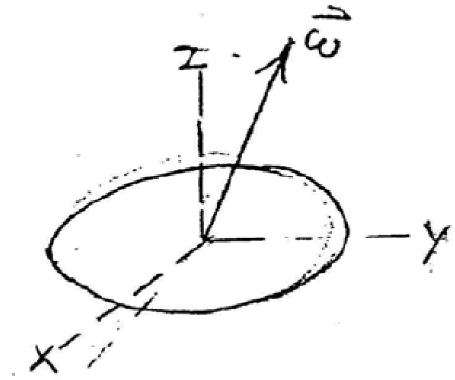
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.01 **Fall Term 2009**

Three Dimensional Rotation Problems

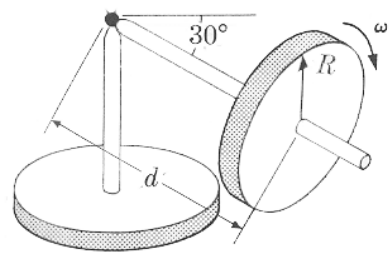
Problem 1 A thin disc of radius R and mass M is instantaneously rotating about its center of mass with angular velocity $\vec{\omega} = \omega_y \hat{j} + \omega_z \hat{k}$. Find the angular momentum about the center of mass of the disc.



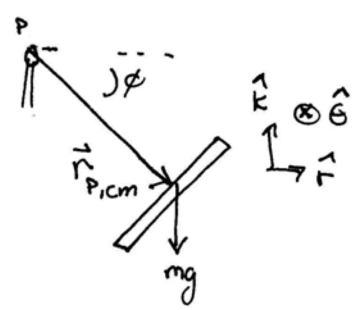
Solution: The moment of inertia about the y-axis passing through the center of mass is $I_y = (1/4)MR^2$. The moment of inertia about the z-axis passing through the center of mass is $I_z = (1/2)MR^2$. Therefore the angular momentum about the center of mass of the disc is given by

$$\vec{L}_{cm} = I_y \omega_y \hat{j} + I_z \omega_z \hat{k} = (1/4)MR^2 \omega_y \hat{j} + (1/2)MR^2 \omega_z \hat{k}. \qquad \textbf{(1)}$$

Problem 2: A gyroscope consists of a uniform disc of mass radius $M = 1.0\ \text{kg}$ and radius $R = 0.2\ \text{m}$. The disc spins with an angular speed $\omega = 400\ \text{rad} \cdot \text{s}^{-1}$ as shown in the figure below. The gyroscope precesses, with its axle at an angle 30° below the horizontal (see figure). The gyroscope is pivoted about a point $d = 0.3\ \text{m}$ from the center of the disc. What is the direction and magnitude of the precessional angular velocity?



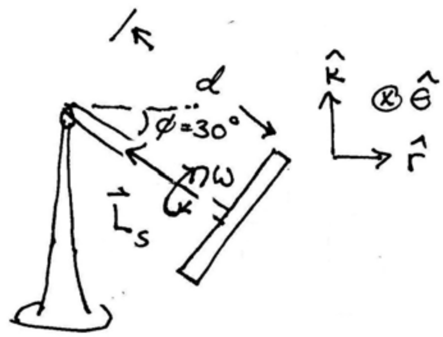
Solution: In this solution, all parameters will be represented symbolically, with numerical values inserted only at the end. Introduce cylindrical coordinates, with the $\hat{\mathbf{k}}$-direction vertical, the $\hat{\mathbf{r}}$-direction horizontal, directed from the pivot to the center of the disc, and the $\hat{\boldsymbol{\theta}}$-direction horizontal and perpendicular to $\hat{\mathbf{r}}$, defined by $\hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \times \hat{\mathbf{r}}$.



The torque with respect to the pivot point is due only to the weight of the disc,

$$\begin{aligned}\vec{\tau}_P &= \vec{r}_{P,cm} \times M\vec{g} = (-d \sin\phi\, \hat{\mathbf{k}} + d \cos\phi\, \hat{\mathbf{r}}) \times Mg(-\hat{\mathbf{k}}) \\ &= d \cos\phi Mg\, \hat{\boldsymbol{\theta}}\end{aligned} . \qquad \textbf{(2)}$$

We can express the total angular momentum of the gyroscope with respect to the pivot point as the sum of two parts, the spin angular momentum and the orbital angular momentum, where the spin angular momentum is parallel to the axle and the orbital angular momentum is vertical.



In these coordinates, the angular momentum about the pivot point is expressed as

$$\vec{L}_P = \vec{r}_{P,cm} \times m\vec{v}_{cm} + \vec{L}_{cm} \,. \qquad \textbf{(3)}$$

We will use the gyroscopic approximation in which $\Omega << \omega$. Then we only need to consider the spin part of the angular momentum about the center of mass

$$\vec{L}_{cm,s} = I_s \omega_s (\sin\phi\, \hat{\mathbf{k}} - \cos\phi\, \hat{\mathbf{r}}) \,. \qquad \textbf{(4)}$$

Because the torque is into the page in the figure above, the angular momentum about the center of mass must change into the page. This requires the gyroscope to precess in a clockwise direction when looking down from above. We are given that the gyroscope precesses uniformly, so $\left|\vec{L}_{cm,s}\right| = I_s \omega_s$ is constant, hence ω is constant. Then as the gyroscopic precesses, the change of the angular momentum about the pivot point is given by

$$\frac{d\vec{L}_P}{dt} = \frac{d\vec{L}_{cm}}{dt} = I_s \omega_s \cos\phi \Omega\, \hat{\theta} \,. \qquad \textbf{(5)}$$

We can now apply the torque equation

$$\vec{\tau}_P = \frac{d\vec{L}_P}{dt} \,. \qquad \textbf{(6)}$$

Substitute Eq. **Error! Reference source not found.** and Eq. (5) into Eq. (6) yielding

$$d \cos\phi Mg\, \hat{\theta} = I_s \omega_s \cos\phi \Omega\, \hat{\theta} \,. \qquad \textbf{(7)}$$

We can now solve for the precessional frequency

$$\Omega = \frac{d \cos\phi Mg}{I_s \omega_s \cos\phi} \,. \qquad \textbf{(8)}$$

For a uniform disc, $I_s = (1/2)MR^2$, and so

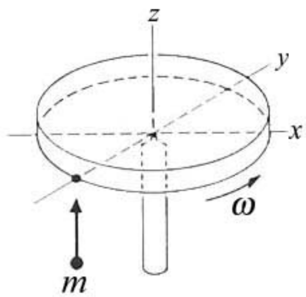
$$\Omega = \frac{2dg}{R^2\omega_s} \quad . \tag{9}$$

Note that this result does not depend on either the mass M of the disc or the angle ϕ that the axle angle makes with respect to the horizontal. Inserting numerical values,

$$\Omega = \frac{2(0.3\,\mathrm{m})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})}{(0.2\,\mathrm{m})^2(400\,\mathrm{rad}\cdot\mathrm{s}^{-1})} = 0.4\,\mathrm{rad}\cdot\mathrm{s}^{-1} \quad . \tag{10}$$

Problem Angular Impulse:

A uniform disc of radius R and mass m, mounted on its center by a universal bearing, rotates originally in a horizontal plane with angular velocity ω shown in the figure below. An object of mass m with speed $v = \omega R / 2$ directed along the *z*-axis collides with the edge of the disc and rebounds with an equal but oppositely directed velocity.



(a) What is the angular momentum of the disc and of the object taken about the center of mass of the disc before the collision?

(b) What angular impulses are imparted to the disc and to the object as a result of the collision?

(c) What is the angular momentum of the disc taken about the center of mass of the disc after the collision?

Solutions:

a) The moment of inertia of the uniform disc about its center of mass is $I_s = (1/2)mR^2$, and so its angular momentum about the center of mass is

$$\vec{L}_{cm,i}^{disk} = I_s \omega\, \hat{\mathbf{k}} = (1/2)mR^2 \omega\, \hat{\mathbf{k}} \,. \qquad \textbf{(11)}$$

The angular momentum of the object with respect to the center of the disc is

$$\vec{L}_{cm,i}^{object} = \vec{r}_{cm,o} \times m\vec{v} = R(-\hat{\mathbf{j}}) \times \frac{m\omega R}{2}(\hat{\mathbf{k}}) = -\frac{m\omega R^2}{2}\hat{\mathbf{i}} \,. \qquad \textbf{(12)}$$

Note that $\left|\vec{L}_{cm,i}^{object}\right| = \left|\vec{L}_{cm,i}^{disk}\right|$. The total angular momentum is then

$$\vec{L}_{cm,i} = \vec{L}_{cm,i}^{object} + \vec{L}_{cm,i}^{disk} = \frac{1}{2}mR^2\omega(-\hat{\mathbf{i}} + \hat{\mathbf{k}}) \,. \qquad \textbf{(13)}$$

(b) The object changes its direction of motion but not its moment arm, and so its angular momentum about the center of the disc changes sign but not direction. The angular impulse imparted to the object by the disc is then

$$\vec{J}_{cm}^{object} \equiv \Delta\vec{L}_{cm}^{object} = \vec{L}_{cm,f}^{object} - \vec{L}_{cm,i}^{object} = \frac{1}{2}mR^2\omega(\hat{\mathbf{i}}) - \frac{1}{2}mR^2\omega(-\hat{\mathbf{i}}) = mR^2\omega\,\hat{\mathbf{i}}\,. \qquad \mathbf{(14)}$$

The angular impulse imparted to the disc is equal in magnitude but opposite in direction from that angular impulse imparted to the object,

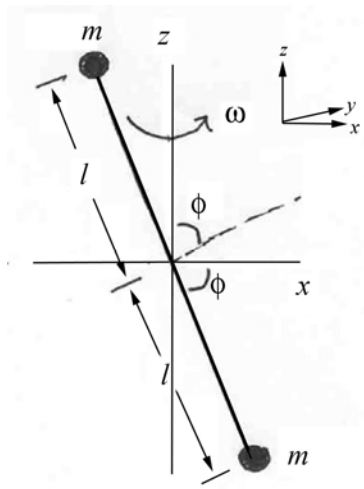
$$\vec{J}_{cm}^{disk} = -\vec{J}_{cm}^{object} = -mR^2\omega\,\hat{\mathbf{i}}\,. \qquad \mathbf{(15)}$$

(c) The final angular momentum of the disc about its center is the sum of the initial angular momentum and the impulse,

$$\vec{L}_{cm,f}^{disk} = \vec{L}_{cm,i}^{disk} + \vec{J}_{cm}^{disk} = (1/2)mR^2\omega\,\hat{\mathbf{k}} - mR^2\omega\,\hat{\mathbf{i}}\,. \qquad \mathbf{(16)}$$

Problem Skewed Rod

Consider a simple rigid body consisting of two particles of mass m separated by a massless rod of length $2l$. The midpoint of the rod is attached to a vertical axis that rotates at angular velocity $\vec{\omega}$ pointing in the positive z-direction. The perpendicular to the rod is skewed at an angle ϕ with respect to the z-axis. At time $t = 0$ the rod lies in the x-z plane. Find the direction and magnitude of the angular momentum about the midpoint of the rod at that instant.



We can simply use $\vec{\mathbf{L}}_{cm} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$ for the angular momentum about the cm for each particle. Denote the upper particle by 1 and the lower particle by 2. Then

$$\vec{\mathbf{L}}_{cm} = \vec{\mathbf{r}}_1 \times \vec{\mathbf{p}}_1 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{p}}_2 = \vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_2 \times m\vec{\mathbf{v}}_2 . \tag{17}$$

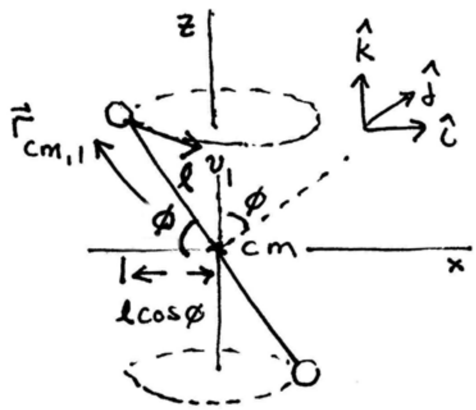
Note that for particle 2, $\vec{\mathbf{r}}_2 = -\vec{\mathbf{r}}_1$ and $\vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_1$, so

$$\vec{\mathbf{L}}_{cm,2} = \vec{\mathbf{r}}_2 \times m\vec{\mathbf{v}}_2 = \vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 = \vec{\mathbf{L}}_{cm,1} . \tag{18}$$

Thus

$$\vec{\mathbf{L}}_{cm} = 2\vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 . \tag{19}$$

Since each particle travels at an angular speed ω in a circular orbit of radius $\ell\cos\phi$, the speed of each particle is given by $v = \ell\cos\phi\,\omega$. We choose a coordinate system shown in the figure below

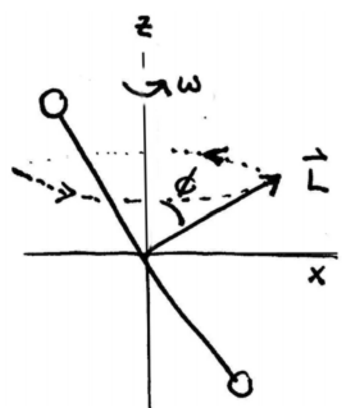


For particle 1: $\vec{\mathbf{r}}_1 = -l\cos\phi\,\hat{\mathbf{i}} + l\sin\phi\,\hat{\mathbf{k}}$ and $\vec{\mathbf{v}}_1 = l\cos\phi\omega\,(-\hat{\mathbf{j}})$. Thus

$$\vec{\mathbf{L}}_{cm} = 2\vec{\mathbf{r}}_1 \times m\vec{\mathbf{v}}_1 = 2(-l\cos\phi\,\hat{\mathbf{i}} + l\sin\phi\,\hat{\mathbf{k}}) \times ml\cos\phi\omega\,(-\hat{\mathbf{j}}). \tag{20}$$

Use the fact that in our right-handed coordinate system $(-\hat{\mathbf{i}}) \times (-\hat{\mathbf{j}}) = \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times (-\hat{\mathbf{j}}) = \hat{\mathbf{i}}$, so the angular momentum about the center of mass for particle 1 is then

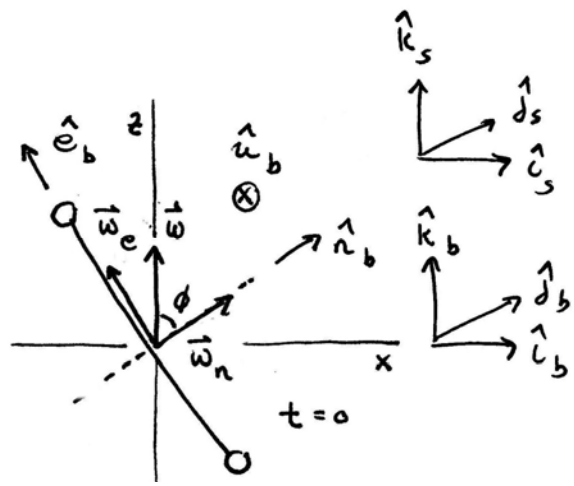
$$\vec{\mathbf{L}}_{cm} = 2ml^2\omega\cos\phi(\cos\phi\;\hat{\mathbf{k}} + \sin\phi\,\hat{\mathbf{i}}). \tag{21}$$



Problem: Principal Axes

For the rotating skewed rod, what are the principal axes. Find the moment of inertia about those axes. Find the components of the angular velocity about those axes. Find the angular momentum about the center of the skewed rod.

Principal Axis: The principal axes are a set of axes that coincide with the symmetry axes of the body. Our principal axes are an axis along the length of the rod and two axes forming a plane perpendicular to the rod. At time $t = 0$, the position of the rod is shown in the figure below.



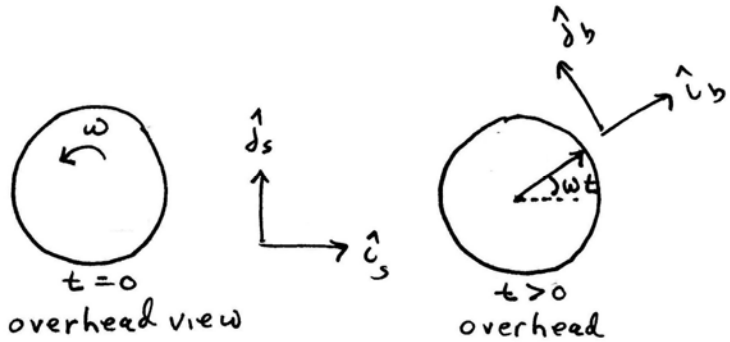
We shall choose three unit vectors that point along these principal axes. We use the subscript b to indicate that it is an axes associated with the body. Choose a unit vector $\hat{\mathbf{e}}_b$ that points from the origin to particle 1. Choose a second unit vector $\hat{\mathbf{n}}_b$ perpendicular to the rod lying in the plane formed by $\hat{\mathbf{e}}_b$ and $\vec{\omega}$, and perpendicular to $\hat{\mathbf{e}}_b$. Choose a third unit vector $\hat{\mathbf{u}}_b$ perpendicular to $\hat{\mathbf{n}}_b$ and $\hat{\mathbf{e}}_b$ pointing into the page of the drawing., to complete our description of the principal axes. These axes are fixed to the body. Note that with respect to the x-y-z axis that are fixed in space, the body principal axes are rotating. We can also choose a Cartesian set of body axes $(\hat{\mathbf{i}}_b, \hat{\mathbf{j}}_b, \hat{\mathbf{k}}_b)$ (they are not the principal axes) that at $t = 0$ coincide with a set of fixed spatial unit vectors $(\hat{\mathbf{i}}_s, \hat{\mathbf{j}}_s, \hat{\mathbf{k}}_s)$.

The unit vectors associated with the principal axes are given by

$$\begin{aligned} \hat{\mathbf{e}}_b &= \sin\phi\, \hat{\mathbf{k}}_b - \cos\phi\, \hat{\mathbf{k}}_b \\ \hat{\mathbf{n}}_b &= \cos\phi\, \hat{\mathbf{k}}_b + \sin\phi\, \hat{\mathbf{k}}_b \\ \hat{\mathbf{u}}_b &= \hat{\mathbf{j}}_b. \end{aligned} \tag{22}$$

At $t > 0$, as the body rotates the body and space z-axes always remain aligned so $\hat{\mathbf{k}}_b = \hat{\mathbf{k}}_s$ however the body unit vectors $(\hat{\mathbf{i}}_b, \hat{\mathbf{j}}_b)$ no longer coincide with the fixed space unit vectors $(\hat{\mathbf{i}}_s, \hat{\mathbf{j}}_s)$. As the body rotates, the components of the unit vectors for the principal axes lying in the x-y plane change according to

$$\begin{aligned} \hat{\mathbf{i}}_b &= \cos\omega t\, \hat{\mathbf{i}}_s + \sin\omega t \hat{\mathbf{j}}_s \\ \hat{\mathbf{j}}_b &= -\sin\omega t\, \hat{\mathbf{i}}_s + \cos\omega t \hat{\mathbf{j}}_s. \end{aligned} \tag{23}$$



So the principal axes unit vectors are given in terms of the fixed space unit vectors by

$$\begin{aligned}
\hat{\mathbf{e}}_b &= \sin\phi\,\hat{\mathbf{k}}_s - \cos\phi\,(\cos\omega t\;\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s)\\
\hat{\mathbf{n}}_b &= \cos\phi\,\hat{\mathbf{k}}_s + \sin\phi\,(\cos\omega t\;\hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s)\\
\hat{\mathbf{u}}_b &= -\sin\omega t\;\hat{\mathbf{i}}_s + \cos\omega t\hat{\mathbf{j}}_s.
\end{aligned} \tag{24}$$

The angular velocity $\vec{\omega} = \omega\,\hat{\mathbf{k}}_b$ can be decomposed into components along the body principal axes,

$$\vec{\omega} = \omega_n\;\hat{\mathbf{n}}_b + \omega_e\hat{\mathbf{e}}_b = \omega\cos\phi\hat{\mathbf{n}}_b + \omega\sin\phi\hat{\mathbf{e}}_b \tag{25}$$

We now use the principal axes theorem that states that the angular momentum about the center of mass for a symmetric body can be written as a sum

$$\vec{\mathbf{L}}_{cm} = I_{cm,n}\;\omega_n\hat{\mathbf{n}}_b + I_{cm,e}\;\omega_e\hat{\mathbf{e}}_b\,. \tag{26}$$

The moment of inertia about the perpendicular to the rod is $I_{cm,n} = 2ml^2$. Also since we are assuming that the rod is massless and that the particles are point-like particles $I_{cm,e} = 0$. So Eq. for the angular momentum about the center of mass becomes

$$\vec{\mathbf{L}}_{cm} = 2ml^2\omega\cos\phi\;\hat{\mathbf{n}}_b\,. \tag{27}$$

At time $t = 0$, using our result that $\hat{\mathbf{n}}_b = (\cos\phi\,\hat{\mathbf{k}}_b + \sin\phi\,\hat{\mathbf{i}}_b)$, the angular momentum about the center of mass is

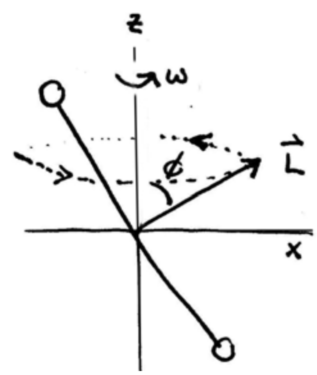
$$\vec{\mathbf{L}}_{cm}(t=0) = 2ml^2\omega\cos\phi(\cos\phi\,\hat{\mathbf{k}}_b + \sin\phi\,\hat{\mathbf{i}}_b)\,, \tag{28}$$

in agreement with our earlier result (Eq. (21)). We can use Eq. (23) to write the angular momentum about the center of mass as a function of time in terms of the fixed space unit vectors

$$\vec{\mathbf{L}}_{cm}(t) = 2ml^2\omega\cos\phi\,\hat{\mathbf{n}}_b = 2ml^2\omega\cos\phi\cos\phi\ \ \hat{\mathbf{k}}_s + 2ml^2\omega\cos\phi\sin\phi(\cos\omega t\ \hat{\mathbf{i}}_s + \sin\omega t\hat{\mathbf{j}}_s). \quad (29)$$

Problem: Torque on Skewed Rod

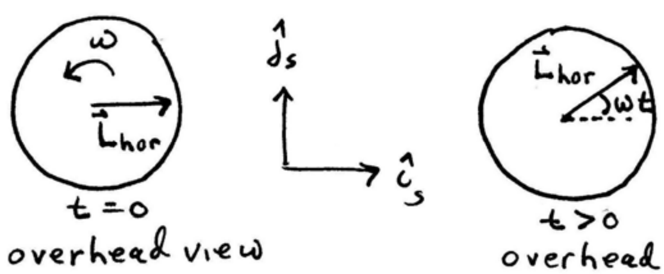
Based on your result for the angular momentum of the skewed rod, calculate the torque about the center of mass for the rod at time $t = 0$ when it lies in the x-z plane .As the rod rotates, the angular momentum vector precessing at an angular speed ω .



At the instant $t = 0$ shown on the left in the figure below (shown from an overhead perspective), the component of the angular momentum about the center of mass that is changing is the x-component of the angular momentum (pointing in the $\hat{\mathbf{i}}_s$ -direction)

$$L_{cm,x} = 2ml^2\omega\cos\phi\sin\phi \quad (30)$$

and the direction of the change of the angular momentum is into the page ($\hat{\mathbf{j}}_s$ -direction).



Hence

$$\frac{d\vec{\mathbf{L}}_{cm}}{dt}(t=0) = 2ml^2\omega\cos\phi\sin\phi\ \omega\ \hat{\mathbf{j}}_s. \quad (31)$$

Because the torque about the center of mass is given by

$$\vec{\tau}_{cm} = \frac{d\vec{\mathbf{L}}_{cm}}{dt} \tag{32}$$

we have at the instant $t = 0$ shown in the figure above on the left

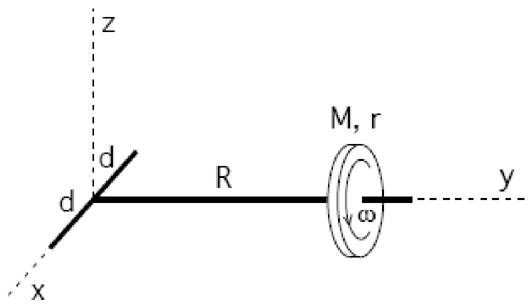
$$\vec{\tau}_{cm}(t = 0) = 2ml^2\omega^2 \cos\phi \sin\phi \,\hat{\mathbf{j}}_s\,. \tag{33}$$

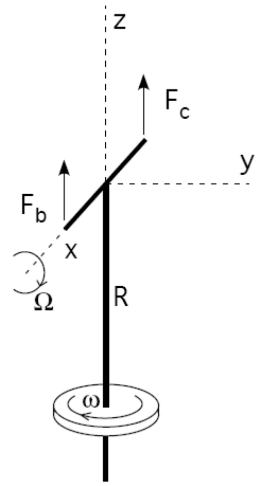
Note that if we want to find at $\vec{\boldsymbol{\tau}}_{cm}(t)$, we can differentiate our expression Eq. (29) using the fact that the $\hat{\mathbf{k}}_s$ -component is constant and find that

$$\vec{\tau}_{cm}(t) = \frac{d\vec{\mathbf{L}}_{cm}(t)}{dt} = 2ml^2\omega \cos\phi \sin\phi(-\omega \sin\omega t\,\hat{\mathbf{i}}_s + \omega\cos\omega t\hat{\mathbf{j}}_s\,) \tag{34}$$

agreeing with our above expression Eq. (33) when we set $t = 0$.

Problem: Rotating Disk Pendulum





A rotating disk pendulum consists of a uniform spinning disk on a shaft attached to a rod pivoted at the origin of the coordinate system. Assume that the rod and shaft are massless. The uniform disk spins on a frictionless bearing. The pendulum is dropped from rest in a horizontal position.

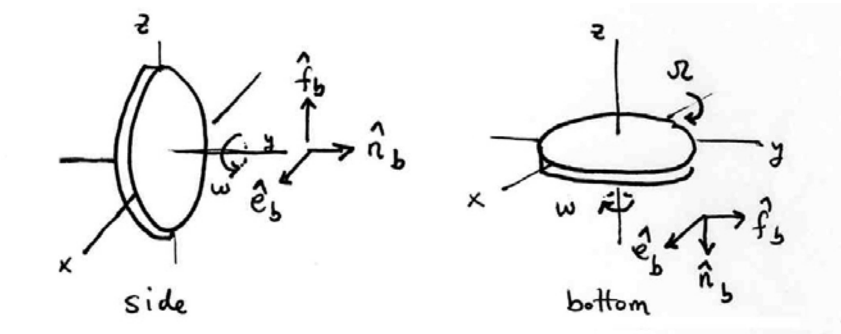
a) Use conservation of energy to find Ω at the bottom of the swing in terms of r, R and g.

b) Find $\vec{L}_0$ about the origin when the pendulum is at the bottom of its swing in terms of Ω, ω, M, r, and R.

c) Find an expression for $F_b - F_c$ in terms of M, r, Ω, ω and d.

Solution:

(a) Because the disk is a symmetric object we can choose three principal axes, (f, e, n) with associated unit vectors $(\hat{\mathbf{f}}_b, \hat{\mathbf{e}}_b, \hat{\mathbf{n}}_b)$ as shown in the figure below. Note that when the disk is in the side position we have chosen $\hat{\mathbf{e}}_b$ to point in the same direction as $\hat{\mathbf{i}}$, $\hat{\mathbf{f}}_b$ points in the same direction as $\hat{\mathbf{k}}$, and $\hat{\mathbf{n}}_b$ points in the same direction as $\hat{\mathbf{j}}$**.** Note that the body axes rotate as the disk moves from the side to the bottom and change their orientation with respect to the fixed x-y-z space axes. So at the bottom of the swing, $\hat{\mathbf{e}}_b$ still points in the same direction as $\hat{\mathbf{i}}$, but $\hat{\mathbf{f}}_b$ points in the same direction as $\hat{\mathbf{j}}$**,** and $\hat{\mathbf{n}}_b$ points in the same direction as $-\hat{\mathbf{k}}$.



Therefore the kinetic energy of rotation about the center of mass for this symmetric disk is given by

$$K_{cm,rot} = \frac{1}{2} I_e \omega_e{}^2 + \frac{1}{2} I_f \omega_f{}^2 + \frac{1}{2} I_n \omega_n{}^2 . \qquad (35)$$

When the disk is released from the side position the angular velocity is only due to the spin of the disk about the n-principal axes (the y-space axes) and so

$$\vec{\omega}_{side} = \omega \hat{\mathrm{n}}_b = \omega \hat{\mathrm{j}} . \qquad (36)$$

The moment of inertia of the disk about the n-principal axes is $I_n = (1/2)Mr^2$. Hence the kinetic energy of rotation about the center of mass when the disk is at the side position is

$$(K_{cm,rot})_{side} = \frac{1}{2} I_n \omega_n{}^2 = \frac{1}{2}(1/2) Mr^2\omega^2 = \frac{1}{4} Mr^2\omega^2 . \qquad (37)$$

Define the zero for potential energy at the bottom of the swing. Then the energy when the disk is released from rest from the side position is given by

$$E_{side} = U_{side} + (K_{cm,rot})_{side} = MgR + \frac{1}{4} Mr^2\omega^2 . \qquad (38)$$

When the disk is at the bottom of the swing, there are two contributions to the angular velocity of the disk,

$$\vec{\omega}_{bottom} = \omega \hat{\mathrm{n}}_b + \Omega(-\hat{\mathbf{e}}_b) = \omega(-\hat{\mathbf{k}}) + \Omega(-\hat{\mathbf{i}}) \qquad (39)$$

where we note that the angular speed ω has not changed since we have assumed that there is no torque acting on the axle of the disk that would slow the disk down. The moment of inertia of the disk about the e-principal axes is $I_e = (1/4)Mr^2$. Hence the kinetic energy of rotation about the center of mass when the disk is at the bottom position is

$$
\begin{aligned}
(K_{cm,rot})_{bot} &= \frac{1}{2} I_e \omega_e{}^2 + \frac{1}{2} I_n \omega_n{}^2 = \frac{1}{2}(1/4) M r^2 \Omega^2 + \frac{1}{2}(1/2) M r^2 \omega^2 \\
&= \frac{1}{8} M r^2 \Omega^2 + \frac{1}{4} M r^2 \omega^2 .
\end{aligned} \tag{40}
$$

When the disk is at the bottom of the swing the center of mass is moving with speed $v_{cm} = R\Omega$, and so the kinetic energy of the motion of the center of mass is

$$
(K_{cm,trans})_{bot} = \frac{1}{2} M v_{cm}{}^2 = \frac{1}{2} M R^2 \Omega^2 . \tag{41}
$$

At the bottom of the swing, the energy of the disk is just the kinetic energy and so using Eqs. (40) and (41) the energy of the disk when it is at the bottom of the swing is given by

$$
\begin{aligned}
E_{bot} = K_{bot} &= (K_{cm,trans})_{bot} + (K_{cm,rot})_{bot} = \frac{1}{2} M R^2 \Omega^2 + \frac{1}{8} M r^2 \Omega^2 + \frac{1}{4} M r^2 \omega^2 \\
&= M\Omega^2 \left(\frac{1}{2} R^2 + \frac{1}{8} r^2 \right) + \frac{1}{4} M r^2 \omega^2 .
\end{aligned} \tag{42}
$$

The pivot forces do no work and so the mechanical energy is conserved. Thus

$$
E_{side} = E_{bot} \tag{43}
$$

which becomes using Eqs. (38) and (42)

$$
MgR + \frac{1}{4} M r^2 \omega^2 = M\Omega^2 \left(\frac{1}{2} R^2 + \frac{1}{8} r^2 \right) + \frac{1}{4} M r^2 \omega^2 . \tag{44}
$$

We can solve Eq. (44) for Ω, the magnitude of the component of the angular velocity about the e-principal axis

$$
\Omega = \sqrt{\frac{2gR}{\left(R^2 + \frac{1}{4} r^2 \right)}} . \tag{45}
$$

(b) When the disk is at the bottom of the swing the angular momentum about the center of mass is

$$
\vec{L}_{cm,bot} = I_n \omega_n \hat{\mathrm{n}}_b + I_e \omega_e \hat{\mathbf{e}}_b = \frac{1}{2} M r^2 \omega (-\hat{\mathbf{k}}) + \frac{1}{4} M r^2 \Omega (-\hat{\mathbf{i}}) . \tag{46}
$$

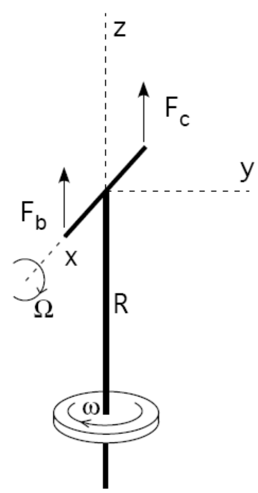
The angular momentum about the origin is then

$$\vec{L}_0 = \vec{r}_{0,cm} \times M\vec{v}_{cm} + \vec{L}_{cm,bot} = RMv_{cm}(-\hat{\mathbf{i}}) + \frac{1}{2}Mr^2\omega(-\hat{\mathbf{k}}) + \frac{1}{4}Mr^2\Omega(-\hat{\mathbf{i}})$$
$$= \frac{1}{2}Mr^2\omega(-\hat{\mathbf{k}}) + M\Omega\left(R^2 + \frac{1}{4}r^2\right)(-\hat{\mathbf{i}}). \tag{47}$$

where we used the fact that $v_{cm} = R\Omega$. We now substitute in our expression for Ω (Eq. (45)) into Eq. (47) and find that

$$\vec{L}_0 = \frac{1}{2}Mr^2\omega(-\hat{\mathbf{k}}) + M\sqrt{\frac{2gR}{\left(R^2 + \frac{1}{4}r^2\right)}}\left(R^2 + \frac{1}{4}r^2\right)(-\hat{\mathbf{i}})$$
$$= \frac{1}{2}Mr^2\omega(-\hat{\mathbf{k}}) + M\sqrt{2gR(R^2 + (1/4)r^2)}(-\hat{\mathbf{i}}). \tag{48}$$

When the disk is at the bottom of the swing, the torque about the origin due to the forces F_b and F_c shown in the figure below is given by



$$(\vec{\tau}_0)_{F_b,F_c} = \vec{r}_{0,F_b} \times \vec{F}_b + \vec{r}_{0,F_c} \times \vec{F}_c = -dF_b\hat{\mathbf{j}} + dF_c\hat{\mathbf{j}} = -d(F_b - F_c)\hat{\mathbf{j}}. \tag{49}$$

At the bottom of the swing the contribution to the angular momentum about the origin that points downward

$$\vec{L}_{0,z} = \frac{1}{2}Mr^2\omega(-\hat{\mathbf{k}}). \tag{50}$$

is changing direction as the disk moves, and the direction of the change is in the $-\hat{\mathbf{j}}$-direction. Therefore the change in angular momentum about the origin due to the contribution that points downward is given by

$$\frac{d\vec{L}_{0,z}}{dt} = \left|\vec{L}_{0,z}\right| \Omega(-\hat{\mathrm{j}}) = \frac{1}{2} M r^2 \omega \Omega(-\hat{\mathrm{j}}) . \tag{51}$$

There may be other torques acting at the pivot that will decrease Ω but that will only produce a change in angular momentum that points in the $-\hat{\mathbf{i}}$-direction, so

$$(\vec{\tau}_0)_{F_b, F_c} = \frac{d\vec{L}_{0,z}}{dt} . \tag{52}$$

Thus we can use substitute Eqs. (49) and (51) into Eq. (52) to find that

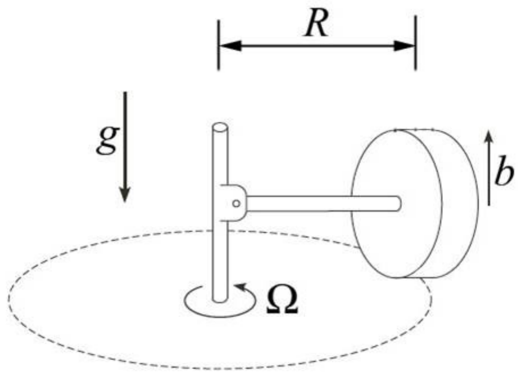
$$-d(F_b - F_c)\hat{\mathbf{j}} = \frac{1}{2} M r^2 \omega \Omega(-\hat{\mathrm{j}}) . \tag{53}$$

which we can now solve for $F_b - F_c$:

$$(F_b - F_c) = \frac{M r^2 \omega \Omega}{2d} . \tag{54}$$

Problem: Grain Mill

In a grain mill, grain is ground by a massive wheel that rolls without slipping in a circle on a flat horizontal surface driven by a vertical shaft. The rolling wheel has radius b and is constrained to roll in a horizontal circle of radius R at angular speed Ω about the vertical axis. Because of the stone's angular momentum, the contact force with the surface can be considerably greater than the weight of the wheel. In this problem, the angular speed Ω about the shaft is such that the contact force between the ground and the wheel is equal to twice the weight. The goal of the problem is to find Ω. Assume that the wheel is closely fitted to the axle so that it cannot tip. Neglect friction and the mass of the axle of the wheel. Express your answer in terms of R, b, M, Ω, and g as needed.



a) How is the angular speed ω of the wheel along the axle related to the angular speed Ω about the shaft?

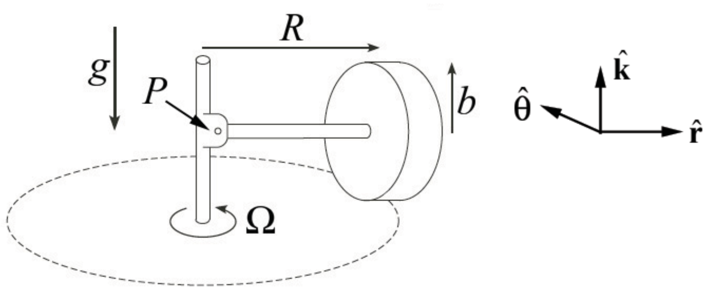
b) What is the horizontal component of the angular momentum vector about the point P in the figure above? Although we have not shown this, for this situation it is correct to compute the horizontal component of the angular momentum by completely ignoring the rotation of the mill wheel about the vertical axis, taking into account only the rotation of the mill wheel about its own axle.

c) Draw a free body force diagram for all the forces acting on the axle–wheel combination.

d) What is the torque about the joint (about the point P in the figure above) due to the forces acting on the axle–wheel combination? Your answer may include any of the given variables R, b, M, Ω and g, and also any forces that you introduced in the force diagram of part (c).

e) Use the torque equation of motion to find the value of Ω that doubles the contact force between the stone and the ground. Your final answer should be expressed in terms of R, b, M and g, as needed.

Solution: First off, here's a figure that shows the pivot point along with some convenient coordinate axes.



For rolling without slipping, the speed of the center of mass of the wheel is related to the angular spin speed by

$$v_{cm} = b\omega \,. \tag{55}$$

Also the speed of the center of mass is related to the angular speed about the vertical axis associated with the circular motion of the center of mass by

$$v_{cm} = R\Omega \,. \tag{56}$$

Therefore equating Eqs. (55) and (56) we have that

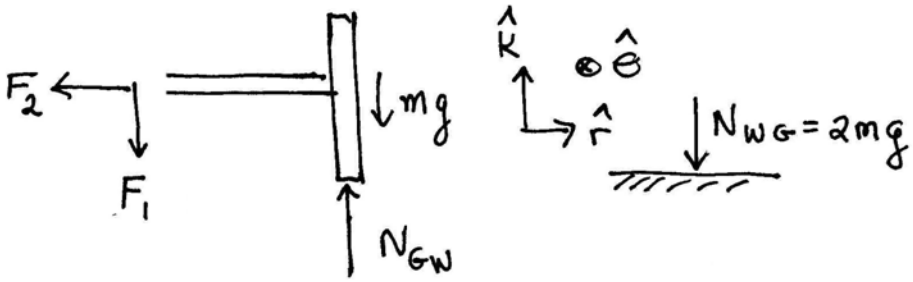
$$\omega = \Omega R / b \,. \tag{57}$$

b) The magnitude of the horizontal component of the angular momentum about the center of mass is the product of the angular velocity ω found in part (a), Equation (57), and the moment of inertia I_{cm} about an axis passing through the center off mass of the wheel perpendicular to the plane of the wheel. Assuming a uniform millwheel, $I_{\text{cm}} = (1/2)Mb^2$, the magnitude of the horizontal component of the angular momentum about the center of mass is

$$L_{\text{cm,h}} = I_{\text{cm}}\omega = \frac{1}{2}Mb^2\omega = \frac{1}{2}\Omega\, MRb \,. \tag{58}$$

The horizontal component of $\vec{\mathbf{L}}_{cm}$ is directed inward, and in vector form $\vec{\mathbf{L}}_{\text{cm}} = L_{\text{cm,h}}\left(-\hat{\mathbf{r}}\right)$ in the above coordinate system.

c) The axle exerts both a force and torque on the wheel, and this force and torque would be quite complicated. That's why we consider the forces and torques on the axle/wheel combination. The normal force of the wheel on the ground is equal in magnitude to $N_{WG} = 2mg$ so the third-law counterpart, the normal force of the ground on the wheel has the same magnitude $N_{GW} = 2mg$. The pivot (or hinge) at point P therefore must exert a force $\vec{F}_{\text{H,A}}$ on the end of the axle that has two components forces an inward force $\vec{F}_2$ to maintain the circular motion and a downward force $\vec{F}_1$ to reflect that the upward normal force is larger in magnitude than the weight.



d) About point P, $\vec{F}_{\mathrm{H,A}}$ exerts no torque. The normal force exerts a torque of magnitude $N_{GW}R = 2mgR$, directed out of the page, or, in vector form, $\vec{\tau}_{P,N} = -2mgR\hat{\boldsymbol{\theta}}$. The weight exerts a toque of magnitude MgR, directed into the page, or, in vector form, $\vec{\tau}_{P,mg} = MgR\hat{\boldsymbol{\theta}}$. So the torque about P is

$$\vec{\tau}_P = \vec{\tau}_{P,N} + \vec{\tau}_{P,mg} = -2mgR\hat{\boldsymbol{\theta}} + MgR\hat{\boldsymbol{\theta}} = -MgR\hat{\boldsymbol{\theta}} . \tag{59}$$

As the wheel rolls, the horizontal component of the angular momentum about the center of mass will rotate, and the inward-directed vector will change in the negative $\hat{\boldsymbol{\theta}}$-direction. Mathematically,

$$\frac{d\vec{L}_{cm,h}}{dt} = \left|\vec{L}_{cm,h}\right|\Omega(-\hat{\boldsymbol{\theta}}) = \frac{1}{2}\Omega\, MRb\Omega(-\hat{\boldsymbol{\theta}}) , \tag{60}$$

where we used Eq. (58) for the magnitude of the horizontal component of the angular momentum about the center of mass. This is consistent with the torque about P pointing out of the page in the above figure. We can now apply the torque condition that

$$\vec{\tau}_P = \frac{d\vec{L}_P}{dt} \tag{61}$$

that becomes using Eqs. (59) and (60)

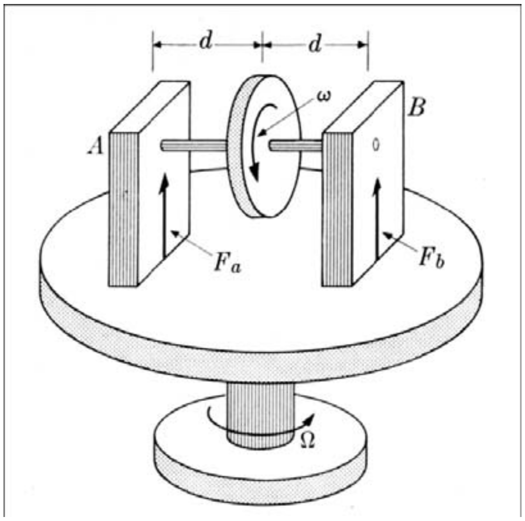
$$MgR(-\hat{\boldsymbol{\theta}}) = \frac{1}{2}\Omega^2 MRb(-\hat{\boldsymbol{\theta}}) \tag{62}$$

We can now solve Eq. (62) for the angular speed about the vertical axis

$$\Omega = \sqrt{\frac{2g}{b}} . \tag{63}$$

Gyroscope on Rotating Platform

A gyroscope consists of a axle of negligible mass and a disk of mass m and radius r mounted on a platform that rotates with angular speed Ω as shown in the figure below. The gyroscope is spinning with a spin angular speed ω. Forces F_a and F_b act on the gyroscopic mounts. The goal of this problem is to find the magnitudes of the forces F_a and F_b. You may assume that the moment of inertia of the gyroscope about an axis passing through the center of mass normal to the plane of the disk is given by I_n.



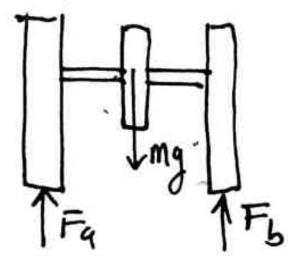
a) Calculate the torque about the center of mass of the gyroscope.

b) Calculate the angular momentum about the center of mass of the gyroscope.

c) Using Newton's Second Law find a relationship between F_a and F_b and the mass m of the gyroscope and g where g is the gravitational constant.

d) Use the torque equation and Newton's Second Law to find expressions for F_a and F_b.

Solution



The vertical forces sum to zero since there is no vertical motion.

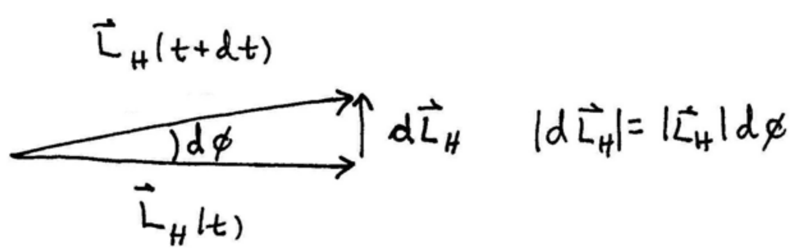
$$F_a + F_b = mg$$

Taking the direction into the board as being positive

$$\tau = F_a d - F_b d = (F_a - F_b)d$$

To complete the expression for the difference of the forces we need a value for τ.

Looking down on the gyroscope from above one has the following view. The $d\vec{L}$ in this diagram is in the same direction as the torque computed in the previous slide. $\vec{L_H}$ is the horizontal component of the angular momentum; the vertical component is not changeing with time.



$$\begin{aligned} |\vec{L_H}| &= I\omega \\ |d\vec{L}| &= |L_H| d\phi \\ \frac{|d\vec{L}_H|}{dt} &= |\vec{L_H}|\frac{d\phi}{dt} = I\omega\Omega \end{aligned}$$

$$
\begin{aligned}
F_a + F_b &= mg \\
F_a - F_b &= |\tau|/d = \frac{|d\vec{L}_H|}{dt}/d = I\omega\Omega/d
\end{aligned}
$$

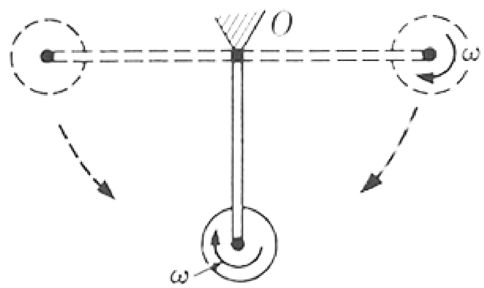
$$
\begin{aligned}
F_a &= (1/2)(mg + I\omega\Omega/d) \\
F_b &= (1/2)(mg - I\omega\Omega/d)
\end{aligned}
$$

Note that if $\Omega = mgd/I\omega$, $F_b = 0$ and one could remove the right hand support. This is just the expression for simple gyroscopic motion.

Problem: ***Angular Momentum and Torque***

Consider a pendulum consisting of a thin bar of length L which can rotate about a horizontal axis through its upper end as shown in the figure below. At its other end the bar supports a disc of radius $R << L$ and mass m which spins with a constant angular velocity $\vec{\boldsymbol{\omega}}$ about a horizontal axis through its center. The mass of the bar may be neglected.

(a) The pendulum is released from its horizontal position. What must be the magnitude and direction of the spin angular velocity of the disc to make the total angular momentum of the pendulum, with respect to the point *O*, be zero at the lowest point of the pendulum? Assume that the pendulum swing is from left to right.



(b) What is the total angular momentum about *O* when it swings from right to left through its lowest point?

Solutions:

Note that we have to be careful with our terminology; an unqualified "*L*" is a length, not an angular momentum.

(a) For zero total angular momentum, the magnitude $\omega I_{\text{disc}} = |\vec{\boldsymbol{\omega}}| I_{\text{disc}}$ of the spin angular momentum must be equal to the magnitude L_{orbital} of the orbital angular momentum,

$$L_{\text{orbital}} = |m\,\vec{\mathbf{r}} \times \vec{\mathbf{v}}| = mLv \tag{2.1}$$

where v is the translational speed of the disc at the bottom of the swing. The most direct way to find the speed is to recognize that since the disc does not change its rotational speed, the change in the disc's kinetic energy is reflected in the speed v, specifically

$$mgL = \frac{1}{2}mv^2 \;\Rightarrow\; v = \sqrt{2gL}\,, \tag{2.2}$$

the same result as if the disc were a non-rotating point mass. Combining Equations (2.1) and (2.2) with the condition that the magnitudes of the spin and orbit angular momenta are equal.

$$\omega = \frac{m\sqrt{2gL^3}}{I_{\mathrm{cm}}}\,. \tag{2.3}$$

From the right-hand rule, the direction of the orbital angular momentum is out of the page in the figure, and so the direction of the spin angular velocity must be into the page, as indicated in the figure.

Although not specified in the problem, if the disc is approximated as being uniform (an approximation at best, due to the presence of the frictionless bearing at the axle), we have that $I_{\mathrm{cm}} = mR^2/2$, and Equation (2.3) becomes

$$\omega = \frac{\sqrt{8gL^3}}{R^2}\,. \tag{2.4}$$

Note that the result, either Equation (2.3) or Equation (2.4), has the proper dimensions of inverse time.

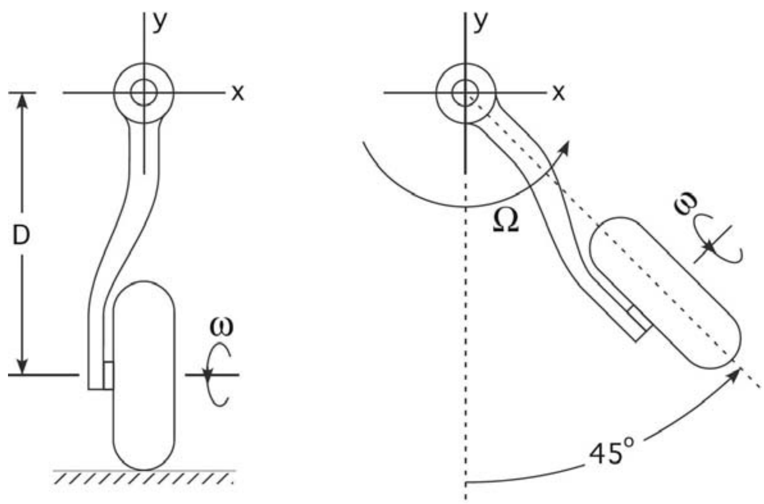
(b) The spin angular momentum is the same, but the orbital angular momentum has changed direction but has the same magnitude. The total angular momentum is twice the spin angular momentum,

$$\left|\vec{\mathbf{L}}_{\mathrm{total}}\right| = 2\omega I_{\mathrm{cm}} \tag{2.5}$$

with direction into the page in the above figure.

Problem Airplane Wheel

The figure shows a landing gear assembly that might be found on a small airplane. After the airplane takes off, the landing gear is retracted into the wing by rotation about the z-axis (which points out of the page in this figure). The wheel continues to spin with a constant angular frequency ω as it is being retracted. When the assembly is at an angle of 45 degrees it is rotating about the z-axis at a constant angular frequency Ω. Neglect all masses except that of the wheel, which has mass M. The wheel has a moment of inertia I_0 about its axle passing through the center of mass and I_d about a diameter.



a) What is the total angular momentum $\vec{L}$ of the wheel about the origin when the assembly is at 45 degrees? Express your answer as a vector with components along the x, y, and z directions.

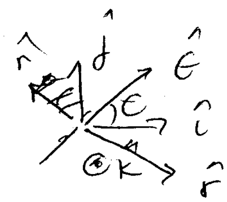
b) Which, if any, of the components you found in a) are changing with time?

c) What is the torque $\vec{\tau}_b$ that must be applied to the landing gear by the bearing at the origin when the assembly is at 45 degrees? Express your answer as a vector with components along the x, y, and z directions. Be sure that your answer would be correct even in the limit $\omega \to 0$.

$$\vec{L}_P = \vec{L}_{orbital} + \vec{L}_{spin}$$

$$= (mD^2 + I_d)\Omega\,\hat{k} + I_0\,\omega\,\hat{\theta}$$

$$\vec{L}_P = (mD^2 + I_d)\Omega\,\hat{k} + I_0\,\omega\,\frac{\sqrt{2}}{2}(\hat{i} + \hat{j})$$

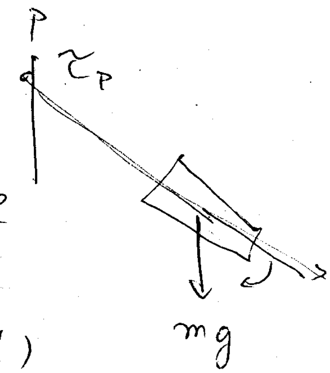


$$\hat{\theta} = \frac{\sqrt{2}}{2}(\hat{i} + \hat{j})$$

$$\hat{r} = \frac{\sqrt{2}}{2}(+\hat{i} + \hat{j})$$

$$\frac{d\vec{L}_P}{dt} = -I_c\,\omega\,\Omega\,\hat{r}$$

$$\frac{d\vec{L}_P}{dt} = -I_0\,\omega\,\Omega\,\frac{\sqrt{2}}{2}(\hat{i} - \hat{j})$$

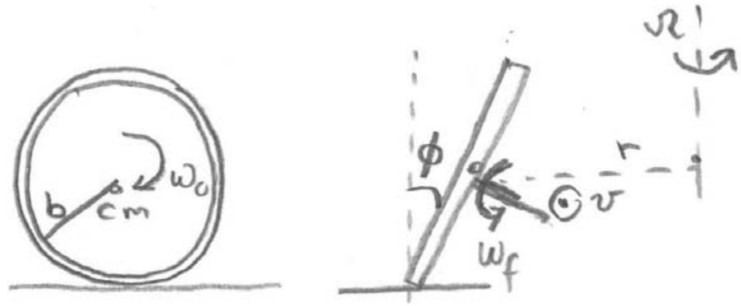


$$\tau_P^{total} = \vec{\tau}_{gear} + \vec{r}_{P,mg} \times m\vec{g} = \frac{d\vec{L}_P}{dt}$$

$$\vec{\tau}_{gear} - Dmg\frac{\sqrt{2}}{2}\hat{k} = +I_0\,\omega\,\Omega\frac{\sqrt{2}}{2}(\hat{j} - \hat{i})$$

$$\vec{\tau}_{gear} = Dmg\frac{\sqrt{2}}{2}\hat{k} + I_0\,\omega\,\Omega\frac{\sqrt{2}}{2}(\hat{j} - \hat{i})$$

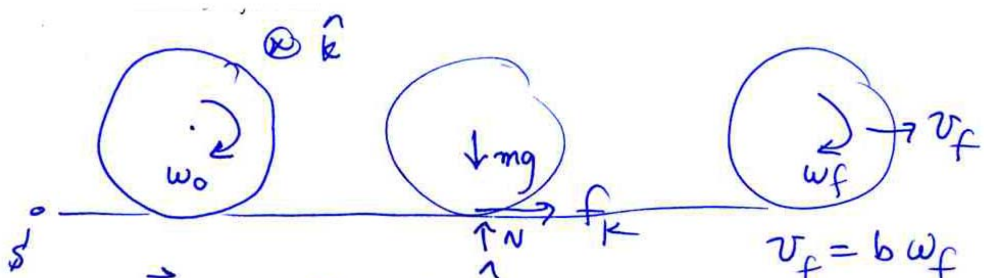
Problem 3: The center of mass of a bicycle wheel is initially at rest and the wheel is spinning with angular velocity ω_0. The wheel starts to skid forward on a level surface until it begins to roll without slipping on a level surface. The moment of inertia of the wheel about the center of mass is $I_{cm} = mb^2$ where b is the radius of the wheel and m is the mass of the wheel.



a) Draw a free body diagram of all the forces acting on the bicycle wheel while it is skidding forward.

b) What is the final angular velocity ω_f of the center of mass of the wheel when it begins to roll without slipping?

c) The wheel hits a bump in the ground causing the wheel to lean over, making a small angle ϕ with respect to a vertical line, and move in a circle of radius $r >> b$ and orbital angular velocity Ω. You may also assume that the final angular velocity $\omega_f >> \Omega$.



$$v_f = b\,\omega_f$$

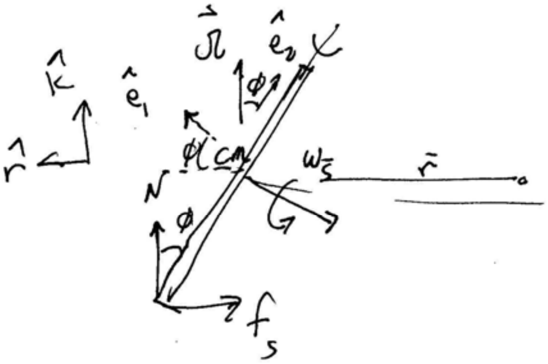
$$\vec{L}_{s,b} = I_{cm}\,\omega_0\,\hat{k}$$

$$\vec{L}_{s,f} = m b v_f\,\hat{k} + I_{cm}\,\omega_f\,\hat{k}$$

$$\vec{L}_{s,b} = \vec{L}_{s,f} \Rightarrow I_{cm}\,\omega_0 = m b v_f + I_{cm}\,\omega_f$$

$$I_{cm} = m b^2 \Rightarrow m b^2 \omega_0 = m b^2 \omega_f + m b^2 \omega_f = 2 m b^2 \omega_f$$

$$\Rightarrow \omega_f = \frac{\omega_0}{2}$$



$$\vec{\Omega} = \Omega\cos\phi\,\hat{e}_2 + \Omega\sin\phi\,\hat{e}_1$$

$$\vec{L}^{\,total}_{cm} = I_{cm,1}\,\omega_{s,1}\,\hat{e}_1 + I_{cm,2}\,\omega_{s,2}\,\hat{e}_2$$

$$\omega_{s,1} = -\omega_s + \Omega\sin\phi$$

$$\omega_{s,2} = \Omega\cos\phi$$

$$I_{cm,1} = mb^2, \quad I_{cm,2} = \tfrac{1}{2}mb^2$$

decompose

$$\vec{L}_{cm} = L_r\,\hat{r} + L_z\,\hat{k}$$

$$\hat{e}_1 = \cos\phi\,\hat{r} + \sin\phi\,\hat{k}$$

$$\hat{e}_2 = \cos\phi\,\hat{k} - \sin\phi\,\hat{r}$$

$$\vec{L}^{\,total}_{cm} = I_{cm,1}(-\omega_s + \Omega\sin\phi)(\cos\phi\,\hat{r} + \sin\phi\,\hat{k}) + I_{cm,2}(\Omega\cos\phi)(\cos\phi\,\hat{k} - \sin\phi\,\hat{r})$$

$$= mb^2(-\omega_s + \Omega\sin\phi)\cos\phi\,\hat{r} + \tfrac{1}{2}mb^2\Omega\cos^2\phi\,\hat{k} + mb^2(-\omega_s + \Omega\sin\phi)\sin\phi\,\hat{k} + -\tfrac{1}{2}mb^2\Omega\cos\phi\sin\phi\,\hat{r}$$

$$= \left(-mb^2\omega_s\cos\phi + \tfrac{1}{2}mb^2\Omega\sin\phi\cos\phi\right)\hat{r} + -mb^2\omega_s\sin\phi + mb^2\Omega\left(\sin^2\phi + \tfrac{1}{2}\cos^2\phi\right)\hat{k}$$

rolling without slipping $\quad b\,\omega_s = v = (b\sin\phi + r)\,\Omega$

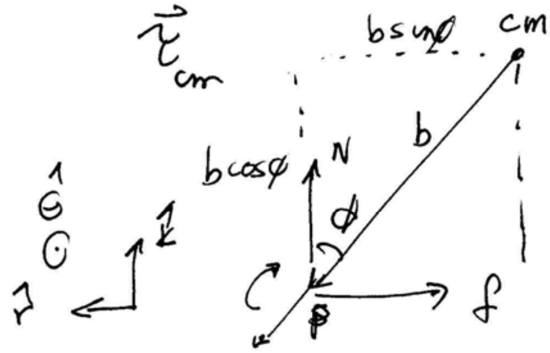
Assumptions: $\quad \omega_s \gg \Omega, \quad b\sin\phi \ll r$

$\Rightarrow \quad b\,\omega_s = r\,\Omega$

$$\vec{L}^{tot\phi}_{cm} = -mb^2\omega_s \cos\phi\, \hat{r} - mb^2\omega_s \sin\phi\, \hat{k}$$

$$\frac{d\vec{L}^{tot\phi}}{}_{cm} = -mb^2\omega_s \cos\phi\, \Omega\, \hat{\theta}$$

$\vec{\tau}_{cm}$



$$\vec{\tau}_{cm} = \vec{r}_{cm,P} \times (\vec{f} + \vec{N})$$

$$= (-b\cos\phi\, \hat{k} + b\sin\phi\, \hat{r}) \times (-f_s\, \hat{r} + N\, \hat{k})$$

$$= b\cos\phi f_s\, \hat{\theta} - b\sin\phi N\, \hat{\theta}$$

force equations:

$\hat{r}$: $\quad -f_s = -mr\Omega^2$

$\hat{\theta}$ $\quad 0 = 0$

$\hat{k}$: $\quad N - mg = 0$

Torque equation

$$\vec{\tau}_{cm} = \frac{d\vec{L}_{cm}}{dt}$$

$$\hat{\theta}: b\cos\phi\, mr\Omega^2 - b\sin\phi\, mg = -mb^2\omega_s\cos\phi\, \Omega$$

$$b\cos\phi\, m r \Omega^2 + m b^2 \omega_s \cos\phi\, \Omega = b \sin\phi\, m g$$

since $\quad b\omega_s = r\Omega \quad \Rightarrow$

$$b\cos\phi\, m r \Omega^2 + m \frac{b^2 r \Omega}{b} \cos\phi\, \Omega = b\sin\phi\, m g$$

$$\cos\phi \left(2 m r b\, \Omega^2 \right) = b \sin\phi\, m g$$

$$r = \frac{\tan\phi\, g}{2\Omega^2}$$