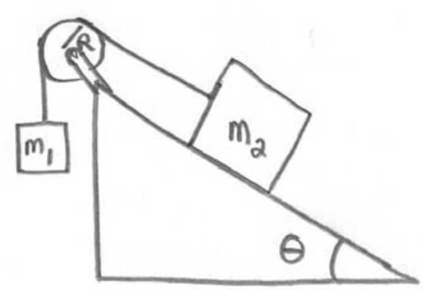
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.012 **Fall Term 2009**

Exam 3 Practice Problems: Solutions

Problem 1 A wheel in the shape of a uniform disk of radius R and mass m_p is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass $I_\mathrm{cm} = (1/2)m_\mathrm{p}R^2$. A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass m_2 that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass m_1 that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle θ. Once the objects are released from rest, the cord moves without slipping around the disk. Find the accelerations of each object, and the tensions in the string on either side of the pulley.



Solution:

For this problem, the wheel is not massless, and so as the blocks move, the wheel moves, and as the blocks accelerate the wheel has an angular acceleration and hence a net torque. The wheel axle is given as being frictionless, so the torque on the wheel is due to the difference between the tensions in the two parts of the rope. Denote these tensions as T_1 and T_2.

The equation of motion for the suspended object is simple,

$$m_1 g - T_1 = m_1 a_1 . \tag{1.1}$$

The equation of motion for the object on the incline involves choosing a coordinate system with one coordinate directed up the incline. The net force in this direction is the tension T_2 and the component of gravity $-m_2 g \sin\theta$, leading to

$$T_2 - m_2 g \sin\theta = m_2 a_2 . \tag{1.2}$$

The relation between the torque on the wheel and its angular acceleration α is

$$\left(T_1 - T_2\right) R = I\alpha . \tag{1.3}$$

At this point, note that Equation (1.1) assumes a positive acceleration downward, Equation (1.2) assumes a positive direction up the incline and Equation (1.3) assumes a positive angular acceleration in the counterclockwise direction in the figure. These direction choices are consistent.

Equations (1.1), (1.2) and (1.3) are five linear equations in five unknowns, and of course we need to impose further conditions. The simplest is $a_1 = a_2 = a$ with the chosen sign convention, indicating that the cord doesn't stretch. If the cord doesn't slip on the wheel, the angular acceleration is $\alpha = a / R$. Equations (1.1), (1.2) and (1.3) then become

$$\begin{aligned} m_1 g - T_1 &= m_1 a \\ T_2 - m_2 g \sin\theta &= m_2 a \\ T_1 - T_2 &= Ia / R^2. \end{aligned} \tag{1.4}$$

In this form, the two tensions may be eliminated by adding the three expressions in (1.4), with the result

$$\begin{aligned} m_1 g - m_2 g \sin\theta = a\left(m_1 + m_2 + I / R^2\right) &= a\left(m_1 + m_2 + m_\mathrm{p} / 2\right) \\ a = g\,\frac{m_1 - m_2 \sin\theta}{m_1 + m_2 + m_\mathrm{p} / 2}. \end{aligned} \tag{1.5}$$

Note that this result does not depend on the radius R. Also note that if $m_2 \sin\theta > m_1$, $a < 0$, and the object on the incline will accelerated down the incline, and the suspended block will accelerate upwards. For any combination of the parameters, $|a| < g$.

The tensions are found by substituting the acceleration found in (1.5) into the respective equations (1.4). There is some algebra involved, but not much;

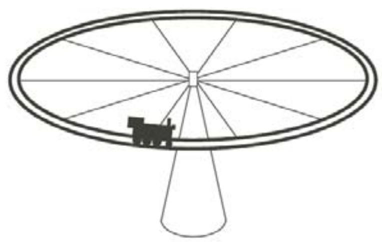
$$\begin{aligned} T_1 &= m_1\left(g - a\right) \\ &= m_1 g \left[1 - \frac{m_1 - m_2 \sin\theta}{m_1 + m_2 + m_\mathrm{p} / 2}\right] \\ &= m_1 g\,\frac{m_2\left(1 + \sin\theta\right) + m_\mathrm{p} / 2}{m_1 + m_2 + m_\mathrm{p} / 2}, \end{aligned} \tag{1.6}$$

$$
\begin{aligned}
T_2 &= m_2 a + m_2 g \sin\theta \\
&= m_2 g \left[\frac{m_1 - m_2 \sin\theta}{m_1 + m_2 + m_{\mathrm{p}}/2} + \sin\theta \right] \\
&= m_2 g \,\frac{m_1\left(1+\sin\theta\right) + m_{\mathrm{p}} \sin\theta / 2}{m_1 + m_2 + m_{\mathrm{p}}/2}.
\end{aligned} \tag{1.7}
$$

Any further simplification might not be useful. However, note that if $m_2 \sin\theta > m_1$, $T_1 > m_1 g$, consistent with the suspended block accelerating upward.

Of course, the relations in (1.4) could be manipulated algebraically to eliminate one tension and the acceleration a in terms of the other tension, but that alternate solution won't be considered here.

Problem 2: Toy Locomotive A toy locomotive of mass m_L runs on a horizontal circular track of radius R and total mass m_T. The track forms the rim of an otherwise massless wheel which is free to rotate without friction about a vertical axis. The locomotive is started from rest and accelerated without slipping to a final speed of v relative to the track. What is the locomotive's final speed, v_f, relative to the floor?



Solution: We begin by choosing our system to consist of the locomotive and the track. Because there are no external torques about the central axis, the angular momentum of the system consisting of the engine and the track is constant about that axis. The initial angular momentum of the system is zero because the locomotive and the track are at rest.

Let $\omega_{T,f}$ denote the final angular speed of the track. In the figure above the locomotive is rotating counterclockwise as seen from above so the track must spin in the clockwise direction. If we choose the positive z-direction to point upward then the final angular momentum of the track is given by

$$\vec{L}_{T,f} = -I_{T,z}\omega_{T,f}\hat{k} = -m_T R^2 \omega_{T,f}\hat{k} \tag{8}$$

where we have assumed that the moment of inertia of the track about an axis passing perpendicularly through the center of the circle forms by the track is $I_{T,z} = m_T R^2$.

The locomotive is moving tangentially with respect to the ground so we can choose polar coordinates and then the final velocity of the locomotive with respect to the ground is

$$\vec{v}_{L,f} = v_f \hat{\theta}. \tag{9}$$

A point on the rim of the track has final velocity

$$\vec{v}_{T,f} = -R\omega_{T,f}\hat{\theta}. \tag{10}$$

Therefore the relative velocity $\vec{v}_{rel} = v\,\hat{\theta}$ of the locomotive to the track is given by the difference of the velocity of the locomotive and the appoint on the rim of the track,

$$\vec{v}_{rel} = \vec{v}_{L,f} - \vec{v}_{T,f} = v_f\hat{\theta} - (-R\omega_{T,f}\hat{\theta}) = (v_f + R\omega_{T,f})\hat{\theta} = v\hat{\theta}, \tag{11}$$

therefore

$$v = v_f + R\omega_{T,f}\,. \tag{12}$$

We can solve the above equation for the final angular speed $\omega_{T,f}$ in terms of the given relative speed and the final speed of the locomotive with respect to the ground yielding

$$\omega_{T,f} = \frac{v - v_f}{R}\,. \tag{13}$$

The final angular momentum of the locomotive with respect to the center of the circle formed by the track when it is moving with speed v_f, relative to the floor is

$$\vec{L}_{L,f} = m_L R v_f \hat{k}\,. \tag{14}$$

Because the angular momentum of the system is constant we have that

$$\vec{0} = \vec{L}_{T,f} + \vec{L}_{L,f} = (-m_T R^2 \omega_{T,f} + m_L R v_f)\hat{k}\,. \tag{15}$$

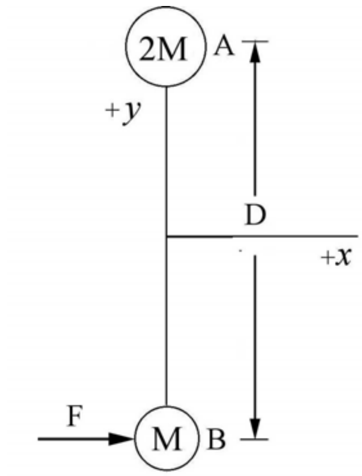
Now substitute Eq. (13) into the z-component of the above equation yielding

$$0 = -m_T R(v - v_f) + m_L R v_f\,. \tag{16}$$

We now solve the above equation to find the final speed of the locomotive relative to the floor

$$v_f = \frac{m_T}{m_T + m_L} v\,. \tag{17}$$

Problem 3 Two point-like objects are located at the points A, and B, of respective masses $M_A = 2M$, and $M_B = M$, as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length D, forming a rigid body. A force of magnitude $F = |\vec{\mathbf{F}}|$ along the x direction is applied to the object at A at $t = 0$ for a short time interval Δt. Neglect gravity. Give all your answers in terms of M, D, F and Δt as needed.



a) Describe qualitatively in words how the system moves after the force is applied: direction, translation and rotation.

b) How far is the center of mass of the system from the object at point B?

c) What is the direction and magnitude of the linear velocity of the center-of-mass after the collision?

d) What is the magnitude of the angular velocity of the system after the collision?

e) Is it possible to apply another force of magnitude *F* along the positive *x*-direction to prevent the system from rotating? Does it matter where the force is applied?

f) Is it possible to apply another force of magnitude *F* in some direction to prevent the center of mass from translating? Does it matter where the force is applied?

Solutions:

a) The center of mass will move to the right in the figure, and the two masses will rotate about the center of mass, counterclockwise in the figure.

b) The distance from the object originally at point B is $M_A D/(M_A + M_B) = (2/3)D$, at a position $y_{cm} = D/3$ in the figure.

c) The magnitude of the linear momentum will be the magnitude $F\Delta t$ and, as in part (a), the direction will be the right. The magnitude of the velocity is then $(F\Delta t)/(3\mathrm{M})$

d) The quickest way to find the angular velocity is to consider the collision in the center of mass frame. In this frame the angular impulse, and hence the magnitude of the angular momentum, is $(F\Delta t)(2/3)\mathrm{D}$. The momentum of inertia about the center of mass is

$$I_{\mathrm{cm}} = (2\mathrm{M})(\mathrm{D}/3)^2 + (\mathrm{M})(2\mathrm{D}/3)^2 = (2/3)\mathrm{MD}^2$$

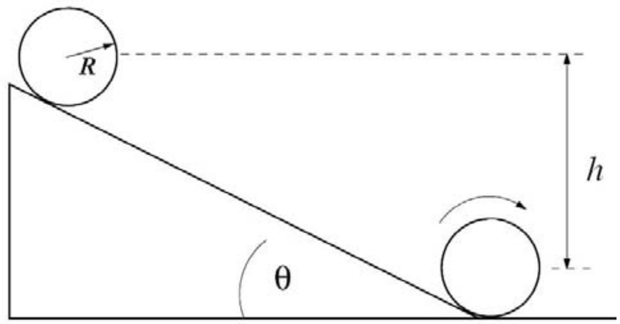
and the magnitude ω_{f} of the final angular momentum is

$$\omega_{\mathrm{f}} = \frac{(F\Delta t)(2/3)\mathrm{D}}{(2/3)\mathrm{MD}^2} = \frac{F\Delta t}{\mathrm{MD}}.$$

e) No. The force additional force would have to be applied at a distance $2\mathrm{D}/3$ above the center of mass, which is not a physical point of the system.

f) An additional force of the same magnitude, in the negative *x* direction, would result in no net force and hence no acceleration of the center of mass.

Problem 4 A hollow cylinder of outer radius R and mass M with moment of inertia about the center of mass $I_{\mathrm{cm}} = MR^2$ starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. The goal of this problem is to find an expression for the smallest possible value of μ_s such that the cylinder rolls without slipping down the incline plane.



a) Draw a free body force diagram showing all the forces acting on the cylinder.

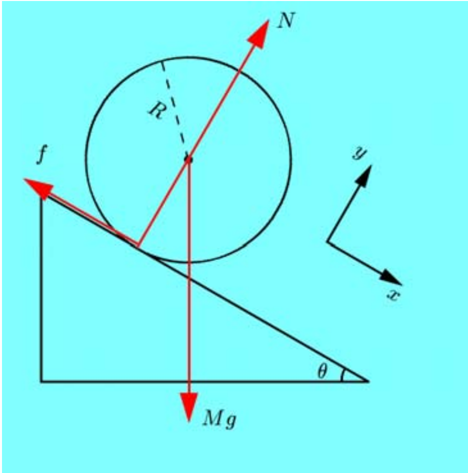
b) Find an expression for both the angular and linear acceleration of the cylinder in terms of M, R, g, θ and h as needed.

c) What is the minimum value for the coefficient of static friction μ_s such that the cylinder rolls without slipping down the incline plane? Express your answer in terms of M, R, g, θ and h as needed.

d) What is the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline? Express your answer in terms of M, R, g, θ and h as needed.

Solutions:

a) The forces are the weight, the normal force and the contact force.



b) With the coordinates system shown, Newton's Second Law, applied in the x- and y-directions in turn, yields

$$
\begin{aligned}
Mg\sin\theta - f &= Ma \\
N - Mg\cos\theta &= 0.
\end{aligned}
$$

The equations above represent two equations in three unknowns, and so we need one more relation; this will come from torque considerations.

Of course, any point could be used for the origin in computing torques, but the "obvious" choice of the center of the cylinder turns out to make things easiest (judgment call, of course). Then, the only force exerting a torque is the friction force, and so we have

$$f\,R = I_{\mathrm{cm}}\alpha = M\,R^2\left(a/R\right) = M\,R\,a$$

where $I_{\mathrm{cm}} = M\,R^2$ and the kinematic constraint for the no-slipping condition $\alpha = a/R$ have been used. This leads to $f = M\,a$, and inserting this into the force equation gives the two relations

$$
\begin{aligned}
f &= \frac{1}{2}Mg\sin\theta \\
a &= \frac{1}{2}g\sin\theta.
\end{aligned}
$$

c) For rolling without slipping, we need $f < \mu_s N$, so we need, using the second force equation above,

$$\mu_s > \frac{1}{2}\tan\theta\,.$$

d) The cylinder rolls a distance $L = h/\sin\theta$ down the incline, and the speed v_f at the bottom is related to the acceleration found in part (b) by

$$\begin{aligned} v_f^2 &= 2aL = 2\left(\frac{1}{2}g\sin\theta\right)\left(h/\sin\theta\right) \\ &= gh. \end{aligned}$$

This result can and should be checked by energy conservation (for rolling without slipping, the friction force does no mechanical work). For the given moment of inertia, the final kinetic energy is

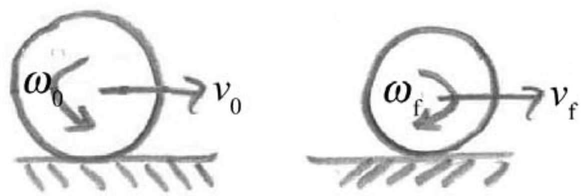
$$\begin{aligned} K_f &= \frac{1}{2}M\,v_f^2 + \frac{1}{2}I_{cm}\omega_f^2 \\ &= \frac{1}{2}M\,v_f^2 + \frac{1}{2}MR^2\left(v_f/R\right)^2 \\ &= M\,v_f^2, \end{aligned}$$

and setting the final kinetic energy equal to the loss of gravitational potential energy leads to the same result for the final speed.

Problem 5 A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0 and backspin with angular speed ω_0, such that $v_0 > R\omega_0$. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = (2/5)mR^2$. Your goal is to determine the speed v_{f} of the bowling ball when it just starts to roll without slipping.

a) Write up your plan for solving this problem. You may find some of the following concepts useful: angular impulse is equal to the change in angular momentum; linear impulse is equal to the change in momentum; Newton's Second Law; torque about the center-of-mass is proportional to the angular acceleration; if the torque about a point is zero, then the angular momentum about that point is constant; etc.

b) What is the speed v_{f} of the bowling ball when it just starts to roll without slipping?



Solution:

a) The easiest approach to solve this problem is to find a fixed point about which the torque is zero and then use the fact that angular momentum is constant about that point. So, if we take the point for determining torques and angular momenta about a point where the friction force exerts no torque, we shouldn't need to know about the nature of the friction force. Accordingly, choose the point to be the original point of contact of the ball with the lane surface. Subsequently, even though the ball has moved, friction will still exert no torque.

b) With respect to the contact point on the ground, the initial and final angular momenta are both the sum of two terms, one representing the motion of the center of mass and the other the rotation ("spin") of the ball;

$$\begin{aligned} L_{\text{initial}} &= mv_0 R - I_{\text{cm}}\omega_0 \\ L_{\text{final}} &= mv_{\text{f}} R + I_{\text{cm}}\omega_{\text{f}} . \end{aligned} \tag{2.1}$$

The problem is now one of basic algebra. For rolling without slipping, $\omega_{\text{f}} = v_{\text{f}} / R$, and the given $I_{\text{cm}} = (2/5)mR^2$ gives

$$v_{\text{f}} = \left(5v_0 - 2\omega_0 R\right)/ 7 . \tag{2.2}$$

It's important to note the signs in the expressions in (2.1). We are given (and the figure certainly implies) that the scalar quantity ω_0, representing backspin, is positive, and so with positive direction for angular momenta being clockwise, the ω_0 term in the initial angular momentum is negative.

This problem may of course be done by considering torques and angular momenta about the center of the ball. The change in linear momentum (watch the signs again) is the impulse

$$\Delta p = m\left(v_{\mathrm{f}} - v_0\right) = -\int f\, dt \tag{2.3}$$

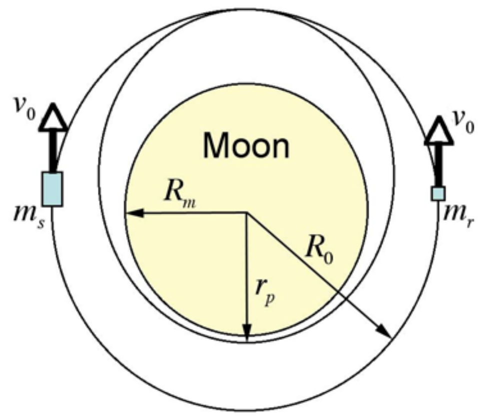
and the change in angular momentum is the angular impulse (the signs are still important)

$$\Delta L = I_{\mathrm{cm}}\left(\omega_f + \omega_0\right) = \int Rf\, dt\,. \tag{2.4}$$

Eliminating the linear impulse $-\int f\, dt$ between Equations (2.3) and (2.4), and using the given $I_{\mathrm{cm}} = (2/5)mR^2$ yields the same result as that in Equation (2.2).

Problem 6 (30 Points)

A lunar mapping satellite of mass m_s is in a circular orbit around the moon, and the orbit has radius $R_0 = 1.5\,R_m$ where R_m is the radius of the moon. A repair robot of mass $m_r < m_s$ is injected into that orbit, but due to a NASA sign error it orbits in the opposite direction. The two collide and stick together in a useless metal mass. The point of this problem is to find whether they create more junk orbiting the moon or crash into the lunar surface. The mass of the moon is denoted by m_m. The universal gravitational constant is denoted by G.



a. What is the initial orbital velocity of the mapping satellite, v_0? Express your answer in terms of R_0, m_m, and G.

b. What is the speed of the space junk (satellite and robot) immediately after the collision? Write it as $f\,v_0$, where you must determine the number f. Express your answer in terms of m_s and m_r.

c. After the collision, the orbit of the space junk has changed. Use conservation of energy and angular momentum to solve for f, R_0, and r_p.

Solution:

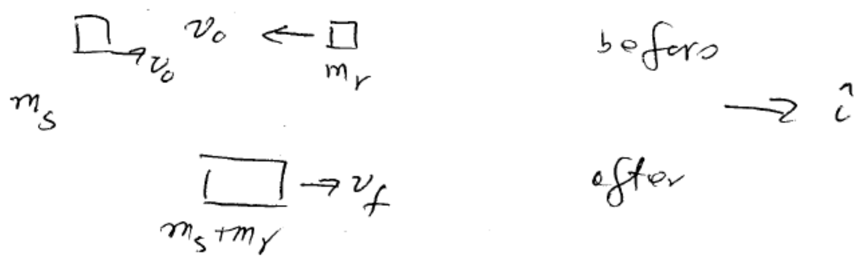
a) The speed of the mapping satellite undergoing uniform circular motion can be found from the force equation,

$$-\frac{Gm_s m_m}{R_0^{\;2}} = -\frac{m_s v_0^{\;2}}{R_0} \qquad \textbf{(5)}$$

So the speed is

$$v_0 = \sqrt{Gm_p / R_0} \qquad \textbf{(6)}$$

b) The momentum flow diagram for the collision is shown below.



Because there are no external forces, the momentum is constant and so

$$m_s v_0 - m_r v_0 = (m_s + m_r) v_f \, . \qquad \textbf{(7)}$$

Thus the speed after the collision is

$$v_f = \frac{m_s - m_r}{m_s + m_r} v_0 = f v_0 \, , \qquad \textbf{(8)}$$

where the ratio of speed after the collision to the speed before the collision is given by the number

$$f = \frac{v_f}{v_0} = \frac{m_s - m_r}{m_s + m_r} \, . \qquad \textbf{(9)}$$

c) After the collision, the energy equation is given by

$$\frac{1}{2}(m_s + m_r) v_f^{\,2} - \frac{G(m_s + m_r) m_p}{R_0} = \frac{1}{2}(m_s + m_r) v_p^{\,2} - \frac{G(m_s + m_r) m_p}{r_p} \, . \qquad \textbf{(10)}$$

Setting $v_f = f v_0$ and simplifying yields

$$\frac{1}{2} f^2 v_0^{\,2} - \frac{G m_p}{R_0} = \frac{1}{2} v_p^{\,2} - \frac{G m_p}{r_p} \, . \qquad \textbf{(11)}$$

The angular momentum equation is

$$(m_s + m_r) R_0 v_f = (m_s + m_r) r_p v_p \, . \qquad \textbf{(12)}$$

Again setting $v_f = fv_0$, Eq. (12) becomes

$$R_0 fv_0 = r_p v_p \tag{13}$$

Eq. (13) implies that $v_p = R_0 fv_0 / r_p$ which we can substitute into Eq. (11)yielding

$$\frac{1}{2} f^2 v_0^2 - \frac{Gm_p}{R_0} = \frac{1}{2}\left(\frac{R_0 fv_0}{r_p}\right)^2 - \frac{Gm_p}{r_p}. \tag{14}$$

Collecting terms yields

$$\frac{1}{2} f^2 v_0^2 \left(1 - \frac{R_0^2}{r_p^2}\right) = Gm_p \left(\frac{1}{R_0} - \frac{1}{r_p}\right). \tag{15}$$

If we assume that at the closest approach $r_p = R_m$, the space junk hits the moon. Then using the values $r_p = R_m$, $v_0 = \sqrt{Gm_p / R_0}$, and $R_0 = (3/2)R_m$, Eq. (15) becomes

$$f^2 \frac{Gm_p}{3R_M}\left(1 - \frac{9R_M^2}{4R_M^2}\right) = Gm_p \left(\frac{2}{3R_M} - \frac{1}{R_M}\right). \tag{16}$$

We can solve Eq. (16) for f:

$$f = \sqrt{4/5}. \tag{17}$$

Problem 7 A wrench of mass m is pivoted a distance l_{cm} from its center of mass and allowed to swing as a physical pendulum. The period for small-angle-oscillations is T.

a) What is the moment of inertia of the wrench about an axis through the pivot?

b) If the wrench is initially displaced by an angle θ_0 from its equilibrium position, what is the angular speed of the wrench as it passes through the equilibrium position?

Solutions:

a) The period of the physical pendulum for small angles is $T = 2\pi\sqrt{I_{\text{cm}} / ml_{\text{cm}} g}$; solving for the moment of inertia,

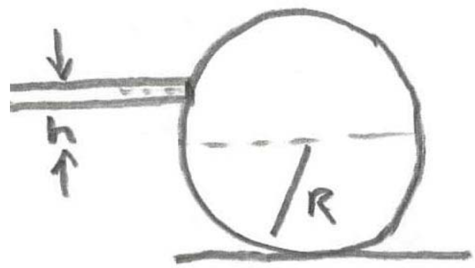
$$I_{\text{cm}} = \frac{T^2 ml_{\text{cm}} g}{4\pi^2}.$$

b) For this part, we are not given a small-angle approximation, and should not assume that θ_0 is a small angle. We will need to use energy considerations, and assume that the pendulum is released from rest.

Taking the zero of potential energy to be at the bottom of the pendulum's swing, the initial potential energy is $U_{\text{initial}} = mgl_{\text{cm}}(1-\cos\theta_0)$ and the final kinetic energy at the bottom of the swing is $U_{\text{final}} = 0$. The initial kinetic energy is $K_{\text{initial}} = 0$ and the final kinetic energy is related to the angular speed ω_{final} at the bottom of the swing by $K_{\text{final}} = (1/2) I_{\text{cm}} \omega_{\text{final}}^2$. Equating initial potential energy to final kinetic energy yields

$$\omega_{\text{final}}^2 = \frac{2mg(1-\cos\theta_0)}{I_{\text{cm}}} = \frac{8\pi^2}{T^2}(1-\cos\theta_0).$$

Problem 8 A spherical billiard ball of uniform density has mass m and radius R, and moment of inertia about the center of mass $I_{\text{cm}} = (2/5)mR^2$. The ball, initially at rest on a table, is given a sharp horizontal impulse by a cue stick that is held an unknown distance h above the centerline (see diagram below). The force applied by the cue to the ball is sufficiently large that you may ignore the friction between the ball and the table during the impulse (as any pool player knows). The ball leaves the cue with a given speed v_0 and an angular velocity ω_0. Because of its initial rotation, the ball eventually acquires a maximum speed of $(9/7)v_0$.



a) Using the fact that the angular impulse on the ball changes the angular momentum, and the linear impulse changes the linear momentum, find an expression for the angular velocity ω_0 of the ball just after the end of the impulse in terms of v_0, R, h and m.

c) Briefly explain why angular momentum is conserved about any point along the line of contact between the ball and the table *after* the impulse.

d) Use conservation of angular momentum about any point along the line of contact between the ball and the table, and your results from part a), to find the ratio h/R.

Solutions:

a) There are several ways to approach this problem. The method presented here avoids any calculation of the force or torque provided by friction, or the details of the force between the cue and the ball. This method will first consider the "collision" between the cue and the ball by taking the collision point as the origin for finding the angular momentum, as the force between the cue and the ball exerts no torque about this point, and we are given that the friction may be ignored during this interaction. After this collision, the angular momentum will be taken about the initial contact point between the ball and the felt. (It should be noted that this method anticipates the answer, which does not involve the coefficient of friction μ_k and also relies on having done Problem 2 above.) It will be helpful to infer, either from the figure and from the fact that $v_f > v_0$, that the ball is given overspin.

b) With respect to the point where the cue is in contact with the ball, note that the rotational angular momentum and the angular momentum due to the motion of the center of mass have different signs; the former is clockwise and the latter is counterclockwise. The sum of these contributions to the angular momenta must sum to zero, and hence have the same magnitude;

$$I_{\text{cm}}\omega_0 = mv_0 h\,. \tag{2.18}$$

While the ball is rolling and slipping, angular momentum is conserved about the contact between the ball and the felt. The initial and final angular momenta are

$$\begin{aligned} L_{\text{initial}} &= mv_0 R + I_{\text{cm}}\omega_0 \\ &= mv_0\left(R+h\right) \\ L_{\text{final}} &= mv_{\text{f}} R + I_{\text{cm}}\omega_{\text{f}} \\ &= mv_{\text{f}} R + \left(2/5\right)\left(mR^2\right)\left(v_{\text{f}}/R\right) \\ &= \left(7/5\right)mv_{\text{f}} R \\ &= \left(9/5\right)mv_0 R, \end{aligned} \tag{2.19}$$

where Equation (2.18) and the given relations $I_{\text{cm}} = \left(2/5\right)mR^2$ and $\left(9/7\right)v_0$ have been used. Setting the initial and final angular momenta equal and solving for h/R gives

$$\frac{h}{R} = \frac{4}{5} \tag{2.20}$$

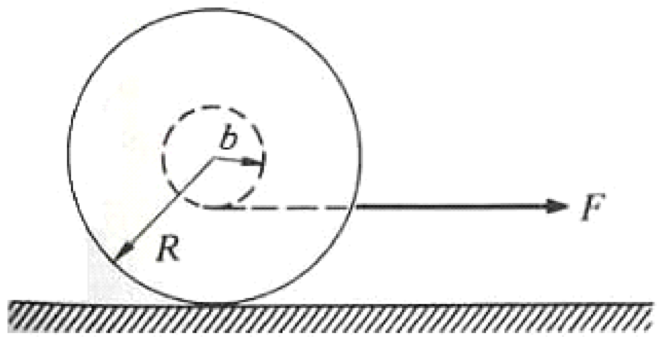
(note that the figure is not quite to scale).

As an alternative, taking the angular momentum after the collision about the center of the ball, note that the time Δt between the moments the ball is struck and when it begins to roll without slipping is $\Delta v/\left(\mu_{\text{k}} g\right)$. But, if the angular momentum is taken about the center of the ball, after the ball is struck the angular impulse delivered to the ball by the friction force is

$$\left(\mu_{\text{k}} mg\right)R\,\Delta t = I_{\text{cm}}\left(\omega_f - \omega_0\right). \tag{2.21}$$

Eliminating Δt between these expressions leads to the same result obtained by equating the first and third expressions in (2.19)

Problem 9 A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia about an axis passing through the center of the Yo-Yo can be approximated by $I_0 = (1/2)mR^2$. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force $\vec{\mathbf{F}}$ to the right as shown in the figure.



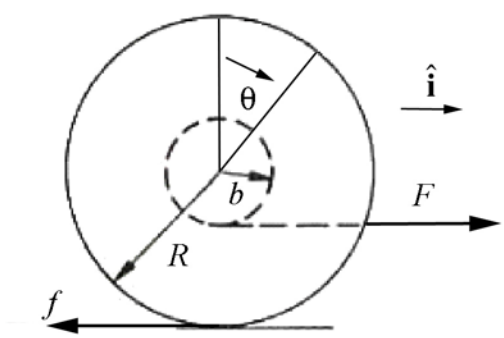
The coefficient of static friction between the Yo-Yo and the table is μ_s.

a) Which way will the Yo-Yo rotate if the string is pulled very gently? If the string is jerked hard, which way will the Yo-Yo rotate?

b) What is the maximum magnitude of the pulling force , $|\vec{\mathbf{F}}|$, for which the Yo-Yo will roll without slipping?

Solution:

The force of friction acting opposite to the applied force is shown in the figure below.



Torque equation:

$$\tau = I\alpha = Rf - bF \tag{2.22}$$

Force equation:

$$F - f = ma \tag{2.23}$$

For rolling without slipping motion along the floor,

$$a = +R\alpha \tag{2.24}$$

(Note that the positive sign is a result of our two choices of sign convention for positive rotation and linear acceleration.)

From (1) and (3),

$$Rf - bF = I\alpha = \frac{Ia}{R} = \frac{1}{2}mR^2\frac{a}{R} = \frac{1}{2}maR \tag{2.25}$$

implies that

$$f - \frac{b}{R}F = \frac{1}{2}ma \tag{2.26}$$

From (2) and (4),

$$F\left(1 - \frac{b}{R}\right) = \frac{3}{2}ma \tag{2.27}$$

implies that

$$a = \frac{2}{3}\frac{F}{m}\left(1 - \frac{b}{R}\right) \tag{2.28}$$

From (2) and (5),

$$f = \frac{F}{3}\left(1 + \frac{2b}{R}\right) \tag{2.29}$$

The frictional force, f , is maximum when $b = R$, and is given by $f = F$ (i.e it equals the applied force). We also know that the maximum possible value of f is given by, $f = \mu_s mg$. So, our assumption of pure clockwise rotation breaks down for $F > \mu_s mg$ and slipping occurs.

a) If the string is pulled very gently, our assumption of pure clockwise rotation holds. And, the Yo-Yo rotates in the forward (clockwise) direction without any slipping.

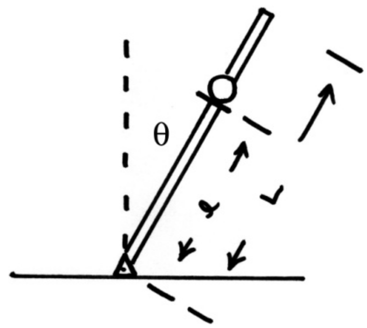
If the string is jerked hard, our assumption of pure rotation in the clockwise direction fails, and slippage occurs. The Yo-Yo rotates in the anti-clockwise direction but still moves forward (by slipping).

b) From the results above Eq. (2.29)

$$F = \frac{3f}{\left(1 + \frac{2b}{R}\right)} \tag{2.30}$$

$$F_{\max} = \frac{3f_{\max}}{\left(1 + \frac{2b}{R}\right)} = \frac{3\mu mg}{\left(1 + \frac{2b}{R}\right)} \tag{2.31}$$

Problem 10 A uniform rod of length d and mass m_r is free to rotate about a pivot at its lower end. A light bead of mass $m_b << m_r$ moves frictionlessly on the rod, but a massless collar fixed on the rod a distance b from the pivot constrains the bead's distance from the pivot to be greater than or equal to b. Initially the rod is at rest, nearly vertical, and the bead is at rest on the collar. The rod is released and falls over.



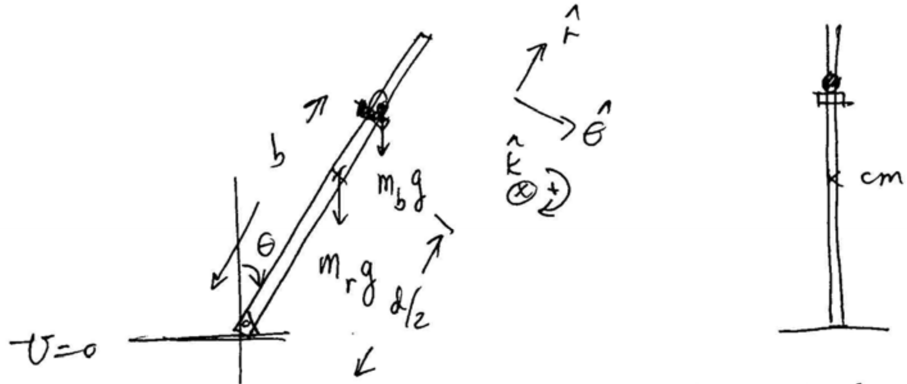
a) Draw force diagrams for the bead, for the rod, and for the system of bead and rod. Indicate clearly your choice of coordinates and unit vectors.

b) What is the total torque on the system of the rod and the bead about the pivot point when the system is at an angle θ with respect to the vertical axis?

c) Using conservation of energy, write down an equation comparing the energy intially with the energy when the system is at an angle θ with respect to the vertical axis.

d) Calculate the the maximum angle, $\theta_{\max}$, that the system makes with the vertical axis when the bead just starts to lose contact with the collar.

e) What is your answer in part d) when $m_b << m_r$?



torque on system of rod and bead about pivot point

$$\frac{d}{2} m_r g \sin\theta + m_b g\, b \sin\theta = I_p \ddot{\theta} \quad \text{where} \qquad (1)$$

$$I_p = \frac{1}{3} m_r d^2 + m_b b^2$$

Conservation of energy

$$E_0 = m_b g b + m_r g \frac{d}{2}$$

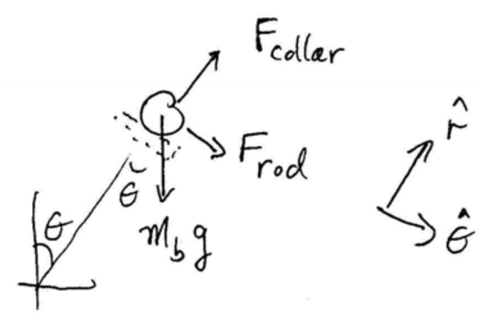
$$E_f = m_b g b \cos\theta + m_r g \frac{d}{2} \cos\theta + \frac{1}{2} I_p \dot{\theta}^2$$

$$E_0 = E_f \Rightarrow$$

$$m_b g b + m_r g \frac{d}{2} = m_b g b \cos\theta + m_r g \frac{d}{2} \cos\theta + \frac{1}{2} I_p \dot{\theta}^2$$

$$\left(m_r g \frac{d}{2} + m_b g b\right)(1 - \cos\theta) = \frac{1}{2} I_p \dot{\theta}^2 \qquad (2)$$

Force diagram on bead



radial equation

$$F_{collar} - m_b g \cos\theta = -m_b b \dot{\theta}^2 \qquad (3)$$

Find θ_{max} s.t. $F_{collar} = 0$

$$-m_b g \cos\theta_{max} = -m_b b \dot{\theta}^2$$

$$\Rightarrow \qquad \dot{\theta}^2 = \frac{g \cos\theta_{max}}{b} \qquad (3a)$$

$$\left(m_r g \frac{d}{2} + m_b g b\right)(1 - \cos\theta_{max}) = \frac{1}{2} I_p \dot{\theta}^2_{max}$$

$$m_r g \frac{d}{2} + m_b g b = \frac{1}{2} \frac{I_p g}{b} \cos\theta_{max} + \left(m_r g \frac{d}{2} + m_b g b\right)\cos\theta_{max}$$

$$\frac{m_r g \frac{d}{2} + m_b g b}{\frac{1}{2} I_p \frac{g}{b} + \left(m_r g \frac{d}{2} + m_b g b\right)} = \cos\theta_{max}$$

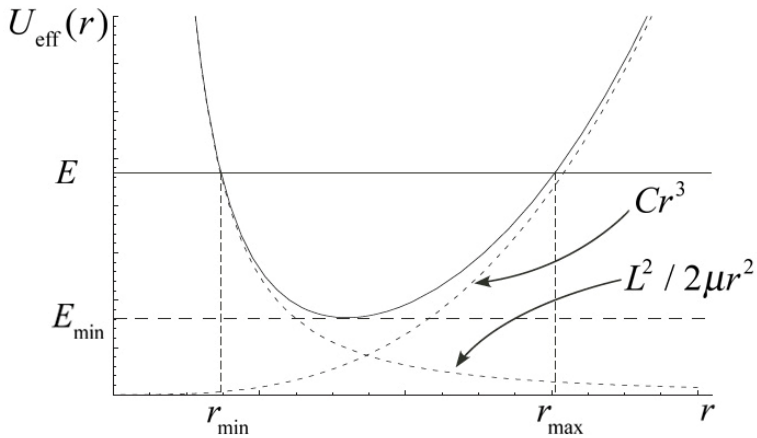
$$\theta_{max} = \cos^{-1}\left(\frac{m_r \frac{d}{2} + m_b b}{\frac{1}{2b}\left(\frac{1}{3} m_r d^2 + m_b b^2\right) + m_r \frac{d}{2} + m_b b}\right)$$

$$\theta_{max} = \cos^{-1}\left(\frac{m_r \frac{d}{2} + m_b b}{\frac{1}{6} m_r \frac{d^2}{b} + m_r \frac{d}{2} + \frac{3}{2} m_b b}\right)$$

let $m_r >> m_b$

$$\theta_{max} \simeq \cos^{-1}\left(\frac{\frac{m_r d}{2}}{\frac{1}{6} m_r \frac{d^2}{b} + m_r \frac{d}{2}}\right) \simeq \cos^{-1}\left(\frac{3b}{d + 3b}\right)$$

Problem 11



The effective potential corresponding to a pair of particles interacting through a central force is given by the expression

$$U_{\mathrm{eff}}(r) = \frac{L^2}{2\mu r^2} + Cr^3 \tag{2.32}$$

where L is the angular momentum, μ is the reduced mass and C is a constant. The total energy of the system is E. The relationship between $U_{\mathrm{eff}}(r)$ and E is shown in the figure, along with an indication of the associated maximum and minimum values of r and the minimum allowed energy $E_{\min}$. In what follows, assume that the center of mass of the two particles is at rest.

a) Find an expression for the radial component $f(r)$ of the force between the two particles. Is the force attractive or repulsive?

b) What is the radius r_0 of the circular orbit allowed in this potential? Express your answer as some combination of L, C, and μ.

c) When E has a value larger than $E_{\min}$, find how rapidly the separation between the particles is changing, dr/dt, as the system passes through the point in the orbit where $r = r_0$. Give your answer in terms of some combination of E, $E_{\min}$, L, C, μ and r_0.

d) Does the relative motion between the particles stop when $r = r_{\max}$? If not, what is the total kinetic energy at that point in terms of some combination of E, L, C, μ, $r_{\max}$ and $r_{\min}$?

Solutions:

a) The effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r)\,, \tag{2.33}$$

and $f(r) = -dU/dr$ for a central force, and so

$$f(r) = -\frac{d}{dr}U(r) = -\frac{d}{dr}Cr^3 = -3Cr^2\,. \tag{2.34}$$

From the figure, $C > 0$, so $f(r) < 0$, a restoring force.

b) The circular orbit will correspond to the minimum effective potential; at this radius the kinetic energy will have no contribution from any radial motion. This minimum effective potential, and hence the radius of the circular orbit, is found from basic calculus and algebra,

$$\begin{aligned} \left[\frac{d}{dr}U_{\text{eff}}(r)\right]_{r=r_0} &= -\frac{L^2}{\mu r_0^3} + 3Cr_0^2 \\ r_0^5 = \frac{L^2}{3\mu C}, \quad r_0 &= \left(\frac{L^2}{3\mu C}\right)^{1/5}. \end{aligned} \tag{2.35}$$

c) Recall that the kinetic energy is

$$K = \frac{L^2}{2\mu r^2} + \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2. \tag{2.36}$$

The difference $E - E_{\text{min}}$ is then found by evaluating U_{eff} at $r = r_0$,

$$E - E_{\text{min}} = \frac{1}{2}\mu\left(\left.\frac{dr}{dt}\right|_{r=r_0}\right)^2, \tag{2.37}$$

or $\left|dr/dt\right| = \sqrt{(2/\mu)(E - E_{\text{min}})}$.

d) No; $dr/dt = 0$, but the kinetic energy, from Equation (2.36), is

$$K_{\text{min}} = \frac{L^2}{2\mu r_{\text{max}}^2}\,. \tag{2.38}$$