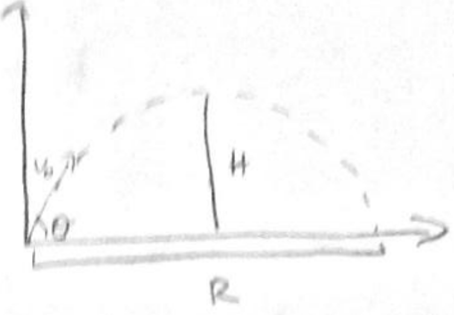
1. Let's find the range first. The projectile, since it starts at the ground, has a time-parametrization of its motion as follows:

(1) $x(t) = v_0 \cos\theta\, t$

(2) $y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$



When the projectile has travelled the range, it is easy to see that the above equations satisfy the following equation:

$$R = v_0 \cos\theta\, t$$

$$0 = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

Since t isn't 0 (since at $t=0$, the projectile hasn't moved), we can divide by t and solve for it in the second equation. Remember we aren't allowed to use t because it isn't given in the problem.

$$t = \frac{2v_0 \sin\theta}{g} \qquad R = v_0 \cos\theta\, t$$

Plugging in, we solve for R. Using simple trig identities, we have:

$$R = v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g}\right) = \frac{v_0^2\, 2\sin\theta\cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

Now let's find the maximum height. We know that by our second equation,

$$H = v_0 \sin\theta t - \frac{1}{2}gt^2$$

This time t is different from the one in the one we used to determine R, obviously. In order to determine it, we look to our velocity equation in the y-direction, which is just the derivative with respect to time of $y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2$:

$$(3) \quad \frac{dy}{dt} = v_y(t) = v_0 \sin\theta - gt$$

When the projectile reaches its apex, its velocity is 0 in the y-direction (not necessarily in the x though). One can verify this by tossing a pencil in the air. The velocity is 0 when the object starts to change direction, which is when it reaches its maximum height.

$$0 = v_0 \sin\theta - gt$$

$$t = \frac{v_0 \sin\theta}{g}$$

$$H = v_0 \sin\theta\, t - \frac{1}{2}gt^2$$

Solving for H:

$$H = \frac{v_0^2 \sin\theta}{2g}$$

This means:

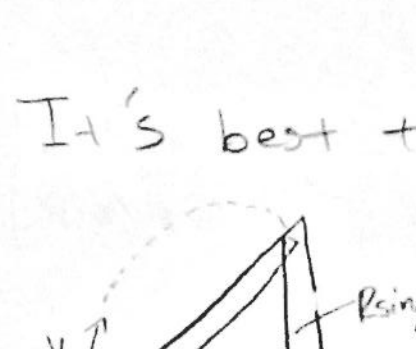
$$\frac{H}{R} = \frac{\left(\frac{v_0^2 \sin^2\theta}{2g}\right)}{\left(\frac{v_0^2 \sin 2\theta}{g}\right)} = \frac{\left(\frac{\sin^2\theta}{2}\right)}{2\sin\theta\cos\theta} = \boxed{\frac{\tan\theta}{4}}$$

Notice how it doesn't depend on v_0 or gravity!

When $H = R$, $\frac{H}{R} = 1$, and so:

$$1 = \frac{\tan\theta}{4} \longrightarrow \boxed{\theta = \arctan(4)}$$

2. It's best to draw a picture first.



Realize that this problem is as if the cannonball were launched with angle $(\theta + \phi)$ and we are trying to maximize R by choosing an appropriate value of θ. Then let's find R as a function of θ.

But upon closer inspection, we see that this range is travelled on the hill when the (x, y) position is $(R\cos\phi, R\sin\phi)$, with the standard assumption that our reference frame has our launch point to be $(0,0)$.

Using our kinematics equations:

$$R\cos\phi = v_0\cos(\theta+\phi)\,t$$

$$R\sin\phi = v_0\sin(\theta+\phi)\,t - \frac{gt^2}{2}$$

Solving for t in the first and plugging into the second:

$$\frac{R\cos\phi}{v_0\cos(\theta+\phi)} = t$$

$$R\sin\phi = v_0\sin(\theta+\phi)\left[\frac{R\cos\phi}{v_0\cos(\theta+\phi)}\right] - \frac{g}{2}\left[\frac{R\cos\phi}{v_0\cos(\theta+\phi)}\right]^2$$

$$\sin\phi = \frac{\sin(\theta+\phi)\cos\phi}{\cos(\theta+\phi)} - \frac{Rg\cos^2\phi}{2v_0^2\cos^2(\theta+\phi)}$$

Multiplying through by $2v_0^2\cos^2(\theta+\phi)$, as we are interested in finding R as a function of θ (ϕ and v_0 are constant) we are left with:

$$2v_0^2\cos^2(\theta+\phi)\sin\phi = 2v_0^2\sin(\theta+\phi)\cos(\theta+\phi)\cos\phi - Rg\cos^2\phi$$

And furthermore solving for R:

$$R = \frac{2v_0^2\sin(\theta+\phi)\cos(\theta+\phi)\cos\phi - 2v_0^2\cos^2(\theta+\phi)\sin\phi}{g\cos^2\phi}$$

$$= \frac{v_0^2}{g\cos^2\phi}\left[\sin(2\theta+2\phi)\cos\phi - 2\cos^2(\theta+\phi)\sin\phi\right]$$

$$= \frac{v_0^2}{g\cos^2\phi}\left[\sin(2\theta+2\phi)\cos\phi - (1+\cos(2\theta+2\phi))\sin\phi\right]$$

$$= \frac{v_0^2}{g\cos^2\phi}\left[\sin(2\theta+2\phi)\cos\phi - \cos(2\theta+2\phi)\sin\phi - \sin\phi\right]$$

$$R = \frac{v_0^2}{g\cos^2\phi}\left[\sin(2\theta+\phi) - \sin\phi\right]$$

(ask yourself: does this make sense when $\phi = 0$, i.e. no hill?)

R is clearly maximized when $\sin(2\theta+\phi)$ is maximized, which is when $1 = \sin(2\theta+\phi)$

That means $\frac{\pi}{2} = 2\theta + \phi$. This means R is maximized when: $\boxed{\theta = \frac{\pi}{4} - \frac{\phi}{2}}$

3. Centrifuges are used in chemistry to separate compounds using rotation. We know that since

our centrifuge is moving with constant speed (the centrifuge is rotating with a constant acceleration magnitude in a circle), the following equation is true:

centripetal acceleration

$$a = \frac{v^2}{R} = \omega^2 R \qquad (\text{as } |v| = R\omega)$$

Then we have:

$$kg = \frac{|v|^2}{R} \longrightarrow |v| = \boxed{\sqrt{Rkg}}$$

Although R isn't given in the problem, this is a mistake. We NEED the radius of motion to solve the problem.

The relationship between the angular frequency f and the angular velocity ω is:

$$\omega = 2\pi f \longrightarrow f = \frac{\omega}{2\pi}$$

To solve for f:

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{a}{R}}}{2\pi} = \boxed{\frac{1}{2\pi}\sqrt{\frac{kg}{R}}}$$

4. This problem is very similar to the previous one, and also really easy, as they are just formulas we should know.

$V = \omega R$, so $\boxed{\omega R}$ is the

$a = \omega^2 R$, so $\boxed{\omega^2 R}$ is the answer

5. Taking derivatives is how we solve this question.

$$x(t) = R\sin\omega t + R\omega t$$

$$v_x(t) = \frac{dx}{dt} = \boxed{R\omega\cos\omega t + R\omega}$$

$$a_x(t) = \frac{d^2x}{dt^2} = \boxed{-R\omega^2 \sin\omega t}$$

$$y(t) = R\cos\omega t + R$$

$$v_y(t) = \frac{dy}{dt} = \boxed{-R\omega\sin\omega t}$$

$$a_y(t) = \frac{d^2y}{dt^2} = \boxed{-R\omega^2\cos\omega t}$$

not really necessary for F=MA or AP Physics, but useful

To find the tangential and normal components of the acceleration vector, we first realise that they must add to the acceleration vector.

$$\vec{a}(t) = -R\omega^2 \sin(\omega t)\hat{i} + -R\omega^2\cos(\omega t)\hat{j} = \vec{a}_N + \vec{a}_T$$

We want to find a_N and a_T, $= a_N\hat{N} + a_T\hat{T}$

which are given by the following formulas:

unit normal vector

unit tangential vector

$$a_N = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|} \qquad a_T = \frac{\vec{v}(t)\cdot\vec{a}(t)}{|\vec{v}(t)|}$$

$$= \frac{\begin{vmatrix} R\omega\cos(\omega t)+R\omega & -R\omega\sin(\omega t) & 0 \\ -R\omega^2\sin(\omega t) & -R\omega^2\cos(\omega t) & 0 \\ i & j & k \end{vmatrix}}{\sqrt{(R\omega\cos(\omega t)+R\omega)^2 + (-R\omega\sin(\omega t))^2}}$$

$$= \frac{|-R^2\omega^3\cos^2\omega t - R^2\omega^3\cos(\omega t) - R^2\omega^3\sin^2(\omega t)|}{\sqrt{2R^2\omega^2 + 2R^2\omega^2\cos\omega t}}$$

$$= \frac{R^2\omega^3(1+\cos\omega t)}{\sqrt{2}\cdot R\omega\sqrt{1+\cos\omega t}}$$

$$= \boxed{\frac{\sqrt{2}}{2}R\omega^2\sqrt{1+\cos\omega t}}$$

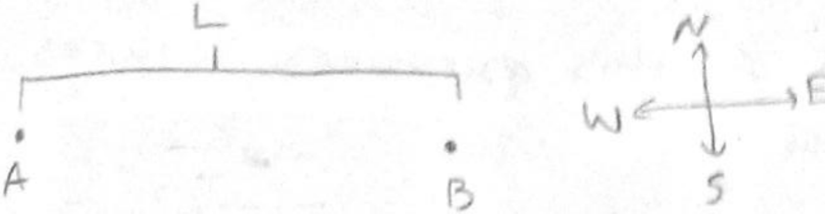
$$= \frac{-(R\omega\cos(\omega t)+R\omega)R\omega^2\sin(\omega t) - R^2\omega^3\sin(\omega t)\cos(\omega t)}{\sqrt{(R\omega\cos(\omega t)+R\omega)^2 + (-R\omega\sin(\omega t))^2}}$$

$$= \frac{R^2\omega^3\sin(\omega t) - 2R^2\omega^3\sin(\omega t)\cos(\omega t)}{\sqrt{2}\cdot R\omega\cdot\sqrt{1+\cos\omega t}}$$

$$= \frac{R\omega^2(\sin(\omega t) - \sin(2\omega t))}{\sqrt{2}\sqrt{1+\cos\omega t}}$$

$$= \boxed{\frac{\sqrt{2}}{2}R\omega^2\left(\frac{\sin\omega t - \sin 2\omega t}{\sqrt{1+\cos\omega t}}\right)}$$

6.



This problem is relative motion. Going east in the first part of the trip, since both velocities are constant, we have:

$$t = \left| \frac{L}{|\vec{v}| + |\vec{u}|} \right| \qquad \xrightarrow{\vec{v}} \quad \xrightarrow{\vec{u}}$$

And going west:

$$t = \left| \frac{-L}{-\vec{v} + u} \right| = \left| \frac{L}{|\vec{v}| - |\vec{u}|} \right| \qquad \xleftarrow{\vec{v}'} \quad \xrightarrow{\vec{u}}$$

$|\vec{v}'| = |\vec{v}|$; they are in opposite directions

So our total time is:

$$t = \frac{L}{|\vec{v}| + |\vec{u}|} + \frac{L}{|\vec{v}| - |\vec{u}|}$$

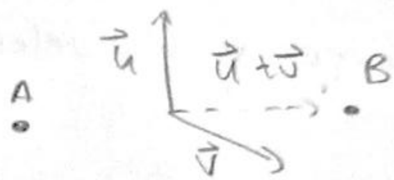
$$= \frac{L(|\vec{v}| - |\vec{u}|) + L(|\vec{v}| + |\vec{u}|)}{(|\vec{v}| + |\vec{u}|)(|\vec{v}| - |\vec{u}|)}$$

$$= \frac{2L|\vec{v}|}{|\vec{v}|^2 - |\vec{u}|^2}$$

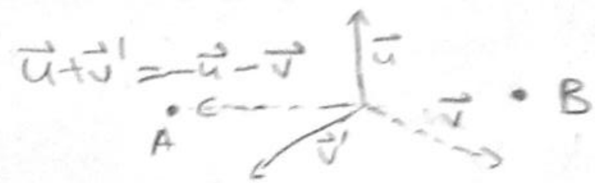
$$= \boxed{\frac{2Lv}{v^2 - u^2}}$$

The second part of this question requires more geometric intuition.

Going east:



Going west:



$\vec{v}'$ is the same velocity as the east velocity v, except a negated x-component

We know that the total time is:

$$t = \left|\frac{L}{|\vec{u}+\vec{v}|}\right| + \left|\frac{L}{|-\vec{u}-\vec{v}|}\right|$$

$$= \left|\frac{2L}{|\vec{u}+\vec{v}|}\right| = \frac{2L}{|\vec{u}+\vec{v}|}$$

$$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} + 2\vec{u}\cdot\vec{v}$$

$$= |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot\vec{v}$$

$$|\vec{u}+\vec{v}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot\vec{v}}$$

$$t = \frac{2L}{\sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot\vec{v}}}$$

Geometrically, $\vec{u}\cdot\vec{v}$ is the dot product of $\vec{u}$ and whatever is parallel to $\vec{u}$ in $\vec{v}$ (which is just $-\vec{u}$!)

$$= \frac{2L}{\sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot(-\vec{u})}}$$

$$= \frac{2L}{\sqrt{|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|^2}} = \frac{2L}{\sqrt{|\vec{v}|^2 - |\vec{u}|^2}} = \boxed{\frac{2L}{\sqrt{v^2 - u^2}}}$$

The solutions to this problemset were written by me, Arun Kannan, If you notice any problems with the solutions or have any questions, please contact me at 2015akannan@tjhsst.edu.