David Tong: Vector Calculus Example Sheet 3

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A note on notation: I try to stick to the notation used in the lecture notes. However, based on space (and how lazy I'm feeling), I may use some shortcuts (i.e. ∂_x instead of $\frac{\partial}{\partial x}$, omit bounds like S or ∂S, etc.)

1)

1.i)

Switch to polar coordinates:

$$\begin{aligned} x &= R\cos(\phi) \\ y &= R\sin(\phi) \\ \mathrm{d}x &= -R\sin(\phi)\,\mathrm{d}\phi \\ \mathrm{d}y &= R\cos(\phi)\,\mathrm{d}\phi \end{aligned}$$

Then we have

$$\begin{aligned} I &= \int_0^{2\pi} \mathrm{d}\phi\left[-(R^2\cos^2\phi)(R\sin\phi)(-R\sin\phi) + (R\cos\phi)(R^2\sin^2\phi)(R\cos\phi)\right] \\ &= \frac{\pi R^4}{2} \end{aligned}$$

For the area integral, we know that $P = -x^2 y$ and $Q = xy^2$ - therefore we also have

$$\begin{aligned} \frac{\partial P}{\partial y} &= -x^2 \\ \frac{\partial Q}{\partial x} &= y^2 \end{aligned}$$

Substituting into Greens theorem (and using $\mathrm{d}A = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$) gives

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^R \mathrm{d}\rho\,\rho(x^2 + y^2) = \frac{\pi R^4}{2}$$

in agreement with our previous result.

1.ii)

By Greens theorem, we have

$$I = \int_A (x^2 + y^2)\,\mathrm{d}A$$

This integral doesn't change if the space is rotated, because $x^2 + y^2$ is simply the distance from the origin, and $\mathrm{d}A$ is independent of the coordinate system.

2)

2.i)

We have that $\nabla \times F = -2z\hat{k}$. Our surface $\mathrm{d}\boldsymbol{S}$ is, in spherical coordinates

$$R^2 \sin(\theta)\boldsymbol{e_r}\,\mathrm{d}\theta\,\mathrm{d}\phi = \sin(\theta)\boldsymbol{e_r}\,\mathrm{d}\theta\,\mathrm{d}\phi$$

since $R = 1$. Using $\hat{k} = \cos\theta$ we have

$$\begin{aligned}(\nabla \times F)\cdot \mathrm{d}\boldsymbol{S} &= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi(-2z\sin\theta\cos\theta) \\ &= -2\pi\end{aligned}$$

The line integral is the rim of the shell, and is parameterized by $\rho = 1, z = 0$ or

$$\boldsymbol{x} = (\cos\phi, \sin\phi, 0) \implies \mathrm{d}\boldsymbol{x} = (-\sin\phi, \cos\phi, 0)\,\mathrm{d}\phi$$

Thus

$$\begin{aligned}\boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x} &= -y\sin\phi - x\cos\phi = -1 \\ \int_{\partial S} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x} &= \int_0^{2\pi} -\mathrm{d}\phi = -2\pi\end{aligned}$$

3)

3.i)

Stokes theorem says

$$\oint_{\partial S} (\boldsymbol{a}\times \boldsymbol{F})\cdot \mathrm{d}\boldsymbol{x} = \int_S \nabla \times (\boldsymbol{a}\times\boldsymbol{F})\cdot \mathrm{d}\boldsymbol{S}$$

Using

$$\begin{aligned}\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) \\ \boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}) \\ \boldsymbol{A}\times\boldsymbol{B} &= -\boldsymbol{B}\times\boldsymbol{A}\end{aligned}$$

we have

$$\begin{aligned}-\boldsymbol{a}\cdot\oint_{\partial S} \mathrm{d}\boldsymbol{x}\times\boldsymbol{F} &= \int_S [\boldsymbol{a}(\nabla\cdot\boldsymbol{F}) - (\boldsymbol{a}\cdot\nabla)\boldsymbol{F}]\,\mathrm{d}\boldsymbol{S} \\ &= \boldsymbol{a}\int_S [(\nabla\cdot\boldsymbol{F})\,\mathrm{d}\boldsymbol{S} - (\mathrm{d}\boldsymbol{S}\cdot\nabla)\boldsymbol{F}]\end{aligned}$$

We can rewrite

$$(\nabla\cdot\boldsymbol{F})\,\mathrm{d}\boldsymbol{S} - (\mathrm{d}\boldsymbol{S}\cdot\nabla)\boldsymbol{F} = -((\mathrm{d}\boldsymbol{S}\times\nabla)\times\boldsymbol{F})$$

Substituting back in, and doing some algebra gives

$$\oint_{\partial S} \mathrm{d}\boldsymbol{x}\times\boldsymbol{F} = \int_S (\mathrm{d}\boldsymbol{S}\times\nabla)\times\boldsymbol{F}$$

as we intended to prove.

3.ii)

Let’s do the line integral first.

1. From origin to (1,0,0) we have $\mathrm{d}\boldsymbol{x} = (1,0,0)\,\mathrm{d}x$, so

$$\int_0^1 \mathrm{d}\boldsymbol{x} \times \boldsymbol{F} = \int_0^1 (1,0,0) \times (x,0,0)\,\mathrm{d}x = 0$$

2. From (1,0,0) to (1,1,0) we have $\mathrm{d}\boldsymbol{x} = (0,1,0)\,\mathrm{d}y$ and $\boldsymbol{F} = (1,y,0)$, so

$$\int_0^1 (0,1,0) \times (1,y,0)\,\mathrm{d}y = (0,0,-1)$$

3. From (1,1,0) to (0,1,0), we have $\mathrm{d}\boldsymbol{x} = (-1,0,0)\,\mathrm{d}x$ and $\boldsymbol{F} = (x,1,0)$, so

$$\int_0^1 (-1,0,0) \times (x,1,0)\,\mathrm{d}x = (0,0,-1)$$

4. From (0,1,0) to origin, we have $\mathrm{d}\boldsymbol{x} = (0,-1,0)\,\mathrm{d}y$ and $\boldsymbol{F} = (0,x,0)$ so

$$\int_0^1 (0,-1,0) \times (0,x,0)\,\mathrm{d}y = 0$$

so the total is $(0,0,-2)$.

For the surface integral we have $\mathrm{d}\boldsymbol{S} = (0,0,1)\,\mathrm{d}x\,\mathrm{d}y$ so

$$\mathrm{d}\boldsymbol{S} \times \nabla = \hat{k} \times \nabla = \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0\right)$$

$$\begin{aligned}\int_S (\mathrm{d}\boldsymbol{S} \times \nabla) \times \boldsymbol{F} &= \int_0^1 \int_0^1 \left(-\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0\right) \times (x,y,z)\,\mathrm{d}x\,\mathrm{d}y \\ &= -2\hat{k}\end{aligned}$$

in agreement with our previous result.

4)

Because $r = 1$ we have that $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$. We also have the normal being in the same direction as $\boldsymbol{x}$, and because $r = 1$ we know that $\boldsymbol{x}$ is already unit sized. Therefore we have that

$$\int_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{x} \cdot \boldsymbol{x}\,\mathrm{d}S = \text{Surface area of sphere} = 4\pi$$

By Stokes theorem we expect that if $\boldsymbol{F} = \nabla \times \boldsymbol{A}$ we should be able to write

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \int_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = 0$$

where the last line comes from the fact that the boundary of a closed surface (like our sphere) is the empty set. However, as calculated above, we clearly got a nonzero value.

We can calculate $\nabla \cdot \boldsymbol{F}$ with the divergence theorem. Because the boundary of a sphere is the empty set, we have that

$$\begin{aligned}\int_V (\nabla \cdot \boldsymbol{F})\,\mathrm{d}V &= \int_{\partial V} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0 \\ &\implies \nabla \cdot \boldsymbol{F} = 0\end{aligned}$$

However, note that we cannot use this to conclude that $\boldsymbol{F} = \nabla \times \boldsymbol{A}$ globally, because the given proof had the requirement that the domain of $\boldsymbol{F}(\boldsymbol{x})$ is $\mathbb{R}^3$, but in our case the domain is $\mathbb{R}^3 \setminus 0$.

5)

I got lazy and put the curl into Wolfram Alpha, which spit out the $\boldsymbol{F}(x)$ given in the previous problem. However, note that this isn't a contradiction because the discontinuity at $x^2 + y^2 = 0$ still exists.

TODO: I also got too lazy to do the surface integral

6)

6.i) E(r)

We have

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_S E \cdot \mathrm{d}S = 4\pi r^2 E(r)$$

By Gauss' law, we know that this equals $\frac{Q}{\varepsilon_0}$ so we have

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\int_V \rho \,\mathrm{d}V}{4\pi\varepsilon_0 r^2}$$

- From $r \leq a$, the charge integral is 0 so $E = 0$.
- For $a < r < b$, the charge integral is

$$\int_a^r \frac{\rho_0}{a} r (4\pi r^2) \,\mathrm{d}r = \frac{\pi\rho_0}{a}(r^4 - a^4)$$
$$\implies E = \frac{\rho_0}{4a\varepsilon_0 r^2}(r^4 - a^4)$$

- For $r \geq b$, the charge is simply (from the previous section)

$$Q = \frac{\pi\rho_0}{a}(b^4 - a^4)$$
$$\implies E = \frac{\rho_0}{4a\varepsilon_0 r^2}(b^4 - a^4)$$

6.ii) $\phi(r)$

It's spherically symmetric, so we test a solution of the form $\nabla^2 r^p = p(p+1)r^{p-2}$.

- For $r \leq a$, this should equal 0 so we have $\phi = \frac{A}{r} + B, r \leq a$.
- For $a < r < b$, this should be proportional to r so we try a solution of the form

$$\kappa r^3 \implies \nabla^2(\kappa r^3) = 12\kappa r = \rho_0 \frac{r}{a}$$
$$\implies \kappa = \frac{\rho_0}{12a}$$

  so we have

$$\phi = \frac{\rho_0}{12a} r^3 + \frac{C}{r} + D$$

- For $r \geq b$ it should take the form $\frac{E}{r} + F$

Now lets impose some conditions. As $r \to \infty, \phi \to 0$ so $F = 0$. We expect $r = 0$ to be non-singular so $A = 0$. We also expect it to be continuous:

$$
\begin{aligned}
B &= \frac{\rho_0 a^2}{12} + \frac{C}{a} + D \\
\frac{E}{b} &= \frac{\rho_0 b^3}{12a} + \frac{C}{b} + D
\end{aligned}
$$

Finally, we expect ϕ' to be continuous at $r = a$ and $r = b$:

$$
\begin{aligned}
0 &= \frac{\rho_0 a}{4} - \frac{C}{a^2} \\
-\frac{E}{b^2} &= \frac{\rho_0 b^2}{4a} - \frac{C}{b^2}
\end{aligned}
$$

Solving gives

$$
\begin{pmatrix} B \\ C \\ D \\ E \end{pmatrix} = \frac{\rho_0}{12a} \begin{pmatrix} 4(a^3 - b^3) \\ 3a^4 \\ -4b^3 \\ 3(a^4 - b^4) \end{pmatrix}
$$

$$
\implies \phi = \begin{cases} \rho_0 \dfrac{a^3 - b^3}{3a} & \text{if } r < a \\ \dfrac{\rho_0}{12a} r^3 + \dfrac{\rho_0 a^3}{4r} - \dfrac{\rho_0 b^3}{3a} & \text{if } a < r < b \\ \dfrac{\rho_0 (a^4 - b^4)}{4ar} & \text{if } r \geq b \end{cases}
$$

You can check yourself that computing $\nabla \phi$ gives E.

7)

7.i) Identities

We have

$$
\nabla \psi = \frac{\partial \psi}{\partial x^i} \boldsymbol{e_i} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x^i} \boldsymbol{e_i} = \psi'(r) \frac{x_i}{r} \boldsymbol{e_i} = \psi'(r) \frac{\boldsymbol{x}}{r}
$$

For the Laplacian, we have

$$
\begin{aligned}
\nabla \cdot (\nabla \psi) &= \frac{\partial}{\partial x^i} \left[\psi'(r) \frac{x_i}{r} \right] \\
&= \frac{\partial \psi'}{\partial x^i} \frac{x_i}{r} + \psi'(r) \frac{\partial}{\partial x^i} \left(\frac{x_i}{r} \right) \\
&= \psi''(r) \frac{x_i^2}{r^2} + \frac{\psi'(r)}{r} - \frac{1}{r} \psi'(r) \frac{{x^i}^2}{r^2} \\
&= \psi''(r) + \frac{3}{r} \psi'(r) - \frac{1}{r} \psi'(r) \\
&= \psi''(r) + \frac{2}{r} \psi'(r)
\end{aligned}
$$

assuming we're in $\mathbb{R}^3$. The Laplacian in spherical coordinates (ignoring all θ and ϕ terms) is

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\psi'(r))$$
$$= \psi''(r) + \frac{2}{r}\psi'(r)$$

in agreement with the result using suffix notation.

7.ii) $\nabla^2\psi = 1$

We try a solution of the form r^p, and note that the result is independent of r so we'll guess κr^2.

$$\nabla^2(\kappa r^2) = 6\kappa = 1 \Longrightarrow \kappa = \frac{1}{6}$$

so we have the general form as

$$\psi(r) = \frac{1}{6}r^2 + \frac{A}{r} + B$$

Alternatively, you could solve this to get the same solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = 1$$

This must be non-singular so $A = 0$, and we must have $\psi(R) = 1 \Longrightarrow B = \frac{5}{6}R^2$. Therefore our full solution is

$$\psi(r) = \frac{1}{6}r^2 + \frac{5}{6}R^2$$

8)

8.i) $\nabla^2\phi = \nabla^2\psi = 0$

We have that

$$\frac{1}{2}(\partial_x + i\partial_y)(\phi + i\psi) = \frac{1}{2}(\partial_x\phi - \partial_y\psi + i(\partial_x\psi + \partial_y\phi))$$
$$= 0$$

For this to equal to zero, the real and imaginary parts must be zero, or

$$\partial_x\phi = \partial_y\psi \quad \Rightarrow \partial_x^2\phi = \partial_x\partial_y\psi$$
$$\partial_y\phi = -\partial_x\psi \Rightarrow \partial_y^2\phi = -\partial_y\partial_x\psi$$

The Laplacian of ϕ is simply $\partial_x^2\phi + \partial_y^2\phi = \partial_x\partial_y\psi - \partial_y\partial_x\psi$. For sufficiently smooth functions, the order of partial derivatives doesn't matter and $\nabla^2\phi = 0$. A similar process holds for ψ.

8.ii) $\phi = c$ perpendicular to $\psi = k$

To show the condition on the curves, we must show that $\nabla\phi \cdot \nabla\psi = 0$, since the gradient gives a normal vector to the curve. We have that

$$\partial_x\phi \cdot \partial_x\psi + \partial_y\phi \cdot \partial_y\psi = (\partial_y\psi)(-\partial_y\phi) + \partial_y\phi \cdot \partial_y\psi = 0$$

thus proving the claim.

8.iii) imaginary shenanigans

We have $f = (x+iy)e^{x+iy} = e^x(x+iy)e^{iy}$. By Euler's formula, we then have that

$$\begin{aligned} f &= e^x(x+iy)(\cos y + i\sin y) \\ &= e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y) \end{aligned}$$

from where it is trivial to pick out $\phi = e^x(x\cos y - y\sin y)$ and $\psi = e^x(y\cos y + x\sin y)$.

9)

9a)

We have $\nabla^2(f(x)e^{\alpha y}) = [f''(x) + \alpha^2 f(x)]e^{\alpha y}$. Setting this equal to zero gives us the equation of harmonic oscillations: $f'' + \alpha^2 f = 0$. This means $f(x) = A\sin(\alpha x + \phi)$, where A and ϕ are constants.

- $\psi(0,y) = 0$ means that we must have $\sin\phi = 0 \Rightarrow \phi = 0$
- $\psi(a,y) = 0$ means that the $\sin(\alpha a)$ term must be zero again, so $\alpha a = \pm\pi \Rightarrow \alpha = \pm\frac{\pi}{a}$. In our case, let's arbitrarily choose the negative solution. Then, our equation is

$$\psi(x,y) = -A\sin\left(\frac{\pi}{a}x\right)e^{-\pi y/a}$$

- Due to our choice of a negative value for α, the condition

$$\lim_{y\to\infty}\psi(x,y) = 0$$

  is satisfied.
- Since $\psi(x,0) = \lambda\sin\left(\frac{\pi}{a}x\right)$ we have that $A = -\lambda$

Therefore, the solution to Laplace's equation for the given boundary conditions is

$$\psi(x,y) = \lambda\sin\left(\frac{\pi}{a}x\right)e^{-\pi y/a}$$

9b)

The Laplacian of $\psi(r,\theta)$ in cylindrical coordinates is

$$\nabla^2\psi(r,\theta) = (\alpha^2 - \beta^2)\rho^{\alpha-2}\cos\beta\theta$$

This is a solution to the Laplacian equation iff $\alpha = \pm\beta$.

9b.i)

Let's assume this function must be non-singular at $r = 0$, so we have $\alpha = \beta$. We have

$$\psi(R,\theta) = AR^\beta\cos\beta\theta = \lambda\cos\theta$$

Therefore $\beta = 1$ and $A = \lambda R^{-1}$ and we have

$$\psi(r,\theta) = \lambda\frac{r}{R}\cos\theta$$

9b.ii)

In order for

$$\lim_{r\to\infty}\psi(r,\theta) = 0$$

we must have $\alpha < 0$, so we'll use $\alpha = -\beta$. Then the condition

$$AR^{-\beta}\cos\beta\theta = \lambda\cos\theta$$

tells us that $\beta = 1$ and $A = \lambda R$ so we have

$$\psi(r,\theta) = \lambda\frac{R}{r}\cos\theta$$

9b.iii)
Recall that the Laplacian solutions are linear, so we can write

$$\psi(x,y) = \left(Ar^{\beta} + Br^{-\beta}\right)\cos\beta\theta$$

We have from the Dirichlet condition that

$$\psi(b,\theta) = \left(Ar^{\beta} + Br^{-\beta}\right)\cos\beta\theta = \lambda\cos 2\theta$$

Therefore, $\beta = 2$ and $Ar^2 + Br^{-2} = \lambda$.

To apply the Neumann condition, note that the normal in polar coordinates is $\hat{r}$. Then we have

$$\begin{aligned}\hat{r}\cdot\nabla\psi(a,\theta) &= \left(2Aa - 2Ba^{-3}\right)\cos 2\theta = 0\\ \implies Aa^4 &= B\end{aligned}$$

Combining with $Ar^2 + Br^{-2} = \lambda$ we have

$$\begin{aligned}A &= \frac{\lambda r^2}{r^4 + a^4}\\ B &= \frac{\lambda r^2}{a^4(r^4 + a^4)}\end{aligned}$$

Substituting back in gives us our equation

$$\psi(r,\theta) = \frac{1}{r^4 + a^4}\left(\lambda r^4 + \frac{1}{a^4}\right)\cos 2\theta$$

10)
We have

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)\mathrm{d}V = \int_V \phi\nabla^2\psi\,\mathrm{d}V - \int_V \psi\nabla^2\phi\,\mathrm{d}V$$

By Greens first identity,

$$\begin{aligned}&\left(\int_{\partial V}\phi\nabla\psi\cdot\mathrm{d}\boldsymbol{S} - \int_V (\nabla\psi)\cdot(\nabla\phi)\,\mathrm{d}V\right) - \left(\int_{\partial V}\psi\nabla\phi\cdot\mathrm{d}\boldsymbol{S} - \int_V (\nabla\psi)\cdot(\nabla\phi)\,\mathrm{d}V\right)\\ &= \int_{\partial V}\left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\mathrm{d}\boldsymbol{S}\end{aligned}$$

11)
Let $\phi \coloneqq \psi_1 - \psi_2$, where ψ_1 and ψ_2 are two different solutions to the problem. We'll try to prove this theorem by contradiction. Note that

$$
\begin{aligned}
-\nabla^2\phi + \phi &= -\nabla^2\psi_1 + \psi_1 - (\nabla^2\psi_2 + \psi_2) \\
&= \rho(\boldsymbol{x}) - \rho(\boldsymbol{x}) \\
&= 0
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
\hat{n}\cdot\nabla\phi &= \hat{n}\cdot\nabla\psi_1 - \hat{n}\cdot\nabla\psi_2 \\
&= f(\boldsymbol{x}) - f(\boldsymbol{x}) \\
&= 0
\end{aligned}
$$

Now, let's multiply Equation 1 by ϕ, and integrate

$$
\begin{aligned}
\int_V (-\nabla^2\phi + \phi)\phi \,\mathrm{d}V = 0 \\
\implies -\int_V \phi\nabla^2\phi \,\mathrm{d}V + \int_V \phi^2 \,\mathrm{d}V = 0
\end{aligned}
$$

By Greens first identity, this equals

$$
\int_V (\nabla\phi)^2 \,\mathrm{d}V - \int_{\partial V} \phi\nabla\phi\cdot\mathrm{d}\boldsymbol{S} + \int_V \phi^2 \,\mathrm{d}V = 0
$$

But since $\mathrm{d}\boldsymbol{S} = \hat{n}\,\mathrm{d}S$ and $\hat{n}\cdot\nabla\phi = 0$, the middle term vanishes and we have

$$
\int_V (\nabla\phi)^2 \,\mathrm{d}V + \int_V \phi^2 \,\mathrm{d}V = 0
$$

Neither of the integrals can be < 0, so both must be zero everywhere, implying $\phi = 0 \Rightarrow \psi_1 = \psi_2$ in direct contradiction of our initial assumption. Hence any solution to this boundary value problem must be unique.

12)

12.i) Uniqueness

Once again, we go looking for a proof by contradiction. Let $\phi = \psi_1 - \psi_2$, where ψ_1 and ψ_2 are two different solutions to the boundary conditions. It follows that $\nabla^2\phi = \nabla^2\psi_1 - \nabla^2\psi_2 = 0$. Then, simply multiply both sides by ϕ and apply Green's first identity:

$$
\begin{aligned}
\int_V \phi\nabla^2\phi \,\mathrm{d}V = 0 \\
\implies \int_V (\nabla\phi)^2 \,\mathrm{d}V - \int_{\partial V} (\phi\nabla\phi)\cdot\hat{n}\,\mathrm{d}S = 0
\end{aligned}
$$

From the boundary condition we have that

$$
\begin{aligned}
(\hat{n}\cdot\nabla\phi)g(\boldsymbol{x}) + \phi &= (\hat{n}\cdot\nabla\psi_1 - \hat{n}\cdot\nabla\psi_2)g(\boldsymbol{x}) + \psi_1 - \psi_2 \\
&= (\hat{n}\cdot\nabla\psi_1)g(\boldsymbol{x}) + \psi_1 - ((\hat{n}\cdot\nabla\psi_2)g(\boldsymbol{x}) + \psi_2) \\
&= f(\boldsymbol{x}) - f(\boldsymbol{x}) \\
&= 0 \\
\implies \hat{n}\cdot\nabla\phi &= -\frac{\phi}{g(\boldsymbol{x})}
\end{aligned}
$$

Substituting this back into our integral, we get

$$\int_V (\nabla\phi)^2 \,\mathrm{d}V + \int_{\partial V} \frac{\phi^2}{g(\boldsymbol{x})} \,\mathrm{d}S = 0$$

We were given the condition $g(\boldsymbol{x}) \geq 0$ on ∂V, so no quantities in this expression can be negative, so they must be zero everywhere. In other words, $\phi = 0 \Rightarrow \psi_1 = \psi_2$, so if a solution exists, then it is unique.

12.ii)

Let's take a spherical solution of the form $\psi = \frac{A}{r} + B$. At $r = 1$, the boundary condition is

$$\begin{aligned}
(\hat{n} \cdot \nabla\psi)(-1) + \psi &= 0 \\
\implies \frac{\partial\psi}{\partial r} &= \psi \\
-\frac{A}{r^2} &= \frac{A}{r} + B \\
B &= -2A
\end{aligned}$$

where in the last line we used $r = 1$. Therefore, we have

$$\psi(\boldsymbol{x}) = A\left(\frac{1}{|\boldsymbol{x}|} - 2\right) \;\forall A \neq 0$$

TODO: this solution is singular at $r = 0$

13)

13.i)

By a rearranging of Greens first identity, we have that

$$\int_V (\nabla v) \cdot (\nabla u) \,\mathrm{d}V = \int_{\partial V} v(\nabla u) \cdot \mathrm{d}\boldsymbol{S} - \int_V v\nabla^2 u \,\mathrm{d}V$$

We have that u is harmonic so $\nabla^2 u = 0$ and the second term on the right is 0. Additionally, we have that $v = 0$ on ∂V so the left term on the right is also 0. Therefore, we have that

$$\int_V (\nabla v) \cdot (\nabla u) \,\mathrm{d}V = 0$$

as we set out to show.

13.ii)

We have that $w = v + u$, so

$$|\nabla w|^2 = |\nabla v + \nabla u|^2 = |\nabla v|^2 + 2(\nabla v) \cdot (\nabla u) + |\nabla u|^2$$

Integrate both sides over a volume V to get

$$\begin{aligned}
\int_V |\nabla w|^2 \,\mathrm{d}V &= \int_V |\nabla v|^2 \,\mathrm{d}V + 2\int_V (\nabla v) \cdot (\nabla u) \,\mathrm{d}V + \int_V |\nabla u|^2 \,\mathrm{d}V \\
&= \int_V |\nabla v|^2 \,\mathrm{d}V + \int_V |\nabla u|^2 \,\mathrm{d}V
\end{aligned}$$

where the last step was due to the expression we proved in part 1. Note that $|\nabla v|^2 \,\mathrm{d}V \geq 0$ always, so we can rewrite this as

$$\int_V |\nabla w|^2 \,\mathrm{d}V \geq \int_V |\nabla u|^2 \,\mathrm{d}V$$

14)

14.i)

We have from the mean value property that $\psi(\boldsymbol{a}) = \frac{1}{4\pi r^2} \int_{\partial V} \psi(\boldsymbol{x}) \,\mathrm{d}S$, where V is the volume of the ball. Our goal is to express $\psi(\boldsymbol{a})$ in terms of the volume V instead of the surface ∂V.

From the definition of a volume integral, we have

$$\int_V \psi(\boldsymbol{x}) \,\mathrm{d}V = \int_0^r \mathrm{d}r' \int_{\partial V} \psi(\boldsymbol{x}) \,\mathrm{d}S$$

From here, notice that $\int_{\partial V} \psi(\boldsymbol{x}) \,\mathrm{d}S = 4\pi r^2 \psi(\boldsymbol{a})$. Plugging it in gives us

$$\begin{aligned}
\int_V \psi(\boldsymbol{x}) \,\mathrm{d}V &= \int_0^r 4\pi r'^2 \psi(\boldsymbol{a}) \,\mathrm{d}r' \\
&= \frac{4}{3}\pi r^3 \psi(\boldsymbol{a}) \\
\implies \psi(\boldsymbol{a}) &= \frac{1}{4\pi r^3/3} \int_V \psi(\boldsymbol{x}) \,\mathrm{d}V
\end{aligned}$$

as we set out to show.

14.ii)

We have

$$\psi(\boldsymbol{a}) = \frac{1}{V} \int_V \psi(\boldsymbol{x}) \,\mathrm{d}V$$

Next, take the gradient and consider the limit as $r \to \infty$:

$$\lim_{r\to\infty} \nabla\psi(\boldsymbol{a}) = \lim_{r\to\infty} \nabla\left(\frac{1}{V} \int_V \psi(\boldsymbol{x}) \,\mathrm{d}V\right)$$

Because $\psi(\boldsymbol{x}) \leq \mathcal{M}$ for some $\mathcal{M}$, the average value approaches a constant as $r \to \infty$ and thus the gradient goes to 0. This holds for arbitrary points $\boldsymbol{a} \in \mathbb{R}^3$.

15)

Consider a small amount of water with mass $\rho \,\mathrm{d}V$. A time $\mathrm{d}t$ later, it will still have a mass $\rho \,\mathrm{d}V = \rho \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}$, assuming water is incompressible and the density stayed approximately constant. Integrating, we then have

$$\begin{aligned}
\int_V \rho \,\mathrm{d}V &= \int_S \rho \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S} \\
\implies \int_V \mathrm{d}V &= \int_S \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}
\end{aligned}$$

The left side is simply V. Then, taking $\frac{\mathrm{d}}{\mathrm{d}t}$ on both sides gives

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{x}\cdot \mathrm{d}\boldsymbol{S} \\ &= \int \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot \mathrm{d}\boldsymbol{S} \\ &= \int \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}\end{aligned}$$

The second result is a generalization of the previous result. The total mass is

$$M = \int_V \rho \,\mathrm{d}V = \int_V \rho\boldsymbol{x}\cdot \mathrm{d}\boldsymbol{S}$$

We can take $\frac{\mathrm{d}}{\mathrm{d}t}$ to see how the mass changes with time. By the chain rule, it follows that

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V \\ &= \int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V + \int_{\partial V} \rho\frac{\partial\boldsymbol{x}}{\partial t}\cdot \mathrm{d}\boldsymbol{S} \\ &= \int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V + \int_{\partial V} \rho\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}\end{aligned}$$

In words, this equation says that in a time δt, the change in the mass of the fluid has contributions from changes in density as well as how much fluid leaves from open surfaces. Note that setting $\rho = \text{constant}$ gives us our first equation back.