le y \le z$, show that

$$x^3 + y^3 + z^3 \ge x^2y + y^2z + z^2x.$$

Solution. Writing the required inequality as

$$x \cdot x^2 + y \cdot y^2 + z \cdot z^3 \ge yx^2 + zy^2 + xz^2,$$

and noting that $x^2 < y^2 < z^2$ (the original numbers are positive), we see that the rearrangement inequality implies the statement we want.

8.11. Suppose a, b, c are positive numbers with $abc = 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a + b + c.$$

Solution. It is difficult to see that Problem 8.11 is a special case of 8.10. But that is a good hint. If you've been struggling with the problem, try again before continuing to read.

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Ready for the solution? Let $x = \sqrt[3]{\frac{a}{b}}$, $y = \sqrt[3]{\frac{b}{c}}$, $z = \sqrt[3]{\frac{c}{a}}$.
(Here is another spot you might want to put the book down.)

Then we have:

$$x^3 + y^3 + z^3 \ge yx^2 + zy^2 + xz^2.$$

Replacing x, y, z with their values in terms of a, b, c, we get:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \sqrt[3]{\frac{b}{c}} \cdot \sqrt[3]{\frac{a^2}{b^2}} + \sqrt[3]{\frac{c}{a}} \cdot \sqrt[3]{\frac{b^2}{c^2}} + \sqrt[3]{\frac{a}{b}} \cdot \sqrt[3]{\frac{c^2}{a^2}};$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \sqrt[3]{\frac{a^2}{bc}} + \sqrt[3]{\frac{b^2}{ac}} + \sqrt[3]{\frac{c^2}{ab}}.$$

Since $abc = 1$, we have, for example

$$\frac{a^2}{bc} = \frac{a^2}{1/a} = a^3,$$

and the right-hand side is just $a + b + c$.

This is a hard problem. Could you have guessed, from the first or second reading, that it would require cube roots?

8.12. Let a, b, c be three positive numbers. Prove that

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ac}{c + a} + \frac{c^2 + ab}{a + b} \geq a + b + c.$$

Solution. The denominators on the left suggest that we take one sequence to be $a \leq b \leq c$. Then $a^2 \leq b^2 \leq c^2$ (the numbers are non-negative), and arguing as in Problem 8.7, we find that

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{b^2}{b + c} + \frac{c^2}{c + a} + \frac{a^2}{a + b}.$$

The required inequality results from adding the expression

$$\frac{bc}{b + c} + \frac{ca}{c + a} + \frac{ab}{a + b}$$

to both sides.

8.13. Let a, b, c be three positive numbers. Prove that

$$3(a^3 + b^3 + c^3) \geq (a + b + c)(a^2 + b^2 + c^2).$$

Solution. The inequality looks like Chebyshev's: a sum on the left and the product of two sums on the right. So the job is to identify the sequences to put this in the form we need. For this, we will want a, b, c to be in a particular order. Luckily, the expressions are symmetric, so without loss of generality, we can assume that $a \leq b \leq c$. Then, since the numbers are positive, we know that $a^2 \leq b^2 \leq c^2$, and it is not hard to see that the result follows from Chebyshev's inequality for (a, b, c) and (a^2, b^2, c^2). It is also not hard to extend this result to longer sequences of the same form. Equality holds exactly when $a = b = c$.

8.14. Suppose $x_1, x_2, \ldots, x_n$ is a sequence of real numbers. Prove that

$$\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^2 \leq \frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}.$$

Solution. As in Problem 8.13, the form of the required inequality suggests Chebyshev's inequality: the left side is the product of two (identical) sequences, and the right-hand side is a single sequence.

In fact, it is not hard to guess that this is exactly Chebyshev's inequality, applied twice to the sequence $x_1, x_2, \ldots, x_n$. We need only assume that $x_1 \leq x_2 \leq \cdots \leq x_n$, which we can because both sides of the inequality are symmetric.

8.15. (Mircea Becheanu, Mathematical Reflections, Problem J138, issue 5, 2009)

Let a, b, c be positive numbers. Prove that

$$\frac{a^3}{b^2 + c^2} + \frac{b^3}{a^2 + c^2} + \frac{c^3}{a^2 + b^2} \geq \frac{a + b + c}{2}.$$

Solution. From symmetry, we can assume that $a \le b \le c$. Then it is not hard to see that $a^2 \le b^2 \le c^2$. From this we can also see that $a^2+b^2 \le a^2+c^2 \le b^2+c^2$: as we go from left to right in this last chain of inequalities, we are increasing one of the addends. It follows that

$$\frac{a^2}{b^2+c^2} \le \frac{b^2}{a^2+c^2} \le \frac{c^2}{a^2+b^2}.$$

Now we can apply Chebyshev's inequality to the sequence above and to the sequence (a, b, c):

$$\begin{aligned}\frac{a^3}{b^2+c^2}+\frac{b^3}{a^2+c^2}+\frac{c^3}{a^2+b^2} &\ge \left(\frac{a+b+c}{3}\right)\left(\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}\right)\\ &\ge \frac{a+b+c}{2}.\end{aligned}$$

8.16. If $a \ge b > 1$, show that $\dfrac{a-2}{a+1} \ge \dfrac{b-2}{b+1}$ and that $\dfrac{a+2}{a-1} \le \dfrac{b+2}{b-1}$.

Solution. Multiplying both sides of the required inequality by the non-negative quantity $(a+1)(b+1)$, we obtain the equivalent inequality

$$ab + a - 2b - 2 \ge ab + b - 2a - 2.$$

This in turn is equivalent to $a - 2b \ge b - 2a$, or $3a \ge 3b$, which is certainly true if $a \ge b$. This proves the first inequality.
To prove the second, we note that neither $a+2$ nor $b+2$ can equal zero, so the required inequality is equivalent to $\dfrac{a-1}{a+2} \ge \dfrac{b-1}{b+2}$. This is proved in the same way as the first result.

8.17. (Po-Ru Loh, Crux Mathematicorum) Suppose a, b, c are three numbers greater than 1 which satisfy

$$\frac{1}{a^2-1}+\frac{1}{b^2-1}+\frac{1}{c^2-1}=1.$$

Show that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \le 1.$$

Solution. From symmetry, we can assume that $a \ge b \ge c$. Then, by Problem 8.16, we have

$$\frac{a-2}{a+1} \ge \frac{b-2}{b+1} \ge \frac{c-2}{c+1}; \quad \frac{a+2}{a-1} \le \frac{b+2}{b-1} \le \frac{c+2}{c-1}.$$

Applying Chebyshev's Inequality,

$$3\left(\frac{a^2-4}{a^2-1}+\frac{b^2-4}{b^2-1}+\frac{c^2-4}{c^2-1}\right)$$

$$\le \left(\frac{a-2}{a+1}+\frac{b-2}{b+1}+\frac{c-2}{c+1}\right)\cdot\left(\frac{a+2}{a-1}+\frac{b+2}{b-1}+\frac{c+2}{c-1}\right).$$

Using the given condition, we obtain

$$\frac{a^2-4}{a^2-1}+\frac{b^2-4}{b^2-1}+\frac{c^2-4}{c^2-1}=3-3\left(\frac{1}{a^2-1}+\frac{1}{b^2-1}+\frac{1}{c^2-1}\right)=0.$$

Since $a,b,c>1$, we obviously have

$$\frac{a+2}{a-1}+\frac{b+2}{b-1}+\frac{c+2}{c-1}>0,$$

which implies that

$$\frac{a-2}{a+1}+\frac{b-2}{b+1}+\frac{c-2}{c+1}\geq 0.$$

This last inequality reduces to $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\leq 1$, and the desired conclusion follows. Equality holds if and only if $a=b=c=2$.

8.18. Suppose x,y,z are positive numbers. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)}$$
$$\geq\left(\frac{x^3+y^3+z^3}{3}\right)\left(\frac{3+x+y+z}{(1+x)(1+y)(1+z)}\right).$$

Solution. From symmetry, we can assume that $x\geq y\geq z$, so that $x^3\geq y^3\geq z^3$. Also,

$$(1+x)(1+y)\geq(1+x)(1+z)\geq(1+y)(1+z),$$

so

$$\frac{1}{(1+y)(1+z)}\geq\frac{1}{(1+z)(1+x)}\geq\frac{1}{(1+x)(1+y)}.$$

We denote by S the left side of the required inequality. Then by Chebyshev's inequality, we have:

$$\begin{aligned}
3S&=3\left(\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)}\right)\\
&\geq(x^3+y^3+z^3)\left(\frac{1}{(1+y)(1+z)}+\frac{1}{(1+z)(1+x)}+\frac{1}{(1+x)(1+y)}\right)\\
&=(x^3+y^3+z^3)\left(\frac{(1+x)+(1+y)+(1+z)}{(1+x)(1+y)(1+z)}\right)\\
&=(x^3+y^3+z^3)\left(\frac{3+x+y+z}{(1+x)(1+y)(1+z)}\right),
\end{aligned}$$

which is equivalent to the result we need.

8.19. (IMO Shortlist 1998) Suppose x,y,z are positive numbers such that $xyz=1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)}\geq\frac{3}{4}.$$

(Note that we have added a condition on x,y,z from Problem 8.18.)

Solution. From the result of Problem 8.18, we have

$$S \geq \left(\frac{x^3+y^3+z^3}{3}\right)\left(\frac{3+x+y+z}{(1+x)(1+y)(1+z)}\right),$$

where again we let S denote the left side of the required inequality. Using the identity $(x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x)$, the inequality reduces to

$$8(x^3+y^3+z^3) \geq 3(x+y)(y+z)(z+x).$$

But $4(x^3+y^3) = 4(x+y)(x^2-xy+y^2)$, and since

$$4x^2-4xy+4y^2 \geq 3(x+y)^2$$

reduces to $(x-y)^2 \geq 0$, we have

$$4(x^3+y^3) \geq (x+y)^3, 4(y^3+z^3) \geq (y+z)^3, 4(z^3+x^3) \geq (z+x)^3.$$

Summing the last three inequalities, we obtain

$$8(x^3+y^3+z^3) \geq (x+y)^3+(y+z)^3+(z+x)^3 \geq 3(x+y)(y+z)(z+x),$$

by the AM-GM inequality.

Chapter 9

The Cauchy-Schwarz Inequality

We begin with an unusual situation: an inequality that concerns four numbers, and with constraints on the numbers. Later we will make the situation much more general.

Example 9.1. Prove that if the numbers (a, b, x, y) satisfy:

$$a^2 + b^2 = 1,$$
$$x^2 + y^2 = 1,$$

then $ax + by \leq 1$. Determine the case for equality.

Solution. Notice that the expression we want to bound $(ax + by)$ almost contains the cross products of the expansion $(a \pm x)^2$, etc. This suggests completing the squares in the given conditions.

We have:

$$a^2 + x^2 + b^2 + y^2 = 2,$$
$$a^2 - 2ax + x^2 + b^2 - 2by + y^2 = 2 - 2(ax + by),$$
$$(a - x)^2 + (b - y)^2 = 2 - 2(ax + by). \tag{9.1}$$

But a square cannot be negative, so we have:

$$2 - 2(ax + by) \geq 0,$$
$$-2(ax + by) \geq -2,$$
$$ax + by \leq 1,$$

(where the inequality has reversed direction).

Equality will occur when the sum of the squares in (9.1) is equal to zero, or when $a = x$ and $b = y$.

Why four numbers? Why those unusual constraints? A geometric interpretation will tell us, and will suggest ways of generalizing the result.

Let us think of a coordinate plane, and the two points (a, b), (x, y). The given conditions tell us that the two points are both 1 unit away from the origin. So they are on the unit circle centered at the origin.

That is, the problem is really about two objects (two points), and not four. But what is the "meaning" of the expression $ax + by$? When do we multiply corresponding coordinates of two points? That is a deeper question, which we will talk about later.

Problems

9.1. State and prove an analogous theorem about two points in three-dimensional space.

Can we continue this into four-dimensional space? Well, most of us cannot picture such a space, but all it means is that we start with two points, each given by four coordinates instead of three, and such that the sum of the squares of the coordinates is 1.

9.2. Write down the statement of a problem analogous to Problem 9.1, but in four-dimensional space. Prove it only if the solution to Problem 9.1 was difficult for you.

Higher Dimensions

Mathematicians have no problem with higher dimensional spaces. All this means is that the points they are talking about are given by $5, 6, \ldots, n$ coordinates. And it turns out that there are good reasons to investigate these higher-dimensional spaces.

So let us help the mathematicians by proving our inequality in general. How do we write down "any dimension" for our points? We use subscripts and dots $(\ldots)$, and this may take some practice.

Lemma 1. For any integer n, if we have two sets of numbers $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ such that:

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 1,$$
$$b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2 = 1,$$

then $a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n \leq 1$ with equality just when $a_i = b_i$ for all i.

Another useful (but at first confusing) set of notations is $\sum$ ("Sigma") notation for sums. Using this notation we can rewrite Lemma 1 as follows.

Lemma 1a. For any integer n, if we have two sets of numbers $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ such that

$$\sum_{i=1}^{n} a_i^2 = 1,$$
$$\sum_{i=1}^{n} b_i^2 = 1$$

then we have $\displaystyle\sum_{i=1}^{n} a_ib_i \leq 1$, with equality just when $a_i = b_i$ for all i.

The proof of Lemma 1 (or 1a, which is the same statement) is completely analogous to the solution of Example 9.1.

Problems

9.3. State and prove an analogous theorem for two points that are not on a unit circle (in two dimensions), but on a circle of radius r, centered at the origin.

9.4. Using dots or $\sum$ notation, state and prove an analogous theorem for two points on an n-dimensional sphere of radius r. If you find this confusing, practice in three dimensions first.

Summation notation can be daunting. One way to feel less intimidated is to write out the actual sum for the cases $n = 3$ or $n = 4$, to see what the letter $\sum$ conceals. You will usually find that it's not really complicated.

More Generally...

What if the points (a, b), (x, y) are not the same distance from the origin? We will do this in two dimensions, and leave the reader to write down the generalization to n dimensions, with dots and $\sum$'s.

Example 9.2. Suppose $a^2 + b^2 = p^2$ and $x^2 + y^2 = q^2$. Find an upper bound for $ax + by$. Determine the case for equality.

Solution. We use a trick called *normalization* to make this situation depend on the situation in Example 9.1.

If $a^2 + b^2 = p^2$, then

$$\frac{a^2}{p^2} + \frac{b^2}{p^2} = 1.$$

Similarly, we have

$$\frac{x^2}{q^2} + \frac{y^2}{q^2} = 1.$$

So we can apply the reasoning of Example 9.1 to the numbers $\frac{a^2}{p^2}, \frac{b^2}{p^2}, \frac{x^2}{q^2}, \frac{y^2}{q^2}$. Or, if you like, to the *points* $\left(\frac{a^2}{p^2}, \frac{b^2}{p^2}\right)$ and $\left(\frac{x^2}{q^2}, \frac{y^2}{q^2}\right)$.

We find:

$$0 \le \left(\frac{a}{p} - \frac{x}{q}\right)^2 + \left(\frac{b}{p} - \frac{y}{q}\right)^2 = 2 - 2\left(\frac{ax}{pq} + \frac{by}{pq}\right),$$
$$2 - 2\left(\frac{ax}{pq} + \frac{by}{pq}\right) \ge 0,$$
$$\frac{ax}{pq} + \frac{by}{pq} \le 1,$$

or $ax + by \le pq$. In other words, we have $(ax + by)^2 \le (a^2 + b^2)(x^2 + y^2)$.

The case for equality is interesting. For equality, we must have:

$$\left(\frac{a}{p} - \frac{x}{q}\right)^2 = 0 \quad \text{and} \quad \left(\frac{b}{p} - \frac{y}{q}\right)^2 = 0.$$

This means that $\frac{a}{p} = \frac{x}{q}$ and $\frac{b}{p} = \frac{y}{q}$. We can rewrite this as $\frac{a}{x} = \frac{p}{q} = \frac{b}{y}$, or simply as $\frac{a}{b} = \frac{x}{y}$.

This is not the usual condition for equality, and again a geometric interpretation will give us some insight. We sometimes think of the coordinate plane as inhabited not by points but by "vectors". For now, you can think of a vector as an arrow with its tail at the origin and and its head at a specific point. But a better idea of a vector will emerge as we look more deeply into the situation.

For example, in physics you may have learned that a vector is a quantity that has "magnitude and direction". For vectors that are arrows pointing from the origin, the direction is given by the angle it makes with the (positive) x-axis, and the magnitude is given by its length. But the concept of "vector" is deeper than this.

Vectors locate points. But they do more: you can do arithmetic and algebra on vectors. For example, if you have a vector pointing, say, to the location $(2, 5)$, you can "stretch" this vector to point, in the same direction, to $(6, 15)$. We say we have *scaled* the vector, or *multiplied it by a scalar*. In this case, the scalar (just a fancy name for a number, to distinguish it from a vector) is 3. To scale a vector (a, b) by a number α, we just multiply each coordinate of the endpoint of the vector by α. Clearly, the endpoints of the new vector and the old one are collinear with the origin (because the "slope" of the first vector and of the second vector are identical).

We snuck in the term "slope of a vector". For two dimensions, this is not hard to define. The slope of the vector pointing to (a, b) is just $\frac{b}{a}$. This is one way to talk about the "direction" of the vector. In higher dimensions, we cannot use this "slang". But the picture stays the same in higher dimensions.

That is, if we scale a vector like this, we just change its length (by stretching or shrinking), without changing its direction. Conversely, if we change a vector's length without changing its direction, we must have scaled it. We can show this easily in two dimensions. Indeed, the old vector and the new vector are collinear, so their slopes are the same. That is, if the old vector is (a, b) and the new vector is (c, d), then $\frac{b}{a} = \frac{d}{c}$. If two fractions are equal, we get one from the other by multiplying the numerator and denominator by the same constant.

More formally, if the slopes of two vectors are equal, we have $\frac{c}{a} = \frac{d}{b} = k$, some constant. Then (equivalently) $d = b \cdot \frac{c}{a} = kb$, and $c = a \cdot \frac{d}{b} = ka$. The vector (a, b) has been scaled by k.

So the condition for equality in Example 9.2 is just that the two vectors (a, b) and (x, y) be collinear. Our arguments must be rephrased for vectors in 3 or more dimensions, but the conclusions remain the same.

One reason why we talk about "vectors" is that they generalize to more than two dimensions.

An Important Problem

This problem is presented in its own section, because it gives the central result of this chapter.

9.5. State the general inequality analogous to that in Problem 9.4, for vectors in n-dimensional space. Use subscripts and $\sum$ notation.

Can you foresee any difficulties in the proof of the general case? Or will it be very much the same as the case $n = 3$?

The Importance of the Problem

The result of Problem 9.5 is a very important inequality. Its importance emerges as we generalize many other results, and grows with our knowledge of these results. It bears the names of two important mathematicians: the Cauchy-Schwarz inequality. (An important generalization of this inequality is sometimes given a triple name: Cauchy-Schwarz-Buniakowski, but you probably will not get to this for some years.)

Some notation: the sums on the right of the Cauchy-Schwarz inequality have a simple geometric interpretation. In two dimensions (for which we usually use x and y as the coordinates of the vector), the expression $\sqrt{x^2 + y^2}$ is simply the length of the vector; that is, the distance from the *point* (x, y) to the origin. In three dimensions, using subscripts, the expression $\sqrt{a_1^2 + a_2^2 + a_3^2}$ is the same thing. So by analogy, in n-dimensions, we define the *length* of a vector $\mathbf{a}$ to be

$$\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

(We often used **boldface** letters for variables denoting vectors.)

We sometimes call the expression in the right above the *absolute value* of the vector $\mathbf{a}$ because it has properties analogous to those of the absolute value of a number. And we even write:

$$\sqrt{\sum_{i=1}^{n} a_i^2} = |\mathbf{a}|.$$

So we can rewrite the Cauchy-Schwarz inequality as:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2.$$

Notice that the absolute values are squared. That way we don't have to bother with radicals. And because squares are never negative, this doesn't change the validity of the inequality.

One more piece of notation, and we'll get back to solving problems. For two vectors $(a_1, a_2, \ldots a_n)$ and $(b_1, b_2, \ldots, b_n)$, the sum on the left above $(a_1b_1 + a_2b_2 + \ldots + a_nb_n)$ is called the *dot product* of the two vectors. If the vectors are $\mathbf{a}$ and $\mathbf{b}$, we write this as $\mathbf{a} \cdot \mathbf{b}$.

We call it a *product* because it turns out to have many (but not all!) properties of ordinary multiplication of numbers. We use the dot notation, rather than some other notation for multiplication, to distinguish it from other useful ways to define multiplication of two vectors.

For now, you can simply read $\mathbf{a} \cdot \mathbf{b}$ as a shorthand for $a_1b_1 + a_2b_2 + \ldots + a_nb_n$. So for now, it just saves time and ink. Later its true meaning will emerge.

So we can write the Cauchy-Schwarz inequality for vectors $\mathbf{a}$ and $\mathbf{b}$ as:

$$(\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2|\mathbf{b}|^2. \tag{9.2}$$

The notation here disguises the difficulties of using this fascinating result, but also highlights its utility, elegance and importance.

Example 9.3. For all real numbers a, b, show that $(a+b)^2 \leq 2(a^2+b^2)$.
Solution. This is open-and-shut, once we introduce Cauchy-Schwarz. Let $\mathbf{a} = (a, b)$ and let $\mathbf{b} = (1, 1)$.

Then we have $(\mathbf{a} \cdot \mathbf{b})^2 = (a+b)^2$. The value of $|\mathbf{a}|^2$ is simply $a^2 + b^2$, and the value of $|\mathbf{b}|^2$ is 2. Rewriting the Cauchy-Schwarz relation, we get just what we need.

The trick of letting one vector simply be $(1, 1)$ (or $(1, 1, 1)$, etc.) is a useful one in working with the Cauchy-Schwarz inequality.
Alternate Solution. We can do this problem without Cauchy-Schwarz, just by multiplying out. We get the equivalent inequality

$$a^2 + 2ab + b^2 \leq 2(a^2 + b^2),$$

which simplifies to

$$0 \leq a^2 - 2ab + b^2 = (a - b)^2,$$

which is certainly true.

Do you like the alternate solution better? Try generalizing it.

This example illustrates a number of points that come up often in work with the Cauchy-Schwarz inequality. One is that a big part of the work is deciding what the two vectors are. There are tricks and hints in every problem to help us with that, but it is a key to solving the problem. Here, we used the technique of letting one vector be simply $(1, 1)$ (or $(1, 1, 1)$, etc.). This technique will come up again.

We have given an alternate solution to Example 9.3, not using Cauchy-Schwarz. For now, the reader may prefer such solutions. But the power of Cauchy-Schwarz will emerge as we consider generalizations to higher dimensions, or ways in which several problems have something in common. We will find that the Cauchy-Schwarz inequality both motivates and organizes computation with vectors.

Problems

Many of the problems that follow can be solved without using the Cauchy-Schwarz inequality. We suggest that even if you find such a solution, you go back and see how the Cauchy-Schwarz inequality can also be used. Then decide, as a matter of taste, which solution you like better.

9.6. For all real numbers a_1, a_2, a_3, a_4, a_5, show that

$$(a_1 + a_2 + a_3 + a_4 + a_5)^2 \leq 5(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2).$$

9.7. Generalize Problem 9.6, using $\sum$ notation.

9.8. For all real numbers a, b, show that

$$(a^2 + b^2)^2 \leq 2(a^4 + b^4).$$

9.9. For all real numbers a, b, show that

$$4a^4 + 4b^4 \geq (a^2 + b^2)(a + b)^2.$$

The problems that follow are often stated for the case of vectors of dimension $n = 3$ or 4. The generalization to n-dimensions, except in a few cases, is left to the reader.

9.10. For any three positive numbers a, b, c, show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a + b + c}.$$

9.11. For three positive numbers a, b, c, show that

$$\sqrt[3]{abc} \geq \frac{3}{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}.$$

That is, the geometric mean of three positive numbers is never less than their harmonic mean. Is this true for n positive numbers?

9.12. Let a, b, c be three positive numbers such that $a+b+c = 1$. Show that

$$\sqrt{9a + 1} + \sqrt{9b + 1} + \sqrt{9c + 1} \leq 6.$$

9.13. Let a, b, c, d be four positive numbers such that $a+b+c+d = 1$. What is the largest possible value of

$$\sqrt{32a + 1} + \sqrt{32b + 1} + \sqrt{32c + 1} + \sqrt{32d + 1}?$$

9.14. Find all triples (a, b, c) of positive numbers such that

$$2\sqrt{a} + 3\sqrt{b} + 6\sqrt{c} = a + b + c = 49.$$

9.15. Find the maximum value of the function $f(x, y, z) = 5x - 6y + 7z$, where x, y, z vary among all real numbers satisfying

$$2x^2 + 3y^2 + 4z^2 = 1. \tag{9.3}$$

9.16. Find the minimum and maximum values of the expression $x + y + z$ if

$$\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6} = 1.$$

9.17. For all positive numbers a, b, c, x, y, z, show that:

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

9.18. For real numbers a, b, c, show that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} \leq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

9.19. For all positive real numbers a, b, c, show that

$$\frac{a}{c} + \frac{c}{b} + \frac{b}{a} \leq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

9.20. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solutions

9.1. State and prove an analogous theorem about two points in three-dimensional space.

Solution. Instead of points on a unit circle, we have points on a unit sphere. That is, we have six numbers a, b, c, x, y, z such that:

$$a^2 + b^2 + c^2 = 1,$$
$$x^2 + y^2 + z^2 = 1.$$

The proof is entirely analogous: we add the two equations and complete the square to find

$$0 \leq (a-x)^2 + (b-y)^2 + (c-z)^2 = 2 - 2(ax + by + cz),$$

so that $ax + by + cz \leq 1$, as before.

9.2. Write down the statement of a problem analogous to Problem 9.1, but in four-dimensional space. Prove it only if the solution to Problem 9.1 was difficult for you.

Solution. For two points (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in four-dimensional space such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1,$$

we have:

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \leq 1,$$

with equality when $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$.

9.3. State and prove an analogous theorem for two points that are not on a unit circle (in two dimensions), but on a circle of radius r, centered at the origin.

Solution. We start with a, b, x, y, four numbers such that $a^2+b^2 = r^2$, $x^2+y^2=r^2$. Proceeding by analogy with Example 9.1, we want to find an upper bound for the expression $ax+by$.

Following the solution to Example 9.1, we have:

$$\begin{aligned} a^2+x^2+b^2+y^2 &= 2r^2, \\ a^2-2ax+x^2+b^2-2by+y^2 &= 2r^2-2ax-2by, \\ 0 \le (a-x)^2+(b-y)^2 &= 2r^2-2ax-2by, \\ ax+by &\le r^2. \end{aligned} \tag{9.4}$$

Equality occurs when the sum of squares in (9.4) is zero, or when $a=x$, $b=y$.

9.4. Using dots or $\sum$ notation, state and prove an analogous theorem for two points on an n-dimensional sphere of radius r. If you find this confusing, practice in three dimensions first.

Solution. The theorem would be the following.

Given two points $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ in n-dimensional space, such that

$$\sum_{i=1}^{n} a_i^2 = r^2, \quad \sum_{i=1}^{n} b_i^2 = r^2$$

(for some real number r), we have:

$$\left(\sum_{i=1}^{n} a_i b_i\right) \le r^2.$$

9.5. State the general inequality analogous to that in Problem 9.4, for vectors in n-dimensional space. Use subscripts and $\sum$ notation.

Solution. For any n, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

And the proof goes through just as before.

9.6. For all real numbers a_1, a_2, a_3, a_4, a_5, show that

$$(a_1+a_2+a_3+a_4+a_5)^2 \le 5(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2).$$

Solution. Yes, we could still multiply out and get a bunch of squares that are non-negative. But if we use Cauchy-Schwarz, the computation is more meaningful. Let $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ and $\mathbf{b} = (1,1,1,1,1)$. Then we have:

$$(\mathbf{a}\cdot\mathbf{b})^2 \le |\mathbf{a}|^2 \cdot |\mathbf{b}|^2,$$

$$(a_1+a_2+a_3+a_4+a_5)^2 \le \left(\sqrt{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2}\right)^2 \left(\sqrt{5}\right)^2$$

and the result is immediate. Even better, the computation organizes itself. The case for equality is not hard: all the numbers must be equal.

9.7. Generalize Problem 9.6, using $\sum$ notation.

Solution. For any n real numbers $a_1, a_2, \dots a_n$, we have

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{k} {a_i}^2.$$

9.8. For all real numbers a, b, show that

$$(a^2 + b^2)^2 \leq 2(a^4 + b^4).$$

Solution. Again we choose $\mathbf{b} = (1, 1)$. This time we let $\mathbf{a} = (a^2, b^2)$. The Cauchy-Schwarz inequality then immediately gives the result we need.

9.9. For all real numbers a, b, show that

$$4a^4 + 4b^4 \geq (a^2 + b^2)(a + b)^2.$$

Solution. The expression on the right looks suspiciously like a dot product. However, it's on the wrong side of the inequality. So trying to match it to a dot product is not going to help. In fact, we can use the result of Problem 9.8 to find that:

$$4a^4 + 4b^4 \geq 2(a^2 + b^2)^2 = 2(a^2 + b^2)(a^2 + b^2),$$

and since $a^2 + b^2 \geq 2ab$ (Example 1.1 in this volume), we have

$$2(a^2 + b^2) \geq a^2 + 2ab + b^2 = (a + b)^2.$$

This leads quickly to our result.

9.10. For any three positive numbers a, b, c. show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a + b + c}.$$

Solution. Typically, the first step in these problems – after recognizing a Cauchy-Schwarz situation – is to identify the two vectors $\mathbf{a}$ and $\mathbf{b}$. In this case, it's easiest to do if we clear fractions:

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(a + b + c) \geq 9.$$

And now we can think of how, for instance $a + b + c$ can represent the square of the absolute value of a vector. The vector we need is just $\mathbf{b} = (\sqrt{a}, \sqrt{b}, \sqrt{c})$. If we let vector $\mathbf{a} = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$, then $|\mathbf{b}|$ is the other factor on the left, and we have a natural fit with Cauchy-Schwarz:

$$(\mathbf{a} \cdot \mathbf{b})^{\mathbf{2}} \leq |\mathbf{a}|^{\mathbf{2}}|\mathbf{b}|^{\mathbf{2}},$$

$$(1 + 1 + 1)^2 \leq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(a + b + c),$$

which is equivalent to the desired result. The condition for equality, in this case, reduces simply to $a = b = c$.

By the way, aren't we lucky that the problem stipulated *positive* numbers, so we could take square roots without any problem? In fact, you can easily concoct a counterexample if some of the given numbers are negative.

Alternate Solution. (without using Cauchy-Schwarz) Clearing fractions in the original problem, then multiplying out, we have:

$$1+1+1+\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\geq 9,$$

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\geq 6.$$

Now each pair of fractions on the left side above is of the form $x+\dfrac{1}{x}$, and we know that for positive x, this expression has a minimal value of 2, with equality when $x=\dfrac{1}{x}$. This proves the required inequality and provides the case for equality.

In general,

$$\sum_{i=1}^{n}\frac{1}{a_i}\geq\frac{n^2}{\left(\displaystyle\sum_{i=1}^{n}a_i\right)}.$$

Note that the result of Problem 9.10 can be written as the *AM-HM inequality*:

$$\frac{a_1+a_2+\ldots+a_n}{n}\geq\frac{n}{\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}}.$$

(Remember the harmonic mean?)

Where does the geometric mean fit into this inequality?

9.11. For three positive numbers a, b, c, show that

$$\sqrt[3]{abc}\geq\frac{3}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}.$$

That is, the geometric mean of three positive numbers is never less than their harmonic mean. Is this true for n positive numbers?

Solution. Rewriting the required inequality as:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}$$

shows this to be a special case of the AM-GM inequality. So equality holds when $a=b=c$.

The proof is exactly the same for n positive numbers.

9.12. Let a, b, c be three positive numbers such that $a+b+c=1$. Show that

$$\sqrt{9a+1}+\sqrt{9b+1}+\sqrt{9c+1}\leq 6.$$

Solution. Since this problem is in a chapter on Cauchy-Schwarz, we try to find two vectors whose dot product is the sum on the left. One trick that we've seen before is to let $\mathbf{a} = (9a+1, 9b+1, 9c+1)$ and simply set $\mathbf{b}$ equal to $(1,1,1)$. And in fact this works:

$$\begin{aligned}&\left(\sqrt{9a+1}\cdot 1+\sqrt{9b+1}\cdot 1+\sqrt{9c+1}\cdot 1\right)^2\\ &\le (9a+1+9b+1+9c+1)(1^2+1^2+1^2)\\ &= 3(3+9(a+b+c)) = 3(3+9) = 36.\end{aligned}$$

Taking the square root of both sides gives us the desired result.

Equality holds when the two vectors have proportional components. In this case, it means simply that $a = b = c = \dfrac{1}{3}$.

9.13. Let a, b, c, d be four positive numbers such that $a+b+c+d = 1$. What is the largest possible value of

$$\sqrt{32a+1}+\sqrt{32b+1}+\sqrt{32c+1}+\sqrt{32d+1}?$$

Solution. We proceed as in Problem 9.12, except we don't have the upper bound. We will find one as we proceed.

As in Problem 9.12, we let

$$\mathbf{a} = \left(\sqrt{32a+1}, \sqrt{32b+1}, \sqrt{32c+1}, \sqrt{32d+1}\right)$$

and $\mathbf{b} = (1,1,1,1)$. From Cauchy-Schwarz, we have:

$$\begin{aligned}&\left(\sqrt{32a+1}+\sqrt{32b+1}+\sqrt{32c+1}+\sqrt{32d+1}\right)^2\\ &\le (32a+1+32b+1+32c+1+32d+1)(4)\\ &= (32(a+b+c+d)+4)(4)\\ &= (32\cdot 1+4)(4) = 144.\end{aligned}$$

So $\left(\sqrt{32a+1}+\sqrt{32b+1}+\sqrt{32c+1}+\sqrt{32d+1}\right)^2 \le 144$, and the given expression is at most 12, with equality (as in Problem 9.12) when $a = b = c = d = \dfrac{1}{4}$.

9.14. Find all triples (a, b, c) of positive numbers such that

$$2\sqrt{a}+3\sqrt{b}+6\sqrt{c} = a+b+c = 49.$$

Solution. It's not an inequality, but the leftmost expression looks like a dot product. So let $\mathbf{a} = (2,3,6)$ and let $\mathbf{b} = (\sqrt{a}, \sqrt{b}, \sqrt{c})$. Then Cauchy-Schwarz say that

$$(2\sqrt{a}+3\sqrt{b}+6\sqrt{c})^2 \le (2^2+3^2+6^2)(a+b+c) = 49(a+b+c) = 49^2.$$

But by the conditions of the given problem, we have

$$(2\sqrt{a}+3\sqrt{b}+6\sqrt{c})^2 = 49^2,$$

so the two terms of the Cauchy-Schwarz inequality are "squeezed", and we must in fact have equality. Now equality holds when the vectors have proportional components. So, for some number x we can write $\sqrt{a} = 2x$, $\sqrt{b} = 3x$, $\sqrt{c} = 6x$, and from the first equation given, we have $4x + 9x + 36x = 49$, $x = 1$, and $(a, b, c) = (4, 9, 36)$. It is easy to check that this is a solution and because our constant x is determined by our logic, this is the only solution.

Alternate Solution. We have

$$a + b + c - 2\left(2\sqrt{a} - 3\sqrt{b} - 6\sqrt{c}\right) = 49 - 2 \cdot 49 = -49.$$

We can complete three squares:

$$a - 4\sqrt{a} + 4 + b - 6\sqrt{b} + 9 + c - 12\sqrt{c} + 36$$

$$= \left(\sqrt{a} - 2\right)^2 + \left(\sqrt{b} - 3\right)^2 + \left(\sqrt{c} - 6\right)^2 = 0.$$

So each of the squares must be 0, which leads to the unique solution $(a, b, c) = (4, 9, 36)$, as before.

9.15. Find the maximum value of the function $f(x, y, z) = 5x - 6y + 7z$, where x, y, z vary among all real numbers satisfying

$$2x^2 + 3y^2 + 4z^2 = 1. \tag{9.5}$$

Solution. This problem is unusual in two ways. First, it asks us for an upper bound. That is, it does not present us with an inequality to prove, but asks us to formulate the inequality. Second, the variables satisfy an unusual condition.

The only thing familiar here is the definition of the function, which looks suspiciously like a dot product of two vectors. Having noticed this, we see that left side of condition (9.5) might be the square of the absolute value of a vector.

Let us try to make good on these suggestions. For (9.5) to be the square of an absolute value, each component of one of the vectors must contain x, y, or z. With a little luck, we can set $\mathbf{a} = (x\sqrt{2}, y\sqrt{3}, 2z)$, so that

$$|\mathbf{a}|^2 = 2x^2 + 3y^2 + 4z^2 = 1.$$

Then we need the corresponding $\mathbf{b}$, to give us the correct dot product. A brief computation gives us

$$\mathbf{b} = \left(\frac{5}{\sqrt{2}}, \frac{-6}{\sqrt{3}}, \frac{7}{2}\right).$$

This works: we have

$$|\mathbf{a}| \cdot |\mathbf{b}| = \left(x\sqrt{2}\right) \cdot \left(\frac{5}{\sqrt{2}}\right) + \left(y\sqrt{3}\right) \cdot \left(\frac{-6}{\sqrt{3}}\right) + (2z) \cdot \left(\frac{7}{2}\right) = 5x - 6y + 7z.$$

Now we must see what Cauchy-Schwarz give us for an upper bound. We have already noted that $|\mathbf{a}|^2 = 1$, and a quick computation shows that $|\mathbf{b}|^2 = \dfrac{25}{2} + 12 + \dfrac{49}{4} = \dfrac{147}{4}$. Thus the Cauchy-Schwarz inequality tells us that:

$$(\mathbf{a}\cdot\mathbf{b})^2 \le |\mathbf{a}|^2|\mathbf{b}|^2,$$
$$(5x - 6y + 7z)^2 \le \frac{147}{4}.$$

So $5x - 6y + 7z \le \dfrac{\sqrt{147}}{2}$ for numbers in the set indicated, with equality when $x\sqrt{2} : y\sqrt{3} : 2z = \dfrac{5}{\sqrt{2}} : -\dfrac{6}{\sqrt{3}} : \dfrac{7}{2}$, or equivalently when $x = \dfrac{5}{2}k$, $y = -2k$, $z = \dfrac{7}{4}k$ for some k. Combining this with condition (9.2) above, we find that

$$\frac{50}{4}k^2 + 12k^2 + \frac{49}{4}k^2 = \frac{147}{4}k^2 = 1$$

and $k = \pm\dfrac{2}{7\sqrt{3}}$.

What accounts for the *two* values of k? We have to look a bit further into the Cauchy-Schwarz inequality to get the most we can out of it. If we know that $P^2 \le Q^2$ for real numbers P and Q, it is jumping the gun a bit to conclude that $P \le Q$.

For example, we know that $(-2)^2 \le (-5)^2$, but -2 is not less than or equal to -5. To solve a quadratic inequality like this, we can use the following general result.

Lemma. *If $P^2 \le Q^2$, then either $-Q \le P \le Q$ or $Q \le P \le -Q$.*

Proof. As with most "conditional" quadratic inequalities, we proceed by factoring:

$$P^2 - Q^2 \le 0,$$
$$(P+Q)(P-Q) \le 0.$$

That is, the product of the two factors on the left is 0 or negative. So the products must have opposite signs. Either:

(A) If $P + Q \ge 0$ (i.e., $P \ge -Q$), then $P - Q \le 0$ (i.e., $P \le Q$), which means that $-Q \le P \le Q$. In particular, Q is non-negative.

or:

(B) If $P + Q \le 0$ (i.e., $P \le -Q$), then $P - Q \ge 0$ (i.e., $P \ge Q$) and $Q \le P \le -Q$. In particular, Q is non-positive.

These observations prove the assertion of the lemma.

We can apply these insights to the Cauchy-Schwarz inequality, with $P = \mathbf{a}\cdot\mathbf{b}$ and $Q = |\mathbf{a}||\mathbf{b}|$. But situation (B) cannot occur, because Q, the product of two absolute values, cannot be negative.

So the Cauchy-Schwarz inequality gives us:

$$-|\mathbf{a}|\cdot|\mathbf{b}| \le (\mathbf{a}\cdot\mathbf{b}) \le |\mathbf{a}||\mathbf{b}|,$$

with equality when $\mathbf{a}$ and $\mathbf{b}$ have proportional components.

For this problem, it is clear that $k = \frac{2}{\sqrt{147}} = \frac{2}{7\sqrt{3}}$ gives a maximum, and $k = -\frac{2}{7\sqrt{3}}$ gives a minimum (which the problem didn't ask for).

A bit of algebra shows that the maximum occurs when

$$(x, y, z) = \left(\frac{5}{7\sqrt{3}}, -\frac{2}{7\sqrt{3}}, \frac{1}{2\sqrt{3}}\right),$$

and a minimum occurs when $\left(-\frac{5}{7\sqrt{3}}, \frac{2}{7\sqrt{3}}, -\frac{1}{2\sqrt{3}}\right)$. The maximum value is $\frac{2}{7\sqrt{3}}$, and the minimum is $-\frac{2}{7\sqrt{3}}$.

9.16. Find the minimum and maximum values of the expression $x + y + z$ if

$$\frac{x^2}{2} + \frac{y^2}{3} + \frac{z^2}{6} = 1.$$

Solution. First let us seek an upper bound on the expression $x + y + z$. This sub-goal suggests Cauchy-Schwarz, and the form of the expression suggests a dot product. If our two vectors are $\mathbf{a}$ and $\mathbf{b}$, we want $\mathbf{a} \cdot \mathbf{b} = x + y + z$, and the given condition on x, y, z suggests that one of the absolute values should be 1. This is not so easy to arrange, but taking our hint from the constants in the given condition, a little experimentation yields $\mathbf{a} = \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{6}}\right)$ and $\mathbf{b} = \left(\sqrt{2}, \sqrt{3}, \sqrt{6}\right)$. Then $\mathbf{a} \cdot \mathbf{b} = x + y + z$. Then we find that $|\mathbf{a}| = 1$ (using the given condition) and $|\mathbf{b}| = \sqrt{11}$. It follows from Cauchy-Schwarz that $x + y + z \leq \sqrt{11}$, with equality when $(x, y, z) = (2, 3, 6)$.

What about the minimal value? Using the lemma of Problem 9.15, it is clear that the minimum value of $x+y+z$ is $-\sqrt{11}$, achieved when $(x, y, z) = (-2, -3, -6)$.

Problems 9.15 and 9.16 have something in common. They seek bounds on a linear function, as the numbers concerned satisfy a quadratic relationship. We can get a bit more insight into such problems (and generalize them) by looking geometrically at the two-dimensional case.

9.17. For all positive numbers a, b, c, x, y, z, show that:

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a + b + c)^2}{x + y + z}.$$

Solution. The left side looks like a dot product, but it's on the wrong side of the inequality to use Cauchy-Schwarz. And the right side, as written, does not look like a dot product at all.

Things look better if we clear fractions (and write the inequality the other way around):

$$(a+b+c)^2 \le \left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)(x+y+z). \tag{9.6}$$

Then the left side looks like the square of a dot product. But what are the two vectors? Our old trick of letting $\mathbf{b}=(1,1,1)$ will not work here. We need vectors whose corresponding components multiply to a, b, c. The right-hand side of (9.6) can give us a hint. If we are to apply Cauchy-Schwarz, this product must be the squares of two absolute values. That suggests $\mathbf{a}=\left(\frac{a}{\sqrt{x}}, \frac{b}{\sqrt{y}}, \frac{c}{\sqrt{z}}\right)$ and $\mathbf{b}=\left(\sqrt{x}, \sqrt{y}, \sqrt{z}\right)$, and this works. We have:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \frac{a}{\sqrt{x}}\cdot\sqrt{x}+\frac{b}{\sqrt{y}}\cdot\sqrt{y}+\frac{c}{\sqrt{z}}\cdot\sqrt{z}=a+b+c,\\
|\mathbf{a}|^2 &= \frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z},\\
|\mathbf{b}|^2 &= x+y+z,
\end{aligned}$$

and the required result is a direct application of the Cauchy-Schwarz inequality.

9.18. For real numbers a, b, c, show that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} \le \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}.$$

Solution. As in Problem 9.17, we must sort out which side of the inequality is a dot product and which is a product of absolute values. In this case, the direction of the inequality gives a good clue. Clearing the fraction gives us:

$$(a+b+c)^2 \le (a^2+b^2+c^2)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right),$$

and this looks right. Clearly we should try $\mathbf{a}=(a,b,c)$. But then $\mathbf{b}$ would have to equal $(1,1,1)$, and this won't work for the second term of the product above. We need two vectors whose dot product is $a+b+c$ and whose absolute values are the terms of the product.

We can try letting $\mathbf{a}=\left(\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right)$ (so that $|\mathbf{a}|^2$ is the second term in the product on the right), and match $\mathbf{b}$ to this value of $\mathbf{a}$.

Again, the "obvious" thing to try is $\mathbf{b}=(a,b,c)$. This again won't work. The trick is in the order of the letters a, b, c as components of $\mathbf{b}$: Let $\mathbf{b}=(b,c,a)$ and all will be well. Indeed, we have $\mathbf{a}\cdot\mathbf{b}=a+b+c$, and the absolute values are as required.

The case for equality is interesting. We need proportional coordinates, or

$$\frac{a}{b^2} = \frac{b}{c^2} = \frac{c}{a^2}.$$

This leads to $a^7 = 1$, so $a = b = c = 1$.

9.19. For all positive real numbers a, b, c, show that

$$\frac{a}{c} + \frac{c}{b} + \frac{b}{a} \leq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Solution. This certainly looks like Cauchy-Schwarz. We can easily find pairs of vectors whose dot product is the term on the left. But what do we make of the term on the right? It looks like the square of an absolute value. But in Cauchy-Schwarz, if the absolute values are squared, then so is the dot product. Also, there's only one term on the right, while Cauchy-Schwarz usually involves two factors.

What's going on? How can we knock this into Cauchy-Schwarz form?

The trick is to square both sides (!):

$$\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)^2 \leq \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right)^2.$$

Now we need two vectors whose dot product is the sum squared on the left, and whose absolute values are both the quantity squared on the right.

We still have our work cut out for us, until we remember the trick from Problem 9.18: use different permutations of an "obvious" choice of vector. Here, we can "factor" the fractions on the left:

$$\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right) = \left(\frac{a}{b} \cdot \frac{b}{c} + \frac{c}{a} \cdot \frac{a}{b} + \frac{b}{c} \cdot \frac{c}{a}\right) = \left(\frac{a}{b}, \frac{c}{a}, \frac{b}{c}\right) \cdot \left(\frac{b}{c}, \frac{a}{b}, \frac{c}{a}\right),$$

the dot product of the last two vectors. And these two vectors clearly have the same absolute value. The result follows.

A quick computation shows that equality holds if and only if $a = b = c$.

9.20. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution. For consistency, we write

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a}{b^2} + \frac{b}{c^2} + \frac{c}{a^2}.$$

We want a dot product on the left and the product of two absolute values on the right. It looks like we have *squares* of absolute values on the right (because there is no radical sign). So we might look for a way to introduce a square of a dot product for the left-hand side. Squaring both sides doesn't seem like a good idea, if we go with the idea that the right-hand side is "already" squared.

We can introduce a square another way, by multiplying both sides by $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. And in fact this works. Our inequality is equivalent to:

$$\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)^2 \leq \left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

And now we seek vectors $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a}\cdot\mathbf{b}$ is the expression that is squared on the left and $|\mathbf{a}|^2|\mathbf{b}|^2$ is the expression on the right.

Here we can use the "rearrangement" trick of Problem 9.18. We let

$$\mathbf{a} = \left(\frac{\sqrt{a}}{b}, \frac{\sqrt{b}}{c}, \frac{\sqrt{c}}{a}\right) \quad \text{and} \quad \mathbf{b} = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$$

and the Cauchy-Schwarz inequality gives us the result immediately.

Note that the requirement in the problem that a, b, c be positive real numbers suggests that try using their square roots in deciding on the two vectors.

Equality holds when $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, which occurs when $a = b = c$.

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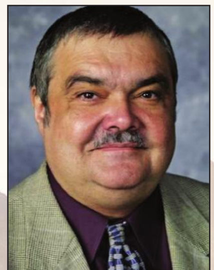


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