MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012 **Fall Term 2009**

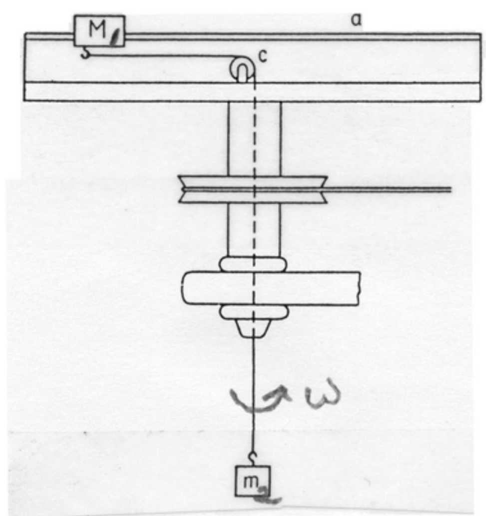
Practice Exam One Solutions

Problem 1 A mass m_1 is constrained to slide along a rod mounted perpendicularly to the axis of rotation of the device. One end of a massless string of length s is attached to the mass m_1, passes over a massless pulley, and is attached at the other end to a suspended mass m_2. The mass m_2 hangs along the central axis of the device. The whole apparatus is rotated with angular velocity ω. Assume the coefficient of static friction is μ.

a) What angular velocity can the device spin such that there is no static friction?

b) What is the minimum velocity that the device can spin so that the mass m_1 does not move radially inward?

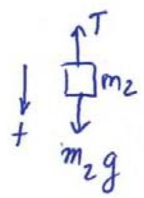
c) What is the maximum velocity that the device can spin so that the mass m_1 does not move radially outward?



A stone of mass m is attached to a string and is being whirled around a vertical circle of radius r. Assume that during this motion the magnitude of the velocity, v, of the stone is constant. If at the top of the circle, the tension in the string is zero, what is the tension in the string at the bottom of the circle?

$\rightarrow$ + inward

a. $\boxed{m_1} \rightarrow T$



$$F_{inward} = m_1 r \omega^2$$

$$T = m_1 r \omega^2$$

$$F_{down} = m a_{down}$$

$$m_2 g - T = 0$$

$$T = m_2 g$$

$$m_2 g = m_1 r \omega^2$$

$$\omega = \sqrt{\frac{m_2}{m_1}\frac{g}{r}}$$

b. $(f_s)_{max} \leftarrow \underset{m_1}{\boxed{\ }} \rightarrow T$

$$T - (f_s)_{max} = m_1 r \omega_{min}^2 \qquad (f_s)_{max} = \mu N = \mu m_1 g$$

$$m_2 g - \mu m_1 g = m_1 r \omega_{min}^2 \Rightarrow \omega_{min} = \sqrt{\frac{m_2}{m_1}\frac{g}{r} - \frac{\mu g}{r}}$$

c. just change friction direction

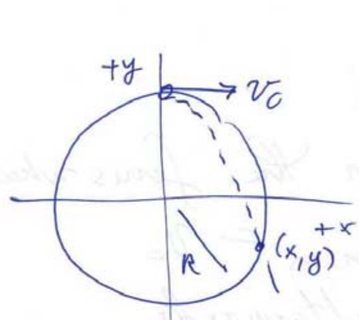
$$\omega_{max} = \sqrt{\frac{m_2}{m_1}\frac{g}{r} + \frac{\mu g}{r}}$$

Problem 2: A person is riding on a circular ferris wheel of radius R which is rotating with a constant angular velocity ω. The person is carrying a water balloon. At the very top of the ride the person let go of the balloon.

a) Express a constraint condition which describes where the balloon hits the wheel in terms of your choice of coordinate system.

b) How long will it take the water balloon to hit the ferris wheel? Leave your answer as a function of R, ω, and g.

c) From your solution in part a) determine the condition for the balloon to hit the wheel.



$$y = R - \tfrac{1}{2} g t^2 \quad (1)$$
$$x = v_0 t \quad (2)$$
$$x^2 + y^2 = R^2 \quad (3)$$
$$v_0 = R\omega$$

(3) yields $\quad v_0^2 t^2 + (R - \tfrac{1}{2} g t^2)^2 = R^2$

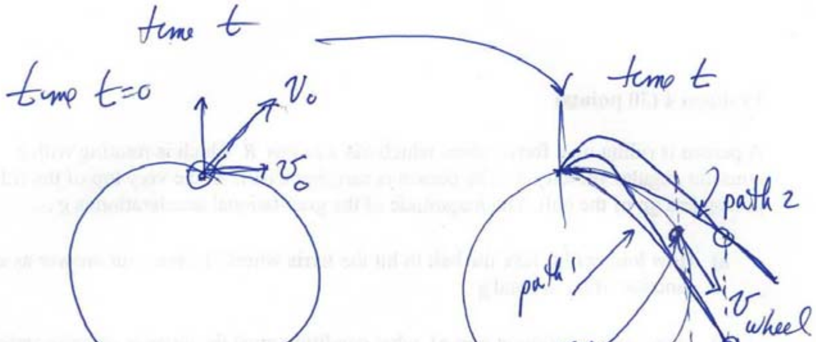
or $\quad v_0^2 t^2 + R^2 - R g t^2 + \tfrac{1}{4} g^2 t^4 = R^2$

$$\Rightarrow \quad \tfrac{1}{4} g^2 t^4 + (v_0^2 - Rg) t^2 = 0$$

$$t^2 = \left(\frac{4(Rg - v_0^2)}{g^2}\right)^{1/2} = \frac{2}{g}(Rg - R^2\omega^2)^{1/2}$$

$$\Rightarrow \quad t = \frac{2}{g}(Rg - v_0^2)^{1/2}, \quad Rg > v_0^2$$

$$\Rightarrow \quad \omega^2 < \frac{g}{R} \qquad R^2\omega^2 = v_0^2 < Rg$$



whether the ball follows path 2 or path 1, the horizontal distance travelled is $x_{ball} = v_0 t$

whereas the person on the ferris wheel
at $t=0$ as $v_{x,\text{person}} = v_0$
but for all times afterwards
$t > 0$ $\quad v_{x,\text{person}}(t) < v_0$
So the horizontal distance traveled by
the person is less than the ball
the ball and person can never
intersect!

Problem 3: (***projectile motion, softball)***

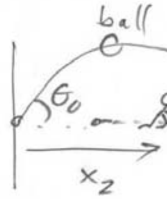
A softball is hit over a third baseman's head. The third baseman, as soon as it is hit, turns around and runs straight back with a constant acceleration of a for a time interval Δt and catches the ball at the same height it left the bat. The third baseman was initially d from home plate.

a) Describe the strategy you have chosen for solving this problem. You may want to consider the following issues. What type of coordinate system will you choose? Where is a good place to choose your origin?

b) What was the initial speed and angle of the softball when it left the bat?

a) Use projectile motion equations for ball, and one dimensional constant acceleration equations for fielder

choose origin where ball is hit



$x_{1,0} = y_{1,0} = 0 \quad x_{2,0} = d_2$

$\vec{v}_0 = v_0 \cos\theta_0 \hat{i} + v_0 \sin\theta_0 \hat{j}$

$x_1 = v_0 \cos\theta_0 t \qquad x_2 = d + \frac{1}{2} a t^2$

$y_1 = v_0 \sin\theta_0 t - \frac{1}{2} g t^2$

final conditions, $y_{2,f} = 0 \quad t_f = \Delta t \quad x_{2,f} = x_{1,f}$

$$v_0 \cos\theta_0 \Delta t = d + \frac{1}{2} a (\Delta t)^2 \Rightarrow v_0 \cos\theta_0 = \frac{d}{\Delta t} + \frac{1}{2} a \Delta t \quad (1)$$

$$0 = v_0 \sin\theta_0 t_f - \frac{g}{2} t_f^2 \Rightarrow v_0 \sin\theta_0 = \frac{1}{2} g \Delta t \quad (2)$$

divide eq (1) by eq (2)

$$\cot\theta_0 = \frac{\frac{d}{\Delta t} + \frac{1}{2} a \Delta t}{\frac{1}{2} g \Delta t} = \frac{2d}{g \Delta t^2} + \frac{a}{g}$$

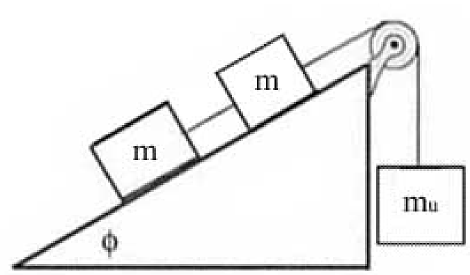
$$\theta_0 = \cot^{-1}\left(\frac{2d}{g\Delta t^2} + \frac{a}{g}\right)$$

square (1) and (2) and add

$$v_0^2 = \left(\left(\frac{d}{\Delta t} + \frac{1}{2} a \Delta t\right)^2 + \left(\frac{1}{2} g \Delta t\right)^2\right)^{1/2}$$

$$v_0 = \left(\frac{d^2}{(\Delta t)^2} + d a + \frac{1}{4}(a^2 + g^2)(\Delta t)^2\right)^{1/2}$$

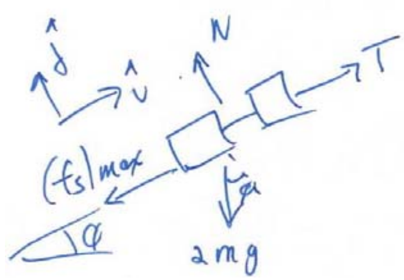
Problem 4: Two identical blocks of given mass m, are attached together by a massless string and constrained to move along a plane that is inclined at a given angle ϕ to the horizontal. The upper block is connected via a second massless inextensible string that passes over a massless pulley, to a third block of unknown mass m_u. Assume the coefficient of static friction between the block and the inclined plane is given as μ_s and the coefficient of kinetic friction is given as μ_k. Assume the gravitational constant is given as g.



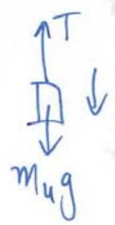
a) What is the mass of the third block such that the blocks **just start slipping up** the inclined plane? Include in your answer a briefly explanation of how you intend to model this problem?

b) Now suppose the string between the two identical blocks is cut. How does your model change? What are the accelerations of the lower block and the upper block on the inclined plane?

model: masses on incline plane just slipping, $a = 0$, $f_s = (f_s)_{max} = \mu_s N$. System: a) two blocks b) block with unknown mass



$\vec{F}$ =	$m\vec{a}$	
$T - (f_s)_{max} - 2mg\sin\phi =$	0	(1)
$N - 2mg\cos\phi =$	0	(2)
$(f_s)_{max} = \mu_s N = 2\mu_s mg\cos\phi$		(3)



$$\vec{F} = m a$$

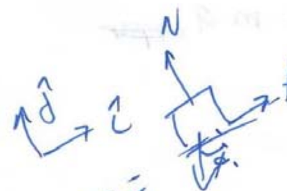
$$m_u g - T = 0 \quad \Rightarrow \quad T = m_u g \quad (4)$$

eq (1) with eq (2), (3), and (4) becomes

$$m_u g - \mu_s 2 m g \cos\phi - 2 m g \sin\phi = 0$$

$$\Rightarrow \quad m_u = 2m(\mu_s \cos\phi + \sin\phi)$$

part 5): model: lower block may slide: three individual objects



$$\vec{F} = m\vec{a}$$

$$f - mg\sin\phi = m a_{lower} \quad (5)$$

$$N - mg\cos\phi = 0 \quad (6)$$

just slipping $f = (f_s)_{max} = \mu_s N = \mu_s mg\cos\phi$

$a_{lower} = 0$

$$\Rightarrow \quad \mu_s mg\cos\phi = mg\sin\phi$$

$$\Rightarrow \quad \mu_s = \tan\phi$$

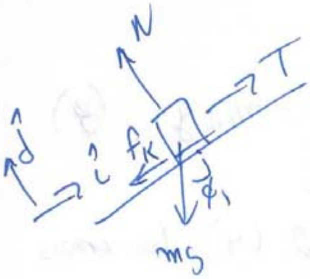
if $\phi > \tan^{-1}(\mu_s)$ slips. Then $f = f_k = \mu_k N = \mu_k mg\cos\phi$

$$\mu_k mg\cos\phi - mg\sin\phi = m a_{lower}$$

$$\Rightarrow \quad a_{lower} = g(\mu_k\cos\phi - \sin\phi) < 0$$

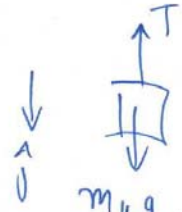
accelerates down inclined plane.

upper block will slip up



$\vec{F}$	$= m\vec{a}$
$T - f_K - mg\sin\phi$	$= m\, a_{upper}$
$N - mg\cos\phi =$	0

$f_K = \mu_k N = \mu_k mg\cos\phi$



$\vec{F}$	$= m_u a_u$
$m_u g - T$	$= m_u a_u$

constraint $a_u = a_{upper} \equiv a$

$$T - \mu_k mg\cos\phi - mg\sin\phi = ma$$

$$m_u \cdot g - T = m_u a$$

add equations

$$m_u g - mg(\mu_k\cos\phi + \sin\phi) = (m + m_u)a$$

$$\Rightarrow a = \frac{m_u g - mg(\mu_k\cos\phi + \sin\phi)}{m + m_u}$$

Problem 5: Consider a planet of mass m_1 in orbit around a extremely massive star of mass m_2. The period of the orbit is T. Assume that there is a uniform distribution of dust, of density ρ throughout the space surrounding the star and extending well beyond the planet with $\frac{4\pi^2}{T^2} > \frac{4}{3}G\pi\rho$. The gravitational effect of this dust cloud is to add an attractive centripetal force, $\vec{\mathbf{F}}_{dust}$, on the planet in addition to the gravitational attraction between the star and the planet. You may neglect any drag forces due to collisions with the dust particles.

a) Briefly describe how you intend to model the motion of the object? What directions are you choosing for analyzing the components of your forces and state why?

b) Find an expression for the radius of the orbit of the planet.

c) If there were no dust present, would the radius of the circular orbit be greater, equal, or less than your result from part a). Briefly explain your reasoning.

Several billion years later, the dust cloud has vanished, but now assume that there is a repulsive force acting on the planet that is given by

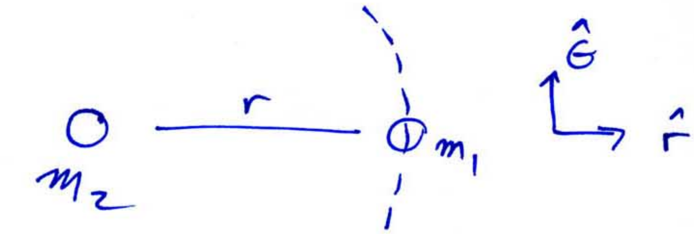
$$\vec{\mathbf{F}}_{repulsive} = \frac{k}{r^3}\hat{\mathbf{r}},$$

in addition to the gravitational force between the star and the planet The constant $k > 0$ and satisfies $G > \frac{2v}{m_2}\sqrt{k/m_1}$.

d) Show that there are two possible circular orbits for the planet that have the same velocity v. Find the radii of these orbits.

Solution: star with uniform dust cloud

circular orbit



dust cloud of density ρ, $m_{dust} = \rho \frac{4}{3}\pi r^3$

$$\vec{F}_{dust} = -\frac{G m_1 m_{dust}}{r^2}\hat{r} = -\frac{G m_1 \rho \frac{4}{3}\pi r^3}{r^2}\hat{r}$$

$$= \frac{-4G}{3} m_1 \rho \pi r \hat{r}$$

$$\vec{F}_{grav} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

$\vec{F}$	$= m\vec{a}$
$-\frac{G m_1 m_2}{r^2} - G m_1 \rho \frac{4}{3}\pi r$	$= -m_1 r \left(\frac{2\pi}{T}\right)^2$

$$\Rightarrow \quad +\frac{G m_2}{r^2} + G\rho\frac{4}{3}\pi r = \frac{r 4\pi^2}{T^2}$$

$$\frac{G m_2}{r^2} = r\left(\frac{4\pi^2}{T^2} - G\rho\frac{4}{3}\pi\right)$$

$$\left(\frac{G m_2}{\left(\frac{4\pi^2}{T^2} - G\rho\frac{4}{3}\pi\right)}\right)^{1/3} = r_{dust}$$

note: $\frac{4\pi^2}{T^2} > G\rho\frac{4}{3}\pi$

with no dust present $\quad r_{\substack{no\\dust}} = \left(\frac{G m_2}{4\pi^2/T^2}\right)^{1/3}$

$$r_{dust} > r_{no\ dust}$$

with repulsive force and gravitation:

$$\vec{F}_{star,grav} \longleftarrow \underset{m_1}{\bigcirc} \longrightarrow F_{repulsive} \qquad \longrightarrow \hat{r}$$

$$-\frac{G m_1 m_2}{r^2} + \frac{k}{r^3} = -\frac{m_1 v^2}{r}$$

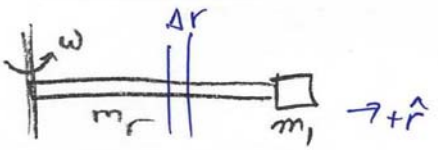
$$-G m_1 m_2 r + k = -m_1 v^2 r^2$$

$$r^2 - \frac{G m_2}{v^2} r + \frac{k}{m_1 v^2} = 0$$

$$r = \left(\frac{G m_2}{v^2} \pm \left(\frac{G^2 m_2^2}{v^4} - \frac{4k}{m_1 v^2}\right)^{1/2}\right) \Big/ 2.$$

note: $\frac{G m_2^2}{v^4} > \frac{4k}{m_1 v^2}$

Problem 6: A uniform rope of mass m and length L is attached to shaft that is rotating at constant angular velocity ω. A second object of mass m_1 is attached to the end of the rope. Find the tension in the rope as a function of distance from the shaft. You may ignore the effect of gravitation.



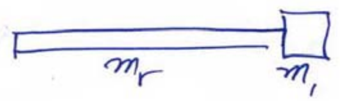
$T(r) \leftarrow \square \rightarrow T(r+\Delta r) \quad \rightarrow +\hat{r}$

F_r	$\Delta m \, a_r$
$T(r+\Delta r) - T(r)$	$= -\Delta m \, r \, \omega^2$

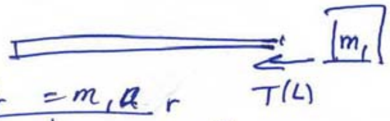
$\Delta m = \lambda \Delta r, \quad \lambda = \frac{m_r}{L}$

$\Delta T = -\lambda \Delta r \, r \omega^2 \qquad \lim_{\Delta r \to 0} \frac{\Delta T}{\Delta r} = -\lambda r \omega^2 \qquad \frac{dT}{dr} = -\lambda r \omega^2$

$$\int_{T(L)}^{T(r)} dT = \int_{r=L}^{r} -\lambda r \omega^2 \, dr \Rightarrow T(r) - T(L) = -\lambda \frac{\omega^2 r^2}{2} + \frac{\lambda \omega^2 L^2}{2}$$



treat as single m^{total} located at center of mass



F_r	$= m_1 a_r$
$-T(L) =$	$-m_1 L \omega^2$

$\Rightarrow T(L) = m_1 L \omega^2$

$$T(r) = T(L) - \frac{m_r}{L}\frac{\omega^2 r^2}{2} + \frac{m_r}{L}\frac{\omega^2}{2} L^2$$

$$= m_1 L \omega^2 - \frac{m_r}{L}\frac{\omega^2 r^2}{2} + \frac{m_r \omega^2}{2} L$$

$$T(r) = \left(m_1 + \frac{m_r}{2}\right) L \omega^2 - \frac{1}{2}\frac{m_r}{L} \omega^2 r^2$$

note $T(0) = \left(m_1 + \frac{m_r}{2}\right) L \, \omega^2 \qquad T(0) = m^{total} R_{cm} \, \omega^2$

since $R_{cm} = \dfrac{\frac{m_r}{2} L + m_1 L}{m_1 + m_2}$