Big Numbers

Jason Bock

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Downloads

https://github.com/JasonBock/BigNumbers
https://github.com/JasonBock/BenchmarkInvestigations
https://github.com/JasonBock/Chudnovsky
https://github.com/JasonBock/Presentations

Overview

- Relevance
- Representation
- Insanity
- Call to Action

Remember…
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Relevance

Why are big numbers so important? What's the fascination with them?

For me, it’s fun! Big numbers have always fascinated me.

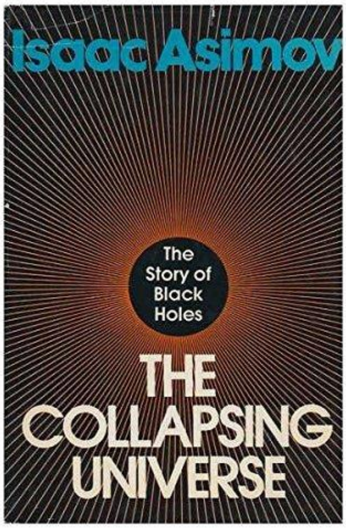
https://www.pexels.com/photo/group-of-people-having-neon-party-1684187/

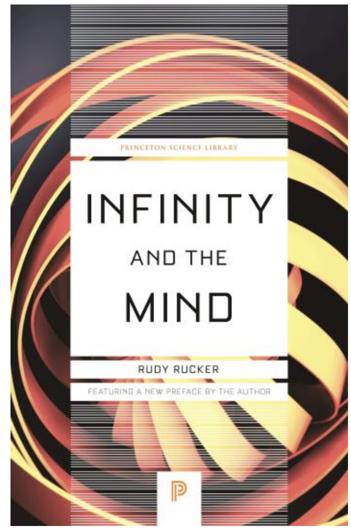
Googol

==

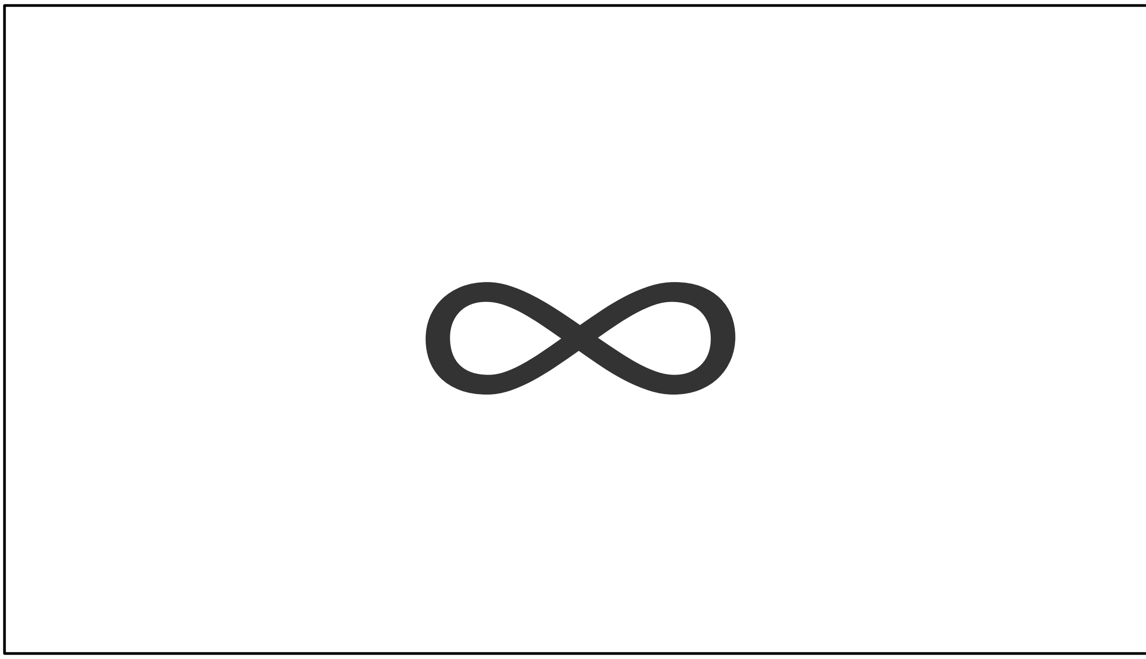
10^{100}

I remember as a kid finding out about bigger and bigger numbers, like "millions", "billions", and then "googol", which is a 1 with 100 zeros after it.





I also read a bunch of books on astronomy and math, and books like this introduced me to real-world “big number” applications, as well as concepts around infinity.



We won't be getting too far into infinity. We're just dealing with finite numbers (mostly).

MULTIPLE-PRECISION MULTIPLICATION

USING THE

NUMBER THEORETIC TRANSFORM

by

Jason R. Bock, B. S.

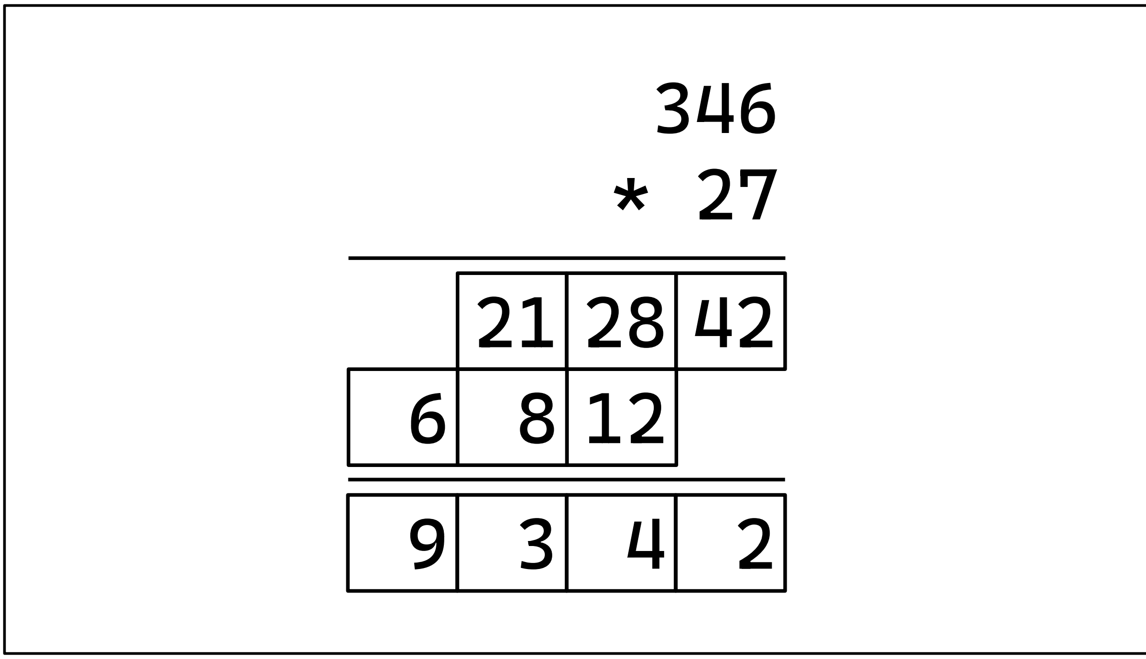
This Thesis Submitted to the Faculty of the
Graduate School, Marquette University, in
Partial Fulfillment of the Requirements for the
Degreee of Master of Science

Milwaukee, Wisconsin

April, 1995

I also had the luck of doing my master’s thesis on multiple precision arithmetic, which was looking at an algorithm to do multiplication faster without losing precision due to round-off errors.

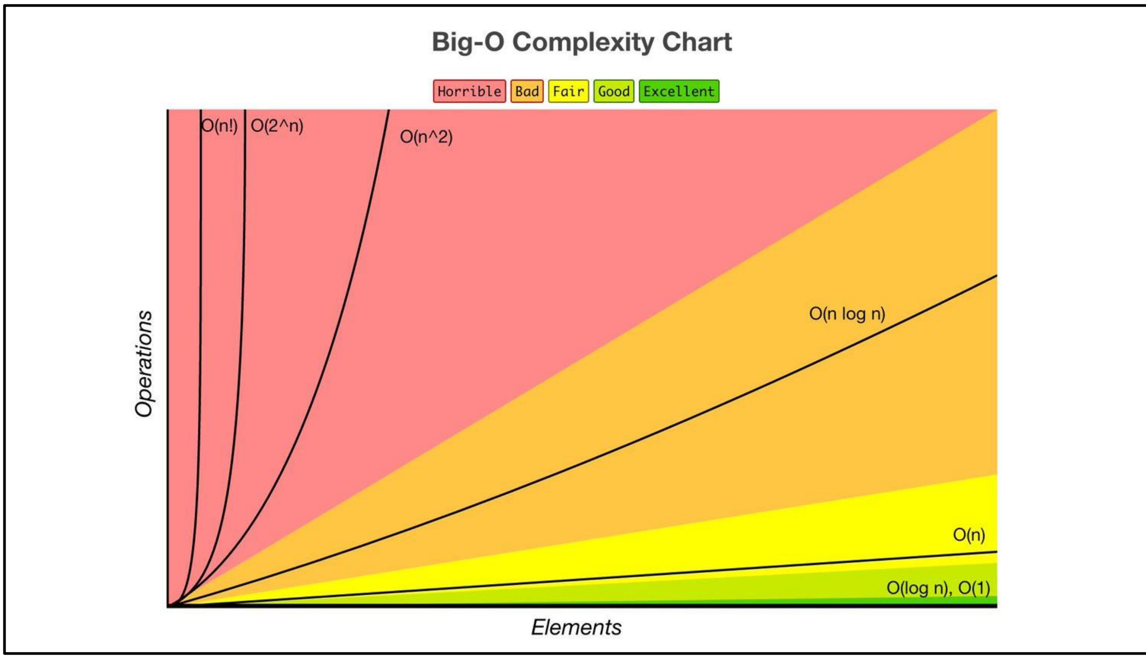
https://eds.b.ebscohost.com/eds/detail/detail?vid=1&sid=3330f1ba-d9c4-4db2-bb07-044d6797a1d8%40sessionmgr102&bdata=JnNpdGU9ZWRzLWxpdmUmc2NvcGU9c2l0ZQ%3d%3d#AN=epmu.theses.4966&db=ir00326a



This is important for performance reasons. Say you wanted to multiply 2 numbers together. The “grade school” method looks like this.

Algorithm	Classification
Grade School	$O(n^2)$
Number Theoretic Transform	$O(n \log n)$
Schönhage–Strassen	$O(n \log n \log \log n)$

In terms of a performance classification, the larger the numbers, the grade school approach is quadratic, whereas the others are logarithmic, and therefore faster.



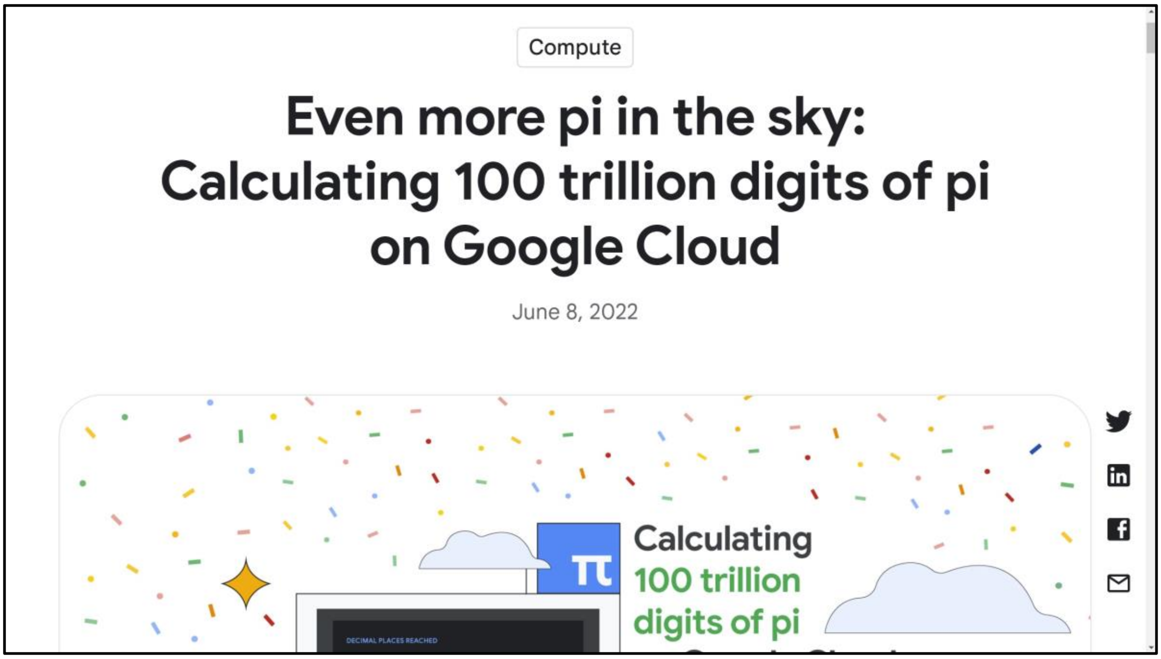
As you can see, logarithmic is far better than quadratic.

https://cdn-images-1.medium.com/max/1200/1*_nsMVEEkIr1CH8aHjTNbzA.jpeg

Appendix B: π to 4980 Decimal Digits

031415926535897932384626433832795028841971693993751058209749445923078
164062862089986280348253421170679821480865132823066470938446095505822
317253594081284811174502841027019385211055596446229489549303819644288
109756659334461284756482337867831652712019091456485669234603486104543
266482133936072602491412737245870066063155881748815209209628292540917
153643678925903600113305305488204665213841469519415116094330572703657
595919530921861173819326117931051185480744623799627495673518857527248

This is important because if you want to calculate something like Pi with a lot of precision, you need to have efficient algorithms.



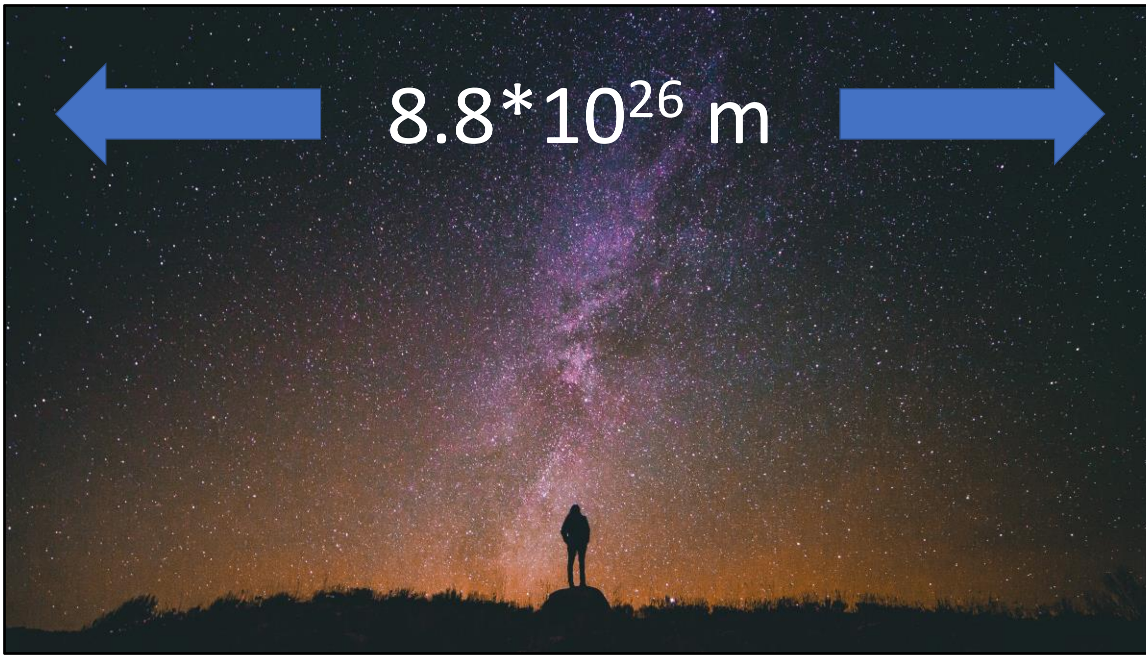
The current world record for the number of PI digits is 100 trillion, announced on 6/8/2022

https://cloud.google.com/blog/products/compute/calculating-100-trillion-digits-of-pi-on-google-cloud

Let's go to the largest size there is: the visible universe. The radius of the universe is about 46 billion light years. Now let me ask a different question: How many digits of pi would we need to calculate the circumference of a circle with a radius of 46 billion light years to an accuracy equal to the diameter of a hydrogen atom (the simplest atom)? **The answer is that you would need 39 or 40 decimal places.** If you think about how fantastically vast the universe is — truly far beyond what we can conceive, and certainly far, far, far beyond what you can see with your eyes even on the darkest, most beautiful, star-filled night — and think about how incredibly tiny a single atom is, you can see that we would not need to use many digits of pi to cover the entire range.

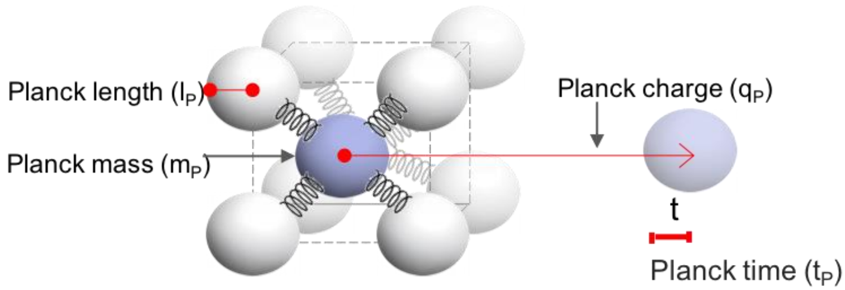
Realistically, though, how “big” do we need to deal with in the “real” world? Arguably, it’s “40 digits of Pi”, and, realistically, it’s 15 for most astronomical journeys.

https://unsplash.com/photos/4Zaq5xY5M_c



The universe is estimated to be 8.8×10^26 m

https://unsplash.com/photos/0LU4vO5iFpM



$$l_p = 1.6 * 10^{-35} \text{ m}$$

One of the smallest measurements is the Planck length.

http://energywavetheory.com/wp-content/uploads/2019/08/The-Planck-constants.png

Total number of Planck lengths = $5.5*10^{61}$ m

So you could say there's a number of Planck lengths, which is a number containing 61 digits.

Since we're talking about atoms, the number of atoms in the universe is approximately 10^80.

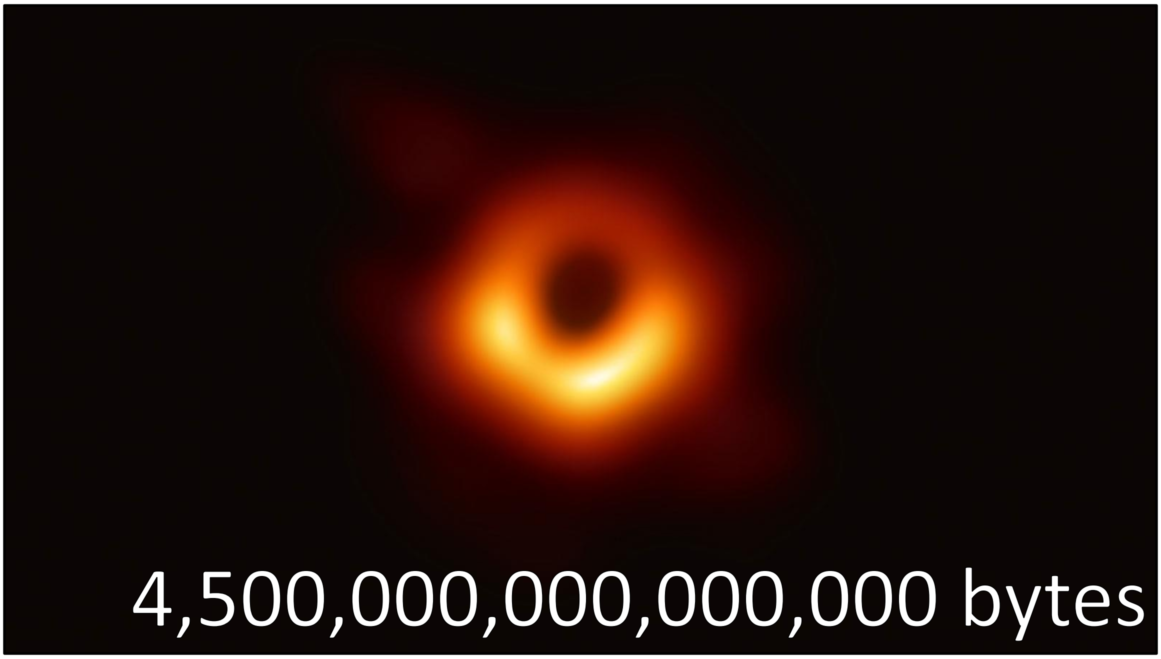
https://unsplash.com/photos/RflgrtzU3Cw

What about something a bit more …. well, not realistic per-se, but more down to earth? If you look at the richest person on earth, that's currently Elon Must, at 219,000,000,000 (2022)

https://www.pexels.com/photo/hand-holding-fan-of-us-dollar-bills-4968663/
https://www.forbes.com/billionaires/

One of the most expensive houses/estates in the US is the Chartwell Estate, at 350 million dollars

https://www.homedit.com/most-expensive-house-in-the-us/
https://www.latfusa.com/media/uploads/2017/08/07/coldwell_banker_chartwell_estate.jpg



The black hole image in M87 took 4.5 petabytes to make.

https://cdn.britannica.com/26/205226-050-B2621B00/Black-hole-M87-centre-evidence-supermassive-black.jpg
https://datamakespossible.westerndigital.com/astronomy-data-petabytes-black-hole-image-messier-87/

Representation

So how do we represent numbers in computers, and how do we make sure we don't lose any information?

Demo: Using Numeric Data Types

Big Numbers

Overflow matters. Consider the Ariane 5 rocket disaster in 1996.

https://www.nasaspaceflight.com/2019/08/arianespace-onward-dual-passenger-ariane-5-launch/

"All it took to explode that rocket less than a minute into its maiden voyage last June, scattering fiery rubble across the mangrove swamps of French Guiana, was a small computer program trying to stuff a 64-bit number into a 16-bit space."

https://youtu.be/PK_yguLapgA?t=93

```
SHA-512("Jason Bock")

==

C12A4E0F61DB33CAEF156A04E37FEB8E043E6B24DD2
F9728F11919955DB269BC0B7B6C4791212CAEA96AAC
571536CA93D88F2279C32A0417F794915025BE4CC1
```

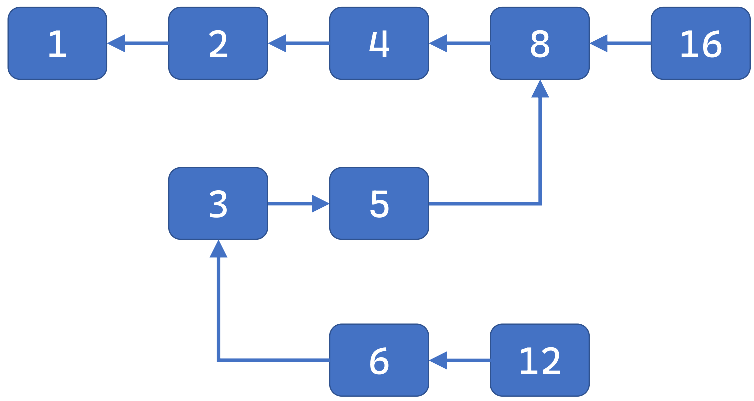
Hashing functions produce “large” numbers, though typically those are stored in raw byte buffers

https://youtu.be/PK_yguLapgA?t=93

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2}, \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

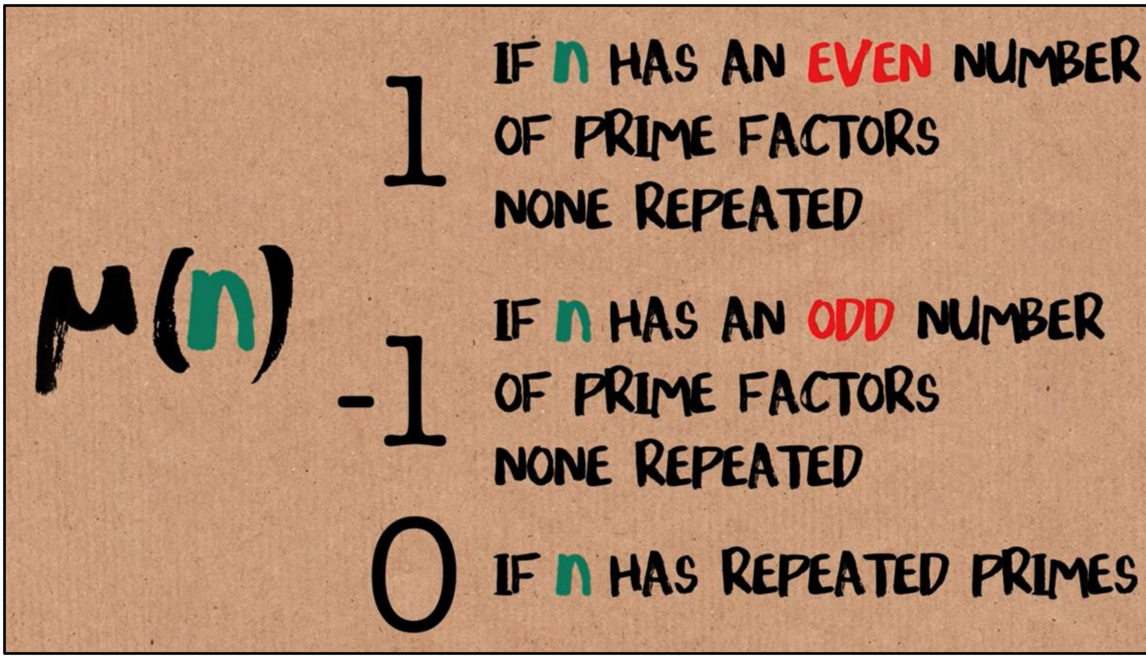
Sometimes we want to test algorithms to see how they behave. The Collatz Conjecture is a prime example of this.

https://en.wikipedia.org/wiki/Collatz_conjecture



As of 2020, the conjecture has been checked by computer for all starting values up to 2^{68}, or approximately $2.95*10^{20}$

Here's a visualization of how some of the numbers converge to 1.



While the general consensus of the Collatz Conjecture is that it is true, we can't rely on experimental calculations with larger and larger numbers. There's a great video of Dr. Holly Kreiger that describes what this is: Merten's conjecture, definitely check it out.

https://www.youtube.com/watch?v=uvMGZb0Suyc

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	−1	−1	0	−1	1	−1	0	0	1

n	11	12	13	14	15	16	17	18	19	20
$\mu(n)$	−1	0	−1	1	1	0	−1	0	−1	0

n	21	22	23	24	25	26	27	28	29	30
$\mu(n)$	1	1	−1	0	0	1	0	0	−1	−1

Here are the first 30 values for u(n)

https://en.wikipedia.org/wiki/M%C3%B6bius_function#Definition

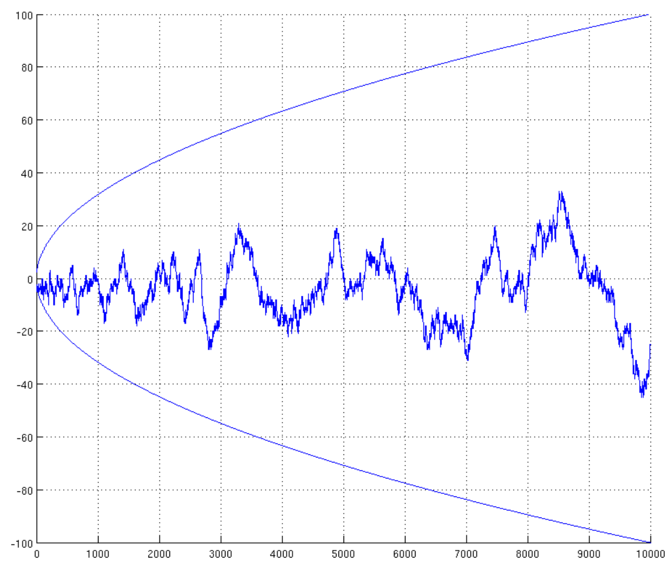
Merten’s function $M(n) = \sum_{k=1}^{n} \mu(k)$

Merten’s conjecture $|M(n)| < \sqrt{n}.$

This is Merten’s function along with the conjecture

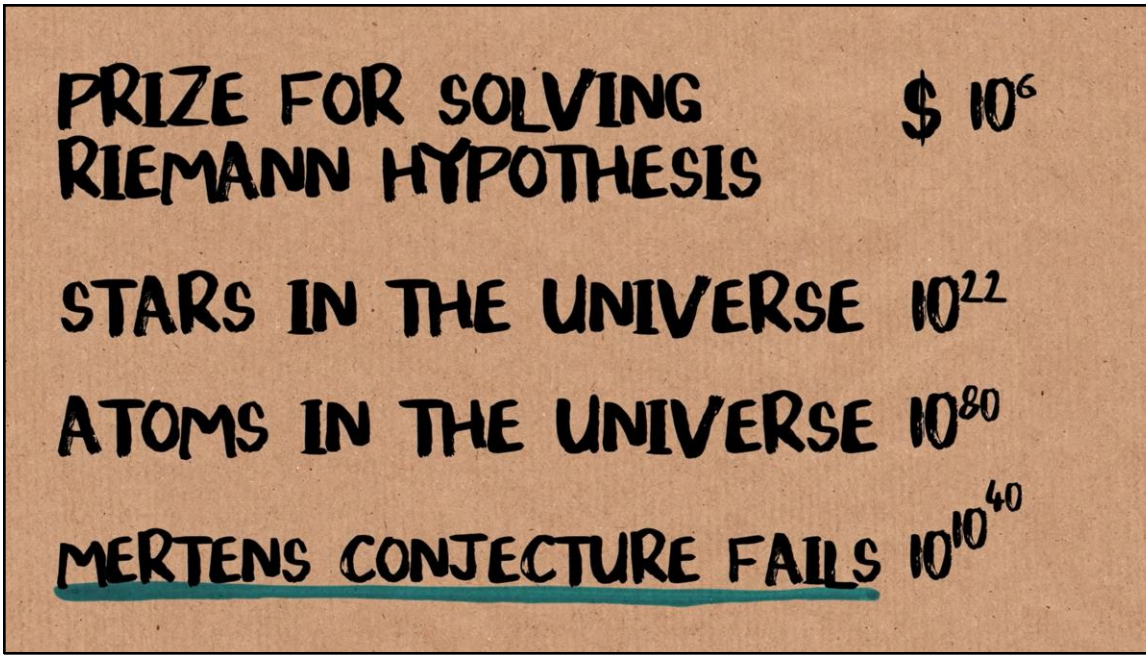
$$M(7) = 1 + -1 + -1 + 0 + -1 + 1 + -1 = -2$$

Here’s M(7)



Here’s a graph of the Mertens conjecture. The conjecture is that the blue line should never cross the parabola.

https://en.wikipedia.org/wiki/Mertens_conjecture

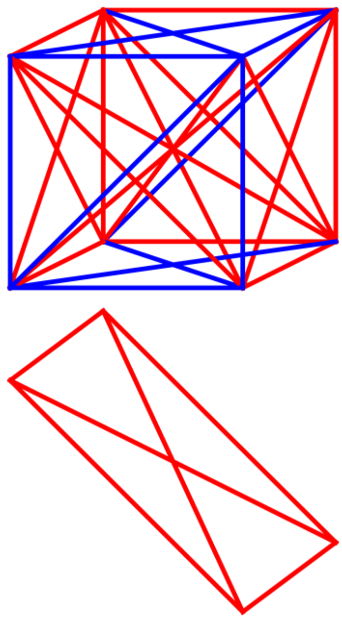


Here's the amazing thing. There is a value that contradicts the Mertens conjecture, but it's unbelievably large. (Technically that number is smaller than that, it's 10^10^23, but that's still huge). Note that the exact value is unknown, but it's "around" that number.

https://www.youtube.com/watch?v=uvMGZb0Suyc
https://twitter.com/hollykrieger/status/1220399809502306304?s=09

Insanity

This is where the insanity starts…



Connect each pair of geometric vertices of an *n*-dimensional hypercube to obtain a complete graph on 2^n vertices. Colour each of the edges of this graph either red or blue. What is the smallest value of *n* for which every such colouring contains at least one single-coloured complete subgraph on four coplanar vertices?

First, let’s talk about Graham’s number. I’m not going to claim to understand this, but…that’s what it is. It’s a problem in Ramsey theory. Basically, you want a plane that does not have the same color of edges. The fun is finding that smallest value for n dimensions.

https://en.wikipedia.org/wiki/Graham%27s_number

$$6 <= N^* <= g(64)$$

The n dimension was originally bounded by Graham to be between 6 and g(64), which is Graham’s number (since then, the bounds have been “improved”). So what is g(64)?

$$
\begin{aligned}
3\uparrow 3 &= 3^3 \\
&= 3*3*3 \\
&= 27
\end{aligned}
$$

First, we have to talk about up-arrow notation.

https://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation

$$
\begin{aligned}
3\uparrow\uparrow 3 &= (3\uparrow(3\uparrow 3)) \\
&= 3^{3^{3}} \\
&= 3^{27} \\
&= 7{,}625{,}597{,}484{,}987
\end{aligned}
$$

First, we have to talk about up-arrow notation.

https://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation

$$3\uparrow\uparrow\uparrow 3 = (3\uparrow\uparrow(3\uparrow\uparrow 3))$$
$$= \text{Gargantuan}$$

First, we have to talk about up-arrow notation.

https://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation

$$a\uparrow^{n}b = a(n \text{ arrows})b$$

$$a\uparrow^{5}b = a\uparrow\uparrow\uparrow\uparrow\uparrow b$$

You can generalize to this, which….is frightening

https://en.wikipedia.org/wiki/Knuth%27s_up-arrow_notation

$$g(1) = 3\uparrow\uparrow\uparrow\uparrow 3$$

So this is the start of g(64), defining g(1)

$$g(2) = 3\uparrow^{g(1)}3$$

g(2) puts g(1) arrows in. This is just….huge. But it doesn't stop there.

$$g(3) = 3\uparrow^{g(2)}3$$
$$g(4) = 3\uparrow^{g(3)}3$$
…
$$g(64) = 3\uparrow^{g(63)}3$$

It's just mind-numbing how big g(64) gets.

$$G = \left.\begin{matrix} 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\cdots\cdots\uparrow}3 \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\uparrow}3 \\ \underbrace{\qquad\qquad\vdots\qquad\qquad} \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\uparrow}3 \\ 3\uparrow\uparrow\uparrow\uparrow 3 \end{matrix}\right\} \text{64 layers}$$

Here’s another way to “visualize” it.

https://en.wikipedia.org/wiki/Graham%27s_number

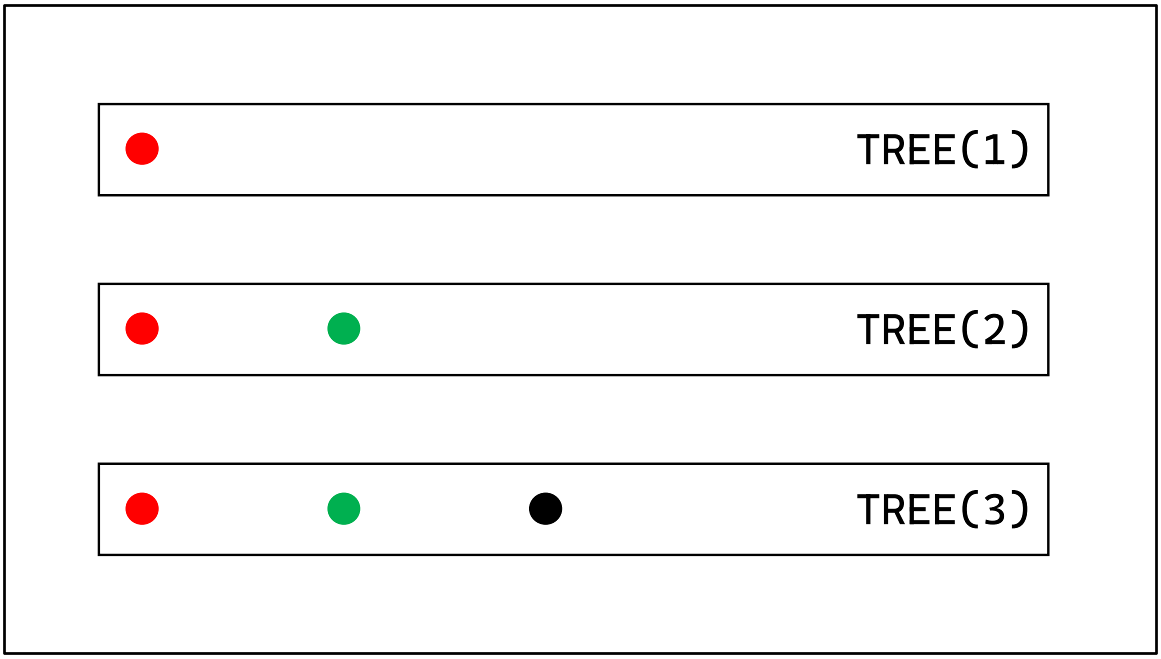
...024259506950647383956574791365193517983345353625214300354012602677162267216041981065226316935518878038814483140652526168785095552646051071172000997092912495443788874960628829117250630010362293491608025459461494578871427832350829421020918258967535604308699380168924988926899510169055919951195027887178308370183402364745488822221615732280101329745092734459450434330090109692802535275183328988446150894042482650181938515625357963961899396790549663800322234872396701848518643905910457562726246419538 7

And yet we know what the last 500 digits are!

https://en.wikipedia.org/wiki/Graham%27s_number

But there is something even horribly bigger. TREE(3). It all has to do with how many "unique" trees exist based on different seeds. The mathematical definition is far more precise than this, but….ignore that for now. It's related to Kruskal's tree theorem.

https://en.wikipedia.org/wiki/Kruskal%27s_tree_theorem
https://www.pexels.com/photo/assorted-color-beans-in-sack-1393382/

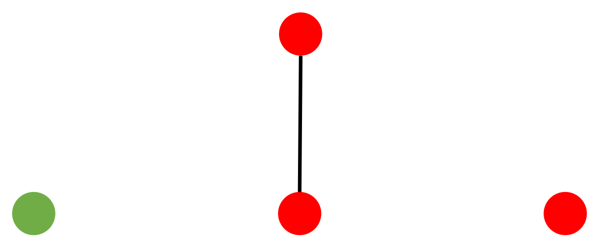


Basically, how many combinations of trees can you make? That’s what TREE(n) is.

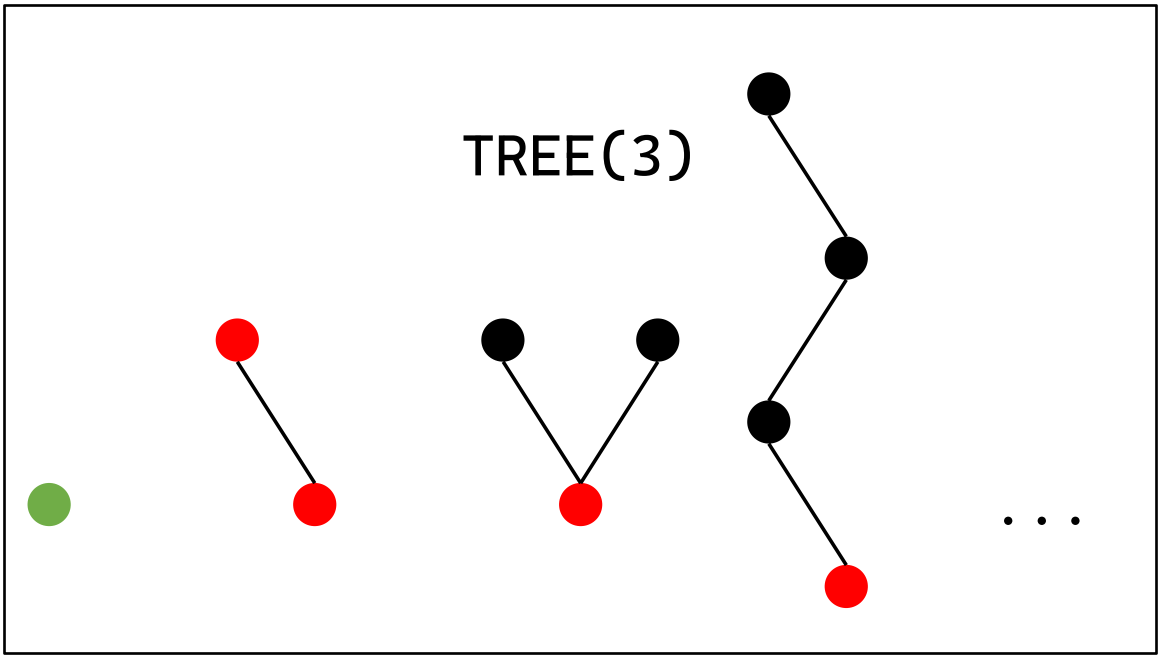
TREE(1)

For TREE(1), we can only draw one tree. Anything else contains the original tree, and that’s not allowed.

TREE(2)



For TREE(2), the maximum is 3.



For TREE(3)…well, the ellipsis at the right tells a huge story.

```
TREE(1) = 1
TREE(2) = 3
TREE(3) = Abandon all hope
```

TREE(3) is so huge, it puts Graham's number to shame. But, it'

Call To Action

Start looking at your code, and think about places where you might overflow. Should you be using a different data type?

https://unsplash.com/photos/afW1hht0NSs

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- https://github.com/JasonBock/Chudnovsky
- https://github.com/JasonBock/Presentations
- References in the notes on this slide

References

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* [A Peek Behind the Black Hole Image’s Petabytes of Data](https://datamakespossible.westerndigital.com/astronomy-data-petabytes-black-hole-image-messier-87/)
* [Pi in the sky: Calculating a record-breaking 31.4 trillion digits of Archimedes’ constant on Google Cloud](https://cloud.google.com/blog/products/compute/calculating-31-4-trillion-digits-of-archimedes-constant-on-google-cloud)
* [How Many Decimals of Pi Do We Really Need?](https://www.jpl.nasa.gov/edu/news/2016/3/16/how-many-decimals-of-pi-do-we-really-need/)
* [Large Number Formats](https://xkcd.com/2319/)
* [Patterns that appear to hold, but don't - 8424432925592889329288197322308900672459420460792433](https://www.youtube.com/watch?v=L4ArlAfKTLA&feature=youtu.be)
* [A Prime Surprise (Mertens Conjecture) -

Numberphile](https://www.youtube.com/watch?v=uvMGZb0Suyc) (This is a great example where a HUGE counterexample nullifies what may seem intuitively correct, there was a correction in the video on the size of the counterexample: https://twitter.com/hollykrieger/status/1220399809502306304?s=09)
* [A Breakthrough in Graph Theory](https://www.youtube.com/watch?v=Tnu_Ws7Llo4) (note the 4^10000 number, which is why the Collatz Conjecture can't be proven with strictly larger numbers)
* Floating Point Numbers
  * [Is floating point math broken?](https://stackoverflow.com/questions/588004/is-floating-point-math-broken)
  * [Mostly harmless: An account of pseudo-normal floating point numbers](https://developers.redhat.com/blog/2021/05/12/mostly-harmless-an-account-of-pseudo-normal-floating-point-numbers)
  * [Floating Point Math](https://0.30000000000000004.com/)
* [64 Bits ought to be enough for anybody!](https://blog.trailofbits.com/2019/11/27/64-bits-ought-to-be-enough-for-anybody/)
* [Why the Sum of Three Cubes Is a Hard Math Problem](https://www.quantamagazine.org/why-the-sum-of-three-cubes-is-a-hard-math-problem-20191105/)
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* [From 1,000,000 to Graham's Number](https://waitbutwhy.com/2014/11/1000000-grahams-number.html)