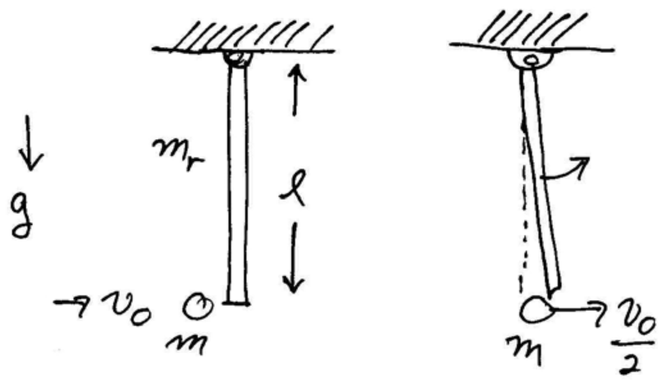
**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
**Department of Physics**

**Physics 8.012** **Fall Term 2009**

# Final Exam Practice Problems

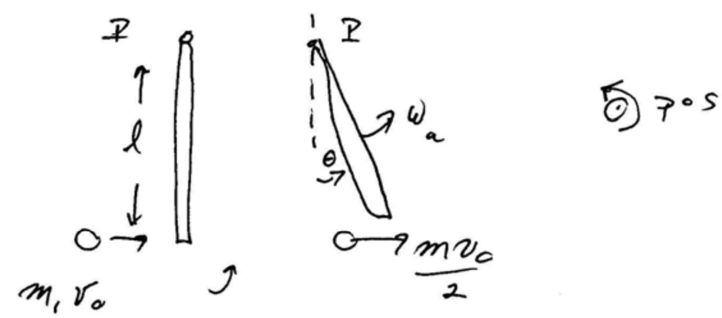
## Problem 1




An object of mass $m$ and speed $v_0$ strikes a rigid uniform rod of length $l$ and mass $m_r$ that is hanging by a frictionless pivot from the ceiling. Immediately after striking the rod, the object continues forward but its speed decreases to $v_0/2$. The moment of inertia of the rod about its center of mass is $I_{cm} = (1/12)m_r l^2$. Gravity acts with acceleration $g$ downward.

a) For what value of $v_0$ will the rod just touch the ceiling on its first swing? You may express your answer in terms of $g$, $m_r$, $m$, and $l$.

b) For what ratio $m_r / m$ will the collision be elastic?




Angular momentum is constant about the pivot point P

pos : $L_{P,0} = L_{P,a}$

$$m v_0 \ell = \frac{m v_0}{2}\ell + I_P\,\omega_a$$

$$I_P = m_r\left(\frac{\ell}{2}\right)^2 + \frac{1}{12} m_r \ell^2 = \frac{1}{3} m \ell^2$$

$$\frac{m v_0}{2}\ell = \frac{1}{3} m_r \ell^2 \omega_a \Rightarrow$$

$$\omega_a = \frac{3}{2}\frac{m}{m_r}\frac{v_0}{\ell} \qquad (1)$$

After the collision, energy is conserved

$$\frac{1}{2} I_P \omega_a^2 = m_r g\, \ell/2 \Rightarrow$$

$$\omega_a^2 = \frac{m_r g \frac{\ell}{2}}{\frac{1}{2}\frac{1}{3} m_r \ell^2} = \frac{3g}{\ell} \qquad (2)$$

$$\Rightarrow \omega_a = \sqrt{\frac{3g}{\ell}} \qquad (2)$$

Combining results

$$\sqrt{\frac{3g}{\ell}} = \frac{3}{2}\frac{m}{m_r}\frac{v_0}{\ell}$$

$$\Rightarrow \qquad v_0 = 2\frac{m_r}{m}\sqrt{\frac{g\ell}{3}} \qquad (3)$$

If the collision is elastic then

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m \left(\frac{v_0}{2}\right)^2 + \frac{1}{2} I_p \omega_a^2$$

$$\frac{3}{8} m v_0^2 = \left(\frac{1}{2}\right)\left(\frac{1}{3} m_r l^2\right)\left(\frac{3g}{l}\right.$$

$$\frac{3}{8} m \ \frac{4}{3} \frac{m_r^2}{m^2} g l = \frac{1}{2} m_r g l$$

$$\Rightarrow \qquad \frac{m_r}{m} = 1$$

**Problem 2**

A particle of mass $m$ moves under an attractive central force of magnitude $F = br^3$ . The angular momentum is equal to $L$.

a) Find the effective potential energy and make sketch of effective potential energy as a function of $r$.

b) Indicate on a sketch of the effective potential the total energy for circular motion.

c) The radius of the particle's orbit varies between $r_0$ and $2r_0$. Find $r_0$.

**Solutions:**

a) The potential energy is, taking the zero of potential energy to be at $r = 0$, is

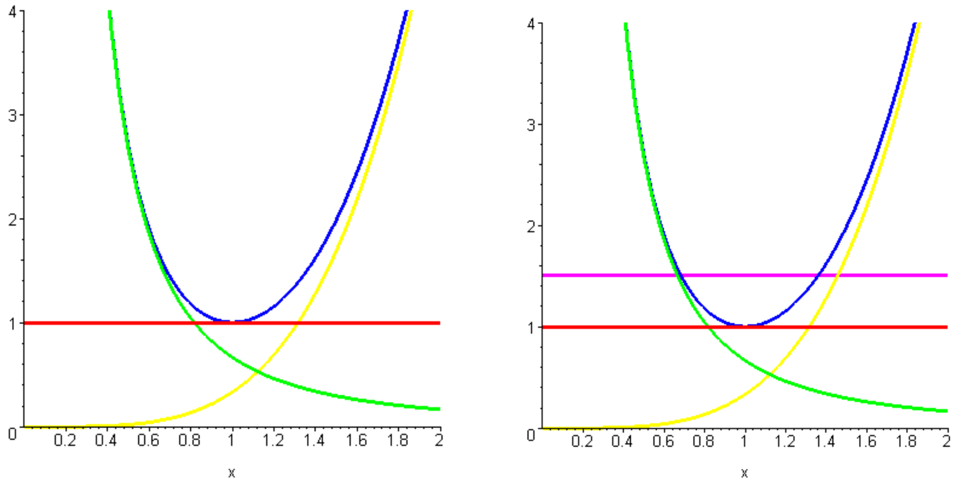
$$U(r) = -\int_0^r \left(-br'^3\right) dr' = \frac{b}{4} r^4$$

and the effective potential is

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} + \frac{b}{4} r^4 .$$

A plot is shown below, including the potential (yellow if seen in color), the term $L^2/2m$ (green) and the effective potential (blue). The minimum effective potential energy is the horizontal line (red). The horizontal scale is in units of the radius of the circular orbit and the vertical scale is in units of the minimum effective potential.

b) See the solution to part (a) above and the plot to the left below.




c) In the left plot, if we could move the red line up until it intersects the blue curve at two point whose value of the radius differ by a factor of $2$, those would be the respective values for $r_0$ and $2r_0$. A graph of this construction (done by computer, of course), showing the corresponding energy as the horizontal magenta is at the right above, and is not part of this problem.

To do this algebraically, we find the value of $r_0$ such that $U_{\text{eff}}\left(r_0\right) = U_{\text{eff}}\,2\left(r_0\right)$. This is

$$\frac{L^2}{mr_0^2} + \frac{b}{4}r_0^4 = \frac{L^2}{m\left(2r_0\right)^2} + \frac{b}{4}\left(2r_0\right)^4.$$

Rearranging and combining terms, and then solving for $r_0$,

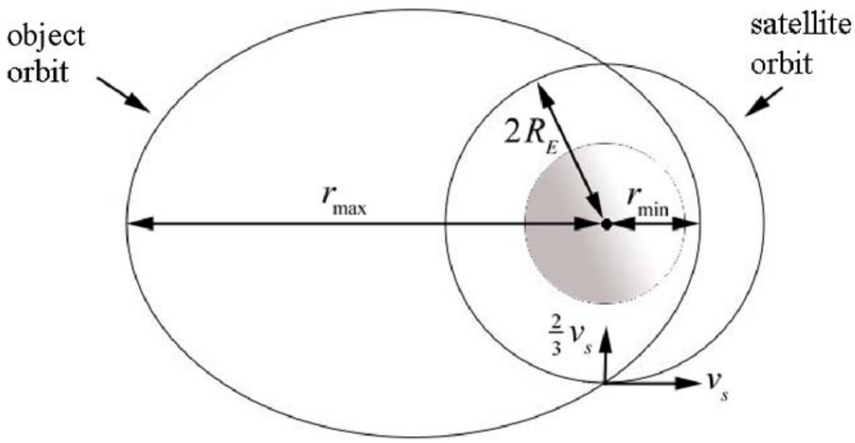
$$\frac{3}{8}\frac{L^2}{m}\frac{1}{r_0^2} = \frac{15}{4}b\,r_0^4$$

$$r_0^6 = \frac{1}{10}\frac{L^2}{mb}.$$

Thus, $r_0 = \left(1/\sqrt{10}\right)r_{\text{circular}}$ (not part of the problem), consistent with the auxiliary figure on the right above.

**Problem 3**

The space shuttle is orbiting the earth with speed $v_s$ in a circular orbit of radius $2R_E$, where $R_E$ is the radius of the Earth. Suppose, in the reference frame of the shuttle, an object of mass $m$ is shot towards the center of Earth at a speed equal to $(2/3)v_s$ (relative to the shuttle). The object moves in an elliptic orbit shown in the figure below. The goal of the problem is to find a quadratic equation whose solutions will give the minimum distance $r_{\min}$ and maximum distance $r_{\max}$ from the center of the Earth attained by the object as it orbits the Earth, expressed in terms of $R_E$. Let $G$ be the gravitational constant and $m_E$ denote the mass of the Earth.




a) Find an expression for the orbital speed $v_s$ of the shuttle in terms of $R_E$, $m$, $G$, and $m_E$ as needed.

b) Find an expression for the speed $v$ of the object relative to the Earth after it is shot from the shuttle in terms of $R_E$, $m$, $G$, and $m_E$ as needed.

c) What is the magnitude of the angular momentum of the object about the center of the Earth? Express your answer in terms of $R_E$, $m$, $G$, and $m_E$ as needed. Is the angular momentum of the object constant when it is in the elliptic orbit? Explain why or why not?

d) What is the magnitude of the energy (kinetic plus potential) of the object about the center of the Earth? Express your answer in terms of $R_E$, $m$, $G$, and $m_E$ as needed. Is the energy of the object constant when it is in the elliptic orbit? Explain why or why not?

e) What are the minimum distance $r_{\min}$ and maximum distance $r_{\max}$ from the center of the Earth attained by the object, expressed in terms of $R_E$? Hint: Use conservation of energy and angular momentum to derive a quadratic equation whose two solutions will give these distances.

Part a) For circular motion, the total force on the shuttle is given by

$$F = \frac{Gmm_E}{\left(2R_E\right)^2} = \frac{Gmm_E}{4R_E^2}$$

But we know that for a circular orbit, $F = mv_s^{\,2} / r$

$$\frac{Gmm_E}{4R_E^2} = \frac{mv_s^2}{2R_E}$$

$$\frac{Gm_E}{2R_E} = v_s^2$$

$$\boxed{v_s = \sqrt{\frac{Gm_E}{2R_E}}}$$

**Part B** The speed is given by

$$v^2 = v_s^2 + \left(\tfrac{2}{3}v_s\right)^2 = \tfrac{13}{9}v_s^2 = \frac{13Gm_E}{18R_E}$$

**Part C** The magnitude of the angular momentum about the center of the earth is only due to the horizontal component of the object's velocity, because the other component is parallel to ***r***. As such

$$L = \left|\vec{L}\right| = \left|\vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}\right| = 2R_E m v_s$$

$$\boxed{L = 2R_E m v_s}$$

It is indeed constant, because there is no external torque about the center of the earth.

**Part d)** The kinetic energy of the object is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{13Gm_E}{18R_E} = \frac{13Gmm_E}{36R_E}$$

The gravitational potential energy of the earth-object system is given by

$$U(2R_E) = -\frac{Gmm_E}{2R_E}$$

And so the total energy is

$$\mathrm{E} = \frac{13Gmm_E}{36R_E} - \frac{Gmm_E}{2R_E}$$

$$\boxed{\mathrm{E} = -\frac{5Gmm_E}{36R_E}}$$

This is also constant, because there is no external work done on the earth-object system nor any non-conservative forces acting inside the system..

Part e) Let the velocity at these points be $v$, and the relevant distances from the centre of the earth *r*. At each of these points, the velocity is perpendicular to ***r***. We can therefore write that at these points

$$L = mvr$$

$$E = \frac{1}{2}mv^2 - \frac{Gmm_E}{r}$$

But energy and angular momentum are conserved, and so we can equate these quantities with the quantities we found before

$$2R_E m v_s = mvr$$

$$\frac{1}{2}mv^2 - \frac{Gmm_E}{r} = -\frac{5Gmm_E}{36R_E}$$

Tidying up:

$$v = \frac{2R_E v_s}{r}$$

$$v^2 - \frac{2Gm_E}{r} = -\frac{5Gm_E}{18R_E}$$

Using the first equation, we can eliminate $v$

$$\frac{4R_E^2 v_s^2}{r^2} - \frac{2Gm_E}{r} = -\frac{5Gm_E}{18R_E}$$

Tidying up:

$$\frac{5Gm_E}{18R_E}r^2 - 2Gm_E r + 4R_E^2 v_s^2 = 0$$

$$\frac{5Gm_E}{18R_E}r^2 - 2Gm_E r + 4R_E^2 \frac{Gm_E}{2R_E} = 0$$

$$5Gm_E r^2 - 36GR_E m_E r + 36R_E^2 Gm_E = 0$$

Solving the quadratic equation

$$r = \frac{36GR_E m_E \pm \sqrt{\left(36GR_E m_E\right)^2 - 4 \cdot 5Gm_E \cdot 36R_E^2 Gm_E}}{2 \cdot 5Gm_E}$$

$$r = \frac{36GR_E m_E \pm R_E m_E G\sqrt{576}}{2 \cdot 5Gm_E}$$

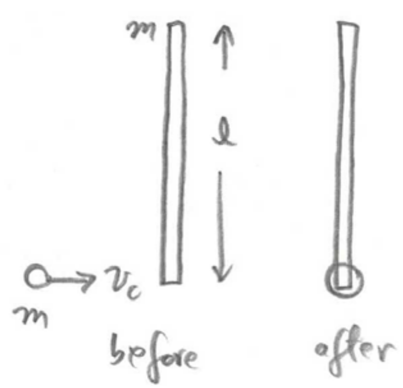
$$r = \frac{36 \pm 24}{10}R_E$$

$$r = 6R_E \qquad \text{or} \qquad r = \tfrac{6}{5}R_E$$

And so

$$\boxed{r_{\text{max}} = 6R_E \qquad\qquad r_{\text{min}} = 6R_E / 5}$$

**Problem 4:** A long narrow uniform stick of length $l$ and mass $m$ lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is $I_{cm}$. A puck (with putty on one side) has the same mass $m$ as the stick. The puck slides without spinning on the ice with a speed of $v_0$ toward the stick, hits one end of the stick, and attaches to it. You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to $I_{cm}$.




a) How far from the midpoint of the stick is the center of mass of the stick-puck combination after the collision?

b) What is the linear velocity of the stick plus puck after the collision?

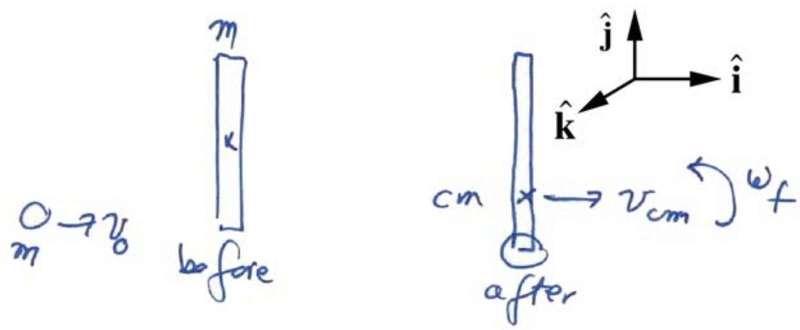
c) Is mechanical energy conserved during the collision? Explain your reasoning.

d) What is the angular velocity of the stick plus puck after the collision?

e) How far does the stick's center of mass move during one rotation of the stick?

**Solution:**

In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass. We note that the mechanical energy is not constant because the puck collides completely inelastically with the stick.




a) With respect to the center of the stick, the center of mass of the stick-puck combination is (neglecting the radius of the puck)

$$d_{\text{cm}} = \frac{m_{\text{stick}} d_{\text{stick}} + m_{\text{puck}} d_{\text{puck}}}{m_{\text{stick}} + m_{\text{puck}}} = \frac{m(0) + m(l/2)}{m + m} = \frac{l}{4}. \tag{0.1}$$

b) During the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty).

Hence, linear momentum is conserved. Initially only the puck had linear momentum $p_0 = mv_0$. After the collision, the center of mass of the system is moving with speed $v_f$. Equating initial and final linear momenta,

$$mv_0 = (2m)v_f \;\Rightarrow\; v_f = \frac{v_0}{2}\ . \tag{0.2}$$

The direction of the velocity is the same as the initial direction of the puck's velocity.

Note that the result of part a) was not needed for part b); if the masses are the same, Equation (0.2) would hold for any mass distribution of the stick.

c) The forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.

d) Choose the center of mass of the stick-puck combination, as found in part a), as the point about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{\text{puck}} \times \vec{\mathbf{p}}_0 = (l/4)(m v_0)\,\hat{\mathbf{k}}\,, \tag{0.3}$$

where $\hat{\mathbf{k}}$ is directed out of the page in the figure above. After the collision, the angular momentum is

$$\vec{\mathbf{L}}_f = I_{\text{cm}'}\omega_f\,\hat{\mathbf{k}}\,, \tag{0.4}$$

where $I_{\text{cm}'}$ is the moment of inertia about the center of mass of the stick-puck combination. This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is $m(l/4)^2$, and so

$$I_{\text{cm}'} = I_{\text{cm}',\,\text{stick}} + I_{\text{cm}',\,\text{puck}} = \left(I_{\text{cm}} + m(l/4)^2\right) + m(l/4)^2 = I_{\text{cm}} + \frac{ml^2}{8}. \tag{0.5}$$

Inserting this expression into Equation (0.4), equating the expressions for $\vec{\mathbf{L}}_0$ and $\vec{\mathbf{L}}_f$ and solving for $\omega_f$ yields

$$\omega_f = \frac{m(l/4)}{I_{\text{cm}} + ml^2/8}v_0\,. \tag{0.6}$$

If the stick is uniform, $I_{\text{cm}} = ml^2/12$ and Equation (0.6) reduces to

$$\omega_f = \frac{6}{5}\frac{v_0}{l}\,. \tag{0.7}$$

It may be tempting to try to calculate angular momentum about the contact point, where the putty hits the stick. If this is done, there is no initial angular momentum, and after the collision the angular momentum will be the sum of two parts, the angular momentum of the center of mass of the stick and the angular moment about the center of the stick,

$$\vec{\mathbf{L}}_f = \vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}} + I_{\text{cm}}\vec{\boldsymbol{\omega}}_f\,. \tag{0.8}$$

There are two crucial things to note: First, the speed of the center of mass is not the speed found in part b); the rotation must be included, so that $v_{\text{cm}} = v_0/2 - \omega_f(l/4)$. Second, the direction of

$\vec{\mathbf{r}}_{\text{cm}} \times \vec{\mathbf{p}}_{\text{cm}}$ with respect to the contact point is, from the right-hand rule, *into* the page, or the $-\hat{\mathbf{k}}$-direction, opposite the direction of $\vec{\boldsymbol{\omega}}_f$. This is to be expected, as the sum in Equation (0.8) must be zero. Adding the $\hat{\mathbf{k}}$-components (the only components) in Equation (0.8),

$$-(l/2)m\big(v_0/2 - \omega_f(l/4)\big) + I_{\text{cm}}\omega_f = 0\,. \tag{0.9}$$

Solving Equation (0.9) for $\omega_f$ yields Equation (0.6).

This alternative derivation should serve two purposes. One is that it doesn't matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It's often a matter of trial and error ("learning by misadventure") to find the "best" way to solve a problem.

e) The time of one rotation will be the same for all observers, independent of choice of origin. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period $T = 2\pi/\omega_f$ and the distance the center of mass moves is

$$\begin{aligned} x_{\text{cm}} &= v_{\text{cm}}T = 2\pi\frac{v_{\text{cm}}}{\omega_f} \\ &= 2\pi\frac{v_0/2}{\left(\dfrac{m(l/4)}{I_{\text{cm}} + ml^2/8}\right)v_0} \\ &= 2\pi\frac{I_{\text{cm}} + ml^2/8}{m(l/2)}. \end{aligned} \tag{0.10}$$

Using $I_{\text{cm}} = ml^2/12$ for a uniform stick gives

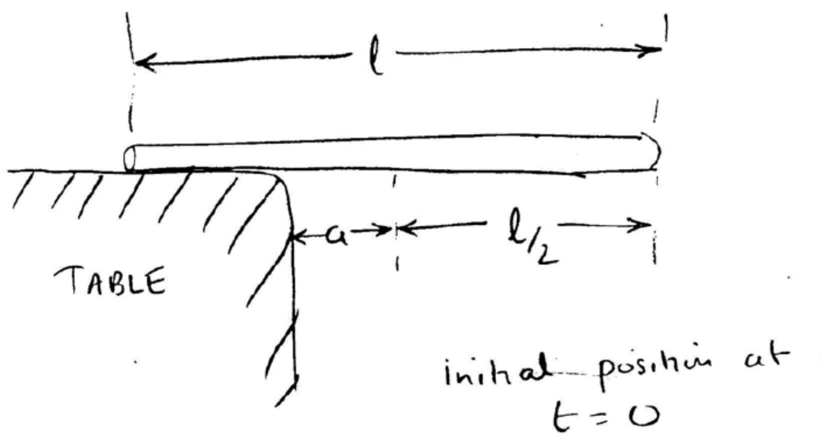
$$x_{\text{cm}} = \frac{5}{6}\pi l\,. \tag{0.11}$$

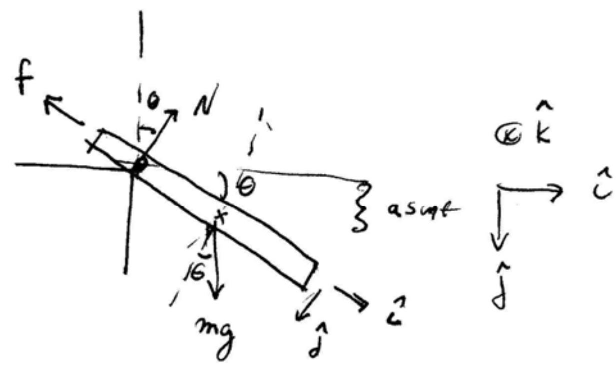
**Problem 5** A uniform rod of mass $m$ and length $l$ is placed on a horizontal table top with its center of mass a distance $a$ from the perpendicular edge as shown in the figure. The rod is released from rest from a horizontal position and begins to rotate about the edge of the table. The coefficient of friction between the rod and the table is $\mu$.

a) Draw a force diagram for the rod when it makes an angle $\theta$ with horizontal before it starts slipping, showing the weight of the rod, the normal and frictional forces.

b) Express the coordinates of the center of mass of the rod in terms of the angle $\theta$.

c) Find the maximum angle the rod attains before slipping begins.

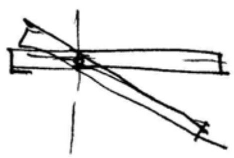





$x = a\cos\theta$

$y = a\sin\theta$

| $\vec{F}$ | $m\vec{a}$ |
|---|---|
| $-f + mg\sin\theta$ | $= -m a\dot\theta^2$ |
| $N - mg\cos\theta =$ | $m a\ddot\theta$ |



$f_{max} = \mu N$

$E = \frac{1}{2} I_p \dot\theta^2 - mga\sin\theta = 0$

$\dot\theta^2 = \frac{2mga\sin\theta}{I_p}$

| $\tau$ | $I_p\ddot\theta$ |
|---|---|
| $mg\cos\theta\, a$ | $= I_p\ddot\theta$ |

$\ddot\theta = \frac{mg\cos\theta\, a}{I_p}$

$-f + mg\sin\theta = -m a \frac{2mga\sin\theta}{I_p}$

$N - mg\cos\theta = m \frac{a^2 mg}{I_p} \cos\theta$

$f = \mu N$

$$-N + \frac{mg\sin\theta}{\mu} = -\frac{ma\,2mg}{I_p}\frac{a\sin\theta}{\mu}$$

$$mg\cos\theta - N = ma^2\frac{mg\cos\theta}{I_p}$$

$$\frac{mg\sin\theta}{\mu} - mg\cos\theta = -\frac{m^2a^2}{I_p}g\left(\cos\theta + 2\frac{\sin\theta}{\mu}\right)$$

$$\left(-\cos\theta + \frac{\sin\theta}{\mu}\right) = -\frac{ma^2}{I_p}\left(\cos\theta + 2\frac{\sin\theta}{\mu}\right)$$

$$\cos\theta\left(\frac{ma^2}{I_p} - 1\right) = -\frac{\sin\theta}{\mu}\left(2\frac{ma^2}{I_p} + 1\right)$$

$$\frac{\mu\left(-\frac{ma^2}{I_p} + 1\right)}{\left(\frac{2ma^2}{I_p} + 1\right)} = \tan\theta$$

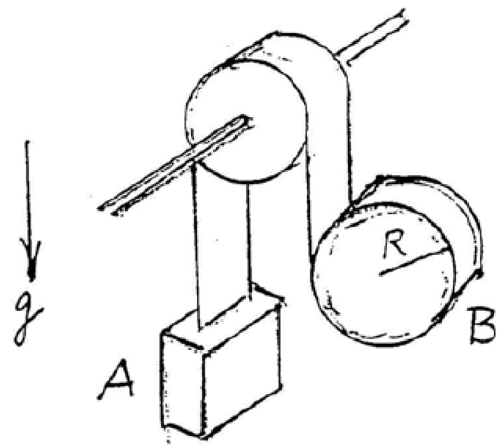
$$I_p = \frac{1}{12}ml^2 + ma^2$$

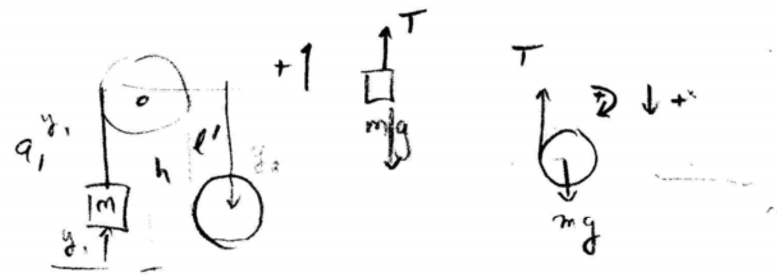
$$\frac{\mu\left(1 - \frac{ma^2}{ma^2 + \frac{1}{12}ml^2}\right)}{\left(1 + \frac{2ma^2}{ma^2 + \frac{1}{12}ml^2}\right)} = \mu\left(\frac{\frac{1}{12}ml^2}{3ma^2 + \frac{1}{12}ml^2}\right) = \tan\theta$$

$$\theta = \tan^{-1}\left(\mu\left(\frac{1}{\frac{36a^2}{l^2} + 1}\right)\right.$$

**Problem 6**

Two equal masses $M$ are suspended from a massless and frictionless pulley, as shown. A is a simple weight. B is a uniform cylinder of radius $R$ around which the tape is wrapped. The system is released from rest. Find the acceleration of A.




$T - mg = m_A a_1$

$RT = I_{cm}\,\alpha$

$l(t) = (h - y_1) + y_2$

$R\,\alpha(t) = -a_1 + a_2$

$\frac{3}{2} T = \frac{1}{2} m_B g - \frac{1}{2} m_B a_1$

$\frac{3}{2}(m_A a_1 + m_A g) = \frac{1}{2} m_B g - \frac{1}{2} m_B a_1$

$3 m_A g - m_B g = -(3 m_A + m_B) a_1$

$$a_1 = \frac{m_B g - 3 m_A g}{(3 m_A + m_B)}$$

$m_B g - T = m_B a_2$

$RT = \frac{1}{2} m_B R^2 \alpha$

$RT = \frac{1}{2} m_B R (a_2 - a_1)$

$T = \frac{1}{2} m_B (a_2 - a_1)$

$T = \frac{1}{2} m_B a_2 - \frac{1}{2} m_B a_1$

$T = \frac{1}{2}(m_B g - T)$

$\frac{3}{2} T = \frac{1}{2} m_B g - \frac{1}{2} m_B a_1$

$\boxed{a_1 = -\frac{1}{2} m g}$