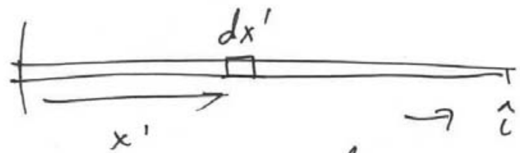
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group Physics 8.012

Problem Set 4 Solutions

Problem 1:

The density of a thin rod of length l varies with distance x from one end as $\lambda = \lambda_0 x^2 / l^2$. Find the position of the center of mass.

chapter 3.1.



$$\lambda(x) = \frac{\rho_0}{l^2} x^2$$

$$\vec{R}_{cm} = X_{cm}\,\hat{i} = \frac{1}{m_{total}} \int_0^l x'\,dm$$

$$dm = \lambda(x')\,dx' = \frac{\rho_0}{l^2} x'^2\,dx'$$

$$m_{total} = \int dm' = \int_{x'=0}^{x'=l} \lambda(x')\,dx'$$

$$m_{total} = \int_0^l \frac{\rho_0}{l^2} x'^2\,dx' = \frac{\rho_0}{l^2}\frac{l^3}{3} = \frac{1}{3}\rho_0 l$$

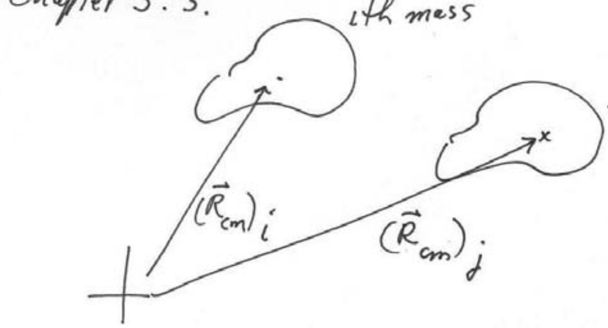
$$X_{cm} = \frac{1}{\frac{1}{3}\rho_0 l} \int_0^l x' \frac{\rho_0}{l^2} x'^2\,dx' = \frac{3}{\rho_0 l}\frac{\rho_0}{l^2}\frac{l^4}{4} = \frac{3}{4} l$$

$$\vec{R}_{cm} = \frac{3}{4} l\,\hat{i}$$

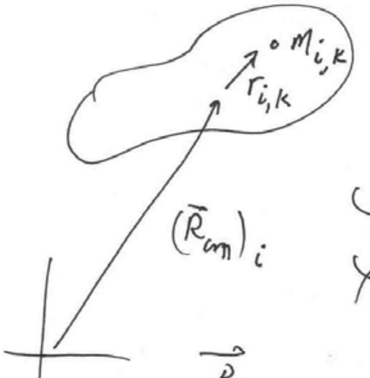
Problem 3:

Suppose that a system consists of several bodies, and that the position of the center of mass of each body is known. Prove that the center of mass of the system can be found by treating each body as a particle concentrated at its center of mass.

chapter 3.3.



$$m_i = \sum_k m_{i,k}$$



ith mass $$(\vec{R}_{cm})_i = \frac{\sum_k m_{i,k}\,\vec{r}_{i,k}}{\sum_k m_{i,k}} \qquad (1)$$

the total center of mass of the system

$$\vec{R}_{cm} = \frac{\sum_{i=1}^{N} \sum_{k=1}^{N} m_{i,k}\,\vec{r}_{i,k}}{m_{total}} \qquad (2)$$

$$m_{total} = \sum_{i=1}^{N} \sum_{k} m_{i,k} = \sum_{i=1}^{N} m_i$$

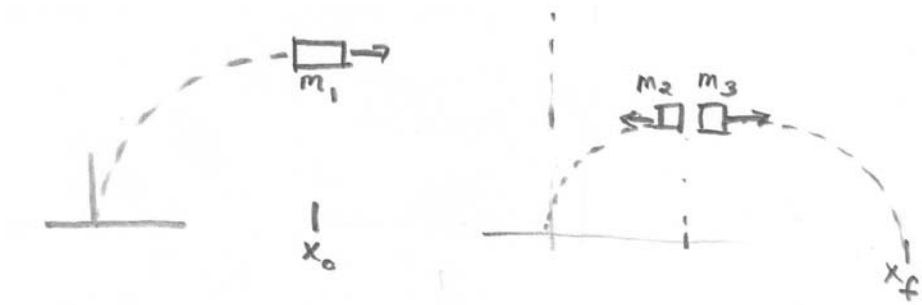
From eq (1) $\Rightarrow$ $\sum_k m_{i,k}\,\vec{r}_{i,k} = \left(\sum_k m_{i,k}\right)\vec{R}_{cm,i}$

Then eq (2) $\Rightarrow$ $\qquad = m_i\,\vec{R}_{cm,i}$

$$\vec{R}_{cm} = \frac{\sum_{i=1}^{N} (\ m_i\ )\,\vec{R}_{cm,i}}{m_{total}} \qquad (3)$$

eq (3) states that each body of mass m_i located at $\vec{R}_{cm,i}$ is a concentrated point mass.

Problem 4: Exploding Projectile An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_0. The projectile breaks into two pieces which fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature.



How far away, x_f, from the original launching point does the larger piece land?

Chapter 3 .4.

t_0

$\underset{m}{\square} \rightarrow v_0$ $\quad$ t_0 picture

$v_1 \leftarrow \underset{m_1}{\square}\ \underset{m_2}{\square} \rightarrow v_2$ $\quad$ t_f picture

$$m_2 = 3m_1 \Rightarrow \begin{cases} m_1 + m_2 = m \\ 4m_1 = m, \ \boxed{m_1 = \frac{m}{4}} \end{cases}$$

Since m_1 is stated to return to exactly the same starting point,

$$v_1 = v_0$$

$\underset{m}{\square} \rightarrow v_0$ $\qquad \rightarrow \hat{i}$

$v_0 \leftarrow \underset{m_1}{\square}\ \ \underset{3m_1}{\square} \rightarrow v_2$

$$\vec{P}_0 = m v_0 \hat{i} \quad , \quad \vec{P}_f = (3m_1 v_2 - m_1 v_0)\hat{i}$$

$$\vec{P}_0 = \vec{P}_f \quad \text{since } (\vec{F}_{ext})_x = 0$$

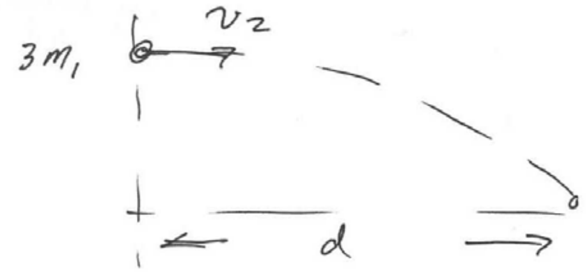
$$m v_0 = 3m_1 v_2 - m_1 v_0$$

$$(m + m_1) v_0 = 3m_1 v_2$$

$$\left(m + \frac{m}{4}\right) v_0 = \frac{3m}{4} v_2 \Rightarrow \boxed{v_2 = \frac{5}{3} v_0}$$

We now use the equations of projectile motion

$$L = v_x t \Rightarrow L = v_0 t$$
$$v_x = v_0$$



the larger projectile takes the same amount of time $t = \frac{L}{v_0}$ to reach the ground

$$d = v_2 t = v_2 \frac{L}{v_0} = \frac{5}{3} v_0 \frac{L}{v_0} = \frac{5}{3} L$$

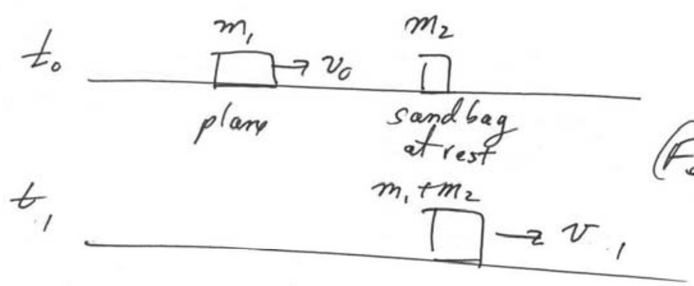
So the larger projectile travels

distance $= d + L = \frac{8}{3} L$ from origin

Problem 6:

A light plane weighing 2500 lb makes an emergency landing on a short runway. With its engine off, it lands on the runway at a speed of 120 ft / sec . A hook on the plane snags a cabal attached to a 250 lb sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is $\mu_k = 0.4$, and if the plane's brakes give an additional retarding force of magnitude 300 lb, how far does the plane go before it comes to a stop?

Chapter 3 6.



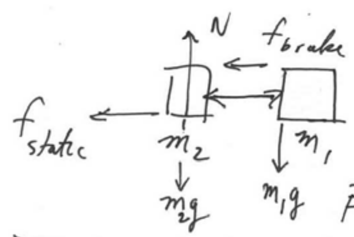
$\rightarrow \hat{i}$

$(F_{ext})_x = 0$ implies

$P_{0,x} = P_{f,x}$

$$m_1 v_0 = (m_1 + m_2) v_1 \Rightarrow v_1 = \frac{m_1 v_0}{m_1 + m_2} = \frac{m_1 g v_0}{(m_1 + m_2) g}$$

$$v_1 = \left(\frac{2{,}500 \text{ lb}}{2{,}500 \text{ lb} + 250 \text{ lb}}\right) 120 \frac{ft}{s} = 1.1 \times 10^2 \frac{ft}{s}$$



$\rightarrow \hat{i}$ $\quad f_{static} = \mu N = \mu m_2 g$

$$\vec{F} = m\vec{a}$$

$$\hat{i}: \quad -f_{static} - f_{brake} = (m_1 + m_2) a_x$$

$$a_x = \frac{-\mu m_2 g - f_{brake}}{m_1 + m_2}$$

$m_2 g = 250$ lbs

$m_1 + m_2 = \frac{2750 \text{ lbs}}{32 \frac{ft}{sec^2}}$

$$= \left(\frac{-(0.4)(250 \text{ lbs}) - 300 \text{ lbs}}{(2750 \text{ lbs})}\right) \frac{32 ft}{sec^2}$$

$$a_x = -4.65 \ ft/sec^2$$

$$d = v_1 t_f + \frac{1}{2} a_x t_f^2$$

$$0 = v_f = v_1 + a_x t_f \Rightarrow t_f = -\frac{v_1}{a_x}$$

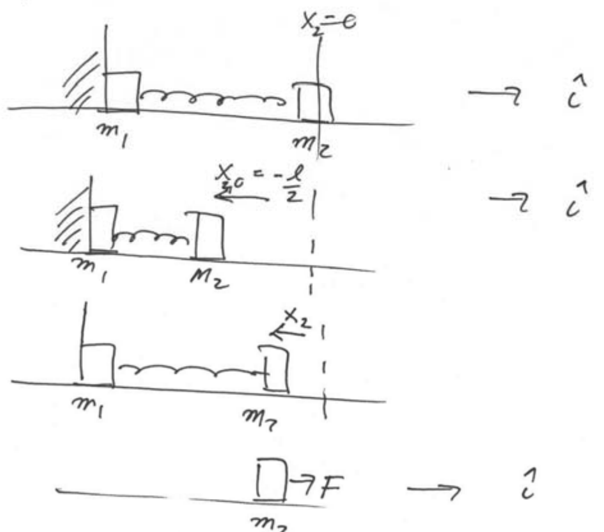
$$\Rightarrow d = \frac{-v_1^2}{a_x} + \frac{1}{2}\frac{v_1^2}{a_x} = -\frac{1}{2}\frac{v_1^2}{a_x}$$

$$d = \left(-\frac{1}{2}\right)\frac{\left(1.1 \times 10^2 \frac{ft}{sec}\right)^2}{-4.65 \frac{ft}{sec^2}} = 1.27 \times 10^3 \ ft$$

Problem 7:

A system is composed of two blocks 1 and 2 of masses m_1 and m_2 respectively that are connected by a massless spring with spring constant k. The blocks slide on a frictionless plane. The unstretched length of the spring is l. Initially the block 2 is held so that the spring is compresses to a length $l/2$ and block 1 is pushed up against a wall. At $t=0$ block 2 is released. Find the motion of the center of mass of the system as a function of time.

chapter 3.7.



$$\vec{F} = m_2 \frac{d^2 x_2}{dt^2}\,\hat{i}$$

$$-k x_2 = m_2 \frac{d^2 x_2}{dt^2}, \quad \text{For } 0<t<t_1$$

Solution $\quad x_2(t) = A\cos\omega t + B\sin\omega t, \quad \omega = \sqrt{\frac{k}{m_2}}$

initial Conditions $\quad x_{2,0} = -\frac{l}{2} = A$

$$v_2(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$v_{2,0} = 0 = \omega B \Rightarrow B = 0$$

$\boxed{x_2(t) = -\frac{l}{2}\cos\omega t}$. The mass m_2 reaches

$x_2 = 0$ when $\omega t_1 = \pi/2 \quad t_1 = \frac{\pi}{2\omega}$

$$v_2(t_1) = +\frac{\omega l}{2}\sin\omega t_1 = +\frac{\omega l}{2}$$

For $t > t_1$, there is no external forces so the center of mass moves at a constant velocity

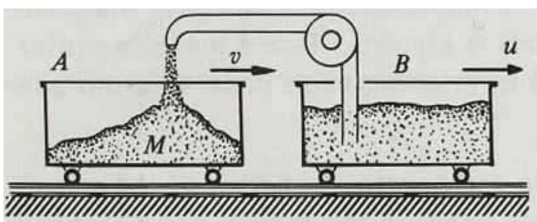
$$\vec{V}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

at $t_1 = \frac{\pi}{2\omega}$, $\vec{v}_1 = 0$,

$$\vec{V}_{cm} = \frac{m_2 v_2(t_1)}{m_1 + m_2}\hat{i} = \frac{m_2 \frac{\omega l}{2}}{m_1 + m_2}\hat{i}$$

Problem 11:

Material is blown into cart A from cart B at a rate of b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity, u as cart B. At the moment of interest, cart A has mass m_A and velocity v.



a) Define the objects that will constitute your system

b) Based on momentum diagrams, derive a differential equation for the velocity v. In particular find an expression for the rate of change of velocity, the instantaneous acceleration, dv/dt.

c) Integrate this equation to find the velocity has a function of time.

Chapter 3.11 There are two ways to solve this

Method 1:

$$\Delta m \to u \qquad\qquad \to \hat{i}$$

$$\boxed{m(t)} \to v \qquad\qquad \boxed{m+\Delta m}\ v+\Delta v$$

$$t \qquad\qquad\qquad t+\Delta t$$

$$0 = (F_{ext})_x = \lim_{\Delta t \to 0} \frac{P_x(t+\Delta t) - P_x(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(m+\Delta m)(v+\Delta v) - (mv+\Delta m\, u)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{\Delta m}{\Delta t}(v-u) + m\frac{\Delta v}{\Delta t} + \underbrace{\frac{\Delta m \Delta v}{\Delta t}}_{\text{ignore}} \right)$$

$$0 = \frac{dm}{dt}(v-u) + m\frac{dv}{dt} \qquad (1)$$

$$m(t) = m_0 + \left(\frac{dm}{dt}\right)t \quad , \quad \frac{dm}{dt} = b$$

$$m(t) = m_0 + bt$$

eq (1) becomes $\boxed{\dfrac{dv}{dt} = -\dfrac{b(v-u)}{m_0+bt}}$

integrating this $\displaystyle\int_{v_0}^{v(t)} \frac{dv}{v-u} = -b\int_{t=0}^{t} \frac{dt}{m_0+bt}$

$$\ln\left(\frac{v-u}{v_0-u}\right) = -\frac{b}{b}\ln\left(\frac{m_0+bt}{m_0}\right) = \ln\left(\frac{m_0}{m_0+bt}\right)$$

$$\frac{v-u}{v_0-u} = \frac{m_0}{m_0+bt} \Rightarrow v(t) = (v_0-u)\left(\frac{m_0}{m_0+bt}\right) + u$$

Problem 12:

A sand-spraying locomotive sprays sand horizontally into a freight car. The locomotive and freight car are not attached. The engineer in the locomotive maintains his speed so that the distance to the freight car is constant. The sand is transferred at a rate $dm/dt = 10\ \text{kg/s}$ with a velocity $u = 5\ m/\text{s}$ relative to the locomotive. The car starts from rest with an initial mass of $2000\ \text{kg}$. Find its speed after $100\ \text{s}$.

Chapter 3.12

$$\boxed{\ }_{\Delta m} \rightarrow v+u \quad \boxed{m} \rightarrow v \qquad\qquad \boxed{\Delta m + m} \rightarrow v + \Delta v$$

$$t \qquad\qquad\qquad\qquad t + \Delta t$$

$$F_{ext} = 0 = \lim_{\Delta t \to 0} \left(\frac{(m+\Delta m)(v+\Delta v) - (mv + \Delta m(v+u))}{\Delta t} \right)$$

$$0 = \lim_{\Delta t \to 0} \left((m+\Delta m)\frac{\Delta v}{\Delta t} - \frac{\Delta m u}{\Delta t} \right) = m\frac{dv}{dt} - \frac{dm}{dt} u$$

$$m\frac{dv}{dt} = \frac{dm}{dt} u = b u \qquad (1)$$

$$m(t) = m_0 + bt$$

$$dv = \frac{b u\, dt}{m_c + bt} \qquad (2)$$

integrate eq (2)

$$\int_{v_0=0}^{v} dv = \int_0^t \frac{b u\, dt}{m_0 + bt} \Rightarrow v = \frac{bu}{b} \ln\left(\frac{m_c + bt}{m_c}\right)$$

$$v(t) = u \ln\left(\frac{m_0 + bt}{m_0}\right) = \left(5\ \frac{kg}{s}\right) \ln\left(\frac{2000\ kg + \left(\frac{10\ kg}{s}\right)(100\ s)}{2000\ kg}\right)$$

$$v(100\ s) = 2.0\ m/s$$