MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group Physics 8.012

Problem Set 5 Solutions

Problem 14 N People jumping off cart

N people, each of mass m_p, stand on a railway flatcar of mass m_c. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction.

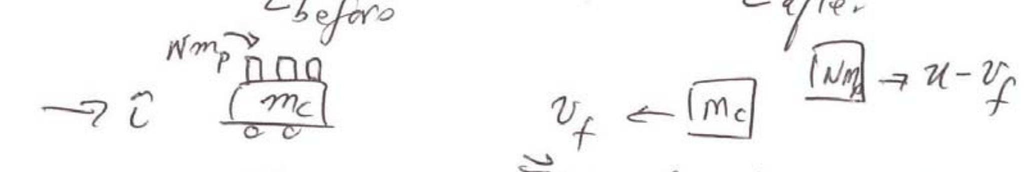
a) What is the final velocity of the car if all the people jump at the same time?

b) What is the final velocity of the car if the people jump off one at a time?

c) Does case a) or b) yield the largest final velocity of the flat car. Give a physical explanation for your answer.

3.14

a) choose reference frame at rest with respect to ground



$v_b = 0$ $\qquad \vec{P}_{after} = (Nm_p(u - v_f) - m_c v_f)\,\hat{i}$

$\vec{P}_{before} = 0\,\hat{i}$

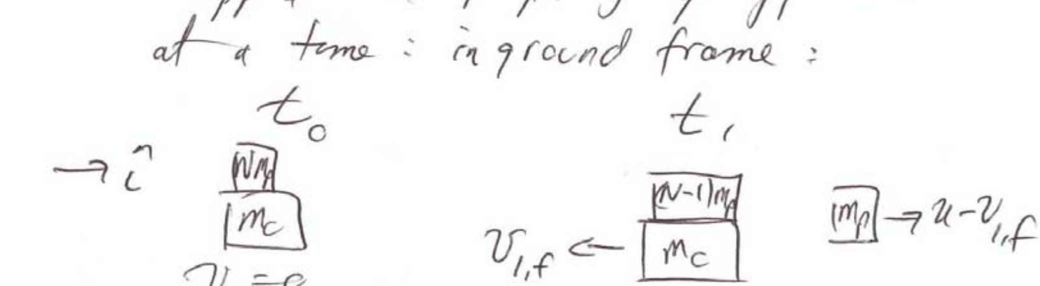
	$\Delta t \vec{F}_{ext}$	$= \Delta \vec{P}$
$\hat{i}:$	0	$= P_{after} - P_{before}$

$$0 = Nm_p(u - v_f) - m_c v_f$$

solve for v_f: $\quad v_f = \dfrac{Nm_p u}{Nm_p + m_c}$

note: $\quad v_f = \dfrac{m^{total}_{people}\, u}{m^{total}_{people} + m^{total}_{cart}} = \dfrac{m^{total}_{people}\, u}{m^{total}}$

b) Suppose the people jump off one at a time: in ground frame:



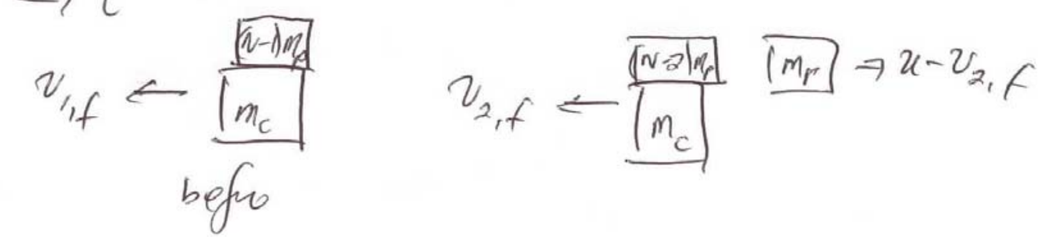
$\hat{i}: \quad 0 = P_1 - P_c = m_p(u - v_{1,f}) - (m_c + (N-1)m_p) v_{1,f}$

solve for $v_{1,f}$:

$$v_{1,f} = \frac{m_p u}{m_c + N m_p} = \frac{m_{person}\, u}{m_{total}}$$

Second person jumps off: ground reference frame

$\rightarrow \hat{i}$



$$\hat{i}: \quad 0 = P_2 - P_1 = \left(m_p(u - v_{2,f}) - (m_c + (N-2)m_p) v_{2,f}\right) - \left(-(m_c + (N-1)m_p) v_{1,f}\right)$$

solve for $v_{2,f}$:

$$m_p u + (m_c + (N-1)m_p) v_{1,f} = (m_c + (N-1)m_p) v_{2,f}$$

$$v_{2,f} = \frac{m_p u}{m_c + (N-1)m_p} + v_{1,f}$$

$$= \frac{m_p u}{m_c + (N-1)m_p} + \frac{m_p u}{m_c + (N)m_p}$$

by induction after the jth person jumps off

$$v_{j,f} = \frac{m_p u}{m_c + (N-(j-1))\, m_p} + v_{j-1,f}$$

in particular when $j = N$

$$v_{N,f} = \frac{m_p u}{m_c + m_p} + v_{N-1,f}$$

$$= \frac{m_p u}{m_c + m_p} + \frac{m_p u}{m_c + 2m_p} + v_{N-2,f}$$

$$= \overbrace{\frac{m_p u}{m_c + m_p} + \frac{m_p u}{m_c + 2m_p} + \dots + \frac{m_p u}{m_c + N m_p}}^{N \text{ terms}}$$

c) Compare the result from part b) to the result from part a)

$$v_f = \frac{N m_p u}{N m_p + m_c} = \overbrace{\frac{m_p u}{m_c + N m_p} + \frac{m_p u}{m_c + N m_p} + \dots \frac{m_p u}{m_c + N m_p}}^{N \text{ terms}}$$

comparing these expressions, note that the denominator in part c) for N-1 terms is greater than the denominator for N-1 terms in part b). Therefore $v_f < v_{N,f}$. The velocity of the cart is slower if they all jump off at once.

The explanation is that when they jump off at once, they are pushing the entire mass of cart and all N people. When they jump off one at a time, each successive person has to push a slightly lighter cart (less people) so the cart recoils faster.

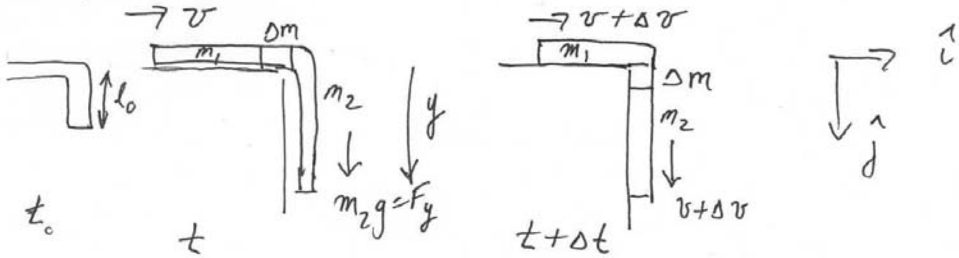
Problem 15:

A rope of mass m and length l lies on a frictionless table, with a short portion l_0 hanging through a hole. Initially the rope is at rest.

a) Find a general differential equation for $x(t)$, the length of rope through the hole.

b) Solve the differential equation with appropriate initial conditions for $y(t)$, the length of rope through the hole.

Chapter 3.15.



$$\hat{j}: \lim_{\Delta t \to 0} \frac{P_y(t+\Delta t) - P_y(t)}{\Delta t} = m_2 g \qquad (1)$$

$$\hat{i}: \lim_{\Delta t \to 0} \frac{P_x(t+\Delta t) - P_x(t)}{\Delta t} = 0 \qquad (2)$$

eq (1) becomes $\quad 0 = \lim\limits_{\Delta t \to 0} \dfrac{(m_2+\Delta m)(v+\Delta v) - m_2 v}{\Delta t} = m_2 g$

$$\lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} v + m_2 \frac{\Delta v}{\Delta t} = m_2 g \quad \Rightarrow$$

$$\frac{dm}{dt} v + m_2 \frac{dv}{dt} = m_2 g \qquad (3)$$

eq (2) becomes $\quad 0 = \lim\limits_{\Delta t \to 0} \dfrac{(m_1(v+\Delta v) - (m_1+\Delta m) v)}{\Delta t}$

$$0 = m_1 \frac{dv}{dt} - \frac{dm}{dt} v \quad \Rightarrow$$

$$m_1 \frac{dv}{dt} = \frac{dm}{dt} v \qquad (4)$$

eq (3) now becomes (using eq (4))

$$(m_1 + m_2) \frac{dv}{dt} = m_2 g \qquad (5)$$

Since $m_2 = \lambda y$ eq (5) becomes

$$m \frac{d^2y}{dt^2} = \lambda y g \qquad (6)$$

where y is the length of rope hanging through the hole at time t. This equation has solution

$$y = Ae^{\gamma t} + Be^{-\gamma t}$$

where $\gamma = \sqrt{\frac{\lambda g}{m}}$

at $t=0$, $y = A + B = l_0 \qquad (7)$

at t. $\frac{dy}{dt} = v_y = \gamma A e^{\gamma t} - \gamma B e^{-\gamma t}$

at $t=0$ $v_y(t=0) = \gamma(A-B) = 0$
since rope starts from rest $\Rightarrow A = B$

So eq (7) $l_0 = 2A \Rightarrow A = \frac{l_0}{2} = B$

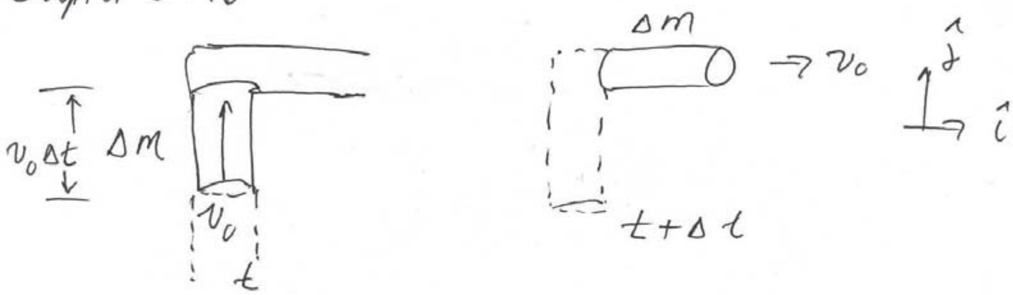
$$\boxed{y = \frac{l_0}{2}\left(e^{\gamma t} + e^{-\gamma t}\right), \quad \gamma = \sqrt{\frac{\lambda g}{m}} = \sqrt{g/l}}$$

$$\frac{\lambda}{m} = \frac{m}{lm} = \frac{1}{l}$$

Problem 16:

Water shoots out of a fire hydrant having nozzle diameter D with nozzle speed V_0. What is the reaction force on the hydrant?

Chapter 3 . 16



$$\Delta m = \rho A v_0 \Delta t$$

$$(F_{ext})_y = \lim_{\Delta t \to 0} \frac{P_y(t+\Delta t) - P_y(t)}{\Delta t}$$

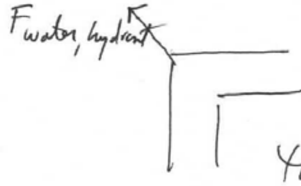
$$(F_{ext})_y = \lim_{\Delta t \to 0}\left(-\frac{\Delta m}{\Delta t} v_0\right) = \lim_{\Delta t \to 0} -\rho \frac{A v_0 \Delta t}{\Delta t} v_0$$

$$(F_{ext})_y = -\rho A v_0^2 \qquad (1)$$

$$(F_{ext})_x = \lim_{\Delta t \to 0} \frac{P_x(t+\Delta t) - P_x(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} v_0 = \rho A v_0^2$$

$$\vec{F}_{water,\, hydrant} = -\vec{F}_{ext} = -\rho A v_0^2 (\hat{i} - \hat{j})$$



Here we assume the water pressure keeps the flow rate constant. The hydrant just changes the direction.

Problem 18:

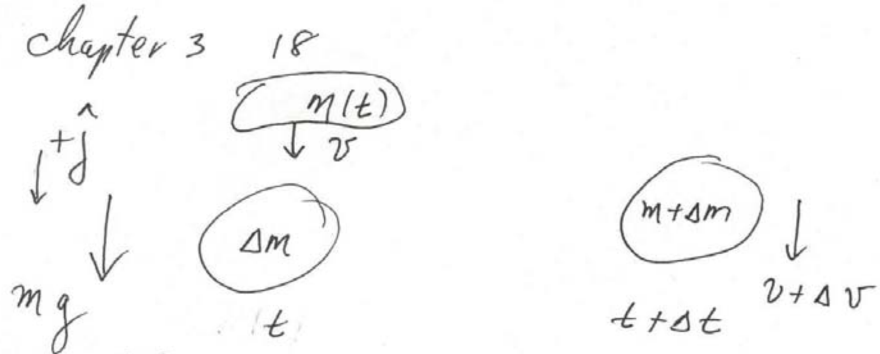
A raindrop of initial mass m_0 starts falling from rest under the influence of gravity. Assume that the raindrop gains mass from the cloud at a rate proportional to the momentum of the raindrop, $dm/dt = kmv$, where m is the instantaneous mass of the raindrop, v is the instantaneous velocity of the raindrop, and k is a constant. You may neglect air resistance.

a) Derive a differential equation for the velocity of the raindrop.

b) Show that the speed of the drop eventually becomes effectively constant and give an expression for the terminal speed.

c) Assume the air resistance is proportional to the square of the velocity. How would air resistance effect the terminal speed?

chapter 3 18



$$(F_{ext})_y = \lim_{\Delta t \to 0} \frac{(m+\Delta m)(v+\Delta v) - m v}{\Delta t}$$

$$mg = m\frac{dv}{dt} + \frac{dm}{dt} v \qquad (1)$$

Assume $\frac{dm}{dt} = k m v \qquad (2)$

Then eq (1) becomes

$$mg = m\frac{dv}{dt} + k m v^2$$

$$\Rightarrow \qquad \frac{dv}{dt} = g - k v^2 \qquad (3)$$

terminal velocity occurs when $\frac{dv}{dt} = 0$ or

$$v_{term} = \sqrt{g/k}$$

We can integrate eq (3)

$$\int_{v_0=0}^{v} \frac{dv}{g-kv^2} = \int_0^t dt = t \quad , \text{ let } a = \frac{k}{g}$$

$$\frac{1}{g}\left(\int_0^v \frac{\frac{1}{2}\,dv}{1-\sqrt{a}\,v} + \int_0^v \frac{\frac{1}{2}\,dv}{1+\sqrt{a}\,v}\right) = t$$

$$t = \frac{1}{2g}\left(\frac{\ln(1-\sqrt{a}\,v)}{-\sqrt{a}}\Big|_0^v + \frac{\ln(1+\sqrt{a}\,v)}{\sqrt{a}}\Big|_0^v \right)$$

$$t = \frac{1}{2g}\left(\ln\left(\frac{1+\sqrt{a}\,v}{1-\sqrt{a}\,v}\right) = -\frac{1}{2g}\ln\left(\frac{1-\sqrt{a}\,v}{1+\sqrt{a}\,v}\right)\right)$$

$$e^{-2gt} = \frac{1-\sqrt{a}\,v}{1+\sqrt{a}\,v}$$

$$(1+\sqrt{a}\,v)\,e^{-2gt} = 1-\sqrt{a}\,v$$

$$\sqrt{a}\,v\,(e^{-2gt}+1) = 1-e^{-2gt}$$

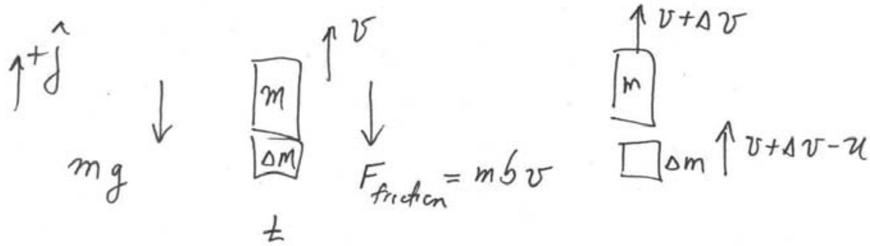
$$v = \frac{1}{\sqrt{a}}\,\frac{1-e^{-2gt}}{1+e^{-2gt}}$$

as $t \to \infty$ $\quad v = \frac{1}{\sqrt{a}} = \frac{1}{\sqrt{K/g}} = \sqrt{g/K}$

Problem 20:

A rocket ascends from rest in a uniform gravitational field by ejecting exhaust with constant speed u relative to the rocket. Assume that the rate at which mass is expelled is given by $dm/dt = \gamma m$, where m is the instantaneous mass of the rocket and γ is a constant. The rocket is retarded by air resistance with a force $F = bmv$ proportional to the instantaneous momentum of the rocket where b is a constant. Find the velocity of the rocket as a function of time.

Chapter 3.20



$$-mg - F_{friction} = \lim_{\Delta t \to 0} \frac{\left(m(v+\Delta v) + \Delta m((v+\Delta v)-u)\right) - (m+\Delta m)v}{\Delta t}$$

$$-mg - mbv = m\frac{dv}{dt} - \frac{dm}{dt}u \qquad (1)$$

Assume $\frac{dm}{dt} = \gamma m$ Then eq (1) becomes

$$-mg - mbv = m\frac{dv}{dt} - \gamma m u$$ or

$$-g - bv = \frac{dv}{dt} - \gamma u$$

$$\frac{dv}{dt} = -g - bv + \gamma u \qquad (2)$$

when $\frac{dv}{dt} = 0 \Rightarrow v = v_{term} \Rightarrow$

$$v_{term} = \frac{\gamma u - g}{b}$$

We can integrate eq (2)

$$\int_{v_0=0}^{v} \frac{dv}{\gamma u - g - bv} = \int_0^t dt$$

$$-\frac{1}{b} \ln\left(\frac{\gamma u - g - bv}{\gamma u - g} \right) = t$$

$$\frac{\gamma u - g - bv}{\gamma u - g} = e^{-bt}$$

$$\boxed{v(t) = \frac{(\gamma u - g)}{b}(1 - e^{-bt})} \qquad (3)$$

$$v(t = \infty) = \frac{\gamma u - g}{b} \equiv v_{term}$$