Massachusetts Institute of Technology
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 11
Maxwell's equations, waves

Due: Wednesday, May 4, 10 AM IN CLASS

Problem 1: Discovery of magnetic charge

You discover magnetic charge. The units of magnetic charge density, μ, are chosen such that $\vec{\nabla} \cdot \vec{B} = 4\pi\mu$.

(a) When this magnetic charge is in motion, there is a "magnetic current density" $\vec{L} = \mu\vec{v}$. In analogy to electric charge density and electric current densities, write down the equation of continuity for magnetic charge.

(b) What do Maxwell's equations become with this new charge?

Hint: The following vector identity may be useful: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ for any $\vec{F}$.

Problem 2: Magnetic field of a moving charge

A charge q moving along the x-axis at constant speed $v \ll c$. When it is at $x = -d$, what is the magnetic field at $(x, y, z) = (0, r, 0)$?

(a) Solve this first using Biot-Savart. (Hint: the current from the moving charge isn't particularly well defined. However, B-S only needs the combination $I\,dl = (dq/dt)\,dl = dq\,(dl/dt) \simeq q_{\text{pt charge}}(dl/dt)$. Sloppy physicist calculus in action!)

(b) Now solve this using displacement current. Look at a circle of radius r centered at the origin and passing through the point $(0, r, 0)$. By symmetry, $\vec{B}$ will be constant on this circle and oriented in the tangential direction. Find a surface which has this circle as a boundary and for which $\int \vec{E} \cdot d\vec{a}$ is simple. Evaluate this flux, apply the "generalized" form of Ampere's law (integral formulation) and you're there.

Note, there's a third way: Lorentz transform from the rest frame electric field. All three answers should agree, at least in the limit $v \ll c$.

Problem 3: General questions

I. $\displaystyle\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{E}\cdot d\vec{a} = 4\pi q_{\text{enclosed}}$ II. $\displaystyle\oiint_{\substack{\text{closed}\\\text{surface}}} \vec{B}\cdot d\vec{a} = 0$ III. $\displaystyle\oint_{\substack{\text{closed}\\\text{loop}}} \vec{E}\cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt}\iint_{\substack{\text{open}\\\text{surface}}} \vec{B}\cdot d\vec{a}$

IV. $\displaystyle\oint_{\substack{\text{closed}\\\text{loop}}} \vec{B}\cdot d\vec{s} = \frac{4\pi}{c} I_{\text{enclosed}} + \frac{1}{c}\frac{d}{dt}\iint_{\substack{\text{open}\\\text{surface}}} \vec{E}\cdot d\vec{a}$

Lorentz Force Equation:

V. $\vec{F}_q = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$

Indicate the number(s) of the Maxwell equation(s) or the Lorentz Force Equation (V.) that can be used to explain the given phenomena:

(a) A coil with a sinusoidal current flowing can levitate above a conducting plate.

(b) The electric field of an isolated point charge drops off like $1/r^2$.

(c) There are no magnetic monopoles.

(d) A conducting disc falls more slowly between the poles of a magnet than does a disc which is an insulator.

(e) The lines of $\vec{B}$ never end.

(f) Iron struck by lightning often becomes magnetized.

(g) There is no magnetic equivalent of a Faraday cage.

(h) All unbalanced charge in a metal is found at the surface under static conditions.

(i) Moving a coil through a magnet generates an electric current in the coil.

(j) Radios can tune in to different frequencies.

(k) A transformer can step up or step down voltage.

P.S. You can skip explaining completely part F for now (we did not discuss magnetization yet!).

Problem 4: Purcell 9.1

If the electric field in free space is $\vec{E} = E_0(\hat{x}+\hat{y})\sin[(2\pi/\lambda)(z+ct)]$, with $E_0 = 2$ statvolts/cm, the magnetic field, not including any static magnetic field, must be what?

Problem 5: Purcell 9.5a

Here is a particular electromagnetic field in free space:

$$\begin{array}{lll} E_x = 0 & E_y = E_0 \sin(kx + \omega t) & E_z = 0 \\ B_x = 0 & B_y = 0 & B_z = -E_0 \sin(kx + \omega t) \end{array}$$

Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

Problem 6: Electromagnetic Plane Waves

Suppose that in the absence of any charges (free space) an electric field exists in the form

$$\vec{E} = E_0 \sin(kz + \omega t)\hat{\imath} + E_0 \cos(kz + \omega t)\hat{\jmath}.$$

Show that $\vec{E}$ satisfies Maxwell's equations provided that a certain magnetic field $\vec{B}(x, y, z, t)$ also exists, and a relation between ω and k is satisfied.

(a) What is the relation between ω and k?

(b) What is $\vec{B}(x, y, z, t)$?

(c) Describe what the electric and magnetic fields look like at the origin as a function of time.

Problem 7: Purcell 9.8

Show that the electromagnetic field described by

$$\begin{aligned} \vec{E} &= E_0 \hat{z} \cos kx \cos ky \cos \omega t \\ \vec{B} &= B_0(\hat{x} \cos kx \sin ky - \hat{y} \sin kx \cos ky) \sin \omega t \end{aligned}$$

will satisfy

$$\begin{array}{ll} \operatorname{curl} \vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t} & \operatorname{div} \vec{E} = 0 \\ \operatorname{curl} \vec{B} = \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t} & \operatorname{div} \vec{B} = 0 \end{array}$$

if $E_0 = \sqrt{2} B_0$ and $\omega = \sqrt{2} ck$. This field can exist inside a square metal box, of dimension π/k in the x and y directions and arbitrary height. What does the magnetic field look like?

Problem 8: Galilean Transformation of Maxwell's Wave Equation

Observers in frame F take 8.022 and derive Maxwell's wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

For simplicity and specificity, assume that $\vec{E} = E(x, t)\,\hat{y}$, and therefore, the wave equation reduces to:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

The goal of this problem is to understand what form the wave equation would have for observers in another inertial frame F' frame moving along the x axis with speed v. The Galilean transformation of coordinates between the two frames is:

$$x' = x - vt$$
$$t' = t$$

(a) Use the chain rule to show that

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}$$

(b) Use the chain rule to show that

$$\frac{\partial E}{\partial t} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'}$$

(c) Use the results of parts (a) and (b) to show that the original wave equation in F transforms to

$$\frac{\partial^2 E}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} = -\frac{2v}{c^2}\frac{\partial^2 E}{\partial x' \partial t'} + \frac{v^2}{c^2}\frac{\partial^2 E}{\partial x'^2}$$

in frame F'.

(d) Show that in frame F' a person computing the speed of waves, V, governed by the modified Maxwell wave equation, would find $V = v \pm c$. You may simply assume that the waves are of the form

$$E(x' \pm Vt')$$

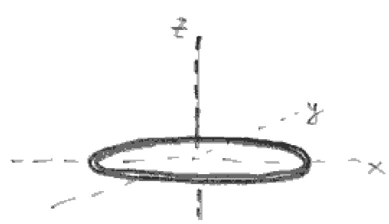
where E is an arbitrary function.

Problem 9: Optional: loop antenna — in SI units

An electromagnetic wave propagating in air has a magnetic field given by

$$B_x = 0 \qquad\qquad B_y = 0 \qquad\qquad B_z = B_0 \cos(\omega t - kx)$$

It encounters a circular loop antenna of radius a centered at the origin $(x, y, z) = (0, 0, 0)$ and lying in the $x - y$ plane. The radius of the antenna $a \ll \lambda$ where λ is the wavelength of the wave. So you can assume that at any time t the magnetic field inside the loop is approximately equal to its value at the center of the loop.



(a) What is the magnetic flux, $\Phi_{\text{mag}}(t) = \iint\limits_{\text{disk}} \vec{B} \cdot d\vec{a}$, through the plane of the loop of the antenna?

The loop has a self-inductance L and a resistance R. Faraday's law for the circuit

$$IR = -\frac{d\Phi_{\text{mag}}}{dt} - L\frac{dI}{dt}.$$

(b) Assume a solution for the current of the form $I(t) = I_0 \sin(\omega t - \phi)$ where ω is the angular frequency of the electromagnetic wave, I_0 is the amplitude of the current, and ϕ is a phase shift between the changing magnetic flux and the current. Find expressions for the constants ϕ and I_0.

(c) What is the magnetic field created at the center of the loop by this current $I(t)$?

Problem 10: Optional — Magnetic monopole: experiments

One way to search for magnetic monopoles is by monitoring the current through a highly conductive (preferably superconducting) loop. Suppose a monopole with magnetic charge s passes through a perfectly conducting circular loop with self-inductance L. The monopole has a constant speed v, perpendicular to the plane of the loop. It approaches from very far away, and then recedes to infinity. Calculate the current I that flows around the loop as a result of the monopole's passage. (Note: experiments of this type have been running for decades, and have produced a few candidate events, but there has been no unambiguous detection.)

Problem 11: The Director's Challenge — Extra credit!!!

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!