DT8248 MATH4853 Introduction to Partial Differential Equations Assignment 1

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Question 3

Determine the general solution of the partial differential equation:

$$(lu - my)\frac{\partial u}{\partial x} + (mx - nu)\frac{\partial u}{\partial y} = ny - lx \quad (1)$$

Where $l,\ m,\ and\ n$ are unkown constants.

Note: This is of the form of a quasi-linear PDE:

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u)$$

Consider the system of ODEs:

$$\begin{aligned}
\frac{dx}{dt} &= lu - my \qquad &(2)\\
\frac{dy}{dt} &= mx - nu \qquad &(3)\\
\frac{du}{dt} &= ny - lx \qquad &(4)
\end{aligned}$$

We are looking for a first integral φ:

$$\begin{aligned}
n\frac{dx}{dt} + l\frac{dy}{dt} + m\frac{du}{dt} &= nlu - nmy + lmx - lnu + mny - mlx\\
\therefore n\frac{dx}{dt} + l\frac{dy}{dt} + m\frac{du}{dt} &= 0\\
\therefore \frac{d[nx + ly + mu]}{dt} &= 0\\
\therefore nx + ly + mu &= K \quad where\ K\ is\ a\ constant\\
\therefore nx + ly + mu &= \varphi(x, y, u) \quad (5)
\end{aligned}$$

Now we need to find a 2nd first integral ψ:

$$\begin{aligned}
x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt} &= xlu - xmy + ymx - ynu + uny - ulx\\
\therefore x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt} &= 0\\
\therefore \frac{d[\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}u^2]}{dt} &= 0\\
\therefore \frac{d\frac{1}{2}[x^2 + y^2 + u^2]}{dt} &= 0\\
\therefore x^2 + y^2 + u^2 &= L \quad where\ L\ is\ a\ constant\\
\therefore x^2 + y^2 + u^2 &= \psi(x, y, u) \quad (6)
\end{aligned}$$

Check that first integrals (5) and (6) are functionally independent first integrals (fifi):

Check that $\nabla\varphi \times \nabla\psi \neq 0$:

$$\begin{aligned}
\nabla\varphi &= n\imath + l\jmath + mk\\
\nabla\psi &= 2x\imath + 2y\jmath + 2uk
\end{aligned}$$

$$\nabla\varphi \times \nabla\psi = \begin{vmatrix} \imath & \jmath & k \\ n & l & m \\ 2x & 2y & 2u \end{vmatrix} = \imath(2lx - 2ym) - \jmath(2nz - 2mx) + k(2ny - 2xl)$$

So $\nabla\varphi \times \nabla\psi \neq 0$, (unless $x = y = u = 0$ or $x = y = z$ and $l = m = n$). So our first integrals ((5) and (6) are fifi.

The general solution to the PDE (1) is:

$$\begin{aligned}
x^2 + y^2 + u^2 &= F(nx + ly + mu)\\
or\ nx + ly + mu &= G(x^2 + y^2 + u^2)
\end{aligned}$$

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