DT8248 MATH4853 Introduction to Partial Differential Equations Assignment 1

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Question 1

Solve the Cauchy problem:

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 1$$

Where $u(x, 0) = 0 \; for \; 0 < x < \infty$

Determine the domain of influence of the initial conditions.

Note: y = 0 along the data curve, so it may not be possible to get y in terms of x for the characteristic curves close to their intersection with the data curve.

Using the parametric approach we get:

$$\frac{dx}{dt} = y \qquad (1)$$
$$\frac{dy}{dt} = -x \qquad (2)$$
$$\frac{du}{dt} = 1 \qquad (3)$$

We need to parameterise the initial conditions. Let s be a parameter along the data curve. Let (s, 0) be a typical point on our data curve. Where $0 < s < \infty$. The initial conditions tell us that:

$u = 0 \quad at\;(s, 0) \qquad when\; 0 < s < \infty.$

$\therefore\; at\; t = 0,\; x = s,\; y = 0,\; and\; u = 0$

or using the alternative approach we get:

$$\frac{dy}{dx} = \frac{-x}{y} \qquad (4)$$
$$\frac{du}{dx} = \frac{1}{y} \qquad (5)$$

Where the initial conditions are: $u = 0, \quad at\;(x, 0)$

Running with the parametric approach:

Substitute (1) into (2):

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = -x$$
$$\therefore \frac{d^2x}{dt^2} + x = 0$$
$$\therefore x = Acost + Bsint \qquad standard\; solution\; to\; the\; 2nd\; order\; ODE$$

When t=0, x = s and y= 0.

$$\therefore s = Acos0 + Bsin0$$
$$\therefore s = A$$
$$\therefore x = scost + Bsint$$
$$\therefore \frac{dx}{dt} = -ssint + Bcost$$
$$\therefore y = -ssint + Bcost$$
$$\therefore 0 = -s(0) + B(1)$$
$$\therefore B = 0$$
$$\therefore y = -ssint \qquad (6)$$
$$\therefore x = scost \qquad (7)$$

Solving (3):

$$\frac{du}{dt} = 1$$
$$\therefore du = dt$$
$$\therefore \int du = \int dt$$
$$\therefore u = t + C$$

When t=0, u=0:

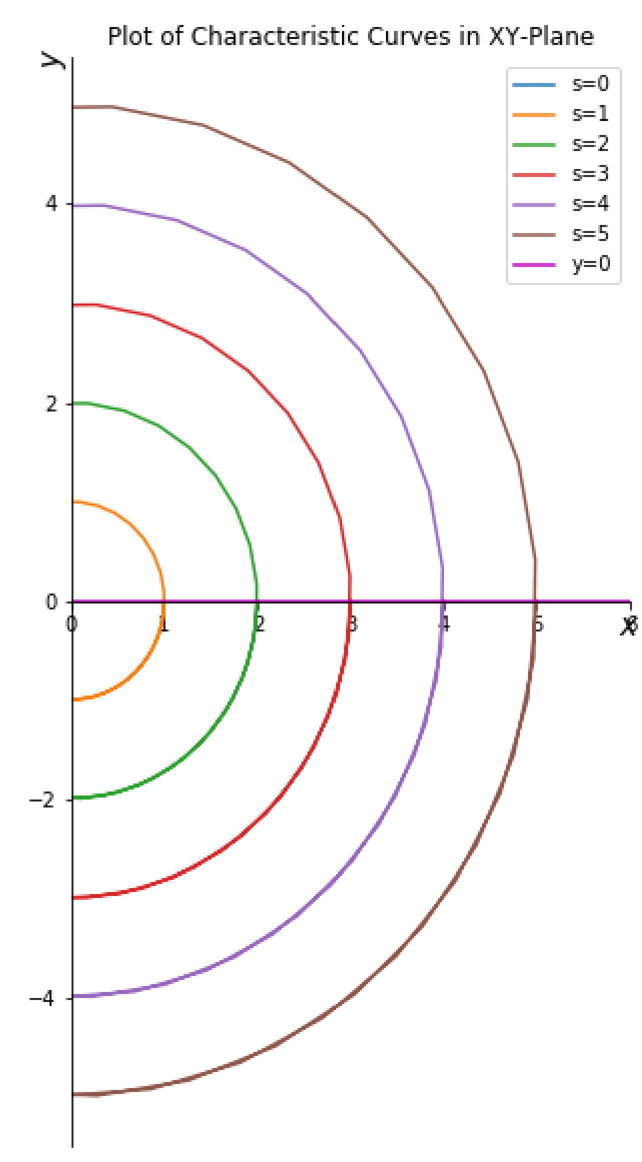
$$u = t + C$$
$$\therefore 0 = 0 + C$$
$$\therefore C = 0$$
$$\therefore u = t \qquad (8)$$

Now will attempt to remove the parameters s and t from equations (6), (7) and (8). Dividing (6) by (7):

$$\frac{y}{x} = \frac{-ssint}{scost}$$
$$\therefore \frac{y}{x} = -tan(t)$$
$$\therefore \frac{y}{x} = -tan(u) \qquad From\;(8)$$
$$\therefore u = \arctan\left(\frac{-y}{x}\right)$$

In [2]:
```
import math
import numpy as np
import matplotlib.pyplot as plt
```

In [12]:
```
t = np.arange(0., 10., 0.2)



fig, ax = plt.subplots(figsize=(5,10))

ax.set_title('Plot of Characteristic Curves in XY-Plane')
ax.set_xlabel('$x$', fontsize=15 )
ax.xaxis.set_label_coords(1, 0.49)
ax.set_ylabel('$y$', fontsize=15 )
ax.yaxis.set_label_coords(0, 1)

# Move left y-axis and bottim x-axis to centre, passing through (0,0)
ax.spines['left'].set_position('zero')
ax.spines['bottom'].set_position('zero')

# Eliminate upper and right axes
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')

# Show ticks in the left and lower axes only
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

ax.set_xlim(0,6)
for s in range (0, 6):
    x = s*np.cos(t)
    y = -s*np.sin(t)
    plt.plot(x,y, label='s='+str(s) )


plt.axhline(y=0, xmin=-5, xmax=5, color='m', label='y=0')

ax.legend(loc='best', fontsize=10)

plt.show()
```



From notes: The initial conditions for a first order linear PDE can restrict the set of points on which the solution exists. The resulting set of points in $\Re^2$ is called the domain of influence.

In the problem above the domain of influence consists of all points that lie on the characteristic curve in the xy-plane which crosses the data curve $y = 0, 0 < x < \infty$.

The characteristic curves have the form $x^2 + y^2 = k$. Where k is a constant. For any point (x_0, y_0) with $x_0 > 0$, we know that it lies on the characteristic curve:

$$x^2 + y^2 = x_0^2 + y_0^2, \qquad where\; k = x_0^2 + y_0^2$$

This does meet the data curve at $(x, 0)$, where $x = x_0$. Therefore the domain of influence of the above is the top right and bottom right quandrants of the xy-plane (i.e. $x > 0$).

In [ ]: