4418 Assignment 2

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[1.1] Consider a resource allocation setting in which n agents have positive additive utilities over m > n indivisible items. Prove or disprove the following statements:

[1.1.1] The allocation that maximizes utilitarian welfare is Pareto-optimal.

- This is **correct** because when maximizes utilitarian welfare is achieved, you can not find another allocation to increase the overall utility, and if the allocation is Pareto-optimal, it means that there has no further allocation methods that can make at least one agent strictly happier(achieve more utility) than before and no agent is less happy than before(achieve less utility).
  If an allocation is maximizes utilitarian welfare but not Pareto-optimal, then there's another allocation that can make at least one agent strictly happier than before and no agent is less happy than before, which means the overall utility will increase, but clearly this is impossible because it conflicts the notion of Maximize utilitarian welfare.
  Therefore, the allocation which maximizes utilitarian welfare is Pareto-optimal.

[1.1.2] If an allocation is Pareto-optimal, it is envy-free.

- This is **incorrect** since the Pareto-optimal is an efficiency concept while the envy-free is a fairness concept, as the example provided in page 21 lecture slides:

[1.1.2]　　　　　　Yijiezhao-z5270589

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X_1 = \{o_1, o_2, o_3\}$. $X_2 = \{o_4\}$.

this situation is Pareto-optimal but not envy-free.

We can see that in this situation, the allocation is Pareto-optimal but not envy-free, thus “If an allocation is Pareto-optimal, it is envy-free” is not correct.

[1.1.3] If n = 2, envy-freeness and proportionality are equivalent.

- This is **correct**, the prove is shown in the diagram below:

[1.1.3] Yijie zhao - z5270589

If envy-freeness satisfies.

for each $i, j \in N$.

$$u_i(X_i) \geq u_i(X_j) \quad \cdots ①$$

And if proportionality. for all $i \in N$

$$u_i(x_i) \geq \frac{u_i(O)}{n} \quad \cdots ②$$

when $n=2$. we can conclude that:

$$u_i(X_i) + u_i(X_j) = u_i(O) \quad \cdots ③$$

and if envy-freeness, we know $u_i(X_i) \geq u_i(X_j)$

and due to ① and ③ we know $\underbrace{u_i(x_i) + u_i(x_i) \geq u_i(O)}$

$$\Downarrow \quad 2 \cdot u_i(x_i) \geq u_i(O)$$

thus we know that $u_i(x_i) \geq \frac{u_i(O)}{2}$.

which implies it also satisfy proportionality when $n=2$. So we can say that if $n=2$. envy-freeness and proportionality are equivalent.

[1.1.4] The sequential allocation algorithm, in which agents arrive in order (1, 2, 3, . . . , n) ∗ and are given a most preferred unallocated item, is strategyproof.

This is **incorrect** because we know that in this sequential allocation algorithm m > n, if there's any agent report the untruthful preference, this agent might receive better outcome than telling the truth, which this example shows the situation:

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[1.1.4] (real preference)

	O_1	O_2	O_3	O_4	O_5	
1	2_5	3_3	4	6_1	3	$\Rightarrow U_1(X_1) = 11$
2	4	4	6_2	4	5_4	$\Rightarrow U_2(X_2) = 11$

If agent 1 report untruthful preference (changed the utility of O_3 from 4 to 7 in this case to pick O_3 firstly)

(changed preference)

	O_1	O_2	O_3	O_4	O_5	
1	2	3_5	$4 \Rightarrow 7_1$	6_3	3	$\Rightarrow u_1(X_1) = 13$
2	4_4	4	6	4	5_2	$\Rightarrow u_2(X_2) = 9$

From this example, we can see that the agent 1 can get a better outcome when it reports the untruthful preference, thus this algorithm is not the strategyproof.

[1.2] Consider the following school choice problem with five students 1, 2, 3, 4, 5 and five schools a, b, c, d, and e with each school having exactly one seat. The preferences of the students are as follows from left to right in decreasing order of preference:

$$\begin{array}{c|ccccc} 1 & e & b & a & c & d \\ 2 & b & a & c & d & e \\ 3 & a & b & c & d & e \\ 4 & a & b & c & d & e \\ 5 & d & b & c & a & e \end{array}$$

The priorities of the schools are as follows from left to right in decreasing order of preference:

$$\begin{array}{c|ccccc} a & 2 & 4 & 3 & 5 & 1 \\ b & 3 & 2 & 4 & 5 & 1 \\ c & 3 & 2 & 4 & 5 & 1 \\ d & 5 & 2 & 4 & 3 & 1 \\ e & 1 & 2 & 3 & 4 & 5 \end{array}$$

[1.2.1] Find the outcome matching of the student proposing deferred acceptance algorithm, showing working out. Prove or disprove that the resultant matching is Pareto-optimal for the students.

[1.2.1] Yijie Zhao - 35-70589

- 3, 4 apply to a, 2 applies b, 5 applies d, 1 applies e.
- a rejects 3 in favour of 4 {(1, e), (2, b), (4, a), (5, d)}
- 3 applies to b.
- b rejects 2 in favour of 3 {(1, e), (3, b), (4, a), (5, d)}
- 2 applies to a.
- a rejects 4 in favour of 2 {(1, e), (3, b), (2, a), (5, d)}
- 4 applies to b.
- b rejects 4. {(1, e), (2, a), (3, b), (5, d)}
- 4 applies to c.
- c in favour of 4 {(1, e), (2, a), (3, b), (4, c), (5, d)}

Now all students has a seat, the final outcome is

{(1, e), (2, a), (3, b), (4, c), (5, d)}

- This **is not Pareto-optimal** because we can change (3, b), (2, a) to (3, a), (2, b). which 3 and 2 are happier but no others less happy, so it is not Pareto-optimal.

[1.2.2] Suppose that initially, student 1 is allocated to school a, student 2 is allocated to school b, student 3 is allocated to school c, student 4 is allocated to school d, and student 5 is allocated to school e. Apply the Top Trading Cycles (TTC) Algorithm with respect to the students' preferences here and find the output, showing working out.

[1.2.2]

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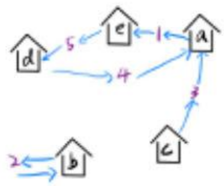
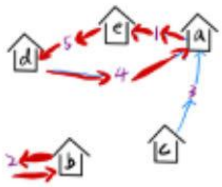
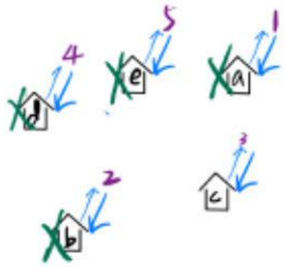
agent	1	2	3	4	5
preference	e	b	a	a	d
	b		b	b	b
	a		c	c	c
				d	a
					e



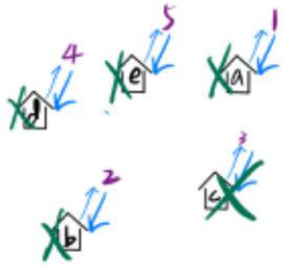
agent	1	2	3	4	5
preference	e	b	a	a	d
	b		b	b	b
	a		c	c	c
				d	a
					e



agent	1	2	3	4	5
preference	e	b	a	a	d
	b		b	b	b
	a		c	c	c
				d	a
					e



agent	1	2	3	4	5
preference	e	b	a	a	d
	b		b	b	b
	a		c	c	c
				d	a
					e



After applied Top Trading Cycles (TTC), we can get the output that: {(1,e), (2,b), (3,c), (4,a), (5,d)}

[1.2.3] Give three reasons, with examples if necessary, to explain why the deferred acceptance algorithm is preferred for school choice over the TTC algorithm.

First reason:

- The deferred acceptance algorithm is **envy-freeness** while TTC algorithm is not, as the answer in [1.2.2], we can see that

[1.2.3] yizhezhou-85[illegible]

According to the answers in [1.2.2], we know that

$$\begin{cases} u_3(x_3) < u_3(x_4) \\ u_3(x_3) < u_3(x_2) \end{cases}$$

These do not satisfy for all $i, j \in N$. $u_i(x_i) \geq u_i(x_j)$.
which in this case TTC is not envy-freeness

Thus to consider envy-freeness in school choice, we prefer the deferred acceptance algorithm over the TTC algorithm.

Second reason:

- The deferred acceptance algorithm can take in account **the preference of school** while the TTC algorithm only considers the preference of students.
  Thus to consider the preference of school in school choice, we prefer the deferred acceptance algorithm over the TTC algorithm.

Third reason:

- The deferred acceptance algorithm in school choice is **strategyproof** while the TTC algorithm is not, which means students can not report untruthful preference to gain better outcome in deferred acceptance but

students can gain better outcome in TTC allocation if they report the untruthful preference.
Thus to consider the strategyproof in school choice, we prefer the deferred acceptance algorithm over the TTC algorithm.

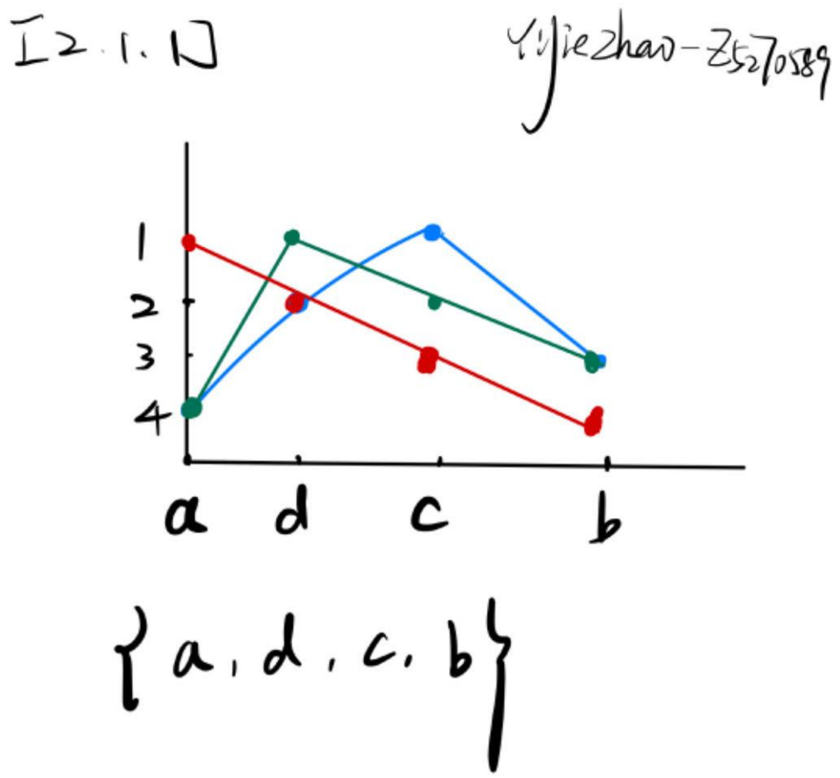
[1.2.4] Give a reason, using an example if necessary, to explain why TTC may be preferred for this school choice setting over the deferred acceptance algorithm.

- The deferred acceptance algorithm is not Pareto-optimal while TTC algorithm is **Pareto-optimal**, as the answer in [1.2.1], we can know that the allocation of deferred acceptance is not Pareto-optimal because we can exchange the school of student 2 and students 3 to make both of them happier while no one is less happy, but we know that the allocation of TTC is Pareto-optimal because you can not make any one happier while no one is less happy.
Thus to consider Pareto-optimal in school choice, we prefer the TTC algorithm over the deferred acceptance algorithm.

Social Choice Theory
[2.1] Consider the following preference profile of voters.

$$\begin{array}{c|l} 1 & c \succ d \succ b \succ a \\ 2 & d \succ c \succ b \succ a \\ 3 & a \succ d \succ c \succ b \end{array}$$

[2.1.1] Prove or disprove that the preference profile is single-peaked with respect to some order of alternatives.



As the picture above shows that the preference profile is single-peaked with respect to the order{a, d, c, b}.

[2.1.2] Prove or disprove that a Condorcet winner exists for the preference profile.

- The Condorcet winner **exists** for the preference profile.
  We can check whether there's an alternative that is pairwise preferred by a majority of voters over every other alternative. And we can quickly find that **d** is the Condorcet winner because d is more preferred than a(1,2 prefer d and 3 prefer a), d is more preferred than b(1,2,3 prefer d than b) and d is more preferred than c(2,3 prefer d and 1 prefer c). Thus we can find that d is

pairwise preferred by a majority of voters over every other alternative(a, b and c), thus there exists the Condorcet winner for the preference profile.

[2.1.3] Compute the pairwise majority graph for the preference profile.

[2.1.3]

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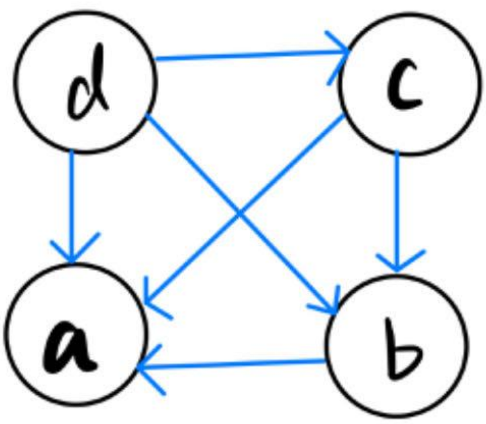
Pairwise majority graph:



[2.1.4] Compute the Top Cycle set for the preference profile.
We know that we have three which is an odd numbers of voters and thus the pairwise majority graph in [2.1.3] is a tournament, and also we know that an alternative is in the Top Cycle set iff it can reach every other alternative by a path in the tournament, based on the diagram in [2.1.3], we can observe that only alternative **d** can reach every other alternative by a path in the tournament, thus d is the only alternative in the Top Cycle set, the Top Cycle set for the preference profile is **{d}**.

[2.2] Let A be a set of alternatives with |A| ≥ 3, and let f : L(A) n → A be a social choice function.
[2.2.1] Show that if f is strategyproof and surjective (onto), then f is Pareto-optimal and monotonic.

- If f is Pareto optimality, it means the alternative will not be chosen if there exists another one that all voters prefer the latter to the former. Now we know that f is Strategyproof, that means no voter can misreport his/her preference to select a more preferred alternative, which in this case each voter will report the true preference, also we know that f is surjective(onto), which means that each alternative is in the consideration and we will select an alternative in each preference profile. Thus in this case we know that if all voters prefer a than b, and they all report the true preference, it’s impossible to choose b rather than a, which this satisfies the b will not be chosen if there exists a that all voters prefer the latter to the former, which f is Pareto-optimal.
- If f is monotonic, it means a chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged). As we talked the property of f is strategyproof and f is onto, thus in this case if previous we return a for the first profile, and a rises in individual ranking(prefer a than b), since we will select an alternative in each preference profile. and all voters will report the true preference, a is still to be chosen, which satisfies the monotonic.
  So if f is strategyproof and surjective (onto), then f is Pareto-optimal and monotonic.

[2.2.2] Hence, prove that f is a dictatorship (i.e. f does not satisfy the non-dictatorial rule). Note. You may not use the Gibbard-Satterthwaite theorem.

- From, [2.2.1], we know that f is Pareto-optimal and monotonic, and we can find that once we have the alternative a which is socially preferred to b, and when a voter change his ranking(from prefer d to c to prefer c to d), finally we will still choose alternative a, and this we know f satisfies Independence of Irrelevant Alternatives, and refer to Arrow's Theorem, we know that any social welfare function for three or more alternatives cannot satisfy all the three axioms: Pareto optimality, Independence of Irrelevant Alternatives and Non-dictatorship, and we know that f satisfies Pareto optimality, Independence of Irrelevant Alternatives, thus it must not Non-dictatorship, so f is a dictatorship.

[2.3] Let f : L(A) n → L(A) be a social welfare function. A function f is said to be unanimous with respect to ≤ if for each x, y ∈ A and each i ∈ N, x ≤i y implies x ≤ y. In other words, if y is favored in every voting profile, then y is the unanimous alternative. Prove that if f satisfies the unanimous property and the independence of irrelevant alternatives axiom, then f is a dictatorship.

- If y is the is the unanimous alternative and y is favored in every voting profile, and if f satisfies the independence of irrelevant alternatives axiom, it means once we have the alternative y which is socially preferred to b, and when a voter change his ranking(from prefer d to c to prefer c to d), finally we will still choose alternative y, which means in this case we know that if all voters prefer y than b, it's impossible to choose b rather than y, which this satisfies the b will not be chosen if there exists y that all voters prefer the latter to the former, which f is Pareto-optimal. Thus according to the Arrow's Theorem, we know that any SWF for three or more alternatives cannot satisfy all the three axioms: Pareto optimality, Independence of Irrelevant Alternatives and Non-dictatorship, and we know that f satisfies Pareto optimality and IIA, thus it must not Non-dictatorship, so f is a dictatorship.