MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012

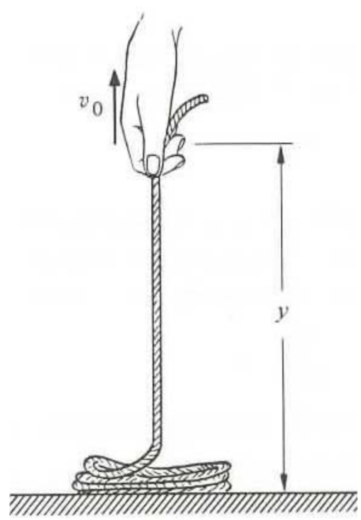
Problem Set 7 Solutions

Readings: (KK) Kleppner, Daniel and Kolenkow, Robert, An Introduction to Mechanics, McGraw Hill, Inc., New York, 1973, Chapter 4.

Problems: Chapter 4: 21, 23, 25, 27, 28, 30

Problem 21:

An unknown rope of linear mass density λ (mass per unit length), is coiled on a smooth horizontal table. On end is pulled straight up with constant speed v_0.

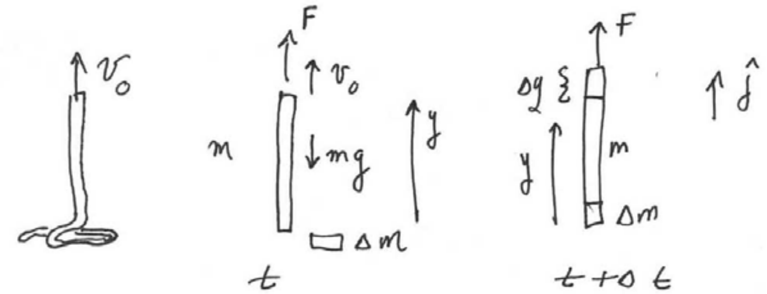


a) Find the force exerted on the end of the rope as a function of the height y of the rope above the table.

b) Compare the power delivered to the rope with the rate of change of the rope's total mechanical energy. Explain whether they should or should not be equal. Remember that the rope is not a rigid body.

4.21

a)



$$\vec{F}_{ext} = \lim_{\Delta t \to 0} \frac{\Delta \vec{p}}{\Delta t}$$

$$F - mg = \frac{(m+\Delta m)v_0 - m v_0}{\Delta t}$$

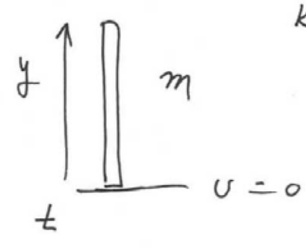
$$F = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} v_0 + mg \qquad (1)$$

Since $\Delta m = \lambda \Delta y$, $m = \lambda y$

$$\frac{dm}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \lambda \frac{\Delta y}{\Delta t} = \lambda v_0$$

eq (1) becomes $F = \lambda(v_0^2 + yg)$

b) Power $= F v_0 = \lambda(v_0^3 + y g v_0)$



$$K = \frac{1}{2} m v_0^2 \quad , \quad U = mg\frac{y}{2} = \frac{\lambda g y^2}{2}$$

$$E_{mech} = \frac{1}{2} m v_0^2 + \frac{\lambda g y^2}{2}$$

$$\frac{dE_{mech}}{dt} = \frac{1}{2}\frac{dm}{dt} v_0^2 + \lambda g y \frac{dy}{dt}$$

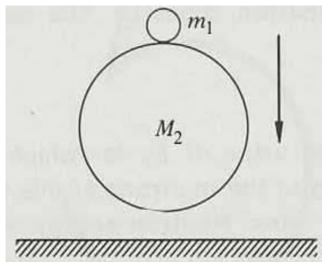
$$\frac{dE_{mech}}{dt} = \frac{1}{2}\lambda v_0^3 + \lambda g y v_0$$

$$\text{Power} - \frac{dE_{mech}}{dt} = \frac{1}{2}\lambda v_0^3$$

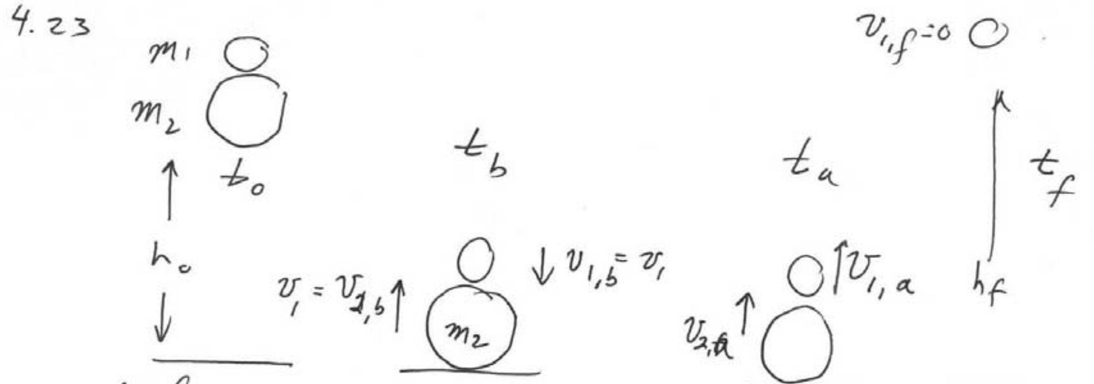
The power by the external force changes the mechanical energy and goes into the internal energy of the rope. Since the rope is not a rigid body, the molecules must be stretched in order to bring em up to speed v_0. Thus $\frac{\text{internal energy}}{\text{time}} = \frac{1}{2}\lambda v_0^3$.

Problem 23:

Two superballs are dropped from a height above the ground. The ball on top has a mass m_1. The ball on the bottom has a mass m_2. Assume that the lower ball collides elastically with the ground. Then as the lower ball starts to move upward, it collides elastically with the upper ball that is still moving downwards. How high will the upper ball rebound in the air? Assume that $m_2 >> m_1$. Hint: consider this collision from an inertial reference frame that moves upward with the same speed as the lower ball has after it collides with ground. What speed does the upper ball have in this reference frame after it collides with the lower ball?



4.23



initial height h_0

m_2 has colliding elastically with ground and is just about to collide with m_1

$K_b = \frac{1}{2} m_1 v_1^2$

after collision

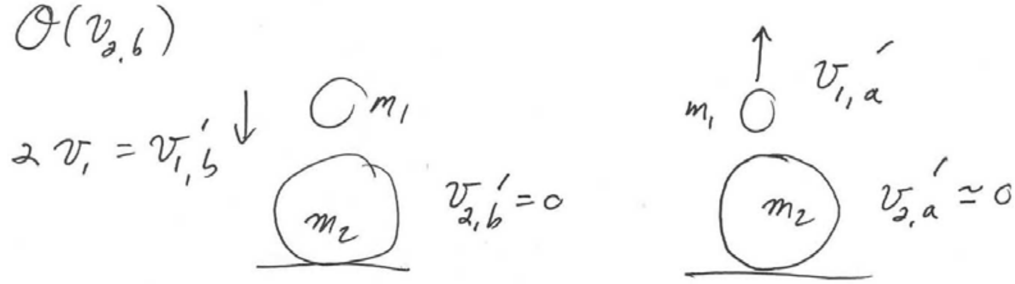
final height

$U_0 = m_1 g h$

$m_1 g h_0 = \frac{1}{2} m_1 v_1^2$

this collision is best analyzed from the reference frame of the observer moving upward with speed $v_{2,b}$, the velocity of m_2 just after it rebounded with the ground

$O(v_{2,b})$



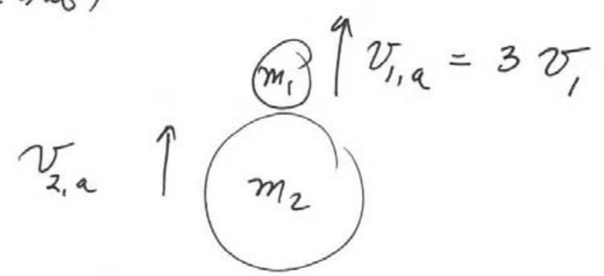
In this reference if we assume that $m_1 << m_2$, then m_2 remains at rest after the collision. Before the collision m_1 has velocity $v_{1,b}' = 2v_1$. Since the collision between m_1 and m_2 is perfectly (nearly) elastic ($m_1 << m_2$), m_1 rebounds with velocity $v_{1,a}' = 2v_1$.

Therefore m_1 goes upwards to a height

$$K_a = \frac{1}{2} m_1 (v_{1,a}')^2 = m_1 g h_f$$

However in the lab frame, m_1 is moving with speed $3v_1$

O(lab)



$$\frac{1}{2} m_1 (3v_1)^2 = m_1 g h_f$$

$$m_1 g h_f = 9 \frac{1}{2} m_1 v_1^2 = 9 m_1 g h_0$$

$$h_f = 9 h_0$$

Problem 25: ***(Elastic collision in two dimensions)***

A proton makes a head-on collision with an unknown particle at rest. The proton rebounds straight back with $4/9$ of its initial kinetic energy. Find the ratio of the mass of the unknown particle to the mass of the proton, assuming that the collision is elastic.

4.25

$\rightarrow \hat{i}$

$\bigcirc \rightarrow v_{1,0}$ $\quad$ $\bigcirc$ $\quad$ t_0

$m_p \equiv m_1$ $\quad$ m_2

$v_{1,f} \leftarrow \bigcirc$ $\quad$ $\bigcirc \rightarrow v_{2,f}$ $\quad$ t_f

m_1 $\quad$ m_2

$$\vec{P}_0 = \vec{P}_f$$

$$m_1 v_{1,0} = -m_1 v_{1,f} + m_2 v_{2,f} \qquad (1)$$

$$K_0 = K_f$$

$$\frac{1}{2} m_1 v_{1,0}^2 = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 \qquad (2)$$

proton rebounds elastically with

$$\frac{1}{2} m_1 v_{1,f}^2 = \frac{4}{9} \frac{1}{2} m_1 v_{1,0}^2 \Rightarrow v_{1,f} = \frac{2}{3} v_{1,0}$$

eq(2) $\Rightarrow$ $\frac{1}{2} m_1 v_{1,0}^2 - \frac{4}{9} \frac{1}{2} m_1 v_{1,0}^2 = \frac{1}{2} m_2 v_{2,f}^2$

$$\Rightarrow \left(\frac{5}{9} \frac{m_1 v_{1,0}^2}{m_2} \right)^{1/2} = v_{2,f} \qquad (3)$$

eq(1) $\Rightarrow$ $m_1 v_{1,0} = -m_1 \frac{2}{3} v_{1,0} + m_2 v_{2,f}$

$$\frac{m_1}{m_2} \frac{5}{3} v_{1,0} = v_{2,f} \qquad (4)$$

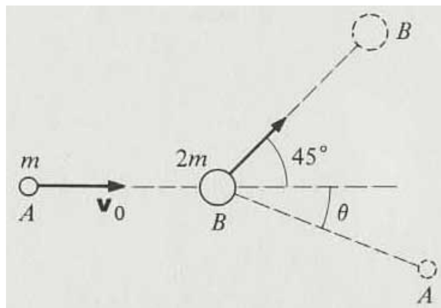
so $(\text{eq}(3))^2 = (\text{eq}(4))^2$ yields

$$\frac{5}{9} \frac{m_1}{m_2} v_{1,0}^2 = \left(\frac{m_1}{m_2} \right)^2 \frac{25}{9} v_{1,0}^2$$

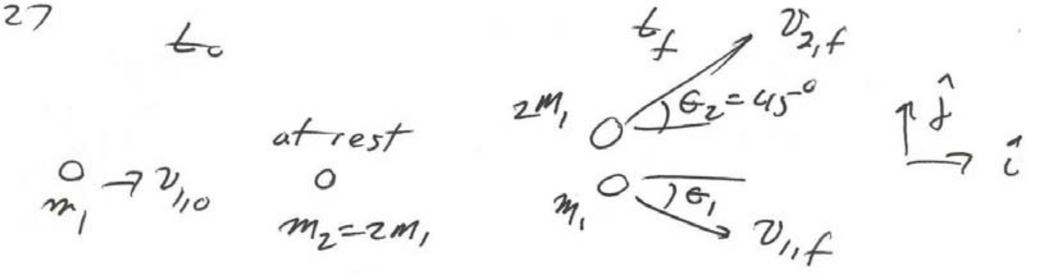
$$\Rightarrow \frac{m_2}{m_1} = 5.$$

Problem 27:

A particle A of mass m is initially moving in the positive x-direction with a speed $v_{A,0}$ and collides elastically with a second particle B of mass $2m$, which is initially at rest.. After the collision the particle A moves with an unknown speed $v_{A,f}$, at an unknown angle $\theta_{A,f}$ with respect to the positive x-direction. After the collision, particle B moves with an unknown speed $v_{B,f}$, at an angle $\theta_{2,f} = 45^o$ with respect to the positive x-direction. Find $\theta_{A,f}$.



4.27



$$\vec{P}_0 = \vec{P}_f$$

$\hat{i}$: $m_1 v_{1,0} = 2m_1 v_{2,f} \cos\theta_2 + m_1 v_{1,f} \cos\theta_1$

$$v_{1,0} = 2v_{2,f}\frac{\sqrt{2}}{2} + v_{1,f}\cos\theta_1 \qquad (1)$$

$\hat{j}$: $0 = 2m_1 v_{2,f} \sin\theta_2 - m_1 v_{1,f} \sin\theta_1$

$$2v_{2,f}\frac{\sqrt{2}}{2} = {}_{1,f}\sin\theta_1 \qquad (2)$$

eq (1) $v_{1,0} - v_{2,f}\sqrt{2} = v_{1,f}\cos\theta_1 \qquad (1a)$

square eq (2) and add to square of eq (1a)

$$2v_{2,f}^2 + (v_{1,0} - v_{2,f}\sqrt{2})^2 = v_{1,f}^2$$

$$2v_{2,f}^2 + v_{1,0}^2 - 2\sqrt{2}\, v_{1,0} v_{2,f} + v_{2,f}^2\, 2 = v_{1,f}^2$$

$$4v_{2,f}^2 - 2\sqrt{2}\, v_{1,0} v_{2,f} + v_{1,0}^2 = v_{1,f}^2 \qquad (3)$$

Conservation of energy

$$\frac{1}{2} m_1 v_{1,0}^2 = \frac{1}{2} 2 m_1 v_{2,f}^2 + \frac{1}{2} m_1 v_{1,f}^2$$

$$v_{1,0}^2 = 2 v_{2,f}^2 + v_{1,f}^2 \qquad (4)$$

Use eq (3) for $v_{1,f}^2$ in eq (4) to yield

$$v_{1,0}^2 = 2 v_{2,f}^2 + 4 v_{2,f}^2 - 2\sqrt{2}\, v_{1,0} v_{2,f} + v_{1,0}^2$$

This becomes

$$v_{2,f}^2 = 2\sqrt{2}\, v_{1,0} v_{2,f}$$

$$\frac{3}{\sqrt{2}} v_{2,f} = v_{1,0} \qquad (5)$$

divide eq (2) by eq (1a)

$$\frac{\sin\theta_1}{\cos\theta_1} = \frac{\sqrt{2}\, v_{2,f}}{v_{1,0} - \sqrt{2}\, v_{2,f}}$$

$$\tan\theta_1 = \frac{(\sqrt{2})\left(\frac{\sqrt{2}}{3} v_{1,0}\right)}{v_{1,0} - \sqrt{2}\frac{\sqrt{2}}{3} v_{1,0}} = \frac{\frac{2}{3} v_{1,0}}{\frac{1}{3} v_{1,0}}$$

$$\tan\theta_1 = 2$$

$$\theta_1 = \tan^{-1}(2) = 63.4^\circ$$

Problem 28:

A thin target of lithium is bombarded by helium nuclei of energy E_0. The lithium nuclei are initially at rest in the target but are essentially unbound. When a helium nucleus enters a lithium nucleus, a nuclear reaction can occur in which the compound nucleus splits apart into a boron nucleus and a neutron. The collision is inelastic, and the final kinetic energy is less than E_0 by $2.8\,\text{MeV}$. ($1\,\text{MeV}=10^6\,\text{eV}=1.6\times10^{-13}\,\text{J}$). The relative masses of the particles are: helium, mass 4; lithium, mass 7; boron mass 10; neutron, mass 1. The reaction can be symbolized

$$^7\text{Li} + ^4\text{He} \rightarrow ^{10}\text{B} + ^1\text{n} - 2.8\,\text{MeV}\,.$$

a) The minimum initial kinetic energy necessary for the reaction to take place is called the threshold energy, $E_{0,\text{threshold}}$. What is $E_{0,\text{threshold}}$ for which neutrons can be produced? What is the energy of the neutrons at this threshold?

b) Show that if the incident kinetic energy falls in the range

$$E_{0,\text{threshold}} < E_0 < E_{0,\text{threshold}} + 0.27\,\text{MeV}\,,$$

the neutrons ejected in the same direction as the incoming helium (forward direction) do not all have the same energy but must have one or the other of two possible energies. By considering the reaction in a reference frame moving with the velocity of the center of mass of the system, explain why there must be two distinct energies.

4.28

$\bigcirc \to v_{He}$ He $\qquad$ $\bigcirc$ Li at rest $\qquad$ $\to v_n$ $\to v_B$ $\qquad$ lab frame

define $K_0 = \frac{1}{2}\, 4m\, v_{He}^2$

$$v_{cm} = \frac{4m\, v_{He}}{4m + 7m} = \frac{4}{11} v_{He}$$

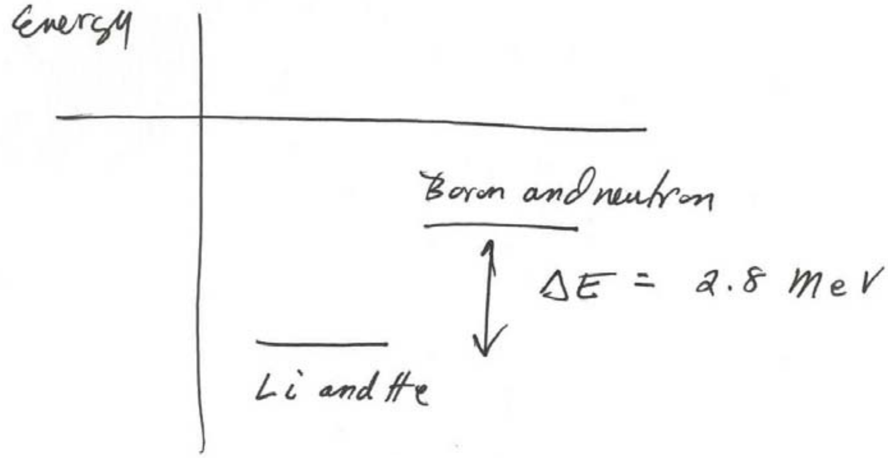
In the center of mass frame

$\bigcirc\left(\frac{4}{11} v_{He}\right)$ $\qquad$ $\vec{v}\,' = \vec{v} - \vec{v}_{cm}$

$$\underset{He}{\bigcirc} \to v_{He} - \frac{4}{11} v_{He} = v'_{He} \qquad \frac{4v_{He}}{11} \leftarrow \underset{Li}{\bigcirc}$$

in center of mass frame	B n at rest	threshold condition

When there is enough initial kinetic energy then in the center of mass frame, a Boron and neutron will be created. The definition of threshold if when the boron and neutron are at rest.



the internal energy of Li and He is lower than the Boron and neutron. So the kinetic energy in the center of mass frame at threshold will be just enough to create the new particles

$$\frac{1}{2} m_{He} v_{He}'^{2} + \frac{1}{2} m_{Li} v_{Li}'^{2} = 2.8 \text{ meV}$$

$$\left(\frac{1}{2}\right)(4m)\left(\frac{7}{11} v_{He}\right)^2 + \frac{1}{2} 7m \left(\frac{4}{11} v_{He}\right)^2 = 2.8 \text{ meV} \quad (1)$$

since $v_{He}' = \frac{7}{11} v_{He}$, $v_{Li}' = \frac{4}{11} v_{He}$

eq (1) simplifies to

$$\frac{1}{2}(4m)\,v_{He}^2\left(\left(\frac{7}{11}\right)^2 + \frac{7\cdot 4}{11^2}\right) = 2.8 \text{ meV}$$

$$\frac{1}{2}(4m)\,v_{He}^2\left(\frac{77}{121}\right) = 2.8 \text{ meV}$$

$$K_0 = \frac{1}{2}(4m)\,v_{He}^2 \qquad \text{so}$$

$$K_0\,\frac{7}{11} = 2.8 \text{ meV}$$

$$\boxed{K_0 = 4.4 \text{ meV}}$$

is the energy in the lab frame which will just create the boron and neutron at rest.

b) if $K_0 > 4.4$ meV then in the center of mass frame

case (1) $n \xleftarrow{(v'_{n,f})_1} \bigcirc \qquad \underset{B}{\bigcirc} \longrightarrow$ $\Rightarrow$ forward direction

case (2) $B \leftarrow \bigcirc \qquad \bigcirc \longrightarrow n$

are both possible final outcomes.

if $(v'_{n,f})_1 < v_{cm}$ in case (1) then in the lab frame

θ (lab) case (1) $\overset{n}{\bigcirc} \rightarrow (v_{n,f})_1$

$\overset{B}{\bigcirc} \rightarrow B$

case (2) $n \bigcirc \rightarrow (v_{n,f})_2$

$B \bigcirc \rightarrow B$

So both neutrons are moving forward but with different velocities

if $(v_{n,f})_1 = v_{cm}$

then the neutron will be at rest in lab frame for case (1)

θ (lab) $\quad \underset{n}{\bigcirc} \quad (v_{n,f})_1 = 0$

$\underset{v_{He}}{\bigcirc} \rightarrow \underset{Li}{\bigcirc} \qquad \underset{B}{\bigcirc} \rightarrow v_{B,f} \quad \rightarrow \hat{i}$

initial $\qquad\qquad$ final

then conservation of momentum

$$\vec{P}_0 = \vec{P}_f$$

$$4m\, v_{He} = 10m\, v_{B,f} \Rightarrow v_{B,f} = \frac{2}{5} v_{He}$$

The energy condition is

$$\frac{1}{2} m_{He} v_{He}^2 = \frac{1}{2} m_B v_B^2 + 2.8 \text{ MeV}$$

$$\frac{1}{2} 4m \, v_{He}^2 = \frac{1}{2} 10m \left(\frac{2}{5} v_{He}\right)^2 + 2.8 \text{ MeV}$$

$$\frac{1}{2} 4m \, v_{He}^2 \left(1 - \frac{2}{5}\right) = 2.8 \text{ MeV}$$

$$K_0 \, \frac{3}{5} = 2.8 \text{ MeV}$$

$$K_0 = (2.8 \text{ MeV})\left(\frac{5}{3}\right) = 4.67 \text{ MeV}$$

So for $4.4 \text{ MeV} \leq K_0 \leq 4.67 \text{ MeV}$
two neutrons will strike in the forward direction.

For $K_c \geq 4.67 \text{ MeV}$

only one neutron will strike in the forward direction (in the other possible outcome the neutron (case 1) is either at rest or moving backward in the lab frame).

Problem 30:

A particle of mass m_1 and velocity $\vec{v}_{1,0}$ by a particle of mass m_2 at rest in the laboratory frame is scattered elastically through a scattering angle Θ in the center of mass frame.

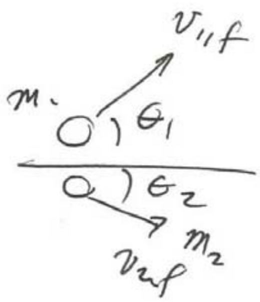
a) Find the final velocity of the incoming particle in the laboratory reference frame.

b) Find the fractional loss of kinetic energy of the incoming particle. Is this the same in every reference frame. Explain.

4.30 Second approach part a)

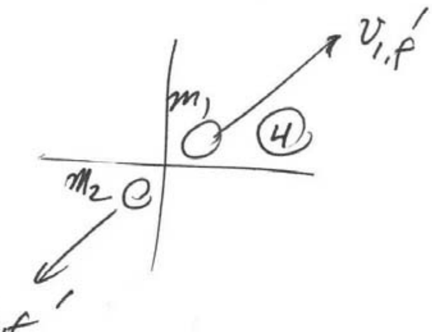
O (lab)

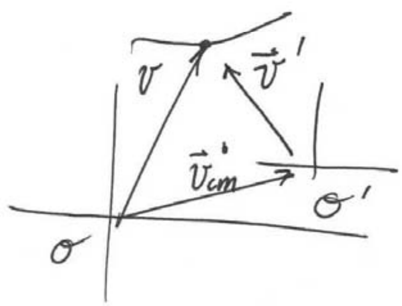
m_1 $\bigcirc \rightarrow v_{1,0}$ $\bigcirc$ m_2



O (cm)

m_1 $\bigcirc \rightarrow v_{1,0}'$ $v_{cm} \leftarrow \bigcirc$ m_2





$$\vec{v}' = \vec{v} - \vec{v}_{cm}$$

$$\vec{v} = \vec{v}' + \vec{v}_{cm}$$

$\hat{i}$: $v_{1,f}\cos\theta_1 = v_{1,f}'\cos 4 + v_{cm}$ (1)

$\hat{j}$: $v_{1,f}\sin\theta_1 = v_{1,f}'\sin 4$ (2)

$$\frac{\text{eq }(2)}{\text{eq }(1)} \Rightarrow \tan\theta_1 = \frac{v_{1,f}'\sin 4}{v_{1,f}'\cos 4 + v_{cm}} \quad (3)$$

squaring and adding eq (1) and eq (2) yields

$$v_{1,f}^2 = (v_{1,f}'\cos\Theta + v_{cm})^2 + (v_{1,f}'\sin\Theta)^2$$

$$v_{1,f}^2 = v_{1,f}'^2 + v_{cm}^2 + 2v_{cm}\, v_{1,f}'\cos\Theta \quad (5)$$

~~Conservation~~ In the center of mass frame

$$\vec{p}\,' = 0$$

Thus

$$m_1 v_{1,0}' = m_2 v_{cm} \quad (3)$$

$$m_1 v_{1,f}' = m_2 v_{2,f}' \quad (4)$$

Conservation of Energy in center of mass frame is

$$\frac{1}{2} m_1 v_{1,0}'^2 + \frac{1}{2} m_2 v_{cm}^2 = \frac{1}{2} m_1 v_{1,f}'^2 + \frac{1}{2} m_2 v_{2,f}'^2$$

Using eq (3) and eq (4)

$$\frac{1}{2} m_1 v_{1,0}'^2 + \frac{1}{2} m_2\left(\frac{m_1}{m_2} v_{1,0}'\right)^2 = \frac{1}{2} m_1 v_{1,f}'^2 + \frac{1}{2} m_2\left(\frac{m_1}{m_2} v_{1,f}'^2\right)$$

$$\frac{1}{2} m_1\left(1 + \frac{m_1}{m_2}\right) v_{1,0}'^2 = \frac{1}{2} m_1\left(1 + \frac{m_1}{m_2}\right) v_{1,f}'^2$$

$$\Rightarrow \quad v_{1,0}' = v_{1,f}'$$

similarly $\quad v_{cm} = v_{2,f}'$

$$v_{cm} = \frac{m_1 v_{1,0}}{m_1 + m_2}$$

$$v_{1,0}{}' = v_{1,0} - v_{cm} = \frac{m_2}{m_1 + m_2} v_{1,0}$$

Thus eq(5) becomes

$$v_{1,f}{}^2 = v_{1,f}{}'^2 + v_{cm}{}^2 + 2 v_{cm} v_{1,f}{}' \cos \Theta$$

becomes

$$= v_{1,0}{}'^2 + \left(\frac{m_1 v_{1,0}}{m_1 + m_2}\right)^2 + \left(\frac{2 m_1 v_{1,0}}{m_1 + m_2}\right) v_{1,0}{}' \cos \Theta$$

$$= \left(\frac{m_2}{m_1 + m_2} v_{1,0}\right)^2 + \left(\frac{m_1 v_{1,0}}{m_1 + m_2}\right)^2 + \frac{2 m_1}{m_1 + m_2} \frac{m_2}{m_1 + m_2} v_{1,0}{}^2 \cos \Theta$$

$$v_{1,f}{}^2 = v_{1,0}{}^2 \left(\frac{(m_1^2 + m_2^2)}{(m_1 + m_2)^2} + \frac{2 m_1 m_2}{(m_1 + m_2)^2} \cos \Theta\right)$$

$$v_{1,f} = \frac{v_{1,0}}{m_1 + m_2} \left(m_1{}^2 + m_2{}^2 + 2 m_1 m_2 \cos \Theta \right)^{1/2}$$

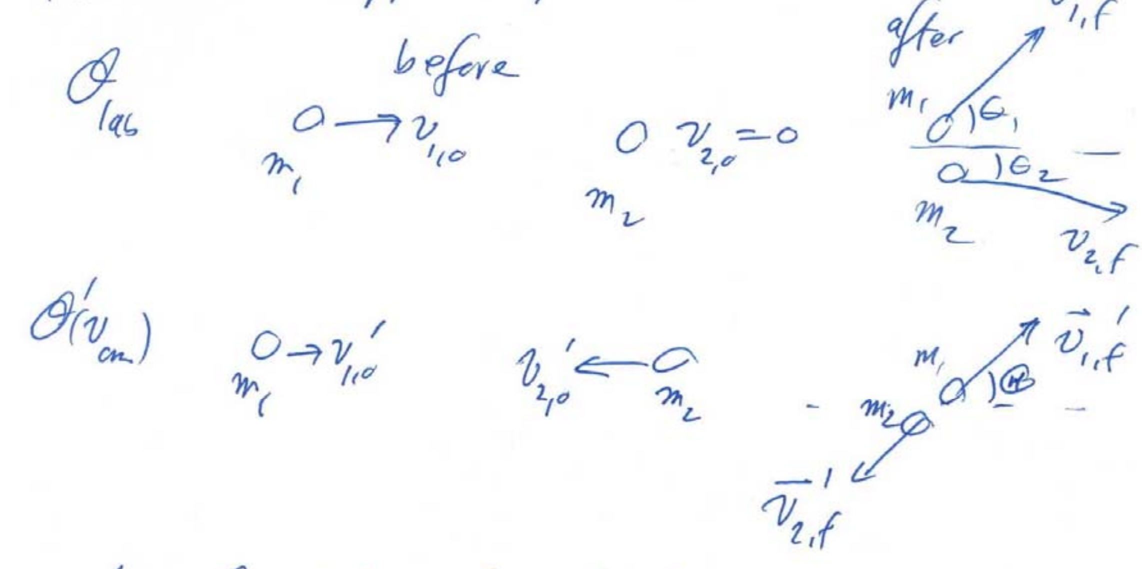
b) $\left(\frac{K_{0,f} - K_{f,1}}{K_{0,}}\right)_1 = \frac{\frac{1}{2} m_1 v_{1,0}^2 - \frac{1}{2} m_1 v_{1,f}^2}{\frac{1}{2} m_1 v_{1,0}^2}$

$$= \frac{v_{1,0}^2 - v_{1,f}^2}{v_{1,0}^2}$$

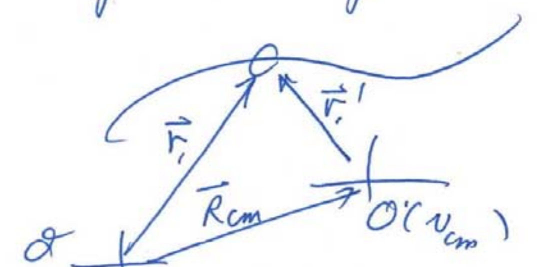
$$= \frac{v_{1,0}^2 - \left(\frac{v_{1,0}^2}{(m_1+m_2)^2} \left(m_1^2 + m_2^2 + 2 m_1 m_2 \cos\Theta \right) \right.}{v_{1,0}^2}$$

$$\Delta K = \frac{2 m_1 m_2}{(m_1+m_2)^2} (1 - \cos\Theta)$$

4.30 first approach part a)



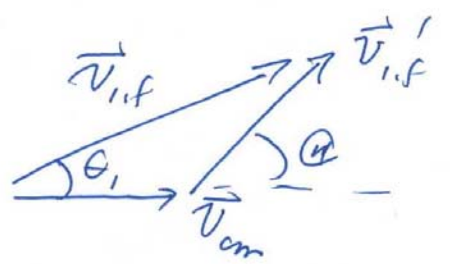
law of addition of velocities



$\vec{r}_1 = \vec{R}_{cm} + \vec{r}_1'$

$\vec{v}_1 = \vec{v}_{cm} + \vec{v}_1'$

So $\vec{v}_{1,f} = \vec{v}_{1,f}' + \vec{v}_{cm}$



law of cosines:

$$v_{1,f}^2 = v_{1,f}'^2 + v_{cm}^2 + 2 v_{1,f}' v_{cm} \cos \theta \quad (0)$$

In the $\mathcal{O}(v_{cm})$

$$v_{1,f}' = v_{1,0}'$$

the particle only changes direction not the magnitude of velocity.

Proof: in $\mathcal{O}(v_{cm})$ conservation of momentum and the fact that $(\vec{P}_T)' = 0$ yields

$$m_1 v_{1,0}' = m_2 v_{2,0}' \quad (1)$$

$$m_1 v_{1,f}' = m_2 v_{2,f}' \quad (2)$$

Conservation of energy

$$\frac{1}{2} m_1 v_{1,0}'^2 + \frac{1}{2} m_2 v_{2,0}'^2 = \frac{1}{2} m_1 v_{1,f}'^2 + \frac{1}{2} m_2 v_{2,f}'^2 \quad (3)$$

use eq (1) and eq (2) in eq (3)

$$\frac{1}{2} m_1 v_{1,0}'^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_2}\right)^2 v_{1,0}'^2 = \frac{1}{2} m_1 v_{1,f}'^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_2}\right)^2 v_{1,f}'^2$$

$$\Rightarrow \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1,0}'^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_{1,f}'^2 \Rightarrow v_{1,0}' = v_{1,f}'$$

So the law of cosines becomes

$$v_{1,f}^2 = v_{1,0}'^2 + V_{cm}^2 + 2 v_{1,0}' \, v_{cm} \cos \Theta \qquad (4)$$

center-of-mass velocity is

$$v_{cm} = \frac{m_1 v_{1,0}}{m_1 + m_2}$$

$$v_{1,0}' = v_{1,0} - v_{cm} = \frac{m_2 v_{1,0}}{m_1 + m_2}$$

So eq (4) becomes

$$v_{1,f}^2 = \left(\frac{m_2 v_{1,0}}{m_1 + m_2}\right)^2 + \left(\frac{m_1 v_{1,0}}{m_1 + m_2}\right)^2 + 2 \frac{m_2 v_{1,0}}{m_1 + m_2} \frac{m_1 v_{1,0}}{m_1 + m_2} \cos \Theta$$

$$= \left(\left(\frac{m_1^2 + m_2^2}{(m_1 + m_2)^2}\right) + \frac{2 m_2^2}{m_1 + m_2} \cos \Theta\right) v_{1,0}^2$$

$$v_{1,f} = \frac{v_{1,0}}{m_1 + m_2} \left(m_1^2 + m_2^2 + 2 m_1 m_2 \cos \Theta \right)^{1/2}$$