MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group
Physics 8.012

Problem Set 8 Solutions

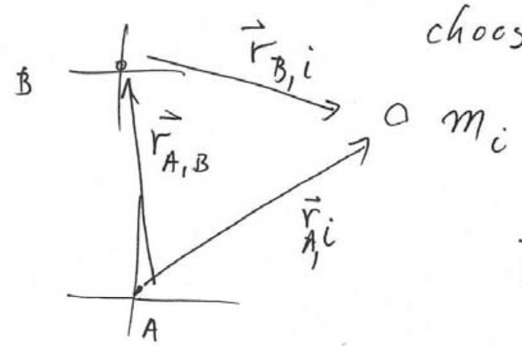
Problems: Week Nine Chapter 6: 1, 2, 4, 6, 7, 10, 13

Problem 1:

a) Show that if the total linear momentum of a system of particles is zero, the angular momentum of the system is the same about all origins. Explain how you may apply this result involving an elastic collision of two rigid bodies.

b) Show that if the total force on a system of particles is zero, the torque on the system is the same about all origins. Explain how you can use this result for static equilibrium problems.

6.1. a) Assume $\vec{P}^T = \sum_i m_i \vec{v}_i = 0$



choose two points A and B with $\vec{r}_{A,B}$ = constant vector

$$\vec{r}_{A,i} = \vec{r}_{A,B} + \vec{r}_{B,i}$$

$$\vec{L}_A = \sum_i \vec{r}_{A,i} \times m_i \vec{v}_i$$

$$= \sum_i (\vec{r}_{A,B} + \vec{r}_{B,i}) \times m_i \vec{v}_i$$

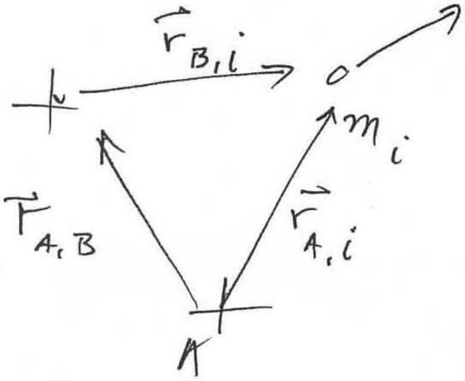
$$= \underbrace{\vec{r}_{A,B} \times \sum_i m_i \vec{v}_i}_{} + \sum_i \vec{r}_{B,i} \times m_i \vec{v}_i$$

Since $\vec{r}_{A,B}$ is the same for each m_i

Since $\sum m_i \vec{v}_i = 0$ and $\vec{L}_B = \sum \vec{r}_{B,i} \times m_i \vec{v}_i$

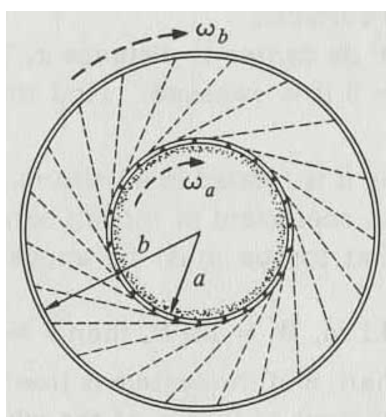
$$\vec{L}_A = \vec{L}_B \qquad \therefore$$

6.1 b) this argument will be identical just substitute $\vec{F}_i$ for $m_i \vec{v}_i$

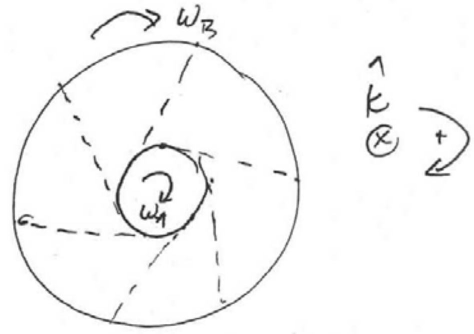


Problem 2:

A drum of mass m_A and radius a rotates freely with initial angular velocity $\omega_{A,0}$. A second drum with mass m_B and radius $b > a$ is mounted on the same axle and is at rest, although it is free to rotate. A thin layer of sand with mass m_S is distributed on the inner surface of the smaller drum. At $t = 0$, small perforations in the inner drum are opened. The sand starts to fly out at a constant rate λ and sticks to the outer drum. Find the subsequent angular velocities of the two drums ω_A and ω_B. Ignore the transit time of the sand.



chapter 6: Problem 2:



We shall apply conservation of angular momentum. The key idea is to decide what masses to put into our system.

When the sand leaves the inner drum through the hole, the sand does not exert any torque on the inner drum. Therefore the angular velocity of the inner drum remains constant. This means that we can ignore the (M_A) mass of the inner drum and just consider the sand and the outer drum

System: m_s and M_B

$$\vec{L}_O^{total} = m_s a^2 \omega_{A,0} \hat{k} + \text{zero}$$

$$\vec{L}_f^{\,total} = (m_s b^2 + m_B b^2)\,\omega_{B,f}\ \hat{k}$$

when the sand collides with the outer drum, then torque on the outer drum is an internal torque. There is no external torque on the system so

$$0 = \vec{\tau}_{ext} = \frac{d\vec{L}^{\,total}}{dt}$$

thus

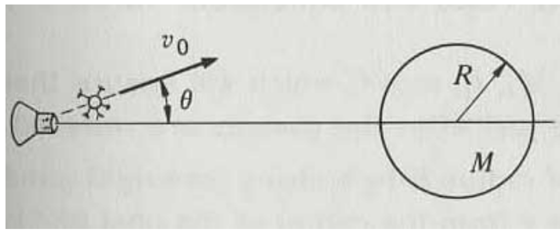
$$\vec{L}_0^{\,total} = \vec{L}_f^{\,total}$$

$$m_s a^2 \omega_{A,0} = (m_s + m_B) b^2 \omega_{B,f}$$

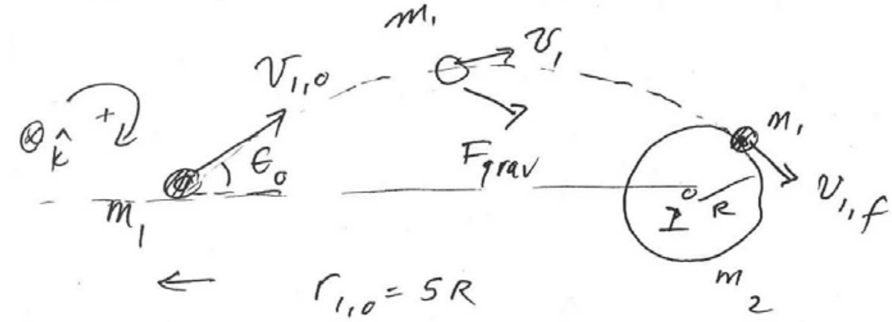
$$\omega_{B,f} = \frac{m_s a^2 \omega_{A,0}}{(m_s + m_B) b^2}$$

Problem 4:

A spaceship is sent to investigate a planet of mass m_p and radius r_p. While hanging motionless in space at a distance $5r_p$ from the center of the planet, the ship fires an instrument package with speed v_0. The package has mass m_i which is much smaller than the mass of the spacecraft. The package is launched at an angle θ with respect to a radial line between the center of the planet and the spacecraft. For what angle θ will the package just graze the surface of the planet.



Chapter 6 Problem 4

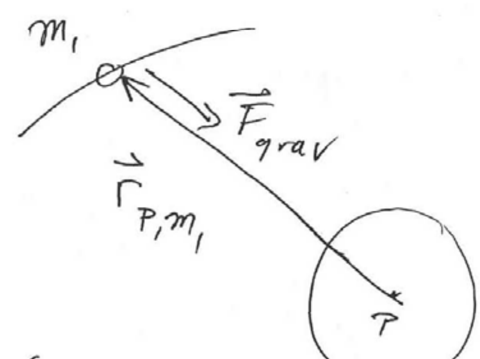


The gravitational force points towards the center of the planet (point P)

The torque about P

$$\vec{\tau}_P = \vec{r}_{P,m_1} \times \vec{F}_{grav,1} = 0$$

Since $\vec{r}_{P,m_1}$ points radially away from the planet



Therefore $\dfrac{d\vec{L}_P}{dt} = 0$

$$r_{1,0} = 5R$$

$$\vec{L}_{P,0} = m_1 \vec{r}_{1,0} \times \vec{v}_{1,0} = m_1 r_{1,0} v_{1,0} \sin\theta_0 \hat{k}$$

$$\vec{L}_{P,f} = m_1 \vec{r}_{1,f} \times \vec{v}_{1,f} = m_1 R v_{1,f} \hat{k}$$

$$m_1 5R v_{1,0} \sin\theta_0 = m_1 R v_{1,f}$$

$$\Rightarrow v_{1,f} = 5 v_{1,0} \sin\theta_0 \qquad (1)$$

The package satisfies conservation of energy

$$\frac{1}{2} m_1 v_{1,0}^2 - G\frac{m_1 m_2}{5R} = \frac{1}{2} m_1 v_{1,f}^2 - G\frac{m_1 m_2}{R} \qquad (2)$$

Substituting $v_{1,f}$ from eq (1) into eq (2) and canceling terms yields

$$v_{1,0}^2 - \frac{2}{5} G\frac{m_2}{R} = 25 v_{1,0}^2 \sin^2\theta_0 - 2\frac{G m_2}{R}$$

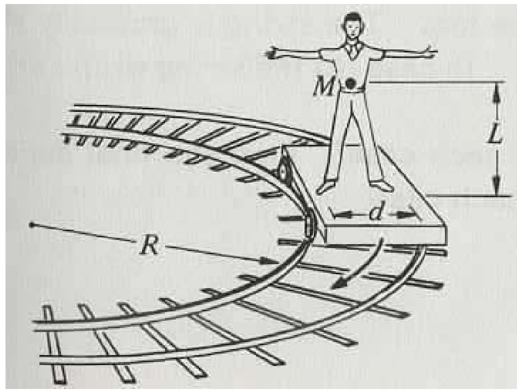
$$v_{1,0}^2 (1 - 25\sin^2\theta_0) = -\frac{4}{5}\frac{G m_2}{R} \Rightarrow$$

$$1 - 25\sin^2\theta_0 = -\frac{4}{5}\frac{G m_2}{R v_{1,0}^2}$$

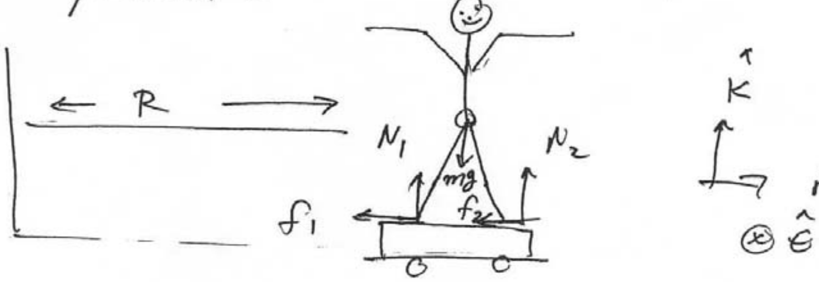
$$\left(\frac{1 + \frac{4}{5}\frac{G m_2}{R v_{1,0}^2}}{25}\right)^{1/2} = \sin\theta_0$$

Problem 6:

A person of mass m is standing on a railroad car which is rounding an unbanked turn of radius R at a speed v. His center of mass is at a height of L above the car midway between his feet which are separated by a distance of d. The man is facing the direction of motion. What is the magnitude of the normal forces on each foot?



Chapter 6 problem 6



the center of mass satisfies

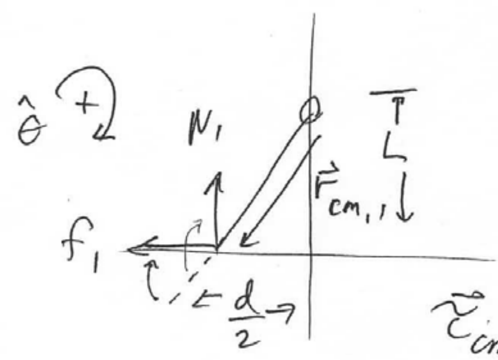
$$\vec{F}^{total} = m^{total}\,\vec{a}_{cm}$$

$\hat{r}$:
$$-(f_1 + f_2) = -\frac{m v^2}{R} \qquad (1)$$

$\hat{k}$:
$$N_1 + N_2 - mg = 0 \qquad (2)$$

the torque about the center of mass is zero since there is no angular acceleration

$\vec{\tau}_{cm} =$	$I_{cm}\,\vec{\alpha}$
$\vec{\tau}_{cm} =$	0 no rotation

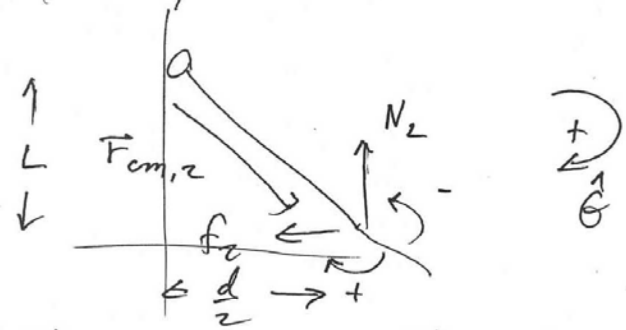


about the inner leg

$$\vec{\tau}_{cm,1} = \vec{r}_{cm,1} \times (\vec{f}_1 + \vec{N}_1)$$

$$\vec{\tau}_{cm,1} = f_1 L + N_1 \frac{d}{2}\ \hat{\theta}$$

the torque about the outer leg



$$\vec{\tau}_{cm,2} = \vec{r}_{cm,2} \times (\vec{f}_2 + \vec{N}_2)$$

$$= \left(Lf_2 - \frac{d}{2} N_2\right)\hat{\theta}$$

$$\vec{\tau}^{\,total}_{cm} = \left(f_1 L + N_1 \frac{d}{2} + Lf_2 - \frac{d}{2} N_2\right)\hat{\theta} = 0$$

$$L(f_1 + f_2) + \frac{d}{2}(N_1 - N_2) = 0 \qquad (3)$$

Using eq (1) for $f_1 + f_2 = \frac{mv^2}{R}$

$$\frac{L\,mv^2}{R} + \frac{d}{2}(N_1 - N_2) = 0$$

$$N_1 - N_2 = -\frac{2L}{d}\frac{mv^2}{R} \qquad (3a)$$

eq (2) $N_1 + N_2 = mg \qquad (2)$

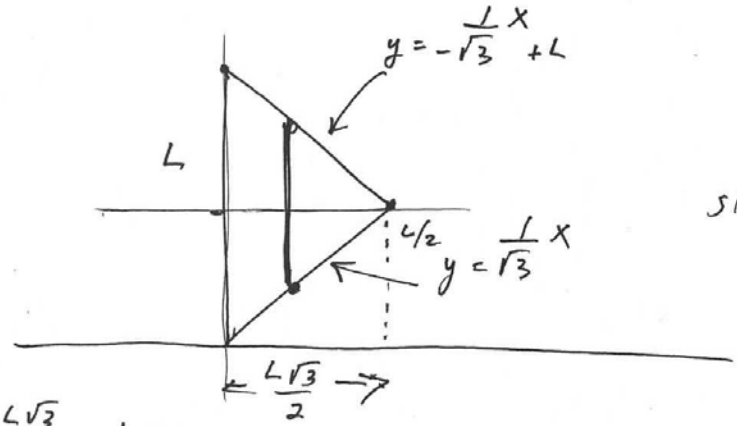
add and solve for $N_1 = \frac{1}{2}\left(mg - \frac{2L}{d}\frac{mv^2}{R}\right)$

subtract and solve for $N_2 = \frac{1}{2}\left(mg + \frac{2L}{d}\frac{mv^2}{R}\right)$

Problem 7:

a) Find the moment of inertia of a thin sheet of metal of mass m in the shape of an isosceles right triangle about an axis that passes through one vertex of the sheet, perpendicular to the plane of the sheet. The length of the two equal sides is s.

b) Find the moment of inertia of a thin sheet of metal of mass m in the shape of an isosceles right equilateral triangle about an axis that passes through the same vertex of the sheet, but aligned along one side of length s (in the plane of the sheet).



$$\text{slope} = \frac{\frac{L}{2}}{\frac{L\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\int_{x=0}^{\frac{L\sqrt{3}}{2}}\int_{y=\frac{1}{\sqrt{3}}x}^{-\frac{1}{\sqrt{3}}x+L} \sigma\, dx\, dy\, (x^2+y^2)$$

$$= \sigma\int_0^{L\sqrt{3}/2} dx\, \left(x^2y + \frac{y^3}{3}\right)\Bigg|_{\frac{1}{\sqrt{3}}x}^{-\frac{1}{\sqrt{3}}x+L}$$

$$= \sigma\int dx\, \left(x^2\left(-\frac{1}{\sqrt{3}}x + L - \frac{1}{\sqrt{3}}x\right) + \left(\left(-\frac{1}{\sqrt{3}}x+L\right)^3 - \frac{1}{\sqrt{3}}x^3\right)\right)$$

$$= \sigma\int dx\, \left(\left(-\frac{2}{\sqrt{3}}x^3 + Lx^2\right) + \left(-\frac{2}{3\sqrt{3}}x^3 + x^2L - \frac{3}{\sqrt{3}}L^2x + L^3\right)\right)$$

$$= \sigma\int dx\left(\frac{-8}{3\sqrt{3}}x^3 + 2Lx^2 - \frac{3}{\sqrt{3}}L^2x + L^3\right)$$

$$= \sigma\left(\frac{-8}{3\sqrt{3}}\,\frac{x^4}{4} + 2L\frac{x^3}{3} - \frac{3}{\sqrt{3}}\,\frac{L^2x^2}{2} + L^3x\right)\Bigg|_0^{\frac{L\sqrt{3}}{2}}$$

$$= \sigma\left(\frac{-8}{3\sqrt{3}} \frac{\left(\frac{L\sqrt{3}}{2}\right)^4}{4} + \frac{2L}{3}\left(\frac{L\sqrt{3}}{2}\right)^3 - \frac{3}{\sqrt{3}} \frac{L^2\left(\frac{L\sqrt{3}}{2}\right)^2}{2} + L^3 \frac{L\sqrt{3}}{2} \right)$$

$$= \sigma\left(\frac{-8\sqrt{3}}{3} \frac{L^4}{64} + \frac{2}{3} L \frac{L^3 3\sqrt{3}}{8} - \sqrt{3} L^2 \frac{L^2 3}{8} + \frac{L^4\sqrt{3}}{2} \right)$$

$$= \sigma \frac{L^4\sqrt{3}}{3}\left(\frac{-8}{64} + \frac{6}{8} - \frac{9}{8} + \frac{3}{2} \right)$$

$$= \sigma \frac{L^4\sqrt{3}}{3}\left(1 \right)$$

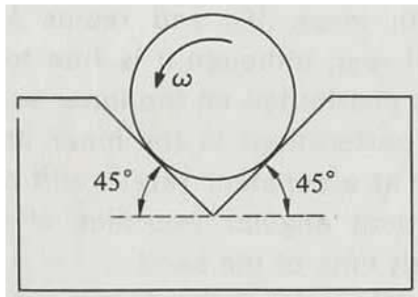
Since $\sigma = \dfrac{4m}{L^2\sqrt{3}}$

$$I_{P,z} = \left(\frac{4m}{L^2\sqrt{3}}\right)\left(\frac{L^4\sqrt{3}}{3}\right)$$

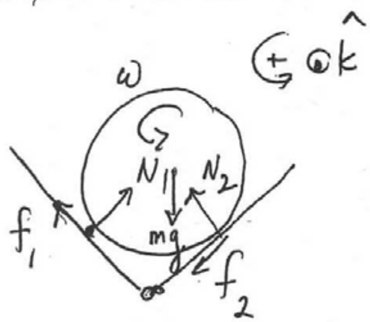
$$= \frac{4}{3} m L^2$$

Problem 10:

A cylinder of mass m and radius R is rotated in a V groove with constant angular velocity ω_0. The coefficient of friction between the cylinder and the surface is μ. What external torque must be applied to the cylinder to keep it rolling?



Chapter 6: Problem 10



If the cylinder moves at constant angular velocity then the total torque about the center of mass is zero

$$\vec{\tau}_{cm,applied} + \vec{\tau}_{cm,1} + \vec{\tau}_{cm,2} = 0$$

where $\vec{\tau}_{cm,1} = \vec{r}_{cm,1} \times \vec{F}_1 = -Rf_1\,\hat{k}$

$\vec{\tau}_{cm,2} = \vec{r}_{cm,2} \times \vec{f}_2 = -Rf_2\,\hat{k}$

The center of the mass is at rest so

$$\vec{F}^{total}_{cm} = m^{total}\vec{a}_{cm} = 0$$

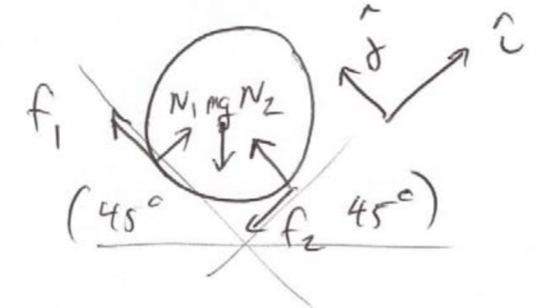
The torque equation

$$\vec{\tau}_{cm,applied} - R(f_1+f_2)\,\hat{k} = 0$$

or $$\vec{\tau}_{cm,applied} = R(f_1+f_2)\,\hat{k}$$

The force equations are noting that

$\cos 45^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$



$$\hat{i}: \quad N_1 - f_2 - mg\frac{\sqrt{2}}{2} = 0$$

$$\hat{j}: \quad N_2 + f_1 - mg\frac{\sqrt{2}}{2}$$

the force law for friction is

$$f_1 = \mu N_1 \qquad f_2 = \mu N_2$$

So the force eq's become

$$N_1 - \mu N_2 = mg\frac{\sqrt{2}}{2} \qquad (2a)$$

$$N_2 + \mu N_1 = mg\frac{\sqrt{2}}{2} \qquad (3a)$$

we first solve these equations for N_1

eq (3a) $\Rightarrow$ $N_2 = mg\frac{\sqrt{2}}{2} - \mu N_1$

Substitute into eq (2a) $\Rightarrow$

$$N_1 - \mu\left(mg\frac{\sqrt{2}}{2} - \mu N_1\right) = mg\frac{\sqrt{2}}{2}$$

$$N_1(1+\mu^2) = mg\frac{\sqrt{2}}{2}(1+\mu)$$

$$N_1 = \frac{\sqrt{2}}{2}mg\frac{(1+\mu)}{(1+\mu^2)}$$

Similarly solve eq (2a) for N_1 and substitute into eq (3a)

$$\mu\left(\mu N_2 + \frac{\sqrt{2}}{2} mg\right) + N_2 = \frac{\sqrt{2}}{2} mg$$

yielding

$$N_2 (1+\mu^2) = \frac{\sqrt{2}}{2} mg (1-\mu)$$

Thus $N_1 = \frac{\sqrt{2}}{2} \frac{(1+\mu)}{1+\mu^2}$

$$N_2 = \frac{\sqrt{2}}{2} \frac{(1-\mu)}{1+\mu^2}$$

Hence

$$N_1 + N_2 = \frac{\sqrt{2}}{2} \frac{1}{(1+\mu^2)} \left((1+\mu) + (1-\mu)\right) = \frac{\sqrt{2}}{1+\mu^2}$$

$$f_1 + f_2 = \mu (N_1 + N_2) = \frac{\mu \sqrt{2}}{1+\mu^2}$$

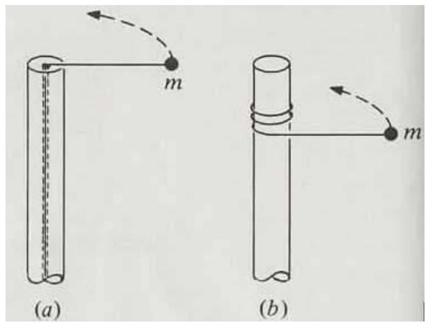
Finally

$$\vec{\tau}_{cm, applied} = R (f_1 + f_2) \hat{k}$$

$$\vec{\tau}_{cm, applied} = \frac{R \mu \sqrt{2}}{1+\mu^2} \hat{k}$$

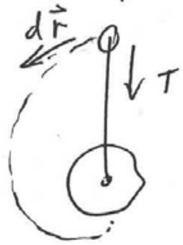
Problem 13:

A body of particle of mass m (treat it as a point like particle) is attached to a post of radius R by a string. Initially it is a distance r_0 from the center of the post and it is moving tangentially with a speed v_0. In case (a) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (b) the string wraps around the outside of the post. What quantities remain constant in each case? Find the final speed of the body when it hits the post for each case.



Chapter 6 problem 13

a)



Since $\vec{T} \cdot d\vec{r} = dW_{n.c} \neq 0$

Energy is not conserved

$\vec{\tau}_{cm} = \vec{r}_{cm,m} \times \vec{T} = 0$

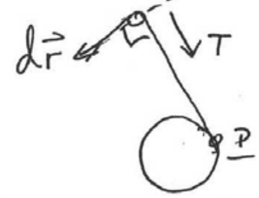
because $\vec{T}$ is a radial force

so $\frac{d\vec{L}_{cm}}{dt} = 0$, $\vec{L}_{cm}$ is conserved.

Momentum is not conserved because

$\vec{F}_{ext} = \vec{T} \neq 0$.

b)



$dW_{n.c} = \vec{T} \cdot d\vec{r} = 0$

because about the point P, the mass is instantaneously moving in a circular orbit so

$\vec{T} \perp d\vec{r}$

thus Energy is conserved.

As the mass moves, $\vec{T}$ does not point towards one specific point so there is no point in which

$\tau_p \neq 0$ for all P

Thus $\frac{d\vec{L}_p}{dt} \neq 0$ and angular momentum is not conserved.

As in part a) momentum is not conserved since $\vec{F}_{ext} = \vec{T} \neq 0$.

part a) Since $\vec{L}_{cm}$ is conserved

$$\vec{L}_{cm,0} = \vec{r}_{cm,0} \times m\vec{v}_0$$

$$\vec{v}_0 = m v_0 \hat{\theta} \quad , \quad \vec{r}_{cm,0} = r_0 \hat{r}$$

$$\vec{L}_{cm,0} = r_0 \hat{r} \times m v_0 \hat{\theta} = m r_0 v_0 \hat{k}$$

$$\vec{L}_{cm,f} = r_0 \hat{r} \times m v_f \hat{\theta} = m r_f v_f \hat{k}$$

since $v_{f,radial} = 0$ (hits post)

(1) Thus $\vec{L}_{cm,0} = \vec{L}_{cm,f} \Rightarrow$

$$m r_0 v_0 = m r_f v_f \Rightarrow v_f = \frac{r_0 v_0}{r_f}$$

part b) $E_0 = E_f \Rightarrow$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_f^2 \Rightarrow v_0 = v_f$$