Requirements

- *Due date*: November 3, 2023 at 11:55PM. This is an individual assignment. This assignment is weighted 15% of the course grade.
- Each student submits one file: `assignment2.pdf`.

Multiagent Resource Allocation

[1.1] Consider a resource allocation setting in which n agents have positive additive utilities over $m > n$ *indivisible* items. Prove or disprove the following statements.

[1.1.1] The allocation that maximises utilitarian welfare is Pareto-optimal.

[1.1.2] If an allocation is Pareto-optimal, it is envy-free.

[1.1.3] If $n = 2$, envy-freeness and proportionality are equivalent.

[1.1.4] The sequential allocation algorithm, in which agents arrive in order $(1, 2, 3, \ldots, n)^*$ and are given a most preferred unallocated item, is strategyproof.

[1.2] Consider the following school choice problem with five students $1, 2, 3, 4, 5$ and five schools a, b, c, d, and e with each school having exactly one seat. The preferences of the students are as follows from left to right in decreasing order of preference:

1	e	b	a	c	d
2	b	a	c	d	e
3	a	b	c	d	e
4	a	b	c	d	e
5	d	b	c	a	e

The priorities of the schools are as follows from left to right in decreasing order of preference:

a	2	4	3	5	1
b	3	2	4	5	1
c	3	2	4	5	1
d	5	2	4	3	1
e	1	2	3	4	5

[1.2.1] Find the outcome matching of the student proposing deferred acceptance algorithm, showing working out. Prove or disprove that the resultant matching is Pareto-optimal for the students.

[1.2.2] Suppose that initially, student 1 is allocated to school a, student 2 is allocated to school b, student 3 is allocated to school c, student 4 is allocated to school d, and student 5 is allocated to school e. Apply the Top Trading Cycles (TTC) Algorithm with respect to the students' preferences here and find the output, showing working out.

Note. *We ignore the school's priorities here.*

[1.2.3] Give three reasons, with examples if necessary, to explain why the deferred acceptance algorithm is preferred for school choice over the TTC algorithm.

[1.2.4] Give a reason, using an example if necessary, to explain why TTC may be preferred for this school choice setting over the deferred acceptance algorithm.

Social Choice Theory

[2.1] Consider the following preference profile of voters.

$$\begin{array}{c|l} 1 & c \succ d \succ b \succ a \\ 2 & d \succ c \succ b \succ a \\ 3 & a \succ d \succ c \succ b \end{array}$$

[2.1.1] Prove or disprove that the preference profile is single-peaked with respect to some order of alternatives.

[2.1.2] Prove or disprove that a Condorcet winner exists for the preference profile.

[2.1.3] Compute the pairwise majority graph for the preference profile.

[2.1.4] Compute the Top Cycle set for the preference profile.

[2.2] Let $\mathcal{A}$ be a set of alternatives with $|\mathcal{A}| \geq 3$, and let $f : \mathcal{L}(\mathcal{A})^n \to \mathcal{A}$ be a social choice function.

[2.2.1] Show that if f is strategyproof and surjective (onto), then f is Pareto-optimal and monotonic.

[2.2.2] Hence, prove that f is a dictatorship (i.e. f does not satisfy the *non-dictatorial* rule).

Note. *You may not use the Gibbard-Satterthwaite theorem.*

[2.3] Let $f : \mathcal{L}(\mathcal{A})^n \to \mathcal{L}(\mathcal{A})$ be a social welfare function. A function f is said to be *unanimous* with respect to $\preceq$ if for each $x, y \in \mathcal{A}$ and each $i \in N$, $x \preceq_i y$ implies $x \preceq y$. In other words, if y is favoured in every voting profile, then y is the *unanimous* alternative.

Prove that if f satisfies the *unanimous* property and the *independence of irrelevant alternatives* axiom, then f is a dictatorship.