DT8248 MATH4853 Introduction to Partial Differential Equations Assignment 1

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Question 4

Assume that the function $u(x,t) is defined implicitly by:

$$u = t^3 + f(x - 4tu + 3t^4) \quad (1)$$

where f is an unknown differentiable function.

(a) Use implicit differentiation to obtain expressions for u_x, and u_t in terms of x, t, and f.

(b) Prove that u satisfies a partial differential equation of the form:

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} = \phi(t)$$

for some function $\phi(t)$ which you should determine.

(c) Use the method of characteristics to determine the general solution to this partial differential equation.

Part(a):

Simplify equation (1), using $\omega = x - 4tu + 3t^4$:

$$\begin{aligned}
u &= t^3 + f(\omega) \\
\therefore u_x &= 0 + \omega_x f'(\omega) \\
since\ \omega_x &= 1 - 4tu_x \\
\therefore u_x &= (1 - 4tu_x)f'(\omega) \\
\therefore u_x + 4tf'(\omega)u_x &= f'(\omega) \\
\therefore u_x &= \frac{f'(\omega)}{1 + 4tf'(\omega)} \qquad (2) \\
u &= t^3 + f(\omega) \\
\therefore u_t &= 3t^2 + \omega_t f'(\omega) \\
since\ \omega_t &= 0 - 4u - 4tu_t + 12t^3 = 12t^3 - 4u - 4tu_t \\
\therefore u_t &= 3t^2 + (12t^3 - 4u - 4tu_t)f'(\omega) \\
\therefore u_t + 4tf'(\omega)u_t &= 3t^2 + (12t^3 - 4u)f'(\omega) \\
\therefore u_t &= \frac{3t^2 + (12t^3 - 4u)f'(\omega)}{1 + 4tf'(\omega)} \\
\therefore u_t &= \frac{3t^2 + [12t^3 - 4(t^3 + f(\omega))]f'(\omega)}{1 + 4tf'(\omega)} \\
\therefore u_t &= \frac{3t^2 + [8t^3 - 4f(\omega))]f'(\omega)}{1 + 4tf'(\omega)} \qquad (3)
\end{aligned}$$

Part(b):

$$\begin{aligned}
\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} &= \phi(t) \\
u_t &= \frac{3t^2 + 8t^3 f'(\omega) - 4f(\omega)f'(\omega)}{1 + 4tf'(\omega)} \\
u_x &= \frac{f'(\omega)}{1 + 4tf'(\omega)} \\
4uu_x &= \frac{4t^3 f'(\omega) + 4f(\omega)f'(\omega)}{1 + 4tf'(\omega)} \\
\therefore \frac{3t^2 + 8t^3 f'(\omega) - 4f(\omega)f'(\omega)}{1 + 4tf'(\omega)} + \frac{4t^3 f'(\omega) + 4f(\omega)f'(\omega)}{1 + 4tf'(\omega)} &= \phi(t) \\
\therefore \frac{3t^2 + 12t^3 f'(\omega)}{1 + 4tf'(\omega)} &= \phi(t) \\
\therefore 3t^2 \frac{1 + 4tf'(\omega)}{1 + 4tf'(\omega)} &= \phi(t) \\
\therefore 3t^2 &= \phi(t)
\end{aligned}$$

Part (c):

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} = 3t^2$$

This leads to the following system of ODEs:

$$\begin{aligned}
\frac{dt}{ds} &= 1 \qquad (4) \\
\frac{dx}{ds} &= 4u \qquad (5) \\
\frac{du}{ds} &= 3t^2 \qquad (6) \\
ds &= \frac{dt}{1} = \frac{dx}{4u} = \frac{du}{3t^2}
\end{aligned}$$

Taking equation (4) and (6) gives:

$$\begin{aligned}
\frac{dt}{1} &= \frac{du}{3t^2} \\
\therefore 3t^2 dt &= du \\
\therefore \int 3t^2 dt &= \int du \\
\therefore u = t^3 + A (7) \\
\therefore \varphi(x,t,u) &= u - t^3 \quad is\ a\ first\ integral\ (7)
\end{aligned}$$

Substitute (7) in (5) and then combine with (4) gives:

$$\begin{aligned}
\frac{dt}{1} &= \frac{dx}{4t^3 + 4A} \\
\therefore (4t^3 + 4A)dt &= dx \\
\therefore \int (4t^3 + 4A)dt &= \int dx \\
\therefore t^4 + 4tA + B &= x \\
\therefore t^4 + 4t(u - t^3) + B &= x \\
\therefore \psi(x,t,u) &= B = x - 4tu + 3t^4 \quad is\ a\ first\ integral\ (8)
\end{aligned}$$

Thus the general solution is:

$$u - t^3 = F(x - 4tu + 3t^4)$$

Check that first integrals (7) and (8) are functionally independent first integrals (fifi):

Check that $\nabla\varphi \times \nabla\psi \neq 0$:

$$\begin{aligned}
\nabla\varphi &= -3t^2 \jmath + k \\
\nabla\psi &= \imath + (12t^3 - 4u)\jmath - 4tk
\end{aligned}$$

$$\nabla\varphi \times \nabla\psi = \begin{vmatrix} \imath & \jmath & k \\ 0 & -3t^2 & 1 \\ 1 & (12t^3 - 4u & -4t \end{vmatrix} = \imath(12t^3 - 12t^3 + 4u) + \jmath + 3t^2 k$$

So $\nabla\varphi \times \nabla\psi \neq 0$, So our first integrals are fifi.

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