MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

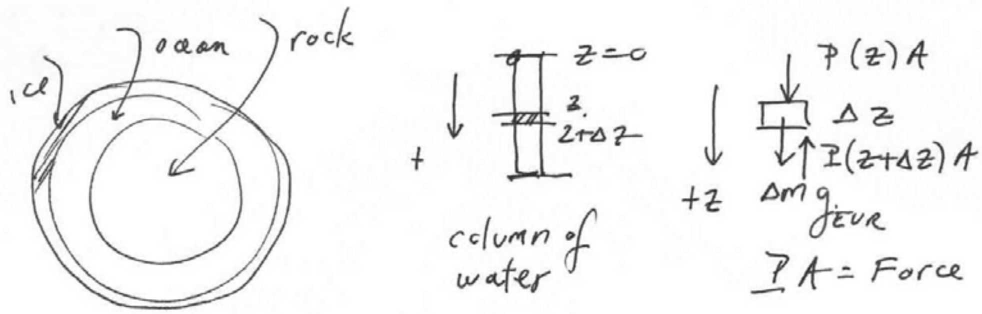
Physics 8.012

Problem Set 1 Solutions

Problem 1: (*Fermi Problem*)

One of the moons of Jupiter, Europa, is reported to have its surface covered by an ocean of water which is 100 km deep. The outermost 8 km are frozen as ice. The radius of Europa is approximately 1/4 the radius of the earth. Estimate the pressure at the bottom of Europa's ocean. (Note: there is some speculation that the combination of internal heat and water makes the ocean of Europa the best candidate in the solar system outside the earth for organized life to evolve.

1.



$$+\hat{z}: \quad P(z)A + \Delta m g - P(z+\Delta z)A = 0$$

$$\Delta m g = \left(P(z+\Delta z) - P(z)\right)A \qquad (1)$$

$$\Delta m = \rho_{H_2O} A \Delta z$$

eq (1) $\Rightarrow$ $$\rho g A \Delta z = \left(P(z+\Delta z) - P(z)\right)A$$

$$\rho_{H_2O}\, g_{EUr} = \frac{dP}{dz}$$

integrate $$\int_{z=0}^{z} \rho_{H_2O}\, g_{EUr}\, dz' = \int_{P_0 \cong 0}^{P(z)} dP$$

$$\rho_{H_2O}\, g_{EUr}\, z = P(z) - P_0$$

$$\boxed{P(z) = \rho g z \quad (2)}$$ pressure decreases with depth z

$$F_{grav,earth} = \frac{G M_E m}{R_E^2}$$

earth

$$F_{grav,europa} = \frac{G m M_{eur}}{R_{eur}^2}$$

europa

$$\frac{F_{grav,earth}}{F_{grav,eur}} = \frac{\dfrac{G M_E m}{R_E^2}}{\dfrac{G m_{eur} m}{R_{eur}^2}} = \frac{R_{Eur}^2}{R_{earth}^2} \frac{M_E}{M_{Eur}}$$

$$\frac{g_{earth}}{g_{Eur}} \simeq \frac{R_{earth}}{R_{Eur}}$$

$$g_{Eur} = \frac{R_{Eur}}{R_{earth}} g_{earth} \simeq \frac{1}{4} g_{earth}$$

assume $4 R_{Eur} \cong R_{earth}$

$$P(z) \cong \rho_{water}\, g_{Eur}\, z \cong \rho_{water} \frac{1}{4} g_{earth}\, z$$

$$P(100\,km) = \left(\frac{10^3\,kg}{m^3}\right)\left(\frac{1}{4}\right)\left(\frac{10\,m}{s^2}\right)\left(10^5\,m\right) \simeq 2.5\times10^8\,Pa$$

$$(P_{atm})_{earth} \simeq \frac{14\,lb}{in^2} = \frac{14\,lb}{in^2}\left(\frac{1\,kg}{2.2\,lb}\right)\left(\frac{9.8\,m}{s^2}\right)\left(\frac{in^2}{(2.54\times10^{-2}\,m)^2}\right)$$

$$(P_{atm})_{earth} \cong 1.0\times10^5\,Pa$$

Problem 2: (*Kinematics-One Dimension*)

A bus leaves a stop at MIT and accelerates at a constant rate for 5 seconds. During this time the bus traveled 25 meters. Then the bus traveled at a constant speed for 15 seconds. Then the driver noticed a red light 18 meters ahead and slams on the brakes. Assume the bus decelerates at a constant rate and comes to a stop some time later just at the light.

a) What was the initial acceleration of the bus?

b) What was the velocity at the bus after 5 seconds?

c) What was the braking acceleration of the bus? Is it positive or negative?

d) How long did the bus brake?

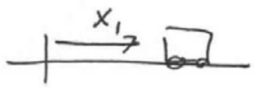
e) What was the distance from the bus stop to the light?

f) Make a graph of the position vs. time for the entire trip.

g) Make a graph of the velocity vs. time for the entire trip.

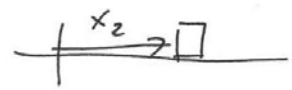
h) Make a graph of the acceleration vs. time for the entire trip.

Problem 2

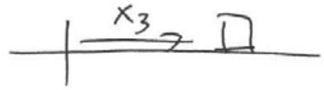
a) 

$$x_1 = \frac{1}{2} a_1 t_1^2 \Rightarrow a_1 = \frac{2x_1}{t_1^2}$$

$$a_1 = \frac{(2)(25\,m)}{(5\,s)^2} = 2\ m/s. \qquad v_1 = a_1 t_1 = \left(\frac{2m}{s^2}\right)(5s) = \frac{10m}{s}$$

b) reset $t=0$; 

$$x_2 = v_1 t_2 = \left(\frac{10m}{s}\right)(15s) = 150m$$

c) reset $t=0$ 

$$x_3 = v_1 t_3 + \frac{1}{2} a_3 t_3^2$$

$$v_3 = a_3 t_3 + v_1$$

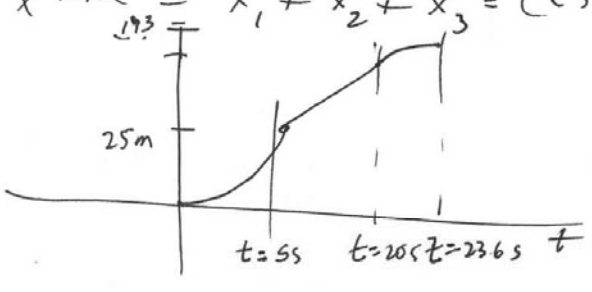
at end $v_3 = 0 \Rightarrow v_1 + a_3 t_3 = 0 \Rightarrow t_3 = -\frac{v_1}{a_3}$

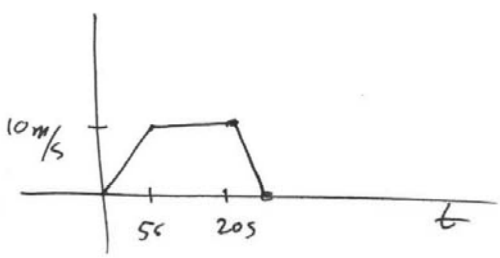
$$\Rightarrow x_3 = v_1\left(\frac{-v_1}{a_3}\right) + \frac{1}{2} a_3 \left(\frac{-v_1}{a_3}\right)^2 = -\frac{1}{2}\frac{v_1^2}{a_3}$$

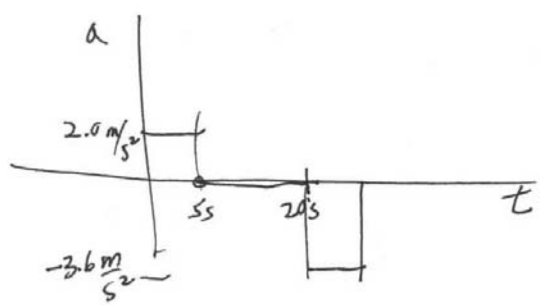
$$a_3 = -\frac{1}{2}\frac{v_1^2}{x_3} = \left(-\frac{1}{2}\right)\left(\frac{(10\,m/s)^2}{18\,m}\right) = -2.8\,\frac{m}{s^2}$$

$$t_3 = \frac{-v_1}{a_3} = \frac{-10\,m/s}{-2.8\,m/s^2} = 3.6\ s$$

$$x_{total} = x_1 + x_2 + x_3 = (25\,m) + (150\,m) + 18m = 193\,m$$







Problem 3: (*Kinematics-One Dimension*)

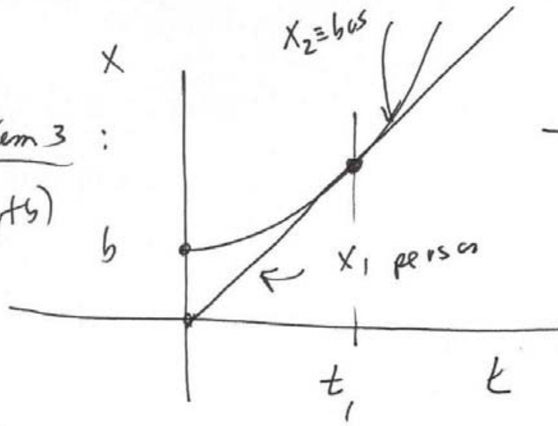
You are a running as fast as you can at a constant velocity, v_p, trying to catch a bus that is at rest at a bus stop. When you are still a distance b away from the bus stop, the bus starts to accelerate at a constant rate a_{bus}.

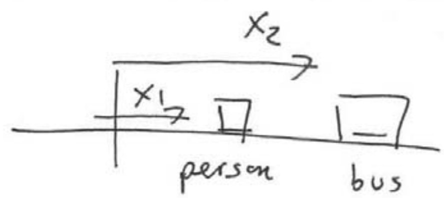
a) What is the minimum velocity that you need to run at in order to just catch the bus?

b) Draw graphs showing the motion of the bus and yourself.

c) How long did it take to catch the bus?

Problem 3 : part b)





$X_2 = b + \frac{1}{2} a t^2$, $v_2 = a t$

$X_1 = v t$, $v_1 = v$

at t_1: $v_2 = v_1 \Rightarrow (1)\ a t = v \Rightarrow t = v/a$

$X_2 = X_1 \quad (2)\ b + \frac{1}{2} a t^2 = v t \Rightarrow$

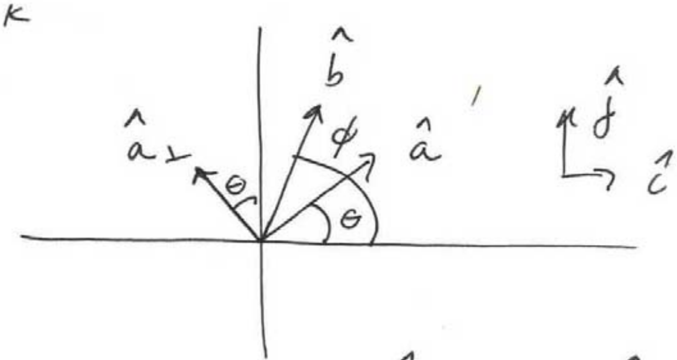
eq. (2) becomes $b + \frac{1}{2} a \left(\frac{v}{a}\right)^2 = (v)\left(\frac{v}{a}\right) \Rightarrow b = \frac{1}{2} \frac{v^2}{a}$

part a) $\boxed{v = (2 b a)^{1/2}}$, $t = \frac{v}{a} = \frac{1}{a}\left(2 b a\right)^{1/2} = \left(\frac{2b}{a}\right)^{1/2}$

Problem 4: K&K 1.7

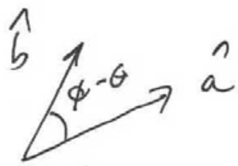
Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the *xy* plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$, and using vector algebra prove that $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Problem #7 KK



$$\hat{b} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}, \quad \hat{a} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{a}_\perp = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$



$$\hat{b}\cdot\hat{a} = \cos(\phi-\theta)\,|\hat{a}|\,|\hat{b}| = \cos(\phi-\theta)$$

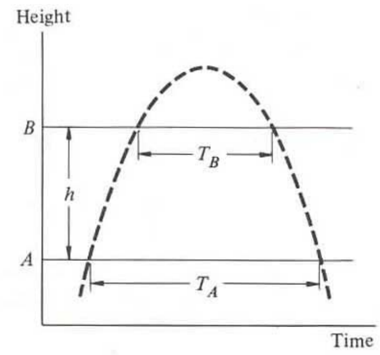
$$= (\cos\phi\,\hat{i} + \sin\phi\,\hat{j})\cdot(+\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$\hat{b}\cdot\hat{a} = \cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

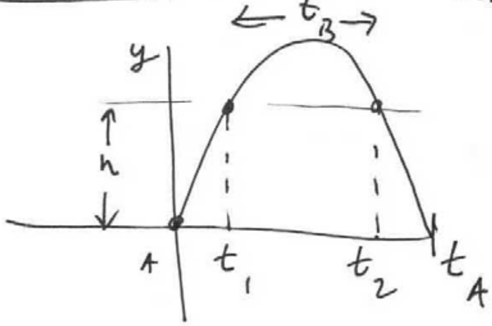
Problem 5: K&K 1.12

The acceleration of gravity can be measured by projecting a body upward and measuring the time that it takes to pass two given points in both directions. Show that if the time the body takes to pass a horizontal line A in both directions is T_A, and the time to go by a second line B is T_B, then, assuming that the acceleration is constant, its magnitude is

$$g = \frac{8h}{T_A^{\ 2} - T_B^{\ 2}}.$$



Problem 12 K.K



choose origin at A

$$t_1 = \frac{t_A - t_B}{2}$$

$$t_2 = t_1 + t_B = \frac{t_A + t_B}{2}$$

$y = h$ at t_1 and t_2

$y = 0$ at $t = 0$ and t_A

$$y = h = v_{y,0}\, t_1 - \tfrac{1}{2} g t_1^2 = v_{y,0}\, t_2 - \tfrac{1}{2} g t_2^2$$

$$0 = v_{y,0}\, t_A - \tfrac{1}{2} g t_A^2 \Rightarrow v_{y,0} = \tfrac{1}{2} g t_A$$

$$h = v_{y,0}\left(\frac{t_A - t_B}{2}\right) - \frac{1}{2} g \left(\frac{t_A - t_B}{2}\right)^2 \qquad (1)$$

$$h = v_{y,0}\left(\frac{t_A + t_B}{2}\right) - \frac{1}{2} g \left(\frac{t_A + t_B}{2}\right)^2$$

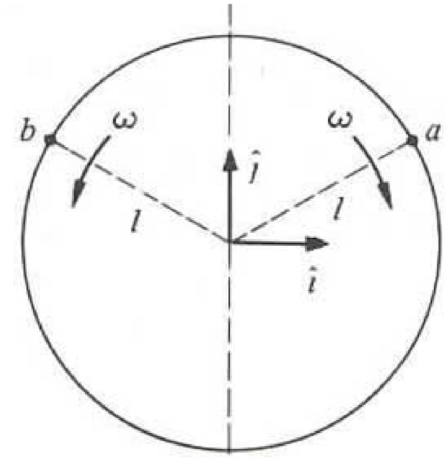
use $v_{y,0} = \frac{1}{2} g t_A$ in eq (1)

$$h = \frac{1}{2} g t_A \left(\frac{t_A - t_B}{2}\right) - \frac{1}{8} g \left(t_A^2 + t_B^2 - 2 t_A t_B\right)$$

$$\Rightarrow h = \frac{1}{8} g t_A^2 - \frac{1}{8} g t_B^2 \Rightarrow \boxed{\frac{8h}{t_A^2 - t_B^2} = g}$$

Problem 6: K&K 1.15

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer's coordinate system.)



a. A point is observed to have velocity $\vec{\mathbf{v}}_A$ relative to coordinate system A. What is its velocity relative to coordinate system B, which is displaced from system A by distance $\vec{\mathbf{R}}$? ($\vec{\mathbf{R}}$ can change in time.)

b. Particles a and b move in opposite directions around a circle with the magnitude of the angular velocity ω, as shown. At $t=0$ they are both at the point $\vec{\mathbf{r}} = l\hat{\mathbf{j}}$, where l is the radius of the circle. Find the velocity of a relative to b.

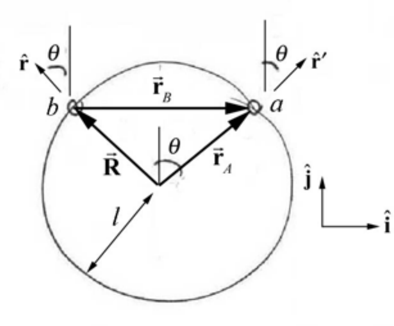
Solution: (a) The position vectors are related by

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}}\,. \tag{1.1}$$

Then velocities are related by the taking derivatives, (law of addition of velocities)

$$\vec{\mathbf{v}}_B = \vec{\mathbf{v}}_A - \vec{\mathbf{V}}\,. \tag{1.2}$$

(b) Let's choose two reference frames; frame B is centered at particle b, and frame A is centered at the center of the circle in the figure below.



Then the relative position vector between the origins of the two frames is given by

$$\vec{\mathbf{R}} = l\,\hat{\mathbf{r}}\,. \tag{1.3}$$

The position vector of particle a relative to frame A is given by

$$\vec{\mathbf{r}}_A = l\,\hat{\mathbf{r}}'\,. \tag{1.4}$$

The position vector of particle b in frame B can be found by substituting Eqs. (1.4) and (1.3) into Eq. (1.1),

$$\vec{\mathbf{r}}_B = \vec{\mathbf{r}}_A - \vec{\mathbf{R}} = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}}\,. \tag{1.5}$$

We can decompose each of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ with respect to the Cartesian unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (see figure)

$$\hat{\mathbf{r}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \tag{1.6}$$

$$\hat{\mathbf{r}}' = \sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\,. \tag{1.7}$$

Then Eq. (1.5) giving the position vector of particle b in frame B becomes

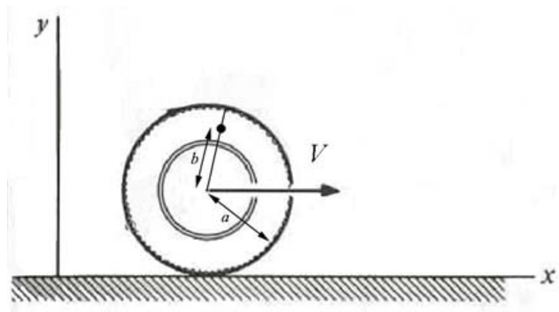
$$\vec{\mathbf{r}}_B = l\,\hat{\mathbf{r}}' - l\,\hat{\mathbf{r}} = l\,(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) - l\,(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) = 2l\sin\theta\,\hat{\mathbf{i}}\,. \tag{1.8}$$

In order to find the velocity vector of particle a in frame B (i.e. with respect to particle b), differentiate Eq. (1.8)

$$\vec{\mathbf{v}}_B = \frac{d}{dt}(2l\sin\theta)\,\hat{\mathbf{i}} = (2l\cos\theta)\frac{d\theta}{dt}\,\hat{\mathbf{i}} = 2\omega l\cos\theta\,\hat{\mathbf{i}}\,. \tag{1.9}$$

Problem 7: K&K 1.19

A bicycle wheel of radius a is rolling in a straight line without slipping at a constant horizontal velocity V. A bead is fixed to a spoke a distance b from the center of the wheel.



a) Find the position and velocity of the bead as a function of time as seen by an observer located at the center of the wheel and moving with the wheel. Make sure you use appropriate unit vectors in your answer.

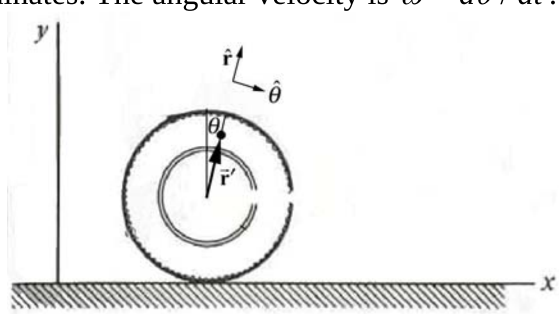
b) What is the position and velocity of the observer at the center of the wheel as seen by an observer fixed to the ground. Assume at $t = 0$ that the center of the wheel is directly over the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

c) What is the relation between the angular velocity of the wheel, ω, and the horizontal velocity, V, of the wheel?

d) Find the position and velocity of the bead as a function of time as seen by the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

Solution:

a) Choose a reference frame with an origin at the center of the wheel, and moving with the wheel. Choose polar coordinates. The angular velocity is $\omega = d\theta / dt$.



Then the bead is undergoing uniform circular motion with the position, velocity, and acceleration given by

$$\vec{\mathbf{r}}' = b\,\hat{\mathbf{r}}, \qquad \vec{\mathbf{v}}' = b\omega\,\hat{\theta}, \qquad \vec{\mathbf{a}}' = -b\omega^2\,\hat{\mathbf{r}} \tag{1.10}$$

Rolling without slipping:

Because the wheel is rolling without slipping, the velocity of a point on the rim of the wheel has speed $v_{rim} = a\omega$. This is equal to the linear speed of the center of mass of the wheel $V_{cm} = V$, thus

$$a\omega = V \tag{1.11}$$

or

$$\omega = V / a \tag{1.12}$$

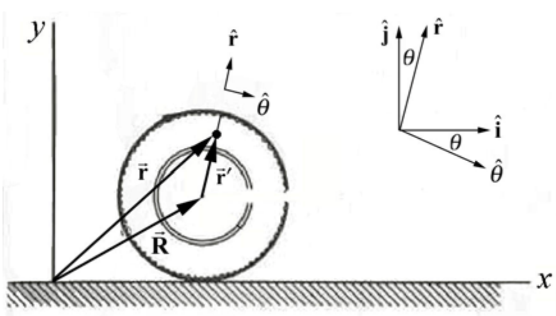
Note that the at $t = 0$, the angle $\theta = \theta_0 = 0$. So the angle grows in time as

$$\theta = \omega t = (V / a)t \tag{1.13}$$

So the velocity and acceleration of the bead with respect to the center of the wheel become

$$\vec{\mathbf{v}}' = (b / a)V\ \hat{\theta}, \quad \vec{\mathbf{a}}' = -(b / a^2)V^2\ \hat{\mathbf{r}} \tag{1.14}$$

b) Define a second reference frame fixed to the ground with choice of origin, Cartesian coordinates and unit vectors as shown in the figure below.



Then the relative position vector of the moving origin of the frame in part (a) to the origin in the second frame is given by

$$\vec{\mathbf{R}} = X\ \hat{\mathbf{i}} + a\ \hat{\mathbf{j}}. \tag{1.15}$$

The relative velocity of the two frames is the derivative

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt} = \frac{dX}{dt}\ \hat{\mathbf{i}} = V\ \hat{\mathbf{i}}. \tag{1.16}$$

Since the center of the wheel is moving at a uniform speed the relative acceleration of the two frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}}\,. \tag{1.17}$$

Note that the at $t = 0$, the angle $\theta = \theta_0 = 0$. So the angle grows in time as

$$\theta = \omega t\,. \tag{1.18}$$

Define the position, velocity, and acceleration in this frame (with respect to the ground) by

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \quad \vec{\mathbf{v}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}}, \quad \vec{\mathbf{a}} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}}\,. \tag{1.19}$$

Then the position vectors are related by

$$\vec{\mathbf{r}} = \vec{\mathbf{R}} + \vec{\mathbf{r}}'\,. \tag{1.20}$$

In order to add these vectors we need to decompose the position vector in the moving frame into Cartesian components,

$$\vec{\mathbf{r}}' = b\,\hat{\mathbf{r}} = b\sin\theta\,\hat{\mathbf{i}} + b\cos\theta\,\hat{\mathbf{j}}\,. \tag{1.21}$$

Then

$$\vec{\mathbf{r}} = \vec{\mathbf{R}} + \vec{\mathbf{r}}' = (X\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}}) + (b\sin\theta\,\hat{\mathbf{i}} + b\cos\theta\,\hat{\mathbf{j}}) = (X\,\hat{\mathbf{i}} + b\sin\theta\,\hat{\mathbf{i}}) + (a + b\cos\theta\,\hat{\mathbf{j}})\,. \tag{1.22}$$

Thus the position components of the bead with respect tot eh ground are given by

$$x = X + b\sin((V/a)t) \tag{1.23}$$

$$y = a + b\cos((V/a)t) \tag{1.24}$$

We can differentiate the position vector to find the velocity

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\big(X + b\sin((V/a)t)\big)\,\hat{\mathbf{i}} + \frac{d}{dt}\big(a + b\cos((V/a)t)\big)\,\hat{\mathbf{j}} \tag{1.25}$$

$$\vec{\mathbf{v}} = \big(V + (b/a)V\cos((V/a)t)\big)\,\hat{\mathbf{i}} - \big((b/a)V\sin((V/a)t)\big)\,\hat{\mathbf{j}} \tag{1.26}$$

Alternatively, we can decompose the velocity of the bead in the moving frame into Cartesian coordinates

$$\vec{\mathbf{v}}' = (b/a)V\left(\cos((V/a)t)\,\hat{\mathbf{i}} - \sin((V/a)t)\,\hat{\mathbf{j}}\right) \tag{1.27}$$

Then velocities are related by the law of addition of velocities

$$\vec{\mathbf{v}} = \vec{\mathbf{V}} + \vec{\mathbf{v}}' \tag{1.28}$$

so

$$\vec{\mathbf{v}} = V\,\hat{\mathbf{i}} + (b/a)V\left(\cos((V/a)t)\,\hat{\mathbf{i}} - \sin((V/a)t)\,\hat{\mathbf{j}}\right) \tag{1.29}$$

$$\vec{\mathbf{v}} = \left(V \ \ + (b/a)V\cos((V/a)t)\right)\hat{\mathbf{i}} - (b/a)\sin((V/a)t)\,\hat{\mathbf{j}} \tag{1.30}$$

in agreement with our previous result.

The acceleration is the same in either frame so

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}' = -(b/a^2)V^2\,\hat{\mathbf{r}} = -(b/a^2)V^2\left(\sin((V/a)t)\,\hat{\mathbf{i}} + \cos((V/a)t)\,\hat{\mathbf{j}}\right). \tag{1.31}$$

Problem 7: K&K 1.20

A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant, $A = (1/\pi)\ \mathrm{m \cdot rad^{-1}}$. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

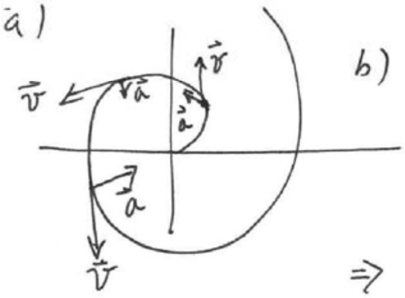
a. Sketch the motion, and indicate the approximate velocity and acceleration at a few points.
b. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}\ \mathrm{rad}$.
c. At what angles do the radial and tangential accelerations have equal magnitude?

Problem 2c: $r = A\theta$, $\theta = \frac{1}{2}\alpha t^2$,

$\dot r = A\dot\theta$, $\dot\theta = \alpha t$

$\ddot r = A\ddot\theta$, $\ddot\theta = \alpha$

a)



b) $a_r = \ddot r - r\dot\theta^2$

$= A\alpha - A\theta(\alpha t)^2 = 0$

$= A\alpha - A\frac{1}{2}\alpha t^2 \alpha^2 t^2 = 0$

$\Rightarrow A\alpha = A\frac{1}{2}\alpha^3 t^4 \Rightarrow$

$1 = \frac{1}{2}\alpha^2 t^4 \Rightarrow t^4 = \frac{2}{\alpha^2}$

$\theta^2 = \frac{1}{4}\alpha^2 t^4 = \frac{1}{4}\alpha^2 \frac{2}{\alpha^2} = \frac{1}{2}$, $\theta = \frac{1}{\sqrt 2}$ rad

c) $a_r = a_{tan} \Rightarrow \ddot r - r\dot\theta^2 = 2\dot r\dot\theta + r\ddot\theta$

$\Rightarrow A\alpha - A\frac{1}{2}\alpha^3 t^4 = 2A\alpha^2 t^2 + A\frac{1}{2}\alpha t^2\alpha$

$\Rightarrow \cancel{A}\ \cancel{\alpha}\ 1 - \frac{1}{2}\alpha^2 t^4 = \frac{5}{2}\alpha t^2$

$\frac{1}{2}\alpha^2 t^4 + \frac{5}{2}\alpha t^2 - 1 = 0$ solve for t^2

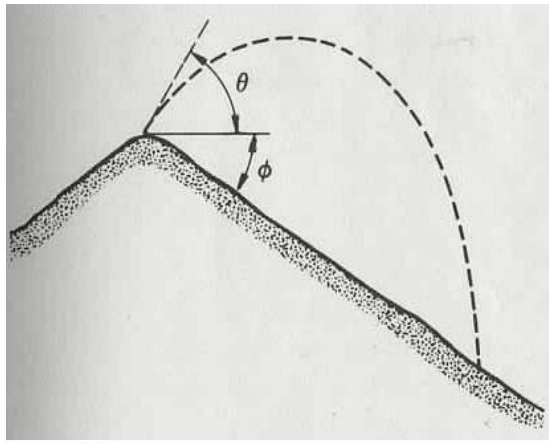
$$t^2 = \frac{-\frac{5}{2}\alpha \pm \left(\left(\frac{5}{2}\alpha\right)^2 - (4)\left(\frac{1}{2}\alpha^2\right)(-1)\right)^{1/2}}{(2)\left(\frac{1}{2}\alpha^2\right)}$$

$$= \frac{-\frac{5}{2}\alpha \pm \left(\frac{25}{4}\alpha^2 + 2\alpha^2\right)^{1/2}}{\alpha^2} = \frac{-\frac{5}{2} \pm \frac{1}{2}(33)^{1/2}}{\alpha}$$

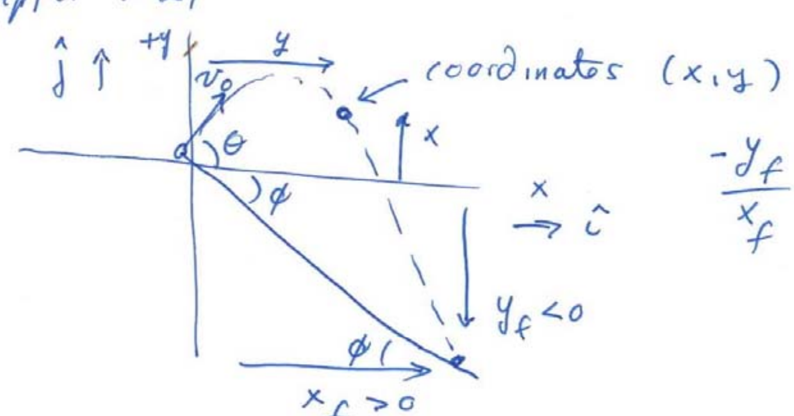
$$\theta = \frac{1}{2}\alpha t^2 = \alpha\left(\frac{1}{2}\right)\frac{\left(\frac{\sqrt{33}}{2} - \frac{5}{2}\right)}{\alpha} = \frac{1}{4}\left(\sqrt{33} - 5\right) \text{ or } -\frac{1}{4}\left(\sqrt{33}+5\right)$$

Problem 9: K&K 1.21

A person throws a rock from the top of a hill. The hill slopes downward uniformly at angle ϕ. At what angle θ from the horizontal should the person throw the rock so that it has the greatest range?


θ
φ

Chapter 1.21



$$\frac{-y_f}{x_f} = \tan\phi \qquad (1)$$

initial conditions: at $t = 0$

$$x_0 = y_0 = 0$$
$$v_{x,0} = v_0 \cos\theta$$
$$v_{y,0} = v_0 \sin\theta$$

equations of motion: at t

$$x = v_{x,0}\,t = v_0 \cos\theta\, t$$
$$y = v_{y,0}\,t - \frac{1}{2} g t^2 = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

final state at t_f

$$x_f = v_0 \cos\theta\, t_f \qquad (2)$$
$$y_f = v_0 \sin\theta\, t_f - \frac{1}{2} g t_f^2 \qquad (3)$$

Substitute y_f from eq (1) into eq (3)

$$-x_f \tan\phi = v_0 \sin\theta\, t_f - \frac{1}{2} g t_f^2 \qquad (3a)$$

Solve eq (2) for $t_f = \dfrac{x_f}{v_0 \cos\theta}$ and substitute into eq (3a)

$$-x_f \tan\phi = \frac{v_0 \sin\theta\, x_f}{v_0 \cos\theta} - \frac{1}{2} g \frac{x_f^2}{v_0^2 \cos^2\theta} \qquad (4)$$

Solve eq (4) for x_f

$$x_f = \frac{2 v_0^2 \cos^2\theta}{g} \left(\tan\theta + \tan\phi \right)$$

$$= \frac{2 v_0^2}{g} \left(\cos\theta \sin\theta + \cos^2\theta \tan\phi \right) \quad (5)$$

The condition that you want to find the angle θ that maximizes the distance x_f is

$$\frac{d x_f}{d\theta} = 0 \quad (6)$$

Thus using eq (5) and eq (6)

$$0 = \frac{2 v_0^2}{g} \frac{d}{d\theta} \left(\frac{1}{2} \sin(2\theta) + \cos^2\theta \tan\phi \right)$$

where $\cos\theta \sin\theta = \frac{\sin(2\theta)}{2}$

Taking derivatives yields

$$0 = \frac{2 v_0^2}{g} \left(\cos(2\theta) + -2 \cos\theta \sin\theta \tan\phi \right)$$

This implies (again using $2\cos\theta\sin\theta = \sin 2\theta$)

$$\cos(2\theta) - \sin 2\theta \tan\phi = 0 \quad (7)$$

Solving eq (7) for cotan 2θ

$$\frac{\cos(2\theta)}{\sin(2\theta)} = \text{cotan}(2\theta) = \tan\phi$$

$$\theta = \frac{1}{2}\,\text{cotan}^{-1}(\tan\phi)$$

When $\phi = 60^\circ$, $\theta = 15^\circ$.