Question 3:

Assume that the Sturm-Liouville system.

$e^{6x}y\,'' + 6e^{6x}y' + 9e^{6x}\lambda y = 0$, where $y(0) = 0 = y(1)$.

This has eiganfunctions $e^{-3x}sin(n\pi x)$ for $n = 1, 2, 3 \ldots$. Determine the eiganfunction expansion for $f(x) = 2x$.

$$
\begin{aligned}
e^{6x}y\,'' + 6e^{6x}y' + 9e^{6x}\lambda y &= 0\\
(e^{6x}y')' + 9e^{6x}\lambda y &= 0\\
\therefore p(x) &= e^{6x}\\
\therefore q(x) &= 0\\
\therefore r(x) &= 9e^{6x}\\
\therefore Assume\ y &= Ae^{rx},\ y' = Are^{rx},\ y\,'' = Ar^2e^{rx}\\
\therefore e^{6x}(Ar^2e^{rx}) + 6e^{6x}(Are^{rx} + 9e^{6x}\lambda(Ae^{rx}) &= 0\\
\therefore r^2 + 6r + 9\lambda &= 0, \quad (\div e^{6x}Ae^{rx})\\
\therefore r &= \frac{-6 \pm \sqrt{36 - 4(1)(9\lambda)}}{2}\\
\therefore r &= -3 \pm 3\sqrt{1-\lambda}
\end{aligned}
$$

Note $p(x) > 0$, and $r(x) > 0$, so this is a SLS.

If $k^2 = 1 - \lambda > 0$:

$$
\begin{aligned}
y &= Be^{(-3+k)x} + Ce^{(-3-k)x}\\
y(0) = 0 &= B + C\\
y(1) = 0 &= Be^{(-3+k)} + Ce^{(-3-k)}\\
0 &= Be^{(-3+k)} - Ce^{(-3-k)}\\
\therefore B &= 0 = C
\end{aligned}
$$

i.e. only trivial solution y = 0.

If $1 - \lambda = 0$:

$$
\begin{aligned}
y &= Be^{-3x} + Cxe^{-3x}\\
y(0) = 0 &= B\\
y(1) = 0 &= Ce^{-3}\\
\therefore B &= 0,\ C = 0
\end{aligned}
$$

i.e. only the trivial solution.

If $-k^2 = 1 - \lambda < 0$, Therefore $r = -3 \pm 3ki$:

$$
\begin{aligned}
y &= e^{-3x}(Acos(3kx) + Bsin(3kx))\\
y(0) = 0 &= A\\
y(1) = 0 &= e^{-3}Bsin(3k)\\
e^{-3} &\neq 0,\ for\ non\ trivial\ solutions\ B \neq 0\\
\therefore sin(3k) &= 0\\
\therefore 3k &= n\pi, \quad n = 1, 2, 3 \ldots\\
\therefore k &= \frac{n\pi}{3}\\
\therefore \lambda &= 1 + \frac{n^2\pi^2}{9}\\
\therefore Eiganfunction,\ y_n &= e^{-3x}sin(n\pi x)
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= 2x\\
f(x) &\sim \sum_{n=1}^{\infty} a_n e^{-3x}sin(n\pi x)\\
a_n &= \frac{\int_a^b f(x)y_n(x)r(x)dx}{\int_a^b (y_n(x))^2 r(x)dx}\\
a_n &= \frac{\int_0^1 2xe^{-3x}sin(n\pi x)(9e^{6x})dx}{\int_0^1 (e^{-3x}sin(n\pi x))^2(9e^{6x})dx}\\
a_n &= \frac{2.9\int_0^1 xe^{3x}sin(n\pi x)dx}{9\int_0^1 e^{-6x}e^{6x}sin^2(n\pi x)dx}\\
a_n &= \frac{2\int_0^1 xe^{3x}sin(n\pi x)dx}{\int_0^1 sin^2(n\pi x)dx}\\
\therefore I_1 &= 2\int_0^1 xe^{3x}sin(n\pi x)dx\\
\therefore I_2 &= \int_0^1 sin^2(n\pi x)dx
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_0^1 sin^2(n\pi x)dx\\
We\ know\ :\ sin^2(u) &= \frac{1}{2}(1 - cos(2u))\\
&= \frac{1}{2}\int_0^1 1 - cos(2n\pi x)dx\\
&= \frac{1}{2}[x - \frac{1}{2n\pi}sin(2n\pi x)]_0^1\\
&= \frac{1}{2}([1 - \frac{1}{2n\pi}sin(2n\pi)] - [0 - \frac{1}{2n\pi}sin(0)])\\
Since\ sin(2n\pi) = 0,\ and\ sin(0) &= 0\\
I_2 &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
e^{ix} &= cos(x) + isin(x)\\
e^{x+iy} &= e^x e^{iy} = e^x(cos(y) + isin(y))\\
Re(e^{x+iy}) &= e^x(cos(y))\\
Im(e^{x+iy}) &= e^x(sin(y))\\
cos(x) &= Re(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-x})\\
sin(x) &= Im(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-x})\\
I_1 &= 2\int_0^1 xe^{3x}sin(n\pi x)dx\\
&= 2\int_0^1 xe^{3x}Im(e^{in\pi x})dx\\
&= 2\int_0^1 xIm(e^{3x+in\pi x})dx\\
&= 2\int_0^1 xIm(e^{(3+in\pi)x})dx
\end{aligned}
$$

Integration by parts is required: $$ \begin {array}{r|r|r|} sign & D & I \ \hline

- & 2x & Im(e^{(3 + i n\pi)x }) \
- & 2 & \frac{1}{3 + i n\pi}Im(e^{(3 + i n\pi)x }) \
- & 0 & (\frac{1}{3 + i n\pi})^2Im(e^{(3 + i n\pi)x })\ \end{array} $$

$$
\begin{array}{ccc}
Sign & D & I\\
+ & 2x & Im(e^{(3+in\pi)x})\\
- & 2 & \frac{1}{3+in\pi}Im(e^{(3+in\pi)x})\\
+ & 0 & (\frac{1}{3+in\pi})^2 Im(e^{(3+in\pi)x})
\end{array}
$$

$$
\begin{aligned}
I_1 &= \left[2x\frac{1}{3+in\pi}Im(e^{(3+in\pi)x}) - 2\left(\frac{1}{3+in\pi}\right)^2 Im(e^{(3+in\pi)x})\right]_0^1\\
\frac{1}{3+in\pi} &= \frac{1}{3+in\pi} * \frac{3-in\pi}{3-in\pi}\\
&= \frac{3-in\pi}{9+n^2\pi^2}\\
\frac{1}{(3+in\pi)^2} &= \frac{1}{9+n^2\pi^2+i6n\pi}\\
&= \frac{1}{9+n^2\pi^2+i6n\pi} * \frac{9+n^2\pi^2-i6n\pi}{9+n^2\pi^2-i6n\pi}\\
&= \frac{9+n^2\pi^2-i6n\pi}{81+54n^2\pi^2+n^4\pi^4}\\
\therefore I_1 &= \left[2xIm(\frac{3-in\pi}{9+n^2\pi^2}e^{(3+in\pi)x}) - 2Im(\frac{9+n^2\pi^2-i6n\pi}{81+54n^2\pi^2+n^4\pi^4}e^{(3+in\pi)x})\right]_0^1\\
&= \left[2xIm(\frac{3-in\pi}{9+n^2\pi^2}e^{3x}(cos(n\pi x) + isin(n\pi x))) - 2Im(\frac{9+n^2\pi^2-i6n\pi}{81+54n^2\pi^2+n^4\pi^4}e^{3x}(cos(n\pi x) + isin(n\pi x)))\right]_0^1\\
&= \left[\frac{e^{3x}}{9+n^2\pi^2}\left(6sin(n\pi x) - 2n\pi cos(n\pi x)\right) + \frac{e^{3x}}{81+54n^2\pi^2+n^4\pi^4}\left(12n\pi cos(n\pi x) - (18+2n^2\pi^2)sin(n\pi x)\right)\right]_0^1\\
Note\ sin(0) &= 0,\ sin(n\pi) = 0,\ cos(0) = 1,\ cos(n\pi) = (-1)^n\\
&= \frac{-2ne^3(-1)^n}{9+n^2\pi^2} + \frac{12n\pi(-1)^n e^3}{81+54n^2\pi^2+n^4\pi^4} + \frac{2n}{9+n^2\pi^2} - \frac{12n\pi}{81+54n^2\pi^2+n^4\pi^4}\\
&= \frac{2n(1-e^3(-1)^n)}{9+n^2\pi^2} + \frac{12n\pi((-1)^n e^3 - 1)}{81+54n^2\pi^2+n^4\pi^4}\\
\therefore a_n &= \frac{I_1}{I_2}\\
&= \frac{\frac{2n(1-e^3(-1)^n)}{9+n^2\pi^2} + \frac{12n\pi((-1)^n e^3-1)}{81+54n^2\pi^2+n^4\pi^4}}{\frac{1}{2}}\\
a_n &= \frac{4n(1-e^3(-1)^n)}{9+n^2\pi^2} + \frac{24n\pi((-1)^n e^3 - 1)}{81+54n^2\pi^2+n^4\pi^4}\\
f(x) &\sim \sum_{n=1}^{\infty} a_n e^{-3x}sin(n\pi x)\\
f(x) &\sim \sum_{n=1}^{\infty}\left[\frac{4n(1-e^3(-1)^n)}{9+n^2\pi^2} + \frac{24n\pi((-1)^n e^3 - 1)}{81+54n^2\pi^2+n^4\pi^4}\right]e^{-3x}sin(n\pi x)
\end{aligned}
$$

Question 1:

Consider the wave equation $u_{tt} - c^2u_{xx} = 0$ for $0 \leq x < \infty$ and $0 \leq t < \infty$, where

i. $u(x, 0) = f(x)$ for $0 \leq x < \infty$,

ii. $u_t(x, 0) = g(x)$ for $0 \leq x < \infty$ and ,

iii. $u_x(0, t) = 0$ for $0 \leq t < \infty$.

Whose solution is given by:

$$
u(x,t) = \begin{cases} \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)d\omega, & \text{if } x - ct \geq 0\\ \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_0^{ct-x} g(\omega)d\omega + \frac{1}{2c}\int_0^{x+ct} g(\omega)d\omega, & \text{if } x - ct < 0 \end{cases}
$$

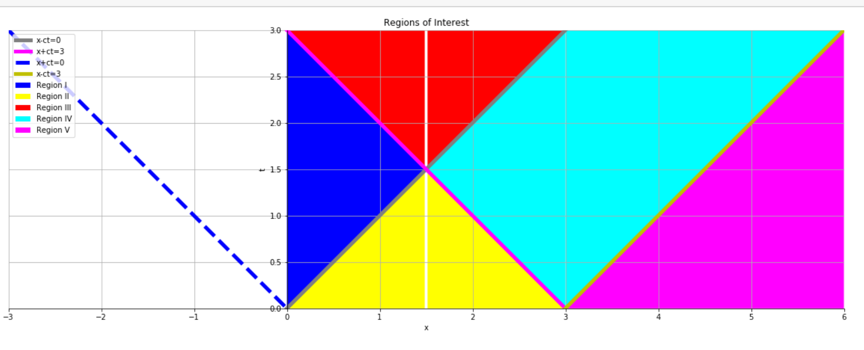
Use the this formula to calculate the solution when:

$$
u(x,0) = xe^{-x},\ for\ x \geq 0,\ and
$$

$$
u_t(x,0) = \begin{cases} (3-x)sinh(x) & for 0 \leq x \leq 3,\\ 0 & for\ x > 3 \end{cases}
$$

In [38]:

```
import matplotlib.pyplot as plt
from numpy import *
import math

x_1 = linspace(0, 3, 100)
x_n1 = linspace(-3, 0, 100)
x_3  = linspace(3, 6, 100)

t_0 = x_1
t_3 = 3 - x_1
t_n0 = -x_n1
t_n3 = x_3 -3

#plt.plot(x_1, t_0, 'g')
#plt.plot(x_1, t_3, 'r')
#plt.plot(x_n1 , t_n0, 'b--')
#plt.plot(x_3 , t_n3, 'y')


fig, ax = plt.subplots(figsize=(20, 10))
ax.set(xlim=(-3, 6), ylim=(0, 3),
               xlabel='x', ylabel='t',
               title='Regions of Interest');
ax.plot(x_1, t_0, 'grey', label='x-ct=0', linewidth=5)
ax.plot(x_1, t_3, 'magenta', label='x+ct=3', linewidth=5)
ax.plot(x_n1 , t_n0, 'b--', label='x+ct=0', linewidth=5)
ax.plot(x_3 , t_n3, 'y',label='x-ct=3', linewidth=5)
ax.set_aspect('equal')
ax.grid(True, which='both')

# set the x-spine (see below for more info on `set_position`)
ax.spines['left'].set_position('zero')

# turn off the right spine/ticks
ax.spines['right'].set_color('none')
ax.yaxis.tick_left()

# set the y-spine
ax.spines['bottom'].set_position('zero')

# turn off the top spine/ticks
ax.spines['top'].set_color('none')
ax.xaxis.tick_bottom()
ax.fill_between(x_1, t_0, t_3, where=t_0 <= t_3, facecolor='blue', interpolate=True, label='Region I')
ax.fill_between(x_1, t_0, 0,  where=t_0 <= t_3, facecolor='yellow', interpolate=True, label='Region II'
)
ax.fill_between(x_1, t_3, 0,  where=t_3 <= t_0, facecolor='yellow', interpolate=True)
ax.fill_between(x_1, t_0, 3,  where=t_0 >= t_3, facecolor='red', interpolate=True, label='Region III')
ax.fill_between(x_1, t_3, 3,  where=t_3 >= t_0, facecolor='red', interpolate=True)
ax.fill_between(x_1, t_0, t_3,  where=t_0 >= t_3, facecolor='cyan', interpolate=True, label='Region IV'
)
ax.fill_between(x_3, t_n3, 3, facecolor='cyan', interpolate=True)
ax.fill_between(x_3, t_n3, 0, facecolor='magenta', interpolate=True, label='Region V')

ax.legend(loc='upper left')
plt.show()
```



Region I: $x - ct < 0$ and $0 \leq x + ct \leq 3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_0^{ct-x} g(\omega)d\omega + \frac{1}{2c}\int_0^{x+ct} g(\omega)d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\int_0^{ct-x}(3-\omega)sinh(\omega)d\omega + \frac{1}{2c}\int_0^{x+ct}(3-\omega)sinh(\omega)d\omega
\end{aligned}
$$

Integration by parts is required:

Sign	D	I
+	(3-w)	sinh(w)
−	-1	cosh(w)
+	0	sinh(w)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\left[(3-\omega)cosh(\omega) + sinh(\omega)\right]_0^{ct-x} + \frac{1}{2c}\left[(3-\omega)cosh(\omega) + sinh(\omega)\right]_0^{x+ct}\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x) - (3-0)cosh(0) - sinh(0)\right]\\
&\quad + \frac{1}{2c}\left[(3-x-ct)cosh(x+ct) + sinh(x+ct) - (3-0)cosh(0) - sinh(0)\right]\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x) - 3\right]\\
&\quad + \frac{1}{2c}\left[(3-x-ct)cosh(x+ct) + sinh(x+ct) - 3\right]\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x)(3-x-ct)cosh(x+ct)\right.\\
&\quad \left. + sinh(x+ct) - 6\right]
\end{aligned}
$$

Region II: $0 \leq x - ct \leq 3$ and $0 \leq x + ct \leq 3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)}) + \frac{1}{2c}\int_{x-ct}^{x+ct}(3-\omega)sinh(\omega)d\omega
\end{aligned}
$$

Integration by parts is required:

Sign	D	I
+	(3-w)	sinh(w)
−	-1	cosh(w)
+	0	sinh(w)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[(3-\omega)cosh(\omega) + sinh(\omega)\right]_{x-ct}^{x+ct}\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[(3-(x+ct))cosh(x+ct) + sinh(x+ct) - (3-(x-ct))cosh(x-ct)\right.\\
&\quad \left. - sinh(x-ct)\right]
\end{aligned}
$$

Region III: $x - ct < 0$ and $x + ct > 3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_0^{ct-x} g(\omega)d\omega + \frac{1}{2c}\int_3^{x+ct} g(\omega)d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)}) + \frac{1}{2c}\int_0^{ct-x}(3-\omega)sinh(\omega)d\omega + \frac{1}{2c}\int_3^{x+ct} 0d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)}) + \frac{1}{2c}\left[(3-\omega)cosh(\omega) + sinh(\omega)\right]_0^{ct-x} + 0\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)}) + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x) - (3-0)cosh(0) - sinh(0)\right]\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)}) + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x) - 3\right]
\end{aligned}
$$

Region IV: $0 \leq x - ct \leq 3$ and $x + ct > 3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)}) + \frac{1}{2c}\int_{x-ct}^{3}(3-\omega)sinh(\omega)d\omega
\end{aligned}
$$

Integration by parts is required:

Sign	D	I
+	(3-w)	sinh(w)
−	-1	cosh(w)
+	0	sinh(w)

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[(3-\omega)cosh(\omega) + sinh(\omega)\right]_{x-ct}^{3}\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[(3-3)cosh(3) + sinh(3) - (3-(x-ct))cosh(x-ct) - sinh(x-ct)\right]\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[sinh(3) - (3-(x-ct))cosh(x-ct) - sinh(x-ct)\right]
\end{aligned}
$$

Region V: $x - ct > 3$ and $x + ct > 3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\omega)d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)}) + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\ d\omega\\
&= \frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)})
\end{aligned}
$$

$$
u(x,t) = \begin{cases}
\frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)} + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x)(3-x-ct)cosh(x+ct) + sinh(x+ct) - 6\right], & \text{if } x-ct<0 \text{ and } 0 \leq x+ct \leq 3 \text{ Region I}\\
\frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[(3-(x+ct))cosh(x+ct) + sinh(x+ct) - (3-(x-ct))cosh(x-ct) - sinh(x-ct)\right], & \text{if } 0 \leq x-ct \leq 3 \text{ and } 0 \leq x+ct \leq 3 \text{ Region II}\\
\frac{1}{2}((x+ct)e^{-(x+ct)} + (ct-x)e^{-(ct-x)}) + \frac{1}{2c}\left[(3-ct+x)cosh(ct-x) + sinh(ct-x) - 3\right], & \text{if } x-ct<0 \text{ and } x+ct>3, \text{ Region III}\\
\frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)} + \frac{1}{2c}\left[sinh(3) - (3-(x-ct))cosh(x-ct) - sinh(x-ct)\right], & \text{if } 0 \leq x-ct \leq 3 \text{ and } x+ct>3, \text{ Region IV}\\
\frac{1}{2}((x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)}), & \text{if } x-ct>3 \text{ and } x+ct>3, \text{ Region V}
\end{cases}
$$

Question 2:

Determine the eigenfunctions and eigenvalues for the following differential equation.

$y'' + 12y' + (36 + \lambda)y = 0$, where $y(0) = 0 = y(2\pi)$.

Assume the following solution $y = Ae^{rx}$, where $A \neq 0$. The differential equation becomes:

$$
\begin{aligned}
&= Ar^2e^{rx} + 12Are^{rx} + (36+\lambda)Ae^{rx} = 0\\
&= r^2 + 12r + (36+\lambda) = 0, \quad \text{Have } \div Ae^{rx}\\
\therefore r &= \frac{-12 \pm \sqrt{12^2 - 4(36+\lambda)(1)}}{2(1)}\\
\therefore r &= \frac{-12 \pm \sqrt{4(36 - (36+\lambda)}}{2}\\
\therefore r &= \frac{-12 \pm 2\sqrt{(-\lambda)}}{2}\\
\therefore r &= -6 \pm \sqrt{(-\lambda)}
\end{aligned}
$$

We have three cases:

1. $\lambda < 0, \therefore -\lambda > 0,$
2. $\lambda = 0,$
3. $\lambda > 0, \therefore -\lambda < 0,$

Case 1:

$\lambda < 0, \therefore -\lambda > 0$

Let $-\lambda = k^2$, where $k > 0$.

So we have $r = -6 \pm k$, which yields solution:

$$
\begin{aligned}
y &= Ae^{(-6+k)x} + Be^{(-6-k)x}\\
&= e^{-6x}(Ae^{kx} + Be^{-kx})\\
\text{Using y(0)} &= 0\\
y(0) &= 0 = A + B\\
\therefore B &= -A\\
\therefore y &= e^{-6x}(Ae^{kx} - Ae^{-kx})\\
&= Ae^{-6x}(e^{kx} - e^{-kx})\\
\text{Using } y(2\pi) &= 0\\
y(2\pi) &= 0 = Ae^{-12\pi}(e^{2\pi k} - e^{-2\pi k})\\
&= Ae^{-12\pi}(e^{4\pi k} - 1) \quad \text{We} \times e^{2\pi k}\\
k > 0 \therefore e^{4\pi k} &> 0,\\
\therefore A &= 0
\end{aligned}
$$

Only the trivial solution with case 1.

Case 2:

$\lambda = 0$

Thus $r = -6$, which yields solution:

$$
\begin{aligned}
y &= Ae^{-6x} + Bxe^{-6x}\\
\text{Using y(0)} &= 0\\
y(0) &= 0 = A\\
\therefore y &= Bxe^{-6x}\\
\text{Using } y(2\pi) &= 0\\
y(2\pi) &= 0 = 2\pi Be^{-12\pi}\\
\therefore B &= 0, \quad \text{As } 2\pi e^{-12\pi} > 0
\end{aligned}
$$

So only the trivial solution with case 2.

Case 3:

$\lambda > 0, \therefore -\lambda < 0$

Let $\lambda = k^2$, where $k > 0$. So we have $r = -6 \pm ik$, which yields solution:

$$
\begin{aligned}
y &= Ae^{(-6+ik)x} + Be^{(-6-ik)x}\\
&= e^{-6x}(Ae^{ikx} + Be^{-ikx})\\
&= e^{-6x}(Dcos(kx) + Csin(kx))\\
&\text{Using y(0)} = 0\\
y(0) &= 0 = (1)(Dcos(0) + Csin(0))\\
\therefore D &= 0\\
y &= Ce^{-6x}sin(kx)\\
&\text{Using } y(2\pi) = 0\\
y(2\pi) &= 0 = Ce^{-12\pi}sin(2\pi k)\\
&\text{We want } C \neq 0, \text{ and we know } e^{-12\pi} \neq 0\\
\therefore sin(2\pi k) &= 0\\
\therefore 2\pi k &= n\pi, \text{ where } n = 1, 2, 3, \ldots\\
\therefore k &= \frac{n}{2}
\end{aligned}
$$

THe eigenfunction is $y_n = C_n sin(n\pi)$, where $n = 1, 2, 3, \ldots$

The eigenvalue is $\lambda_n = k^2 = \frac{n^2}{4}$

In [ ]: