Lagrangian Mechanics
Lecture Problem Solutions
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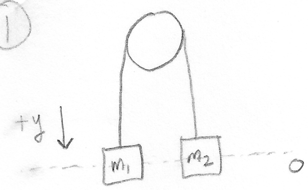
(1)



Mass m_1

Position: $y\,\hat{j}$

Velocity: $\dot{y}\,\hat{j}$

$v^2 = \dot{y}^2$

Mass m_2

Position: $-y\,\hat{j}$

Velocity: $-\dot{y}\,\hat{j}$

$v^2 = \dot{y}^2$

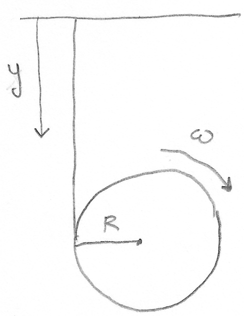
$$\mathcal{L} = T - U$$
$$= \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{y}^2 - (m_2 g y - m_1 g y)$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}}$$

$$g(m_1 - m_2) = \frac{d}{dt}(m_1 \dot{y} + m_2 \dot{y})$$

$$\boxed{\ddot{y} = \frac{g(m_1 - m_2)}{m_1 + m_2}}$$

(2)



Position: $y\,\hat{j}$

Velocity: $\dot{y}\,\hat{j}$

$v^2 = \dot{y}^2$

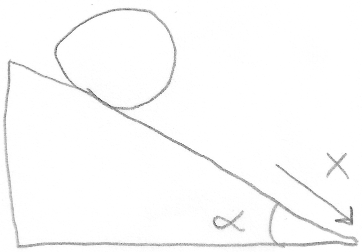
$$\mathcal{L} = T - U$$
$$= \left(\frac{1}{2} m \dot{y}^2 + \frac{1}{2} I \omega^2\right) + mgy$$
$$= \frac{1}{2} m \dot{y}^2 + \frac{1}{2}\left(\frac{1}{2} m R^2\right)\left(\frac{\dot{y}^2}{R^2}\right) + mgy$$
$$= \frac{3}{4} m \dot{y}^2 + mgy$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}}$$

$$mg = \frac{d}{dt}\left(\frac{3}{2} m \dot{y}\right)$$

$$\boxed{\ddot{y} = \frac{2}{3} g}$$

(3)



Position: $x\,\hat{i}$

Velocity: $\dot{x}\,\hat{i}$

$$\mathcal{L} = T - U$$
$$= \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \omega^2\right) + mgx\sin\alpha$$
$$= \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{x}^2 + mgx\sin\alpha$$

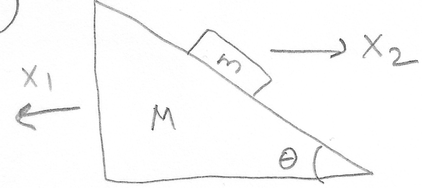
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$mg\sin\alpha = \frac{d}{dt}\left(\left(m + \frac{I}{R^2}\right)\dot{x}\right)$$

$$\boxed{\ddot{x} = \frac{mg\sin\alpha}{m + \frac{I}{R^2}}}$$

(4)

Relative distance between = $x_1 + x_2$



Mass M

Position: $x_1\,\hat{i}$

Velocity: $\dot{x}_1\,\hat{i}$

Mass m

Position: $x_2\,\hat{i} - (x_1 + x_2)\tan\theta\,\hat{j}$

Velocity: $\dot{x}_2\,\hat{i} - (\dot{x}_1 + \dot{x}_2)\tan\theta\,\hat{j}$

$v^2 = \dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta$

$$\mathcal{L} = T - U$$
$$= \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m \left[\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta\right] + mg(x_1 + x_2)\tan\theta$$

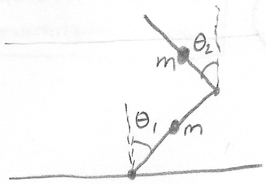
$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} \qquad \frac{\partial \mathcal{L}}{\partial x_2} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_2}$$

$$mg\tan\theta = \frac{d}{dt}\left(M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right)$$

$$mg\tan\theta = \frac{d}{dt}\left(m\dot{x}_2 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right)$$

$$\ddot{x}_1 = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

(5)



Bottom mass:

Position: $r\sin\theta_1 \hat{i} + r\cos\theta_1 \hat{j}$

Velocity: $r\cos\theta_1 \dot{\theta}_1 \hat{i} - r\sin\theta_1 \dot{\theta}_1 \hat{j}$ $\quad V^2 = r^2\dot{\theta}_1^2$

Top mass:

Position: $(2r\sin\theta_1 - r\sin\theta_2)\hat{i} + (2r\cos\theta_1 + r\cos\theta_2)\hat{j}$

Velocity: $(2r\cos\theta_1\dot{\theta}_1 - r\cos\theta_2\dot{\theta}_2)\hat{i} + (-2r\sin\theta_1\dot{\theta}_1 - r\sin\theta_2\dot{\theta}_2)\hat{j}$

$$V^2 = r^2((2\cos\theta_1\dot{\theta}_1 - \cos\theta_2\dot{\theta}_2)^2 + (-2\sin\theta_1\dot{\theta}_1 - \sin\theta_2\dot{\theta}_2)^2) = r^2(2\dot{\theta}_1 - \dot{\theta}_2)^2$$

$\cos\theta \approx 1 \quad \sin\theta \approx 0$!!

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}m(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) - mgr(3\cos\theta_1 + \cos\theta_2) \qquad \cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$= \frac{1}{2}m(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) - mgr(4 - \frac{3}{2}\theta_1^2 - \frac{1}{2}\theta_2^2)$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} \qquad \frac{\partial\mathcal{L}}{\partial\theta_2} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}$$

at instant $\theta_1 = 0 \quad \theta_2 = \varepsilon$

$$\frac{3g}{r}\theta_1 = 5\ddot{\theta}_1 - 2\ddot{\theta}_2 \qquad \frac{g}{r}\theta_2 = -2\ddot{\theta}_1 + \ddot{\theta}_2$$

$$\ddot{\theta}_1 = \frac{2g\epsilon}{r} \qquad \ddot{\theta}_2 = \frac{5g\epsilon}{r}$$