Massachusetts Institute of Technology
Experimental Study Group

Physics 8.012, Fall 2010

Problem Set 5 Solutions

Due: Friday, October 15

Reading: Kleppner and Kolenkow, *An Introduction to Mechanics*, Chapter Three

Problem 1: K&K 3.14: Two people jumping off cart

Problem

N people, each of mass m_p, stand on a railway flatcar of mass m_c. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

(a) What is the final velocity of the car if all the people jump at the same time?

(b) What is the final velocity of the car if the people jump off one at a time?

(c) Does case (a) or (b) yield the largest final velocity of the flat car? Give a physical explanation for your answer.

Solution

a) Since the car undergoes no acceleration after the people leave, u is relative to the final speed. $(Nm_p + m_c)v = Nm_p u$, so $v = \frac{Nm_p u}{Nm_p + m_c}$

b) Let $v(k)$ denote the velocity after k people have jumped. Then $v(0) = 0$. Then $(m_c + (N - k + 1)m_p)v(k-1) = (m_c + (N-k)m_p)v(k) + (v(k) - u)m_p = (m_c + (N-k+1)m_p)v(k) - um_p$. Then $v(k) = \frac{(m_c+(N-k+1)m_p)v(k-1)+um_p}{m_c+(N-k+1)m_p} = v(k-1) + \frac{um_p}{m_c+(N-k+1)m_p}$. Then $v(k) = \sum_{i=1}^{k} \frac{um_p}{m_c+(N-i+1)m_p} = \sum_{i=0}^{k-1} \frac{um_p}{m_c+(N-i)m_p}$.

c)

$$
\begin{aligned}
v(0) &= 0 \\
v(1) &= \frac{um_p}{m_c + Nm_p} \\
v(2) &= um_p\left(\frac{1}{m_c + Nm_p} + \frac{1}{m_c + (N-1)m_p}\right)
\end{aligned}
$$

Since the fractions are increasing in b), b) has the larger final velocity. This is because in a), each person is changing the momentum of the car, with all the people on it, while in b), each successive person has to change the momentum of less mass.

Problem 2: K&K 3.15

Problem

A rope of mass m and length l lies on a frictionless table, with a short portion l_0 hanging through a hole. Initially the rope is at rest.

(a) Find a general differential equation for $y(t)$, the length of rope through the hole.

(b) Solve the differential equation with appropriate initial conditions for $y(t)$, the length of rope through the hole.

Solution

(a) Assume the rope is of uniform mass. Then $m\frac{d^2y}{dt^2} = \frac{y(t)}{l}mg$, so $\frac{d^2y}{dt^2} = y(t)\frac{g}{l}$.

(b) $y(t) = Ae^{\alpha t}+Be^{-\alpha t}$, so $y(t) = Ae^{\sqrt{\frac{g}{l}}t}+Be^{-\sqrt{\frac{g}{l}}t}$. Since $\dot{y}(0) = 0$, $\sqrt{\frac{g}{l}}t\left(Ae^{\sqrt{\frac{g}{l}}t} - Be^{-\sqrt{\frac{g}{l}}t}\right) = 0$, so $e^{-\sqrt{\frac{g}{l}}t}\left(Ae^{2\sqrt{\frac{g}{l}}t} - B\right)$, so $B = Ae^{2\sqrt{\frac{g}{l}}t}$. Then $y(t) = A\left(e^{\sqrt{\frac{g}{l}}t} + e^{2\sqrt{\frac{g}{l}}t}e^{-\sqrt{\frac{g}{l}}t}\right) = A\left(e^{\sqrt{\frac{g}{l}}t} + e^{\sqrt{\frac{g}{l}}t}\right) = 2Ae^{\sqrt{\frac{g}{l}}t}$. Since $y(0) = l_0$, $2A = l_0$, so $y(t) = l_0e^{\sqrt{\frac{g}{l}}t}$.

Problem 3: K&K 3.16

Problem

Water shoots out of a fire hydrant having nozzle diameter D with nozzle speed V_0. What is the reaction force on the hydrant?

Solution

Let ρ be the density of water.

$$\begin{aligned}
\frac{dp}{dt} &= \lim_{\Delta t \longrightarrow 0} \frac{(m+\Delta m)V_0 - mV_0}{\Delta t} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{\rho A V_0 \Delta t V_0}{\Delta t} \\
&= \rho A V_0^2 \\
&= \rho \pi r^2 V_0^2 \\
&= \rho \frac{\pi D^2}{4} V_0^2 \\
&= \frac{\rho \pi D^2 V_0^2}{4}
\end{aligned}$$

Problem 4: K&K 3.18

Problem

A raindrop of initial mass m_0 starts falling from rest under the influence of gravity. Assume that the raindrop gains mass from the cloud at a rate proportional to the momentum of the raindrop, $dm/dt = kmv$, where m is the instantaneous mass of the raindrop, v is the instantaneous velocity of the raindrop, and k is a constant. You may neglect air resistance.

(a) Derive a differential equation for the velocity of the raindrop.

(b) Show that the speed of the drop eventually becomes effectively constant and give an expression for the terminal speed.

(c) Assume the air resistance is proportional to the square of the velocity. How would air resistance effect the terminal speed?

Solution

a) $mg = F = \frac{dp}{dt} = \lim_{t\to 0} \frac{(m+\Delta m)(v+\Delta v)-mv}{\Delta t} = m\frac{dv}{dt} + v\frac{dm}{dt}$. Since $\frac{dm}{dt} = kmv$, $mg = m\frac{dv}{dt} + kmv^2$, so $g = \frac{dv}{dt} + kv^2$, so $\frac{dv}{dt} = g - kv^2$.

b) If $kv^2 > g$, $\frac{dv}{dt} < 0$, and if $kv^2 > g$, $\frac{dv}{dt} > 0$, so kv^2 limits to g. Then $v_{\text{terminal}} = \sqrt{\frac{g}{k}}$.

c) It would decrease it, if the resistance were proportional to mass. As it is, it will make the rain drop never reach terminal velocity, but it will not change the terminal velocity. $mg - av^2 = F = m\frac{dv}{dt} + kmv^2$, so $mg = m\frac{dv}{dt} + (km+a)v^2$, so $\frac{dv}{dt} = g - \left(k + \frac{a}{m(t)}\right)v^2 = g - \left(k + \frac{a}{m(t)}\right)v^2$. Since $\frac{dm}{dt} = \frac{dm}{dz}\frac{dz}{dt} = \frac{dm}{dz}v = kmv$, $\frac{dm}{dz} = km$. Then $m = m_0 e^{kz}$.

$$
\begin{aligned}
\frac{dv}{dt} &= \frac{dv}{dz}\frac{dz}{dt} \\
&= g - \left(k + \frac{a}{m(t)}\right)v^2 \\
&= g - \left(k + \frac{a}{m_0 e^{kz}}\right)v^2 \\
g &= \frac{dv}{dz}v + \left(k + \frac{a}{m_0 e^{kz}}\right)v^2
\end{aligned}
$$

Let $w(z) = v(z)^2$. Then $\frac{dw}{dz} = 2v\frac{dv}{dz}$.

$$
\begin{aligned}
g &= \frac{1}{2}\frac{dw}{dz} + w\left(k + \frac{a}{m_0 e^{kz}}\right) \\
2g &= \frac{dw}{dz} + 2w\left(k + \frac{a}{m_0 e^{kz}}\right)
\end{aligned}
$$

Using Mathematica, (and adjusting the bounds of integration to be more appropriate)

$$
\begin{aligned}
w(z) &= C_1 e^{-2\int_1^z \left(k + \frac{a}{m_0 e^{kz_1}}\right) dz_1} + e^{-2\int_0^z \left(k + \frac{a}{m_0 e^{kz_1}}\right) dz_1} 2g \int_0^z e^{2\int_0^{z_1}\left(k + \frac{a}{m_0 e^{kz_2}}\right) dz_2} dz_1 \\
&= C_1 e^{-2\left(kz + \frac{a}{m_0}\int_0^z e^{-kz_1}\, dz_1\right)} + e^{-2\left(kz + \frac{a}{m_0}\int_0^z e^{-kz_1}\, dz_1\right)} 2g \int_0^z e^{2\left(kz_1 + \frac{a}{m_0}\int_0^{z_1} e^{-kz_2}\, dz_2\right)} dz_1 \\
&= C_1 e^{-2\left(kz + \frac{a}{m_0}\cdot\frac{e^{-kz}}{-k}\right)} + e^{-2\left(kz + \frac{a}{m_0}\cdot\frac{e^{-kz}}{-k}\right)} 2g \int_0^z e^{2\left(kz_1 + \frac{a}{m_0}\cdot\frac{e^{-kz_1}}{-k}\right)} dz_1 \\
&= C_1 e^{2\left(\frac{ae^{-kz}}{m_0 k} - kz\right)} + e^{2\left(\frac{ae^{-kz}}{m_0 k} - kz\right)} 2g \int_0^z e^{2\left(kz_1 - \frac{ae^{-kz_1}}{m_0 k}\right)} dz_1
\end{aligned}
$$

Since $w(0) = 0$,

$$\begin{aligned} w(0) = 0 &= C_1 e^{2\left(\frac{ae^0}{m_0 k} - 0\right)} + e^{2\left(\frac{ae^0}{m_0 k} - 0\right)} 2g \int_0^0 e^{2\left(kz_1 - \frac{ae^{-kz_1}}{m_0 k}\right)} \, dz_1 \\ &= C_1 e^{\frac{2a}{m_0 k}} \end{aligned}$$

Then $C_1 = 0$.

$$\begin{aligned} w(z) &= e^{2\left(\frac{ae^{-kz}}{m_0 k} - kz\right)} 2g \int_0^z e^{2\left(kz_1 - \frac{ae^{-kz_1}}{m_0 k}\right)} \, dz_1 \\ v(z) &= \sqrt{2g} e^{\frac{ae^{-kz}}{m_0 k} - kz} \sqrt{\int_0^z e^{2\left(kz_1 - \frac{ae^{-kz_1}}{m_0 k}\right)} \, dz_1} \end{aligned}$$

Problem 5: K&K 3.20

Problem

A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u relative to the rocket. Assume that the rate at which mass is expelled is given by $dm/\, dt = \gamma m$, where m is the instantaneous mass of the rocket and γ is a constant. The rocket is retarded by air resistance with a force $F = bmv$ proportional to the instantaneous momentum of the rocket where b is a constant. Find the velocity of the rocket as a function of time.

Solution

$$
\begin{aligned}
-bmv - mg &= \frac{(m - \, dm)(v + \, dv) + \, dm(v-u) - mv}{dt} \\
&= m\frac{dv}{dt} - v\frac{dm}{dt} + \frac{dm}{dt}(v-u) \\
&= m\frac{dv}{dt} - u\gamma m \\
-bv - g &= \frac{dv}{dt} - u\gamma \\
\frac{dv}{dt} &= -bv + u\gamma - g \\
\frac{-1}{b}\frac{dv}{dt} &= v + \frac{g - u\gamma}{b} \\
dt &= \frac{-1}{b}\frac{dv}{v + \frac{g-u\gamma}{b}} \\
\int_0^t dt &= \frac{-1}{b}\int_0^v \frac{dv}{v + \frac{g-u\gamma}{b}} \\
t &= \frac{-1}{b}\left(\ln\left(v + \frac{g-u\gamma}{b}\right) - \ln\left(\frac{g-u\gamma}{b}\right)\right) \\
-bt &= \ln\left(\frac{bv + g - u\gamma}{g - u\gamma}\right) \\
e^{-bt} &= \frac{bv + g - u\gamma}{g - u\gamma} \\
bv + g - u\gamma &= (g - u\gamma)e^{-bt} \\
bv &= (g - u\gamma)e^{-bt} - g + u\gamma \\
v(t) &= \frac{g - u\gamma}{b}e^{-bt} + \frac{u\gamma - g}{b}
\end{aligned}
$$