MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012, Fall 2010

Problem Set 9 Solutions

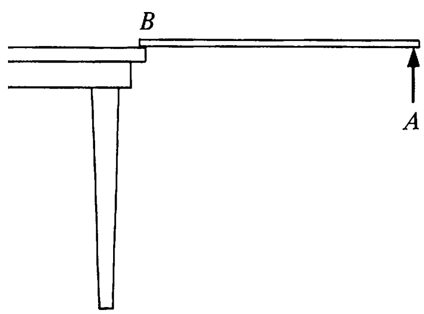
Due: Friday, November 19

Reading: Kleppner and Kolenkow, *An Introduction to Mechanics*, Chapter Six

Problem 1: K&K 6.14

Problem

A uniform stick of mass m and length l is suspended horizontally with end B at the edge of a table and the other end A is held by hand. Point A is suddenly released. At the instant after release:



(a) What is the torque about the end B on the table?

(b) What is the angular acceleration about the end B on the table?

(c) What is the vertical acceleration of the center of mass?

(d) What is the vertical component of the hinge force at B? Does the hinge force have a horizontal component at the instant after release?

Solution

(a) The center of mass is at a distance $l/2$ from the table, and so the torque, treating it as a point mass, is $mgl/2$. Alternatively, $\vec{\tau} = \int_0^l \lambda(x)gx\,dx = \frac{\lambda l^2 g}{2} = mlg/2$.

(b) The angular acceleration is

$$\begin{aligned}
\frac{mlg/2}{\int_0^l \lambda r^2\,dr} &= \frac{mlg/2}{\frac{m}{l}\int_0^l r^2\,dr} \\
&= \frac{mlg/2}{\frac{m}{l}\int_0^l r^2\,dr} \\
&= \frac{mlg/2}{\frac{m}{l}\cdot\frac{l^3}{3}} \\
&= \frac{mlg/2}{m\frac{l^2}{3}} \\
&= \frac{g/2}{\frac{l}{3}} \\
&= \frac{3g}{2l}
\end{aligned}$$

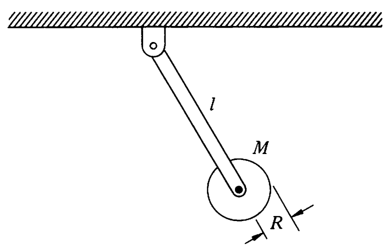
(c) The vertical acceleration is $r\alpha = \frac{3gl}{4l} = \frac{3}{4}g$.

(d) The vertical component of the hinge force is $m\frac{g}{4}$. The hinge force has no horizontal component, because the center of mass is not accelerating in the horizontal direction.

Problem 2: K&K 6.18

Problem

A physical pendulum consists of a disc of radius R and mass m_d fixed at the end of a rod of mass m_r and length l.



(a) Find the period of the pendulum.

(b) How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin?

Solution

(a) Since the pendulum is a rigid body, and the the center of mass is at length $\frac{m_d l + m_r l/2}{m_d + m_r}$, the forces can be taken to act at this point. Let θ be the angle from the vertical. Then $\tau = \frac{m_d l + m_r l/2}{m_d + m_r}(m_d + m_r)g\sin\theta = (m_d l + m_r l/2)g\sin\theta$. Then $\frac{dL}{dt} = (m_d l + m_r l/2)g\sin\theta$.

Theorem 2.1 (Parallel Axis Theorem)**.** *Suppose ℓ_1 and ℓ_2 are parallel axes, and ℓ_1 is through the center of mass, and $\vec{d} = d\hat{d}$ is the vector from ℓ_2 and ℓ_1. Suppose that s is a point on ℓ_1 and*

$s' = s + \vec{d}$ is a point on ℓ_2. Then, for any rigid object of total mass m_T, rotating about the fixed axis ℓ_2,

$$I_{\ell_1} + m_T d^2 = I_{\ell_2} \tag{1}$$

Proof.

$$\begin{aligned}
(\vec{L}_{s'})_{\ell_2} &= \sum_i (\vec{r}_{s'i} \times m_i \vec{v}_i)_{\ell_2} \\
&= \sum_i (\vec{r}_{s'i} \times m_i(\vec{\omega}_i \times \vec{r}_{s'i}))_{\ell_2} \\
&= \sum_i (\vec{r}_{s'i} \times m_i(\omega_i r_{\ell_2 i} \hat{v}))_{\ell_2}
\end{aligned}$$

We may discard the non-vertical components because we are concerned only with the angular momentum about ℓ_1. Then

$$\begin{aligned}
(\vec{L}_{s'})_{\ell_2} &= \sum_i \vec{r}_{\ell_2 i} \times m_i(\omega_i r_{\ell_2 i} \hat{v}) \\
&= \vec{\omega} \sum_i \vec{r}^{\,2}_{\ell_2 i} m_i \\
&= \vec{\omega} I_{\ell_2} \\
&= \vec{\omega} \sum_i r^2_{\ell_2 i} m_i \\
&= \vec{\omega} \sum_i (\vec{d} + r_{\ell_1 i})^2 m_i \\
&= \vec{\omega} \sum_i (d^2 + r^2_{\ell_1 i} + 2\vec{d} \cdot \vec{r}_{\ell_1 i}) m_i \\
&= \vec{\omega} \left(d^2 \sum_i m_i + \sum_i r^2_{\ell_1 i} m_i + 2\vec{d} \cdot \sum_i \vec{r}_{\ell_1 i} m_i \right) \\
&= \vec{\omega} \left(d^2 m_T + I_{\ell_1} + 2\vec{d} \cdot \vec{0} \right) \\
&= \vec{\omega} \left(d^2 m_T + I_{\ell_1} \right)
\end{aligned}$$

$$I_{\ell_2} = I_{\ell_1} + d^2 m_T$$

□

$\alpha = \ddot{\theta} = \frac{dL/\,dt}{I}$.

$$\begin{aligned}
I &= m_r \int_0^l r^2\,dr + m_d l^2 + \int_0^R \frac{m_d 2\pi r}{\pi R^2} r^2\,dr \\
&= m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d 2}{R^2} \int_0^R r^3\,dr \\
&= m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d 2 R^4}{4R^2} \\
&= m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d R^2}{2}
\end{aligned}$$

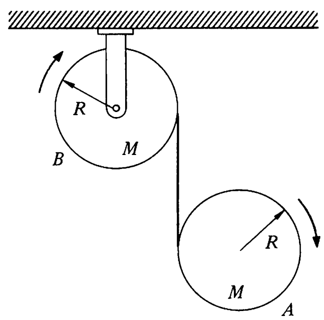
Then $\ddot{\theta} = \frac{-(m_d l + m_r l/2) g \sin\theta}{m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d R^2}{2}}$. Using the small angle approximation of $\sin\theta \approx \theta$, $\frac{d^2\theta}{dt^2} = -\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d R^2}{2}} \theta$. Then $\omega = \sqrt{\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d R^2}{2}}}$ and the period is $\frac{2\pi}{\sqrt{\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2 + \frac{m_d R^2}{2}}}}$.

(b) If the disc is allowed to rotate freely, then the disc acts as a point mass on the end of the rod. Then $I = m_r \frac{l^3}{3} + m_d l^2$, so $\ddot{\theta} = \frac{-(m_d l + m_r l/2) g \sin\theta}{m_r \frac{l^3}{3} + m_d l^2}$. Using the small angle approximation of $\sin\theta \approx \theta$, $\frac{d^2\theta}{dt^2} = -\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2} \theta$. Then $\omega = \sqrt{\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2}}$ and the period is $\frac{2\pi}{\sqrt{\frac{(m_d l + m_r l/2) g}{m_r \frac{l^3}{3} + m_d l^2}}}$.

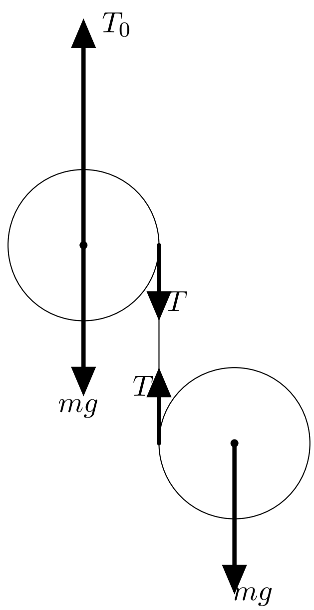
Problem 3: K&K 6.24

Problem

A drum A of mass m and radius R is suspended from a drum B also of mass m and radius R, which is free to rotate about its axis. The suspension is in the form of a massless metal tape wound around the outside of each drum, and free to unwind. Gravity is directed downwards. Both drums are initially at rest. Find the initial acceleration of drum A, assuming that it moves straight down.



Solution



Since the tape is massless and rigid, the tension on A is the same as the tension on B. Let ℓ be the length of the tape. Then $\frac{d\ell}{dt} = \omega_B R + \omega_A R = v_A$. Then $\ddot{\ell} = R(\alpha_A + \alpha_B) = a_A = g - \frac{T}{m}$. Since $\vec{\tau} = \vec{r} \times \vec{F}$, and $I_{\text{disc}} = \int_0^R r^2 \frac{2\pi r m_T}{\pi R^2}\, dr = \frac{2m_T}{R^2} \cdot \frac{R^4}{4} = \frac{m_T R^2}{2}$,

$$\begin{aligned}
R(\alpha_A + \alpha_B) &= g - \frac{T}{m} \\
R\frac{\tau_A + \tau_B}{mR^2/2} &= g - \frac{T}{m} \\
2 \cdot \frac{\tau_A + \tau_B}{mR} &= g - \frac{T}{m} \\
2 \cdot \frac{TR + TR}{mR} &= g - \frac{T}{m} \\
\frac{4T}{m} &= g - \frac{T}{m} \\
T &= \frac{1}{5} mg
\end{aligned}$$

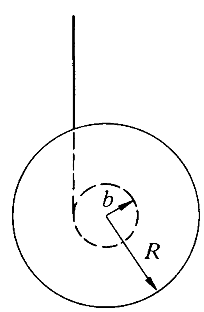
Then $a_A = \frac{4}{5} g$.

Problem 4: K&K 6.29

Problem

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected.

The Yo-Yo is released from rest.



(a) What is the tension in the cord as the Yo-Yo descends and as it ascends?

(b) The center of the Yo-Yo descends a distance h before the string is fully unwound. Use conservation of energy to find the angular velocity of the Yo-Yo when it reaches its lowest point.

(c) What happens to the Yo-Yo at the bottom of the string?

(d) Assuming it reverses direction with uniform angular velocity, find the average force on the string while the Yo-Yo turns around.

Solution

(a) Let y be the position of the Yo-Yo, with the up direction and clockwise direction positive, and let ℓ be the length of the string which is unwound. Then $\frac{dy}{dt} = -\frac{d\ell}{dt} = -\omega b$. Since

$\ddot{y} = -g + \frac{T}{m} = -\alpha b$ and $\alpha = \frac{\tau}{I} = \frac{Tb}{mR^2/2}$, $-g + \frac{T}{m} = -\frac{Tb^2}{mR^2/2}$. Then $mg = T\left(1 + \frac{2b^2}{R^2}\right)$, so $T = \frac{mg}{1+\frac{2b^2}{R^2}} = \frac{mgR^2}{R^2+2b^2}$.

Going up, $\frac{dy}{dt} = -\frac{d\ell}{dt} = \omega b$. Since $\ddot{y} = -g + \frac{T}{m} = \alpha b$ and $\alpha = \frac{\tau}{I} = \frac{Tb}{mR^2/2}$, $-g + \frac{T}{m} = \frac{Tb^2}{mR^2/2}$. Then $mg = T\left(1 - \frac{2b^2}{R^2}\right)$, so $T = \frac{mg}{1-\frac{2b^2}{R^2}} = \frac{mgR^2}{R^2-2b^2}$.

(b) Since $\omega_f b = v_f$. Then $mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}mb^2\omega_f^2 + \frac{1}{4}mR^2\omega_f^2$. Then $\omega_f = \sqrt{\frac{2gh}{b^2+R^2/2}}$.

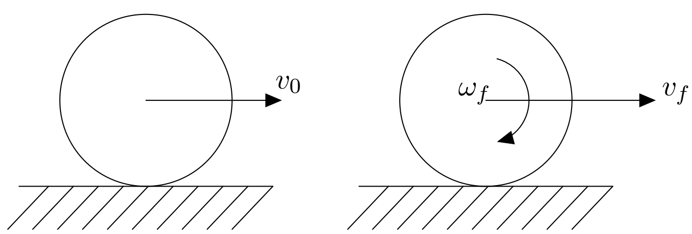
(c) Assuming there is a fixed point, the Yo-Yo rotates about its point of contact with the string. (FIX)

(d) Assuming that the speed remains constant, the average tension is $\frac{1}{\pi}\int_0^\pi \frac{mv_f^2}{R} + \sin\theta mg = \frac{mv_f^2}{b} + 2mg$.

Problem 5: K&K 6.30

Problem

A bowling ball of mass m and radius R is initially thrown down an alley with an initial velocity v_0 and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. What is the velocity of the bowling ball when its just start to roll without slipping.



Solution

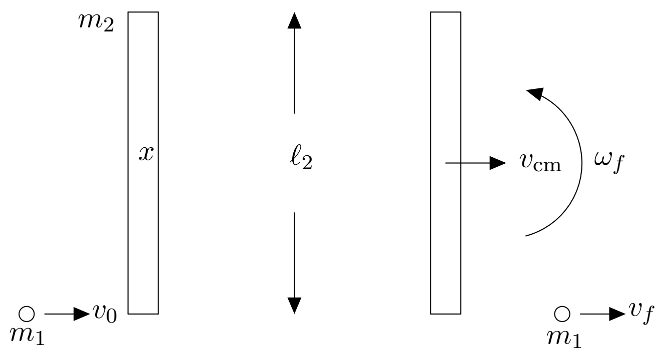
The final angular velocity is such that $\omega_f R = v_f$. The force of friction is such that $m\dot{v} = F_{fr}$ and $R \times F_{fr} = I\dot{\omega}$. Then

$$\begin{aligned}
\int_0^{t_f} m\dot{v}\,dt &= \int_0^{t_f} \frac{I}{R}\dot{\omega}\,dt \\
m(v_f - v_0) &= \frac{I}{R}(\omega_f - \omega_0) \\
m(v_f - v_0) &= \frac{I}{R}\omega_f \\
m(v_f - v_0) &= \frac{Iv_f}{R^2} \\
v_f &= \frac{Iv_f}{mR^2} + v_0
\end{aligned}$$

Problem 6: K&K 6.37

Problem

A hockey puck of mass m_1 slides along ice with a velocity v_0 and strikes one end of a stick lying on the ice of length l_2 and mass m_2. The center of mass of the stick moves with an unknown magnitude v_{cm}. The stick also rotates about the center of mass with unknown angular velocity ω_f. The puck continues to move in the same straight line as before it hit the stick with velocity v_f. Assume the ice is frictionless and there is no loss of mechanical energy during the collision.



(a) Write down the equation for conservation of momentum.

(b) Write down the equation for conservation of energy.

(c) Is there any external torques acting on the system consisting of the puck and the stick? Write down the equation for conservation of angular momentum about a convenient point.

(d) Find the velocity of the center of mass of the stick.

(e) Find the velocity of the puck after the collision.

(f) Find the angular velocity of the stick after the collision.

Solution

(a) For linear momentum, $m_1 v_0 = m_1 v_f + m_2 v_{cm}$.

(b) $\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_{cm}^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_{cm}^2 + \frac{1}{2}m_2 \frac{l^3}{12}\omega_f^2$.

(c) No. The moment of inertia of the stick is $m_2 \int_{-l_2/2}^{l_2/2} r^2\,dr = m_2\left(\frac{l_2^3}{3\cdot 2^3} + \frac{l_2^3}{3\cdot 2^3}\right) = m_2\frac{l_2^3}{12}$. For angular momentum about a point on the line of motion of the center of mass of the stick, $\frac{l_2}{2}m_1 v_0 = \frac{l_2}{2}m_1 v_f + I\omega_f = \frac{l_2}{2}m_1 v_f + m_2\frac{l_2^3}{12}\omega_f$. Alternatively, for angular momentum about a point on the line of motion of the puck, $0 = I\omega_f + \frac{l_2}{2}m_2 v_{cm} = m_2\frac{l_2^3}{12}\omega_f + \frac{l_2}{2}m_2 v_{cm}$. That is, $m_2\frac{l_2^3}{12}\omega_f = -\frac{l_2}{2}m_2 v_{cm}$, or $-\frac{l_2^2}{6}\omega_f = v_{cm}$

(d) Let $V = \frac{m_1}{m_1+m_2}v_0$ be the velocity of the center of mass of the system. Then $-V = -(v_{cm} - V)$. Then $v_{cm} = 2V$, so $v_{cm} = \frac{2m_1}{m_1+m_2}v_0$.

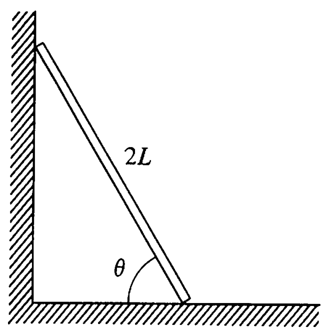
(e) Let $V = \frac{m_1}{m_1+m_2}v_0$ be the velocity of the center of mass of the system. Then $(v_0-V) = -(v_f-V)$. Then $v_f = 2V - v_0 = v_0\left(\frac{2m_1}{m_1+m_2} - 1\right) = v_0\frac{m_1-m_2}{m_1+m_2}$

(f) Since $-\frac{l_2^2}{6}\omega_f = v_{cm}$, $\omega_f = -v_{cm}\frac{6}{l_2^2} = -\frac{12m_1}{l_2^2(m_1+m_2)}$

Problem 7: K&K 6.41

Problem

A plank of length $2l$ leans against a wall. The mass of the plank is m which is uniformly distributed. The plank is initially inclined at an angle θ with respect to the horizontal. It starts to slip downward without friction.



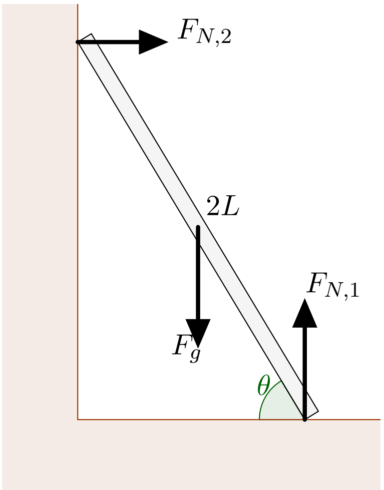
(a) Draw a force diagrams showing all the forces acting on the plank. What is the condition that the plank just starts to slip from the wall.

(b) Is the mechanical energy of the plank conserved as it slips down the wall?

(c) What equations arise from the conditions for static equilibrium for both forces and torque? Think about which point to compute the torque about.

(d) Show that the top of the plank loses contact with the wall when it is two-thirds of its initial height against the wall. Hint: only a single variable and its derivatives are needed to describe the motion of the system. Consider the motion of the center of mass of the plank.

Solution

(a) 

The plank starts to slip when $F_{N,2}$ is 0.

(b) Yes.

(c) Let (x, y) be the coordinates of the center of mass. Then $x = l\cos\theta$ and $y = l\sin\theta$. Then the equations of linear motion are

$$F_{N,2} = m\ddot{x}$$

and

$$F_{N,1} - mg = m\ddot{y}.$$

Since the normal forces do no work, $\frac{1}{2}mv_f^2 + \frac{1}{2}I_{cm}\ddot{\theta}^2 = -mg\Delta h$, so

$$\frac{1}{2}m(\ddot{x}^2 + \ddot{y}^2) + \frac{1}{2}I_{cm}\ddot{\theta}^2 = mg(y_0 - y).$$

To calculate torque about the center of mass, $\vec{\tau} = I\vec{\alpha} = \circlearrowleft F_{N,1}\cos\theta + \circlearrowright F_{N,2}\sin\theta$. Then $\vec{\tau} = \circlearrowleft m((\ddot{y}+g)\cos\theta - \ddot{x}\sin\theta)$. Since $I_{cm} = \int_{-l}^{l} \frac{m}{2l} r^2\, dr = m\frac{l^2}{3}$, $-\left(m\frac{l^2}{3}\right)\ddot{\theta} = m((\ddot{y}+g)\cos\theta - \ddot{x}\sin\theta)$. Since $F_{N,2} = 0$, $\ddot{x} = 0$, so

$$-\frac{l^2}{3}\ddot{\theta} = (\ddot{y} + g)\cos\theta$$

. FIX

The point at which the horizontal contact force is zero requires that $\ddot{x} = 0$. Then FIX What equations arise from the conditions for static equilibrium for both forces and torque? Think about which point to compute the torque about.

(d) FIX Show that the top of the plank loses contact with the wall when it is two-thirds of its initial height against the wall. Hint: only a single variable and its derivatives are needed to describe the motion of the system. Consider the motion of the center of mass of the plank.