Massachusetts Institute of Technology
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 10
Maxwell's equations, waves

Due: Wednesday, May 4th 10 am IN CLASS

Problem 1: Discovery of magnetic charge

You discover magnetic charge. The units of magnetic charge density, μ, are chosen such that $\vec{\nabla} \cdot \vec{B} = 4\pi\mu$.

(a) When this magnetic charge is in motion, there is a "magnetic current density" $\vec{L} = \mu\vec{v}$. In analogy to electric charge density and electric current densities, write down the equation of continuity for magnetic charge.

(b) What do Maxwell's equations become with this new charge? Hint: this vector identity may be useful.. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ for any $\vec{F}$.

Problem 2: Magnetic field of a moving charge

A charge q moving along the x-axis at constant speed $v \ll c$. When it is at $x = -d$, what is the magnetic field at $(x, y, z) = (0, r, 0)$?

(a) Solve this first using Biot-Savart. (Hint: the current from the moving charge isn't particularly well defined. However, B-S only needs the combination $Idl = (dq/dt)dl = dq(dl/dt) \simeq q_{\text{pt charge}}(dl/dt)$. Sloppy physicist calculus in action!)

(b) Now solve this using displacement current. Look at a circle of radius r centered at the origin and passing through the point $(0, r, 0)$. By symmetry, $\vec{B}$ will be constant on this circle and oriented in the tangential direction. Find a surface which has this circle as a boundary and for which $\int \vec{E} \cdot d\vec{a}$ is simple. Evaluate this flux, apply the "generalized" form of Ampere's law (integral formulation) and you're there.

Note, there's a third way: Lorentz transform from the rest frame electric field. All three answers should agree, at least in the limit $v \ll c$.

Problem 3: General questions

I. $\displaystyle\oiint_{\substack{closed\\surface}} \vec{\mathbf{E}}\cdot d\vec{\mathbf{a}} = 4\pi q_{enclosed}$

II. $\displaystyle\oiint_{\substack{closed\\surface}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{a}} = 0$

III. $\displaystyle\oint_{\substack{closed\\loop}} \vec{\mathbf{E}}\cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt}\iint_{\substack{open\\surface}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{a}}$

IV. $\displaystyle\oint_{\substack{closed\\loop}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \frac{4\pi}{c} I_{enclosed} + \frac{1}{c}\frac{d}{dt}\iint_{\substack{open\\surface}} \vec{\mathbf{E}}\cdot d\vec{\mathbf{a}}$

Lorentz Force Equation:

V. $\vec{\mathbf{F}}_q = q\vec{\mathbf{E}} + q\dfrac{\vec{\mathbf{v}}}{c}\times\vec{\mathbf{B}}$

Figure 1: Equations

Indicate the number(s) of the Maxwell equation(s) or the Lorentz Force Equation (V.) that can be used to explain the given phenomena:

A. A coil with a sinusoidal current flowing can levitate above a conducting plate.

B. The electric field of an isolated point charge drops off like $1/r^2$.

C. There are no magnetic monopoles.

D. A conducting disc falls more slowly between the poles of a magnet than does a disc which is an insulator.

E. The lines of $\vec{B}$ never end.

F. Iron struck by lightning often becomes magnetized.

G. There is no magnetic equivalent of a Faraday cage.

H. All unbalanced charge in a metal is found at the surface under static conditions.

I. Moving a coil through a magnet generates an electric current in the coil

J. Radios can tune in to different frequencies.

K. A transformer can step up or step down voltage.

Figure 2: Maxwell's equations

Problem 4: Purcell 9.1

9.1 If the electric field in free space is $\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \sin (2\pi/\lambda)(z + ct)$, with $E_0 = 2$ statvolts/cm, the magnetic field, not including any static magnetic field, must be what?

Figure 3: Purcell 9.1

Problem 5: Purcell 9.5a

9.5 Here is a particular electromagnetic field in free space:

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$

$$B_x = 0 \qquad B_y = 0 \qquad B_z = -E_0 \sin(kx + \omega t)$$

(*a*) Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

Figure 4: Purcell 9.5

Problem 6: EM waves

Problem 7: ***Electromagnetic Plane Waves.*** Suppose that in the absence of any charges (free space) an electric field exists in the form

$$\vec{\mathbf{E}} = E_0 \sin(kz + \omega t)\,\hat{\mathbf{i}} + E_0 \cos(kz + \omega t)\,\hat{\mathbf{j}}.$$

Show that $\vec{\mathbf{E}}$ satisfies Maxwell's equations provided that a certain magnetic field $\vec{\mathbf{B}}(x, y, z, t)$ also exists, and a relation between ω and k is satisfied.

a) What is the relation between ω and k?

b) What is $\vec{\mathbf{B}}(x, y, z, t)$?

c) Describe what the electric and magnetic fields look like at the origin as a function of time.

Figure 5: Waves

Problem 7: Purcell 9.8

9.8 Show that the electromagnetic field described by

$$\mathbf{E} = E_0\hat{\mathbf{z}} \cos kx \cos ky \cos \omega t$$

$$\mathbf{B} = B_0(\hat{\mathbf{x}} \cos kx \sin ky - \hat{\mathbf{y}} \sin kx \cos ky) \sin \omega t$$

will satisfy Eqs. 16 if $E_0 = \sqrt{2}B_0$ and $\omega = \sqrt{2}ck$. This field can exist inside a square metal box, of dimension π/k in the x and y directions and arbitrary height. What does the magnetic field look like?

Figure 6: Wave in a box

Problem 8: Galilean Transformation of Maxwell's Wave Equation

Observers in frame F take 8.022 and derive Maxwell's wave equation:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

For simplicity and specificity, assume that $\vec{E} = E(x,t)\,\hat{y}$, and therefore, the wave equation reduces to:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

The goal of this problem is to understand what form the wave equation would have for observers in another inertial frame F' frame moving along the x axis with speed v. The Galilean transformation of coordinates between the two frames is:

$$x' = x - vt$$

$$t' = t$$

a. Use the chain rule to show that

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}$$

b. Use the chain rule to show that

$$\frac{\partial E}{\partial t} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'}$$

c. Use the results of parts (a) and (b) to show that the original wave equation in F transforms to

$$\frac{\partial^2 E}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} = -\frac{2v}{c^2}\frac{\partial^2 E}{\partial x'\partial t'} + \frac{v^2}{c^2}\frac{\partial^2 E}{\partial x'^2}$$

in frame F'.

d. Show that in frame F' a person computing the speed of waves, V, governed by the modified Maxwell wave equation, would find $V = v \pm c$. You may simply assume that the waves are of the form

$$E(x' \pm Vt')$$

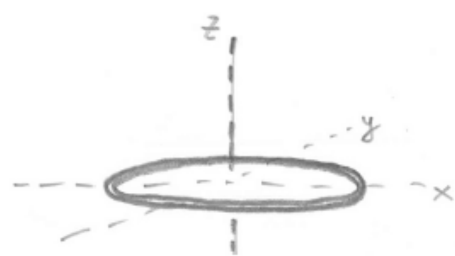
where E is an arbitrary function.

Problem 9: Optional: loop antenna

Problem 9: ***Loop Antenna.*** An electromagnetic wave propagating in air has a magnetic field given by

$$B_x = 0 \qquad B_y = 0 \qquad B_z = B_0 \cos(\omega t - kx) .$$

It encounters a circular loop antenna of radius a centered at the origin $(x, y, z) = (0,0,0)$ and lying in the x-y plane. The radius of the antenna $a << \lambda$ where λ is the wavelength of the wave. So you can assume that at any time t the magnetic field inside the loop is approximately equal to its value at the center of the loop.



a) What is the magnetic flux, $\Phi_{mag}(t) \equiv \iint_{disk} \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$, through the plane of the loop of the antenna?

The loop has a self-inductance L and a resistance R. Faraday's law for the circuit is

$$IR = -\frac{d\Phi_{mag}}{dt} - L\frac{dI}{dt} .$$

b) Assume a solution for the current of the form $I(t) = I_0 \sin(\omega t - \phi)$ where ω is the angular frequency of the electromagnetic wave, I_0 is the amplitude of the current, and ϕ is a phase shift between the changing magnetic flux and the current.. Find expressions for the constants ϕ and I_0.

c) What is the magnetic field created at the center of the loop by this current $I(t)$?

Figure 7: Antenna

Problem 10: Optional — Magnetic monopole: experiments

Magnetic monopoles: experiment. [EXTRA CREDIT, 15 bonus points] One way to search for magnetic monpoles is by monitoring the current through a highly conductive (preferably superconducting) loop. Suppose a monopole with magnetic charge s passes through a perfectly conducting circular loop with self-inductance L. The monopole has a constant speed v, perpendicular to the plane of the loop. It approaches from very far away, and then recedes to infinity. Calculate the current I that flows around the loop as a result of the monopole's passage.

(Note: experiments of this type have been running for decades, and have produced a few candidate events, but there has been no unambiguous detection.)

Figure 8: Magnetic monopole

Problem 11: The Director's Challenge — Extra credit!!!

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!