MASSACHUSETTS INSTITUTE OF TECHNOLOGY
EXPERIMENTAL STUDY GROUP

PROBLEM SET 4: CONDUCTORS AND CAPACITORS
Due date: Sunday, February 27th at 10:00 pm

1. Purcell 3.1: Charges in a spherical conductor

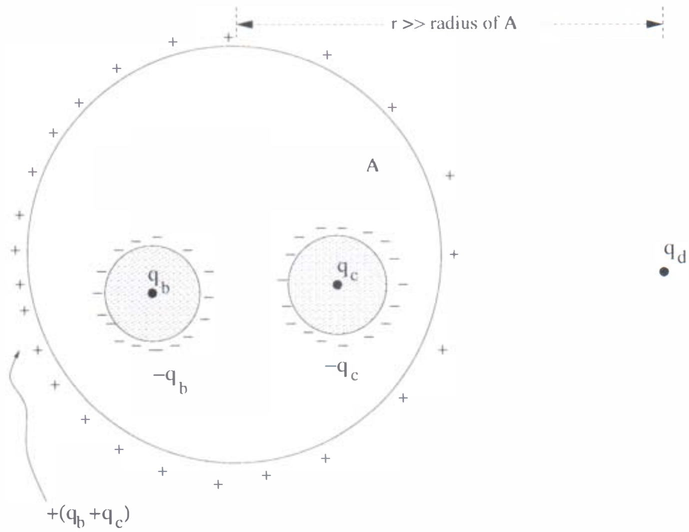


Figure 1: A spherical conductor with two spherical cavities. This figure assumes that q_b and q_c are positive, but in the problem they can be positive or negative.

A charge q_b lies in the *center* of one cavity; hence it induces a surface charge of $-q_b$ on the spherical boundary of the cavity. This distribution is *strictly* uniform due to the symmetry. The same argument applies to the charge q_c and the induced surface charge $-q_c$.

Due to the spherical symmetry the induced charges exert no force on q_c and q_d. Therefore the forces on q_c and q_d are strictly

$$F_b = F_c = 0$$

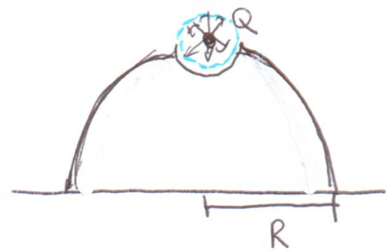
Since the spherical conductor A is totally electrically neutral, a surface charge of $+(q_b + q_c)$ distributes on the outer spherical shell of A. This charge is not distributed uniformly, but since q_d is far away from A, this surface charge distribution is *approximately* uniform. We can treat the sphere like a point source, therefore

$$F_d = F_A \approx \frac{q_d(q_b + q_c)}{r^2}$$

approximately and depending on r being large compared to radius of A.

$$\vec{F_A} = -\vec{F_d}$$

2) (Purcell 3.3)



$R = ?$

- $\Phi_{TOTAL} = \oint_A \bar{E} \cdot d\bar{A} = 0$ because the surface contains no charges

- $\Phi_{side} = 0$ because there $\hat{E} \perp \hat{n}$ everywhere

- $\Phi_{Top} = \dfrac{-\oint_{sphere} \bar{E} \cdot d\bar{A}}{2} = \dfrac{-\frac{Q}{\cancel{\varepsilon^2}} 4\pi \cancel{\varepsilon^2}}{2} = -2\pi Q$

The sign minus is due to the fact that $\hat{n}$ points towards

$E = \dfrac{Q}{R^2}$ close to Q we can neglect the plane

We can consider an infinitesimal sphere ($\varepsilon \to 0$). The flux doesn't depend on ε.

- $\Phi_{bottom} = \int_0^R E_{plane} \, 2\pi r' \, dr' =$

We derived E on the plane in class

$$= \int_0^R \frac{2Qh}{(h^2+r'^2)^{3/2}} 2\pi r' \, dr' = \left[-\frac{4\pi Q h}{(h^2+r'^2)^{1/2}} \right]_0^R =$$

$$= 4\pi Q - \frac{4\pi Q h}{(h^2+R^2)^{1/2}}$$

- $\Phi_{TOTAL} = \Phi_{Top} + \Phi_{bottom} = 0$

$$-2\pi Q + 4\pi Q - \frac{4\pi Q h}{(h^2+R^2)^{1/2}} = 0$$

$$(h^2+R^2)^{1/2} = 2h$$

$$h^2 + R^2 = 4h^2 \longrightarrow R = \sqrt{3}\, h$$

3. Purcell 3.5: Work pulling a charge away from a conducting plane

By definition of work, we conclude that the *second student* is correct. Before calculating it explicitly, let's try to understand the difference between these two methods. The work required to separate to infinite distance two charges Q and -Q is $Q^2/2h$, the absolute value of the electric potential energy. Recall how we made it: we always assume *one of the two charges is fixed*, since moving it would cost "extra" energy.

That is not our case here — we have an image charge problem. Remember that the image charge is always located symmetrically on the other side of the conducting plane. When we move the real charge, the image charge moves simultaneously. It is equivalent to a two-charge system in which some mysterious hand is moving the image charge at the same rate as we move the real charge. Consequentely, we only need to do work corresponding to half of the "total" potential energy.

The result we expect is $W = (1/2)(Q^2/2h)$. Let's verify this:

$$W = \int_h^\infty \frac{Q^2}{(2z)^2} dz = \frac{Q^2}{4h}.$$

Aside: You might ask who kindly offers this mysterious hand that "does" the other "half" of "work"? Your intuition is correct – nobody. Remember that the solution for the image charge only applies to the region above the plane, $z > 0$, not everywhere. So in the *real* charge-plane configuration, the actual *total* potential energy is not $-Q^2/2h$, but only half, since $E = 0$ for $z < 0$. This means that the work done to move Q away from the conducting plane is exactly the potential energy of the system; no more work is need in real configuration. The extra "half" of work done by the mysterious hand is just because we included an extra imaginary amount of potential energy in the charge - image charge configuration.

4. Purcell 3.8: Three conducting plates

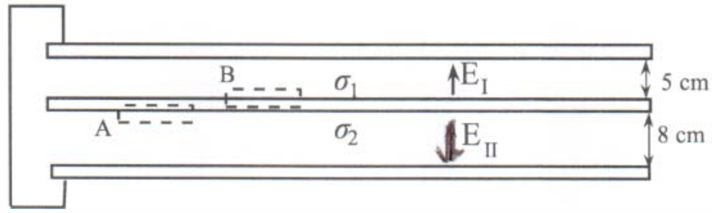


Figure 3: Charge Density on Three Conducting Planes

Solution. Using Gauss's law for the Gaussian "pillbox" A (see Figure 3) we get,

$$E_{II} = -4\pi\sigma_2$$

Similarly, for "pillbox" B we get,

$$E_I = 4\pi\sigma_1$$

Note that we have used the fact that the electric field is zero inside a conductor to obtain the above equations. Further, the top and bottom plates are at the same potential as they are connected by a wire. This implies

$$\int_{\text{bot}}^{\text{top}} \vec{E} \cdot \vec{ds} = 0 = 5E_I + 8E_{II}$$

Using the results from equations (17) and (18) in the above equation we get,

$$5\sigma_1 - 8\sigma_2 = 0$$

Further, the total charge density on the isolated plate is 10 esu.cm^{-2}. Hence,

$$\sigma_1 + \sigma_2 = 10$$

Solving equations (20) and (21) we get,

$$\sigma_1 = \frac{80}{13} \text{ esu.cm}^{-2} \text{ and}$$

$$\sigma_2 = \frac{50}{13} \text{ esu.cm}^{-2}$$

5. Purcell 3.17: Design a spherical capacitor

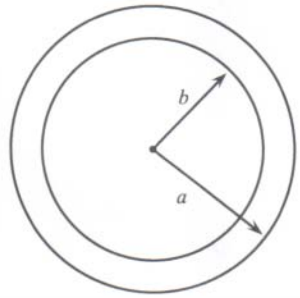


Figure 4: (a) Sketch of a spherical capacitor

Solution. Figure (4) shows a sketch of a spherical capacitor. The inner sphere has radius b and carries a charge Q which is uniformly distribute over the surface. The outer sphere has a radius a and carried a charge $-Q$. The electric field in the region $a < r < b$ is given by,

$$\vec{E} = \frac{Q}{r^2}\hat{r}$$

This can be determined using Gauss's law. The electric field vanishes outside the spherical capacitor and inside the inner spherical shell. The potential difference between the two spheres can be found by integrating along a radial line:

$$\varphi(a) - \varphi(b) = -\int_b^a \frac{Q}{r^2}dr = Q\left(\frac{1}{a} - \frac{1}{b}\right) \tag{25}$$

Hence, the capacitance of the spherical capacitance is given by

$$C = \left|\frac{Q}{\Delta\varphi_{ab}}\right| = \frac{ab}{a-b} \tag{26}$$

The energy stored in the capacitor is

$$U = \frac{1}{2}C\Delta\varphi_{ab}^2 = \frac{Q^2}{2}\left(\frac{1}{b} - \frac{1}{a}\right) \tag{27}$$

Note that, $Q = b^2E(b) \leq b^2E_0$ where E_0 is the maximum allowed electric field at the surface of the inner sphere (the constraint set in the problem). Therefore,

$$U \leq U_{up} = \frac{E_0^2}{2}\left(b^3 - \frac{b^4}{a}\right) \tag{28}$$

The potential energy is maximum when $b^3 - b^4/a$ is maximum. This happens when

$$\frac{dU_{up}}{db} = 0 = \frac{E_0^2}{2}\left(3b^2 - 4\frac{b^3}{a}\right) \tag{29}$$

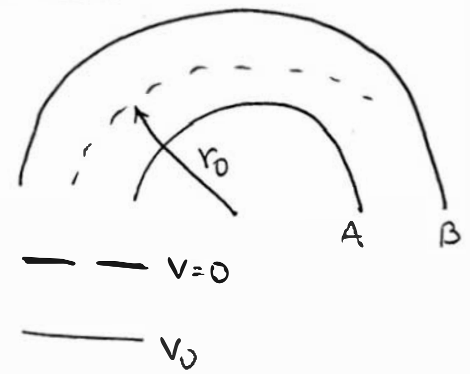
Hence, the potential energy is maximum when the electric field at the inner surface is maximum and when

$$b = \frac{3}{4}a$$

The maximum potential energy is

$$U_{\max} = \frac{27}{512}a^3E_0^2 \tag{30}$$

P. 3.19



Show that for circular motion if

$$\left.\begin{aligned} V_B &= 2V_0 \ln\left(\frac{b}{r_0}\right) \\ V_A &= 2V_0 \ln\left(\frac{a}{r_0}\right) \end{aligned}\right\} (1)$$

Note that the potential between the electrodes is:

$$V(r) = 2V_0 \ln\left(\frac{r}{r_0}\right) \qquad (2)$$

thus the electric field is

$$E_r = -\frac{\partial V(r)}{\partial r} = -\frac{2V_0}{r} \qquad (3)$$

If $V_0 > 0$, the ions must be positive

They enter the semicircle with velocity v

$$\frac{1}{2}mv^2 = qV_0 \quad \text{(energy gained)} \qquad (4)$$

They have to perform circular motion with this velocity

$$\frac{mv^2}{r} = F = q|E| \qquad (5)$$

but $mv^2 = 2qV_0 \rightarrow$

$$\frac{2qV_0}{r} = q|E|$$

$$\rightarrow |E| = \frac{2V_0}{r} \qquad (6)$$

Just as predicted in (3).

7. Purcell 10.1 : Make a capacitor

b) For example:
http://www.sciencebuddies.org/science-fair-projects/project_ideas/Elec_p049.shtml

7a) (Purcell 10.1)

- In SI units

$$C = \frac{\varepsilon A}{S}$$

- $\varepsilon = 2.3\,\varepsilon_0 = 2.3 \times 8.85 \times 10^{-12} \left(\frac{C^2}{m^2\,N}\right) \to \frac{F}{m}$

$S = 0.001$ inch $= 0.00254$ cm $= 2.54 \times 10^{-5}$ m

$C = 0.05\,\mu F$

$\downarrow$

$$A = \frac{sC}{\varepsilon} = 0.062 \text{ m}^2$$

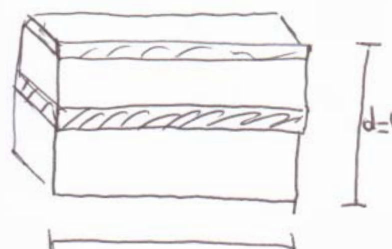
Since $w = 2.25$ inch $\simeq 0.05$ m

$\downarrow$

$$\ell = \frac{A}{w} \simeq 1.24 \text{ m}$$

- Take 1.24 m of polyethylene and alluminum tape.

Cut them in half, stack them, roll them



0.62 m = $6.2 \times 10^{+1}$ cm

DRAWINGS OUT OF SCALE!!

$$\pi r_0^2 \sim d \cdot \frac{\ell}{2}$$

$r_0 \sim 0.4$ cm

8) (Purcell 10.14)

- Without the dielectric

$$C_0 = \frac{A}{4\pi(d/2)}$$, with a dielectric $C_D = \mathcal{E} C_0$

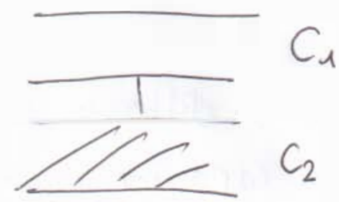
-  $=$ 

FIG. 1

FIG. 2
Two capacitors in series
What is the capacity of two capacitors in series?

$$C = \frac{Q}{|\Delta V|}$$

FIG. 1 the potential can be found by integrating the electric field along a straight line from the top to the bottom plate

$$\Delta V = -\int_0^{2d} E\, ds = -\Delta V_0 - \Delta V_D = -E_0 d - E_D d =$$

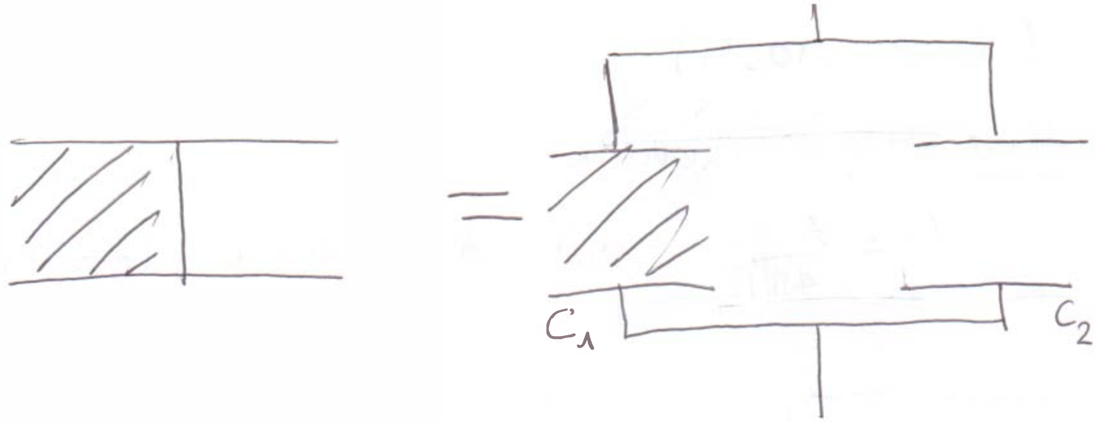
$$= -\frac{Q}{A} 4\pi d - \frac{Q}{A}\frac{4\pi}{\mathcal{E}} d$$

$$\frac{Q}{|\Delta V|} = \frac{1}{\frac{4\pi d}{A} + \frac{4\pi d}{A\mathcal{E}}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$C_1 = 2C_0$
$C_2 = 2C_D$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{2\mathcal{E} C_0^2}{C_0 + \mathcal{E} C_0} = \frac{2\mathcal{E}}{1+\mathcal{E}} C_0$$

FIG. 2 For capacitors connected in series the equivalent capacitance is $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$



The potential difference on each part of the capacitor is the same. The system is equivalent to a system composed by two capacitors connected in parallel

$$Q_1 = C_1 V$$
$$Q_2 = C_2 V$$
$$Q_1 + Q_2 = (C_1 + C_2) V$$
$$Q = (C_1 + C_2) V$$

$$C_2 = C_0/2$$
$$C_1 = \varepsilon C_0/2$$

$$\downarrow$$

$$C = C_1 + C_2 =$$

$$= \varepsilon \frac{C_0}{2} + \frac{C_0}{2} = \frac{\varepsilon + 1}{2} C_0$$

8) (Purcell 10.6)

- $d = 3\text{ m} = 300\text{ cm}$

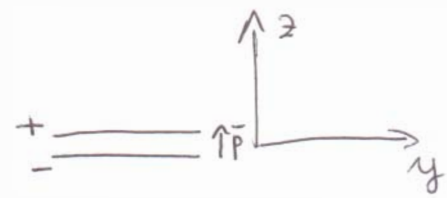
$C = \dfrac{A}{4\pi s} \rightarrow A \simeq 4700\ \text{cm}^2$

$\sqrt{A} \simeq 69\text{ cm} << d \rightarrow$ we are far away from the capacitor compared to its dimensions.

The capacitor is neutral $\rightarrow$ the monopole component is zero.

The next important term is the dipole

$Q = CV = 1500\text{ esu}$



The dipole $\bar{p}$ is:

$$\bar{p} = qs\hat{z}$$

In the previous pset we found the electric field of a dipole

$z = r\cos\theta$
$y = r\sin\theta$

$$\bar{E} = \frac{2qs\cos\theta}{r^3}\hat{r} + \frac{qs\sin\theta}{r^3}\hat{\theta} =$$

$\hat{\theta} = -\sin\theta\,\hat{z} + \cos\theta\,\hat{y}$
$\hat{r} = \cos\theta\,\hat{z} + \sin\theta\,\hat{y}$
$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$
$\sin\theta\,\hat{\theta} = \cos\theta\,\hat{r} - \hat{z}$

$$= \frac{3qs(\hat{z}\cdot\hat{r})\hat{r} - \boxed{qs\hat{z}}}{r^3}$$

a) $\bar{r} = 300\text{ cm}\,\hat{y}$

$$\bar{E} = -\frac{qs}{r^3}\hat{z} = -\frac{1500\times 1.5}{(300)^3}\hat{z} = -8.3\times 10^{-5}\,\frac{\text{esu}}{\text{cm}^2}\hat{z}$$

$\longrightarrow$

b) $\bar{r} = 300 \quad \hat{z}$

$$\bar{E} = \frac{2p}{r^3}\,\hat{z} = 16.7 \times 10^{-5}\ \frac{esu}{cm^2}\ \hat{z}$$