Data Driven Reduced Order Modeling

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Abstract

Reduced Order Modeling (ROM) can be applied to high-dimensional problems to reduce the computational cost of finding solutions. A high fidelity model was created for several partial differential equations of physical significance. Using data from these models, accurate reduced order models were constructed.

1 Main Objectives

1. Make online computation possible in computationally challenging problems.
2. Speed up analysis of problems with many degrees of freedom.
3. Produce an understanding of physical processes which are currently obscured by complexity.

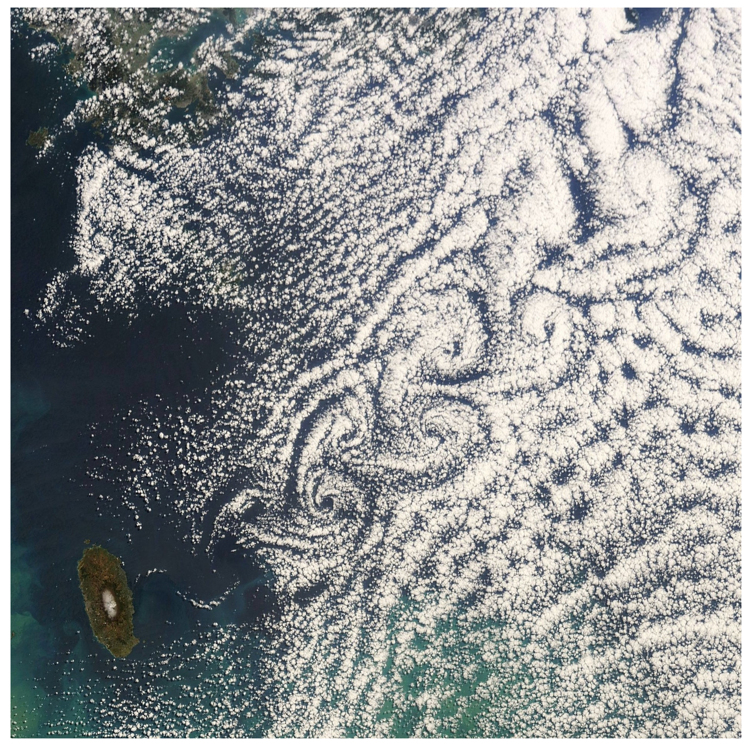


Figure 1: Atmospheric von Karman Vortex Street [?]

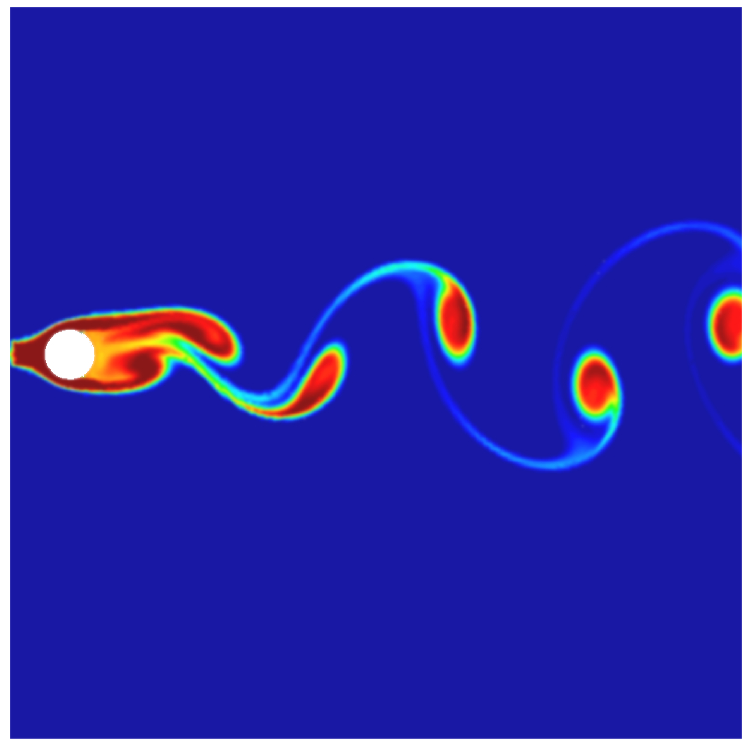


Figure 2: Simulated von Karman Vortex Street [?]

2 Introduction

- Linear Convection and Diffusion Equation describes movement and diffusion of a wave in one dimension.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$\vec{u}^{\,n+1} = \mathcal{L}(\vec{u}^{\,n}) = \mathbf{B}\vec{u}^{\,n}$$

- Viscous Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

$$\vec{u}^{\,n+1} = \mathcal{L}(\vec{u}^{\,n}) + \mathcal{N}(\vec{u}^{\,n})$$

- Black-Scholes Model prices European options.

$$\frac{\partial w}{\partial t} = rw - rx\frac{\partial w}{\partial x} - \frac{1}{2}v^2x^2\frac{\partial^2 w}{\partial x^2}$$

3 Methods

3.1 Full Order/High Fidelity Model

3.1.1 Linear Convection and Diffusion Equation

Finite Difference Scheme:

$$\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\vdots$$

$$\frac{\partial \vec{u}_i}{\partial t} = \left(\frac{\nu}{\Delta x^2} - \frac{c}{2\Delta x}\right)u_{i+1} - \left(\frac{2}{\Delta x^2}\right)u_i + \left(\frac{\nu}{\Delta x^2} + \frac{c}{2\Delta x}\right)u_{i-1}$$

$$\begin{bmatrix}
(-\frac{2}{\Delta x^2}) & (\frac{\nu}{\Delta x^2} - \frac{c}{2\Delta x}) & 0 & \dots & 0 & (\frac{\nu}{\Delta x^2} + \frac{c}{2\Delta x}) & 0 \\
(\frac{\nu}{\Delta x^2} + \frac{c}{2\Delta x}) & (-\frac{2}{\Delta x^2}) & (\frac{\nu}{\Delta x^2} - \frac{c}{2\Delta x}) & 0 & \dots & & 0 \\
0 & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & 0 \\
0 & & \dots & 0 & (\frac{\nu}{\Delta x^2} + \frac{c}{2\Delta x}) & (-\frac{2}{\Delta x^2}) & (\frac{\nu}{\Delta x^2} - \frac{c}{2\Delta x}) \\
0 & (\frac{\nu}{\Delta x^2} - \frac{c}{2\Delta x}) & 0 & \dots & 0 & (\frac{\nu}{\Delta x^2} + \frac{c}{2\Delta x}) & (-\frac{2}{\Delta x^2})
\end{bmatrix}$$

3.1.2 Viscous Burger's Equation

Newton's Method:

$$\mathbf{J}\left(\vec{u}^{\,n+1} - \vec{u}^{\,n}\right) - \mathbf{R}\left(\vec{u}^{\,n}\right) = 0$$

3.2 Reduced Order Model

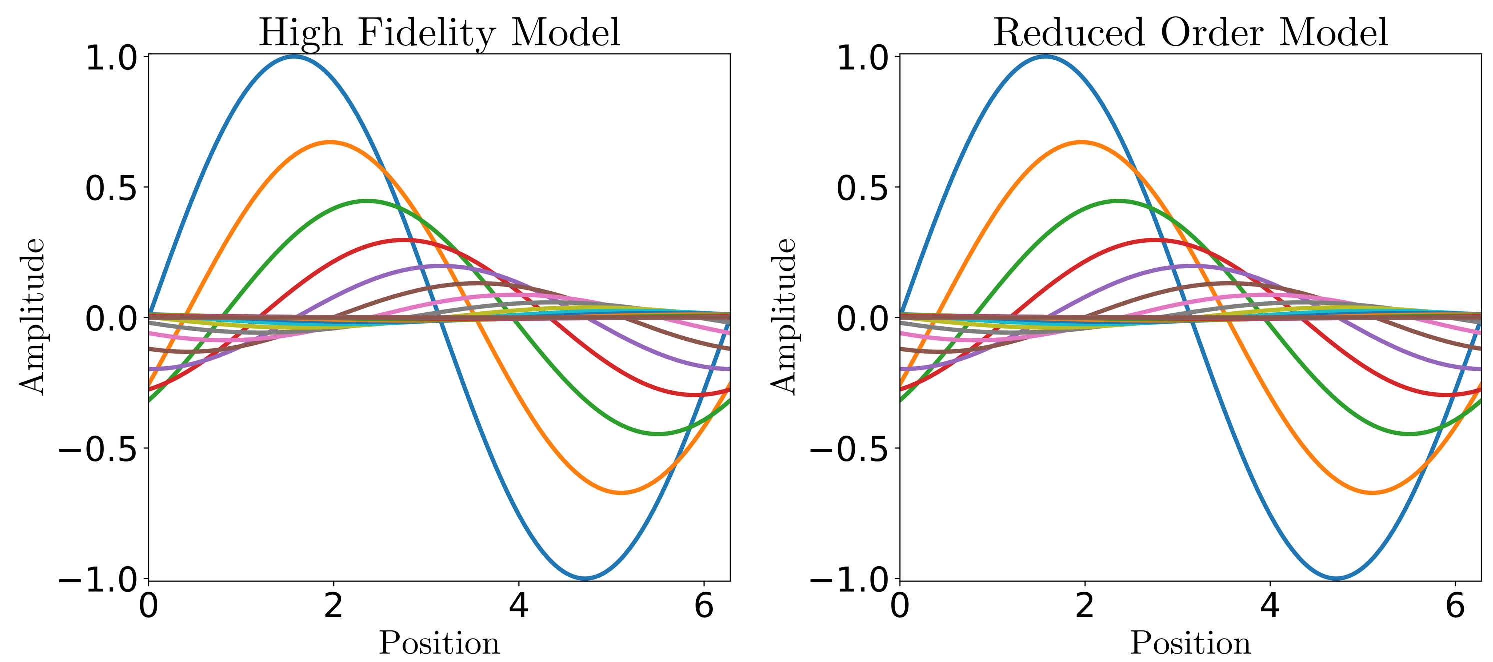
The efficiences and benefits of ROM are realized by reducing the complexity of the underlying mathematics of the problem. This can be achieved by reducing the rank of matrices involved in evolving the system; this can be done with **singular value decomposition** (SVD).

$$\min\{||\mathbf{X} - \tilde{\mathbf{X}}||_F\}\ \ s.t.\ \ \mathrm{rank}(\tilde{\mathbf{X}}) \ll \mathrm{rank}(\mathbf{X})$$

$$\tilde{\mathbf{X}} = U\Sigma V^T$$

The number of modes used to create the model is equivalent to the choice of singular values. In the case of linear convection and diffusion, five modes were used to model with nearly no error.

4 Results



Preliminary timing tests show computational speed-up of about 50%, but this may not stay constant as the complexity of the problem is varied.

5 Forthcoming Research

- Model Black-Scholes Equation
- Merton Model

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma_s^2 s^2\frac{\partial^2 u}{\partial s^2} + (r - \mu\xi)s\frac{\partial u}{\partial s} - (r+\mu)u + \mu\int_0^\infty u(sy,\tau)p(y)dy$$

- Implement hyper-reduction
  - Discrete Empirical Interpolation Method (DEIM)

6 Acknowledgements

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