DT8248 MATH4853 Introduction to Partial Differential Equations Assignment 1

Lecturer: Colum Watt

Student Name: Jason Borland

Student Number: D17129310

Question 2

Consider the semi-linear, first order, partial differential equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = c(x, y, u) \quad (*)$$

Define $\omega(r, \theta)$ by $\omega(r, \theta) = u(rcos(\theta), rsin(\theta))$ for $r > 0$ and $\theta \in \Re$.

(a) Express $\frac{\partial \omega}{\partial r}$ and $\frac{\partial \omega}{\partial \theta}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

(b) Rewrite (*) in terms of $w, r, and \theta$.

(c) Rewrite the Cauchy problem:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 1$$

Where $u(x, 0) = 0 \; for \; 0 < x < \infty$ in terms of $w, r, and \theta$ and hece solve it.

(d) In terms of $r \; and \; \theta$, determine the domain of influence of the initial conditions.

Part (a):

In the semi-linear, 1st order PDE (*), u is a function of x and y. (i.e. u(x,y)).

If $\omega(r, \theta) = u(rcos(\theta), rsin(\theta))$, therefore $x = rcos(\theta)$ and $y = rsin(\theta)$ then

$$\begin{aligned}
\frac{\partial \omega}{\partial r} &= \frac{\partial[u(x(r,\theta), y(r,\theta))]}{\partial r} \\
\frac{\partial \omega}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
\frac{\partial \omega}{\partial r} &= \frac{\partial[u(rcos(\theta), rsin(\theta))]}{\partial r} \\
\frac{\partial \omega}{\partial r} &= \cos(\theta)\frac{\partial u}{\partial x} + \sin(\theta)\frac{\partial u}{\partial y} \quad (1)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\frac{\partial \omega}{\partial \theta} &= \frac{\partial[u(x(r,\theta), y(r,\theta))]}{\partial \theta} \\
\frac{\partial \omega}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\
\frac{\partial \omega}{\partial \theta} &= \frac{\partial[u(rcos(\theta), rsin(\theta))]}{\partial \theta} \\
\frac{\partial \omega}{\partial \theta} &= -r\sin(\theta)\frac{\partial u}{\partial x} + r\cos(\theta)\frac{\partial u}{\partial y} \quad (2)
\end{aligned}$$

Part (b):

If you multiply (1) by $r\sin(\theta)$ and (2) by $\cos(\theta)$ then add the results:

$$\begin{aligned}
r\sin(\theta)\frac{\partial \omega}{\partial r} &= r\sin(\theta)\cos(\theta)\frac{\partial u}{\partial x} + r\sin^2(\theta)\frac{\partial u}{\partial y} && (3) \\
\cos(\theta)\frac{\partial \omega}{\partial \theta} &= -r\cos(\theta)\sin(\theta)\frac{\partial u}{\partial x} + r\cos^2(\theta)\frac{\partial u}{\partial y} && (4) \\
r\sin(\theta)\frac{\partial \omega}{\partial r} + \cos(\theta)\frac{\partial \omega}{\partial \theta} &= r(\sin^2(\theta) + \cos^2(\theta))\frac{\partial u}{\partial y} && \\
r\sin(\theta)\frac{\partial \omega}{\partial r} + \cos(\theta)\frac{\partial \omega}{\partial \theta} &= r\frac{\partial u}{\partial y} && (3) + (4) = (5) \\
\sin(\theta)\frac{\partial \omega}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial \omega}{\partial \theta} &= \frac{\partial u}{\partial y} && (6)
\end{aligned}$$

If you multiply (1) by $r\cos(\theta)$ and (2) by $\sin(\theta)$ then subtract the results:

$$\begin{aligned}
r\cos(\theta)\frac{\partial \omega}{\partial r} &= r\cos^2(\theta)\frac{\partial u}{\partial x} + r\sin(\theta)\sin(\theta)\frac{\partial u}{\partial y} && (7) \\
\sin(\theta)\frac{\partial \omega}{\partial \theta} &= -r\sin^2(\theta)\frac{\partial u}{\partial x} + r\sin(\theta)\cos(\theta)\frac{\partial u}{\partial y} && (8) \\
r\cos(\theta)\frac{\partial \omega}{\partial r} - \sin(\theta)\frac{\partial \omega}{\partial \theta} &= r(\cos^2(\theta) + \sin^2(\theta))\frac{\partial u}{\partial x} && \\
r\cos(\theta)\frac{\partial \omega}{\partial r} - \sin(\theta)\frac{\partial \omega}{\partial \theta} &= r\frac{\partial u}{\partial x} && (7) + (8) = (9) \\
\cos(\theta)\frac{\partial \omega}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial \omega}{\partial \theta} &= \frac{\partial u}{\partial x} && (10)
\end{aligned}$$

Now using (6), (10), $x = rcos(\theta)$, $y = rsin(\theta)$ and substitute into (*):

$$\begin{aligned}
rsin(\theta)\left(\cos(\theta)\frac{\partial \omega}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial \omega}{\partial \theta}\right) - rcos(\theta)\left(\sin(\theta)\frac{\partial \omega}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial \omega}{\partial \theta}\right) &= c(rcos(\theta), rsin(\theta), \omega(r, \theta)) \\
\left(r\sin(\theta)\cos(\theta) - r\sin(\theta)\cos(\theta)\right)\frac{\partial \omega}{\partial r} - \left(\sin^2(\theta) + \cos^2(\theta)\right)\frac{\partial \omega}{\partial \theta} &= c(rcos(\theta), rsin(\theta), \omega(r, \theta)) \\
-\frac{\partial \omega}{\partial \theta} &= c(rcos(\theta), rsin(\theta), \omega(r, \theta)) \quad (11)
\end{aligned}$$

Part (c):

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 1$$

Where $u(x, 0) = 0 \; for \; 0 < x < \infty$

Rewriting initial conditions:

$$\omega(r, \theta) = u(r\cos(\theta), r\sin(\theta)) \qquad = u(r\cos(0), r\sin(0)) = 0$$

So $\omega(r, 0) = 0$, where $0 < r\cos(\theta) < \infty$.

Therefore $r > 0$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Rewriting the PDE:

$$-\frac{\partial \omega}{\partial \theta} = 1$$

This leads to the following:

$$\begin{aligned}
ds = \frac{dr}{0} &= \frac{d\theta}{-1} = \frac{d\omega}{1} \\
\therefore d\omega &= -d\theta \\
\therefore \int d\omega &= \int -d\theta \\
\therefore \omega &= -\theta + A \\
\therefore \omega &= -\theta \qquad (12)
\end{aligned}$$

Note $A = 0$ from initial conditions.

Part(d):

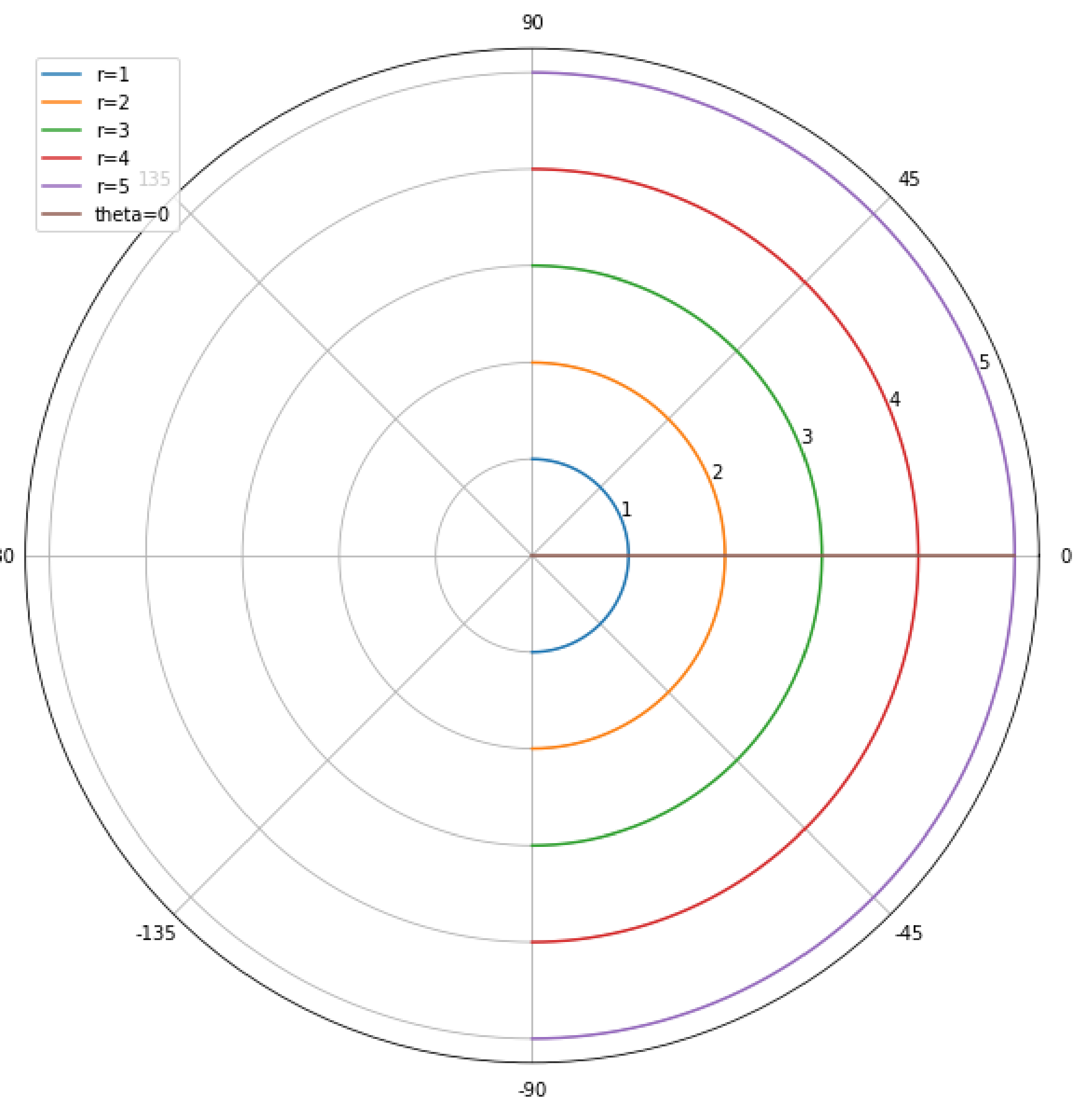
The domain of influence of the initial conditions is the top-right and bottom-right quadrants of the xy-plane:

$r > 0$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

In [ ]:

In [36]:

```
import numpy as np
import matplotlib.pyplot as plt
import math

fig, ax1 = plt.subplots(figsize=(10, 10), subplot_kw=dict(polar=True))

# creating an array containing the
# radian values
rads = np.arange(-np.pi/2, (np.pi/2), 0.01)
l = np.size(rads)
# plotting the circle
for r in range (1,6):
    radii = [r]*l
    ax1.plot(rads, radii, label='r='+str(r))

ax1.set_xticklabels([0, 45, 90, 135, 180, -135, -90, -45 ])


theta = [0,0]
r = [0, radii[0]]
ax1.plot(theta, r, label='theta=0')
ax1.legend(loc='best', fontsize=10)

plt.show()
```



From notes: The initial conditions for a first order linear PDE can restrict the set of points on which the solution exists. The resulting set of points in $\Re^2$ is called the domain of influence.

In the problem above the domain of influence consists of all points that lie on the characteristic curve in the $r\theta$-plane which crosses the data curve $\theta = 0, 0 < r < \infty$.

The characteristic curves have the form $r = k$. Where k is a constant. For any point (r_0, θ_0) with $r_0 > 0$, we know that it lies on the characteristic curve:

$$r = r_0, \quad where \; k = r_0$$

This does meet the data curve at $(r, 0)$, where $r = r_0$. Therefore the domain of influence of the above is the top right and bottom right quandrants of the $r\theta$-plane (i.e. $r > 0 \; and \; \frac{-\pi}{2} < \theta < \frac{\pi}{2}$).

In [ ]: