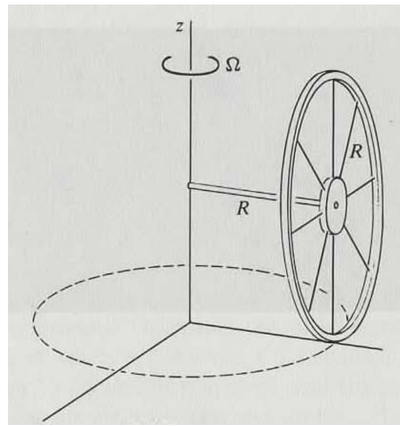
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012 **Fall Term 2009**

Chapters Seven and Eight Problem Set 11 Solutions

Problems: Chapter 7: 1, 3, 5, 8 and Chapter 8: 1, unnumbered, 10, 12,

Problem 7.1: A thin hoop of mass m and radius R rolls without slipping about the z axis. It is supported by an axle of length R through its center. The hoop circles around the z axis with angular speed Ω. (Note: the moment of inertia of a hoop for an axis along its diameter is $(1/2)mR^2$.)



a) What is the instantaneous angular velocity $\vec{\omega}$ of the hoop? Specify the direction and magnitude.
b) What is the angular momentum $\vec{\mathbf{L}}$ of the hoop about a point where the axle meets the z axis? Is $\vec{\mathbf{L}}$ parallel to $\vec{\omega}$?

a) The ring has two contributions to the angular velocity. As the wheel rolls around the circle with angular velocity Ω, there is a spin velocity $\vec{\omega}_{spin} = \omega_s(-\hat{r})$ and the orbit velocity $\vec{\omega}_{orbital} = \Omega\,\hat{k}$. The rolling without slipping condition is

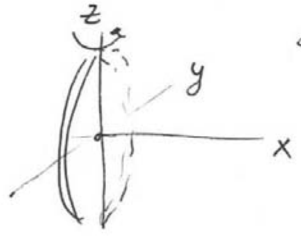
$$R\Omega = R\omega_s \Rightarrow \Omega = \omega_s.$$

Thus $\vec{\omega}_{total} = \omega_s(-\hat{r}) + \Omega\hat{k} = \Omega(-\hat{r}+\hat{k})$

b) The angular momentum about the center of mass is

$$\vec{L}_{cm} = I_{cm}\,\omega_s(-\hat{r}) + I_{cm,z}\,\Omega\,\hat{k}$$

The $I_{cm} = mR^2$. The moment of inertial about the z-axis can be



determined by the fact that

$$I_{z,cm} + I_{y\,cm} = \int dm\,(z^2+y^2)$$
$$= I_x = mR^2$$

Since $I_{z,cm} = I_{y\,cm} \Rightarrow 2I_{z,cm} = mR^2$, $I_{z,cm} = \frac{mR^2}{2}$

So the angular momentum about the z-axis passing through the origin

$$I_{z,o} = I_{z,cm} + mR^2 = \frac{3}{2} mR^2$$

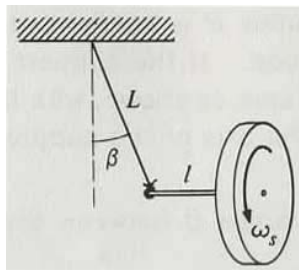
thus $\vec{L}_{z,o} = I_{cm}\,\omega_s(-\hat{r}) + \frac{3}{2} mR^2 \Omega\,\hat{k}$

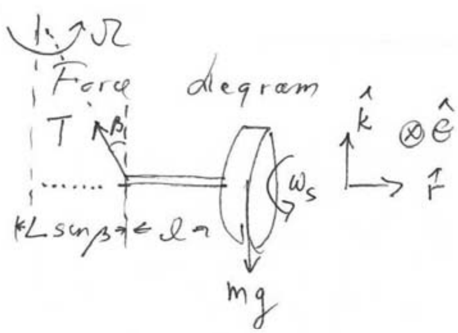
$$\vec{L}_{z,o} = mR^2\Omega\left(-\hat{r} + \frac{3}{2}\hat{k}\right)$$

$\vec{L}_{z,o}$ is not parallel to $\vec{\omega}$

Problem 7.3:

A gyroscope wheel is at one end of an axle of length l. The other end of the axle is suspended from a string of length L. The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass m and moment of inertia about its center of mass I_{cm}. Its spin angular velocity is ω_s. Neglect the mass of the shaft and the mass of the string. Find the angle β that the string makes with the vertical. Assume that β is so small that approximations like $\sin\beta \cong \beta$ are justified.





$$\vec{L}_{cm} = I_{cm}\,\omega_s\,\hat{r}$$
$$= I_0\,\omega_s\,\hat{r}$$
$$\frac{d\vec{L}_{cm}}{dt} = I_0\,\omega_s\,\Omega\,\hat{\theta}$$

$$\vec{F} = m\vec{a}$$

$$\hat{r}: \quad -T\sin\beta = -m(L\sin\beta + l)\,\Omega^2 \qquad (1)$$

$$\hat{k}: \quad +T\cos\beta - mg = 0 \qquad (2)$$

$\vec{\tau}_{cm}$	$= \dfrac{d\vec{L}_{cm}}{dt}$
$\hat{\theta} \quad lT\cos\beta$	$= I_0\,\omega_s\,\Omega \qquad (3)$

For the small angle approximation

$$\cos\beta \simeq 1$$
$$\sin\beta \simeq \beta \qquad (4)$$

eq (1) becomes $\quad +T\beta \cong m(L\beta+\ell)\Omega^2 \quad (1a)$

eq (2) becomes $\quad T \simeq mg \quad (2a)$

eq (3) becomes $\quad \ell T \simeq I_0 \omega_s \Omega \quad (3a)$

Substituting (2a) into (3a)

$$\ell mg \simeq I_0 \omega_s \Omega \Rightarrow \Omega = \frac{\ell mg}{I_0 \omega_s} \quad (3b)$$

Using eq 3b and eq (2a) in eq (1a) yields

$$mg\beta \simeq m(L\beta+\ell)\left(\frac{\ell mg}{I_0\omega_s}\right)^2$$

Solve for β

$$\beta\left(mg - (mL)\left(\frac{\ell mg}{I_0\omega_s}\right)^2\right) = m\ell\left(\frac{\ell mg}{I_0\omega_s}\right)^2$$

$$\beta \cong \frac{(\ell)\left(\frac{\ell mg}{I_0\omega_s}\right)^2}{g - L\left(\frac{\ell mg}{I_0\omega_s}\right)^2}$$

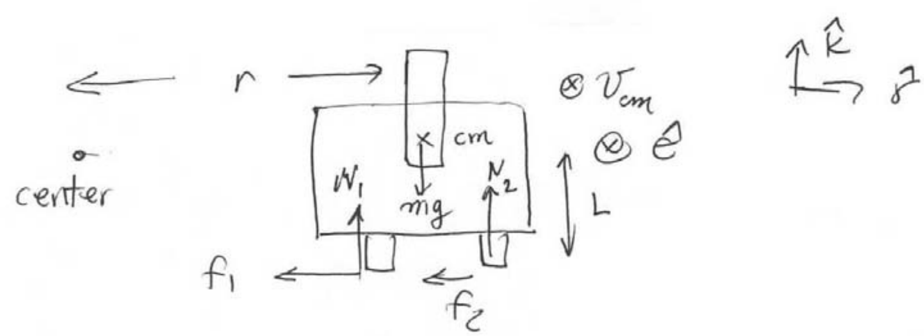
Problem 7.5: When an automobile rounds a curve at high speed, the loading (weight distribution) on the wheels is markedly changed. For sufficiently high speeds the loading on the inside wheel goes to zero, at which point the car starts to roll over. The tendency can be avoided by mounting a large spinning flywheel on the car.

a) In what direction should the flywheel be mounted, and what should be the sense of rotation, to help equalize the loading? (Be sure that your method works for cars turning in either direction.)

b) Show that for a disk-shaped flywheel of mass m_w and radius R, the requirement for equal loading is that the angular velocity of the flywheel, ω_s, is related to the velocity of the car v by

$$\omega_s = 2v\frac{m_T L}{m_w r^2}$$

where m_T is the total mass of the car and flywheel, and L is the height of the center of mass of the car (including the flywheel) above the road. Assume the road is unbanked.



$$\vec{F} = m_c \vec{a}$$

$$\hat{r}: \quad -(f_1 + f_2) = -m_c r\Omega^2 \qquad (1)$$

In order to turn the flywheel a torque must be applied to the wheel since its angular momentum is changing direction

$$\vec{\tau}_{flywheel} = \frac{d\vec{L}_{cm}}{dt},$$

The car is applying this torque

to the flywheel. Therefore the wheel applies an equal and opposite torque to the car

$$\vec{\tau}_{car} = -\vec{\tau}_{flywheel}$$

This is the torque that should stabilize the car s.t.

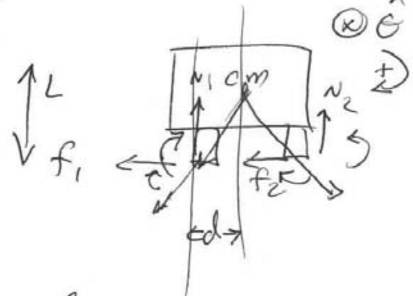
$$N_1 = N_2 \qquad \text{equal loading.}$$

The total torque about the center of mass of the car is zero

$$\vec{\tau}_{cm} = 0$$

$$\hat{\theta}: (f_1+f_2)L + (N_1-N_2)d + \vec{\tau}_{car} = 0 \qquad (2)$$

if $N_1 = N_2$ we have



$$(f_1+f_2)L\hat{\theta} = -\vec{\tau}_{car}$$

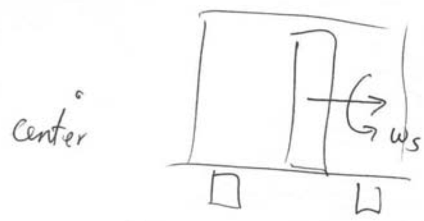
$$(f_1+f_2)L\hat{\theta} = \vec{\tau}_{flywheel}$$

Therefore the dir of the $\vec{\tau}_{flywheel}$ is $+\hat{\theta}$

note from eq (1)

$$m_c r \Omega^2 L\,\hat{\theta} = \vec{\tau}_{flywheel} \qquad (2a)$$

This means that the spin points away from the center



$$\vec{L}_{s,cm} = I_{cm}\,\omega_s\,\hat{r}$$

$$\frac{d\vec{L}_{s,cm}}{dt} = I_{cm}\,\omega_s\,\Omega\,(\hat{\theta})$$

$\vec{\tau}_{flywheel}$	$= \frac{d\vec{L}_{cm}}{dt}$
$\hat{\theta}:\ \tau_{flywheel}\,\hat{\theta}$	$= I_{cm}\,\omega_s\,\Omega\,\hat{\theta}$ (3)

So eq (3) becomes using eq (2a)

$$m_c\, r\Omega^2 L = I_{cm}\,\omega_s\,\Omega$$

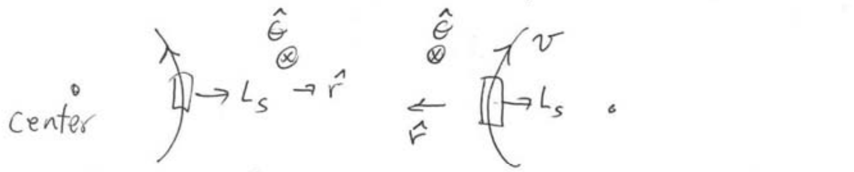
$$\Rightarrow \quad \omega_s = \frac{m_c\, r\Omega\, L}{I_{cm}} \qquad (4)$$

now $r\Omega = v_{cm}$ and $I_{cm} = \frac{1}{2} m_w R^2$

so eq (4) becomes

$$\omega_s = \frac{2\, m_c\, v_{cm}\, L}{m_w R^2}$$

Seen from overhead this works for both directions of turning

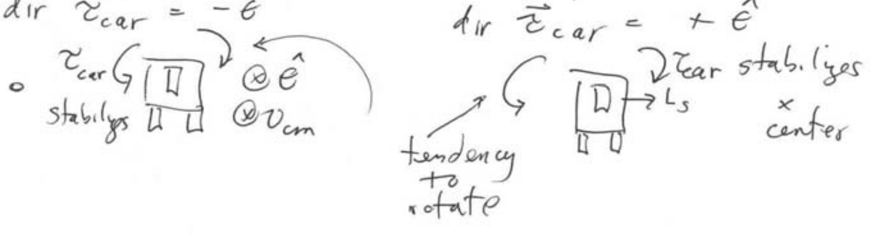


dir $\vec{\tau}_{flywheel} = \hat{\theta}$

dir $\vec{\tau}_{car} = -\hat{\theta}$

dir $\vec{\tau}_{flywheel} = -\hat{\theta}$

dir $\vec{\tau}_{car} = +\hat{\theta}$

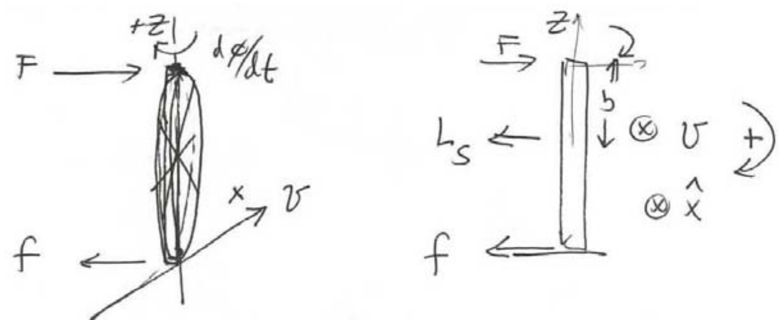


Problem 7.8:

A child's hoop of mass m and radius b rolls in a straight line with velocity v. Its top is given a light tap with a stick at right angles to the direction of motion. The impulse of the blow is I.

a) Show that this results in a deflection of the line of rolling by an angle $\phi = I/mv$, assuming that the gyroscopic approximation holds and neglecting friction with the ground.

b) Show that the gyroscopic approximation is valid provided $F << mv^2/b$, where F is the peak applied force.



the $\vec{\tau}_{cm} = bF\hat{x}$ (into page)

this means that

$$\frac{d\vec{L}_{cm}}{dt} = I_{cm}\,\omega_s \frac{d\phi}{dt}\,\hat{x}$$

where $\frac{d\phi}{dt}$ is the rate that the hoop is rotating about the z-axis

$$\vec{\tau}_{cm} = \frac{d\vec{L}_{cm}}{dt} \Rightarrow$$

$$bF = I_{cm}\,\omega_s \frac{d\phi}{dt} \qquad (1)$$

$$\Rightarrow \frac{bF\,dt}{I_{cm}\,\omega_s} = d\phi$$

We can integrate this equation

$\mathcal{J} = \text{Impulse} = \int F\,dt$. Thus

$$\frac{\int b F\,dt}{I_{cm}\,\omega_s} = \phi$$

$$\phi = \frac{\mathcal{J} b}{I_{cm}\,\omega_s} = \frac{\mathcal{J} b}{m b^2 \omega_s} = \frac{\mathcal{J}}{m b \omega_s} = \frac{\mathcal{J}}{m v}$$

where $v = b\,\omega_s$

b) The gyroscopic approximation is

that the $L_s >> L_z$ $\quad (2)$

$$L_s = I_{cm}\omega_s = m b^2 \omega_s = m \frac{b^2 v}{b} = m b v \quad (3)$$

The angular momentum about the z-axis

$$L_z = I_z \frac{d\phi}{dt}$$

$I_z = \frac{1}{2} m b^2$ (see problem 7.1) Thus

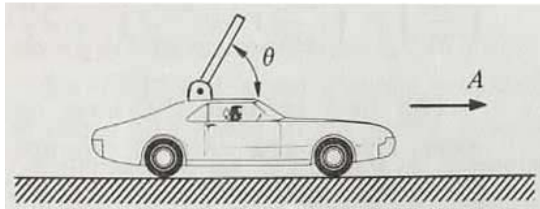
$$L_z = \frac{1}{2} m b^2 \frac{d\phi}{dt}$$

by eq (1) $\quad L_z = \frac{1}{2} m b^2 \frac{bF}{I_{cm}\,\omega_s} = \frac{1}{2} \frac{bF}{\omega_s} = \frac{1}{2}\frac{b^2 F}{v}$

Thus eq (2) $\quad L_s >> L_z$ becomes

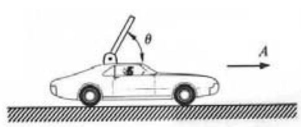
$m b v >> \frac{1}{2}\frac{b^2 F}{v}$ or $2\frac{m v^2}{b} >> F$.

Problem 8.1 A uniform thin rod of length L and mass m is pivoted at one end. The pivot is attached to the top of a car accelerating at rate A.



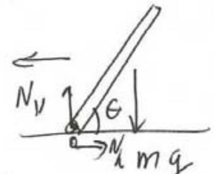
a) What is the equilibrium value of the angle θ between the rod and the top of the car?

b) Suppose that the rod is displaced a small angle ϕ from equilibrium. What is its motion for small ϕ?

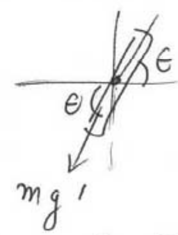


The force diagram on the rod

$F_{eff} = mA \leftarrow$



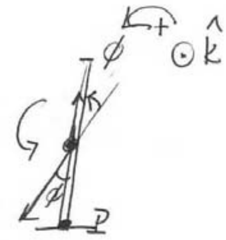
implies that gravity has changed direction



$$\tan\theta = \frac{mg}{mA} = \frac{g}{A}, \quad \theta = \tan^{-1}\left(\frac{g}{A}\right)$$

So the pivot forces are balanced by the "effective gravity").

b) $$g' = (g^2 + A^2)^{1/2}$$



$$\vec{\tau}_P = I\,\vec{\alpha} \quad \text{where } I_p = \frac{1}{3}mL^2$$

The torque about the pivot is

$$\vec{\tau}_p = mg' \frac{L}{2} \sin\phi \ \hat{k} \qquad \text{so}$$

$$mg' \frac{L}{2} \sin\phi = \frac{1}{3} m L^2 \ddot{\phi} \hat{k} \quad , \text{let} \sin\phi \simeq \phi$$

so $\qquad \ddot{\phi} = \frac{3}{2} \frac{g'}{L} \phi$

$$\phi = \phi_0 \, e^{\pm \gamma t} \qquad \text{hyperbolic motion}$$

where $\qquad \gamma = \sqrt{\frac{3}{2} g'/L}$

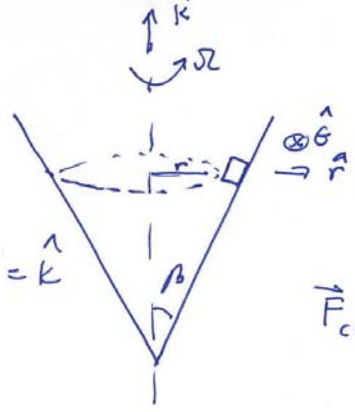
Problem unnumbered:

A particle of mass m slides without friction on the inside of a cone. The axis of the cone is vertical and gravity points downward. The apex half-angle of the cone is β. The cone is rotating about the vertical axis with angular velocity $\vec{\Omega} = \Omega \hat{k}$. The particle travels in a circular orbit with radius r in the horizontal plane with a constant but unknown speed $(v_\theta)_{rot} = r\omega_{rot}$ as measured in the rotating reference frame.

a) What is the direction of the Coriolis force $\vec{F}_{cor} = -2m\vec{\Omega} \times \vec{v}_{rot}$?

b) What is the direction of the centrifugal force $\vec{F}_{cent} = -m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$?

c) What is the angular velocity ω_{rot} of the particle in the rotating reference frame?



$\hat{r} \times \hat{\theta} = \hat{k}$

$\vec{\Omega} = \Omega \hat{k}$

$\vec{v} = r\omega\hat{\theta}$

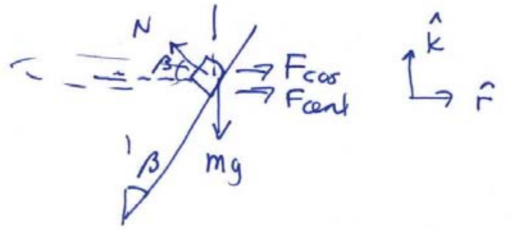
$\vec{r} = r\hat{r}$

$\vec{F}_{cor} = -2m\vec{\Omega}\times\vec{v} = -2m(\Omega\hat{k}\times r\omega\hat{\theta})$

$= +2m\Omega r\omega\hat{r}$

$\vec{F}_{cent} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -m\Omega\hat{k}\times(\Omega\hat{k}\times r\hat{r})$

$= -m\Omega\hat{k}\times\Omega r\hat{\theta} = m\Omega^2 r\hat{r}$



	$\vec{F} =$	$m\vec{a}$
$\hat{r}$:	$F_{cor} + F_{cent} - N\cos\beta =$	$-mr\omega^2$
$\hat{k}$:	$N\sin\beta - mg$	$= 0$

$\Rightarrow \quad N = \dfrac{mg}{\sin\beta}$

$F_{cor} + F_{cent} - N\cos\beta = -mr\omega^2 \quad \Rightarrow$

$2m\Omega r\omega + m\Omega^2 r - \dfrac{mg\cos\beta}{\sin\beta} = -mr\omega^2 \quad \Rightarrow$

$mr\omega^2 + 2m\Omega r\omega + m\Omega^2 r - mg\,\mathrm{cotan}\,\beta = 0$

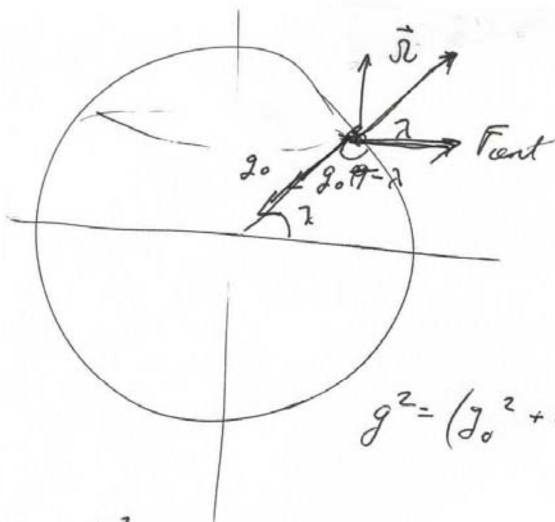
$\omega^2 + 2\Omega\omega + (\Omega^2 - \frac{g}{r}\mathrm{cotan}\,\beta) = 0$

$\omega = \dfrac{-2\Omega \pm (4\Omega^2 - 4(\Omega^2 - \frac{g}{r}\mathrm{cotan}\beta))^{1/2}}{2} = -\Omega \pm \left(\frac{g}{r}\mathrm{cotan}\beta\right)^{1/2}$

choose positive square root $\quad \omega = -\Omega + \left(\frac{g}{r}\mathrm{cotan}\beta\right)^{1/2}$

Problem 8.10:

The acceleration due to gravity measured in an earthbound coordinate system is denoted by g. However, because of the earth's rotation, g from the true acceleration due to gravity g_0. Assuming that the earth is perfectly round, with radius R_e and angular velocity Ω_e, find g as a function of latitude λ. (Assuming the earth to be round is actually not justified; the contributions to the variation of g due to the polar flattening is comparable to the effect calculated here.)



$$\vec{g} = \vec{g}_0 + \vec{F}_{cent}$$

$$\Omega^2 r \cos\lambda$$

$$F_{cent} = m r \cos\lambda\, \Omega^2$$

$$g^2 = \left(g_0^2 + 2\vec{g}_0 \cdot \frac{\vec{F}_{cent}}{m} + \frac{F_{cent}^2}{m^2}\right)^{1/2}$$

$$g = \left(g_0^2 + 2 g_0 F_{cent} \cos(\pi - \lambda) + F_{cent}^2\right)^{1/2}$$

$$= \left(g_0^2 - 2 g_0 F_{cent} \cos\lambda + F_{cent}^2\right)^{1/2}$$

let $r = R_e$

$$g = \left(g_0^2 - 2 g_0 \Omega^2 R_e \cos^2\lambda + \Omega^4 R_e^2 \cos^2\lambda\right)^{1/2}$$

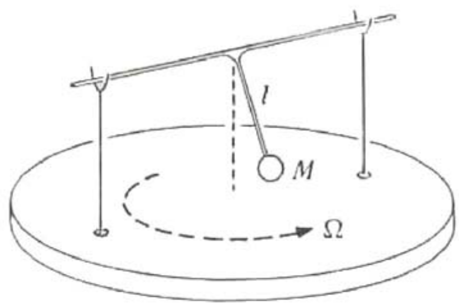
let $x = R_e \Omega^2 / g_0$

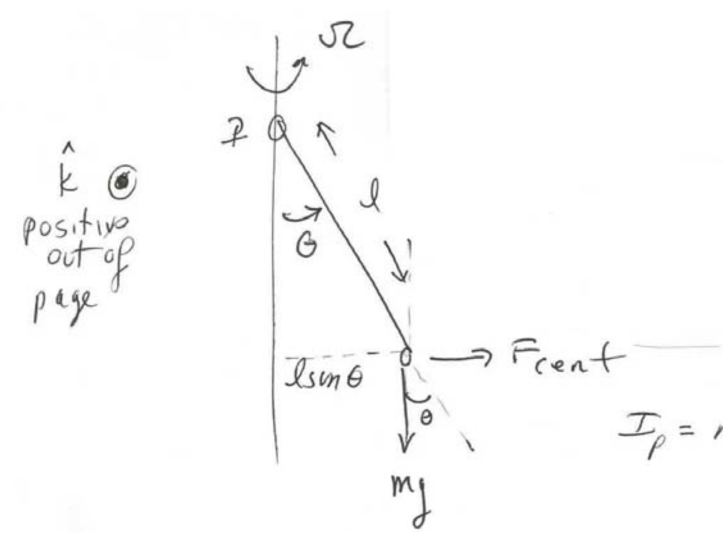
$$= g_0\left(1 - 2x\cos^2\lambda + x^2\cos^2\lambda\right)^{1/2}$$

$$= g_0\left(1 - (2x - x^2)\cos^2\lambda\right)^{1/2}$$

Problem 8.12:

A pendulum is rigidly fixed to an axle held by two supports so that it can only swing in a plane perpendicular to the axle. The pendulum consists of a mass m attached to a massless rod of length l. The supports are mounted on a platform which rotates with constant angular velocity Ω. Find the pendulum's frequency assuming the amplitude is small.





$$I_p = ml^2$$

the torque about the pivot point is

$$\vec{\tau}_p = I_p \vec{\alpha}$$

$$\hat{k}: \quad -mgl\sin\theta + F_{cent}\, l\cos\theta = I_p \ddot{\theta} \quad (1)$$

the centrifugal effective force

$$F_{cent} = m(l\sin\theta)\Omega^2$$

For small angles : $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$$eq(1) \Rightarrow -mgl\theta + ml^2\theta\Omega^2 \simeq ml^2\ddot{\theta}$$

$$\ddot{\theta} + \left(\frac{g}{l} - \Omega^2\right)\theta \simeq 0$$

$$\omega = \left(\frac{g}{l} - \Omega^2\right)^{1/2}$$ if $\Omega^2 > \frac{g}{l}$ the motion is no longer harmonic.