Massachusetts Institute of Technology
Experimental Study Group
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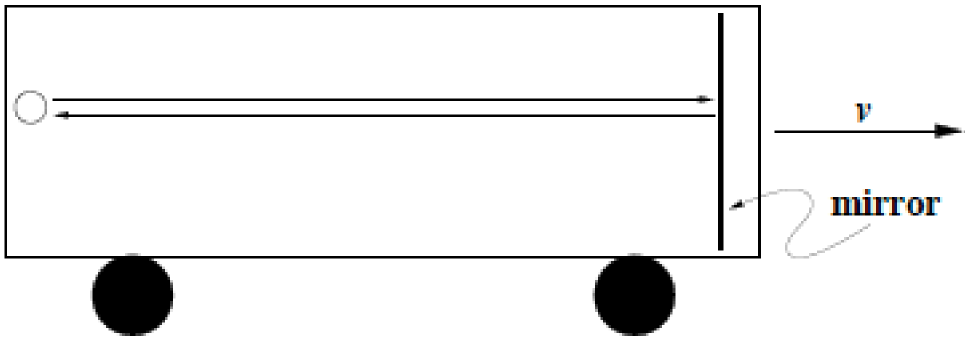
Solution Special relativity

1.

Let's move our light source to the back of the train. We flash it in the forward direction, where it reflects off a mirror at the front of the train. The light then returns to the light source, where we measure it. This is a way to measure the length of the railway car.



The two events of interest in this case are the emission of light by the bulb, and its reception after it bounces back to us.

Let's examine what's going on in the train's reference frame first. If the elapsed time between emission of the light and reception is Δt_{train}, we would infer that the length of the train is

$$\Delta x_{\text{train}} = c\Delta t_{\text{train}}/2$$

How do things look in the station frame? We break down the light travel into two pieces: emission to the mirror, then mirror back to reception. Let's say it takes a time $\Delta t_{\text{station, 1}}$ for the first piece — emission to the mirror. An interval $\Delta t_{\text{station, 2}}$ passes following reflection to travel back to reception. If the length of the train as measured in the station is $\Delta x_{\text{station}}$, then the travel times in each bounce is given by

$$\begin{aligned}\Delta t_{\text{station},1} &= (\Delta x_{\text{station}} + v\Delta t_{\text{station},1})/c \\ \Delta t_{\text{station},2} &= (\Delta x_{\text{station}} - v\Delta t_{\text{station},2})/c\end{aligned}$$

The time from emission to mirror is longer since the mirror is moving in the same direction as the light — the light has to chase the mirror on the first piece of the round trip, which takes extra time. On the second piece, the back of the train is rushing towards the light, so it takes less time.

We rearrange these expressions to isolate the travel times:

$$\begin{aligned}\Delta t_{\text{station},1} &= \frac{\Delta x_{\text{station}}}{c - v} \\ \Delta t_{\text{station},2} &= \frac{\Delta x_{\text{station}}}{c + v}\end{aligned}$$

Then, we add them together to get the total travel time:

$$\begin{aligned}
\Delta t_{\text{station}} &\equiv \Delta t_{\text{station},1} + \Delta t_{\text{station},2} \\
&= \frac{2\Delta x_{\text{station}}/c}{1 - v^2/c^2} \\
&= 2\gamma^2 \Delta x_{\text{station}}/c
\end{aligned}$$

From our discussion of time dilation, we also know that $\Delta t_{\text{station}} = \gamma \Delta t_{\text{train}} = 2\gamma \Delta x_{\text{train}}/c$. Combining this with our result for $\Delta t_{\text{station}}$, we get

$$\gamma \Delta x_{\text{train}} = \gamma^2 \Delta x_{\text{station}}$$

or

$$\Delta x_{\text{station}} = \Delta x_{\text{train}}/\gamma$$

Since $\gamma \geq 1$, this tells us that *moving objects are contracted*: the train's length according to observers that see it moving is less than the length measured in the "rest frame".

2 Invariant interval: A quantity that is left unchanged by Lorentz transformations is a called a "Lorentz invariant". Consider two events described in the laboratory frame by (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2). Show that

$$\Delta s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \tag{1}$$

is a Lorentz invariant.

Let's label events 1 and 2 in the laboratory frame by (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2), and in the boosted frame by (t'_1, x'_1, y'_1, z'_1) and (t'_2, x'_2, y'_2, z'_2). For simplicity suppose boosted direction is along positive x-axis and the boost frame is of relative velocity v to the laboratory frame. The transformation law for event 1 reads

$$\begin{aligned} ct'_1 &= \gamma(ct_1 - \beta x_1) \\ x'_1 &= \gamma(x_1 - \beta ct_1) \\ y'_1 &= y_1 \\ z'_1 &= z_1, \end{aligned} \tag{2}$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. For event 2 the transformation is simply to rewrite all subscripts "1" in eq.(2) to subscript "2". The spacetime interval Δs^2 in boosted frame then becomes

$$\begin{aligned} \Delta s'^2 &= -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \\ &= -(ct'_2 - ct'_1)^2 + (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 \\ &= -\gamma^2[(ct_2 - ct_1) - \beta(x_2 - x_1)]^2 + \gamma^2[(x_2 - x_1) - \beta(ct_2 - ct_1)]^2 \\ &\quad +(y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= -\gamma^2(1-\beta^2)(ct_2 - ct_1)^2 + \gamma^2(1-\beta^2)(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= \Delta s^2. \end{aligned} \tag{3}$$

So Δs^2 is a Lorentz invariant.

3 Galilean tranformations: Prior to special relativity, people related coordinates between different frames with the "Galilean transformation" – clocks in different reference frame tick at the same rate, spatial positions are shifted by a term that depends on the relative velocity just as you would expect. For example, for frames that are moving with respect to each other in the x direction, we would have

$$\begin{aligned} t' &= t \\ x' &= x - vt \end{aligned} \tag{4}$$

Using the binomial expansion on γ, show that for small v/c the Lorentz tranformations reduce to the Galilean tranformations. At what value of v does the next term in the expansion change the x transformation by 1%?

The Lorentz transformation reads in eq.(2) without subscripts "1" for the sake of generality. Note that as $\beta = v/c \ll 1$,

$$\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2 + \cdots, \tag{5}$$

where "$\cdots$" denotes terms in the order $\mathcal{O}(\beta^4)$. Then Lorentz transformation reduces to be

$$\begin{aligned} ct' &= (ct - \beta x)(1 + \frac{1}{2}\beta^2) = ct + \mathcal{O}(\beta); \\ x' &= (x - \beta ct)(1 + \frac{1}{2}\beta^2) = x - vt + \mathcal{O}(\beta^2). \end{aligned} \tag{6}$$

or $t' = t$, $x' = x - vt$. This is exactly Galilean transformation. The next term in the expansion of x transformation is $(1/2)x\beta^2$, so it changes by a rate $((1/2)x\beta^2)/x = (1/2)\beta^2 = 1\%$. This gives $v = 0.14c \simeq 4 \times 10^7 m/s$, beyond which, that means, the relativistic effect cannot be ignored and Newton Mechanics is not quite valid anymore.

4 Transforming velocities: A bullet is fired with velocity $\vec{u'}$ in the (x', y') plane of a moving frame F'. Frame F' moves with speed v in the $+x$ direction with respect to the laboratory frame F.

(a) Find the angle that the velocity vector makes with x axis of the lab frame.

The velocity transforms as

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \tag{7}$$

$$\vec{u}_\perp = \frac{\vec{u'}_\perp}{\gamma_v(1 + \frac{vu'_x}{c^2})}, \tag{8}$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$; u_x and $\vec{u}_\perp$ are respectively the component of velocity $\vec{u}$ in lab frame F in x direction and in the direction normal to x. Similar notations with primes are for velocity $\vec{u'}$ in moving frame F'. Let the angle that $\vec{u}$ makes with respect to x-axis in F be θ, and that $\vec{u'}$ makes with respect to x'-axis in F' be θ'. So $u_x = u\cos\theta$, $|\vec{u}_\perp| = u\sin\theta$, and similarly for u'_x and $\vec{u'}_\perp$. Let $u = |\vec{u}|$, and $u' = |\vec{u'}|$.

$$\tan\theta = \frac{|\vec{u}_\perp|}{u_x} = \frac{u'\sin\theta'}{\gamma_v(u'\cos\theta' + v)} \tag{9}$$

$$u = \sqrt{u_x^2 + |\vec{u}_\perp|^2} = \frac{\sqrt{u'^2 + v^2 + 2u'v\cos\theta' - (u'v\sin\theta'/c)^2}}{1 + \frac{u'v}{c^2}\cos\theta'}. \tag{10}$$

(Eq.(10) is for the use of part (c).) Or for our problem,

$$\theta = \arctan\left(\frac{u'\sin\theta'}{\gamma_v(u'\cos\theta' + v)}\right).$$

(b) What is this angle in the limit $|\vec{u'}| = c$? Does anything wierd happen?

Plug in $u' = c$ to eq.(9).

$$\tan\theta = \frac{\tan\theta'}{\gamma_v(1 + \frac{v}{c}\sec\theta')}. \tag{11}$$

We observe that $\theta' \neq \theta$ generically except when $v \to 0$ or $\theta' = 0$. This means that observers in different frame will observe different orientation of light if their relative velocity is not in the same direction of the emitted light. Nothing particularly wierd happens — we just get a slightly simpler version of the formula.

(c) Show that when $|\vec{u'}| = c$, $|\vec{u}| = c$ – the speed of light is the same in both frames.

Plugging $u' = c$ to eq.(10):

$$\begin{aligned}
u &= \frac{\sqrt{c^2 + v^2 + 2cv\cos\theta' - v^2\sin^2\theta'}}{1 + v\cos\theta'/c} \\
&= \frac{\sqrt{c^2 + 2cv\cos\theta' + v^2(1 - \sin^2\theta')}}{1 + v\cos\theta'/c} \\
&= \frac{\sqrt{c^2 + 2cv\cos\theta' + v^2\cos^2\theta'}}{1 + v\cos\theta'/c} \\
&= \frac{\sqrt{(c + v\cos\theta')^2}}{1 + v\cos\theta'/c} \\
&= \frac{c + v\cos\theta'}{1 + v\cos\theta'/c} = c\ .
\end{aligned}$$

5 "Beating the speed of light": ... The idea is as follows. We make a cart roll across the floor with speed v. We put a smaller cart on top of that cart, and roll it with speed v with respect to the first cart, and in the same direction as the first cart. We put a third cart on this second cart; it rolls with speed v with respect to the second cart. We put a fourth cart ... you get the idea. Your uncle claims that there is some n at which the cart must be going faster than the speed of light.
(a) Prove him wrong. Using mathematical induction, prove that if $v < c$, then $v_n < c$, where v_n is the velocity of the nth cart. Show that this holds even for extremely large n.

Eq.(7) gives the recurrence relation of v_{n+1} (the $(n+1)$th cart's velocity with respect to the floor) in terms of v_n and v, i.e. we think of the floor as the lab frame and nth cart as the moving frame with boost velocity v_n; the $(n+1)$th cart moves with velocity v with repect to the nth cart. (When you apply eq.(7), be careful of the meaning of notations since v in eq.(7) means boost velocity.) The recurrence relation is

$$v_{n+1} = \frac{v + v_n}{1 + v\, v_n/c^2}. \tag{12}$$

Now let's prove by induction that, assuming $v < c$, then $v_n < c$ for all n. Firstly we observe that $v_1 = v < c$ by assumption. Then, suppose $v_n < c$ holds, show that $v_{n+1} < c$: let $\beta_v = v/c$, $\beta_n = v_n/c$, and so on.

$$\begin{aligned} \beta_{n+1}^2 = (\frac{v_{n+1}}{c})^2 &= \left(\frac{\beta_v + \beta_n}{1 + \beta_v \beta_n}\right)^2 \\ &= 1 - \frac{(1-\beta_v^2)(1-\beta_n^2)}{(1+\beta_v\beta_n)^2} \\ &< 1. \end{aligned} \tag{13}$$

Therefore, by mathematical induction we deduce that $v_n < c$ for *all* n if $v < c$, including for extremely large n.

(b) Calculate the value of v_n given v and n.

Define $\beta_n = v_n/c = \tanh(x_n)$; so $\beta_v = v/c = v_1/c = \tanh(x_1)$.

$$\begin{aligned} \tanh(x_{n+1}) &= \frac{\tanh(x_1) + \tanh(x_n)}{1 + \tanh(x_1)\tanh(x_n)} \\ &= \tanh(x_1 + x_n). \end{aligned} \tag{14}$$

Since $\tanh(x)$ is a monotonically increasing function, the recurrence relation simplifies to be $x_n = x_{n-1} + x_1$ for $n \geq 2$. This gives

$$x_n = n x_1 \tag{15}$$

$$v_n = c\tanh(n x_1) = c\tanh(n\,\mathrm{arctanh}(v/c)). \tag{16}$$

As $n \to \infty$, $x_n \to \infty$, $\tanh(x_n) \to 1$. In other words, v limits to the speed of light, but never beats it.

6 Energy-momentum identity.

Start from the fact that $E = \gamma_u mc^2$, $\vec{p} = \gamma_u m\vec{u}$ where $\gamma_u = (1 - u^2/c^2)^{-1/2}$.

$$\begin{aligned}
p^2c^2 + m^2c^4 &= m^2c^4(1 + \frac{\gamma_u^2 u^2}{c^2}) \\
&= m^2c^4(1 + \frac{u^2/c^2}{1 - u^2/c^2}) \\
&= \gamma_u^2 m^2 c^4 \\
&= E^2. \qquad (17)
\end{aligned}$$

7 Transformation of fields: A very large sheet of charge lies in the $x - y$ plane of the frame F. The charge per unit area of this sheet is σ. In the frame F', this sheet moves to the right with speed v.

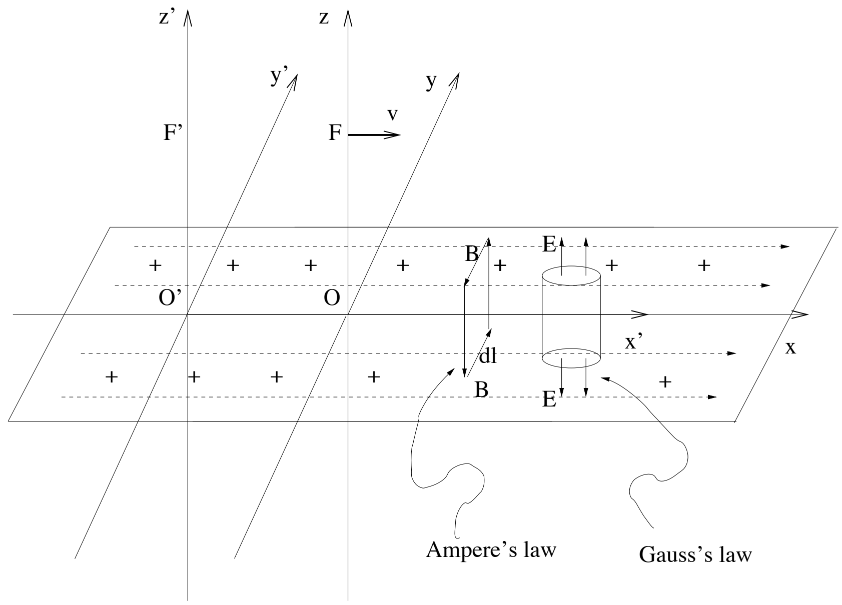


Figure 1: A sheet of charge of density σ stays at rest in frame F, and moves with velocity v along positive x-axis in frame F'. It has density $\sigma' = \gamma\sigma$ in F'.

(a) What is the electric field in the rest frame (above and below the sheet)?

Refer to figure 1. In rest frame F, we just apply Gauss's law.

$$2EA = 4\pi\sigma A, \tag{18}$$

$$E = 2\pi\sigma. \tag{19}$$

Note that the direction of $\vec{E}$ is upward in $z > 0$ and downward in $z < 0$ (if we assume the charge is positive). So

$$\vec{E} = 2\pi\sigma\ Sign(z)\hat{z}, \tag{20}$$

where $Sign(z) = 1,\ z > 0;\ -1,\ z < 0$.

(b) What is the electric field in the frame F' (above and below the sheet)?

We shall follow the same derivation as in part (a). But be careful of the charge density. If the sheet moves in positive x-axis, $dx' = dx/\gamma,\ dy' = dy$, so $\sigma' = \gamma\sigma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. So we have

$$\vec{E'} = 2\pi\sigma'\ Sign(z)\hat{z} = 2\pi\gamma\sigma\ Sign(z)\hat{z}. \tag{21}$$

(c) What is the magnetic field in the frame F' (above and below the sheet)?

The current on the sheet (due to the moving of the charge) *per unit length* is $dI/dl = \sigma' v$ in frame F', where dl is a length normal to the current flow. You can achieve this result by considering that during a time dt, some charges of an area of $vdt\,dl$ cross the line dl. Due to the symmetry, the magnetic field at $z > 0$ is along negative y-axis, and at $z < 0$ along positive y-axis, so that the circulation of the loop in Figure 1 observes right-hand rule with the direction of current flow. Apply Ampere's law:

$$\begin{aligned} 2B'dl &= \frac{4\pi}{c}\sigma' v dl, && (22) \\ B' &= \frac{2\pi}{c}\sigma' v = \frac{2\pi}{c}\gamma\sigma v. && (23) \end{aligned}$$

Considering the direction,

$$\vec{B'} = -\frac{2\pi}{c}\gamma\sigma v\, Sign(z)\hat{y}. \quad (24)$$

(d) Show that the results of (b) and (c) are consistent with the general Lorentz transformations for electric and magnetic fields, Eq. (60) of Purcell Chapter 6.

A little caution is that in eq.(60) of Purcell, F moves in negative x-axis seen from F'; so we should change $\beta \to -\beta$ in eq.(60).

$$\begin{aligned} E'_{\parallel} &= E_{\parallel} = 0; && (25) \\ \vec{E'}_{\perp} &= \vec{E'} = 2\pi\gamma\sigma\, Sign(z)\hat{z} && \\ &= \gamma\vec{E} = \gamma(\vec{E}_{\perp} - \vec{\beta}\times\vec{B}_{\perp}); && (26) \\ B'_{\parallel} &= B_{\parallel} = 0; && (27) \\ \vec{B'}_{\perp} &= \vec{B'} = -\frac{2\pi}{c}\gamma\sigma v\, Sign(z)\hat{y} && \\ &= \gamma(\vec{B}_{\perp} + \vec{\beta}\times\vec{E}_{\perp}). && (28) \end{aligned}$$

Note that $\vec{B} = 0$ and $\vec{E} = \vec{E}_{\perp}$ in frame F, $\vec{\beta} = \beta\hat{x}$, $\hat{x}\times\hat{z} = -\hat{y}$.

(a) Suppose we used $E = 3 \times 10^{20}$ eV $= 48$ J to throw a baseball of mass 0.14 kg. The rest mass energy of the baseball is $m_{bb}c^2 = 1.26 \times 10^{16}$ J. Compare to $E = 48$ J, this is huge! This means that the baseball thrown with E is highly non relativistic, the kinetic energy T of the baseball is going to be very close to $T_{NR} = \frac{1}{2}m_{bb}v^2$. All of E goes into kinetic energy of the ball, so $T_{NR} = E$, and

$$v = \left(\frac{2E}{m_{bb}}\right)^{1/2} = 26.18m/s$$

It is not at all necessary, but we could try to be fully relativistic, in which case we would set $E = (\gamma - 1)m_{bb}c^2$. We could then solve for γ, and then β, determining the velocity of the ball. This calculation, however, is difficult to carry out accurately, because γ is very close to 1: $\gamma = 1.00000000000000381$. Unless your calculator keeps more than 15 significant figures, there is no way to distinguish this number from 1. If we want to see the very small relativistic corrections, it is best to analytically solve the equation $E = (\gamma - 1)m_{bb}c^2$ for v, finding

$$v = \left(\frac{2E}{m_{bb}}\right)^{1/2} \frac{\sqrt{1 + \frac{E}{2m_{bb}c^2}}}{1 + \frac{E}{m_{bb}c^2}} \ . \tag{5.1}$$

One can then see that the fractional correction is of order

$$\frac{E}{m_{bb}c^2} \approx \frac{48 \text{ J}}{1.26 \times 10^{16} \text{ J}} \approx 4 \times 10^{-15} \ .$$

Thus, there is absolutely no need to take these corrections into account.

(b) The cosmic ray particle, in its rest frame, sees the Milky Way careening towards it. The distance across the Milky Way is Lorentz contracted. The cosmic ray particle sees a width of

$$l = \frac{l_0}{\gamma} = \frac{10^5 ly}{\gamma} \tag{5.2}$$

What is γ? The total energy of a particle is given by

$$E = \gamma mc^2$$

so

$$\gamma = \frac{E}{mc^2} \tag{5.3}$$

Again, $E = 3\times10^{20}eV$, but now it is assumed to be the total energy of the cosmic ray particle. The cosmic ray particle is a proton, of rest mass energy $E_0 = 938MeV = 9.38\times10^{8}eV << E = 3\times10^{20}eV$. The proton is highly relativistic. Doing the math gives

$$\gamma = \frac{3\times10^{20}eV}{9.38\times10^{8}eV} = 3.20\times10^{11}$$

This goes into (5.2), giving the width of the Milky Way as seen by the cosmic ray particle to be

$$l = \frac{10^5 ly}{3.20\times10^{11}} = 3.125\times10^{-7}ly = 2.96\times10^{9}m$$

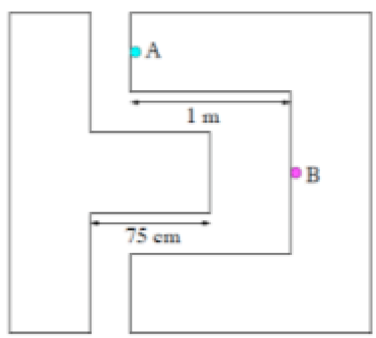
The time it takes the cosmic ray particle to cross the Milky Way, as measured in its rest frame, is

$$
\begin{aligned}
t = \frac{l}{v} &= \frac{10^5 ly}{c\beta\gamma} \\
&= \frac{3.156 \times 10^{12} s}{\beta\gamma} \\
&= \frac{3.156 \times 10^{12} s}{\left(\sqrt{1 - \frac{1}{\gamma^2}}\right)\gamma} \\
&= \frac{3.156 \times 10^{12} s}{\sqrt{\gamma^2 - 1}} \\
&= \frac{3.156 \times 10^{12} s}{\sqrt{3.20^2 \times 10^{22} - 1}} \\
&\approx \frac{3.156 \times 10^{12} s}{3.20 \times 10^{11}} \\
t &= 9.86 s
\end{aligned}
$$

9.

A U-shaped structure made of ultrastrong steel contains a detonator. The depth of the U is 1 meter in its own rest frame. If the activation switch (at point B) is hit while the detonator is armed, a tremendous explosion goes off. However, there is a disarming switch at point A. When the switch at A is hit, a signal is sent (at the speed of light) to B, disarming the detonator. If point B is hit after the disarming signal has been received, then the explosion does not happen. Bear in mind that the disarming signal cannot arrive sooner than the light travel time from A to B. If point B doesn't "know" that point A has been hit, the explosion clearly must occur.

A T-shaped structure made of the same steel fits inside the U; the long arm of the T is 75 cm in its rest frame. When both structures are at rest in our laboratory, the T cannot reach far enough to hit the activation switch; instead, the disarmament switch is pressed.



The detonator is armed. The T structure is moved far to the left and accelerated to a speed $v = \sqrt{3}c/2$. It zooms in and smashes into the U.

(a) [3 pts] Compute the length of the T's arm in the rest frame of the U. Which switch is hit first, A or B?

(b) [3 pts] Compute the depth of the U in the rest frame of the T. Which switch is hit first, A or B?

(c) [9 pts] *Does the explosion happen or not?* Clearly, it cannot happen in one rest frame and not the other! Resolve the apparent contradiction in the two viewpoints.

Hint: Bodies cannot be perfectly rigid in special relativity. Information can only travel through the body at some finite speed, the speed of sound. This speed cannot exceed that of light (and is typically far, far smaller).

When $\beta = \frac{\sqrt{3}}{2}$, $\gamma = 2$.

(a) An observer in the rest frame of U sees that the arm of T is contracted. He or she sees an arm of length $l_U = \frac{l_T}{\gamma} = \frac{.75m}{2} = .375m$. The arm is not long enough to reach the sensor at B. It hits A first, then, and the bomb does not go off.

(b) An observer in the rest frame of T sees that the depth of U is contracted. He or she sees a depth $d_T = \frac{d_U}{\gamma} = \frac{1m}{2} = .5m$. But the arm in this frame has a length of 0.75 m, so it cannot ever hit A. Therefore it hits B, and the bomb does go off.

(c) What's going on here? The hint suggests that we think about what it means for a body to be rigid.

If we smash a block of material very hard into a wall, it compresses. This compression occurs on a timescale on the order of magnitude of the length of the block divided by the speed of sound in that material. This timescale $\tau_{compress}$ tells us about how long it takes the back end of the block to "receive" the news that the front end has stopped moving. Now, the block is moving with some speed v. There is another timescale relevant to the problem, which we can denote by τ_v. This timescale is related to how long it takes the body to pass by a point – it is the length of the body divided by the speed at which it is moving.

Let's compare the two timescales. If the block is moving very slowly (slower than the speed of sound in that material), the material has time to communicate to the back end of the block (via sound waves, i.e. pressure) that the front end has hit the wall and stopped moving. In this case, $\tau_{compress} << \tau_v$, and the compression is very slight, and the block is approximately a perfect rigid body (though in reality I don't think any material is perfectly rigid – there is always some compression). However, if the block is moving very fast (faster than the speed of sound in that material), the material does not have time to communicate to the back end of the block that the front end has stopped moving. In fact, the back end will keep moving, the material compresses greatly. If the block is moving at a speed orders of magnitude larger than the speed of sound (i.e. relativistically), the material will compress so much that the back end hits the wall as well as the front end – this is what the hint means by "bodies cannot be perfectly rigid in relativity".

Let's apply this to what we did in parts (a) and (b). In both parts, we assumed that both the T and the U were perfectly rigid. However, a body can only be considered rigid when it is at rest or nearly at rest($\tau_{compress} << \tau_v \rightarrow \infty$).

In part (a), this means that the U can be treated as a rigid body, but the T can not, because it is moving faster than the speed of sound in its material. So, although the cap of the T hits sensor A and stops moving, this event cannot influence what the tip of the T is doing. There is not enough time to communicate to the tip that the cap has hit the U, so the tip keeps moving until it hits sensor B. This is another event. (Note that we are treating the tip of the T and the cap of the T as seperate objects, since they cannot communicate to each other via sound waves). The bomb will go not off if these two events are causally connected, but it will if they are not.

Suppose the cap of the T hits sensor A at $t_A = 0$ and $x_A = 0$. Sensor B is located at $x_B = 1m$. Since the arm of the T is 0.375 m in this frame, the tip of the T must travel $1m - 0.375m = 0.625m$ after the cap hits A. Therefore, the time that sensor B is hit by the tip of the arm is $t_B = \frac{0.625m}{v} =$, for $v = \frac{\sqrt{3}}{2}c = 2.5 \times 10^{-9}s$. Has the disarming signal from A arrived yet? The signal from A is sent at $t = 0$, and must travel a distance of 1m at the speed of light. This takes a time of $t_{disarm} = \frac{1m}{c} = 3.33 \times 10^{-9}s$. But $\tau_{disarm} > t_B$ means that the disarming signal does not reach point B in time to dismantly the bomb! On a spacetime diagram, event B lies outside the light cone of event A. The detonator goes off!

In part (b), it is the arm of T that is rigid, and U that is not. It is still true that the arm hits point B, but now we know that the part of U with the A sensor will keep moving towards the cap of the T because the material of U does not have time to react. We can calculate the space time positions of events A and B in the primed frame by using the Lorentz transformation. Since $t_A = x_A = 0$, $t'_A = x'_A = 0$ also. However,

$$\begin{aligned} ct'_B &= \gamma(ct_B - \beta x_B) \\ &= 2(\frac{0.625}{\beta} - \beta \times 1m) \\ &= -0.288675s \end{aligned} \tag{6.1}$$

$$\begin{aligned} x'_B &= \gamma(x_B - \beta ct_B) \\ &= 0.75 \qquad \text{as expected} \end{aligned} \tag{6.2}$$

Event B (the tip of the arm of T hitting sensor B) happens before event A (the cap of T triggering sensor A) in the rest frame of T. The detonator goes off.