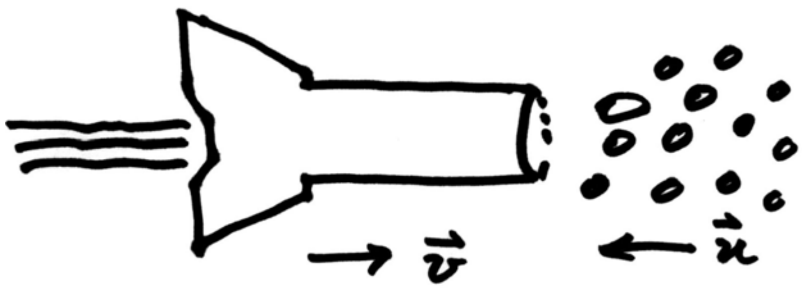
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group
Physics 8.012

Practice Exam 2 Solutions

Problem 1: Momentum Transfer Space Junk

A spacecraft of cross-sectional area *A*, proceeding along the positive *x*-direction, enters an asteroid storm at time $t=0$, in which the asteroid mass density is ρ and the average asteroid velocity is $\vec{\mathbf{u}} = -u\,\hat{\mathbf{i}}$ in the negative *x*-direction. As the spacecraft proceeds through the storm, all of the asteroids that hit the spacecraft stick to it.



a) Suppose that at time t the velocity of the spacecraft is $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$ in the positive x-direction, and its mass is m. Further, suppose that in an interval Δt, the mass of the spacecraft increases by an amount Δm. Given that there are no external forces, using conservation of momentum find an equation for the change of the spacecraft velocity Δv, in terms of Δm, u, and v?

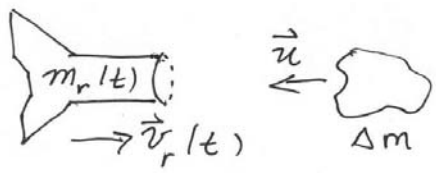
b) When the spacecraft enters the asteroid storm, the magnitude of its velocity and mass are v_0 and m_0, respectively. Integrate your differential equation in part a) to find the velocity v of the spacecraft when the mass is m.

c) Find an expression for the mass of the asteroids Δm that sticks to the spacecraft within the time interval Δt? (Hint: consider the volume of asteroids swept up by the spacecraft in time Δt).

d) When the spacecraft enters the asteroid storm, the magnitude of its velocity and mass are v_0 and m_0, respectively. What is the mass of the spacecraft at time t? (Use your results from parts c) and b).)

1. space-junk:

before:
time t



$\rightarrow \hat{i}$

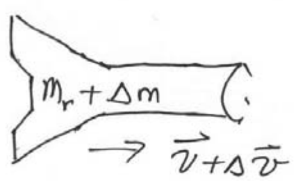
$$\vec{P}_{before} = m_r(t)\vec{v}_r(t) + \Delta m\,\vec{u}$$

$$= m_r(t)\, v_r(t)\,\hat{i} - \Delta m\, u\,\hat{i}$$

where $\vec{v}_r(t) = v_r(t)\,\hat{i}$, $v_r(t) > 0$

$\vec{u}(t) = -u\,\hat{i}$, $u > 0$

after
time $t+\Delta t$



$\rightarrow \hat{i}$

$$\vec{P}_{final} = (m_r + \Delta m)(v + \Delta v)\,\hat{i}$$

no external forces, so momentum is constant

$$m_r v_r - \Delta m\, u = (m_r + \Delta m)(v + \Delta v)$$

$$\hat{i}: \; 0 = \lim_{\Delta t \to 0} \frac{(m_r + \Delta m)(v_r + \Delta v) - (m_r v_r - \Delta m\, u)}{\Delta t}$$

$$0 = \lim_{\Delta t \to 0} \left(m_r \frac{\Delta v}{\Delta t} + \frac{\Delta m}{\Delta t}(v_r + u) + \underbrace{\frac{\Delta m\, \Delta v}{\Delta t}}_{\text{vanishes}} \right)$$

$$0 = m_r \frac{dv_r}{dt} + \frac{dm}{dt}(v_r + u) \qquad (1)$$

mess equation: $\frac{dm_r}{dt} = \frac{dm}{dt}$

Eq (1) becomos

$$0 = m_r \frac{dv_r}{dt} + \frac{dm_r}{dt}(v_r + u)$$

$$m_r \, dv_r = -dm_r (v_r + u)$$

$$\int_{v_{r,0} \equiv v_0}^{v_r(t)} \frac{dv_r}{v_r + u} = \int_{m_{r,0} \equiv m_0}^{m_r(t)} -\frac{dm_r}{m_r}$$

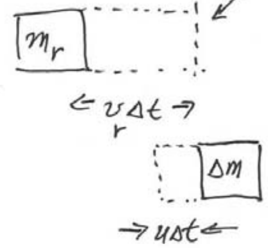
$$\ln\left(\frac{v_r(t) + u}{v_0 + u}\right) = -\ln\left(\frac{m_r(t)}{m_0}\right) = \ln\left(\frac{m_0}{m_r(t)}\right)$$

$$\frac{v_r(t) + u}{v_0 + u} = \frac{m_0}{m_r(t)}$$

$$v_r(t) = (v_0 + u)\frac{m_0}{m_r(t)} - u \qquad (2)$$

Mass Conservation

t



$(\text{volume})_1 = (v_r \Delta t) A$

$A =$ cross sectional area

$(\text{Volume})_2 = (u \Delta t) A$

In time interval Δt, rocket sweeps out $(\text{Volume})_1 = (v_r \Delta t) A$. During the same interval a $(\text{volume})_2 = (u \Delta t) A$ of space junk enters the rocket. So the total mass acquired by the rocket in time interval Δt is

$$\Delta m = \rho\left((\text{Volume})_1 + (\text{Volume})_2\right)$$
$$= \rho (v_r + u) \Delta t \, A$$

Thus

$$\frac{dm}{dt} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho (v_r + u) A \qquad (3)$$

From Eq (2)

$$v_r + u = (v_0 + u)\frac{m_c}{m_r(t)}$$

Then Eq (3) becomes

$$\frac{dm_r}{dt} = \frac{dm}{dt} = \rho A (v_0 + u)\frac{m_c}{m_r}$$

We can integrate this

$$\int_{m_0}^{m_r(t)} m_r \, dm_r = \int \rho A (v_0 + u)\, dt$$

$$\frac{1}{2}(m_r(t))^2 - \frac{1}{2} m_0^2 = \rho A (v_0 + u)\, t \, m_0$$

$$m_r(t) = \left(m_0^2 + 2\rho A (v_0 + u)\, t\right)^{1/2}$$

$$m_r(t) = \left(m_0^2 + 2\rho A (v_0 + u)\, m_0 \, t\right)^{1/2}$$

Problem 2: Momentum transfer: Boat and fire hose

A burning boat of initial mass m_0 is initially at rest, and slides without negligible resistance on the Charles River. A fireman stands on the Harvard Bridge and sprays water onto the boat. The water leaves the fire hose with a velocity u at a rate α (measured in $\mathrm{kg}\cdot\mathrm{s}^{-1}$). Assume the motion of the boat and the water jet are horizontal, and gravity does not play any role. Also assume that the mass change of the boat is only due to the water jet and all the water from the jet is added to the boat.

a) In a time interval $[t, t+\Delta t]$, an amount of water Δm hits the boat. Choose a system. Is the total momentum constant in your system? Write down a differential equation that results from the analysis of the momentum changes inside your system.

b) Integrate the differential equation you found in part a), to find the velocity $v(m)$ as a function of the increasing mass m of the boat.

c) What is the linear density λ ($\mathrm{kg}\cdot\mathrm{m}^{-1}$) of the water jet? What is the mass flow dm/dt ($\mathrm{kg}\cdot\mathrm{s}^{-1}$) which hits the boat?

d) Using your results from parts b) and c) , calculate the mass in the boat $m(t)$ as a function of time (express your answer in terms of m_0, u, α, and t.

(a)

$$P(t) = mv + u\Delta m$$

$$P(t+\Delta t) = (m+\Delta m)(v+\Delta v) = mv + m\Delta v + \Delta m v + \cancel{\Delta m \Delta v}$$ neglig

$$P(t+\Delta t) - P(t) = 0 \Rightarrow mv + m\Delta v + \Delta m v - mv - u\Delta m = 0$$

$$\boxed{m\Delta v + \Delta m(v-u) = 0}$$

(b)

$$m\,dv + dm\,(v-u) = 0$$

$$m\,dv = dm\,(u-v)$$

$$\int_0^v \frac{dv}{u-v} = \int_{m_0}^m \frac{dm}{m} \Rightarrow -\ln(u-v)\Big|_0^v = \ln m\Big|_{m_0}^m$$

$$-\left[\ln(u-v) - \ln u\right] = \ln\frac{m}{m_0}$$

$$\ln\left(\frac{u}{u-v}\right) = \ln\left(\frac{m}{m_0}\right)$$

$$\frac{u}{u-v} = \frac{m}{m_0} \Rightarrow -mu + mv = -m_0 u$$

$$\boxed{v = u\left(1 - \frac{m_0}{m}\right)}$$

(c) $\lambda =$ linear density of water jet $(kg/m) = \dfrac{\alpha\ (kg/sec)}{u\ (m/sec)}$

$$\boxed{\lambda = \frac{\alpha}{u}}$$

mass flow (kg/sec) which hits boat/sec $= \dfrac{dm}{dt} = \lambda(u-v) = \dfrac{\alpha}{u}\left(u - u\left(1 - \dfrac{m_0}{m}\right)\right)$

$$\boxed{\frac{dm}{dt} = \alpha\left(1 - \frac{v}{u}\right) = \alpha \cdot \frac{m_0}{m}}$$

(d) $\dfrac{dm}{dt} = \alpha \dfrac{m_0}{m}$

$$\int_{m_0}^m m\,dm = \int_0^t \alpha m_0\,dt$$

$$\frac{1}{2}(m^2 - m_0^2) = \alpha m_0 t$$

$$m^2 = m_0^2 + 2\alpha m_0 t$$

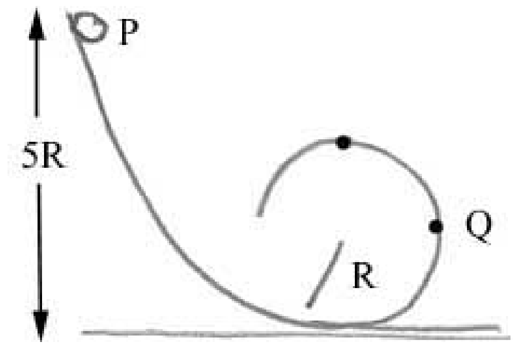
$$\boxed{m = \sqrt{m_0^2 + 2\alpha m_0 t}}$$

Problem 3: Energy

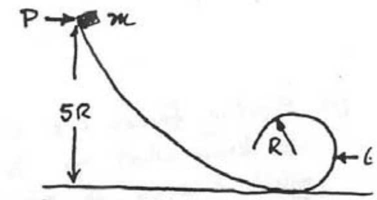
The frictionless track shown in the sketch has a final loop that consists of three-quarters of a circle of radius R. A small block of mass m is released from rest at the point P, a distance $5R$ above the ground.

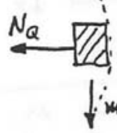
a) What is the force exerted by the track on the block at the point Q, at the right most edge of the track?

b) At what height above the ground should the block be released so that the force exerted by the track at the top of the final loop is equal to the block's weight?



(a) We know the path followed by the block: it just follows the track. Let us examine a FBD at point Q:





We are asked for the normal force N_Q. At point Q, N_Q is the only force with a component towards the center of the circle, so applying NII we get

$$N_Q = ma = m\frac{v_Q^2}{R}$$ where v_Q is the speed of the block at point Q.

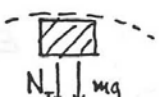
Since the track is frictionless, v is easily obtained from energy conservation:

$$E = \frac{1}{2}\cancelto{0}{m v_P^2} + mg(5R) = \frac{1}{2}m v_Q^2 + mg(R)$$

$$\text{or } v_Q^2 = 8gR$$

Thus $\boxed{N_Q = \frac{m v_Q^2}{R} = 8mg}$

b) Part (b) is just like part (a); here's the FBD at the top:



As in part (a), we write down NII which now involves both N_T and mg:

$$N_T + mg = \frac{m v_T^2}{R}$$

where v_T is the speed at the top. Again using conservation of E:

$$E = \frac{1}{2}\cancelto{0}{m v_P^2} + mg(h) = \frac{1}{2}m v_T^2 + mg(2R)$$ where h is the release height, or

$$v_T^2 = g(2h - 4R)$$

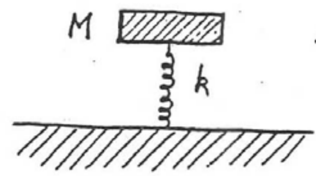
We are told that h should be such that $N_T = mg$ so

$$N_T + mg = 2mg = \frac{m v_T^2}{R} = 2mg\left(\frac{h}{R} - 2\right).$$ Solving, we get

$$\boxed{h = 3R}$$

Problem 4: Harmonic Oscillation

A vertical spring, of an unstretched length y_0 and spring constant k, supports a block of mass m. At time $t = 0$, a small bird of mass m_0 lands on the block with negligible incoming velocity. The block now moves downward.

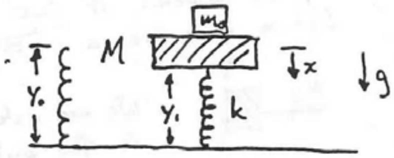


a) If the block is initially a height y_1 above the ground, what is the lowest height it reaches? You may neglect any friction.

b) At what time is this minimum height reached? Again, neglect friction.

c) If friction were not negligible. The block and bird would oscillate with diminishing amplitude, until they eventually came to rest at time t_f. Calculate the total work done by friction during the time interval $[0, t_f]$.

(a) Spring forces are conservative, but until we know what y_1 is we can't know how much energy is stored in the spring.



Thus let us first figure out y_1, the equilibrium point when M rests on the spring. The equilibrium will exist when M is not accelerating, i.e. has zero net force. The downward force of gravity, Mg, must be balanced by an upward spring force $kx = k(y_0-y_1)$. Thus:

$$k(y_0-y_1) = Mg \quad \text{or} \quad y_1 = y_0 - \frac{M}{k}g$$

Now, when the bird lights on the block the initial kinetic energy of each is zero and the potential energy in the spring is $\frac{1}{2}kx^2$:

$$E_i = 0 + \frac{1}{2}kx^2 + (M+m_0)gy_1 = \frac{1}{2}k(y_0-y_1)^2 + (M+m_0)gy_1$$

At the point of maximum compression, the KE is again zero so

$$E_f = 0 + \frac{1}{2}kx_f'^2 + (M+m)gy_f = \frac{1}{2}k(y_0-y_f)^2 + (M+m)gy_f$$

Since energy is conserved $E_f = E_i$ and $\frac{1}{2}k(y_0-y_{min})^2 = \frac{1}{2}\frac{M^2g^2}{k}$

$$\frac{1}{2}k(y_0-y_1)^2 + (M+m)gy_1 = \frac{1}{2}k(y_0-y_{min})^2 + (M+m)gy_{min}$$

or $\frac{1}{2}k\left[(y_0-y_1)^2-(y_0-y_{min})^2\right] + (M+m)g(y_1-y_{min}) = 0$

or $\frac{1}{2}k\left[y_1^2 - 2y_0y_1 + 2y_0y_{min} - y_{min}^2\right] + (M+m)g(y_1-y_{min}) = 0$

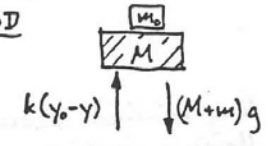
or $\frac{1}{2}k\left[\cancel{(y_1-y_{min})}(y_1+y_{min}-2y_0)\right] + (M+m)g\cancel{(y_1-y_{min})} = 0$

$$y_1 + y_{min} - 2y_0 = -\frac{2(M+m)}{k}g$$

$$\boxed{y_{min} = 2y_0 - y_1 - \frac{2(M+m)}{k}g = y_0 - \frac{(M+2m)}{k}g}$$

In order to figure out the <u>time</u> for compression, we have to go beyond energy and look at FBD's and Newton's laws:

<u>FBD</u>



<u>NII</u> $\quad k(y_0-y) - (M+m)g = (M+m)a = (M+m)\ddot{y}$

or $\quad \ddot{y} + \frac{k}{M+m_0} y - \frac{k}{M+m_0} y_0 + g = 0$

This equation looks a lot like the SHM equation, and in fact if we change variables from y to $x' = y - y_0 + \frac{M+m_0}{k} g$ we get that

$$\ddot{x}' + \frac{k}{M+m_0} x' = 0$$

This has solution $x'(t) = A\cos\left(\sqrt{\frac{k}{M+m_0}}\, t + \varphi\right)$. Now at the moment that the bird lands
That the bird lands $x(t) = \text{maximum}$ and $v(t) = 0$; at the moment of maximum compression $x(t) = \text{minimum}$ and $v(t) = 0$, so the time elapsed is exactly half a period of this sinusoidal motion:

$$\boxed{\Delta t = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \pi\sqrt{\frac{M+m_0}{k}}}$$

(c) We can calculate the work done by friction without knowing the details by calculating the change in energy. The initial energy is

$$E_i = \tfrac{1}{2}k(y_0-y_1)^2 + (M+m)g\,y_1 = \tfrac{1}{2}k\frac{M^2g^2}{k^2} + (M+m)g\left(y_0 - \frac{M}{k}g\right)$$

The final position is $x' = 0$ from part (b), i.e. $y_f = y_0 - \frac{M+m_0}{k}g$
So

$$E_f = \tfrac{1}{2}k(y_0-y_f)^2 + (M+m_0)g\,y_f = \tfrac{1}{2}k\frac{(M+m)^2g^2}{k^2} + (M+m)g\left(y_0 - \frac{M+m_0}{k}g\right)$$

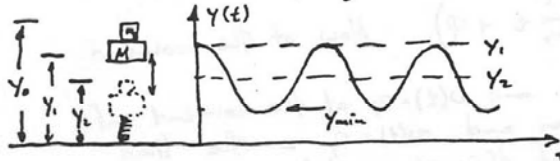
The work is then

$$W = E_f - E = \tfrac{1}{2}k\left[(M+m)^2 - M^2\right]\frac{g^2}{k^2} + (M+m)g\left[y_0 - \frac{M+m_0}{k}g - y_0 + \frac{M}{k}g\right]$$

$$= \frac{g^2}{2k}\left(2Mm_0 + m_0^2\right) - (M+m_0)g\,\frac{m_0 g}{k}$$

$$\boxed{W = -\frac{g^2}{2k}m_0^2}$$

The previous derivation is very long-winded because it assumes that we know nothing about springs. Here's a quicker way ...The mass is at equilibrium when the bird lands. The extra mass of the bird will cause the spring to oscillate around its new equilibrium. Since the initial velocity is zero, the amplitude is just the difference between these points. The displacements to the new equilibria are just the amounts that the spring must compress to balance gravity:



$$\left.\begin{aligned} y_0 - y_1 &= \frac{Mg}{k} \\ y_1 - y_2 &= \frac{m_0 g}{k} \end{aligned}\right\} y_2 = y_0 - \frac{(M+m)g}{k}$$

$$y_{min} = y_2 - (y_1 - y_2) = y_0 - \frac{(M+2m_0)g}{k}$$

(b) The system is oscillating about equilibrium with forces $-kx$ and $(M+m)g$. Since we know that it is executing SHM, even without a FBD we know that N II will give us $(M+m)\ddot{x} = -kx + (M+m)g$ and that we will be solving $\ddot{x} + \frac{k}{M+m}x = 0$. This gives us $\omega^2 = \frac{k}{M+m}$ and the time to execute half a period is

$$\boxed{\Delta t = \frac{\pi}{\omega} = \pi\sqrt{\frac{M+m}{k}}}$$

(c) The amplitude of the spring's motion is $y_1 - y_2 = \frac{m_0 g}{k} = A$. Any spring which is oscillating about an equilibrium point has energy $\frac{1}{2}kA^2$.

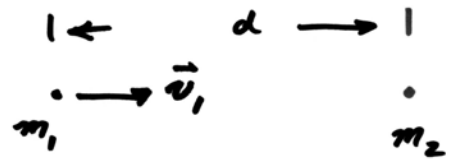
Thus $\quad E_i = \frac{1}{2}kA^2 = \frac{1}{2}k\frac{m_0^2 g^2}{k^2} = \frac{m_0^2 g^2}{2k}$

As above the final energy is zero (amplitude is zero), so

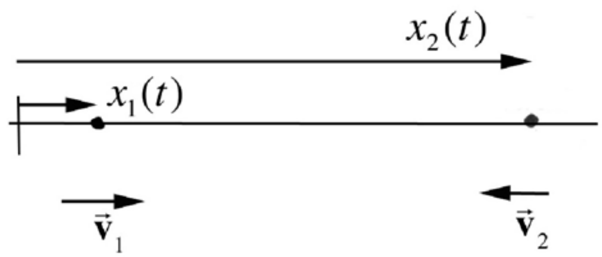
$$\boxed{W = E_f - E_i = -\frac{m_0^2 g^2}{2k}}$$

Problem 5: Center of Mass

Two small particles of mass m_1 and mass m_2 attract each other with a force that varies at the inverse cube of their separation. At time t_0, m_1 has velocity $\vec{\mathbf{v}}_{1,0}$ directed towards m_2, which is at rest a distance d away. At time t_1, the particles collide. How far does m_1 travel in time interval $t_1 - t_0$?



Solution: Let's consider the particle 2 (and everything in it) and the particle 1 as the system. The system is isolated in that there are no other external forces acting on it. Therefore the center of mass of the system does not accelerate, hence the velocity of the center of mass is constant. Choose an origin at the location of the particle 1 at $t = t_0$ as shown in the figure below.



Let $\vec{\mathbf{r}}_1(t) = x_1(t)\,\hat{\mathbf{i}}$ denote the position vector of the particle 1 at time t. Let $\vec{\mathbf{r}}_2(t) = x_2(t)\,\hat{\mathbf{i}}$ denote the position vector of the particle 2 at time t. At time $t = t_0$, the particle 1 has position vector $\vec{\mathbf{r}}_1(t = t_0) = \vec{\mathbf{0}}$ and the particle 2 has position vector $\vec{\mathbf{r}}_2(t = t_0) = d\,\hat{\mathbf{i}}$. So at $t = t_0$ the center of mass is located at

$$\vec{\mathbf{R}}_{cm}(t = t_0) = \frac{m_1\,\vec{\mathbf{r}}_1 + m_2\,\vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2\,d}{m_1 + m_2}\,\hat{\mathbf{i}}\,.$$

The velocity of the center of mass is the derivative with respect to time

$$\vec{\mathbf{V}}_{cm} = \frac{m_1\,\vec{\mathbf{v}}_1 + m_2\,\vec{\mathbf{v}}_2}{m_1 + m_2}\,.$$

The center of mass velocity remains constant because there are no external forces acting on the system. At time $t = t_0$, the particle 2 is at rest $\vec{\mathbf{v}}_2(t = t_0) = \vec{\mathbf{0}}$ and the particle 1 has velocity $\vec{\mathbf{v}}_1(t = t_0) = v_{1,0}\hat{\mathbf{i}}$, the center of mass velocity is

$$\vec{\mathbf{V}}_{cm} = \frac{m_1 v_{1,0}}{m_1 + m_2} \hat{\mathbf{i}} .$$

The position of the center of mass as a function of time is therefore

$$\vec{\mathbf{R}}_{cm}(t) = \vec{\mathbf{R}}_{cm}(t = t_0) + \vec{\mathbf{V}}_{cm}(t - t_0) = \left(\frac{m_2 d + m_1 v_{1,0}(t - t_0)}{m_1 + m_2} \right) \hat{\mathbf{i}} .$$

The collision takes place at the location of the center of mass at the time $t = t_1$

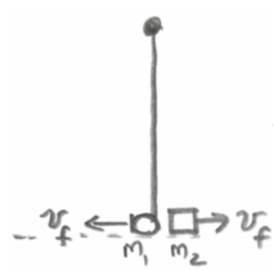
$$\vec{\mathbf{R}}_{cm}(t_1) = \left(\frac{m_2 d + m_1 v_{1,0}(t_1 - t_0)}{m_1 + m_2} \right) \hat{\mathbf{i}}$$

Problem 6: ***Pendulums and Collisions***

A simple pendulum consists of a bob of mass $m_1 = 0.4\,kg$ that is suspended by a massless string. The bob is pulled out and released from a height $h_0 = 0.2\,m$ as measured from the bottom of the swing and swings downward in a circular orbit. At the bottom of the swing, the bob collides with a block of mass m_2 that is initially at rest on a frictionless table. Assume the pivot point is frictionless.

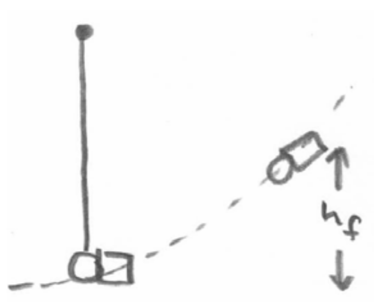
a) What is the velocity of the bob immediately before the collision at the bottom of the swing

b) Assume the collision is perfectly elastic. The block moves along the table and the bob moves in the opposite direction but with the same speed as the block. What is the mass, m_2, of the block?



c) Suppose the collision is completely inelastic due to some putty that is placed on the block. What is the velocity of the combined system immediately after the collision? (Assume that the putty is massless.)

d) After the completely inelastic collision, the bob and block continue in circular motion. What is the maximum height, h_f, that the combined system rises after the collision?



Solution:

a) The mechanical energy of the bob (system includes the Earth) is constant between when it is released and the bottom of the swing. We can use

$$(1/2)\, m_1 v_{1,0}^2 = m_1 g h_0 \tag{0.0.1}$$

to calculate the speed of the bob at the low point of the swing just before the collision,

$$v_{1,0} = \sqrt{2 g h_0} \tag{0.0.2}$$

b) Consider just the bob as the system. Although tension in the string and the gravitation force are now acting as external forces, both are particular to the motion of the bob, If we additionally assume that the collision is nearly instantaneous, then the momentum is constant in the direction of the bob's motion.

$$m_1 v_{1,0} = m_2 v_{2,f} - m_1 v_{1,f}\,. \tag{0.0.3}$$

Since the bob and block were assumed to have the same speeds after the collision, rebounded with the same speed as it had before the collision $v_f \equiv v_{2,f} = v_{1,f}$, rewrite Eq. (0.0.3) as

$$m_1 v_{1,0} = (m_2 - m_1) v_f\,. \tag{0.0.4}$$

Solve Eq. (0.0.4) for the speed of the bob after the collision

$$v_f = v_{1,0}\, m_1 / (m_2 - m_1) \tag{0.0.5}$$

The kinetic energy of the bob before the collision is equal to the kinetic energy of the bob and the block after the collision

$$(1/2)\, m_1 v_{1,0}^2 = (1/2)(m_1 + m_2) v_f^2 \tag{0.0.6}$$

Substitute Eq. (0.0.5) into Eq. (0.0.6) yielding

$$m_1 v_{1,0}^2 = (m_1 + m_2)\left(\frac{m_1}{m_2 - m_1}\right)^2 v_{1,0}^2\,. \tag{0.0.7}$$

Canceling the common factor of $m_1 v_{1,0}^2$ from both sides of Eq. (0.0.7) and rearranging gives

$$(m_2 - m_1)^2 = (m_1 + m_2)\, m_1 \,. \qquad (0.0.8)$$

Expanding the square and canceling m_1^2 yields

$$m_2 (m_2 - 3\, m_1) = 0\,, \qquad (0.0.9)$$

So the block has mass

$$m_2 = 3 m_1\,, \qquad (0.0.10)$$

and the final speed is

$$v_f = v_{1,0} \,/\, 2 = \sqrt{g\, h_0 \,/\, 2}\,. \qquad (0.0.11)$$

c) The bob and block stick together and move with a speed v'_f after the collision. The external forces are still perpendicular to the motion, and if we assume that the collision time is negligible, then the momentum in the direction of the motion is constant,

$$m_1\, v_{1,0} = (m_1 + m_2)\, v'_f \qquad (0.0.12)$$

So the speed immediately after the collision is (recalling that $m_2 = 3 m_1$)

$$v'_f = m_1\, v_{1,0} \,/\, (m_1 + m_2) = (3/4) v_{1,0} \qquad (0.0.13)$$

Using Eq. (0.0.10) and Eq. (0.0.2) in Eq. (0.0.13) yields

$$v'_f = (3/4) v_{1,0} = (3/4)\sqrt{2 g h_0}\,. \qquad (0.0.14)$$

d) The change in kinetic energy of the bob and block due to the collision in part c) is given by

$$\Delta K = K_{\text{after}} - K_{\text{before}} = (1/2)\big(m_1 + m_2\big) {v'_f}^2 - (1/2)\, m_1 v_{1,0}{}^2 \qquad (0.0.15)$$

Use Eq. (0.0.13) in Eq. (0.0.15),

$$\Delta K = (1/2)\big(m_1 + m_2\big)(m_1{}^2\, v_{1,0}{}^2 \,/\, (m_1 + m_2)^2)^2 - (1/2)\, m_1 v_{1,0}{}^2 \qquad (0.0.16)$$

Collecting terms in Eq. (0.0.16) yields

$$\Delta K = (1/2)(m_1 v_{1,0}{}^2)\left((\frac{m_1}{m_1+m_2} - 1)\right) = -(1/2)\frac{m_1 m_2}{m_1+m_2} v_{1,0}{}^2 \qquad (0.0.17)$$

The ratio of the change in kinetic energy to the kinetic energy before the collision is independent of the speed of the bob before the collision, and since $m_2 = 3m_1$ is given by

$$\frac{\Delta K}{K_{\text{before}}} = -\frac{m_2}{m_1+m_2} = -3/4 \qquad (0.0.18)$$

d) After the collision, the tension is acting on the bob-block system but the tension force is perpendicular to the motion so it does no work on the bob-block -(Earth) system so the mechanical energy is constant,

$$(1/2)(m_1+m_2)v_f'^{\,2} = (m_1+m_2)g\,h_f' \,. \qquad (0.0.19)$$

Use Eq. (0.0.13) in Eq. (0.0.19) to yield

$$(1/2)\,m_1 v_{1,0}{}^2 \frac{m_1}{(m_1+m_2)^2} = g\,h_f' \,. \qquad (0.0.20)$$

Use Eq. (0.0.1) for the kinetic energy of the bob before the collision in Eq. (0.0.20),

$$m_1\,g\,h_0 \frac{m_1}{(m_1+m_2)^2} = g\,h_f' \,. \qquad (0.0.21)$$

We can now solve for the height the bob and block rise until they first come to rest, (recalling that $m_2 = 3m_1$)

$$h_f' = \frac{m_1{}^2}{(m_1+m_2)^2} h_0 = h_0/16 \qquad (0.0.22)$$

Problem 7: Small Oscillations

The force on a particle of mass m_1 due to the interaction with a second particle of mass m_2 a distance r away is given by

$$\vec{\mathbf{F}} = \varepsilon(e^{-2(r-r_0)/r_0} - e^{-r/r_0})\hat{\mathbf{r}}$$

where ε and r_0 are positive and $\hat{\mathbf{r}}$ is a unit vector in the direction from the particle of mass m_2 to the particle of mass m_1.

a) What is $U(r)$, the potential energy, when the particles are a distance r apart?

b) Sketch $U(r)$.

c) For what value of r is the force zero?

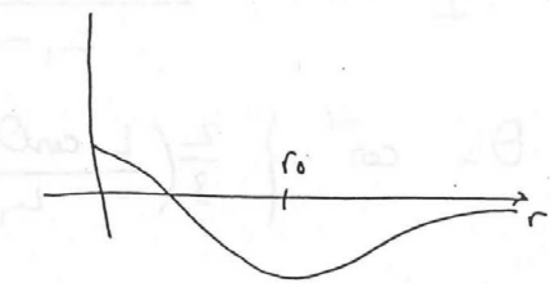
d) Calculate the angular frequency of small oscillations about the value of r at which the force is zero.

a.) $U(r) = -\int \vec{F}\cdot d\vec{r} = -\int_{\infty}^{r} \varepsilon \left[e^{-(2r-r_0)/r_0} - e^{-r/r_0} \right] dr$

$= -\varepsilon \left[e^{-(2r-r_0)/r_0} \left(-\frac{r_0}{2}\right) - e^{-r/r_0}(-r_0) \right] \Big|_{\infty}^{r}$

$= \boxed{\varepsilon r_0 \left[\frac{1}{2} e^{-\frac{(2r-r_0)}{r_0}} - e^{-r/r_0} \right]}$

b.)



c.) $\vec{F} = 0$ when $e^{-(2r-r_0)/r_0} - e^{-r/r_0} = 0$

$\Rightarrow 2r - r_0 = r$

or $\boxed{r = r_0}$

d.) $\frac{d^2U}{dr^2} = -\frac{d}{dr}F(r) = -\varepsilon \left[\frac{-2}{r_0} e^{-(2r-r_0)/r_0} + \frac{1}{r_0} e^{-r/r_0} \right]$

$\frac{d^2U}{dr^2}\Big|_{r=r_0} = \frac{\varepsilon}{e r_0}$

$\omega = \sqrt{\frac{d^2U/dr^2|_{r=r_0}}{\mu}}$; $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$\boxed{\omega = \sqrt{\frac{\varepsilon}{e r_0} \frac{(m_1 + m_2)}{m_1 m_2}}}$