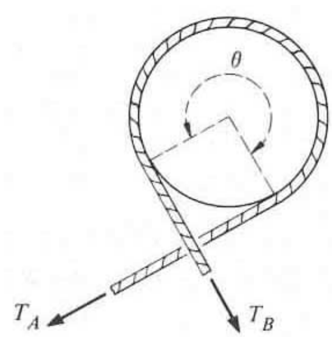
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012

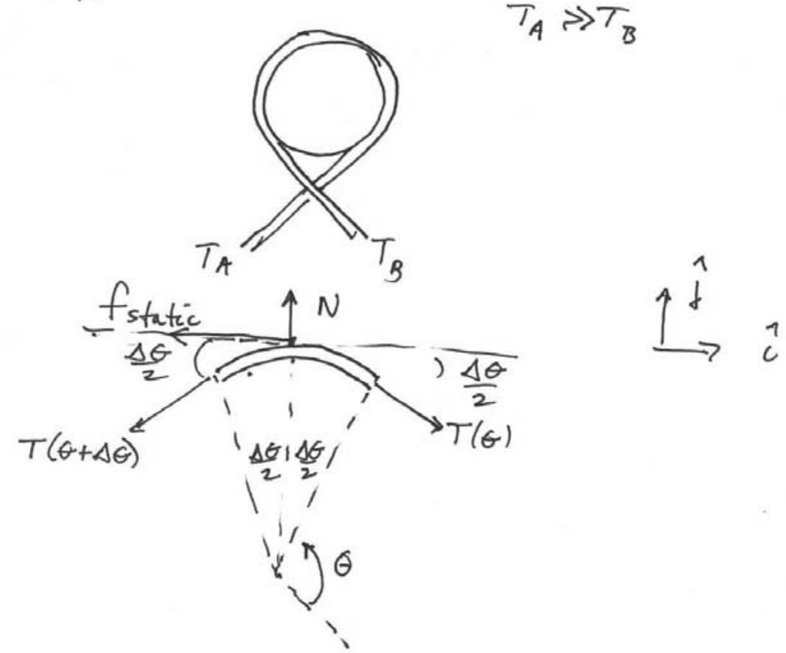
Problem Set 3 Solutions

Problem 24:

A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing shows about three fourths turn). The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B. Can you show that $T_B = T_A e^{-\mu_s \theta}$, where μ_s is the coefficient of static friction and θ is the total angle subtended by the rope on the drum?



Chapter 2: 24.



Since $T_A \gg T_B$ the rope would tend to slip to the right. Thus motion does not occur due to static friction with static friction

$$(f_{static}) = (f_{static})_{max} = \mu N \qquad (1)$$

equal to its maximum value i.e. the 'just slipping condition. The rope is in static equilibrium so

$$\vec{a}_{rope} = 0$$

Thus the force decomposition yields

	$\vec{F} =$	$m\bar{a}$
$\hat{i}:$	$-f_{static} + T(\theta)\cos(\frac{\Delta\theta}{2}) - T(\theta+\Delta\theta)\cos(\frac{\Delta\theta}{2})$	$= 0$ (2)

$$\hat{j}: -T(\theta)\sin\frac{\Delta\theta}{2} - T(\theta+\Delta\theta)\sin\frac{\Delta\theta}{2} + N = 0 \quad (3)$$

define $\Delta T = T(\theta+\Delta\theta) - T(\theta)$

Use the small angle approximations

$$\cos(\tfrac{\Delta\theta}{2}) \simeq 1$$

$$\sin(\tfrac{\Delta\theta}{2}) \simeq \frac{\Delta\theta}{2}$$

Then eq (2) becomes

$$-\mu N + T - (T+\Delta T) = 0 \quad (2a)$$

$$\Rightarrow \quad \Delta T = -\mu N \quad (2b)$$

Eq (3) becomes

$$-T\frac{\Delta\theta}{2} - (T+\Delta T)\frac{\Delta\theta}{2} + N = 0 \quad (3a)$$

which becomes

$$-T\Delta\theta - \Delta T\frac{\Delta\theta}{2} + N = 0 \quad (3b)$$

We ignore second order terms $\Delta T\frac{\Delta\theta}{2} \simeq 0$

since they vanish in the limit as $\Delta\theta \to 0$. So eq (3b) becomes

$$N = T\Delta\theta \qquad (3c)$$

Combining this with eq (2b), we get

$$-\frac{\Delta T}{\mu} = T\Delta\theta$$

$$\frac{\Delta T}{\Delta\theta} = -\mu T$$

or in the $\lim_{\Delta\theta \to 0} \frac{\Delta T}{\Delta\theta} = -\mu T$

$$\boxed{\frac{dT}{d\theta} = -\mu T} \qquad (4)$$

We can solve this by integration

$$\int_{T_A}^{T_B} \frac{dT}{T} = \int_{\theta=0}^{\theta} -\mu \, d\theta$$

note at $\theta = 0$, the tension is T_A and at θ, the tension is T_B.

$$\ln\left(\frac{T_B}{T_A}\right) = -\mu\theta$$

$$\frac{T_B}{T_A} = e^{-\mu\theta} \Rightarrow \boxed{T_B = T_A e^{-\mu\theta}}$$

Extra Problem: The Gravitational Field of a Spherical Shell of Matter

Consider a spherical shell of radius R of mass m_s, that is uniformly distributed over the shell with mass per unit area $\sigma = \dfrac{m_s}{4\pi R^2}$. Show that

1) The gravitational force on a mass m placed outside a spherical shell of matter of uniform surface mass density σ is the same force that would arise if all the mass of the shell were placed at the center of the sphere.

2) The gravitational force on a mass m placed inside a spherical shell of matter is zero.

$$\vec{\mathbf{F}}_{m,s}(r) = \begin{cases} -G\dfrac{mm_s}{r^2}\hat{\mathbf{r}}, & r > R \\ \vec{\mathbf{0}}, & r < R \end{cases}$$

where $\hat{\mathbf{r}}$ is the unit vector located at the position of mass m and pointing radially away from the center of the shell.

Now consider a solid uniform sphere of mass m_T and radius R.

3) Show that the gravitational force on a body of mass m, uniform mass density $\rho = m_T / \dfrac{4}{3}\pi R^3$. located a distance $r < R$, is due solely to the mass lying at a distance , $r' \le r$, measured from the center of the sphere, and is given by $\vec{\mathbf{F}}_{m,s}(r) = -(4\pi/3)Gm\rho r\hat{\mathbf{r}}, \quad r < R$.

Proof:

Let dm be an infinitesimal mass element on the shell. Let da be an infinitesimal area element . Choose spherical coordinates as shown in figure 1. Then the infinitesimal area element on the surface of the shell is given by

$$da = R\sin\theta d\theta d\phi .$$

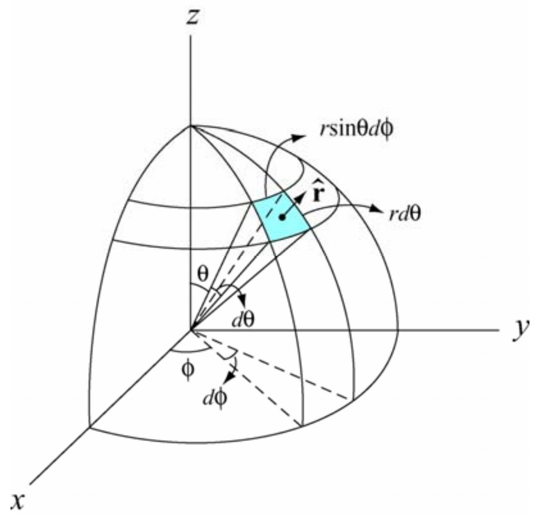


Figure 1: infinitesimal area element

Then the mass contained in that element is

$$dm = \sigma da = \sigma R^2 \sin\theta d\theta d\phi .$$

The contribution from dm to the gravitational force at the mass m that lies outside the shell has a component pointing in the $\hat{\mathbf{k}}$ direction and a radial component pointing towards the z-axis. By symmetry there is another dm on the opposite side of the shell, which will produce a gravitational force, which exactly cancels the radial component of the force pointing towards the z-axis. When we integrate the area element with respect to the angle $\left[0 \le \varphi \le 2\pi\right]$, we form a ring. So the force of the ring on the mass m has only a component in the $\hat{\mathbf{k}}$ direction. Again from figure 2 we see that

$$\left(d\vec{\mathbf{F}}_{m,s}\right)_z \equiv dF_z\hat{\mathbf{k}} = -G\frac{mdm}{r^2}\cos\alpha\hat{\mathbf{k}}$$

where

$$r^2 = R^2 + z^2 - 2Rz\cos\theta \qquad \text{and} \qquad \cos\alpha = \frac{z - z'}{r} = \frac{z - R\cos\theta}{r}$$

Thus

$$dF_z = -G\frac{mdm}{r^2}\cos\alpha = -G\frac{m_s m}{4\pi R^2}\frac{R^2\sin\theta d\theta d\phi\left(z - R\cos\theta\right)}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{3/2}} .$$

Let $u = z - R\cos\theta$. Then $du = R\sin\theta d\theta$ and $r^2 = R^2 + z^2 - 2Rz\cos\theta = R^2 + 2uz - z^2$.

Then

$$dF_z = -G\frac{m_s m}{4\pi R^2}\frac{Rudud\phi}{\left(R^2+2uz-z^2\right)^{3/2}}.$$

When $\theta = 0$, $u = z - R$; when $\theta = \pi$, $u = z + R$. Thus

$$F_z = \int_{u=z-R}^{u=z+R}\int_{\phi=0}^{\phi=2\pi} dF_z = -G\frac{m_s m}{4\pi R}\int_{u=z-R}^{u=z+R}\int_{\phi=0}^{\phi=2\pi}\frac{udud\phi}{\left(R^2+2uz-z^2\right)^{3/2}}$$

The double integral

$$F_z = -G\frac{m_s m}{4\pi R}\int_{u=z-R}^{u=z+R}\int_{\phi=0}^{\phi=2\pi}\frac{udud\phi}{\left(R^2+2uz-z^2\right)^{3/2}} = -2\pi G\frac{m_s m}{4\pi R}\int_{u=z-R}^{u=z+R}\frac{udu}{\left(R^2+2uz-z^2\right)^{3/2}}$$

because the integrand is independent of ϕ.

Then the remaining integral becomes

$$F_z = -G\frac{m_s m}{2R}\int_{u=z-R}^{u=z+R}\frac{udu}{\left(R^2+2uz-z^2\right)^{3/2}} = -G\frac{m_s m}{2R}\frac{1}{2z^2}\left(\sqrt{R^2+2uz-z^2}-\frac{\left(z^2-R^2\right)}{\sqrt{R^2+2uz-z^2}}\right)\Bigg|_{u=z-R}^{u=z+R}$$

$$F_z = -G\frac{m_s m}{2R}\frac{1}{2z^2}\left(\left(z+R-\frac{z^2-R^2}{z+R}\right)-\left(\sqrt{R^2-2zR+z^2}-\frac{\left(z^2-R^2\right)}{\sqrt{R^2-2zR+z^2}}\right)\right)$$

Now there is a subtlety. Since $\sqrt{R^2-2zR+z^2}$ is always positive, we have two special cases:

$$\sqrt{R^2-2zR+z^2} = \begin{Bmatrix} z-R, & z>R \\ R-z, & z<R \end{Bmatrix}$$

Then for $z > R$:

$$F_z = -G\frac{m_s m}{2R}\frac{1}{2z^2}\left(\left(z+R-\frac{(z+R)(z-R)}{z+R}\right)-\left(z-R-\frac{(z+R)(z-R)}{z-R}\right)\right)$$

Simplifying this yields

$$F_z = -G\frac{m_s m}{2R}\frac{1}{2z^2}4R = -G\frac{m_s m}{z^2}$$

And for $z < R$:

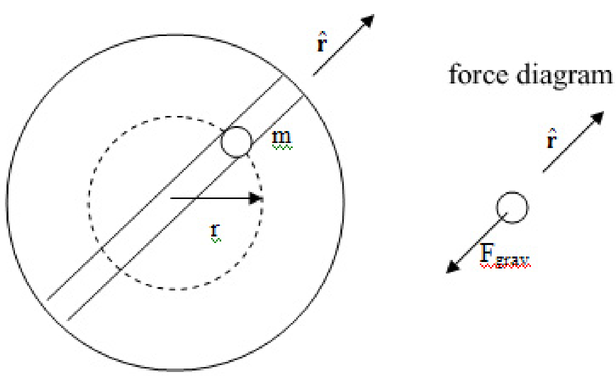
$$F_z = -G\frac{m_s m}{2R}\frac{1}{2z^2}\left(\left(z+R-\frac{(z+R)(z-R)}{z+R}\right)-\left(R-z-\frac{(z+R)(z-R)}{R-z}\right)\right) = 0$$

$$F_z = \begin{Bmatrix} -G\dfrac{m_s m}{z^2}, & z > R \\ 0 \quad , & z < R \end{Bmatrix}$$

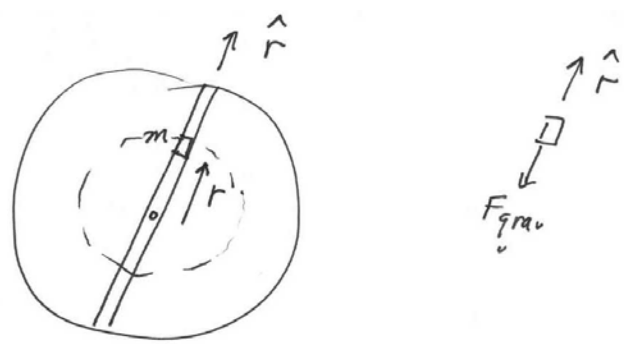
This proves the result that the gravitational force inside the shell is zero and the gravitational force outside the shell is equivalent to putting all the mass at the center of the shell.

Problem 26: Simple Harmonic Motion: Tunnel through the earth

Suppose a tunnel is drilled through the earth passing through the center. The earth has radius R, total mass m and uniform mass density $\rho = m / \frac{4}{3}\pi R^3$. If an object is released at one end of the tunnel, how long will it take the object to return to the starting point? You may assume that the earth is a uniformly dense sphere, and you must neglect all friction and any effects due to the earth's rotation.



Problem 26:



F_{grav} on the mass m is

just $\vec{F}_{grav} = -G \dfrac{m \, m_{enclosed}}{r^2} \hat{r}$

where $m_{enclosed} = \rho \frac{4}{3}\pi r^3$

with $\rho = \dfrac{m_{earth}}{\frac{4}{3}\pi R^3}$

Newton's 2nd Law $\vec{F} = m\vec{a}$ becomes

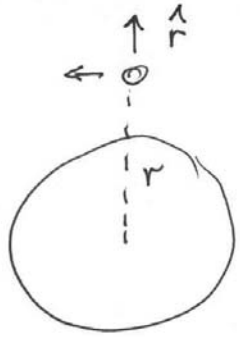
$$\hat{r}: \quad -\frac{G\, m\, \rho \frac{4}{3}\pi r^3}{r^2} = m\frac{d^2r}{dt^2}$$

$$\frac{d^2r}{dt^2} + G\rho\frac{4}{3}\pi\, r = 0$$

This is the equation for simple harmonic motion with

$$\omega = \sqrt{G\rho\tfrac{4}{3}\pi}\ , \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{G\rho\frac{4}{3}\pi}}$$

satellite
orbiting
earth



$$\vec{F} = m\vec{a}$$

$$\hat{r}: \quad -\frac{G m M_{earth}}{r^2} = -m r\left(\frac{2\pi}{T}\right)^2$$

solving for the period T

$$T^2 = \frac{(2\pi)^2 r^3}{G M_{earth}} = \frac{(2\pi)^2 r^3}{G \rho \frac{4}{3}\pi R_e^3}$$

when $r \simeq R_e$ (low orbit)

$$T = \frac{2\pi}{\sqrt{G\rho\frac{4}{3}\pi}}$$

which is the same period as the mass oscillating through the center of the earth.

Problem 28: (*Circular motion: banked turn)*

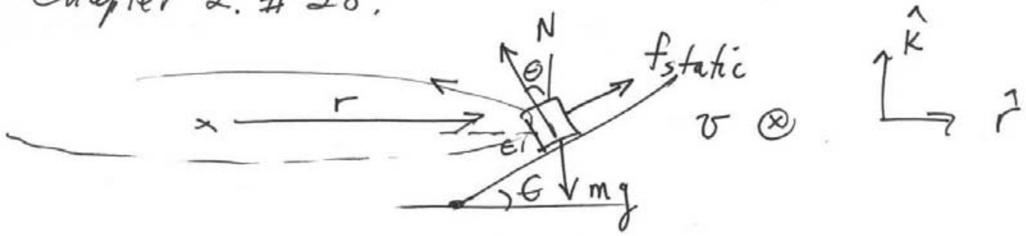
A car of mass m is going around a circular turn of radius R which is banked at an angle θ with respect to the ground. Assume there is a coefficient of static friction μ_s between the wheels and the road. Let g be the magnitude of the acceleration due to gravity. You may neglect kinetic friction. In each part below show your force diagrams.

a) Derive an expression for the minimum velocity necessary to keep the car moving in a circle without slipping down the embanked turn. Express your answer in terms of the given quantities.

b) Derive an expression for the maximum velocity necessary to keep the car moving in a circle without slipping up the embanked turn. Express your answer in terms of the given quantities.

c) Derive an expression for the velocity necessary to keep the car moving in a circle without slipping up or down the embanked turn such that the static friction force vanishes. Express your answer in terms of the given quantities.

Chapter 2. # 28.



when $v_{min} < v < v_0$, the car would slide down the banked turn if static friction didnt hold it up. v_0 is the velocity such that $f_{static} = 0$, at $v = v_0$.

	$\vec{F} =$	$m\vec{a}$	
$\hat{r}$:	$-N\sin\theta + f_{static}\cos\theta =$	$-\dfrac{mv^2}{r}$	(1)
$\hat{k}$:	$N\cos\theta + f_{static}\sin\theta - mg =$	0	(2)

$$f_{static} = \mu N \quad \text{when } v = v_{min}$$

So eq (1) becomes

$$-N\sin\theta + \mu N\cos\theta = -\frac{mv_{min}^2}{r} \qquad (1a)$$

eq (2) becomes

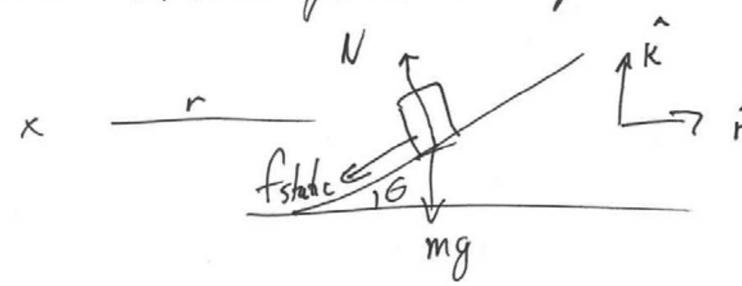
$$N\cos\theta + \mu N\sin\theta = mg \qquad (2a)$$

dividing thes equations yelds

$$-\frac{\sin\theta + \mu\cos\theta}{\cos\theta + \mu\sin\theta} = -\frac{v_{min}^2}{rg}$$

$$v_{min} = \left(rg\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right)\right)^{1/2}$$

When $v_0 < v < v_{max}$, the car would tend to slide up the inclined plane so static friction points down



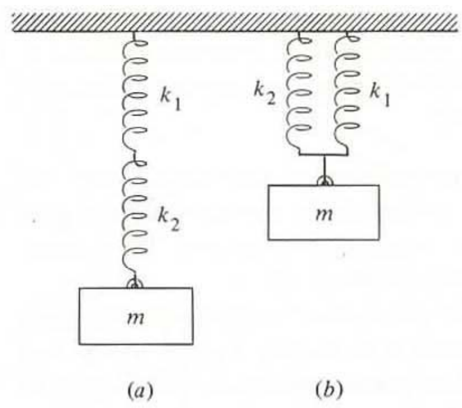
$$f_{static} = \mu N \quad \text{when} \quad v = v_{max}$$

the analysis is identical to the previous case except for the change in sign of f_{static} so

$$v_{max} = \left(rg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)\right)^{1/2}$$

Problem 31:

Find the frequency of oscillation of a block of mass m suspending by two springs having spring constants k_1 and k_2,

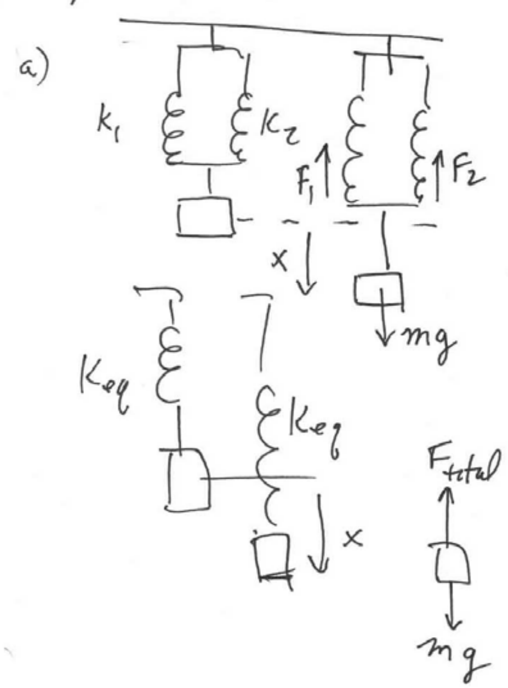


a) when they are each attached to the wall and the block (in parallel);

b) when spring 1 is attached to the wall and one end of spring 2, and the other end of spring 2 is attached to the block (in series).

chapter 2. 31

a)



When the springs are parallel, both springs are displaced by the same distance x.

The force on the mass $F^{total} = F_1 + F_2$ adds
$F_1 = k_1 x$, $F_2 = k x_2$

we replace the spring by an equivalent spring with spring constant k_{eq} and the total force $F^{total} = F_1 + F_2$

We define K_{eq} by

$$F^{total} = K_{eq}\, x$$

Thus

$$F^{total} = F_1 + F_2 \quad \text{becomes}$$

$$K_{eq}\, x = k_1 x + k_2 x$$

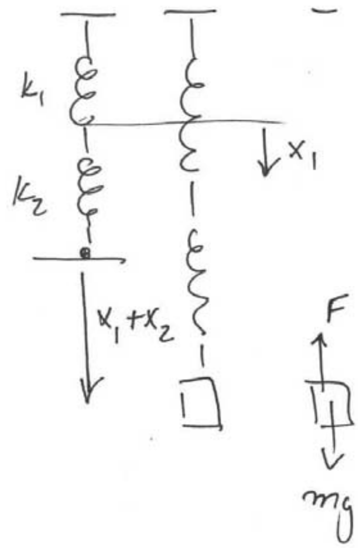
$$K_{eq} = k_1 + k_2$$

Thus

$$w = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{k_1 + k_2}{m}}$$

angular frequency of oscillation

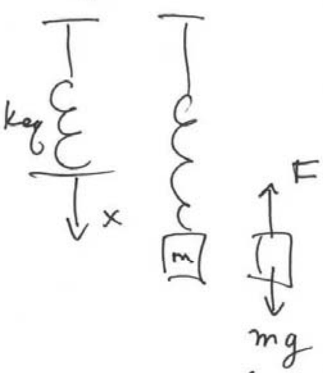
b) When the springs are in series, each spring stretches giving a total stretch



$$x = x_1 + x_2$$

Once again we replace the two springs with an equivalent spring with

$$K_{eq} x = F$$



Now the springs are all held under the same tension. This means that

$$F_1 = F_2 = F$$

where $F_1 = k x_1$ and $F_2 = k x_2$, $F = K_{eq} x$

Since $x = x_1 + x_2$

$$\frac{F}{K_{eq}} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow$$

$$\frac{1}{K_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$w = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{k_1 k_2}{(k_1+k_2)}\frac{1}{m}}$$

Problem 34: A body of mass m whirls around on a string which passes through a fixed ring located at the center of the circular motion. The string is held by a person who pulls the string downward with a constant velocity of magnitude V so that the radial distance to the body decreases. Initially the body is a distance r_0 from the center and is revolving with angular velocity ω_0. You may neglect the effect of gravity.

a) Draw a free body force diagram for body. Identify which coordinates you will choose and draw the unit vectors associated with your coordinate system on your free body diagram.

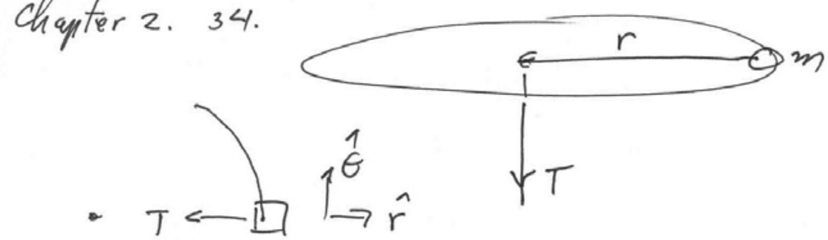
b) Using Newton's second laws, derive a differential equation for the angular velocity ω.

c) Solve the differential equation for ω by integration techniques and using the initial conditions, find an expression for the angular velocity as a function of time, $\omega(t)$.

d) Find the force needed to pull the string.

e) Compute the quantity, $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$, (this is called the *angular momentum*). Does this quantity change in time as the body moves in a spirally inward motion?

Chapter 2. 34.



The mass is moving inward so $\dot{r} \neq 0$, if the string is pulled in at a constant rate $\dot{r} = v =$ constant

$$r = r_0 - v t \quad , \quad 0 \leq t \leq \frac{r_0}{v}$$

$$\dot{r} = -v, \quad \ddot{r} = 0$$

	$\vec{F} =$	$m\vec{a}$	
$\hat{r}$:	$-T =$	$m(\ddot{r} - r\omega^2)$	(1)

becomes $\quad T = m r \omega^2 \quad$ (1a)

$\hat{\theta}$: $\quad 0 = m(2\dot{r}\omega + r\dot{\omega}) \quad$ (2)

We can solve eq (2)

$$2\frac{dr}{dt}\omega = -r\frac{d\omega}{dt}$$

$$\frac{2\,dr}{r} = -\frac{d\omega}{\omega}$$

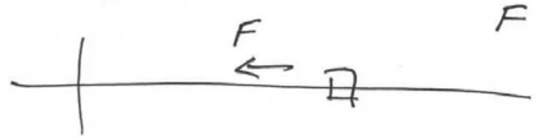
integrate: $\displaystyle\int_{r_0}^{r(t)} \frac{2\,dr}{r} = -\int_{\omega_0 =}^{\omega(t)} \frac{d\omega}{\omega}$

$$2\ln\left(\frac{r(t)}{r_0}\right) = -\ln\left(\frac{\omega(t)}{\omega_0}\right) \Rightarrow$$

$$\left(\frac{r(t)}{r_0}\right)^2 = \frac{\omega_0}{\omega(t)} \Rightarrow \boxed{\omega(t) = \frac{r_0^2\,\omega_0}{(r(t))^2} = \frac{r_0^2\,\omega_0}{(r_0 - vt)^2}}$$

Problem 36: A particle of mass m moving along a straight line is acting on by a velocity dependent retarding force (one always directed against the motion) of magnitude $F = be^{\alpha v}$, where b and α are constants and v is the magnitude of the velocity (speed). At $t = 0$, the particle is moving with speed v_0. Find the velocity as a function of time t, $v(t)$.

Chapter 2: Problem 36



$$F = b e^{\alpha v}$$

$\to \hat{i}$

	$\vec{F}$	=	$m\vec{a}$
$\hat{i}$:	$-be^{\alpha v}$	=	$m\dfrac{dv}{dt}$

$$-\frac{b}{m}\,dt = \frac{dv}{e^{\alpha v}} = e^{-\alpha v}\,dv$$

integrate
$$\int_{t=0}^{t} -\frac{b}{m}\,dt = \int_{v_0}^{v(t)} e^{-\alpha v}\,dv$$

$$-\frac{b}{m}t = \left.\frac{e^{-\alpha v}}{-\alpha}\right|_{v_0}^{v(t)} = -\frac{1}{\alpha}\left(e^{-\alpha(v(t))} - e^{-\alpha v_0}\right)$$

$$\frac{\alpha b}{m}t + e^{-\alpha v_0} = e^{-\alpha v}$$

take natural ln of both sides

$$\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right) = -\alpha v$$

$$v = -\frac{1}{\alpha}\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)$$

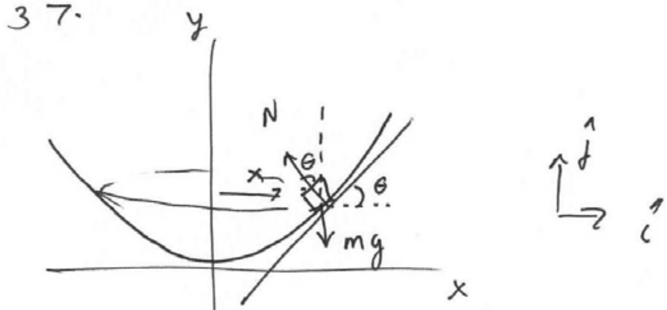
$$v(t) = \frac{1}{\alpha}\ln\left(\frac{1}{\frac{\alpha b}{m}t + e^{-\alpha v_0}}\right)$$

Problem 37: The Eureka Hovercraft Corporation wanted to hold hovercraft races as an advertising stunt. The hovercraft supports itself by blowing air downward, and has a big fixed propeller on the top deck for forward propulsion. Unfortunately it has no steering equipment, so that pilots found that making high speed turns was very difficult. The company decided to overcome this problem by designing a bowl shaped track in which the hovercraft, once up tp speed, would coast along in a circular path with no need to steer. They hired an engineer to design and build the track, and when he finished, he hastily left the country. When the company held their first race, they found to their dismay that the craft took exactly the same time T to circle the track, no matter what speed.

a) Explain why the time is independent of the speed.

b) Find the equation for the height of the cross section of the bowl in terms of the period to complete one revolution T.

Chapter 2 . 37.



$$\vec{F} = m\vec{a}$$

$\hat{i}$: $-N\sin\theta = -m x \omega^2 \Rightarrow N\sin\theta = m x \omega^2$

$\hat{j}$: $N\cos\theta - mg = 0 \qquad N\cos\theta = mg$

divide these eq's. $\tan\theta = \dfrac{x\omega^2}{g}$

Constraint condition: $\dfrac{dy}{dx} = \tan\theta$

$$\frac{dy}{dx} = \frac{x\,\omega^2}{g}$$

integrate: $\displaystyle\int_0^y dy = \int_0^x \frac{x\,\omega^2}{g}\,dx = \frac{\omega^2}{g}\int_0^x x\,dx$

The condition that the craft has the same T for any speed is the statement that ω = constant hence we pull it out of the integral yielding

$$\boxed{y = \frac{1}{2}\frac{\omega^2}{g}x^2} = \boxed{\frac{1}{2}\frac{4\pi^2}{T^2 g}x^2}$$