CHAPTER 1.1

Homotopy theories and model categories

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Contents

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1. Introduction

This paper is an introduction to the theory of "model categories", which was developed by Quillen in [22] and [23]. By definition a model category is just an ordinary category with three specified classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*, which satisfy a few simple axioms that are deliberately reminiscent of properties of topological spaces. Surprisingly enough, these axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory. The machinery can then be used immediately in a large number of different settings, as long as the axioms are checked in each case. Although many of these settings are geometric (spaces (§8), fibrewise spaces (3.11), G-spaces [11], spectra [5], diagrams of spaces [10] ...), some of them are not (chain complexes (§7), simplicial commutative rings [24], simplicial groups [23] ...). Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language. What is the suspension of an augmented commutative algebra? One of incidental appeals of Quillen's theory (to a topologist!) is that it both makes a question like this respectable and gives it an interesting answer (11.3).

We have tried to minimize the prerequisites needed for understanding this paper; it should be enough to have some familiarity with CW-complexes, with chain complexes, and with the basic terminology associated with categories. Almost all of the material we present is due to Quillen [22], but we have replaced his treatment of suspension functors and loop functors by a general construction of homotopy pushouts and homotopy pullbacks in a model category. What we do along these lines can certainly be carried further. This paper is not in any sense a survey of everything that is known about model categories; in fact we cover only a fraction of the material in [22]. The last section has a discussion of some ways in which model categories have been used in topology and algebra.

Organization of the paper. Section 2 contains background material, principally a discussion of some categorical constructions (limits and colimits) which come up almost immediately in any attempt to build new objects of some abstract category out of old ones. Section 3 gives the definition of what it means for a category $\mathbf{C}$ to be a model category, establishes some terminology, and sketches a few examples. In §4 we study the notion of "homotopy" in $\mathbf{C}$ and in §5 carry out the construction of the homotopy category $\mathrm{Ho}(\mathbf{C})$. Section §6 gives $\mathrm{Ho}(\mathbf{C})$ a more conceptual significance by showing that it is equivalent to the "localization" of $\mathbf{C}$ with respect to the class of weak equivalences. For our purposes the "homotopy theory" associated to $\mathbf{C}$ is the homotopy category $\mathrm{Ho}(\mathbf{C})$ together with various related constructions (§10).

Sections 7 and 8 describe in detail two basic examples of model categories, namely the category **Top** of topological spaces and the category $\mathbf{Ch}_R$ of nonnegative chain complexes of modules over a ring R. The homotopy theory of **Top** is of course familiar, and it turns out that the homotopy theory of $\mathbf{Ch}_R$ is what is usually called homological algebra. Comparing these two examples helps explain why Quillen called the study of model categories "homotopical algebra" and thought of it as a generalization of homological algebra. In §9 we give a criterion for a pair of functors between two model categories to induce equivalences between the associated homotopy categories; pinning down the meaning of "induce" here leads to the definition of derived functor. Section 10 constructs homotopy pushouts and homotopy pullbacks in an arbitrary model category in terms of derived functors. Finally, §11 contains some concluding remarks, sketches some applications of homotopical algebra, and mentions a way in which the theory has developed since Quillen.

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2. Categories

In this section we review some basic ideas and constructions from category theory; for more details see [17]. The reader might want to skip this section on first reading and return to it as needed.

2.1. Categories. We will take for granted the notions of *category*, *subcategory*, *functor* and *natural transformation* [17, I]. If $\mathbf{C}$ is a category and X and Y are objects of $\mathbf{C}$, we will assume that the morphisms $f : X \to Y$ in $\mathbf{C}$ form a *set* $\mathrm{Hom}_{\mathrm{C}}(X, Y)$, rather than a class, a collection, or something larger. These morphisms are also called *maps* or *arrows* in $\mathbf{C}$ from X to Y. Some categories that come up in this paper are:

(i) the category **Set** whose objects are sets and whose morphisms are functions from one set to another,

(ii) the category **Top** whose objects are topological spaces and whose morphisms are continuous maps,

(iii) the category $\mathbf{Mod}_R$ whose objects are left R-modules (where R is an associative ring with unit) and whose morphisms are R-module homomorphisms.

2.2. Natural equivalences. Suppose that $F, F' : \mathbf{C} \to \mathbf{D}$ are two functors, and that t is a natural transformation from F to F'. The transformation t is called a *natural equivalence* [17, p. 16] if the morphism $t_X : F(X) \to F'(X)$ is an isomorphism in $\mathbf{D}$ for each object X of $\mathbf{C}$. The functor F is said to be an *equivalence of categories* if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that the composites FG and GF are related to the appropriate identity functors by natural equivalences [17, p. 90].

2.3. Full and faithful. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *full* (resp. *faithful*) if for each pair (X, Y) of objects of $\mathbf{C}$ the map

$$\mathrm{Hom}_{\mathrm{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{D}}(F(X), F(Y))$$

given by F is an epimorphism (resp. a monomorphism) [17, p. 15]. A *full subcategory* $\mathbf{C}'$ of $\mathbf{C}$ is a subcategory with the property that the inclusion functor $i : \mathbf{C}' \to \mathbf{C}$ is full (the functor i is always faithful). A full subcategory of $\mathbf{C}$ is determined by the objects in $\mathbf{C}$ which it contains, and we will sometimes speak of the full subcategory of $\mathbf{C}$ *generated* by a certain collection of objects.

2.4. Opposite category. If $\mathbf{C}$ is a category then the *opposite category* $\mathbf{C}^{\mathrm{op}}$ is the category with the same objects as $\mathbf{C}$ but with one morphism $f^{\mathrm{op}} : Y \to X$ for each morphism $f : X \to Y$ in $\mathbf{C}$ [17, p. 33]. The morphisms of $\mathbf{C}^{\mathrm{op}}$ compose according to the formula $f^{\mathrm{op}}g^{\mathrm{op}} = (gf)^{\mathrm{op}}$. A functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is the same thing as what is sometimes called a *contravariant functor* $\mathbf{C} \to \mathbf{D}$. For example, for any category $\mathbf{C}$ the assignment $(X, Y) \mapsto \mathrm{Hom}_{\mathbf{C}}(X, Y)$ gives a functor

$$\mathrm{Hom}_{\mathbf{C}}(-,-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set} \quad .$$

2.5. Smallness and functor categories. A category $\mathbf{D}$ is said to be *small* if the collection $\mathrm{Ob}(\mathbf{D})$ of objects of $\mathbf{D}$ forms a set, and *finite* if $\mathrm{Ob}(\mathbf{D})$ is a finite set and $\mathbf{D}$ has only a finite number of morphisms between any two objects. If $\mathbf{C}$ is a category and $\mathbf{D}$ is a small category, then there is a *functor category* $\mathbf{C}^{\mathrm{D}}$ in which the objects are functors $F : \mathbf{D} \to \mathbf{C}$ and the morphisms are natural transformations; this is also called the category of *diagrams in* $\mathbf{C}$ *with the shape of* $\mathbf{D}$. For example, if $\mathbf{D}$ is the category $\{a \to b\}$ with two objects and one nonidentity morphism, then the objects of $\mathbf{C}^{\mathrm{D}}$ are exactly the morphisms $f : X(a) \to X(b)$ of $\mathbf{C}$ and a map $t : f \to g$ in $\mathbf{C}^{\mathrm{D}}$ is a commutative diagram

$$\begin{array}{ccc} X(a) & \xrightarrow{t_a} & Y(a) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ X(b) & \xrightarrow{t_b} & Y(b) \end{array} \quad .$$

In this case $\mathbf{C}^{\mathrm{D}}$ is called the *category of morphisms* of $\mathbf{C}$ and is denoted $\mathbf{Mor}(\mathbf{C})$.

2.6. Retracts. An object X of a category $\mathbf{C}$ is said to be a *retract* of an object Y if there exist morphisms $i : X \to Y$ and $r : Y \to X$ such that $ri = \mathrm{id}_X$. For example, in the category $\mathbf{Mod}_R$ an object X is a retract of Y if and only if there exists a module Z such that Y is isomorphic to $X \oplus Z$. If f and g are morphisms of $\mathbf{C}$, we will say that f is a retract of g if the object of $\mathbf{Mor}(\mathbf{C})$ represented by f is a retract of the object of $\mathbf{Mor}(\mathbf{C})$ represented by g (see the proof of the next lemma for a picture of what this means).

Lemma 2.7. *If g is an isomorphism in $\mathbf{C}$ and f is a retract of g, then f is also an isomorphism.*

Proof. Since f is a retract of g, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

in which the composites ri and $r'i'$ are the appropriate identity maps. Since g is an isomorphism, there is a map $h : Y' \to Y$ such that $hg = \mathrm{id}_Y$ and $gh = \mathrm{id}_{Y'}$. It is easy to check that $k = rhi'$ is the inverse of f. $\square$

2.8. Adjoint functors. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be a pair of functors. An *adjunction* from F to G is a collection of isomorphisms

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathrm{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathrm{C}}(X, G(Y)), \quad X \in \mathrm{Ob}(\mathbf{C}),\ \ Y \in \mathrm{Ob}(\mathbf{D})$$

natural in X and Y, i.e., a collection which gives a natural equivalence (2.2) between the two indicated Hom-functors $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ (see 2.4). If such an adjunction exists we write

$$F : \mathbf{C} \Longleftrightarrow \mathbf{D} : G$$

and say that F and G are *adjoint functors* or that (F, G) is an *adjoint pair*, F being the *left adjoint* of G and G the *right adjoint* of F. Any two left adjoints of G (resp. right adjoints of F) are canonically naturally equivalent, so we speak of "the" left adjoint or right adjoint of a functor (if such a left or right adjoint exists) [17, p. 81]. If $f : F(X) \to Y$ (resp. $g : X \to G(Y)$), we denote its image under the bijection $\alpha_{X,Y}$ by $f^\sharp : X \to G(Y)$ (resp. $g^\flat : F(X) \to Y$).

2.9. Example. Let $G : \mathbf{Mod}_R \to \mathbf{Set}$ be the forgetful functor which assigns to each R-module its underlying set. Then G has a left adjoint $F : \mathbf{Set} \to \mathbf{Mod}_R$ which assigns to each set X the free R-module generated by the elements of X. The functor G does not have a right adjoint.

2.10. Example. Let $G : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor which assigns to each topological space X its underlying set. Then G has a left adjoint, which is the functor which assigns to each set Y the topological space given by Y with the discrete topology. The functor G also has a right adjoint, which assigns to each set Y the topological space given by Y with the indiscrete topology (cf. [17, p. 85]).

2.11. Colimits

We introduce the notion of the colimit of a functor. Let $\mathbf{C}$ be a category and $\mathbf{D}$ a small category. Typically, $\mathbf{C}$ is one of the categories in 2.1 and $\mathbf{D}$ is from the following list.

2.12. Shapes of colimit diagrams.

(i) A category with a set $\mathcal{I}$ of objects and no nonidentity morphisms.

(ii) The category $\mathbf{D} = \{a \leftarrow b \rightarrow c\}$, with three objects and the two indicated nonidentity morphisms.

(iii) The category $\mathbf{Z}^+ = \{0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots\}$ with objects the nonnegative integers and a single morphism $i \rightarrow j$ for $i \leqslant j$.

There is a diagonal or "constant diagram" functor

$$\Delta : \mathbf{C} \rightarrow \mathbf{C}^{\mathrm{D}},$$

which carries an object $X \in \mathbf{C}$ to the constant functor $\Delta(X) : \mathbf{D} \rightarrow \mathbf{C}$ (by definition, this "constant functor" sends each object of $\mathbf{D}$ to X and each morphism of $\mathbf{D}$ to id_X). The functor Δ assigns to each morphism $f : X \rightarrow X'$ of $\mathbf{C}$ the constant natural transformation $t(f) : \Delta_X \rightarrow \Delta_{X'}$ determined by the formula $t(f)_d = f$ for each object d of $\mathbf{D}$.

2.13. Definition. Let $\mathbf{D}$ be a small category and $F : \mathbf{D} \rightarrow \mathbf{C}$ a functor. A *colimit* for F is an object C of $\mathbf{C}$ together with a natural transformation $t : F \rightarrow \Delta(C)$ such that for every object X of $\mathbf{C}$ and every natural transformation $s : F \rightarrow \Delta(X)$, there exists a unique map $s' : C \rightarrow X$ in $\mathbf{C}$ such that $\Delta(s')t = s$ [17, p. 67].

Remark. The universal property of a colimit implies as usual that any two colimits for F are canonically isomorphic. If a colimit of F exists we will speak of "the" colimit of F and denote it $\mathrm{colim}(F)$. The colimit is sometimes called the direct limit, and denoted $\varinjlim F$, $\varinjlim^{\mathrm{D}} F$ or $\mathrm{colim}^{\mathrm{D}} F$. Roughly speaking, $\Delta(\mathrm{colim}(F))$ is the constant diagram which is most efficient at receiving a map from F, in the sense that any map from F to a constant diagram extends uniquely over the universal map $F \rightarrow \Delta(\mathrm{colim}(F))$.

2.14. Remark. A category $\mathbf{C}$ is said to *have all small* (resp. *finite*) *colimits* if $\mathrm{colim}(F)$ exists for any functor F from a small (resp. finite) category $\mathbf{D}$ to $\mathbf{C}$. The categories $\mathbf{Set}$, $\mathbf{Top}$ and $\mathbf{Mod}_R$ have all small colimits. Suppose that $\mathbf{D}$ is a small category and $F : \mathbf{D} \rightarrow \mathbf{Set}$ is a functor. Let U be the disjoint union of the sets which appear as values of F, i.e., let U be the set of pairs (d, x) where $d \in \mathrm{Ob}(\mathbf{D})$ and $x \in F(d)$. Then $\mathrm{colim}(F)$ is the quotient of U with respect to the equivalence relation "$\sim$" generated by the formulas $(d, x) \sim (d', F(f)(x))$, where $f : d \rightarrow d'$ is a morphism of $\mathbf{D}$. If $F : \mathbf{D} \rightarrow \mathbf{Top}$ is a functor, then $\mathrm{colim}(F)$ is an analogous quotient space of the space U which is the disjoint union of the spaces appearing as values of F. If $F : \mathbf{D} \to \mathbf{Mod}_R$ is a functor, then $\operatorname{colim}(F)$ is an analogous quotient module of the module U which is the direct sum of the modules appearing as values of F.

Remark. If $\operatorname{colim}(F)$ exists for every object F of $\mathbf{C}^{\mathrm{D}}$, an argument from the universal property (2.13) shows that the various objects $\operatorname{colim}(F)$ of $\mathbf{C}$ fit together into a functor $\operatorname{colim}(-)$ which is left adjoint to Δ:

$$\operatorname{colim} : \mathbf{C}^{\mathrm{D}} \Longleftrightarrow \mathbf{C} : \Delta \ .$$

We will now give some examples of colimits [17, p. 64].

2.15. Coproducts. Let $\mathbf{D}$ be the category of 2.12(i), so that a functor $X : \mathbf{D} \to \mathbf{C}$ is just a collection $\{X_i\}_{i \in \mathcal{I}}$ of objects of $\mathbf{C}$. The colimit of X is called the *coproduct* of the collection and written $\coprod_i X_i$ or, if $\mathcal{I} = \{0, 1\}$, $X_0 \coprod X_1$. If $\mathbf{C}$ is **Set** or **Top** the coproduct is disjoint union; if $\mathbf{C}$ is $\mathbf{Mod}_R$, coproduct is direct sum. If $\mathcal{I} = \{0, 1\}$, then the definition of colimit (2.13) gives natural maps $\mathrm{in}_0 : X_0 \to X_0 \coprod X_1$ and $\mathrm{in}_1 : X_1 \to X_0 \coprod X_1$; given maps $f_i : X_i \to Y$ $(i = 0,\, 1)$ there is a unique map $f : X_0 \coprod X_1 \to Y$ such that $f \cdot \mathrm{in}_i = f_i$ $(i = 0,\, 1)$. The map f is ordinarily denoted $f_0 + f_1$.

2.16. Pushouts. If $\mathbf{D}$ is the category of 2.12(ii), then a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram $X(a) \leftarrow X(b) \to X(c)$ in $\mathbf{C}$. In this case the colimit of X is called the *pushout* P of the diagram $X(a) \leftarrow X(b) \to X(c)$. It is the result of appropriately gluing $X(a)$ to $X(c)$ along $X(b)$. The definition of colimit gives a natural commutative diagram

$$\begin{array}{ccc} X(b) & \xrightarrow{i} & X(c) \\ {\scriptstyle j}\downarrow & & {\scriptstyle j'}\downarrow \\ X(a) & \xrightarrow{i'} & P \end{array} .$$

Any diagram isomorphic to a diagram of this form is called a *pushout diagram*; the map i' is called the *cobase change* of i (along j) and the map j' is called the *cobase change* of j (along i). Given maps $f_a : X(a) \to Y$ and $f_c : X(c) \to Y$ such that $f_a j = f_c i$, there is a unique map $f : P \to Y$ such that $fj' = f_c$ and $fi' = f_a$.

2.17. Sequential colimits. If $\mathbf{D}$ is the category of 2.12(iii), a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram of the following form

$$X(0) \to X(1) \to \cdots \to X(n) \to \cdots$$

in $\mathbf{C}$; this is called a *sequential direct system* in $\mathbf{C}$. The colimit of this direct system is called the sequential colimit of the objects $X(n)$, and denoted $\mathrm{colim}_n X(n)$. If $\mathbf{C}$ is one of the categories $\mathbf{Set}$, $\mathbf{Top}$ or $\mathbf{Mod}_R$ and each one of the maps $X(n) \to X(n{+}1)$ is an inclusion, then $\mathrm{colim}_n X(n)$ can be interpreted as an increasing union of the $X(n)$; if $\mathbf{C} = \mathbf{Top}$ a subset of this union is open if and only if its intersection with each $X(n)$ is open.

2.18. Limits

We next introduce the notion of the limit of a functor [17, p. 68]. This is strictly dual to the notion of colimit, in the sense that a limit of a functor $F : \mathbf{D} \to \mathbf{C}$ is the same as a colimit of the "opposite functor" $F^{\mathrm{op}} : \mathbf{D}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$. From a logical point of view this may be everything there is to say about limits, but it is worthwhile to make the construction more explicit and work out some examples.

Let $\mathbf{C}$ be a category and $\mathbf{D}$ a small category. Typically, $\mathbf{C}$ is as before (2.1) and $\mathbf{D}$ is one of the following.

2.19. Shapes of limit diagrams.

(i) A category with a set $\mathcal{I}$ of objects and no nonidentity morphisms.

(ii) The category $\mathbf{D} = \{a \to b \leftarrow c\}$, with three objects and the two indicated nonidentity morphisms.

Let $\Delta : \mathbf{C} \to \mathbf{C}^{\mathrm{D}}$ be as before (2.11) the "constant diagram" functor.

2.20. Definition. Let $\mathbf{D}$ be a small category and $F : \mathbf{D} \to \mathbf{C}$ a functor. A *limit* for F is an object L of $\mathbf{C}$ together with a natural transformation $t : \Delta(L) \to F$ such that for every object X of $\mathbf{C}$ and every natural transformation $s : \Delta(X) \to F$, there exists a unique map $s' : X \to L$ in $\mathbf{C}$ such that $t\Delta(s') = s$.

Remark. The universal property of a limit implies as usual that any two limits for F are canonically isomorphic. If a limit of F exists we will speak of "the" limit of F and denote it $\lim(F)$. The limit is sometimes called the inverse limit, and denoted $\varprojlim F$, $\varprojlim_{\mathrm{D}} F$ or $\lim_{\mathrm{D}} F$. Roughly speaking, $\Delta(\lim(F))$ is the constant diagram which is most efficient at originating a map to F, in the sense that any map from a constant diagram to F lifts uniquely over the universal map $\Delta(\mathrm{colim}(F)) \to F$.

2.21. Remark. A category $\mathbf{C}$ is said to *have all small* (resp. *finite*) *limits* if $\lim(F)$ exists for any functor F from a small (resp. finite) category $\mathbf{D}$ to $\mathbf{C}$. The categories $\mathbf{Set}$, $\mathbf{Top}$ and $\mathbf{Mod}_R$ have all small limits. Suppose that $\mathbf{D}$ is a small category and $F : \mathbf{D} \to \mathbf{Set}$ is a functor. Let P be the product of the sets which appear as values of F, i.e., let U be the set of pairs (d, x) where $d \in \mathrm{Ob}(\mathbf{D})$ and $x \in F(d)$, $q : U \to \mathrm{Ob}(\mathbf{D})$ the map with $q(d, x) = d$, and P the set of all functions $s : \mathrm{Ob}(\mathbf{D}) \to U$

such that qs is the identity map of $\mathrm{Ob}(\mathbf{D})$. For $s \in P$ write $s(d) = (d, s_1(d))$, with $s_1(d) \in F(d)$. Then $\lim(F)$ is the subset of P consisting of functions s which satisfy the equation $s_1(d') = F(f)(s_1(d))$ for each morphism $f : d \to d'$ of $\mathbf{D}$. If $F : \mathbf{D} \to \mathbf{Top}$ is a functor, then $\lim(F)$ is the corresponding subspace of the space P which is the product of the spaces appearing as values of F. If $F : \mathbf{D} \to \mathbf{Mod}_R$ is a functor, then $\lim(F)$ is the corresponding submodule of the module U which is the direct product of the modules appearing as values of F.

Remark. If $\lim(F)$ exists for every object F of $\mathbf{C}^{\mathrm{D}}$, an argument from the universal property (2.20) shows that various objects $\lim(F)$ of $\mathbf{C}$ fit together into a functor $\lim(-)$ which is right adjoint to Δ:

$$\Delta : \mathbf{C} \Longleftrightarrow \mathbf{C}^{\mathrm{D}} : \lim \ .$$

We will now give two examples of limits [17, p. 70].

2.22. Products. Let $\mathbf{D}$ be the category of 2.19(i), so that a functor $X : \mathbf{D} \to \mathbf{C}$ is just a collection $\{X_i\}_{i \in \mathcal{I}}$ of objects of $\mathbf{C}$. The limit of X is called the *product* of the collection and written $\prod_i X_i$ or, if $\mathcal{I} = \{0, 1\}$, $X_0 \times X_1$ (the notation "$X_0 \prod X_1$" is more logical but seems less common). If $\mathbf{C}$ is $\mathbf{Set}$ or $\mathbf{Top}$ the product is what is usually called direct product or cartesian product. If $\mathcal{I} = \{0, 1\}$ then the definition of limit (2.20) gives natural maps $\mathrm{pr}_0 : X_0 \times X_1 \to X_0$ and $\mathrm{pr}_1 : X_0 \prod X_1 \to X_1$; given maps $f_i : Y \to X_i$ ($i = 0,\ 1$) there is a unique map $f : Y \to X_0 \times X_1$ such that $\mathrm{pr}_i \cdot f = f_i$ ($i = 0,\ 1$). The map f is ordinarily denoted (f_0, f_1).

2.23. Pullbacks. If $\mathbf{D}$ is the category of 2.19(ii), then a functor $X : \mathbf{D} \to \mathbf{C}$ is a diagram $X(a) \to X(b) \leftarrow X(c)$ in $\mathbf{C}$. In this case the limit of X is called the *pullback* P of the diagram $X(a) \to X(b) \leftarrow X(c)$. The definition of limit gives a natural commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{i'} & X(c) \\ {\scriptstyle j'}\downarrow & & {\scriptstyle j}\downarrow \\ X(a) & \xrightarrow{i} & X(b) \end{array} \quad .$$

Any diagram isomorphic to a diagram of this form is called a *pullback diagram*; the map i' is called the *base change* of i (along j) and the map j' is called the *base change* of j (along i). Given maps $f_a : Y \to X(a)$ and $f_c : Y \to X(c)$ such that $if_a = jf_c$, there is a unique map $f : Y \to P$ such that $i'f = f_c$ and $j'f = f_a$.

2.24. Some remarks on limits and colimits

An object $\emptyset$ of a category $\mathbf{C}$ is said to be an *initial object* if there is exactly one map from $\emptyset$ to any object X of $\mathbf{C}$. Dually, an object $*$ of $\mathbf{C}$ is said to be a *terminal* *object* if there is exactly one map $X \to *$ for any object X of $\mathbf{C}$. Clearly initial and terminal objects of $\mathbf{C}$ are unique up to canonical isomorphism. The proof of the following statement just involves unravelling the definitions.

Proposition 2.25. *Let $\mathbf{C}$ be a category, $\mathbf{D}$ the empty category (i.e. the category with no objects), and $F : \mathbf{D} \to \mathbf{C}$ the unique functor. Then $\operatorname{colim}(F)$, if it exists, is an initial object of $\mathbf{C}$ and $\lim(F)$, if it exists, is a terminal object of $\mathbf{C}$.*

Suppose that $\mathbf{D}$ is a small category, that $X : \mathbf{D} \to \mathbf{C}$ is a functor, and that $F : \mathbf{C} \to \mathbf{C}'$ is a functor. If $\operatorname{colim}(X)$ and $\operatorname{colim}(FX)$ both exist, then it is easy to see that there is a natural map $\operatorname{colim}(FX) \to F(\operatorname{colim} X)$. Similarly, if $\lim(F)$ and $\lim(FX)$ both exist, then it is easy to see that there is a natural map $F(\lim X) \to \lim(FX)$. The functor F is said to *preserve colimits* if whenever $X : \mathbf{D} \to \mathbf{C}$ is a functor such that $\operatorname{colim}(X)$ exists, then $\operatorname{colim}(FX)$ exists and the natural map $\operatorname{colim}(FX) \to F(\operatorname{colim} X)$ is an isomorphism. The functor F is said to *preserve limits* if the corresponding dual condition holds [17, p 112]. The following proposition is a formal consequence of the definition of an adjoint functor pair.

Proposition 2.26. *[17, p. 114–115] Suppose that*

$$F : \mathbf{C} \Longleftrightarrow \mathbf{C}' : G$$

is an adjoint functor pair. Then F preserves colimits and G preserves limits.

Remark. Proposition 2.26 explains why the underlying set of a product (2.22) or pullback (2.23) in the category $\mathbf{Mod}_R$ or $\mathbf{Top}$ is the same as the product or pullback of the underlying sets: in each case the underlying set (or forgetful) functor is a right adjoint (2.9, 2.10) and so preserves limits, e.g. products and pullbacks. Conversely, 2.26 pins down why the forgetful functor G of 2.9 cannot possibly be a left adjoint or equivalently cannot possibly have a right adjoint: G does not preserve colimits, since for instance it does not take coproducts of R-modules (i.e. direct sums) to coproducts of sets (i.e. disjoint unions).

We will use the following proposition in §10.

Lemma 2.27. *[17, p. 112] Suppose that $\mathbf{C}$ has all small limits and colimits and that $\mathbf{D}$ is a small category. Then the functor category $\mathbf{C}^{\mathrm{D}}$ also has small limits and colimits.*

Remark. In the situation of 2.27 the colimits and limits in $\mathbf{C}^{\mathrm{D}}$ can be computed "pointwise" in the following sense. Suppose that $X : \mathbf{D}' \to \mathbf{C}^{\mathrm{D}}$ is a functor. Then for each object d of $\mathbf{D}$ there is an associated functor $X_d : \mathbf{D}' \to \mathbf{C}$ given by the formula $X_d(d') = (X(d'))(d)$. It is not hard to check that for each $d \in \operatorname{Ob}(\mathbf{D})$ there are natural isomorphisms $(\operatorname{colim} X)(d) \cong \operatorname{colim}(X_d)$ and $(\lim X)(d) \cong \lim(X_d)$.

3. Model categories

In this section we introduce the concept of a model category and give some examples. Since checking that a category has a model category structure is not usually trivial, we defer verifying the examples until later (§7 and §8).

3.1. Definition. Given a commutative square diagram of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{g} & Y \end{array} \tag{3.2}$$

a *lift* or *lifting* in the diagram is a map $h : B \to X$ such that the resulting diagram with five arrows commutes, i.e., such that $hi = f$ and $ph = g$.

3.3. Definition. A *model category* is a category **C** with three distinguished classes of maps:

(i) *weak equivalences* ($\xrightarrow{\sim}$),
(ii) *fibrations* ($\twoheadrightarrow$), and
(iii) *cofibrations* ($\hookrightarrow$)

each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an *acyclic fibration* (resp. *acyclic cofibration*). We require the following axioms.

MC1 Finite limits and colimits exist in **C** (2.14, 2.21).
MC2 If f and g are maps in **C** such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.
MC3 If f is a retract of g (2.6) and g is a fibration, cofibration, or a weak equivalence, then so is f.
MC4 Given a commutative diagram of the form 3.2, a lift exists in the diagram in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.
MC5 Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and (ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

Remark. The above axioms describe what in [22] is called a “closed” model category; since no other kind of model category comes up in this paper, we have decided to leave out the word “closed”. In [22] Quillen uses the terms “trivial cofibration” and “trivial fibration” instead of “acyclic cofibration” and “acyclic fibration”. This conflicts with the ordinary homotopy theoretic use of “trivial fibration” to mean a fibration in which the total space is equivalent to the product of the base and fibre; in geometric examples of model categories, the “acyclic fibrations” of 3.3 usually turn out to be fibrations with a trivial fibre, so that the total space is equivalent to the base. We have followed Quillen's later practice in using the word "acyclic". The axioms as stated are taken from [23].

3.4. Remark. By **MC1** and 2.25, a model category $\mathbf{C}$ has both an initial object $\emptyset$ and a terminal object $*$. An object $A \in \mathbf{C}$ is said to be *cofibrant* if $\emptyset \to A$ is a cofibration and *fibrant* if $A \to *$ is a fibration. Later on, when we define the homotopy category $\mathrm{Ho}(\mathbf{C})$, we will see that $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, B)$ is in general a quotient of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ only if A is cofibrant and B is fibrant; if A is not cofibrant or B is not fibrant, then there are not in general a sufficient number of maps $A \to B$ in $\mathbf{C}$ to represent every map in the homotopy category.

The factorizations of a map in a model category provided by **MC5** are not required to be functorial. In most examples (e.g., in cases in which the factorizations are constructed by the small object argument of 7.12) the factorizations can be chosen to be functorial.

We now give some examples of model categories.

3.5. Example. (see §8) The category **Top** of topological spaces can be given the structure of a model category by defining $f : X \to Y$ to be

(i) a *weak equivalence* if f is a weak homotopy equivalence (8.1)

(ii) a *cofibration* if f is a retract (2.6) of a map $X \to Y'$ in which Y' is obtained from X by attaching cells (8.8), and

(iii) a *fibration* if f is a Serre fibration (8.2).

With respect to this model category structure, the homotopy category $\mathrm{Ho}(\mathbf{Top})$ is equivalent to the usual homotopy category of CW-complexes (cf. 8.4).

The above model category structure seems to us to be the one which comes up most frequently in everyday algebraic topology. It puts an emphasis on CW-structures; every object is fibrant, and the cofibrant objects are exactly the spaces which are retracts of generalized CW-complexes (where a "generalized CW-complex" is a space built up from cells, without the requirement that the cells be attached in order by dimension.) In some topological situations, though, weak homotopy equivalences are not the correct maps to focus on. It is natural to ask whether there is another model category structure on **Top** with respect to which the "weak equivalences" are the ordinary homotopy equivalences. There is a beautiful paper of Strom [26] which gives a positive answer to this question. If B is a topological space, call a subspace inclusion $i : A \to B$ a *closed Hurewicz cofibration* if A is a closed subspace of B and i has the homotopy extension property, i.e., for every space Y a lift (3.1) exists in every commutative diagram

$$\begin{array}{ccc} B \times 0 \cup A \times [0,1] & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times [0,1] & \longrightarrow & * \end{array} .$$

Call a map $p : X \to Y$ a *Hurewicz fibration* if p has the homotopy lifting property, i.e., for every space A a lift exists in every commutative diagram

$$\begin{array}{ccc} A \times 0 & \longrightarrow & X \\ \downarrow & & {\scriptstyle p}\downarrow \\ A \times [0,1] & \longrightarrow & Y \end{array} .$$

3.6. Example. [26] The category **Top** of topological spaces can be given the structure of a model category by defining a map $f : X \to Y$ to be

(i) a *weak equivalence* if f is a homotopy equivalence,
(ii) a *cofibration* if f is a closed Hurewicz cofibration, and
(iii) a *fibration* if f is a Hurewicz fibration.

With respect to this model category structure, the homotopy category Ho(**Top**) is equivalent to the usual homotopy category of topological spaces.

Remark. The model category structure of 3.6 is quite different from the one of 3.5. For example, let W be the "Warsaw circle"; this is the compact subspace of the plane given by the union of the interval $[-1, 1]$ on the y-axis, the graph of $y = \sin(1/x)$ for $0 < x \leqslant 1$, and an arc joining $(1, \sin(1))$ to $(0, -1)$. Then the map from W to a point is a weak equivalence with respect to the model category structure of 3.5 but not a weak equivalence with respect to the model category structure of 3.6.

It turns out that many purely algebraic categories also carry model category structures. Let R be a ring and $\mathbf{Ch}_R$ the category of nonnegatively graded chain complexes over R.

3.7. Example. (see §7) The category $\mathbf{Ch}_R$ can be given the structure of a model category by defining a map $f : M \to N$ to be

(i) a *weak equivalence* if f induces isomorphisms on homology groups,
(ii) a *cofibration* if for each $k \geqslant 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective R-module (§7.1) as its cokernel, and
(iii) a fibration if for each $k \geqslant 1$ the map $f_k : M_k \to N_k$ is an epimorphism.

The cofibrant objects in $\mathbf{Ch}_R$ are the chain complexes M such that each M_k is a projective R-module. The homotopy category $\mathrm{Ho}(\mathbf{Ch}_R)$ is equivalent to the category whose objects are these cofibrant chain complexes and whose morphisms are ordinary chain homotopy classes of maps (cf. proof of 7.3).

Given a model category, it is possible to construct many other model categories associated to it. We will do quite a bit of this in §10. Here are two elementary examples.

3.8. Example. Let $\mathbf{C}$ be a model category. Then the opposite category $\mathbf{C}^{\mathrm{op}}$ (2.4) can be given the structure of a model category by defining a map $f^{\mathrm{op}} : Y \to X$ to be

(i) a *weak equivalence* if $f : X \to Y$ is a weak equivalence in $\mathbf{C}$,
(ii) a *cofibration* if $f : X \to Y$ is a fibration in $\mathbf{C}$,
(iii) a *fibration* if $f : X \to Y$ is a cofibration in $\mathbf{C}$.

3.9. Duality. Example 3.8 reflects the fact that the axioms for a model category are self-dual. Let P be a statement about model categories and P^* the dual statement obtained by reversing the arrows in P and switching "cofibration" with "fibration". If P is true for all model categories, then so is P^*.

Remark. The duality construction in 3.9 corresponds via 3.5 or 3.6 to what is usually called "Eckmann-Hilton" duality in ordinary homotopy theory. Since there are interesting true statements P about the homotopy theory of topological spaces whose Eckmann-Hilton dual statements P^* are not true, it must be that there are interesting facts about ordinary homotopy theory which cannot be derived from the model category axioms. Of course this is something to be expected; the axioms are an attempt to codify what all homotopy theories might have in common, and just about any particular model category has additional properties that go beyond what the axioms give.

If $\mathbf{C}$ is a category and A is an object of $\mathbf{C}$, the under category [17, p. 46] (or comma category) $A{\downarrow}\mathbf{C}$ is the category in which an object is a map $f : A \to X$ in $\mathbf{C}$. A morphism in this category from $f : A \to X$ to $g : A \to Y$ is a map $h : X \to Y$ in $\mathbf{C}$ such that $hf = g$.

3.10. Remark. Let $\mathbf{C}$ be a model category and A an object of $\mathbf{C}$. Then it is possible to give $A{\downarrow}\mathbf{C}$ the structure of a model category by defining $h : (A \to X) \to (A \to Y)$ in $A{\downarrow}\mathbf{C}$ to be

(i) a *weak equivalence* if $h : X \to Y$ is a weak equivalence in $\mathbf{C}$,
(ii) a *cofibration* if $h : X \to Y$ is a cofibration in $\mathbf{C}$, and
(iii) a *fibration* if $h : X \to Y$ is a fibration in $\mathbf{C}$.

Remark. Let **Top** have the model category structure of 3.6 and as usual let $*$ be the terminal object of **Top**, i.e., the space with one point. Then $*{\downarrow}\mathbf{Top}$ is the category of pointed spaces, and an object X of $*{\downarrow}\mathbf{Top}$ is cofibrant if and only if the basepoint of X is closed and nondegenerate [25, p. 380]. Thus (3.7) from the point of view of model categories, having a nondegenerate basepoint is for a space what being projective is for a chain complex!

3.11. Remark. In the situation of 3.10, we leave it to the reader to define the *over category* $\mathbf{C}{\downarrow}A$ and describe a model category structure on it. If $\mathbf{C}$ is the category of spaces (3.5 and 3.6), the model category structure on $\mathbf{C}{\downarrow}A$ is related to fibrewise homotopy theory [15].

In the remainder of this section we make some preliminary observations about model categories.

3.12. Lifting Properties. A map $i : A \to B$ is said to have the *left lifting property* (LLP) with respect to another map $p : X \to Y$ and p is said to have the *right lifting property* (RLP) with respect to i if a lift exists (3.1) in any diagram of the form 3.2.

Proposition 3.13. *Let* $\mathbf{C}$ *be a model category.*

(i) *The cofibrations in* $\mathbf{C}$ *are the maps which have the LLP with respect to acyclic fibrations.*

(ii) *The acyclic cofibrations in* $\mathbf{C}$ *are the maps which have the LLP with respect to fibrations.*

(iii) *The fibrations in* $\mathbf{C}$ *are the maps which have the RLP with respect to acyclic cofibrations.*

(iv) *The acyclic fibrations in* $\mathbf{C}$ *are the maps which have the RLP with respect to cofibrations.*

Proof. Axiom **MC4** implies that an (acyclic) cofibration or an (acyclic) fibration has the stated lifting property. In each case we need to prove the converse. Since the four proofs are very similar (and in fact statements (iii) and (iv) follow from (i) and (ii) by duality), we only give the first proof. Suppose that $f : K \to L$ has the LLP with respect to all acyclic fibrations. Factor f as a composite $K \hookrightarrow L' \overset{\sim}{\twoheadrightarrow} L$ as in **MC5**(i), so $i : K \to L'$ is a cofibration and $p : L' \to L$ is an acyclic fibration. By assumption there exists a lifting $g : L \to L'$ in the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{i} & L' \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow\sim \\ L & \xrightarrow{\mathrm{id}} & L \end{array}$$

This implies that f is a retract of i:

$$\begin{array}{ccccc} K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{\mathrm{id}} & K \\ {\scriptstyle f}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ L & \xrightarrow{g} & L' & \xrightarrow{p} & L. \end{array}$$

By **MC3** we conclude that f is a cofibration. $\square$

Remark. Proposition 3.13 implies that the axioms for a model category are overdetermined in some sense. This has the following practical consequence. If we are trying to set up a model category structure on some given category and have chosen the fibrations and the weak equivalences, then the class of cofibrations is pinned down by property 3.13(i). Dually, if we have chosen the cofibrations and weak equivalences, the class of fibrations is pinned down by property 3.13(iii). Verifying the axioms then comes down in part to checking certain consistency conditions.

Proposition 3.14. *Let* **C** *be a model category.*

(i) *The class of cofibrations in* **C** *is stable under cobase change (2.16).*
(ii) *The class of acyclic cofibrations is stable under cobase change.*
(iii) *The class of fibrations is stable under base change (2.23).*
(iv) *The class of acyclic fibrations is stable under base change.*

Proof. The second two statements follow from the first two by duality (3.9), so we only prove the first and indicate the proof of the second. Assume that $i : K \hookrightarrow L$ is a cofibration, and pick a map $f : K \to K'$. Construct a pushout diagram (cf. **MC1**):

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow \\ L & \xrightarrow{g} & L'. \end{array}$$

We have to prove that j is a cofibration. By (i) of the previous proposition it is enough to show that j has the LLP with respect to an acyclic fibration. Let $p : E \to B$ be an acyclic fibration and consider a lifting problem

$$\begin{array}{ccc} K' & \xrightarrow{a} & E \\ {\scriptstyle j}\downarrow & & {\scriptstyle p}\downarrow \\ L' & \xrightarrow{b} & B \end{array} \quad . \tag{3.15}$$

Enlarge this to the following diagram

$$\begin{array}{ccccc} K & \xrightarrow{f} & K' & \xrightarrow{a} & E \\ {\scriptstyle i}\downarrow & & & & {\scriptstyle p}\downarrow \\ L & \xrightarrow{g} & L' & \xrightarrow{b} & B \end{array} \quad .$$

Since i is a cofibration, there is a lifting $h : L \to E$ in the above diagram. By the universal property of pushouts, the maps $h : L \to E$ and $a : K' \to E$ induce the desired lifting in 3.15. The proof of part (ii) is analogous, the only difference being that we need to invoke 3.13(ii) instead of 3.13(i). □

4. Homotopy Relations on Maps

In this section **C** is some fixed model category, and A and X are objects of **C**. Our goal is to exploit the model category axioms to construct some reasonable homotopy relations on the set $\mathrm{Hom}_{\mathbf{C}}(A, X)$ of maps from A to X. We first study a notion of *left homotopy*, defined in terms of *cylinder objects*, and then a dual (3.9) notion of *right homotopy*, defined in terms of *path objects*. It turns out (4.21) that the two notions coincide in what will turn out to be the most important case, namely if A is cofibrant and X is fibrant.

4.1. Cylinder objects and left homotopy

4.2. Definition. A *cylinder object* for A is an object $A \wedge I$ of $\mathbf{C}$ together with a diagram (**MC1**, 2.15):

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the folding map $\mathrm{id}_A + \mathrm{id}_A : A \coprod A \to A$ (2.15). A cylinder object $A \wedge I$ is called

(i) a *good cylinder object*, if $A \coprod A \to A \wedge I$ is a cofibration, and

(ii) a *very good cylinder object*, if in addition the map $A \wedge I \to A$ is a (necessarily acyclic) fibration.

If $A \wedge I$ is a cylinder object for A, we will denote the two structure maps $A \to A \wedge I$ by $i_0 = i \cdot \mathrm{in}_0$ and $i_1 = i \cdot \mathrm{in}_1$ (cf. 2.15).

4.3. Remark. By **MC5**, at least one very good cylinder object for A exists. The notation $A \wedge I$ is meant to suggest the product of A with an interval (Quillen even uses the notation "$A \times I$" for a cylinder object). However, a cylinder object $A \wedge I$ is not necessarily the product of A with anything in $\mathbf{C}$; it is just an object of $\mathbf{C}$ with the above formal property. An object A of $\mathbf{C}$ might have many cylinder objects associated to it, denoted, say, $A \wedge I$, $A \wedge I'$,..., etc. We do not assume that there is some preferred natural cylinder object for A; in particular, we do not assume that a cylinder object can be chosen in a way that is functorial in A.

Lemma 4.4. *If A is cofibrant and $A \wedge I$ is a good cylinder object for A, then the maps $i_0, i_1 : A \to A \wedge I$ are acyclic cofibrations.*

Proof. It is enough to check this for i_0. Since the identity map $\mathrm{id}_A : A \to A$ factors as $A \xrightarrow{i_0} A \wedge I \xrightarrow{\sim} A$, it follows from **MC2** that i_0 is a weak equivalence. Since $A \coprod A$ is defined by the following pushout diagram (2.16)

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ {\scriptstyle\text{cofibration}}\downarrow & & {\scriptstyle\mathrm{in}_0}\downarrow \\ A & \xrightarrow{\mathrm{in}_1} & A \coprod A \end{array}$$

it follows from 3.14 that the map in_0 is a cofibration. Since i_0 is thus the composite

$$A \xrightarrow{\mathrm{in}_0} A \coprod A \to A \wedge I,$$

of two cofibrations, it itself is a cofibration. $\square$

Definition. Two maps $f, g : A \to X$ in $\mathbf{C}$ are said to be *left homotopic* (written $f \overset{l}{\sim} g$) if there exists a cylinder object $A \wedge I$ for A such that the sum map $f + g : A \coprod A \to X$ (2.15) extends to a map $H : A \wedge I \to X$, i.e. such that there exists a map $H : A \wedge I \to X$ with $H(i_0 + i_1) = f + g$. Such a map H is said to be a *left homotopy* from f to g (via the cylinder object $A \wedge I$). The left homotopy is said to be *good* (resp. *very good*) if $A \wedge I$ is a good (resp. very good) cylinder object for A.

Example. Let $\mathbf{C}$ be the category of topological spaces with the model category structure described in 3.5. Then one choice of cylinder object for a space A is the product $A \times [0, 1]$. The notion of left homotopy with respect to this cylinder object coincides with the usual notion of homotopy. Observe that if A is not a CW-complex, $A \times [0, 1]$ is not usually a good cylinder object for A.

4.5. Remark. If $f \overset{l}{\sim} g$ via the homotopy H, then it follows from **MC2** that the map f is a weak equivalence if and only if g is. To see this, note that as in the proof of 4.4 the maps i_0 and i_1 are weak equivalences, so that if $f = Hi_0$ is a weak equivalence, so is H and hence so is $g = Hi_1$.

Lemma 4.6. *If $f \overset{l}{\sim} g : A \to X$, then there exists a good left homotopy from f to g. If in addition X is fibrant, then there exists a very good left homotopy from f to g.*

Proof. The first statement follows from applying **MC5**(i) to the map $A \coprod A \to A \wedge I$, where $A \wedge I$ is the cylinder object in some left homotopy from f to g. For the second, choose a good left homotopy $H : A \wedge I \to X$ from f to g. By **MC5**(i) and **MC2**, we may factor $A \wedge I \xrightarrow{\sim} A$ as

$$A \wedge I \overset{\sim}{\hookrightarrow} A \wedge I' \overset{\sim}{\twoheadrightarrow} A.$$

Applying **MC4** to the following diagram

$$\begin{array}{ccc} A \wedge I & \xrightarrow{H} & X \\ \downarrow & & \downarrow \\ A \wedge I' & \longrightarrow & * \end{array}$$

gives the desired very good homotopy $H' : A \wedge I' \to X$. $\square$

Lemma 4.7. *If A is cofibrant, then $\overset{l}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

Proof. Since we can take A itself as a cylinder object for A, we can take f itself as a left homotopy between f and f. Let $s : A \coprod A \to A \coprod A$ be the map which switches factors (technically, $s = \mathrm{in}_1 + \mathrm{in}_0$). The identity $(g+f) = (f+g)s$ shows that if $f \overset{l}{\sim} g$, then $g \overset{l}{\sim} f$. Suppose that $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$. Choose a good (4.6) left homotopy $H : A \wedge I \to X$ from f to g (i.e. $Hi_0 = f, Hi_1 = g$) and a good left homotopy $H' : A \wedge I' \to X$ from g to h (i.e. $H'i'_0 = g, H'i'_1 = h$). Let $A \wedge I''$ be the pushout of the following diagram:

$$A \wedge I \underset{\sim}{\xleftarrow{i_1}} A \underset{\sim}{\xrightarrow{i'_0}} A \wedge I' \ .$$

Since the maps $i_1 : A \to A \wedge I$ and $i'_0 : A \to A \wedge I'$ are acyclic cofibrations, it follows from 3.14 and the universal property of pushouts (2.16) that $A \wedge I''$ is a cylinder object for A. Another application of 2.16 to the maps H and H' gives the desired homotopy $H'' : A \wedge I'' \to X$ from f to h. □

Let $\pi^l(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by left homotopy.

4.8. Remark. The word "generated" in the above definition of $\pi^l(A, X)$ is important. We will sometimes consider $\pi^l(A, X)$ even if A is not cofibrant; in this case left homotopy, taken on its own, is not necessarily an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

Lemma 4.9. *If A is cofibrant and $p : Y \to X$ is an acyclic fibration, then composition with p induces a bijection:*

$$p_* : \pi^l(A, Y) \to \pi^l(A, X), \quad [f] \mapsto [pf].$$

Proof. The map p_* is well defined, since if $f, g : A \to Y$ are two maps and H is a left homotopy from f to g, then pH is a left homotopy from pf to pg. To show that p_* is onto, choose $[f] \in \pi^l(A, X)$. By **MC4**(i), a lift $g : A \to Y$ exists in the following diagram:

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & {\scriptstyle p}\downarrow\sim \\ A & \xrightarrow{f} & X \end{array} \quad .$$

Clearly $p_*[g] = [pg] = [f]$. To prove that p_* is one to one, let $f, g : A \to Y$ and suppose that $pf \overset{l}{\sim} pg : A \to X$. Choose (4.6) a good left homotopy $H : A \wedge I \to X$

from pf to pg. By **MC4**(i), a lifting exists in the following diagram

$$\begin{array}{ccc} A\coprod A & \xrightarrow{f+g} & Y \\ \downarrow & & p\downarrow\sim \\ A\wedge I & \xrightarrow{H} & X. \end{array}$$

and gives the desired left homotopy from f to g. □

Lemma 4.10. *Suppose that X is fibrant, that f and g are left homotopic maps $A \to X$, and that $h : A' \to A$ is a map. Then $fh \overset{l}{\sim} gh$.*

Proof. By 4.6, we can choose a very good left homotopy $H : A \wedge I \to X$ between f and g. Next choose a good cylinder object for A':

$$A' \coprod A' \overset{j}{\hookrightarrow} A' \wedge I \overset{\sim}{\to} A'.$$

By **MC4**, there is a lifting $k : A' \wedge I \to A \wedge I$ in the following diagram:

$$\begin{array}{ccccc} A'\coprod A' & \xrightarrow{h\coprod h} & A\coprod A & \xrightarrow{i} & A\wedge I \\ j\downarrow & & & & \sim\downarrow \\ A'\wedge I & \xrightarrow{\sim} & A' & \xrightarrow{h} & A \end{array}.$$

It is easy to check that Hk is the desired homotopy. □

Lemma 4.11. *If X is fibrant, then the composition in $\mathbf{C}$ induces a map:*

$$\pi^l(A', A) \times \pi^l(A, X) \to \pi^l(A', X), \qquad ([h], [f]) \mapsto [fh].$$

Proof. Note that we are not assuming that A is cofibrant, so that two maps $A \to X$ which represent the same element of $\pi^l(A, X)$ are not necessarily directly related by a left homotopy (4.8). Nevertheless, it is enough to show that if $h \overset{l}{\sim} k : A' \to A$ and $f \overset{l}{\sim} g : A \to X$ then fh and gk represent the same element of $\pi^l(A', X)$. For this it is enough to check both that $fh \overset{l}{\sim} gh : A' \to X$ and that $gh \overset{l}{\sim} gk : A' \to X$. The first homotopy follows from the previous lemma. The second is obtained by composing the homotopy between h and k with g. □

4.12. Path objects and right homotopies

By duality (3.9), what we have proved so far in this section immediately gives corresponding results "on the other side".

Definition. A *path object* for X is an object X^I of $\mathbf{C}$ together with a diagram (2.22)

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

which factors the diagonal map $(\mathrm{id}_X, \mathrm{id}_X) : X \to X \times X$. A path object X^I is called

(i) a *good path object*, if $X^I \to X \times X$ is a fibration, and

(ii) a *very good path object*, if in addition the map $X \to X^I$ is a (necessarily acyclic) cofibration.

4.13. Remark. By **MC5**, at least one very good path object exists for X. The notation X^I is meant to suggest a space of paths in X, i.e., a space of maps from an interval into X. However a path object X^I is not in general a function object of any kind; it is just some object of $\mathbf{C}$ with the above formal property. An object X of $\mathbf{C}$ might have many path objects associated to it, denoted X^I, $X^{I'}$,..., etc.

We denote the two maps $X^I \to X$ by $p_0 = \mathrm{pr}_0 \cdot p$ and $p_1 = \mathrm{pr}_1 \cdot p$ (cf. 2.22).

Lemma 4.14. *If X is fibrant and X^I is a good path object for X, then the maps $p_0, p_1 : X^I \to X$ are acyclic fibrations.*

Definition. Two maps $f, g : A \to X$ are said to be right homotopic (written $f \overset{r}{\sim} g$) if there exists a path object X^I for X such that the product map $(f, g) : A \to X \times X$ lifts to a map $H : A \to X^I$. Such a map H is said to be a right homotopy from f to g (via the path object X^I). The right homotopy is said to be *good* (resp.*very good*) if X^I is a good (resp. very good) path object for X.

Example. Let the category of topological spaces have the structure described in 3.5. Then one choice of path object for a space X is the mapping space $\mathrm{Map}([0, 1], X)$.

Lemma 4.15. *If $f \overset{r}{\sim} g : A \to X$ then there exists a good right homotopy from f to g. If in addition A is cofibrant, then there exists a very good right homotopy from f to g.*

Lemma 4.16. *If X is fibrant, then $\overset{r}{\sim}$ is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$.

Let $\pi^r(A, X)$ denote the set of equivalence classes of $\mathrm{Hom}_{\mathbf{C}}(A, X)$ under the equivalence relation generated by right homotopy.

Lemma 4.17. *If X is fibrant and $i : A \to B$ is an acyclic cofibration, then composition with i induces a bijection:*

$$i^* : \pi^r(B, X) \to \pi^r(A, X)\ .$$

Lemma 4.18. *Suppose that A is cofibrant, that f and g are right homotopic maps from A to X, and that $h : X \to Y$ is a map. Then $hf \overset{r}{\sim} hg$.*

Lemma 4.19. *If A is cofibrant then the composition in $\mathbf{C}$ induces a map $\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y)$.*

4.20. Relationship between left and right homotopy

The following lemma implies that if A is cofibrant and X is fibrant, then the left and right homotopy relations on $\mathrm{Hom}_{\mathbf{C}}(A, X)$ agree.

Lemma 4.21. *Let $f, g : A \to X$ be maps.*

(i) *If A is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*

(ii) *If X is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

4.22. Homotopic maps. If A is cofibrant and X is fibrant, we will denote the identical right homotopy and left homotopy equivalence relations on $\mathrm{Hom}_{\mathbf{C}}(A, X)$ by the symbol "$\sim$" and say that two maps related by this relation are *homotopic*. The set of equivalence classes with respect to this relation is denoted $\pi(A, X)$.

Proof of 4.21. Since the two statements are dual, we only have to prove the first one. By 4.6 there exists a good cylinder object

$$A \coprod A \xrightarrow{i_0+i_1} A \wedge I \xrightarrow{j} A$$

for A and a homotopy $H : A \wedge I \to X$ from f to g. By 4.4 the map i_0 is an acyclic cofibration. Choose a good path object (4.13)

$$X \xrightarrow{q} X^I \xrightarrow{(p_0,p_1)} X \times X$$

for X. By **MC4** it is possible to find a lift $K : A \wedge I \to X^I$ in the diagram

$$\begin{array}{ccc} A & \xrightarrow{qf} & X^I \\ i_0 \downarrow & & \downarrow (p_0,p_1) \\ A \wedge I & \xrightarrow{(fj,H)} & X \times X \end{array} .$$

The composite $Ki_1 : A \to X^I$ is the desired right homotopy from f to g. □

4.23. Remark. Suppose that A is cofibrant, X is fibrant, $A \wedge I$ is some *fixed* good cylinder object for A and X^I is some *fixed* good path object for X. Let $f, g : A \to X$ be maps. The proof of 4.21 shows that $f \sim g$ if and only if $f \overset{r}{\sim} g$ via the fixed path object X^I. Dually, $f \sim g$ if and only if $f \overset{l}{\sim} g$ via the fixed cylinder object $A \wedge I$.

We will need the following observation later on.

Lemma 4.24. *Suppose that $f : A \to X$ is a map in $\mathbf{C}$ between objects A and X which are both fibrant and cofibrant. Then f is a weak equivalence if and only if f has a homotopy inverse, i.e., if and only if there exists a map $g : X \to A$ such that the composites gf and fg are homotopic to the respective identity maps.*

Proof. Suppose first that f is a weak equivalence. By **MC5** we can factor f as a composite

$$A \underset{\sim}{\overset{q}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X \tag{4.25}$$

in which by **MC2** the map p is also a weak equivalence. Because $q : A \to C$ is a cofibration and A is fibrant, an application of **MC4** shows that there exists a left inverse for q, i.e. a morphism $r : C \to A$ such that $rq = \mathrm{id}_A$. By lemma 4.17, q induces a bijection $q^* : \pi^r(C, C) \to \pi^r(A, C)$, $[g] \mapsto [gq]$. Since $q^*([qr]) = [qrq] = [q]$, we conclude that $qr \overset{r}{\sim} 1_C$ and hence that r is a two-sided right (equivalently left) homotopy inverse for q. A dual argument (3.9) gives a two-sided homotopy inverse of p, say s. The composite rs is a two sided homotopy inverse of $f = pq$.

Suppose next that f has a homotopy inverse. By **MC5** we can find a factorization $f = pq$ as in 4.25. Note that the object C is both fibrant and cofibrant. By **MC2**, in order to prove that f is a weak equivalence it is enough to show that p is a weak equivalence. Let $g : X \to A$ be a homotopy inverse for f, and $H : X \wedge I \to X$ a homotopy between fg and id_X. By **MC4** we can find a lift $H' : X \wedge I \to C$ in the diagram

$$\begin{array}{ccc} X & \xrightarrow{qg} & C \\ i_0 \downarrow & & p \downarrow \\ X \wedge I & \xrightarrow{H} & X \end{array} .$$

Let $s = H'i_1$, so that $ps = \mathrm{id}_X$. The map q is a weak equivalence, so by the result just proved above q has a homotopy inverse, say r. Since $pq = f$, composing on the right with r gives $p \sim fr$ (4.11). Since in addition $s \sim qg$ by the homotopy H', it follows (4.11, 4.19) that

$$sp \sim qgp \sim qgfr \sim qr \sim \mathrm{id}_C \ .$$

By 4.5, then, sp is a weak equivalence. The commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{\mathrm{id}_C} & C & \xrightarrow{\mathrm{id}_C} & C \\ \downarrow p & & \downarrow sp & & \downarrow p \\ X & \xrightarrow{s} & C & \xrightarrow{p} & X \end{array}$$

shows that p is a retract (2.6) of sp, and hence by **MC3** that the map p is a weak equivalence. $\square$

5. The homotopy category of a model category

In this section we will use the machinery constructed in §4 to give a quick construction of the *homotopy category* $\mathrm{Ho}(\mathbf{C})$ associated to a model category $\mathbf{C}$.

We begin by looking at the following six categories associated to $\mathbf{C}$.

$\mathbf{C}_c$ - the full (2.3) subcategory of $\mathbf{C}$ generated by the cofibrant objects in $\mathbf{C}$.
$\mathbf{C}_f$ - the full subcategory of $\mathbf{C}$ generated by the fibrant objects in $\mathbf{C}$.
$\mathbf{C}_{cf}$ - the full subcategory of $\mathbf{C}$ generated by the objects of $\mathbf{C}$ which are both fibrant and cofibrant.
$\pi\mathbf{C}_c$ - the category consisting of the cofibrant objects in $\mathbf{C}$ and whose morphisms are right homotopy classes of maps (see 4.19).
$\pi\mathbf{C}_f$ - the category consisting of fibrant objects in $\mathbf{C}$ and whose morphisms are left homotopy classes of maps (see 4.11).
$\pi\mathbf{C}_{cf}$ - the category consisting of objects in $\mathbf{C}$ which are both fibrant and cofibrant, and whose morphisms are homotopy classes (4.22) of maps.

These categories will be used as tools in defining $\mathrm{Ho}(\mathbf{C})$ and constructing a canonical functor $\mathbf{C} \to \mathrm{Ho}(\mathbf{C})$. For each object X in $\mathbf{C}$ we can apply **MC5**(i) to the map $\emptyset \to X$ and obtain an acyclic fibration $p_X : QX \overset{\sim}{\twoheadrightarrow} X$ with QX cofibrant. We can also apply **MC5**(ii) to the map $X \to *$ and obtain an acyclic cofibration $i_X : X \overset{\sim}{\hookrightarrow} RX$ with RX fibrant. If X is itself cofibrant, let $QX = X$; if X is fibrant, let $RX = X$.

Lemma 5.1. *Given a map $f : X \to Y$ in $\mathbf{C}$ there exists a map $\tilde{f} : QX \to QY$ such that the following diagram commutes:*

$$\begin{array}{ccc} QX & \xrightarrow{\tilde{f}} & QY \\ p_X \downarrow \sim & & p_Y \downarrow \sim . \\ X & \xrightarrow{f} & Y \end{array}$$

The map $\tilde{f}$ depends up to left homotopy or up to right homotopy only on f, and is a weak equivalence if and only if f is. If Y is fibrant, then $\tilde{f}$ depends up to left homotopy or up to right homotopy only on the left homotopy class of f.

Proof. We obtain $\tilde{f}$ by applying **MC4** to the diagram:

$$\begin{array}{ccc} \emptyset & \longrightarrow & QY \\ \downarrow & & \sim \downarrow p_Y \\ QX & \xrightarrow{f \cdot p_X} & Y \end{array} \quad .$$

The statement about the uniqueness of $\tilde{f}$ up to left homotopy follows from 4.9. For the statement about right homotopy, observe that QX is cofibrant, and so by 4.21(i) two maps with domain QX which are left homotopic are also right homotopic. The weak equivalence condition follows from **MC2**, and the final assertion from 4.11. $\square$

5.2. Remark. The uniqueness statements in 5.1 imply that if $f = \mathrm{id}_X$ then $\tilde{f}$ is right homotopic to id_{QX}. Similarly, if $f : X \to Y$ and $g : Y \to Z$ and $h = gf$, then $\tilde{h}$ is right homotopic to $\tilde{g}\tilde{f}$. Hence we can define a functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ sending $X \to QX$ and $f : X \to Y$ to the right homotopy class $[\tilde{f}] \in \pi^r(QX, QY)$.

The dual (3.9) to 5.1 is the following statement.

Lemma 5.3. *Given a map $f : X \to Y$ in $\mathbf{C}$ there exists a map $\bar{f} : RX \to RY$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow \sim & & i_Y \downarrow \sim \\ RX & \xrightarrow{\bar{f}} & RY. \end{array}$$

The map $\bar{f}$ depends up to right homotopy or up to left homotopy only on f, and is a weak equivalence if and only if f is. If X is cofibrant, then $\bar{f}$ depends up to right homotopy or up to left homotopy only on the right homotopy class of f.

5.4. Remark. The uniqueness statements in 5.3 imply that if $f = \mathrm{id}_X$ then $\bar{f}$ is left homotopic to id_{RX}. Moreover, if $f : X \to Y$ and $g : Y \to Z$ and $h = gf$, then $\bar{h}$ is left homotopic to $\bar{g}\bar{f}$, Hence we can define a functor $R : \mathbf{C} \to \pi\mathbf{C}_f$ sending $X \to RX$ and $f : X \to Y$ to the left homotopy class $[\bar{f}] \in \pi^l(RX, RY)$.

Lemma 5.5. *The restriction of the functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ to $\mathbf{C}_f$ induces a functor $Q' : \pi\mathbf{C}_f \to \pi\mathbf{C}_{cf}$. The restriction of the functor $R : \mathbf{C} \to \pi\mathbf{C}_f$ to $\mathbf{C}_c$ induces a functor $R' : \pi\mathbf{C}_c \to \pi\mathbf{C}_{cf}$.*

Proof. The two statements are dual to one another, and so we will prove only the second. Suppose that X and Y are cofibrant objects of $\mathbf{C}$ and that $f, g : X \to Y$ are maps which represent the same map in $\pi\mathbf{C}_c$; we must show that $Rf = Rg$. It is enough to do this in the special case $f \overset{r}{\sim} g$ in which f and g are directly related by a right homotopy; however in this case it is a consequence of 5.3. $\square$

5.6. Definition. *The homotopy category* $\mathrm{Ho}(\mathbf{C})$ of a model category $\mathbf{C}$ is the category with the same objects as $\mathbf{C}$ and with

$$\mathrm{Hom}_{\mathrm{Ho(C)}}(X, Y) = \mathrm{Hom}_{\pi\mathrm{C}_{cf}}(R'QX, R'QY) = \pi(RQX, RQY)\ .$$

5.7. Remark. There is a functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ which is the identity on objects and sends a map $f : X \to Y$ to the map $R'Q(f) : R'Q(X) \to R'Q(Y)$. If each of the objects X and Y is both fibrant and cofibrant, then by construction the map $\gamma : \mathrm{Hom}_{\mathrm{C}}(X,Y) \to \mathrm{Hom}_{\mathrm{Ho(C)}}(X,Y)$ is surjective and induces a bijection $\pi(X,Y) \cong \mathrm{Hom}_{\mathrm{Ho(C)}}(X,Y)$.

It is natural to ask whether or not dualizing the definition of $\mathrm{Ho}(\mathbf{C})$ by replacing the composite functor $R'Q$ by $Q'R$ would give anything different. The answer is that it would not; rather than prove this directly, though, we will give a symmetrical construction of the homotopy category in the next section. There are some basic observations about $\mathrm{Ho}(\mathbf{C})$ that will come in handy later on.

Proposition 5.8. *If f is a morphism of $\mathbf{C}$, then $\gamma(f)$ is an isomorphism in* $\mathrm{Ho}(\mathbf{C})$ *if and only if f is a weak equivalence. The morphisms of* $\mathrm{Ho}(\mathbf{C})$ *are generated under composition by the images under γ of morphisms of $\mathbf{C}$ and the inverses of images under γ of weak equivalences in $\mathbf{C}$.*

Proof. If $f : X \to Y$ is a weak equivalence in $\mathbf{C}$, then $R'Q(f)$ is represented by a map $f' : RQ(X) \to RQ(Y)$ which is also a weak equivalence (see 5.1 and 5.3); by 4.24, then, the map f' has an inverse up to left or right homotopy and represents an isomorphism in $\pi\mathbf{C}_{cf}$. This isomorphism is exactly $\gamma(f)$. On the other hand, if $\gamma(f)$ is an isomorphism then f' has an inverse up to homotopy and is therefore a weak equivalence by 4.24; it follows easily that f is a weak equivalence.

Observe by the above that for any object X of $\mathbf{C}$ the map $\gamma(i_{QX})\gamma(p_X)^{-1}$ in $\mathrm{Ho}(\mathbf{C})$ is an isomorphism from X to $RQ(X)$. Moreover, for two objects X and Y of $\mathbf{C}$, the functor γ induces an epimorphism (5.7)

$$\mathrm{Hom}_{\mathrm{C}}(RQ(X), RQ(Y)) \to \mathrm{Hom}_{\mathrm{Ho(C)}}(RQ(X), RQ(Y)) \ .$$

Consequently, any map $f : X \to Y$ in $\mathrm{Ho}(\mathbf{C})$ can be represented as a composite

$$f = \gamma(p_Y)\gamma(i_{QY})^{-1}\gamma(f')\gamma(i_{QX})\gamma(p_X)^{-1}$$

for some map $f' : RQ(X) \to RQ(Y)$ in $\mathbf{C}$. $\square$

Proposition 5.8 has the following simple but useful consequence.

Corollary 5.9. *If F and G are two functors* $\mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ *and $t : F\gamma \to G\gamma$ is a natural transformation, then t also gives a natural transformation from F to G.*

Proof. It is necessary to check that for each morphism h of $\mathrm{Ho}(\mathbf{C})$ an appropriate diagram $D(h)$ commutes. By assumption $D(h)$ commutes if $h = \gamma(f)$ or $h = \gamma(g)^{-1}$ for some morphism f in $\mathbf{C}$ or weak equivalence g in $\mathbf{C}$. It is easy to check that if $h = h_1h_2$, the $D(h)$ commutes if $D(h_1)$ commutes and $D(h_2)$ commutes. The lemma then follows from the fact (5.8) that any map of $\mathrm{Ho}(\mathbf{C})$ is a composite of maps of the form $\gamma(f)$ and $\gamma(g)^{-1}$. $\square$

Lemma 5.10. *Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ be a functor taking weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$. If $f \overset{l}{\sim} g : A \to X$ or $f \overset{r}{\sim} g : A \to X$, then $F(f) = F(g)$ in $\mathbf{D}$.*

Proof. We give a proof assuming $f \overset{l}{\sim} g$, the other case is dual. Choose (4.6) a good left homotopy $H : A \wedge I \to X$ from f to g, so that $A \wedge I$ is a good cylinder object for A:

$$A \coprod A \overset{i_0+i_1}{\hookrightarrow} A \wedge I \xrightarrow[\sim]{w} A \ .$$

Since $wi_0 = wi_1 (= \mathrm{id}_A)$, we have $F(w)F(i_0) = F(w)F(i_1)$. Since w is a weak equivalence, the map $F(w)$ is an isomorphism and it follows that $F(i_0) = F(i_1)$. Hence $F(f) = F(H)F(i_0)$ is the same as $F(g) = F(H)F(i_1)$. ☐

Proposition 5.11. *Suppose that A is a cofibrant object of $\mathbf{C}$ and X is a fibrant object of $\mathbf{C}$. Then the map $\gamma : \mathrm{Hom}_{\mathbf{C}}(A, X) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$ is surjective, and induces a bijection $\pi(A, X) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$.*

Proof. By 5.10 and 5.8 the functor γ identifies homotopic maps and so induces a map $\pi(A, X) \to \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi(RA, QX) & \longrightarrow & \pi(A, X) \\
\gamma \downarrow & & \gamma \downarrow \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(RA, QX) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(A, X)
\end{array}$$

in which the horizontal arrows are induced by the pair (i_A, p_X). By 5.8 the lower horizontal map is a bijection, while by 4.9 and 4.17 the upper horizontal map is a bijection. As indicated in 5.7, the left-hand vertical map is also a bijection. The desired result follows immediately. ☐

6. Localization of Categories

In this section we will give a conceptual interpretation of the homotopy category of a model category. Surprisingly, this interpretation depends *only* on the class of weak equivalences. This suggests that in a model category the weak equivalences carry the fundamental homotopy theoretic information, while the cofibrations, fibrations, and the axioms they satisfy function mostly as tools for making various constructions (e.g., the constructions later on in §10). This also suggests that in putting a model category structure on a category, it is most important to focus on picking the class of weak equivalences; choosing fibrations and cofibrations is a secondary issue.

6.1. Definition. Let $\mathbf{C}$ be a category, and $W \subseteq \mathbf{C}$ a class of morphisms. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be *a localization of* $\mathbf{C}$ *with respect to* W if

(i) $F(f)$ is an isomorphism for each $f \in W$, and

(ii) whenever $G : \mathbf{C} \to \mathbf{D}'$ is a functor carrying elements of W into isomorphisms, there exists a unique functor $G' : \mathbf{D} \to \mathbf{D}'$ such that $G'F = G$.

Condition 6.1(ii) guarantees that any two localizations of $\mathbf{C}$ with respect to W are canonically isomorphic. If such a localization exists, we denote it by $\mathbf{C} \to W^{-1}\mathbf{C}$.

Example. Let $\mathbf{Ab}$ be the category of abelian groups, and W the class of morphisms $f : A \to B$ such that $\ker(f)$ and $\operatorname{coker}(f)$ are torsion groups. Let $\mathbf{D}$ be the category with the same objects, but with $\mathrm{Hom}_{\mathbf{D}}(A, B) = \mathrm{Hom}_{\mathbf{Ab}}(\mathbf{Q} \otimes A, \mathbf{Q} \otimes B)$. Define $F : \mathbf{Ab} \to \mathbf{D}$ to be the functor which sends an object A to itself and a map f to $\mathbf{Q} \otimes f$. It is an interesting exercise to verify directly that F is the localization of $\mathbf{Ab}$ with respect to W [12, p. 15].

Theorem 6.2. *Let $\mathbf{C}$ be a model category and $W \subseteq \mathbf{C}$ the class of weak equivalences. Then the functor $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ is a localization of $\mathbf{C}$ with respect to W.*

More informally, Theorem 6.2 says that if $\mathbf{C}$ is a model category and $W \subseteq \mathbf{C}$ is the class of weak equivalences, then $W^{-1}\mathbf{C}$ exists and is isomorphic to $\mathrm{Ho}(\mathbf{C})$.

Proof of 6.2. We have to verify the two conditions in 6.1 for γ. Condition 6.1(i) is proved in 5.8. For 6.1(ii), suppose given a functor $G : \mathbf{C} \to \mathbf{D}$ carrying weak equivalences to isomorphisms. We must construct a functor $G' : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ such that $G'\gamma = G$, and show that G' is unique. Since the objects of $\mathrm{Ho}(\mathbf{C})$ are the same as the objects of $\mathbf{C}$, the effect of G' on objects is obvious. Pick a map $f : X \to Y$ in $\mathrm{Ho}(\mathbf{C})$, which is represented by a map $f' : RQ(X) \to RQ(Y)$, well defined up to homotopy (4.22). Observe by 5.10 that $G(f')$ depends only on the homotopy class of f', and therefore only on f. Define $G'(f)$ by the formula

$$G'(f) = G(p_Y)G(i_{QY})^{-1}G(f')G(i_{QX})G(p_X)^{-1} \ .$$

It is easy to check that G' is a functor, that is, respects identity maps and compositions. If f is the image of a map $h : X \to Y$ of $\mathbf{C}$, then (5.1 and 5.3) after perhaps altering f' up to right homotopy we can find a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{i_{QX}} & RQ(X) \\ {\scriptstyle h}\downarrow & & {\scriptstyle \tilde{h}}\downarrow & & {\scriptstyle f'}\downarrow \\ Y & \xleftarrow{p_Y} & QY & \xrightarrow{i_{QY}} & RQ(Y) \end{array} .$$

Applying G to this diagram shows that $G'(f) = G(h)$ and thus that G' extends G, that is, $G'\gamma = G$. The uniqueness of G' follows immediately from the second statement in 5.8. $\square$

7. Chain complexes

Suppose that R is an associative ring with unit and let $\mathbf{Mod}_R$ denote the category of left R-modules. Recall that the category $\mathbf{Ch}_R$ of (nonnegatively graded) chain complexes of R-modules is the category in which an object M is a collection

$\{M_k\}_{k\geqslant 0}$ of R-modules together with boundary maps $\partial : M_k \to M_{k-1}$ ($k \geqslant 1$) such that $\partial^2 = 0$. A morphism $f : M \to N$ is a collection of R-module homomorphisms $f_k : M_k \to N_k$ such that $f_{k-1}\partial = \partial f_k$. In this section we will construct a model category structure (7.2) on $\mathbf{Ch}_R$ and give some indication (7.3) of how the associated homotopy theory is related to homological algebra.

7.1. Preliminaries. For an object M of $\mathbf{Ch}_R$, define the k-dimensional cycle module $\mathrm{Cy}_k(M)$ to be M_0 if $k = 0$ and $\ker(\partial : M_k \to M_{k-1})$ if $k > 0$. Define the k-dimensional boundary module $\mathrm{Bd}_k(M)$ to be $\mathrm{image}(\partial : M_{k+1} \to M_k)$. There are *homology functors* $\mathrm{H}_k : \mathbf{Ch}_R \to \mathbf{Mod}_R$ ($k \geqslant 0$) given by $\mathrm{H}_k M = \mathrm{Cy}_k(M)/\mathrm{Bd}_k(M)$ (we think of these homology groups as playing the role for chain complexes that homotopy groups do for a space). A chain complex M is *acyclic* if $\mathrm{H}_k M = 0$ ($k \geqslant 0$). Recall that an R-module P is said to be *projective* [6] if the following three equivalent conditions hold:

(i) P is a direct summand of a free R-module,
(ii) every epimorphism $f : A \to P$ of R-modules has a right inverse, or
(iii) for every epimorphism $A \to B$ of R-modules, the induced map

$$\mathrm{Hom}_{\mathbf{Mod}_R}(P, A) \to \mathrm{Hom}_{\mathbf{Mod}_R}(P, B)$$

is an epimorphism.

The first goal of this section is to prove the following result.

Theorem 7.2. *Define a map $f : M \to N$ in $\mathbf{Ch}_R$ to be*
(i) *a weak equivalence if the map f induces isomorphisms $\mathrm{H}_k M \to \mathrm{H}_k N$ ($k \geqslant 0$),*
(ii) *a cofibration if for each $k \geqslant 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective R-module as its cokernel, and*
(iii) *a fibration if for each $k > 0$ the map $f_k : M_k \to N_k$ is an epimorphism.*
Then with these choices $\mathbf{Ch}_R$ is a model category.

After proving this we will make the following calculation. If A is an R-module, let $K(A, n)$ ($n \geqslant 0$) denote the object M of $\mathbf{Ch}_R$ with $M_n = A$ and $M_k = 0$ for $k \neq n$ (these are the chain complex analogues of Eilenberg-Mac Lane spaces).

Proposition 7.3. *For any two R-modules A and B and nonnegative integers m, n there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}_R)}(K(A, m), K(B, n)) \cong \mathrm{Ext}_R^{n-m}(A, B)\ .$$

Here Ext_R^k is the usual Ext functor from homological algebra [6]. We take it to be zero if $k < 0$.

7.4. Proof of **MC***1*–**MC3**

We should first note that the classes of weak equivalences, fibrations and cofibrations clearly contain all identity maps and are closed under composition. It is easy to see that limits and colimits in $\mathbf{Ch}_R$ can be computed degreewise, so that **MC1** follows from the fact that $\mathbf{Mod}_R$ has all small limits and colimits. Axiom **MC2** is clear. Axiom **MC3** follows from the fact that in $\mathbf{Mod}_R$ a retract of an isomorphism, monomorphism or epimorphism is another morphism of the same type (cf. 2.7). It is also necessary to observe that a retract (i.e. direct summand) of a projective R-module is projective.

7.5. Proof of **MC4***(i)*

We need to show that a lift exists in every diagram of chain complexes:

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & & \sim\downarrow{\scriptstyle p} \\ B & \xrightarrow{h} & Y, \end{array} \tag{7.6}$$

in which i is a cofibration and p is an acyclic fibration. By the definition of fibration, p_k is onto for $k > 0$. But since $(p_0)_* : H_0(X) \to H_0(Y)$ is an isomorphism, an application of the five lemma [17, p. 198] shows that p_0 is also onto. Hence there is a short exact sequence of chain complexes

$$0 \to K \to X \to Y \to 0$$

and it follows from the associated long exact homology sequence [6] [25, p. 181] that K is acyclic.

We will construct the required map $f_k : B_k \to X_k$ by induction on k. It is easy to construct a plausible map f_0, since, by 7.1 and the definition of cofibration, the module B_0 splits up to isomorphism as a direct sum $A_0 \oplus P_0$, where P_0 is a projective module; the map f_0 is chosen to be g_0 on the factor A_0 and any lifting $P_0 \to X_0$ of the given map $P_0 \to Y_0$ on the factor P_0. Assume that $k > 0$ and that for $j < k$ maps $f_j : B_j \to X_j$ with the following properties have been constructed:

(i) $\partial f_j = f_{j-1}\partial \quad 1 \leqslant j < k$,

(ii) $p_j f_j = h_j \quad 0 \leqslant j < k$,

(iii) $f_j i_j = g_j \quad 0 \leqslant j < k$.

Proceeding as for $k = 0$ we can write $B_k \cong A_k \oplus P_k$ and construct a map $\tilde{f}_k : B_k \to X_k$ with properties (ii) and (iii) above. Let $\mathcal{E} : B_k \to X_{k-1}$ be the difference map $\partial\tilde{f}_k - f_{k-1}\partial$, so that the map $\mathcal{E}$ measures the failure of $\tilde{f}_k$ to satisfy (i). Then

(a) $\partial \cdot \mathcal{E} = 0$ because f_{k-1} satisfies (i),

(b) $p_{k-1} \cdot \mathcal{E} = 0$ because $p_k\tilde{f}_k = h_k$ commutes with ∂, and

(c) $\mathcal{E} \cdot i_k = 0$ because $\tilde{f}_k i_k = g_k$ commutes with ∂.

It follows that $\mathcal{E}$ induces a map

$$\mathcal{E}' : B_k/i_k(A_k) \cong P_k \to \mathrm{Cy}_{k-1}(K) \ .$$

However, the chain complex K is acyclic and so the boundary map $K_k \to \mathrm{Cy}_{k-1}(K)$ is an epimorphism. Since P_k is a projective, $\mathcal{E}'$ can be lifted to a map $\mathcal{E}'' : P_k \to K_k$, which, after precomposition with the surjection $B_k \to P_k$ and postcomposition with the injection $K_k \to X_k$, gives a map $\mathcal{E}''' : B_k \to X_k$. It is straightforward to check that setting $f_k = \tilde{f}_k - \mathcal{E}'''$ gives a map $B_k \to X_k$ which satisfies all conditions (i)–(iii). This allows the induction to continue. □

7.7. Proof of **MC4***(ii)*

This depends on a definition and a few lemmas. Suppose that A is an R-module. For $n \geqslant 1$ define the object $D_n(A)$ of $\mathbf{Ch}_R$ to be the chain complex with

$$D_n(A)_k = \begin{cases} 0 & k \neq n,\ n-1 \\ A & k = n,\ n-1; \end{cases}$$

The boundary map $D_n(A)_n \to D_n(A)_{n-1}$ is the identity map of A. The letter "D" in this notation stands for "disk".

Lemma 7.8. *Let A be an R-module and M an object of $\mathbf{Ch}_R$. Then the map*

$$\mathrm{Hom}_{\mathbf{Ch}_R}(D_n(A), M) \to \mathrm{Hom}_{\mathbf{Mod}_R}(A, M_n)$$

which sends f to f_n is an isomorphism.

This is obvious by inspection. In fact, the functor $D_n(-)$ is left adjoint to the functor from $\mathbf{Ch}_R$ to $\mathbf{Mod}_R$ which sends M to M_n.

7.9. Remark. Lemma 7.8 immediately implies that if A is a projective R-module then $D_n(A)$ is what might be called a "projective chain complex", in the sense that if $p : M \to N$ is an epimorphism of chain complexes (or even an epimorphism in degrees $\geqslant 1$), then any map $D_n(A) \to N$ lifts over p to a map $D_n(A) \to M$. Similarly, any chain complex sum of the form $\oplus_i D_{n_i}(A_i)$ is a "projective chain complex" as long as each A_i is a projective R-module.

Lemma 7.10. *Suppose that P is an acyclic object of $\mathbf{Ch}_R$ such that each P_k is a projective R-module. Then each module $\mathrm{Cy}_k P$ $(k \geqslant 0)$ is projective, and P is isomorphic as a chain complex to the sum $\oplus_{k \geqslant 1} D_k(\mathrm{Cy}_{k-1}P)$.*

Proof. For $k \geqslant 1$ let $P^{(k)}$ be the chain subcomplex of P which agrees with P above degree $k-1$, contains $\mathrm{Bd}_{k-1}P$ in degree $k-1$, and vanishes below degree $k-1$.

The acyclicity condition gives isomorphisms $P^{(k)}/P^{(k+1)} \cong D_k(\mathrm{Cy}_{k-1}P)$. It is clear that $\mathrm{Cy}_0(P) = P_0$ is a projective R-module, and so by 7.9 there is an isomorphism $P = P^{(1)} \cong P^{(2)} \oplus D_1(\mathrm{Cy}_0 P)$. Since any direct factor of a projective R-module is projective, it follows that $P^{(2)}$ is a chain complex which satisfies the conditions of the lemma but vanishes in degree 0. Repeating the above argument in degree 1 gives an isomorphism $P^{(2)} \cong P^{(3)} \oplus D_2(\mathrm{Cy}_1 P)$. The proof is now completed by continuing along these lines. □

7.11. Remark. Lemma 7.10 implies that if P is an acyclic object of $\mathbf{Ch}_R$ with the property that each P_k is a projective R-module, then P is a "projective chain complex" in the sense of 7.9.

Now we are ready to handle **MC4**(ii). We need to show that a lift exists in every diagram of the form 7.6 in which i is an acyclic cofibration and p is a fibration. By the definition of cofibration, the map i is a monomorphism of chain complexes and the cokernel P of i is a chain complex with the property that each P_k is a projective R-module. By the long exact homology sequence [6] associated to the short exact sequence

$$0 \to A \to B \to P \to 0$$

of chain complexes, P is acyclic. It follows from 7.11 that P is a "projective chain complex" in the sense of 7.9, so that B is isomorphic to the direct sum $A \oplus P$, and the desired lift can be obtained by using the map g on the factor A and, as far as the other factor is concerned, picking any lift $P \to X$ of the given map $P \to Y$. □

7.12. The small object argument

It is actually not hard to prove **MC5** in the present case by making very elementary constructions. We have decided, however, to give a more complicated proof that works in a variety of circumstances. This proof depends on an argument, called the "small object argument", that is due to Quillen and is very well adapted to producing factorizations with lifting properties. For the rest of this subsection we will assume that $\mathbf{C}$ is a category with all small colimits.

Given a functor $B : \mathbf{Z}^+ \to \mathbf{C}$ (ii) and an object A of $\mathbf{C}$, the natural maps $B(n) \to \operatorname{colim} B$ induce maps $\mathrm{Hom}_{\mathrm{C}}(A, B(n)) \to \mathrm{Hom}_{\mathrm{C}}(A, \operatorname{colim} B)$ which are compatible enough for various n to give a canonical map (2.17)

$$\operatorname{colim}_n \mathrm{Hom}_{\mathrm{C}}(A, B(n)) \to \mathrm{Hom}_{\mathrm{C}}(A, \operatorname{colim}_n B(n)) \ . \tag{7.13}$$

7.14. Definition. An object A of $\mathbf{C}$ is said to be *sequentially small* if for every functor $B : \mathbf{Z}^+ \to \mathbf{C}$ the canonical map 7.13 is a bijection.

7.15. Remark. A set is sequentially small if and only if it is finite. An R-module is sequentially small if it has a finite presentation, i.e., it is isomorphic to the cokernel of a map between two finitely generated free R-modules. An object M of $\mathbf{Ch}_R$ is sequentially small if only a finite number of the modules M_k are non zero, and each M_k has a finite presentation.

Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i\in\mathcal{I}}$ be a *set* of maps in $\mathbf{C}$. Suppose that $p : X \to Y$ is a map in $\mathbf{C}$, and suppose that we desire to factor p as a composite $X \to X' \to Y$ in such a way that the map $X' \to Y$ has the RLP (3.12) with respect to all of the maps in $\mathcal{F}$. Of course we could choose $X' = Y$, but the secondary goal is to find a factorization in which X' is as close to X as reasonably possible. We proceed as follows. For each $i \in \mathcal{I}$ consider the set $S(i)$ which contains all pairs of maps (g, h) such that the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle p}\downarrow \\ B_i & \xrightarrow{h} & Y \end{array} . \tag{7.16}$$

We define the *Gluing Construction* $G^1(\mathcal{F}, p)$ to be the object of $\mathbf{C}$ given by the pushout diagram

$$\begin{array}{ccc} \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} A_i & \xrightarrow{+_i +_{(g,h)} g} & X \\ \coprod f_i \downarrow & & {\scriptstyle i_1}\downarrow \\ \coprod_{i\in\mathcal{I}} \coprod_{(g,h)\in S(i)} B_i & \xrightarrow{+_i +_{(g,h)} h} & G^1(\mathcal{F}, p) \end{array} .$$

This is reminiscent of a "singular complex" construction; we are gluing a copy of B_i to X along A_i for *every* commutative diagram of the form 7.16. As indicated, there is a natural map $i_1 : X \to G^1(\mathcal{F}, p)$. By the universal property of colimits, the commutative diagrams 7.16 induce a map $p_1 : G^1(\mathcal{F}, p) \to Y$ such that $p_1 i_1 = p$. Now repeat the process: for $k > 1$ define objects $G^k(\mathcal{F}, p)$ and maps $p_k : G^k(\mathcal{F}, p) \to Y$ inductively by setting $G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$ and $p_k = (p_{k-1})_1$. What results is a commutative diagram

$$\begin{array}{ccccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & G^2(\mathcal{F}, p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_k} & G^k(\mathcal{F}, p) & \longrightarrow & \cdots \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & & {\scriptstyle p_k}\downarrow & & \\ Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots & \xrightarrow{=} & Y & \xrightarrow{=} & \cdots \end{array} .$$

Let $G^\infty(\mathcal{F}, p)$, the Infinite Gluing Construction, denote the colimit (2.17) of the upper row in the above diagram; there are natural maps $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ such that $p_\infty i_\infty = p$.

Proposition 7.17. *In the above situation, suppose that for each $i \in \mathcal{I}$ the object A_i of $\mathbf{C}$ is sequentially small. Then the map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ has the RLP (3.12) with respect to each of the maps in the family $\mathcal{F}$.*

Proof. Consider a commutative diagram which gives one of the lifting problems in question:

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^\infty(\mathcal{F},p) \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle p_\infty}\downarrow \\ B_i & \xrightarrow{h} & Y \end{array} .$$

Since A_i is sequentially small, there exists an integer k such that the map g is the composite of a map $g' : A_i \to G^k(\mathcal{F},p)$ with the natural map $G^k(\mathcal{F},p) \to G^\infty(\mathcal{F},p)$. Therefore the above commutative diagram can be enlarged to another one

$$\begin{array}{ccccccc} A_i & \xrightarrow{g'} & G^k(\mathcal{F},p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathcal{F},p) & \longrightarrow & G^\infty(\mathcal{F},p) \\ {\scriptstyle f_i}\downarrow & & {\scriptstyle p_k}\downarrow & & {\scriptstyle p_{k+1}}\downarrow & & {\scriptstyle p_\infty}\downarrow \\ B_i & \xrightarrow{h} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y \end{array}$$

in which the composite all the way across the top row is g. However, the pair (g', h) contributes itself as an index in the construction of $G^{k+1}(\mathcal{F},p)$ from $G^k(\mathcal{F},p)$; what it indexes is in fact a gluing of B_i to $G^k(\mathcal{F},p)$ along A_i. By construction, then, there exists a map $B_i \to G^{k+1}(\mathcal{F},p)$ which makes the appropriate diagram commute. Composing with the map $G^{k+1}(\mathcal{F},p) \to G^\infty(\mathcal{F},p)$ gives a lifting in the original square. □

7.18. Proof of **MC5**

For $n \geqslant 1$, let D^n (the "n-disk") denote the chain complex $D_n(R)$ (7.7) and for $n \geqslant 0$ let S^n (the "n-sphere") denote the chain complex $K(R,n)$ (7.3). There is an evident inclusion $j_n : S^{n-1} \to D_n$ which is the identity on the copy of R in degree $(n-1)$. Let D^0 denote the chain complex $K(R,0)$, let S^{-1} denote the zero chain complex, and let $j_0 : S^{-1} \to D^0$ be the unique map. Note that the chain complexes D^n and S^n are sequentially small (7.15).

The following proposition is an elementary exercise in diagram chasing.

Proposition 7.19. *A map $f : X \to Y$ in $\mathbf{Ch}_R$ is*

(i) *a fibration if and only if it has the RLP with respect to the maps $0 \to D^n$ for all $n \geqslant 1$, and*

(ii) *an acyclic fibration if and only if it has the RLP with respect to the maps $j_n : S^{n-1} \to D^n$ for all $n \geqslant 0$.*

To verify **MC5**(i), let $f : X \to Y$ be the map to be factored, and let $\mathcal{F}$ be the set of maps $\{j_n\}_{n \geqslant 0}$. Consider the factorization of f provided by the small object argument (7.12):

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F},f) \xrightarrow{p_\infty} Y .$$

It is immediate from 7.17 and 7.19 that p_∞ is an acyclic fibration, so what we have to check is that i_∞ is a cofibration. This is essentially obvious; in each degree n, $G^{k+1}(\mathcal{F}, f)$ is by construction the direct sum of $G^k(\mathcal{F}, f)$ with a (possibly large) number of copies of R; passing to the colimit shows that $G^\infty(\mathcal{F}, f)_n$ is similarly the direct sum of X_n with copies of R.

The proof of **MC5**(ii) is very similar: let $f : X \to Y$ be the map to be factored, let $\mathcal{F}'$ be the set of maps $\{0 \to D_n\}_{n\geqslant 1}$ and consider the factorization of f provided by the small object argument:

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}', f) \xrightarrow{p_\infty} Y \ .$$

Again it is immediate from 7.17 and 7.19 that p_∞ is a fibration. We leave it to the reader to check that i_∞ in this case is an acyclic cofibration. □

Proof of 7.3. We will only treat the case in which $m = 0$ and $n > 0$; the general case is similar. Use **MC5**(i) to find a weak equivalence $P \to K(A, 0)$, where P is some cofibrant object of $\mathbf{Ch}_R$. There are bijections

$$\mathrm{Hom}_{\mathrm{Ho(C)}}(K(A, 0), K(B, n)) \cong \mathrm{Hom}_{\mathrm{Ho(C)}}(P, K(B, n)) \cong \pi(P, K(B, n))$$

where the first comes from the fact (5.8) that the map $P \to K(A, 0)$ becomes an isomorphism in $\mathrm{Ho}(\mathbf{C})$, and the second is from 5.11. Let X denote the good path object for $K(B, n)$ given by

$$X_i = \begin{cases} B \oplus B & i = n \\ B & i = n - 1 \\ 0 & \text{otherwise} \end{cases}$$

with boundary map $X_n \to X_{n-1}$ sending (b_0, b_1) to $b_1 - b_0$. The path object structure maps $q : K(B, n) \to X$ and $p_0, p_1 : X \to K(B, n)$ are determined in dimension n by the formulas $q(b) = (b, b)$ and $p_i(b_0, b_1) = b_i$. According to 4.23, two maps $f, g : P \to K(B, n)$ represent the same class in $\pi(P, K(B, n))$ if and only if they are related by right homotopy with respect to X, that is, if and only if there is a map $H : P \to X$ such that $p_0H = f$ and $p_1H = g$.

In the language of homological algebra, P is a projective resolution of A. A map $f : P \to K(B, n)$ amounts by inspection to a module map $f_n : P_n \to B$ such that $f_n\partial = 0$. Two maps $f, g : P \to K(B, n)$ are related by a right homotopy with respect to X if and only if there exists a map $h : P_{n-1} \to B$ such that $h\partial = f_n - g_n$. A comparison with the standard definition of $\mathrm{Ext}^*_R(A, -)$ in terms of a projective resolution of A [6] now shows that $\pi(P, K(B, n))$ is in natural bijective correspondence with $\mathrm{Ext}^n_R(A, B)$. □

8. Topological spaces

In this section we will construct the model category structure 3.5 on the category **Top** of topological spaces.

8.1. Definition. A map $f : X \to Y$ of spaces is called a *weak homotopy equivalence* [25, p. 404] if for each basepoint $x \in X$ the map $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geqslant 1$.

8.2. Definition. A map of spaces $p : X \to Y$ is said to be a *Serre fibration* [25, p. 375] if, for each CW-complex A, the map p has the RLP (3.12) with respect to the inclusion $A \times 0 \to A \times [0, 1]$.

Proposition 8.3. *Call a map of topological spaces*

(i) *a* weak equivalence *if it is a weak homotopy equivalence,*

(ii) *a* fibration *if it is a Serre fibration, and*

(iii) *a* cofibration *if it has the LLP with respect to acyclic fibrations (i.e. with respect to each map which is both a Serre fibration and a weak homotopy equivalence).*

Then with these choices **Top** *is a model category.*

After proving this we will make the following calculation.

Proposition 8.4. *Suppose that A is a CW-complex and that X is an arbitrary space. Then the set* $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(A, X)$ *is in natural bijective correspondence with the set of (conventional) homotopy classes of maps from A to X.*

Remark. In the model category structure of 8.3, every space is weakly equivalent to a CW-complex.

We will need two facts from elementary homotopy theory (cf. 7.19). Let D^n ($n \geqslant 1$) denote the topological n-disk and S^n ($n \geqslant 0$) the topological n-sphere. Let D^0 be a single point and S^{-1} the empty space. There are standard (boundary) inclusions $j_n : S^{n-1} \to D^n$ ($n \geqslant 0$).

Lemma 8.5. *[14, Theorem 3.1, p. 63] Let $p : X \to Y$ be a map of spaces. Then p is a Serre fibration if and only if p has the RLP with respect to the inclusions $D^n \to D^n \times [0, 1]$, $n \geqslant 0$.*

Lemma 8.6. *Let $p : X \to Y$ be a map of spaces. Then the following conditions are equivalent:*

(i) *p is both a Serre fibration and a weak homotopy equivalence,*

(ii) *p has the RLP with respect to every inclusion $A \to B$ such that (B, A) is a relative CW-pair, and*

(iii) *p has the RLP with respect to the maps $j_n : S^{n-1} \to D^n$ for $n \geqslant 0$.*

This is not hard to prove with the arguments from [25, p. 376]. We will also need a fact from elementary point-set topology.

Lemma 8.7. *Suppose that*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

is a sequential direct system of spaces such that for each $n \geqslant 0$ the space X_n is a subspace of X_{n+1} and the pair (X_{n+1}, X_n) is a relative CW-complex [25, p. 401]. Let A be a finite CW-complex. Then the natural map (7.13)

$$\mathrm{colim}_n \mathrm{Hom}_{\mathbf{Top}}(A, X_n) \to \mathrm{Hom}_{\mathbf{Top}}(A, \mathrm{colim}_n X_n)$$

is a bijection (of sets).

8.8. Remark. In the situation of 8.7, we will refer to the natural map $X_0 \to \mathrm{colim}_n X_n$ as a *generalized relative CW inclusion* and say that $\mathrm{colim}_n X_n$ *is obtained from X_0 by attaching cells*. It follows easily from 8.6 that any such generalized relative CW inclusion is a cofibration with respect to the model category structure described in 8.3. There is a partial converse to this.

Proposition 8.9. *Every cofibration in* **Top** *is a retract of a generalized relative CW inclusion.*

8.10. Proof of **MC1**–**MC3**. It is easy to see directly that the classes of weak equivalences, fibrations and cofibrations contain all identity maps and are closed under composition. Axiom **MC1** follows from the fact that **Top** has all small limits and colimits (2.14, 2.21). Axiom **MC2** is obvious. For the case of weak equivalences, **MC3** follows from functoriality and 2.6. The other two cases of **MC3** are similar, so we will deal only with cofibrations. Suppose that f is a retract of a cofibration f'. We need to show that a lift exists in every diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{b} & Y \end{array} \tag{8.11}$$

in which p is an acyclic fibration. Consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i} & A' & \xrightarrow{r} & A & \xrightarrow{a} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & & & {\scriptstyle p}\downarrow \\ B & \xrightarrow{j} & B' & \xrightarrow{s} & B & \xrightarrow{b} & Y \end{array}$$

in which maps i, j, r and s are retraction constituents. Since f' is a cofibration, there is a lifting $h : B' \to X$ in this diagram. It is now easy to see that hj is the desired lifting in the diagram 8.11. □

The proofs of **MC4**(ii) and **MC5**(ii) depend upon a lemma.

Lemma 8.12. *Every map $p : X \to Y$ in* **Top** *can be factored as a composite $p_\infty i_\infty$, where $i_\infty : X \to X'$ is a weak homotopy equivalence which has the LLP with respect to all Serre fibrations, and $p_\infty : X' \to Y$ is a Serre fibration.*

Proof. Let $\mathcal{F}$ be the set of maps $\{D^n \times 0 \to D^n \times [0,1]\}_{n \geqslant 0}$. Consider the Gluing Construction $G^1(\mathcal{F}, p)$ (see 7.12). It is clear that $i_1 : X \to G^1(\mathcal{F}, p)$ is a relative CW inclusion and a deformation retraction; in fact, $G^1(\mathcal{F}, p)$ is obtained from X by taking (many) solid cylinders and attaching each one to X along one end. It follows from the definition of Serre fibration that the map i_1 has the LLP with respect to all Serre fibrations. Similarly, for each $k \geqslant 1$ the map $i_{k+1} : G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p)$ is a homotopy equivalence which has the LLP with respect to all Serre fibrations. Consider the factorization

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}, p) \xrightarrow{p_\infty} Y$$

provided by the Infinite Gluing Construction. It is immediate that i_∞ has the LLP with respect to all Serre fibrations: given a lifting problem, one can inductively find compatible solutions on the spaces $G^k(\mathcal{F}, p)$ and then use the universal property of colimit to obtain a solution on $G^\infty(\mathcal{F}, p) = \mathrm{colim}_k G^k(\mathcal{F}, p)$. The proof of Proposition 7.17 shows that p_∞ has the RLP with respect to the maps in $\mathcal{F}$ and so (8.5) is a Serre fibration; it is only necessary to observe that although the spaces D^n are not in general sequentially small, they are (8.7) small with respect to the particular sequential colimit that comes up here. Finally, by 8.7 any map of a sphere into $G^\infty(\mathcal{F}, p)$ or any homotopy involving such maps must actually lie in $G^k(\mathcal{F}, p)$ for some k; it follows that i_∞ is a weak homotopy equivalence because (by the remarks above) each of the maps $X \to G^k(\mathcal{F}, p)$ is a weak homotopy equivalence. □

Proof of **MC5**. Axiom **MC5**(ii) is an immediate consequence of 8.12. The proof of **MC5**(i) is similar to the proof of 8.12. Let p be the map to be factored, let $\mathcal{F}$ be the set

$$\mathcal{F} = \{j_n : S^{n-1} \to D^n\}_{n \geqslant 0}$$

and consider the factorization $p = p_\infty i_\infty$ of p provided by the Infinite Gluing Construction $G^\infty(\mathcal{F}, p)$. By 8.6 each map $i_{k+1} : G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p)$ has the LLP with respect to Serre fibrations which are weak homotopy equivalences; by induction and a colimit argument the map i_∞ has the same LLP and so by definition is a cofibration. By 8.7 and the proof of 7.17, the map p_∞ has the RLP with respect to all maps in the set $\mathcal{F}$, and so (8.6) is a Serre fibration and a weak equivalence.

Proof of **MC4**. Axiom **MC4**(i) is immediate from the definition of cofibration. For **MC4**(ii) suppose that $f : A \to B$ is an acyclic cofibration; we have to show that f has the LLP with respect to fibrations. Use 8.12 to factor f as a composite pi, where p is a fibration and i is weak homotopy equivalence which has the LLP with respect to all fibrations. Since $f = pi$ is by assumption a weak homotopy equivalence, it is clear that p is also a weak homotopy equivalence. A lift $g : B \to A'$ exists in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ f\downarrow & & p\downarrow\sim \\ B & \xrightarrow{\mathrm{id}} & B \end{array} \tag{8.13}$$

because f is a cofibration and p is an acyclic fibration. (Recall that by definition every cofibration has the LLP with respect to acyclic fibrations). This lift g expresses the map f as a retract (2.6) of the map i. The argument in 8.10 above can now be used to show that the class of maps which have the LLP with respect to all Serre fibrations is closed under retracts; it follows that f has the LLP with respect to all Serre fibrations because i does. □

Proof of 8.4. Since A is cofibrant (8.6) and X is fibrant, the set $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Top})}(A, X)$ is naturally isomorphic to $\pi(A, X)$ (see 5.11). It is also easy to see from 8.6 that the product $A \times [0, 1]$ is a good cylinder object for A. By 4.23, two maps $f, g : A \to X$ represent the same element of $\pi(A, X)$ if and only if they are left homotopic via the cylinder object $A \times [0, 1]$, in other words, if and only if they are homotopic in the conventional sense. □

Proof of 8.9. Let $f : A \to B$ be a cofibration in **Top**. The argument in the proof of **MC5(i)** above shows that f can be factored as a composite pi, where $i : A \to A'$ is a generalized relative CW inclusion and $p : A' \to B$ is an acyclic fibration. Since f is a cofibration, a lift $g : B \to A'$ exists in the resulting diagram 8.13, and this lift g expresses f as a retract of i. □

9. Derived functors

Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ a functor. In this section we define the *left and right derived functors* of F; if they exist, these are functors

$$LF,\ RF : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$$

which, up to natural transformation on one side or the other, are the best possible approximations to an "extension of F to $\mathrm{Ho}(\mathbf{C})$", that is, to a factorization of F through $\gamma : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$. We give a criterion for the derived functors to exist, and study a condition under which a pair of adjoint functors (2.8) between two model categories induces, via a derived functor construction, adjoint functors between the associated homotopy categories. The homotopy pushout and homotopy pullback functors of §10 will be constructed by taking derived functors of genuine pushout or pullback functors.

9.1. Definition. Suppose that $\mathbf{C}$ is a model category and that $F : \mathbf{C} \to \mathbf{D}$ is a functor. Consider pairs (G, s) consisting of a functor $G : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ and natural transformation $s : G\gamma \to F$. A *left derived functor* for F is a pair (LF, t) of this type which is universal from the left, in the sense that if (G, s) is any such pair, then there exists a unique natural transformation $s' : G \to LF$ such that the composite natural transformation

$$G\gamma \xrightarrow{s' \circ \gamma} (LF)\gamma \xrightarrow{t} F \tag{9.2}$$

is the natural transformation s.

Remark. A *right derived functor* for F is a pair (RF, t), where $RF : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ is a functor and $t : F \to (RF)\gamma$ is a natural transformation with the analogous property of being "universal from the right".

Remark. The universal property satisfied by a left derived functor implies as usual that any two left derived functors of F are canonically naturally equivalent. Sometimes we will refer to LF as the left derived functor of F and leave the natural transformation t understood. If F takes weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$, then there is a functor $F' : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ with $F' = F\gamma$ (6.2), and it is not hard to see that in this case F' itself (with the identity natural transformation $t : F'\gamma \to F$) is a left derived functor of F. The next proposition shows that sometimes LF exists even though a functor F' as above does not.

Proposition 9.3. *Let $\mathbf{C}$ be a model category and $F : \mathbf{C} \to \mathbf{D}$ a functor with the property that $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects in $\mathbf{C}$. Then the left derived functor (LF, t) of F exists, and for each cofibrant object X of $\mathbf{C}$ the map*

$$t_X : LF(X) \to F(X)$$

is an isomorphism.

The proof depends on a lemma, which for future purposes we state in slightly greater generality than we actually need here.

Lemma 9.4. *Let $\mathbf{C}$ be a model category and $F : \mathbf{C}_c \to \mathbf{D}$ (§5) a functor such that $F(f)$ is an isomorphism whenever f is an acyclic cofibration between objects of $\mathbf{C}_c$. Suppose that $f, g : A \to B$ are maps in $\mathbf{C}_c$ such that f is right homotopic to g in $\mathbf{C}$. Then $F(f) = F(g)$.*

Proof. By 4.15 there exists a right homotopy $H : A \to B^I$ from f to g such that B^I is a very good path object for B. Since the path object structure map $w : B \to B^I$ is then an acyclic cofibration and B by assumption is cofibrant, it follows that B^I is cofibrant and hence that $F(w)$ is defined and is an isomorphism. The rest of the proof is identical to the dual of the proof of 5.10. First observe that there are equalities $F(p_0)F(w) = F(p_1)F(w) = F(\mathrm{id}_B)$ and then use the fact that $F(w)$ is an isomorphism to cancel $F(w)$ and obtain $F(p_0) = F(p_1)$. The equality $F(f) = F(g)$ then follows from applying F to the equalities $f = p_0H$ and $g = p_1H$. $\square$

Proof of 9.3. By Lemma 9.4, F identifies right homotopic maps between cofibrant objects of $\mathbf{C}$ and so induces a functor $F' : \pi\mathbf{C}_c \to \mathbf{D}$. By assumption, if g is a morphism of $\pi\mathbf{C}_c$ which is represented by a weak equivalence in $\mathbf{C}$ then $F'(g)$ is an isomorphism. Recall from 5.2 that there is a functor $Q : \mathbf{C} \to \pi\mathbf{C}_c$ with the property (5.1) that if f is a weak equivalence in $\mathbf{C}$ then $g = Q(f)$ is a right homotopy class which is represented by a weak equivalence in $\mathbf{C}$. It follows that the composite functor $F'Q$ carries weak equivalences in $\mathbf{C}$ into isomorphisms in $\mathbf{D}$. By the universal property (6.2) of $\mathrm{Ho}(\mathbf{C})$, the composite $F'Q$ induces a functor $\mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$, which we denote LF. There is a natural transformation $t : (LF)\gamma \to F$ which assigns to each X in $\mathbf{C}$ the map $F(p_X) : LF(X) = F(QX) \to F(X)$. If X is cofibrant then $QX = X$ and the map t_X is the identity; in particular, t_X is an isomorphism.

We now have to show that the pair (LF, t) is universal from the left in the sense of 9.1. Let $G : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ be a functor and $s : G\gamma \to F$ a natural transformation. Consider a hypothetical natural transformation $s' : G \to LF$, and construct (for each object X of $\mathbf{C}$) the following commutative diagram which in the horizontal direction involves the composite of $s' \circ \gamma$ and t;

$$\begin{array}{ccccc} G(QX) & \xrightarrow{s'_{QX}} & LF(QX) & \xrightarrow{t_{QX}=\mathrm{id}} & F(QX) \\ {\scriptstyle G(\gamma(p_X))}\downarrow & & \downarrow{\scriptstyle LF(\gamma(p_X))=\mathrm{id}} & & \downarrow{\scriptstyle F(p_X)} \\ G(X) & \xrightarrow{s'_X} & LF(X) & \xrightarrow{t_X=F(p_X)} & F(X) \end{array}$$

If s' is to satisfy the condition of 9.1, then the composite across the top row of this diagram must be equal to s_{QX}, which gives the equality $s'_X = s_{QX}G(\gamma p_X)^{-1}$ and proves that there is at most one natural transformation s' which satisfies the required condition. However, it is obvious that setting $s'_X = s_{QX}G(\gamma p_X)^{-1}$ does give a natural transformation $G\gamma \to (LF)\gamma$, and therefore (5.9) it also gives a natural transformation $G \to LF$. ☐

9.5. Definition. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. A *total left derived functor* $\mathbf{L}F$ for F is a functor

$$\mathbf{L}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$$

which is a left derived functor for the composite $\gamma_{\mathrm{D}} \cdot F : \mathbf{C} \to \mathrm{Ho}(\mathbf{D})$. Similarly, a *total right derived functor* $\mathbf{R}F$ for F is a functor $\mathbf{R}F : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ which is a right derived functor for the composite $\gamma_{\mathrm{D}} \cdot F$.

Remark. As usual, total left or right derived functors are unique up to canonical natural equivalence.

9.6. Example. Let R be an associative ring with unit, and $\mathbf{Ch}_R$ the chain complex model category constructed in §7. Suppose that M is a *right* R-module, so that $M\otimes-$

gives a functor $F : \mathbf{Ch}_R \to \mathbf{Ch}_{\mathbf{Z}}$. Proposition 9.3 can be used to show that the total derived functor $\mathbf{L}F$ exists (see 9.11). Let N be a left R-module and $K(N,0)$ (cf. 7.3) the corresponding chain complex. The final statement in 9.3 implies that $\mathbf{L}F(K(N,0))$ is isomorphic in $\mathrm{Ho}(\mathbf{Ch}_{\mathbf{Z}})$ to $F(P)$, where P is any cofibrant chain complex with a weak equivalence $P \xrightarrow{\sim} K(N,0)$. Such a cofibrant chain complex P is exactly a projective resolution of N in the sense of homological algebra, and so we obtain natural isomorphisms

$$\mathrm{H}_i\mathbf{L}F(K(N,0)) \cong \mathrm{Tor}_i^R(M,N) \quad i \geqslant 0$$

where $\mathrm{Tor}_i^R(M,-)$ is the usual i'th left derived functor of $M \otimes_R -$. This gives one connection between the notion of total derived functor in 9.5 and the standard notion of derived functor from homological algebra.

Theorem 9.7. *Let $\mathbf{C}$ and $\mathbf{D}$ be model categories, and*

$$F : \mathbf{C} \Longleftrightarrow \mathbf{D} : G$$

a pair of adjoint functors (2.8). Suppose that

(i) *F preserves cofibrations and G preserves fibrations.*

Then the total derived functors

$$\mathbf{L}F : \mathrm{Ho}(\mathbf{C}) \Longleftrightarrow \mathrm{Ho}(\mathbf{D}) : \mathbf{R}G$$

exist and form an adjoint pair. If in addition we have

(ii) for each cofibrant object A of $\mathbf{C}$ and fibrant object X of $\mathbf{D}$, a map $f : A \to G(X)$ is a weak equivalence in $\mathbf{C}$ if and only if its adjoint $f^\flat : F(A) \to X$ is a weak equivalence in $\mathbf{D}$,

then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories.

Remark. In this paper we will not use the last statement of 9.7, but this criterion for showing that two model categories have equivalent homotopy categories is used heavily by Quillen in [23]. There are various other structures associated to a model category besides its homotopy category; these include fibration and cofibration sequences [22], Toda brackets [22], various homotopy limits and colimits (§10), and various function complexes [9]. All such structures that we know of are preserved by adjoint functors that satisfy the two conditions above.

9.8. Remark. Condition 9.7(i) is equivalent to *either* of the following two conditions:

(i′) G preserves fibrations and acyclic fibrations.

(i″) F preserves cofibrations and acyclic cofibrations.

Assume, for instance, that F preserves acyclic cofibrations. Let $f : A \to B$ be an acyclic cofibration in $\mathbf{C}$ and $g : X \to Y$ a fibration in $\mathbf{D}$. Suppose given the commutative diagram on the left together with its "adjoint" diagram (2.8) on the right:

$$\begin{array}{ccc} A & \xrightarrow{u} & G(X) \\ {\scriptstyle f}\downarrow & & {\scriptstyle G(g)}\downarrow \\ B & \xrightarrow{v} & G(Y) \end{array} \qquad \qquad \begin{array}{ccc} F(A) & \xrightarrow{u^\flat} & X \\ {\scriptstyle F(f)}\downarrow & & {\scriptstyle g}\downarrow \\ F(B) & \xrightarrow{v^\flat} & Y \end{array} \quad .$$

Since F preserves acyclic cofibrations, a lift $w : F(B) \to X$ exists in the right-hand diagram. Its adjoint $w^\sharp : B \to G(X)$ is then a lift in the left-hand diagram. It follows that $G(g)$ has the RLP with respect to all acyclic cofibrations in $\mathbf{C}$, and therefore by 3.13(iii) that $G(g)$ is a fibration. This gives 9.7(i). Running the argument in reverse and using 3.13(ii) shows the converse: if G preserves fibrations then F preserves acyclic cofibrations.

The proof of 9.7 depends on a lemma that is also useful in verifying the hypotheses of 9.3.

Lemma 9.9. *(K. Brown) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

Proof. Let $f : A \to B$ be a weak equivalence in $\mathbf{C}$ between cofibrant objects. By **MC5**(i) we can factor the coproduct (2.15) map $f + \mathrm{id}_B : A \coprod B \to B$ as a composite pq, where $q : A \coprod B \to C$ is a cofibration and $p : C \to B$ is an acyclic fibration. It follows from the fact that A and B are cofibrant (cf. 4.4) that the composite maps $q \cdot \mathrm{in}_0 : A \to C$ and $q \cdot \mathrm{in}_1 : B \to C$ are cofibrations. Since $pq \cdot \mathrm{in}_i$ is a weak equivalence for $i = 0$, 1 and p is a weak equivalence, it is clear from **MC2** that $q \cdot \mathrm{in}_i$ is a weak equivalence, $i = 0$, 1. By assumption, then $F(q \cdot \mathrm{in}_0)$, $F(q \cdot \mathrm{in}_1)$ and $F(pq \cdot \mathrm{in}_1) = F(\mathrm{id}_B)$ are weak equivalences in $\mathbf{D}$. It follows that the maps $F(p)$ and hence $F(pq \cdot \mathrm{in}_0) = F(f)$ are also weak equivalences. □

Proof of 9.7. In view of 9.8, 9.9 and the dual (3.9) of 9.9, Proposition 9.3 and its dual guarantee that the total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ exist. Since F is a left adjoint it preserves colimits (2.26) and therefore (2.25) initial objects. Since G is a right adjoint it preserves limits and therefore terminal objects. It then follows as in 9.8 that F carries cofibrant objects in $\mathbf{C}$ into cofibrant objects in $\mathbf{D}$, and that G carries fibrant objects in $\mathbf{D}$ into fibrant objects in $\mathbf{C}$.

Suppose that A is a cofibrant object in $\mathbf{C}$ and that X is a fibrant object in $\mathbf{D}$. We will show that the adjunction isomorphism $\mathrm{Hom}_{\mathbf{C}}(A, G(X)) \cong \mathrm{Hom}_{\mathbf{D}}(F(A), X)$ respects the homotopy equivalence relation (4.21) and gives a bijection

$$\pi(A, G(X)) \cong \pi(F(A), X) \ . \tag{9.10}$$

If $f, g : A \to G(X)$ represent the same class in $\pi(A, G(X))$, then f is left homotopic to g via a left homotopy $H : A \wedge I \to G(X)$ in which the cylinder object $A \wedge I$ is good (4.6) and hence cofibrant (4.4). It then follows from 9.8(i$''$) that $F(A \wedge I)$ is a cylinder object for $F(A)$ and hence that $H^\flat : F(A \wedge I) \to X$ is a left homotopy between $f^\flat$ and $g^\flat$. Thus $f^\flat \sim g^\flat$. A dual argument with right homotopies shows that if $f^\flat \sim g^\flat$ then $f \sim g$ and establishes the isomorphism 9.10.

Let Q be the construction of 5.2 for $\mathbf{C}$ and S the construction of 5.4 for $\mathbf{D}$. (We have temporarily changed the letter denoting this functor from "R" to "S" in order to avoid confusion with the notation for right derived functors). In view of the construction of $\mathbf{L}F$ and $\mathbf{R}G$ given by the proof of 9.3 and its dual, the isomorphism 9.10 gives for every object A of $\mathbf{C}$ and object X of $\mathbf{D}$ a bijection

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ho(C)}}(A, \mathbf{R}G(X)) &\xrightarrow{(\gamma p_A)^*} \mathrm{Hom}_{\mathrm{Ho(C)}}(QA, G(SX)) \\ &\cong \mathrm{Hom}_{\mathrm{Ho(D)}}(F(QA), SX) \xrightarrow{((\gamma i_X)^{-1})_*} \mathrm{Hom}_{\mathrm{Ho(D)}}(\mathbf{L}F(A), X)\ .\end{aligned}$$

It is clear that this bijection gives a natural equivalence of functors from $\mathbf{C}^{\mathrm{op}} \times \mathbf{D}$ to **Sets**, and the argument of 5.9 shows that it also gives a natural equivalence of functors $\mathrm{Ho}(\mathbf{C})^{\mathrm{op}} \times \mathrm{Ho}(\mathbf{D}) \to$ **Sets**. This provides the adjunction between $\mathbf{L}F$ and $\mathbf{R}G$.

Suppose that condition (ii) is satisfied. Let A be an cofibrant object of $\mathbf{C}$. The map $i^\sharp_{F(A)} : A \to G(SF(A))$ is then a weak equivalence in $\mathbf{C}$ because its adjoint $i_{F(A)} : F(A) \to SF(A)$ is a weak equivalence in $\mathbf{D}$. Let

$$\epsilon_A = \mathrm{id}^\sharp_{\mathbf{L}F(A)} : A \to \mathbf{R}G(\mathbf{L}F(A))$$

denote the map in $\mathrm{Ho}(\mathbf{C})$ which is adjoint to the identity map of $\mathbf{L}F(A)$ in $\mathrm{Ho}(\mathbf{D})$. It follows from the above constructions that ϵ_A is an isomorphism. Since every object of $\mathrm{Ho}(\mathbf{C})$ is isomorphic to A for a cofibrant object A of $\mathbf{C}$, we conclude that ϵ_A is an isomorphism for any object A of $\mathrm{Ho}(\mathbf{C})$ and thus that the composite $(\mathbf{R}G)(\mathbf{L}F)$ is naturally equivalent to the identity functor of $\mathrm{Ho}(\mathbf{C})$. A dual argument shows that the composite $(\mathbf{L}F)(\mathbf{R}G)$ is naturally equivalent to the identity functor of $\mathrm{Ho}(\mathbf{D})$. This proves that $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories. □

9.11. Example. Let $F : \mathbf{Ch}_R \to \mathbf{Ch}_{\mathbf{Z}}$ be the functor of 9.6. In order to use 9.3 to show that the total derived functor $\mathbf{L}F$ exists, it is necessary to show that F carries weak equivalences between cofibrant objects to weak equivalences. By 9.9 it is enough to check this for acyclic cofibrations between cofibrant objects. Let $i : A \to B$ be a acyclic cofibration between cofibrant objects in $\mathbf{Ch}_R$. The quotient B/A is then an acyclic chain complex which satisfies the hypotheses of 7.10, so that by 7.11 there is an isomorphism $B \cong A \oplus (B/A)$ and (7.10) a further isomorphism between B/A and a direct sum of chain complexes of the form $D_k(P)$. Since F respects direct sums we conclude that $F(B)$ is isomorphic to the direct sum of $F(A)$ with a number of chain complexes of the form $F(D_k(P))$. By inspection $F(D_k(P))$ is acyclic, and so $F(i)$ is a weak equivalence.

10. Homotopy pushouts and homotopy pullbacks

The constructions in this section are motivated by the fact that pushouts and pullbacks are not usually well-behaved with respect to homotopy equivalences. For example, in the category **Top** of topological spaces, let D^n ($n \geqslant 1$) denote the n-disk, $j_n : S^{n-1} \to D^n$ the inclusion of the boundary $(n-1)$-sphere, and $*$ the one-point space. There is a commutative diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{j_n} & S^{n-1} & \xrightarrow{j_n} & D^n \\ \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array} \tag{10.1}$$

in which all three vertical arrows are homotopy equivalences. The pushout (2.16) or colimit of the top row is homeomorphic to S^n, the pushout of the bottom row is the space "$*$", and the map $S^n \to *$ induced by the diagram is not a homotopy equivalence.

Faced with diagram 10.1, a seasoned topologist would probably say that the pushout of the top row has the "correct" homotopy type and invoke the philosophy that to give a pushout homotopy significance the maps involved should be replaced if necessary by cofibrations. In this section we work in an arbitrary model category $\mathbf{C}$ and find a conceptual basis for this philosophy. The strategy is this. Let $\mathbf{D}$ be the category $\{a \leftarrow b \to c\}$ of 2.12 and $\mathbf{C}^{\mathrm{D}}$ the category of functors $\mathbf{D} \to \mathbf{C}$ (2.5). An object of $\mathbf{C}^{\mathrm{D}}$ is pushout data

$$X(a) \leftarrow X(b) \to X(c)$$

in $\mathbf{C}$ and a morphism $f : X \to Y$ is a commutative diagram

$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow & & {\scriptstyle f_c}\downarrow \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array} . \tag{10.2}$$

The pushout or colimit construction gives a functor $\mathrm{colim} : \mathbf{C}^{\mathrm{D}} \to \mathbf{C}$. We will construct a model category structure on $\mathbf{C}^{\mathrm{D}}$ with respect to which a weak equivalence is a map f whose three components (f_a, f_b, f_c) are weak equivalences in $\mathbf{C}$. As 10.1 illustrates, in this setting the functor colim(–) is not usually homotopy invariant (i.e., does not usually carry weak equivalences in $\mathbf{C}^{\mathrm{D}}$ to weak equivalences is $\mathbf{C}$) and so colim(–) does not directly induce a functor $\mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \to \mathrm{Ho}(\mathbf{C})$. However, it turns out that colim(–) does have a *total left derived functor* (9.5)

$$\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \to \mathrm{Ho}(\mathbf{C})$$

which in a certain sense (9.1) is the best possible homotopy invariant approximation to colim(–). We will call $\mathbf{L}$colim the *homotopy pushout functor*; it is left adjoint to the functor

$$\mathrm{Ho}(\Delta) : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C}^{\mathrm{D}})$$

induced by the "constant diagram" (2.11) construction $\Delta : \mathbf{C} \to \mathbf{C}^{\mathrm{D}}$. By 9.3, computing $\mathbf{L}\mathrm{colim}(X)$ for a diagram X involves computing $\mathrm{colim}(X')$, where X' is a cofibrant object of $\mathbf{C}^{\mathrm{D}}$ which is weakly equivalent to X. It turns out that finding such a cofibrant X' involves replacing $X(b)$ by a cofibrant object and replacing the maps $X(b) \to X(a)$ and $X(b) \to X(c)$ by cofibrations, and so in the end what we do is more or less recover, in this abstract setting, the standard philosophy. In fact, it becomes clear (see 9.6) that this philosophy is no different from the philosophy in homological algebra that a cautious practitioner should usually replace a module by a projective resolution before, for instance, tensoring it with something.

Working dually gives a construction of the *homotopy pullback functor*. At the end of the section we make a few remarks about more general homotopy colimits or limits in $\mathbf{C}$.

10.3. Remark. In the above situation, there is a natural functor $\mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \to \mathrm{Ho}(\mathbf{C})^{\mathrm{D}}$, but this functor is usually *not* an equivalence of categories (and much of the subtlety of homotopy theory lies in this fact). Consequently, the homotopy pushout functor $\mathbf{L}\mathrm{colim}$ does *not* provide "pushouts in the homotopy category", that is, it is *not* a left adjoint to constant diagram functor

$$\Delta_{\mathrm{Ho}(\mathrm{C})} : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})^{\mathrm{D}} \ .$$

10.4. Homotopy pushouts

Let $\mathbf{C}$ be a model category, $\mathbf{D}$ be the category $\{a \leftarrow b \to c\}$ above, and $\mathbf{C}^{\mathrm{D}}$ the category of functors $\mathbf{D} \to \mathbf{C}$. Given a map $f : X \to Y$ of $\mathbf{C}^{\mathrm{D}}$ as in 10.2, let $\partial_b(f)$ denote $X(b)$ and define objects $\partial_a(f)$ and $\partial_c(f)$ of $\mathbf{C}$ by the pushout diagrams

$$\begin{array}{ccc} X(b) & \longrightarrow & X(a) \\ {\scriptstyle f_b}\downarrow & & \downarrow \\ Y(b) & \longrightarrow & \partial_a(f) \end{array} \qquad \begin{array}{ccc} X(b) & \longrightarrow & X(c) \\ {\scriptstyle f_b}\downarrow & & \downarrow \\ Y(b) & \longrightarrow & \partial_c(f) \end{array} \ . \tag{10.5}$$

The commutative diagram 10.2 induces maps $i_a(f) : \partial_a(f) \to Y(a)$, $i_b(f) : \partial_b(f) \to Y(b)$, and $i_c(f) : \partial_c(f) \to Y(c)$.

Proposition 10.6. *Call a morphism $f : X \to Y$ in $\mathbf{C}^{\mathrm{D}}$*

(i) *a weak equivalence, if the morphisms f_a, f_b and f_c are weak equivalences in $\mathbf{C}$,*

(ii) *a fibration if the morphisms f_a, f_b and f_c are fibrations in $\mathbf{C}$, and*

(iii) *a cofibration if the maps $i_a(f)$, $i_b(f)$ and $i_c(f)$ are cofibrations in* $\mathbf{C}$.
Then these choices provide $\mathbf{C}^{\mathrm{D}}$ *with the structure of a model category.*

Proof. Axiom **MC1** follows from 2.27. Axiom **MC2** and the parts of **MC3** dealing with weak equivalences and fibrations are direct consequences of the corresponding axioms in $\mathbf{C}$. It is not hard to check that if f is a retract of g, then the maps $i_a(f)$, $i_b(f)$ and $i_c(f)$ are respectively retracts of $i_a(g)$, $i_b(g)$ and $i_c(g)$, so that the part of **MC3** dealing with cofibrations is also a consequence of the corresponding axiom for $\mathbf{C}$. For **MC4**(i), consider a commutative diagram

$$\begin{array}{ccc} (A(a) \leftarrow A(b) \rightarrow A(c)) & \longrightarrow & (X(a) \leftarrow X(b) \rightarrow X(c)) \\ {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow \\ (B(a) \leftarrow B(b) \rightarrow B(c)) & \longrightarrow & (Y(a) \leftarrow Y(b) \rightarrow Y(c)) \end{array}$$

in which f is a cofibration and p is an acyclic fibration. This diagram consists of three slices:

$$\begin{array}{ccc} A(a) & \longrightarrow & X(a) \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle p_a}\downarrow \\ B(a) & \longrightarrow & Y(a), \end{array} \qquad \begin{array}{ccc} A(b) & \longrightarrow & X(b) \\ {\scriptstyle f_b}\downarrow & & {\scriptstyle p_b}\downarrow \\ B(b) & \longrightarrow & Y(b), \end{array} \qquad \begin{array}{ccc} A(c) & \longrightarrow & X(c) \\ {\scriptstyle f_c}\downarrow & & {\scriptstyle p_c}\downarrow \\ B(c) & \longrightarrow & Y(c) \end{array} .$$

Since f is a cofibration and p is an acyclic fibration, we can obtain the desired lifting in the middle slice by applying **MC4**(i) in $\mathbf{C}$; this lifting induces maps $u : \partial_a(f) \to X(a)$ and $v : \partial_c(f) \to X(c)$. Liftings in the other two slices can now be constructed by applying **MC4**(i) in $\mathbf{C}$ to the squares

$$\begin{array}{ccc} \partial_a(f) & \xrightarrow{u} & X(a) \\ {\scriptstyle i_a(f)}\downarrow & & {\scriptstyle p_a}\downarrow \\ B(a) & \longrightarrow & Y(a) \end{array} \qquad \begin{array}{ccc} \partial_c(f) & \xrightarrow{v} & X(c) \\ {\scriptstyle i_c(f)}\downarrow & & {\scriptstyle p_c}\downarrow \\ B(c) & \longrightarrow & Y(c) \end{array}$$

in which each left-hand arrow is a cofibration. The proof of the second part of **MC4**(ii) is analogous; in this case the fact that the maps $i_c(f)$ and $i_a(f)$ are acyclic cofibrations follows easily from the fact that the class of acyclic cofibrations in $\mathbf{C}$ is closed under cobase change (3.14).

To prove **MC5**(ii), suppose that we have a morphism $f : A \to B$. Use **MC5**(ii) in $\mathbf{C}$ to factor the map $f_b : A(b) \to B(b)$ as $A(b) \overset{\sim}{\hookrightarrow} Y \twoheadrightarrow B(b)$. Let X be the pushout of the diagram $A(a) \leftarrow A(b) \to Y$ and Z the pushout of $Y \leftarrow A(b) \to A(c)$. There is a commutative diagram

$$\begin{array}{ccccc} A(a) & \longleftarrow & A(b) & \longrightarrow & A(c) \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ X & \longleftarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ B(a) & \longleftarrow & B(b) & \longrightarrow & B(c) \end{array}$$

in which the lower outside vertical arrows are constructed using the universal property of pushouts. Now use **MC5**(ii) in $\mathbf{C}$ again to factor the lower outside vertical arrows as $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow B(a)$ and $Z \overset{\sim}{\hookrightarrow} Z' \twoheadrightarrow B(c)$. It is not hard to see that the object $X' \leftarrow Y \rightarrow Z'$ of $\mathbf{C}^{\mathrm{D}}$ provides the intermediate object for the desired factorization of f. The proof of **MC5**(i) is similar. □

Proposition 10.7. *The adjoint functors*

$$\mathrm{colim} : \mathbf{C}^{\mathrm{D}} \Longleftrightarrow \mathbf{C} : \Delta$$

satisfy condition (i) of Theorem 9.7. Hence the total derived functors $\mathbf{L}\mathrm{colim}$ *and* $\mathbf{R}\Delta$ *exist and form an adjoint pair*

$$\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \Longleftrightarrow \mathrm{Ho}(\mathbf{C}) : \mathbf{R}\Delta.$$

Proof. This is clear from 9.8, since the functor Δ preserves both fibrations and acyclic fibrations. □

This completes the construction of the *homotopy pushout functor* $\mathbf{L}\mathrm{colim} : \mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \to \mathrm{Ho}(\mathbf{C})$. According to 9.3, $\mathbf{L}\mathrm{colim}(X)$ is isomorphic to $\mathrm{colim}(X)$ if X is a cofibrant object of $\mathbf{C}^{\mathrm{D}}$; in general $\mathbf{L}\mathrm{colim}(X)$ is isomorphic to $\mathrm{colim}(X')$ for any cofibrant object X' of $\mathbf{C}^{\mathrm{D}}$ weakly equivalent to X.

10.8. Homotopy pullbacks

The following results on homotopy pullbacks are dual (3.9) to the above ones on homotopy pushouts, so we state them without proof.

Let $\mathbf{C}$ be a model category, let $\mathbf{D}$ be the category $\{a \to b \leftarrow c\}$, and $\mathbf{C}^{\mathrm{D}}$ the category of functors $\mathbf{D} \to \mathbf{C}$. Given a map $f : X \to Y$ of $\mathbf{C}^{\mathrm{D}}$

$$\begin{array}{ccccc} X(a) & \longrightarrow & X(b) & \longleftarrow & X(c) \\ {\scriptstyle f_a}\downarrow & & {\scriptstyle f_b}\downarrow & & {\scriptstyle f_c}\downarrow \\ Y(a) & \longrightarrow & Y(b) & \longleftarrow & Y(c) \end{array}, \tag{10.9}$$

let $\delta_b(f)$ denote $X(b)$ and define objects $\delta_a(f)$ and $\delta_c(f)$ of $\mathbf{C}$ by the pullback diagrams

$$\begin{array}{ccc} \delta_a(f) & \longrightarrow & X(b) \\ \downarrow & & {\scriptstyle f_b}\downarrow \\ Y(a) & \longrightarrow & Y(b) \end{array} \qquad \begin{array}{ccc} \delta_c(f) & \longrightarrow & X(b) \\ \downarrow & & {\scriptstyle f_b}\downarrow \\ Y(c) & \longrightarrow & Y(b) \end{array} . \tag{10.10}$$

The commutative diagram 10.9 induces maps $p_a(f) : X(a) \to \delta_a(f)$, $p_b(f) : X(b) \to \delta_b(f)$, and $p_c(f) : X(c) \to \delta_c(f)$.

Proposition 10.11. *Call a morphism $f : X \to Y$ in $\mathbf{C}^{\mathrm{D}}$*

(i) *a weak equivalence, if the morphisms f_a, f_b and f_c are weak equivalences in* $\mathbf{C}$*,*

(ii) *a cofibration if the morphisms f_a, f_b and f_c are cofibrations in* $\mathbf{C}$*, and*

(iii) *a fibration if the maps $p_a(f)$, $p_b(f)$ and $p_c(f)$ are fibrations in* $\mathbf{C}$*.*

Then these choices provide $\mathbf{C}^{\mathrm{D}}$ with the structure of a model category.

Proposition 10.12. *The adjoint functors*

$$\Delta : \mathbf{C}^{\mathrm{D}} \Longleftrightarrow \mathbf{C} : \lim$$

satisfy condition (i) of Theorem 9.7. Hence the total derived functors $\mathbf{R}\mathrm{lim}$ *and* $\mathbf{L}\Delta$ *exist and form an adjoint pair*

$$\mathbf{L}\Delta : \mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \Longleftrightarrow \mathrm{Ho}(\mathbf{C}) : \mathbf{R}\mathrm{lim}\ .$$

This completes the construction of the *homotopy pullback functor* $\mathbf{R}\mathrm{lim} : \mathrm{Ho}(\mathbf{C}^{\mathrm{D}}) \to \mathrm{Ho}(\mathbf{C})$. According to 9.3, $\mathbf{R}\mathrm{lim}(X)$ is isomorphic to $\lim(X)$ if X is a fibrant object of $\mathbf{C}^{\mathrm{D}}$; in general $\mathbf{R}\mathrm{lim}(X)$ is isomorphic to $\lim(X')$ for any fibrant object X' of $\mathbf{C}^{\mathrm{D}}$ weakly equivalent to X.

10.13. Other homotopy limits and colimits

Say that a category $\mathbf{D}$ is *very small* if it satisfies the following conditions

(i) $\mathbf{D}$ has a finite number of objects,

(ii) $\mathbf{D}$ has a finite number of morphisms, and

(iii) there exists an integer N such that if

$$A_0 \xrightarrow{f_1} A_1 \to \cdots \to A_n$$

is a string of composable morphisms of $\mathbf{D}$ with $n > N$, then some f_i is an identity morphism.

Propositions 10.6 and 10.11 can be generalized to give two distinct model category structures on the category $\mathbf{C}^{\mathrm{D}}$ whenever $\mathbf{D}$ is very small. These structures share the same weak equivalences (and therefore have isomorphic homotopy categories) but they differ in their fibrations and cofibrations. One of these structures is adapted to constructing $\mathbf{L}\mathrm{colim}$ and the other to constructing $\mathbf{R}\mathrm{lim}$. We leave this as an interesting exercise for the reader. The generalization of 10.6(iii) is as follows. For each object d of $\mathbf{D}$, let ∂d denote the full subcategory of $\mathbf{D}{\downarrow}d$ (3.11) generated by all the objects *except* the identity map of d. There is a functor $j_d : \partial d \to \mathbf{D}$ which sends an object $d' \to d$ of ∂d to the object d' of $\mathbf{D}$. If X is an object of $\mathbf{C}^{\mathrm{D}}$, let $X|_{\partial d}$ denote the composite of X with j_d and let $\partial_d(X)$ denote the object of $\mathbf{C}$ given by $\mathrm{colim}(X|_{\partial d})$. There is a natural map $\partial_d(X) \to X(d)$. If $f : X \to Y$ is a map of $\mathbf{C}^{\mathrm{D}}$, define $\partial_d(f)$ by the pushout diagram

$$\begin{array}{ccc} \partial_d(X) & \longrightarrow & X(d) \\ \downarrow & & \downarrow \\ \partial_d(Y) & \longrightarrow & \partial_d(f) \end{array}$$

and observe that there is a natural map $i_d(f) : \partial_d(f) \to Y(d)$. Then the generalization of 10.6(iii) is the condition that the map $i_d(f)$ be a cofibration for every object d of $\mathbf{D}$.

Suppose that $\mathbf{D}$ is an arbitrary small category. It seems unlikely that $\mathbf{C}^{\mathrm{D}}$ has a natural model category structure for a general model category $\mathbf{C}$. However, $\mathbf{C}^{\mathrm{D}}$ does have a model category structure if $\mathbf{C}$ is the category of simplicial sets (11.1) [3, XI, §8]. The arguments of §8 can be used to construct a parallel model category structure on $\mathbf{Top}^{\mathrm{D}}$. In these special cases the homotopy limit and colimit functors have been studied by Bousfield and Kan [3]; they deal explicitly only with the case of simplicial sets, but the topological case is very similar.

11. Applications of model categories

In this section, which is less self-contained than the rest of the paper, we will give a sampling of the ways in which model categories have been used in topology and algebra. For an exposition of the theory of model categories from an alternate point of view see [16]; for a slightly different approach to axiomatic homotopy theory see, for example, [1].

11.1. Simplicial Sets. Let $\mathbf{\Delta}$ be the category whose objects are the ordered sets $[\mathbf{n}] = \{0, 1, \ldots, n\}$ $(n \geqslant 0)$ and whose morphisms are the order-preserving maps between these sets. (Here "order-preserving" means that $f(i) \leqslant f(j)$ whenever $i \leqslant j$). The category $s\mathbf{Set}$ of *simplicial sets* is defined to be the category of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$; the morphisms, as usual (2.5), are natural transformations. Recall from 2.4 that a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is the same as a contravariant functor $\mathbf{\Delta} \to \mathbf{Set}$. For an equivalent but much more explicit description of what a simplicial set is see [18, p. 1]. If X is a simplicial set it is customary to denote the set $X([\mathbf{n}])$ by X_n and call it the *set of n-simplices* of X.

A simplicial set is a combinatorial object which is similar to an abstract simplicial complex with singularities. In an abstract simplicial complex [21, p. 15] [25, p. 108], for instance, an n-simplex has $(n+1)$ distinct vertices and is determined by these vertices; in a simplicial set X, an n-simplex $x \in X_n$ does have $n+1$ "vertices" in X_0 (obtained from x and the $(n+1)$ maps $[\mathbf{n}] \to [\mathbf{0}]$ in $\mathbf{\Delta}^{\mathrm{op}}$) but these vertices are not necessarily distinct and they in no way determine x. Let Δ_n denote the standard topological n-simplex, considered as the space of formal convex linear combinations of the points in the set $[\mathbf{n}]$. If Y is a topological space, it is possible to construct an associated simplicial set $\mathrm{Sing}(Y)$ by letting the set of n-simplices $\mathrm{Sing}(Y)_n$ be the set of all continuous maps $\Delta_n \to Y$; this is a set-theoretic precursor of the singular chain complex of Y. The functor Sing : $\mathbf{Top} \to s\mathbf{Set}$ has a left adjoint, which sends a simplicial set X to a space $|X|$ called the *geometric realization* of X [18, Ch. III]; this construction is a generalization of the geometric realization construction for simplicial complexes. Call a map $f : X \to Y$ of simplicial sets

(i) a *weak equivalence* if $|f|$ is a weak homotopy equivalence (8.1) of topological spaces,

(ii) a *cofibration* if each map $f_n : X_n \to Y_n$ $(n \geqslant 0)$ is a monomorphism, and

(iii) a *fibration* if f has the RLP with respect to acyclic cofibrations (equivalently, f is a *Kan fibration* [18, §7]).

Quillen [22] proves that with these definitions the category $s\mathbf{Set}$ is a model category. He also shows that the adjoint functors

$$|?| : s\mathbf{Set} \Longleftrightarrow \mathbf{Top} : \mathrm{Sing}$$

satisfy both conditions of Theorem 9.7 and so induce an equivalence of categories $\mathrm{Ho}(s\mathbf{Set}) \to \mathrm{Ho}(\mathbf{Top})$ (this is of course with respect to the model category structure on $\mathbf{Top}$ from §8). This shows that the category of simplicial sets is a good category of algebraic or combinatorial "models" for the study of ordinary homotopy theory.

11.2. Simplicial Objects. There is an obvious way to extend the notion of simplicial set: if $\mathbf{C}$ is a category, the category $s\mathbf{C}$ of *simplicial objects in* $\mathbf{C}$ is defined to be the category of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$ (with natural transformations as the morphisms). The usual convention, if $\mathbf{C}$ is the category of groups, for instance, is to call an object of $s\mathbf{C}$ a "simplicial group". The category $\mathbf{C}$ is embedded in $s\mathbf{C}$ by the "constant diagram" functor (2.11) and in dealing with simplicial objects it is common to identify $\mathbf{C}$ with its image under this embedding. Suppose that $\mathbf{C}$ has an "underlying set" or forgetful functor $U : \mathbf{C} \to \mathbf{Set}$ (cf. 2.9). Call a map $f : X \to Y$ in $s\mathbf{C}$

(i) a *weak equivalence* if $U(f)$ is a weak equivalence in $s\mathbf{Set}$,

(ii) a *fibration* if $U(f)$ is a fibration in $s\mathbf{Set}$, and

(iii) a *cofibration* if f has the LLP with respect to acyclic fibrations.

In [22, Part II, §4] Quillen shows that in all common algebraic situations (e.g., if $\mathbf{C}$ is the category of groups, abelian groups, associative algebras, Lie algebras, commutative algebras, ...) these choices give $s\mathbf{C}$ the structure of a model category; he also characterizes the cofibrations [22, Part II, p. 4.11].

Consider now the example $\mathbf{C} = \mathbf{Mod}_R$. It turns out that there is a normalization functor $N : s\mathbf{Mod}_R \to \mathbf{Ch}_R$ [18, §22] which is an equivalence of categories and translates the model category structure on $s\mathbf{Mod}_R$ above into the model category structure on $\mathbf{Ch}_R$ from §7. Thus the homotopy theory of $s\mathbf{Mod}_R$ is ordinary homological algebra over R. For a general category $\mathbf{C}$ there is no such normalization functor, and so it is natural to think of an object of $s\mathbf{C}$ as a substitute for a chain complex in $\mathbf{C}$, and consider the homotopy theory of $s\mathbf{C}$ as homological algebra, or better homotopical algebra, over $\mathbf{C}$. This leads to the conclusion (11.1) that homotopical algebra over the category of sets is ordinary homotopy theory!

11.3. Simplicial commutative rings. Let $\mathbf{C}$ be the category of commutative rings. In [24] Quillen uses the model category structure on $s\mathbf{C}$ which was described above in order to construct a cohomology theory for commutative rings (now called André-Quillen cohomology). This has been studied extensively by Miller [19] and Goerss [13] because of the fact that if X is a space the André-Quillen cohomology of $\mathrm{H}^*(X;\mathcal{F}_p)$ plays a role in various unstable Adams spectral sequences associated to X. In this way the homotopical algebra of the commutative ring $\mathrm{H}^*(X;\mathcal{F}_p)$ leads back to information about the homotopy theory of X itself; this is parallel to the way in which, if Y is a spectrum, the homological algebra of $\mathrm{H}^*(Y;\mathcal{F}_p)$ as a module over the Steenrod algebra leads to information about the homotopy theory of Y.

We can now answer a question from the introduction. Suppose that k is a field. Let $\mathbf{C}$ be the category of commutative augmented k-algebras and let R be an object of $\mathbf{C}$. Recall that $\mathbf{C}$ can be identified with a subcategory of $s\mathbf{C}$ by the constant diagram construction. Topological intuition suggests that the *suspension* ΣR of R should be the homotopy pushout (§10) of the diagram $* \leftarrow R \rightarrow *$, where $*$ is a terminal object in $s\mathbf{C}$. Since this terminal object is k itself, ΣR should be the homotopy pushout in $s\mathbf{C}$ of $k \leftarrow R \rightarrow k$. It is not hard to compute this; up to homotopy ΣR is given by the bar construction [19, Section 5] [13, p. 51] and the i'th homotopy group of the underlying simplicial set of ΣR is $\mathrm{Tor}_i^R(k,k)$.

11.4. Rational homotopy theory. A simplicial set X is said to be 2-reduced if X_i has only a single point for $i<2$. Call a map $f: X \to Y$ between 2-reduced simplicial sets

(i) a *weak equivalence* if $\mathrm{H}_*(|f|;\mathbf{Q})$ is an isomorphism,
(ii) a *cofibration* if each map $f_k : X_k \to Y_k$ is a monomorphism, and
(iii) a *fibration* if f has the RLP with respect to acyclic cofibrations.

In [23], Quillen shows that these choices give a model category structure on the category $s\mathbf{Set}_2$ of 2-reduced simplicial sets. A differential graded Lie algebra X over $\mathbf{Q}$ is said to be 1-reduced if $X_0 = 0$. Call a map $f : X \to Y$ between 1-reduced differential graded Lie algebras over $\mathbf{Q}$

(i) a *weak equivalence* if $\mathrm{H}_*(f)$ is an isomorphism,
(ii) a *fibration* if $f_k : X_k \to Y_k$ is surjective for each $k > 1$, and
(iii) a *cofibration* if f has the LLP with respect to acyclic fibrations.

These choices give a model category structure on the category $\mathbf{DGL}_1$ of 1-reduced differential graded Lie algebras over $\mathbf{Q}$. By repeated applications of Theorem 9.7, Quillen shows [23] that the homotopy categories $\mathrm{Ho}(s\mathbf{Set}_2)$ and $\mathrm{Ho}(\mathbf{DGL}_1)$ are equivalent. It is not hard to relate the category $s\mathbf{Set}_2$ to the category $\mathbf{Top}_1$ of 1-connected topological spaces (there is a slight difficulty in that $\mathbf{Top}_1$ is not closed under colimits or limits and so cannot be given a model category structure). What results is a specific way in which objects of $\mathbf{DGL}_1$ can be used to model the rational homotopy types of 1-connected spaces. For a dual approach based on differential graded algebras see [4] and for an attempt to eliminate some denominators [7]. There is a large amount of literature in this area.

11.5. Homology localization. Let h_* be a homology theory on the category of spaces which is represented in the usual way by a spectrum. Call a map $f : X \to Y$ in $s\mathbf{Set}$

(i) a *weak h_*-equivalence* if $h_*(|f|)$ is an isomorphism,

(ii) an *h_*-cofibration* if f is a cofibration with respect to the conventional model category structure (11.1) on $s\mathbf{Set}$, and

(iii) an *h_*-fibration* if f has the RLP with respect to each map which is both a weak h_*-equivalence and an h_*-cofibration.

Bousfield shows [2, Appendix] that these choices give a model category structure on $s\mathbf{Set}$, called, say the h_*-structure. The hardest part of the proof is verifying **MC5**(ii). Bousfield does this by an interesting generalization of the small object argument (7.12). He first shows that there is a *single* map $i : A \to B$ which is both a weak h_*-equivalence and a h_*-cofibration, such that f is a h_*-fibration if and only if f has the RLP with respect to i. (Actually he finds a set $\{i_\alpha\}$ of such test maps, but there is nothing lost in replacing this set by the single map $\coprod_\alpha i_\alpha$.) Now the domain A of i is potentially quite large, and so A is not necessarily sequentially small. However, if η is the cardinality of the set $\coprod_n A_n$ of simplices of A, the functor $\mathrm{Hom}_{s\mathrm{Set}}(A, –)$ does commute with colimits indexed by transfinite ordinals of cofinality greater than η. Bousfield then proves **MC5**(ii) by using the general idea in the proof of 7.17 but applying the gluing construction $G(\{i\}, –)$ transfinitely; this involves applying the gluing construction itself at each successor ordinal, and taking a colimit of what has come before at each limit ordinal.

Let **Ho** denote the conventional homotopy category of simplicial sets (11.1). Say that a simplicial set X is h_*-local if any weak h_*-equivalence $f : A \to B$ induces a bijection $\mathrm{Hom}_{\mathrm{Ho}}(B, X) \to \mathrm{Hom}_{\mathrm{Ho}}(A, X)$. It is not hard to show that a simplicial set which is fibrant with respect to the h_*-structure above is also h_*-local. It follows that using **MC5**(ii) (for the h_*-structure) to factor a map $X \to *$ as a composite $X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow *$ gives an *h_*-localization* construction on $s\mathbf{Set}$, i.e, gives for any simplicial set X a weak h_*-equivalence $X \to X'$ from X to an h_*-local simplicial set X'. Since the factorization can be done explicitly with a (not so) small object argument, we obtain an h_*-localization functor on $s\mathbf{Set}$. It is easy to pass from this to an analogous h_*-localization functor on **Top**. These functors extract from a simplicial set or space exactly the fraction of its homotopy type which is visible to the homology theory h_*.

11.6. Feedback. We conclude by describing a way to apply the theory of model categories to itself (see [8] and [9]). The intuition behind this application is the idea that almost any simple algebraic construction should have a (total) derived functor (§9), even, for instance, the localization construction (§6) which sends a pair $(\mathbf{C}, W)$ to the localized category $W^{-1}\mathbf{C}$. In fact it *is* possible to construct a total left derived functor of $(\mathbf{C}, W) \mapsto W^{-1}\mathbf{C}$, although this involves using Proposition 9.3 in a "meta" model category in which the objects themselves are categories enriched over simplicial sets [17, p. 181]! If $\mathbf{C}$ is a model category with weak equivalences W, let $L(\mathbf{C}, W)$ denote the result of applying this derived functor to the pair $(\mathbf{C}, W)$. The object $L(\mathbf{C}, W)$ is a category enriched over simplicial sets (or, with the help of the geometric realization functor, a category enriched over topological spaces) with the same collection of objects as $\mathbf{C}$. For any pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$ there is a natural bijection

$$\pi_0 \mathrm{Hom}_{L(\mathrm{C},W)}(X, Y) \cong \mathrm{Hom}_{\mathrm{Ho(C)}}(X, Y)$$

which exhibits the set $\mathrm{Hom}_{\mathrm{Ho(C)}}(X, Y)$ as just the lowest order invariant of an entire simplicial set or space of maps from X to Y which is created by the localization process. The homotopy types of these "function spaces" $\mathrm{Hom}_{L(\mathrm{C},W)}(X, Y)$ can be computed by looking at appropriate simplicial resolutions of objects of $\mathbf{C}$ [9, §4]; these function spaces seem to capture most if not all of the higher order structure associated to $\mathbf{C}$ which was envisaged and partially investigated by Quillen [22, part I, p. 0.4] [22, part I, §2, §3].

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