DT8248 MATH4854 Introduction to PDEs Assiginment 2

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Question 1:

Use separation of variables to solve the problem:

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) = xy$$

where $u(0, y) = -y^2$ for $y \in \mathbb{R}$

Solution: We have two options

$$\begin{aligned} u(x, y) &= f(x)g(y) \quad (1) \\ u(x, y) &= f(x) + g(y) \quad (2) \end{aligned}$$

Try Option 2:

$$\begin{aligned} u(x, y) &= f(x) + g(y) \quad (1) \\ \therefore \frac{\partial u}{\partial x} &= f'(x) \\ \therefore \frac{\partial u}{\partial y} &= g'(y) \\ \therefore (\frac{\partial u}{\partial x}\Big)\Big(\frac{\partial u}{\partial y}\Big) &= (f'(x))(g'(y)) = xy \\ \therefore f'(x)g'y) &= xy \\ \therefore \frac{f'(x)}{x} &= \frac{y}{g'(y)} \end{aligned}$$

Since the LHS depends on x alone, and the RHS depends on y alone, therefore:

$$\begin{aligned} \frac{f'(x)}{x} &= \lambda \quad (3) \\ \frac{y}{g'(y)} &= \lambda \quad (4) \end{aligned}$$

Solving (3):

$$\begin{aligned} \frac{f'(x)}{x} &= \lambda \quad (3) \\ \therefore f'(x) &= \lambda x \\ \therefore f(x) &= \frac{\lambda x^2}{2} + A \quad (5) \end{aligned}$$

Solving (4):

$$\begin{aligned} \frac{y}{g'(y)} &= \lambda \quad (4) \\ \therefore g'(y) &= \frac{y}{\lambda} \\ \therefore g(y) &= \frac{y^2}{2\lambda} + B \quad (6) \end{aligned}$$

Therefore using (5) and (6) to sub into (2):

$$\begin{aligned} u(x, y) &= f(x) + g(y) \quad (2) \\ \therefore u(x, y) &= \frac{\lambda x^2}{2} + A\frac{y^2}{2\lambda} + B \\ \therefore u(x, y) &= \frac{\lambda x^2}{2} + \frac{y^2}{2\lambda} + C \quad C = A + B \\ \therefore u(0, y) &= 0 + \frac{y^2}{2\lambda} + C = -y^2 \\ \therefore C &= 0\, and\, \lambda = \frac{-1}{2} \\ \therefore u(x, y) &= \frac{-x^2}{4} - y^2 \end{aligned}$$

Question 2:

Use the parametric form of the Lagrange-Charpit equations to solve the problem:

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) = xy$$

where $u(0, y) = -y^2$ for $y \in \mathbb{R}$

First lets parameterise the initial conditions: At $t = 0,\ x = 0,\ y = s$ and $u = -s^2$.

Our PDE can be written as: $F(x, y, u, p, q) = pq - xy = 0$ (1).

The Lagrange-Charpit equations in parametric form are:

$$\begin{aligned} \frac{dx}{dt} &= F_p = q \quad (2) \\ \frac{dy}{dt} &= F_p = p \quad (3) \\ \frac{du}{dt} &= pF_p + qF_q = pq + qp = 2pq \quad (4) \\ \frac{dp}{dt} &= -F_x - pF_u = -(-y) - p(0) = y \quad (5) \\ \frac{dq}{dt} &= -F_y - qF_u = -(-x) - q(0) = x \quad (6) \\ dt &= \frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{y} = \frac{dq}{x} \end{aligned}$$

From (2) and (6), we get:

$$\begin{aligned} \frac{dx}{dt} &= q \quad (2) \\ \therefore \frac{d^2x}{dx^2} &= \frac{dq}{dt} \\ \therefore \frac{d^2x}{dx^2} &= x \quad from\,(6) \\ \therefore x &= Acosht + Bsinht \quad from\, solutions\, to\, standard\, ODEs \\ \therefore 0 &= A(1) + B(0) \quad from\, initial\, conditions \\ \therefore A &= 0,\ B =? \\ \therefore x &= Bsinht \quad (7) \\ \therefore \frac{dx}{dt} &= Bcosht = q \quad (8) \end{aligned}$$

Similarly from (3) and (5):

$$\begin{aligned} \frac{dy}{dt} &= p \quad (3) \\ \therefore \frac{d^2y}{dx^2} &= \frac{dp}{dt} \\ \therefore \frac{d^2y}{dx^2} &= y \quad from\,(5) \\ \therefore y &= Ccosht + Dsinht \quad from\, solutions\, to\, standard\, ODEs \\ \therefore s &= C(1) + D(0) \quad from\, initial\, conditions \\ \therefore C &= s, D =? \\ \therefore y &= scosht + Dsinht \quad (9) \\ \therefore \frac{dy}{dt} &= ssinht + Dcosht = p \quad (10) \end{aligned}$$

Using Equation (1) and (7), (8), (9) and (10) :

$$\begin{aligned} pq &= xy \\ \therefore (ssinht + Dcosht)(Bcosht) &= Bsinht(scosht + Dsinht) \\ \therefore sB(cosht)(sinht) + BDcosh^2t &= sB(cosht)(sinht) + BDsinh^2t \\ \therefore BD &= 0,\ B \neq 0,\ so\ D = 0 \\ \therefore y &= scosht \quad (11) \\ \therefore \frac{dy}{dt} &= ssinht = p \quad (12) \end{aligned}$$

Now we will substitute (12) and (8) into (4):

$$\begin{aligned} \frac{du}{dt} &= 2pq \\ \therefore \frac{du}{dt} &= 2(ssinh(t))(Bcosh(t)) \quad where\ s,\ and\ B\, are\, constants \\ \therefore u &= 2Bs\int sinh(t)cosh(t)dt = 2Bs\int \frac{sinh(2t)}{2}dt \\ \therefore u &= \frac{sBcosh(2t)}{2} + E \\ \therefore -s^2 &= \frac{sBcosh2(0)}{2} + E \\ \therefore -s^2 &= \frac{sB}{2} + E \\ \therefore B &= -2s,\ E = 0 \\ \therefore x &= -2ssinh(t) \\ \therefore y &= scosh(t) \\ \therefore x^2 &= 4s^2sinh^2(t) \\ \therefore y^2 &= s^2cosh^2(t) \\ \therefore u &= -s^2cosh(2t) = -s^2(cosh^2(t) + sinh^2(t)) \\ \therefore u &= -\frac{x^2}{4} - y^2 \end{aligned}$$

Please note I have made some assumptions about constants which I am uncomfortable with. I believe there is probably more than one solution to this PDE.

Question 3:

Determine the general solution of the equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x} - 6\frac{\partial^2 u}{\partial y^2} = 0$$

By factorising the relevent linear differential operator.

Looking for factorisation of the form:

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\right)\left(C\frac{\partial}{\partial x} + D\frac{\partial}{\partial y}\right)u = 0$$

The process of finding the constants A, B, C and D can be greatly simplified if we simply factorise the characteristic equation $r^2 - r - 6$, which is the following $(r + 2)(r - 3)$. Thus we have the following:

$$\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)u = 0$$

Which gives us the following system of ODEs to solve:

$$\begin{aligned} v &= \frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} \quad (1) \\ 0 &= \frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial y} \quad (2) \end{aligned}$$

We now solve Equation 2 by using the Method of Characteristics:

$$\begin{aligned} \frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial y} &= 0 \quad (2) \\ \therefore \frac{dy}{dx} &= \frac{b}{a} = \frac{2}{1} = 2 \quad (3) \\ \therefore \frac{dv}{dx} &= \frac{c}{a} = \frac{0}{1} \quad (4) \end{aligned}$$

Solving Equation 3:

$$\begin{aligned} \frac{dy}{dx} &= 2 \quad (3) \\ \therefore y &= 2x + k \\ \therefore k &= y - 2x \end{aligned}$$

Solving Equation 4:

$$\begin{aligned} \frac{dv}{dx} &= 0 \quad (4) \\ \therefore v &= f(k) \\ \therefore v &= f(y - 2x) \quad (5) \end{aligned}$$

Using (5) in (1) to solve:

$$\begin{aligned} v &= \frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = f(y - 2x) \\ \therefore \frac{dy}{dx} &= \frac{b}{a} = \frac{-3}{1} = -3 \quad (6) \\ \therefore \frac{du}{dx} &= \frac{c}{a} = \frac{f(y - 2x)}{1} \quad (7) \end{aligned}$$

Solving (6):

$$\begin{aligned} \frac{dy}{dx} &= \frac{b}{a} = \frac{-3}{1} = -3 \quad (6) \\ \therefore y &= -3x + L \\ \therefore L &= y + 3x \end{aligned}$$

Solving (7):

$$\begin{aligned} \frac{du}{dx} &= f(y - 2x) \quad (7) \\ \therefore u &= \int f(y - 2x)dx + g(L) \quad where\, g\, is\, an\, arbitary\, function \\ \therefore u &= \int f(y - 2x)dx + g(L) = \int f(w)(-2)dw + g(L) \quad w = y - 2x \\ \therefore u &= -2\int f(w)dw + g(L) \\ \therefore u &= h(w) + g(L) \quad where\, h\, is\, an\, arbitary\, function \\ \therefore u &= h(y - 2x) + g(y + 3x) \end{aligned}$$

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