MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012, Fall 2010

Problem Set 1

Handed out: September 8 **Due: Friday, September 17**

Reading: Chapter One: Kleppner and Kolenkow, *An Introduction to Mechanics*

Problem 1: Fermi Problem (This problem is hard and should be a challenge.)

Problem

One of the moons of Jupiter, Europa, is reported to have its surface covered by an ocean of water which is 100 km deep. The outermost 8 km are frozen as ice. The radius of Europa is approximately 1/4 the radius of the earth. Estimate the pressure at the bottom of Europa's ocean. (Note: there is some speculation that the combination of internal heat and water makes the ocean of Europa the best candidate in the solar system outside the earth for organized life to evolve.)

Solution

Pressure at bottom of ocean? Force / Area. $(7.5 \cdot 10^{21}$ N$)/(2.7 \cdot 10^{13}$ m$^2) \approx 3 \cdot 10^8$ N / m^2

- Force: Mass $\cdot$ Acceleration (due to gravity). $(3 \cdot 10^{21}$ kg$) \cdot (2.5$m / s$^2) \approx 7.5 \cdot 10^{21}$ N
  - Mass: density $\cdot$ volume. $(10^{12}$ kg / km$^3) \cdot (3 \cdot 10^9$ km$^3) \approx 3 \cdot 10^{21}$ kg
    - Density: the density of ice is about the same as that of water, 1 g / cm^3, or 10^{12} kg / km^3
    - Volume: The volume is the difference of the volumes of the spheres, or $\frac{4}{3}\pi(R^3 - r^3)$, or (approximating π as 3), $4((1600 \text{ km})^3 - (1600 \text{ km})^3) \approx 3 \cdot 10^9$ km^3
  - Acceleration due to gravity: Since mass is linear in volume, and volume is cubic in radius, Europa has about $\frac{1}{4^3}$ the mass of earth, and since the force of gravity is linear in mass / the square of the radius, the force of gravity on Europa is about $\frac{4^2}{4^3} = \frac{1}{4}$ that on earth, so it's about 2.5 m/s^2 Alternatively, by dimensional analysis, the acceleration due to gravity is linear in m^{-1}, so the acceleration due to gravity on Europa is about $\frac{1}{4}$ that of earth.
- Area: $4\pi r^2$. Let $\pi \approx 3$, so area is about $2.7 \cdot 10^{13}$ m^2
  - Radius: $\frac{1}{4}$ radius of earth, less 100 km, so about 1500 km (since, according to google, the radius of earth is 6378.1 km)

Alternatively: Pressure $= \frac{F}{A} = \frac{mg}{A} = \frac{\rho V g}{A} = \frac{\rho h A g}{A} = \rho h g = (10^{12}$ kg / km$^3)(100$ km$)(2.5$ m / s$^2) \approx 3 \cdot 10^8$ N / m^2

- Density: the density of ice is about the same as that of water, 1 g / cm^3, or 10^{12} kg / km^3
- Height: the height is about 100 km
- Acceleration due to gravity: Since mass is linear in volume, and volume is cubic in radius, Europa has about $\frac{1}{4^3}$ the mass of earth, and since the force of gravity is linear in mass / the square of the radius, the force of gravity on Europa is about $\frac{4^2}{4^3} = \frac{1}{4}$ that on earth, so it's about 2.5 m/s^2 Alternatively, by dimensional analysis, the acceleration due to gravity is linear in m^{-1}, so the acceleration due to gravity on Europa is about $\frac{1}{4}$ that of earth.

Problem 2: Kinematics-One Dimension

Problem

A bus leaves a stop at MIT and accelerates at a constant rate for 5 seconds. During this time the bus traveled 25 meters. Then the bus traveled at a constant speed for 15 seconds. Then the driver noticed a red light 18 meters ahead and slams on the brakes. Assume the bus decelerates at a constant rate and comes to a stop some time later just at the light.

a) What was the initial acceleration of the bus?

b) What was the velocity at the bus after 5 seconds?

c) What was the braking acceleration of the bus? Is it positive or negative?

d) How long did the bus brake?

e) What was the distance from the bus stop to the light?

f) Make a graph of the position vs. time for the entire trip.

g) Make a graph of the velocity vs. time for the entire trip.

h) Make a graph of the acceleration vs. time for the entire trip.

Solution

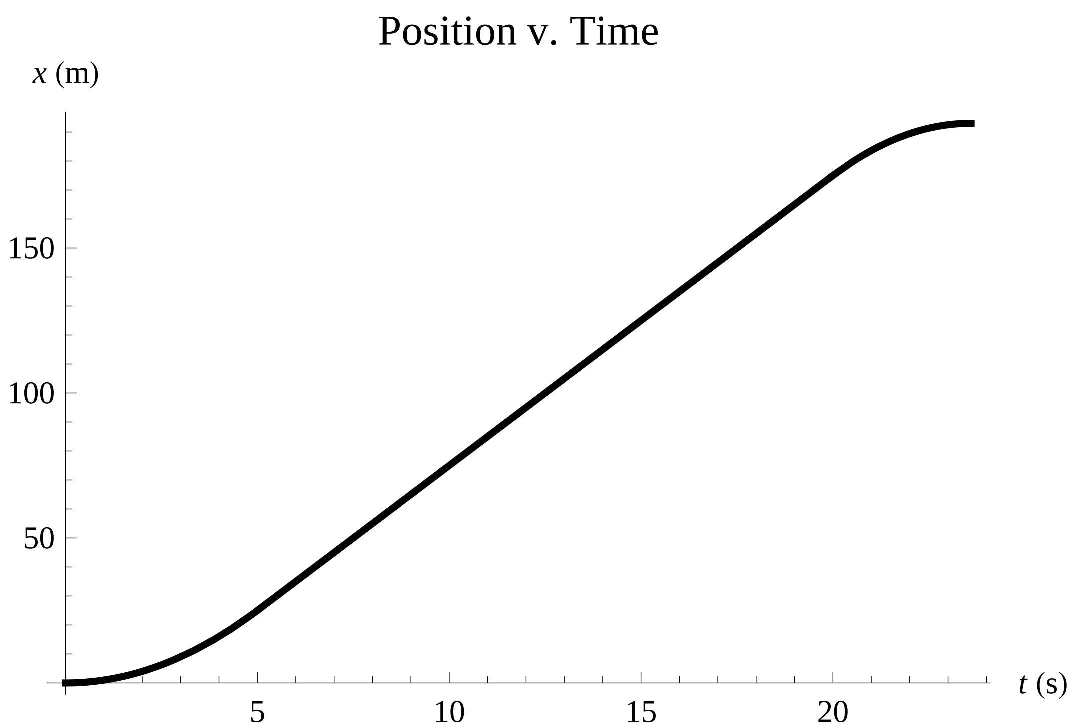
a) Since $\displaystyle\int_0^5 \int_0^t at't = 25$ m, $\frac{1}{2}a \cdot 5^2 = 25$, so 2 m / s^2.

b) 10 m / s

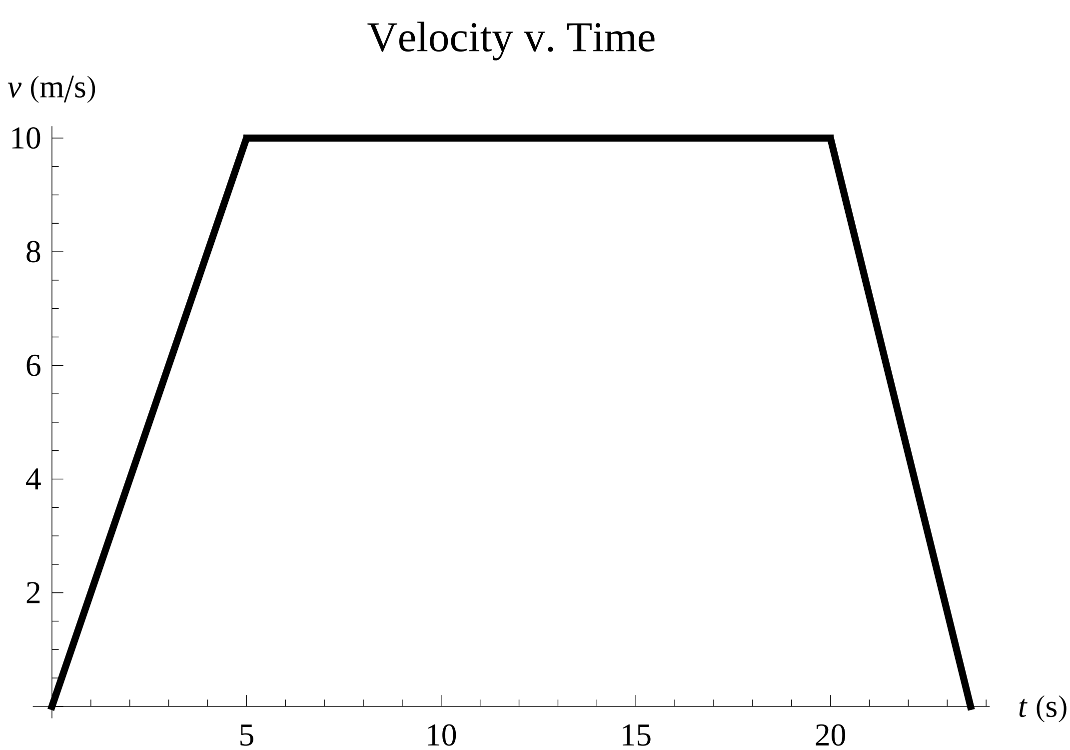
c) $\frac{1}{2}at^2 = 18$ m, and $at = 10$ m / s, so $t = 3.6$ s, and $a = 2.\overline{7}$ m / s^2. It is negative.

d) 3.6 s

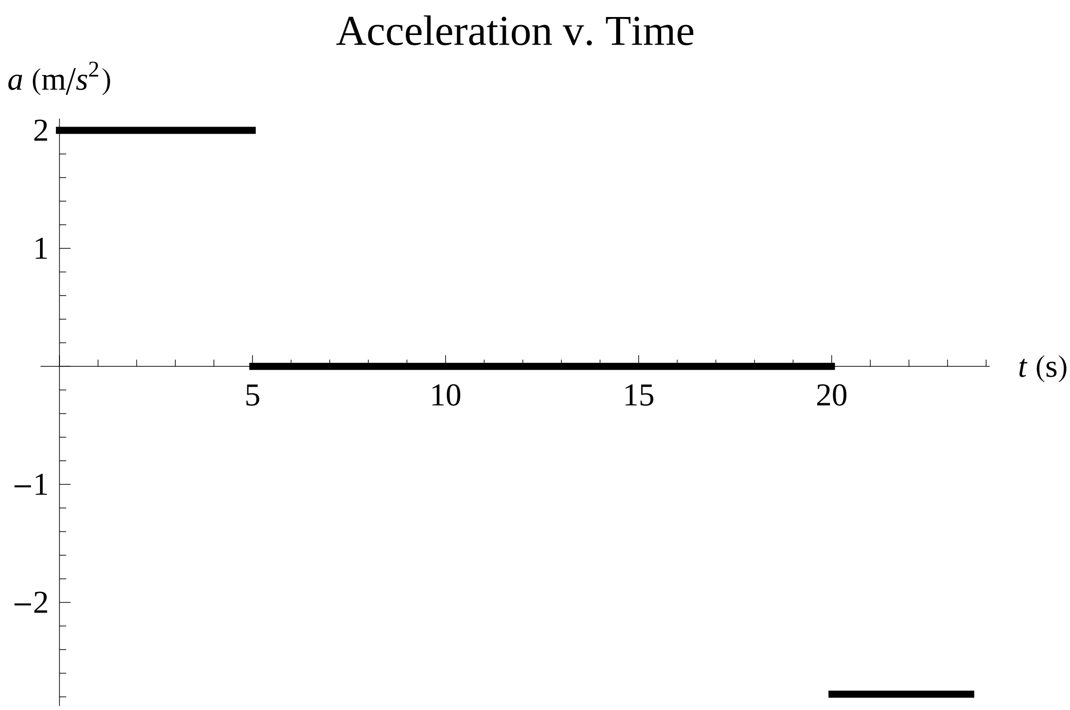
e) $25 + 10 \cdot 15 + 18 = 193$ m


Position v. Time
x (m)
150
100
50
t (s)
5
10
15
20

f)


Velocity v. Time
v (m/s)
10
8
6
4
2
t (s)
5
10
15
20

g)

h) 

Problem 3: Kinematics-One Dimension

Problem

You are a running as fast as you can at a constant velocity, v_p, trying to catch a bus that is at rest at a bus stop. When you are still a distance b away from the bus stop, the bus starts to accelerate at a constant rate a_{bus}.

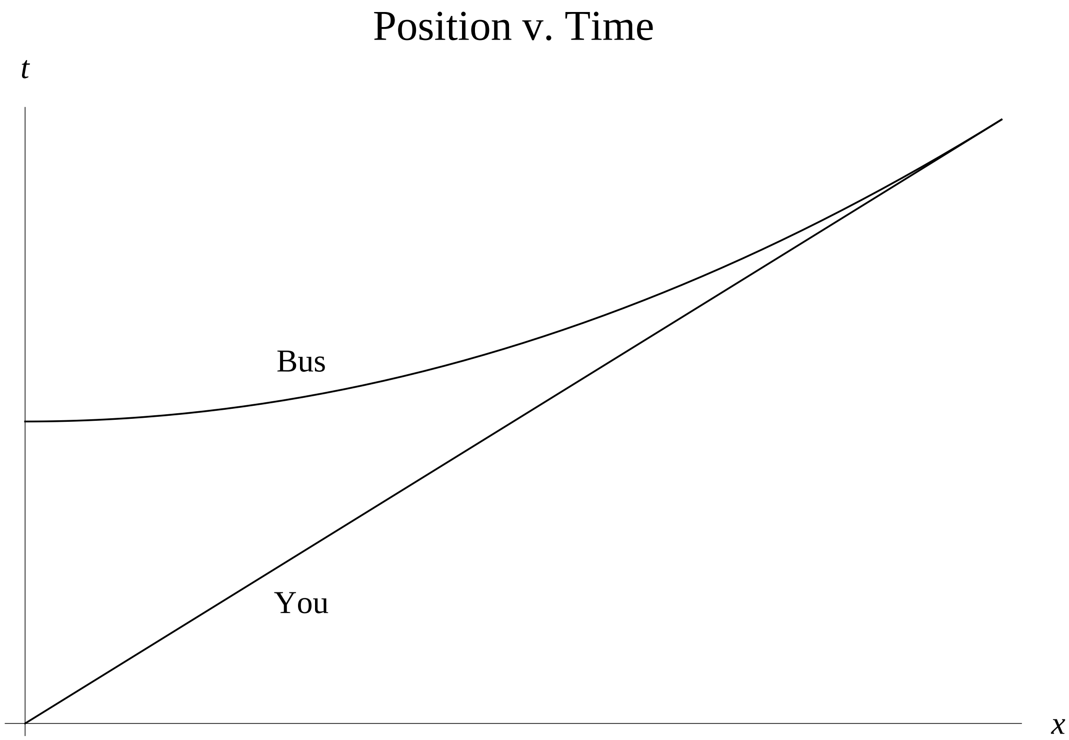
a) What is the minimum velocity that you need to run at in order to just catch the bus?

b) Draw graphs showing the motion of the bus and yourself.

c) How long did it take to catch the bus?

Solution

a) If you just catch the bus after time t, then $v_p = a_{bus}t$. If you catch the bus after time t, $v_pt = b + \frac{1}{2}a_{bus}t^2 = b + \frac{1}{2}v_pt$, so $t = \sqrt{2\frac{b}{a_{bus}}}$. Then $v_p = \sqrt{2ba_{bus}}$.

b) 

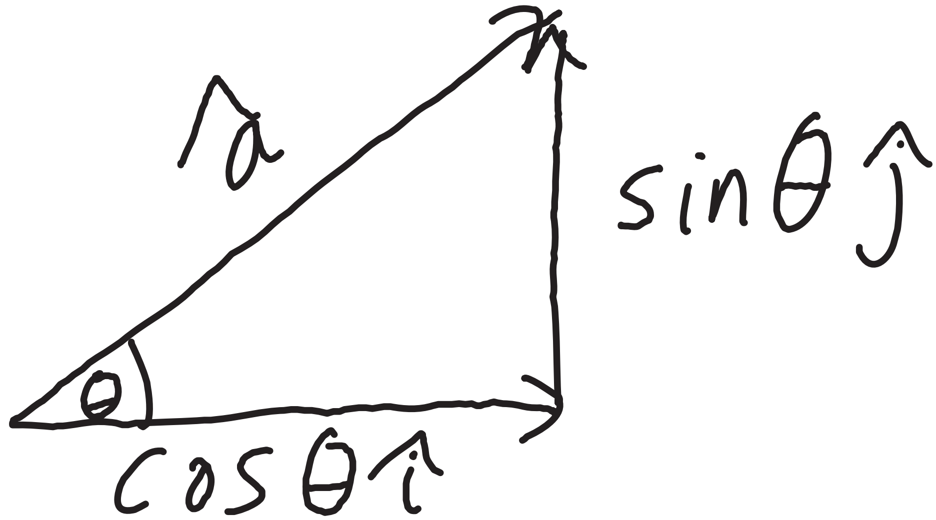
c) $\sqrt{2\frac{b}{a_{bus}}}$

Problem 4: K&K 1.7

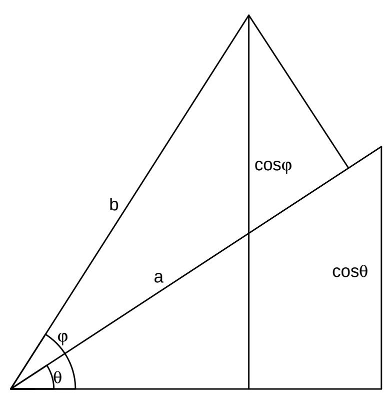
Problem

Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the xy plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$, and using vector algebra prove that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Solution



As seen in the diagram, by the definition of cos and sin, $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$. By relabeling, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$.



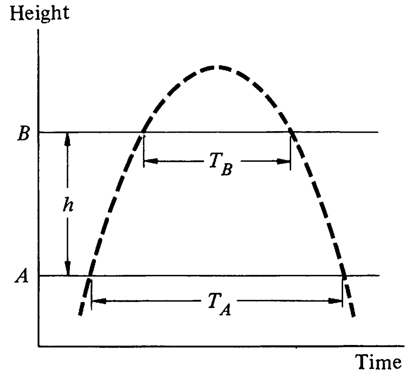
By the definition of cos, $\cos(\phi - \theta) = \vec{a} \cdot \vec{b}$ (since $\vec{a}$ and $\vec{b}$ are unit vectors). Then $\cos(\theta - \phi) = \cos(\phi - \theta) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Problem 5: K&K 1.12

Problem

The acceleration of gravity can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions. Show that if the time the body takes to pass a horizontal line A in both directions is T_A , and the time to go by a second line B is T_B, then, assuming that the acceleration is constant, its magnitude is

$$g = \frac{8h}{T_A^2 - T_B^2}$$



Solution

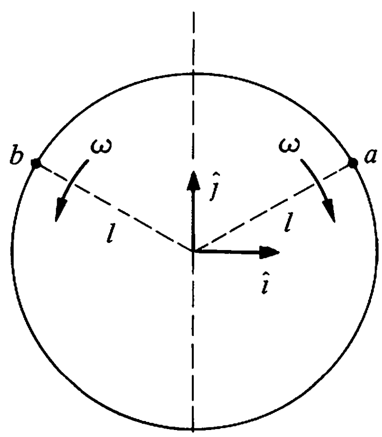
Since acceleration is constant downwards, $\frac{\partial^2 y}{\partial t^2} = g$. Then $y(t) = y_0 + v_0 t + \frac{1}{2}gt^2$. It is given that for some t, $y\left(t + \frac{T_B}{2}\right) = y\left(t - \frac{T_B}{2}\right)$, and $y\left(t + \frac{T_A}{2}\right) - y\left(t + \frac{T_B}{2}\right) = h$. Translate the function such that this $t = 0$. Then $v_0 \frac{T_B}{2} + \frac{1}{2}g\left(\frac{T_B}{2}\right)^2 = -v_0 \frac{T_B}{2} + \frac{1}{2}g\left(\frac{T_B}{2}\right)^2$. Equivalently, $v_0 = 0$.

Then, the other conditions states that $\frac{1}{2}g\left(\left(\frac{T_A}{2}\right)^2 - \left(\frac{T_B}{2}\right)^2\right) = h$. Solving for g, $g = \frac{8h}{T_A^2 - T_B^2}$.

Problem 6: K&K 1.15

Problem

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.)



a. A point is observed to have velocity $\vec{v}_A$ relative to coordinate system A. What is its velocity relative to coordinate system B, which is displaced from system A by distance $\vec{R}$? ($\vec{R}$ can change in time.)

b. Particles a and b move in opposite directions around a circle with angular velocity ω, as shown. At $t = 0$ they are both at the point $\vec{r} = l\hat{j}$, where l is the radius of the circle. Find the velocity of a relative to b.

Solution

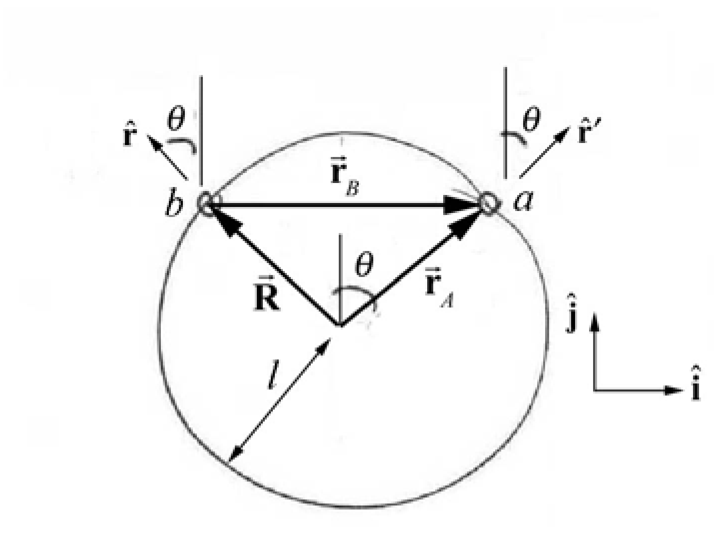
a. The position vectors are related by

$$\vec{r}_B = \vec{r}_A - \vec{R}.$$

Then velocities are related by the taking derivatives, (law of addition of velocities)

$$\vec{v}_B = \vec{v}_A - \vec{V}.$$

b. Let's choose two reference frames; frame B is centered at particle b, and frame A is centered at the center of the circle in the figure below.



Then the relative position vector between the origins of the two frames is given by

$$\vec{R} = l\hat{r}.$$

The position vector of particle a relative to frame A is given by

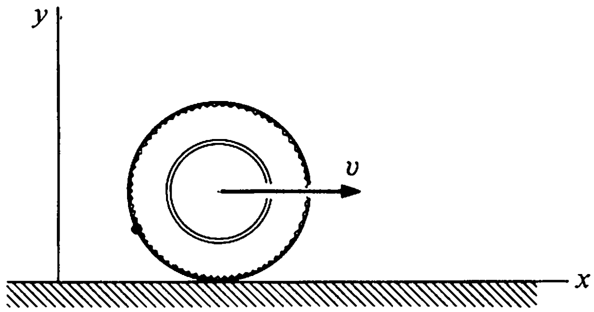
$$\vec{r}_A = l\vec{r}'.$$

The position vector of particle b in frame B can be found by substituting Eqs. (1.4) and (1.3) into Eq. (1.1), B A r = r ?R = l r? ? l r ? ? ? . (1.5) We can decompose each of the unit vectors r and ? r with respect to the Cartesian unit vectors i and j (see figure) r = ?sin? i + cos? j (1.6) r? = sin? i + cos? j . (1.7) Then Eq. (1.5) giving the position vector of particle b in frame B becomes (sin cos) (sin cos) 2 sin B r? = l r? ? l r = l ? i + ? j ? l ? ? i + ? j = l ? i . (1.8) In order to find the velocity vector of particle a in frame B (i.e. with respect to particle b), differentiate Eq. (1.8) (2 sin) (2 cos) 2 cos B $\vec{a} = l\hat{r}_a$, $\vec{b} = l\hat{r}_b$. $\hat{r}_a = \cos(\omega t)\hat{j} + \sin(\omega t)\hat{i}$ and $\hat{r}_b = \cos(\omega t)\hat{j} - \sin(\omega t)\hat{i}$. The velocity of a relative to b, $\vec{a}_b$, is $\frac{\partial \vec{v}_A}{\partial t} - \frac{\partial \vec{b}}{\partial t} = l\left(\frac{\partial \hat{r}_a}{\partial t} - \frac{\partial \hat{r}_b}{\partial t}\right)$. Since it is circular motion, $\frac{\partial \hat{r}_b}{\partial t} = -\omega(\sin(\omega t)\hat{j} + \cos(\omega t)\hat{i})$ and $\frac{\partial \hat{r}_a}{\partial t} = \omega(-\sin(\omega t)\hat{j} + \cos(\omega t)\hat{i})$. Then $\vec{a}_b = l\omega\left(-\sin(\omega t)\hat{j} + \cos(\omega t)\hat{i} + \sin(\omega t)\hat{j} + \cos(\omega t)\hat{i}\right) = 2l\omega\cos(\omega t)\hat{i}$.

Problem 7: K&K 1.19

Problem

A bicycle wheel of radius a is rolling in a straight line without slipping at a constant horizontal velocity V. A bead is fixed to a spoke a distance b from the center of the wheel.



a) Find the position and velocity of the bead as a function of time as seen by an observer located at the center of the wheel and moving with the wheel. Make sure you use appropriate unit vectors in your answer.

b) What is the position and velocity of the observer at the center of the wheel as seen by an observer fixed to the ground. Assume at $t = 0$ that the center of the wheel is directly over the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

c) What is the relation between the angular velocity of the wheel, ω, and the horizontal velocity, V, of the wheel?

d) Find the position and velocity of the bead as a function of time as seen by the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

Solution

a) The angular velocity of the wheel, $\omega = 2\pi \frac{V}{2\pi a} = \frac{V}{a}$. Then the position of the bead is given by $\vec{r} = b\cos\left(\frac{V}{a}t\right)\hat{\imath} + b\sin\left(\frac{V}{a}t\right)\hat{\jmath}$. $\frac{\partial \vec{r}}{\partial t} = -b\frac{V}{a}\sin\left(\frac{V}{a}t\right)\hat{\imath} + b\frac{V}{a}\cos\left(\frac{V}{a}t\right)\hat{\jmath}$

b) $\frac{\partial \vec{x}}{\partial t} = V\hat{\imath}$ and $\vec{x} = Vt\hat{\imath}$

c) $\omega = 2\pi \frac{V}{2\pi a} = \frac{V}{a}$

d) $\vec{r}_G = \left(b\cos\left(\frac{V}{a}t\right) + Vt\right)\hat{\imath} + b\sin\left(\frac{V}{a}t\right)\hat{\jmath}$ and $\frac{\partial \vec{r}_G}{\partial t} = \left(-b\frac{V}{a}\sin\left(\frac{V}{a}t\right) + V\right)\hat{\imath} + b\frac{V}{a}\cos\left(\frac{V}{a}t\right)\hat{\jmath}$

Problem 8: K&K 1.20

Problem

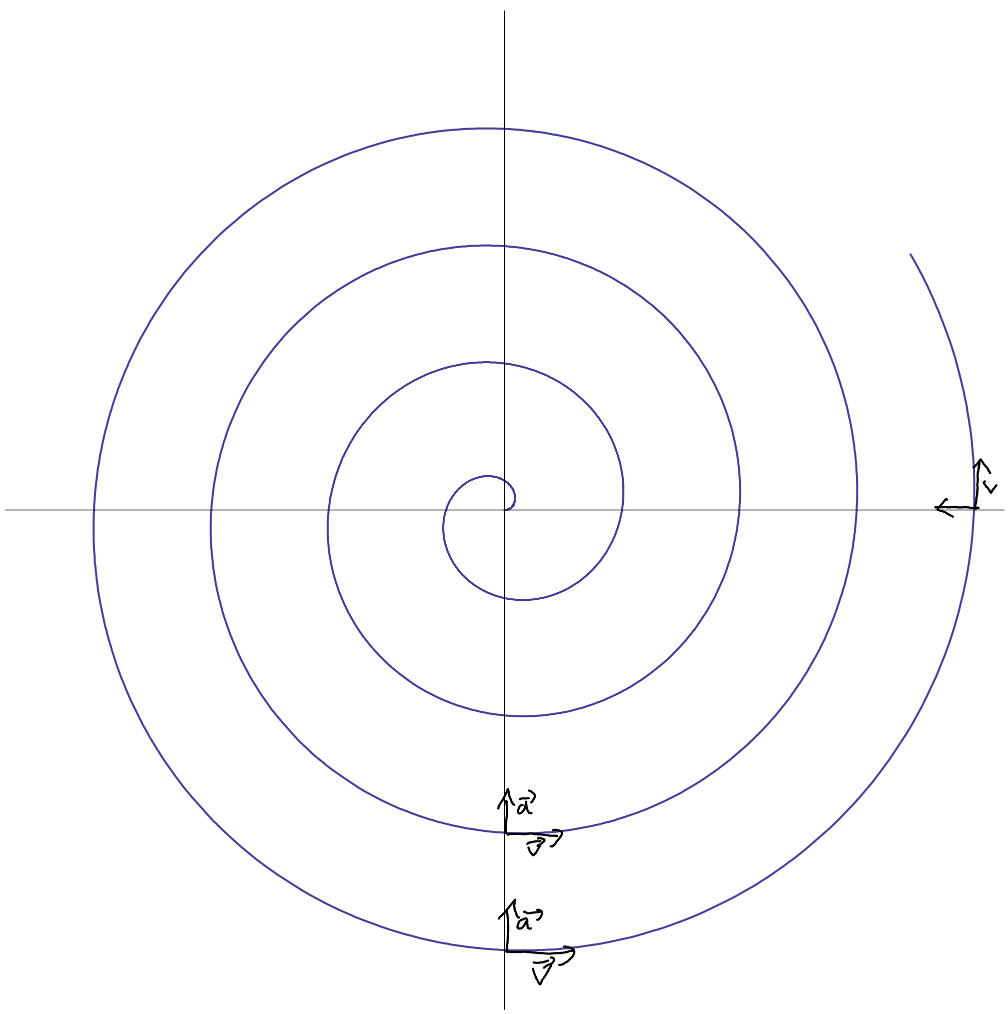
A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant, $A = (1/\pi)$ m $\cdot$ rad^{-1}. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

a. Sketch the motion, and indicate the approximate velocity and acceleration at a few points.

b. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

c. At what angles do the radial and tangential accelerations have equal magnitude?

Solution



a.

b.

$$\vec{r}(t) = A\frac{\alpha t^2}{2}\hat{r}$$
$$\vec{\theta}(t) = \frac{\alpha t^2}{2}\hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial t} = \alpha t\hat{\theta}$$
$$\frac{\partial \hat{\theta}}{\partial t} = -\alpha t\hat{r}$$

$$\begin{aligned}\frac{\partial \vec{r}}{\partial t} &= \frac{A\alpha}{2}\left(2t\hat{r} + t^2\frac{\partial \hat{r}}{\partial t}\right)\\ &= \frac{A\alpha}{2}\left(2t\hat{r} + \alpha t^3\hat{\theta}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \vec{r}}{\partial t^2} &= \frac{A\alpha}{2}\frac{\partial}{\partial t}\left(2t\hat{r} + \alpha t^3\hat{\theta}\right)\\ &= \frac{A\alpha}{2}\left(2\hat{r} + 2t\frac{\partial \hat{r}}{\partial t} + 3\alpha t^2\hat{\theta} + \alpha t^3\frac{\partial \hat{\theta}}{\partial t}\right)\\ &= \frac{A\alpha}{2}\left(2\hat{r} + 2t\alpha t\hat{\theta} + 3\alpha t^2\hat{\theta} + \alpha t^3(-\alpha t\hat{r})\right)\\ &= \frac{A\alpha}{2}\left(2\hat{r} + 2\alpha t^2\hat{\theta} + 3\alpha t^2\hat{\theta} - \alpha^2 t^4\hat{r}\right)\\ &= \frac{A\alpha}{2}\left(\left(2 - 4\left(\frac{\alpha t^2}{2}\right)^2\right)\hat{r} + 5\alpha t^2\hat{\theta}\right)\\ &= A\alpha\left(\left(1 - 2\theta^2\right)\hat{r} + 5\theta\hat{\theta}\right)\\ &= A\alpha\left(\left(1 - 2\cdot\frac{1}{2}\right)\hat{r} + \frac{5}{\sqrt{2}}\hat{\theta}\right)\\ &= \frac{5A\alpha}{\sqrt{2}}\hat{\theta}\end{aligned}$$

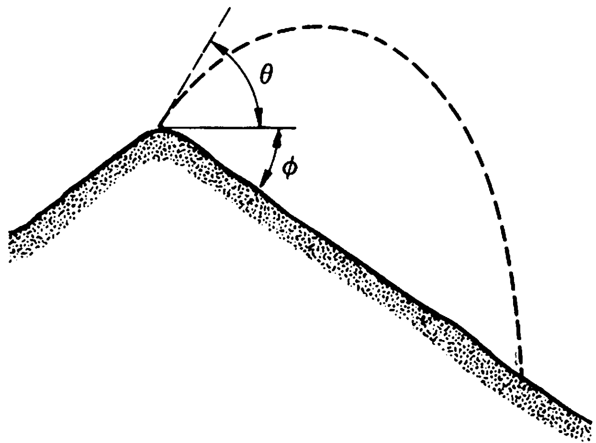
Since all of the coefficient of $\hat{r}$ is 0, the radial acceleration is 0.

c. The radial and tangential accelerations have equal magnitudes if $1 - 2\theta^2 = 5\theta$. That is, $\theta \equiv \frac{-5\pm\sqrt{33}}{4} \pmod{2\pi}$.

Problem 9: K&K 1.21

Problem

A person throws a rock from the top of a hill. The hill slopes downward uniformly at angle ϕ. At what angle θ from the horizontal should the person throw the rock so that it has the greatest range?



Solution

The height of the hill, $h_h(x)$, is $-x\tan\phi$. The acceleration of the rock, $\frac{\partial^2\vec{r}(t)}{\partial t^2}$, is $-g\hat{j}$. The velocity of the rock, $\frac{\partial\vec{r}(t)}{\partial t}$, is $(v_0\sin\theta-gt)\hat{j}+v_0\cos\theta\hat{i}$. The position of the rock, $\vec{r}(t)$, is $\left(v_0t\sin\theta-\frac{1}{2}gt^2\right)\hat{j}+v_0t\cos\theta\hat{i}$. The rock stops when $v_0t\sin\theta-\frac{1}{2}gt^2=-v_0t\cos\theta\tan\phi$, or $v_0\sin\theta-\frac{1}{2}gt=-v_0\cos\theta\tan\phi$. Solving for t, $t=\frac{2v_0\sin\theta+2v_0\cos\theta\tan\phi}{g}$. The maximal range occurs when $v_0t\cos\theta=v_0\frac{2v_0\sin\theta\cos\theta}{g}+v_0\frac{2v_0\cos^2\theta\tan\phi}{g}=\frac{v_0^2\sin 2\theta+2v_0^2\cos^2\theta\tan\phi}{g}$ is maximal, which occurs when $\frac{2v_0^2\cos 2\theta+4v_0^2\cos\theta\sin\theta\tan\phi}{g}=0$, or $\cos 2\theta\cos\phi+\sin 2\theta\sin\phi=\cos(\phi-2\theta)=0$. This is true when $\phi-2\theta=2k\pi\pm\frac{\pi}{4}$ for $k\in\mathbb{Z}$. Then, the optimal angle is $\frac{\phi}{2}-\frac{\pi}{8}$.