Chpater14

The Black-Scholes-Merton Model

THE LOGNORMAL PROPERTY

The model of stock price behavior used by Black, Scholes, and Merton is the model we developed in Chapter 13. It assumes that percentage changes in the stock price in a very short period of time are normally distributed. Define

μ: Expected return on stock per year

σ: Volatility of the stock price per year.

The mean and standard deviation of the return in time Δt are approximately $\mu\,\Delta t$ and $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t)$$

THE LOGNORMAL PROPERTY

Property 1 of Wiener Process

Property 1. *The change Δz during a small period of time Δt is*

$$\Delta z = \epsilon\sqrt{\Delta t}$$

where ϵ has a standard normal distribution $\phi(0, 1)$.

From 13.6

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz$$

We combine them together

$$\left.\begin{array}{l} \dfrac{ds}{s} = \mu\, dt + \sigma\, dz \Rightarrow \dfrac{\Delta s}{s} = \mu\Delta t + \sigma\Delta z \;\; (\textbf{discrete version}) \\ \Delta z = \epsilon\sqrt{\Delta t} \end{array}\right\} \rightarrow \begin{array}{l} \dfrac{\Delta S}{S} = \mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t} \\ \Delta S = \mu S\,\Delta t + \sigma S\epsilon\sqrt{\Delta t} \end{array}$$

THE LOGNORMAL PROPERTY

We know that $\boldsymbol{\epsilon}$ *is distributed* $\boldsymbol{\phi(0,1)}$

$$Var\left(\mu\,\Delta t + \sigma\epsilon\sqrt{\Delta t}\right) = \sigma^2 \Delta \mathrm{t} Var(\epsilon) = \sigma^2 t$$

$$E\,(\mu\Delta t + \sigma\epsilon\sqrt{\Delta t}) = \mu\Delta t + 0 = \mu\Delta t$$

$$\frac{\Delta S}{S} \sim \phi(\mu\,\Delta t,\ \sigma^2 \Delta t)$$

THE LOGNORMAL PROPERTY

Define

$$G = \ln S$$

Since

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \qquad \frac{\partial G}{\partial t} = 0$$

Then we put all these into Ito's lemma equation

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

THE LOGNORMAL PROPERTY

Then we can get

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dz$$

G = lnS follows a generalized Wiener process, and the drift rate is $\left(\mu - \frac{\sigma^2}{2}\right)$. The variance is σ^2.

$$L_n S \sim \phi\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right)$$

THE LOGNORMAL PROPERTY

The change in lnS between time 0 and some future time T is therefore normally distributed.

$$\ln S_T - \ln S_0 = \Delta \ln S \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

(This is because of the additivity of normal distribution)

$$\ln \frac{S_T}{S_0} \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

$$LnS_T = \Delta \ln S + LnS_0$$

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

THE LOGNORMAL PROPERTY

Consider a stock with an initial price of $40, an expected return of 16% per annum, and a volatility of 20% per annum.

$$S_0 = 40, \mu = 16\%, \sigma = 20\%$$

From equation (15.3), the probability distribution of the stock price S_T in 6 months' time is given by .

$$\ln S_T \sim \phi[\ln 40 + (0.16 - 0.2^2/2) \times 0.5,\ 0.2^2 \times 0.5]$$

$$\ln S_T \sim \phi(3.759,\ 0.02)$$

There is a 95% probability that a normally distributed variable has a value within 1.96 standard deviations of its mean. In this case, the standard deviation is $\sqrt{0.02} = 0.141$.Hence, with 95% confidence ,

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

THE LOGNORMAL PROPERTY

$$e^{3.759-1.96\times0.141} < S_T < e^{3.759+1.96\times0.141}$$

$$32.55 < S_T < 56.56$$

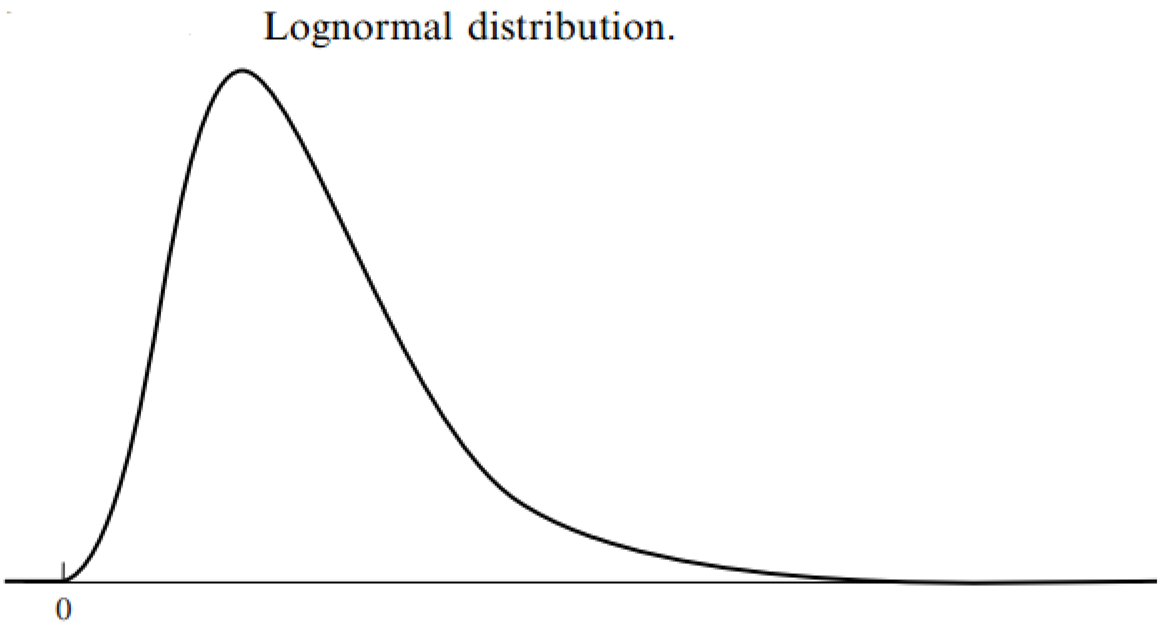
Thus, there is a 95% probability that the stock price in 6 months will lie between 32.55 and 56.56.

THE LOGNORMAL PROPERTY

Log-normal distribution:

In probability theory, a log-normal (or lognormal) distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed.

https://en.wikipedia.org/wiki/Log-normal_distribution



THE LOGNORMAL PROPERTY

Figure 14.1 illustrates the shape of a lognormal distribution. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different.

$$\mathrm{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = (\mathrm{E}[X])^2 (e^{\sigma^2} - 1) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

THE LOGNORMAL PROPERTY

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

The expectation is

$$e^{lnS_0+\left(\mu-\frac{\sigma^2}{2}\right)T+\frac{\sigma^2 T}{2}} = e^{lnS_0+\mu T} = S_0 e^{\mu T}$$

The variance is

$$\left(e^{\sigma^2 T}-1\right)e^{2\left[lnS_0+\left(\mu-\frac{\sigma^2}{2}\right)T\right]+\sigma^2 T} = \left(e^{\sigma^2 T}-1\right)e^{2lnS_0+2\mu T}$$

$$= S_0^2 e^{2\mu T}\left(e^{\sigma^2 T}-1\right)$$

BSM PRICING FORMULAS

Using risk-neutral valuation to derive the Black–Scholes–Merton formulas. Consider a European call option. The expected value of the option at maturity in a risk neutral world is:

$$\hat{E}[\max(S_T - K,\ 0)] \quad \textbf{(K is the strike price)}$$

where, as before, $\hat{E}$ denotes the expected value in a risk-neutral world. From the risk-neutral valuation argument, the European call option price c is this expected value discounted at the risk-free rate of interest, that is,

$$c = e^{-rT}\,\hat{E}[\max(S_T - K,\ 0)]$$

BSM PRICING FORMULAS

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right]$$

$$E(S_T) = S_0 e^{\mu T}$$

In risk-neutral world, $\mu = r$

$$\hat{E}(S_T) = S_0 e^{rT}$$

S.D of $\ln S_T$ is equal to $\sigma\sqrt{T}$

If V is lognormally distributed and the standard deviation of $\ln V$ is w, then

$$E[\max(V-K,\ 0)] = E(V)N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w}$$

$$d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

and E denotes the expected value.

BSM PRICING FORMULAS

Here V = S_T**,** $\hat{E}(S_T) = S_0 e^{rT}$**, so it becomes**

$$E[\max(S_T - K, 0)] = E(S_T)N(d_1) - KN(d_2)$$

$$\hat{E}(S_T) = S_0 e^{rT}$$

Bring it into $c = e^{-rT}\hat{E}[\max(S_T - K,\ 0)]$

We get

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

BSM PRICING FORMULAS

$$d_1 = \frac{\ln[E(V)/K] + w^2/2}{w} \qquad d_2 = \frac{\ln[E(V)/K] - w^2/2}{w}$$

Here V = S_T $\omega = \sigma\sqrt{T}$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

BSM PRICING FORMULAS

Now we get

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

How to get put option formula?

BSM PRICING FORMULAS

If we are buying a put now (at time 0), suppose we decide also to sell a call and buy a share of stock. It will cost

$$P_0 - C_0 + S_0$$

So I have to borrow money from the bank to pay this cost, and the risk-free rate of Interest is r. At maturity,

$$P_T - C_T + S_T \qquad ①$$

We know that

$$P_T = \max(K - S_T, 0) \qquad C_T = \max(S_T - K, 0)$$

BSM PRICING FORMULAS

If $S_T > K$ the equation ① is equal to

$$0 - (S_T - K) + S_T = K$$

If $S_T < K$ the equation ① is equal to

$$(K - S_T) - 0 + S_T = K$$

And because of the no-arbitrage argument, at maturity, the

$$(P_0 - C_0 + S_0)e^{rt} = K$$

$$P_0 - C_0 + S_0 = Ke^{-rt}$$

This is called put-call parity.

BSM PRICING FORMULAS

$$P_0 - C_0 + S_0 \ = Ke^{-rt}$$

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$\Downarrow$$

$$\begin{aligned} P_0 \ &= C_0 - S_0 + Ke^{-rt} = S_0 N(d_1) - Ke^{-rt} N(d_2) + Ke^{-rt} - S_0 \\ &= Ke^{-rt}\big(1 - N(d_2)\big) - S_0(1 - N(d_1)) \end{aligned}$$

Because the symmetry of normal distribution

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

BSM PRICING FORMULAS

Shaded area represents $N(x)$.

BSM PRICING FORMULAS

Understanding $N(d_1)$ and $N(d_2)$

The term $N(d_2)$ in equation (14.20) has a fairly simple interpretation. It is the probability that a call option will be exercised in a risk-neutral world. The $N(d_1)$ term is not quite so easy to interpret. The expression $S_0N(d_1)e^{rT}$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero. The strike price is only paid if the stock price is greater than K and as just mentioned this has a probability of $N(d_2)$. The expected payoff in a risk-neutral world is therefore

$$S_0N(d_1)e^{rT} - KN(d_2)$$

Present-valuing this from time T to time zero gives the Black–Scholes–Merton equation for a European call option:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

BSM PRICING FORMULAS

For another interpretation, note that the Black–Scholes–Merton equation for the value of a European call option can be written as

$$c = e^{-rT}N(d_2)[S_0e^{rT}N(d_1)/N(d_2) - K]$$

The terms here have the following interpretation:

e^{-rT}: Present value factor

$N(d_2)$: Probability of exercise

$e^{rT}N(d_1)/N(d_2)$: Expected percentage increase in stock price in a risk-neutral world if option is exercised

K: Strike price paid if option is exercised.

PROPERTIE OF THE BSM FORMULAS

We now consider some extreme situations, When the stock price, S0, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

$$S_0 - Ke^{-rt}$$

This is, in fact, the call price given by equation (14.20) because, when S0 becomes very large, both d1 and d2 become very large, and $N(d_1)$ and $N(d_2)$ become close to 1.0. When the stock price becomes very large, the price of a European put option, p, approaches zero. This is consistent with equation (14.21) because $N(d_1)$ and $N(d_2)$ are both close to zero in this case.

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

PROPERTIE OF THE BSM FORMULAS

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to S_0e^{rT} at time T and the payoff from a call option is

$$\max(S_0e^{rT} - K,\ 0)$$

Discounting at rate r, the value of the call today is

$$e^{-rT}\max(S_0e^{rT} - K,\ 0) = \max(S_0 - Ke^{-rT},\ 0)$$

PROPERTIE OF THE BSM FORMULAS

To show that this is consistent with equation (14.20), consider first the case where $S_0 > Ke^{-rT}$. This implies that $\ln(S_0/K) + rT > 0$. As σ tends to zero, d_1 and d_2 tend to $+\infty$, so that $N(d_1)$ and $N(d_2)$ tend to 1.0 and equation (14.20) becomes

$$c = S_0 - Ke^{-rT}$$

When $S_0 < Ke^{-rT}$, it follows that $\ln(S_0/K) + rT < 0$. As σ tends to zero, d_1 and d_2 tend to $-\infty$, so that $N(d_1)$ and $N(d_2)$ tend to zero and equation (14.20) gives a call price of zero. The call price is therefore always $\max(S_0 - Ke^{-rT}, 0)$ as σ tends to zero. Similarly, it can be shown that the put price is always $\max(Ke^{-rT} - S_0, 0)$ as σ tends to zero.

CUMULATIVE NORMAL DISTRIBUTION FUNCTION

Using the NORMSDIST function in excel to calculate N(x)

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

$$d_1 = \frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = \frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}} = 0.6278$$

and

$$Ke^{-rT} = 40e^{-0.05} = 38.049$$

CUMULATIVE NORMAL DISTRIBUTION FUNCTION

Hence, if the option is a European call, its value c is given by

$$c = 42N(0.7693) - 38.049N(0.6278)$$

If the option is a European put, its value p is given by

$$p = 38.049N(-0.6278) - 42N(-0.7693)$$

Using the NORMSDIST function in Excel gives

$$N(0.7693) = 0.7791, \qquad N(-0.7693) = 0.2209$$
$$N(0.6278) = 0.7349, \qquad N(-0.6278) = 0.2651$$

so that

$$c = 4.76, \qquad p = 0.81$$

Ignoring the time value of money, the stock price has to rise by \$2.76 for the purchaser of the call to break even. Similarly, the stock price has to fall by \$2.81 for the purchaser of the put to break even.