MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012

Problem Set 6 Solutions

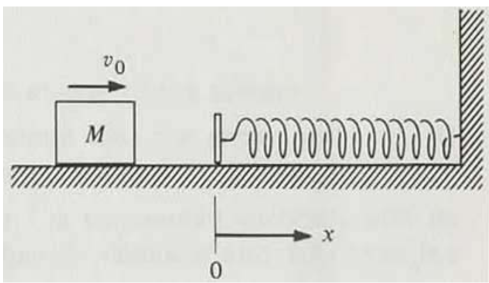
Readings: (KK) Kleppner, Daniel and Kolenkow, Robert, An Introduction to Mechanics, McGraw Hill, Inc., New York, 1973, Chapter 4.

Problems:

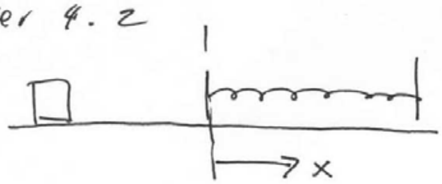
Chapter 4: 2, 5, 7, 9, 10, 12, 13

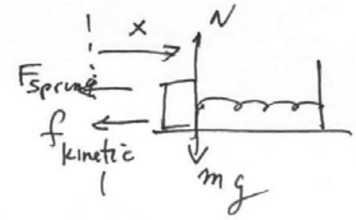
Problem 2:

A block of mass m slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $u_k = bx$, where b is a constant. What is the change in mechanical energy when the block has first come momentarily to rest?

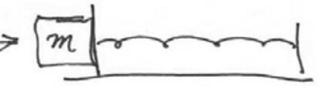


Chapter 4.2



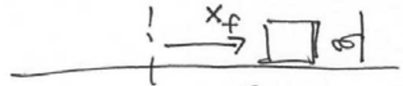


$t_0 = 0$ $\quad v_0 \rightarrow$



$E_0 = \frac{1}{2} m v_0^2$

t_f



$E_f = \frac{1}{2} k x_f^2$

$v_f = 0$

$$W^{nc} = -\int_0^{x_f} f_k \, dx \quad , \quad \mu = b x \quad \text{model}$$

$$f_k = \mu N = b x m g$$

$$W^{nc} = -\int_0^{x_f} m g b \, x \, dx = -m g b \frac{x_f^2}{2}$$

ΔE =	$W^{n.c.}$
$\frac{1}{2} k x_f^2 - \frac{1}{2} m v_0^2$	$= -\frac{1}{2} m g b x_f^2$ $\qquad (1)$

Solve for x_f: $\quad \frac{1}{2} x_f^2 (k + m g b) = \frac{1}{2} m v_0^2$

$$x_f = \left(\frac{m v_0^2}{(k + m g b)} \right)^{1/2}$$

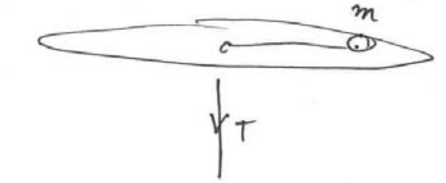
$$W^{nc} = -\frac{1}{2} m g b x_f^2 = -m g b \left(\frac{\frac{1}{2} m v_0^2}{(k + m g b)} \right)$$

. This is the loss of mechanical energy.

Problem 5:

A body of mass m whirls around on a string which passes through a fixed ring located at the center of the circular motion. The string is held by a person who pulls the string downward with a constant velocity of magnitude V so that the radial distance to the body decreases from an initial distance r_0 to a final distance r_f from the center. The body has an initial angular velocity ω_0. You may neglect the effect of gravity. Show that the work done in pulling the string equals the increase in kinetic energy of the body.

chapter 4.5



Assume ~~s~~tring is pulled inward s.t.
$\dot{r} = V = \text{constant}$

$\dot{i}$ $\quad T \longleftarrow (m) \quad \longrightarrow +\hat{r}$

$$\vec{F} = m\vec{a}$$

$$\hat{r}: \quad -T = -mr\omega^2 \qquad (1)$$

$$\hat{\theta}: \quad 0 = m(2\dot{r}\omega + r\dot{\omega}) \qquad (2)$$

$$W^{n.c} = \int_{r_0}^{r_f} \vec{T} \cdot d\vec{r} = -\int_{r_0}^{r_f} T\,dr$$

$$W^{n.c.} = -\int_{r_0}^{r_f} mr\omega^2\,dr \qquad (3)$$

Now ω is a function of r. To see this, Consider ~~the~~ the tangential force eq.

$$0 = 2\dot{r}\omega + r\dot{\omega}$$

$$\Rightarrow \quad 2\frac{dr}{dt}\omega = -\frac{r\,d\omega}{dt} \Rightarrow \frac{2\,dr}{r} = -\frac{d\omega}{\omega} \quad (4)$$

Eq (4) can be integrated

$$\int_{r_0}^{r} \frac{2\,dr'}{r'} = -\int_{\omega_0}^{\omega} \frac{d\omega'}{\omega'}$$

$$2\ln\left(\frac{r}{r_0}\right) = \ln\left(\frac{\omega_0}{\omega}\right) \Rightarrow \left(\frac{r}{r_0}\right)^2 = \frac{\omega_0}{\omega}$$

$$\Rightarrow \quad r^2\omega = r_0{}^2\omega_0 \qquad (5)$$

Equation (5) shows that the quantity

$$r^2\omega = \text{constant. or } \omega = \frac{r_0^2\omega_0}{r^2}$$

So eq (3) becomes

$$W^{n.c} = -\int_{r_0}^{r_f} (m r')\left(\frac{r_0^4 w_0^2}{r'^4}\right) dr'$$

$$W^{n.c} = -\int_{r_0}^{r_f} m r_0^4 w_0^2 \frac{dr'}{r'^3} = \left.\frac{m r_0^4 w_0^2}{2 r^2}\right|_{r_0}^{r_f}$$

$$= \frac{m r_0^4 w_0^2}{2 r_f^2} - \frac{m r_0^2 w_0^2}{2} \qquad (6)$$

From eq (5) : $r_f^2 w_f = r_0^2 w_0$

So eq (6) becomes

$$W^{n.c} = \frac{1}{2} m \frac{r_f^4 w_f^2}{r_f^2} - \frac{1}{2} m r_0^2 w_0^2 \qquad (7)$$

$$W^{n.c} = \frac{1}{2} m r_f^2 w_f^2$$

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the $\vec{v} = \dot{r}\,\hat{r} + r\omega\,\hat{\theta}$ with $v^2 = (\dot{r}^2 + r^2\omega^2)$

So the initial kinetic energy + final K.E. are

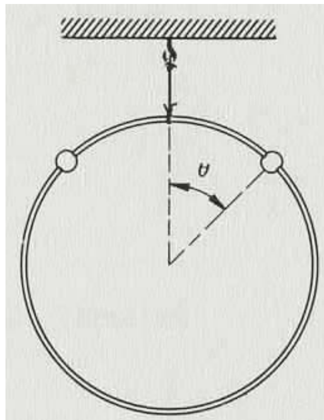
$$K_0 = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 w_0^2$$

$$K_f = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_f^2 w_f^2 . \quad \text{Thus}$$

$$K_f - K_0 = \frac{1}{2} m r_f^2 w_f^2 - \frac{1}{2} m r_0^2 w_0^2 = W^{n.c.}$$

Problem 7:

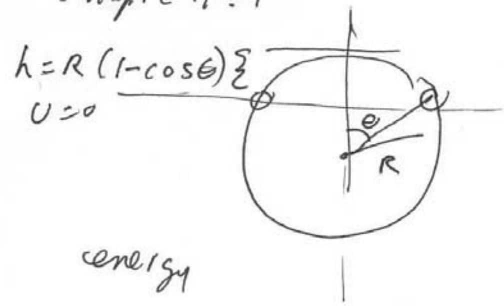
A ring of mass m_r hangs from a thread, and two identical beads of mass m_b slide on it without friction. The beads are released simultaneously from the top of the ring and slide down opposite sides.



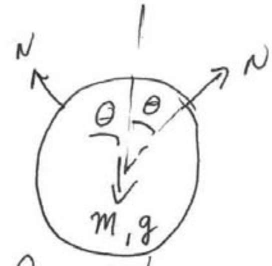
a) Draw free body force diagrams for the ring and the beads. What direction is the force of the bead on the ring pointing? Does it change has the bead moves. Can you still proceed with an analysis using Newton's second Law if you are not sure which way this force points? Try to find a physical explanation for the direction of this force.

b) Show that the ring will start to rise if $m_b > (3/2)m_r$, and find the angle θ with respect to the vertical direction that this occurs.

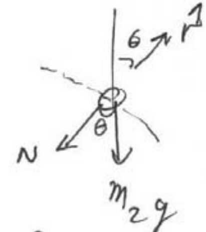
Chapte 4.7



energy



force diagram on ring



force diagram on bead

$\vec{F}_1 =$	$m_1 \vec{a}_1$
$2N\cos\theta - m_1 g =$	0 condition that the ring just starts to rise

$\Rightarrow \quad N = \dfrac{m_1 g}{2\cos\theta} \quad (1)$

$\vec{F}_2 =$	$m_2 \vec{a}_2$
$\hat{r} \quad -N - m_2 g\cos\theta =$	$-\dfrac{m_2 v^2}{R}$

$\Rightarrow \quad N + m_2 g\cos\theta = \dfrac{m_2 v^2}{R} \quad (2)$

$E_0 =$	E_f for single bead
$R m_2 g(1-\cos\theta) =$	$\frac{1}{2} m_2 v^2 \quad (3)$

substituting eq (1) into eq (2) yields

$$\frac{m_1 g}{2\cos\theta} + m_2 g\cos\theta = \frac{m_2 v^2}{R} \qquad (4)$$

solving eq (3) for $m_2 v^2$ and substituting into eq (4) yields

$$\frac{m_1 g}{2\cos\theta} + m_2 g\cos\theta = 2m_2 g(1-\cos\theta) \quad (5)$$

This simplifies to

$$m_1 g + 2m_2 g\cos^2\theta = 4m_2 g\cos\theta - 4m_2 g\cos^2\theta$$

or

$$m_1 g = 4m_2 g\cos\theta - 6m_2 g\cos^2\theta$$

which is a quadratic eq for $\cos^2\theta$

$$\cos^2\theta - \frac{2}{3}\cos\theta + \frac{m_1}{6m_2} = 0 \quad (6)$$

solving

$$\cos\theta = \frac{\frac{2}{3} \pm \left(\frac{4}{9} - \frac{2}{3}\frac{m_1}{m_2}\right)^{1/2}}{2} \quad (6a)$$

when $m_1 = 0$ choose pos. root to get

$$\cos\theta = \frac{2}{3}$$

Since $\left(\frac{4}{9} - \frac{2}{3}\frac{m_1}{m_2}\right) > 0$ or we would have an imaginary root

$$\frac{2}{3} > \frac{m_1}{m_2}$$

$$m_2 > \frac{3}{2}m_1 \quad (7)$$

Problem 9:

Consider the exothermic reaction (final kinetic energy is greater than the initial kinetic energy).

$$H + H \rightarrow H_2 + 5ev$$

Two hydrogen atoms collide and produce a diatomic hydrogen molecule. However, when hydrogen atoms collide in free space they simply bounce apart. The reason is that it is impossible to satisfy the laws of conservation of energy and momentum in a simple two body collision which releases energy.

a) Can you prove this? Try to analyse this collision in a reference frame moving with the velocity of the center of mass of the system.

b) Can this two body reaction take place if the temperature is dramatically lowered to near zero degrees Kelvin? Try to give an physical explanation for your answer.

Chapter 4. 9

$$H + H \rightarrow H_2 + 5\,eV$$

is an exothermic reaction. This means that $K_f > K_0$. Since

$$\Delta K = K_f - K_0 = W^{nc}$$

we have that $W^{nc} = 5\,eV$

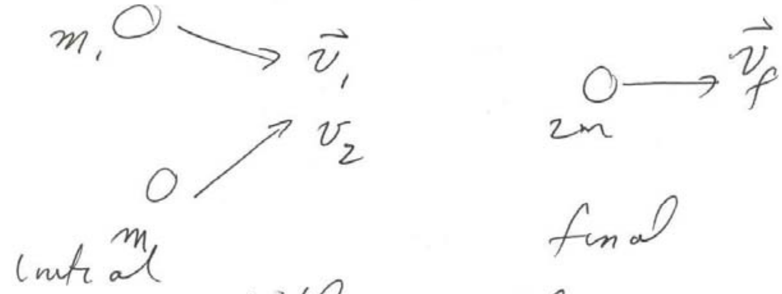
<u>First Argument</u>: In the center of mass frame with $(\vec{p}^{\,total})_{cm} = 0$, the collision looks like

initial: $\underset{m}{O} \rightarrow \vec{v}\,' \quad \vec{v} \leftarrow \underset{m}{O}$

final: $\underset{2m}{O} \quad \vec{v}_f\,' = 0 \Rightarrow K_f' = 0$

The final velocity in the cm frame must be zero because $(\vec{p}_f^{\,total})' = 0$. However $K_f' > K_0'$ since the reaction is exothermic so an exothermic reaction in which a two-body collision results in only one final body is impossible unless there is a second final body to carry momentum away.

Second Argument: In the lab frame



$$\vec{p}_0^{\,total} = \vec{p}_f^{\,total}$$

$$m(\vec{v}_1 + \vec{v}_2) = 2m\,\vec{v}_f$$

$$\Rightarrow \quad \vec{v}_f = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$$

$K_f = K_0 + 5\,eV$ becomes

$$\frac{1}{2}(2m)(\vec{v}_f \cdot \vec{v}_f) = \frac{1}{2} m\, \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m\, \vec{v}_2 \cdot \vec{v}_2 + 5\,eV$$

$$\left(\frac{1}{2}(\vec{v}_1 + \vec{v}_2) \cdot \frac{1}{2}(\vec{v}_1 + \vec{v}_2)\right) = \frac{1}{2}\vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2}\vec{v}_2 \cdot \vec{v}_2 + \frac{5\,eV}{m}$$

$$\frac{1}{4}\vec{v}_1 \cdot \vec{v}_1 + \frac{1}{4}\vec{v}_2 \cdot \vec{v}_2 + \frac{1}{2}\vec{v}_1 \cdot \vec{v}_2 = \frac{1}{2}\vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2}\vec{v}_2 \cdot \vec{v}_2 + \frac{5\,eV}{m}$$

$$0 = \frac{1}{4}(\vec{v}_1 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 - 2\vec{v}_1 \cdot \vec{v}_2) + \frac{5\,eV}{m}$$

$$0 = \underbrace{\frac{1}{4}(\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2)}_{\text{this is always positive since } \vec{A} \cdot \vec{A} > 0} + \underbrace{\frac{5\,eV}{m}}_{\text{positive}}$$

but this is impossible so reaction cannot happen.

part b) The reaction can take place a low temperatures because of a quantum mechanical resonance state called the Feshback resonance. Two hydrogen atoms can combine to form a resonant state with a very high orbital angular momentum (l=32) with the electrons in spin up and spin down. A second state with approximately the same energy consisting of two free hydrogen atoms with a lower orbital angular momentum (l=31) but the electrons are in spin up states. There are some hyperfine states associated with the second state that if tuned by magnetic fields can have the same energy as the resonance. So the physical state is a linear combination of these states. At high temperatures this possible channel has a rather low probability but at very low temperatures it dominates the collision process. Wolfgang Ketterlee was the first to create this state here at MIT., now it used all the time in Bose-Einstein Condensates. Thus a third hydrogen atom can scatter with the Feshback resonance creating a stable H_2 molecule.

Problem 10:

A block of mass m_b on a horizontal table is connected to one end of a spring with spring constant k. The other end of the spring is attached to a wall. The block is set in motion so that it oscillates about its equilibrium point with amplitude A_0.

a) What is the period of the motion?

A lump of sticky putty of mass m_p is dropped onto the block. The putty sticks without bouncing. The putty hits the block at the instant when the velocity of the block is zero.

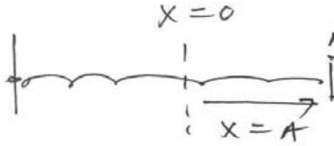
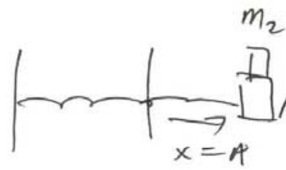
b) Find the new period, the new amplitude, and the change in mechanical energy of the system.

The putty is removed and the block is set in motion. This time the putty is dropped and hits the block at the instant the block has its maximum velocity.

c) Find the new period, the new amplitude, and the change in mechanical energy of the system.

Chapter 4.10.

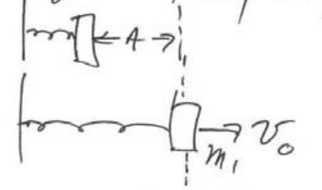
a. 

before  $v_1 = 0$ after 

Since the block was not moving there is no change in energy due to this totally inelastic collision. Hence $E_0 = E_f$ implies that the amplitude will not change in the next cycle.

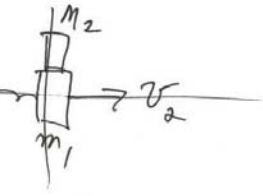
The new period $T'_{new} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m_1+m_2}}}$

$t=0$ $v_0 = 0$ fully compressed

b) before t_1  before the collision

energy is conserved so $\frac{1}{2} m v_0^2 = \frac{1}{2} k A^2$

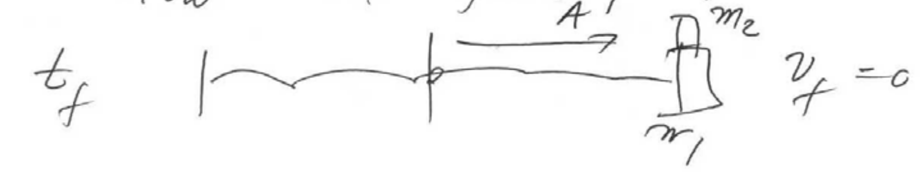
$$A = \sqrt{\frac{m_1}{k}}\, v_0$$

after t_2  $\vec{p}(t_1) = \vec{p}(t_2)$

$m_1 v_0 = (m_1 + m_2) v_2$

$$v_2 = \frac{m_1 v_0}{m_1 + m_2}$$

Now the final amplitude



$$E(t_2) = E(t_f)$$

$$\frac{1}{2}(m_1+m_2)v_2^2 = \frac{1}{2}kA'^2$$

$$\Rightarrow \quad A' = \sqrt{\frac{m_1+m_2}{k}}\, v_2 = \left(\sqrt{\frac{m_1+m_2}{k}}\right)\left(\frac{m_1 v_0}{m_1+m_2}\right)$$

$$A' = \left(\sqrt{\frac{m_1}{k}}\, v_0\right)\sqrt{\frac{m_1}{m_1+m_2}} = A\sqrt{\frac{m_1}{m_1+m_2}}$$

Where A is the original amplitude.

$$T' = \frac{2\pi}{\omega_{new}} = \frac{2\pi}{\sqrt{\frac{k}{m_1+m_2}}}$$

The change in energy is

$$\Delta E = \frac{1}{2}kA'^2 - \frac{1}{2}kA^2 = \frac{1}{2}kA^2\left(\frac{m_1}{m_1+m_2} - 1\right)$$

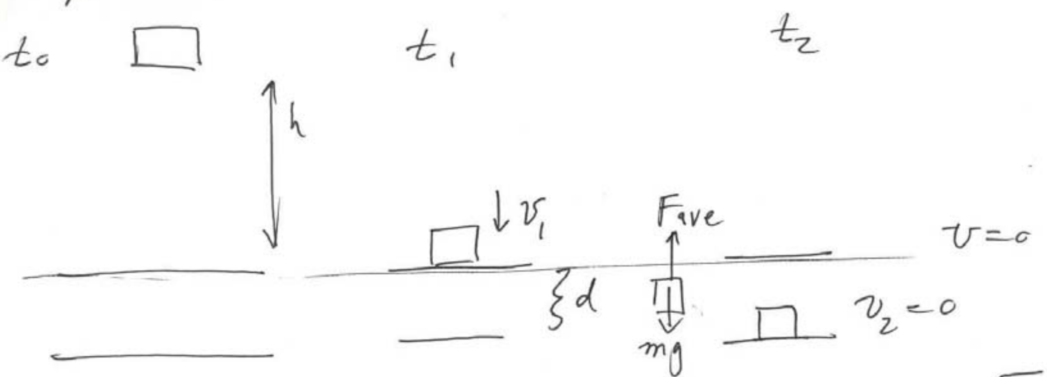
$$= \frac{1}{2}kA^2\left(\frac{-m_2}{m_1+m_2}\right) = -E_{original}\frac{m_2}{m_1+m_2}$$

$$\boxed{\frac{\Delta E}{E_{original}} = -\frac{m_2}{m_1+m_2}}$$

Problem 12:

During the Second World War the Russians. Lacking sufficient parachutes for airborne operations, occasionally dropped soldiers inside bales of hay onto snow. The human body can survive an average pressure of $30\ \text{lbs/in}^2$. Suppose that the lead plane drops a dummy bale equal in weight to a loaded one from an altitude of $150\ \text{ft}$., and that the pilot observes that it sinks about $2\ \text{ft}$ into the snow. If the weight of an average soldier is $144\ \text{lbs}$ and his effective area is $5\ \text{ft}^2$, is it safe to drop the men?

chapter 4.12



$$E_0 = mgh \qquad E_1 = \frac{1}{2} m v_1^2 \quad , \quad E_0 = E_1 \Rightarrow v_1 = \sqrt{2gh}$$

$$E_2 = -mgd$$

$$E_2 - E_1 = W^{n.c} = -F_{ave}\,\Delta r$$

$$-mgd - \frac{1}{2} m v_1^2 = -F_{ave}\, d \qquad \text{or}$$

$$-mgd - mgh = -F_{ave}\, d$$

$$F_{ave} = mg\left(1 + \frac{h}{d}\right)$$

$$\text{Pressure} = \frac{F_{ave}}{A} = \frac{mg}{A}\left(1 + \frac{h}{d}\right) = \frac{144\ lb}{5\ ft^2}\left(1 + \frac{150\ ft}{2\ ft}\right)$$

$$= 2.19 \times 10^3\ \frac{lb}{ft^2}\ \frac{1 ft^2}{144 in^2} \simeq 15\ \frac{lb}{in^2}$$

So it seems safe! Who's first?

Problem 13:

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6,12 potential

$$U(r) = \varepsilon\left[(r_0/r)^{12} - 2(r_0/r)^6\right].$$

a) Show that the radius at the potential minimum is r_0 , and that the depth of the potential well is ε.

b) Find the angular frequency of small oscillations about the stable equilibrium position for two identical atoms of mass m bound to each other by the Lennard-Jones interaction.

Chapter 4.13

$$U = \varepsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right]$$

a. $$\frac{dU}{dr} = \varepsilon\left((r_0)^{12}(-12)r^{-13} - 2r_0^{6}(-6)r^{-7}\right) = 0$$

becomes $$\frac{12\, r_0^{12}}{r^{13}} = \frac{12\, r_0^{6}}{r^{7}} \Rightarrow r = r_0$$

b. The reduced mass $\mu = \frac{m^2}{2m} = \frac{m}{2}$. The

$$\omega = \sqrt{\frac{\left.\frac{d^2U}{dr^2}\right|_{r=r_0}}{\mu}}$$ ang. frequency of small oscillations

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = \left.\varepsilon\left((-12)(-13)r_0^{12}r^{-14} - 2r_0^{6}(-6)(-7)r^{-8}\right)\right|_{r=r_0}$$

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = \varepsilon\, 12 r_0^{6}\left(\frac{13 r_0^{6}}{r^{14}} - \frac{7}{r^{8}}\right) = \frac{72\,\varepsilon}{r_0^{2}}$$

$$\omega = \sqrt{\frac{\frac{72\varepsilon}{r_0^2}}{m/2}} = \frac{12}{r_0}\sqrt{\frac{\varepsilon}{m}}$$