

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 2 Solutions
Gauss's law and electric potential

Due: Sunday, February 13

Problem 0: Line integrals; work; units and dimensional analysis (Optional)

Problem

A force $\vec{F} = A(y^2\hat{x} + 2x^2\hat{y})$ is acting on a particle which is initially at the origin of the (x, y) coordinate system. We transport the particle on a square path defined by the points $(0, 0)$, $(0, l)$, (l, l) , $(l, 0)$, $(0, 0)$. The constant A is positive.

- (a) Suppose we work in SI units: the coordinates (x, y) are measured in meters, so that the particle moves l meters along each leg of the path; the force is measured in Newtons. What must be the units of A ? Express in terms of kg, m, and s.
- (b) Suppose we work in cgs units: the coordinates (x, y) are measured in centimeters, and the force is measured in dynes. What must be the units of A ? Express in terms of gm, cm, and s.
- (c) How much work does the force do when the particle travels around the path? (Your answer does not depend on the choice of units: express it in terms of the constants A and l , which are assumed to have units built into them.) Is this a conservative force?
- (d) If we place a particle right at the origin, the total force is zero, so it will just stay there. Is this a stable situation? Give any argument you please (mathematical, physical, intuitive) to justify the stability (or instability) of this situation.

Solution

- (a) A must have the units $[\text{Force}]/[\text{Length}]^2$. Since Newtons are kg m/s^2 , A must be in $\text{kg}/(\text{m s}^2)$.
- (b) A must be in $\text{g}/(\text{cm s}^2)$.
- (c) We use $W = \int \vec{F} \cdot d\vec{s}$, where the integral is taken along the specified path:

$$W = \int_0^l \vec{F}(x, y=0) \cdot \hat{x} dx + \int_0^l \vec{F}(x=l, y) \cdot \hat{y} dy + \int_l^0 \vec{F}(x, y=l) \cdot \hat{x} dx + \int_l^0 \vec{F}(x=0, y) \cdot \hat{y} dy.$$

Each of these integrals is simple to evaluate:

$$\begin{aligned}\int_0^l \vec{F}(x, y=0) \cdot \hat{x} dx &= A \int_0^l (0 + 2x^2 \hat{y}) \cdot \hat{x} dx = 0, \\ \int_0^l \vec{F}(x=l, y) \cdot \hat{y} dy &= A \int_0^l (y^2 \hat{x} + 2l^2 \hat{y}) \cdot \hat{y} dy = 2Al^3, \\ \int_l^0 \vec{F}(x, y=l) \cdot \hat{x} dx &= -A \int_0^l (l^2 \hat{x} + 2x^2 \hat{y}) \cdot \hat{x} dx = -Al^3, \\ \int_l^0 \vec{F}(x, y=0) \cdot \hat{y} dy &= -A \int_0^l (y^2 \hat{x} + 0) \cdot \hat{y} dy = 0.\end{aligned}$$

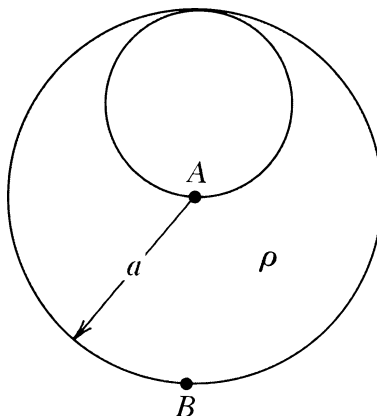
Adding, the total work done is $W = Al^3$. Since this is a closed path and the work done around this path is nonzero this is *not* a conservative force.

- (d) This is clearly *not* stable! In a stable situation, if the particle were to be displaced a small distance in any direction, there would be a *restoring force* that would tend to push it back towards the origin. In the case, when the particle is slightly displaced, the force points away from the origin. This force law instead drives a “runaway”. Any tiny perturbing force would be enough to cause the particle to fall out of its stable point.

Problem 1: Purcell 1.16

Problem

The sphere of radius a was filled with positive charge at uniform density ρ . Then a smaller sphere of radius $a/2$ was carved out, as shown in the figure, and left empty. What are the direction and magnitude of the electric field at A ? At B ?



Solution

Refer to Figure 1. The hollow sphere (our case) is equivalent to the combination of a solidly charged sphere of uniform density ρ (case 1) and a half-radius sphere of uniform density $-\rho$ (case 2). Then we can read off the electric field at point A and point B from the figure.

$$\begin{aligned}E_A &= \frac{\rho \frac{4}{3} \pi (a/2)^3}{(a/2)^2} \\ &= \frac{2}{3} \pi \rho a.\end{aligned}$$

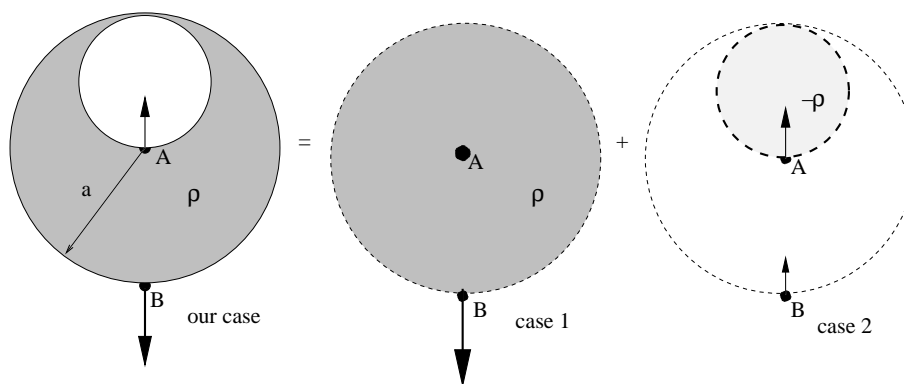


Figure 1: A hollow sphere and the equivalent “decomposition”.

The direction of \vec{E}_A is upward.

$$E_B = \frac{\rho \frac{4}{3}\pi a^3}{a^2} - \frac{\rho \frac{4}{3}\pi (a/2)^3}{(3a/2)^2}$$

$$= \frac{34}{27}\pi\rho a.$$

The direction of \vec{E}_B is downward.

Problem 2: Purcell 1.17

Problem

- A point charge q is located at the center of a cube of edge length d . What is the value of $\int \vec{E} \cdot d\vec{a}$ over one face of the cube?
- The charge q is moved to one corner of the cube. What is now the value of the flux of \vec{E} through each of the faces of the cube?

Solution

- Since the point charge q is located at the center of a cube, this problem respects the rotational symmetry. Thus the flux $\int \vec{E} \cdot d\vec{a}$ is the same for all six faces of the cube. Gauss's law therefore gives

$$\int_{\text{one face}} \vec{E} \cdot d\vec{a} = \frac{1}{6} \oint \vec{E} \cdot d\vec{a}$$

$$= \frac{1}{6} \times 4\pi q$$

$$= \frac{2}{3}\pi q.$$

- q is now at a corner of the cube. Imagine we assemble seven other identical cubes together as in fig.2(b). In the new big cube of edge length $2d$, the charge q is again located in its center. Due to the rotational symmetry, we conclude the fluxes through three faces ABCD, AEGC,

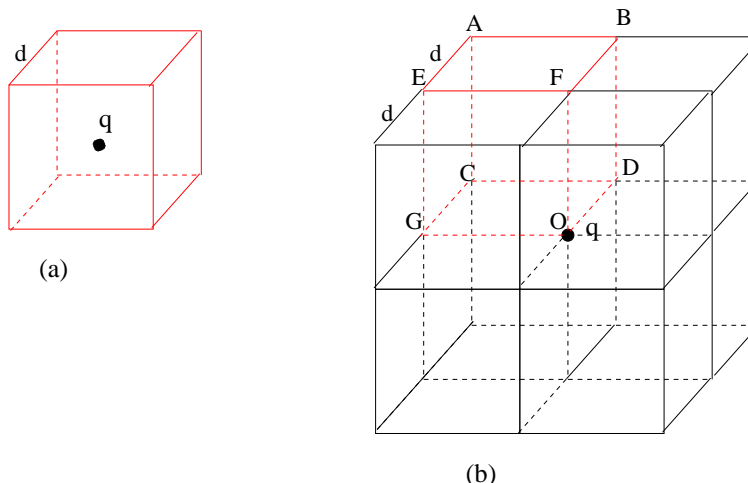


Figure 2: A point charge q at the center (a) and at a corner (b) of a cube (the upper-left-back octant of the larger cube; red in color version).

AEFB are the same and each is equal to one-quarter of the flux through one face of the big cube. Therefore

$$\begin{aligned} \int_{ABCD} \vec{E} \cdot d\vec{a} &= \int_{AEFG} = \int_{AEFB} = \frac{1}{4} \times \frac{2}{3} \pi q \\ &= \frac{1}{6} \pi q. \end{aligned}$$

The fluxes through the other faces, OFEG, OFBD, ODCD are zero since the electric field is parallel to these faces, so $\vec{E} \cdot d\vec{a} = 0$.

Problem 3: Purcell 1.31

Problem

A charged soap bubble experiences an outward electrical force on every bit of its surface. Given the total charge Q on a bubble of radius R , what is the magnitude of the resultant force tending to pull any hemispherical half of the bubble away from the other half? (Should this force divided by $2\pi R$ exceed the surface tension of the soap film interesting behavior might be expected!)

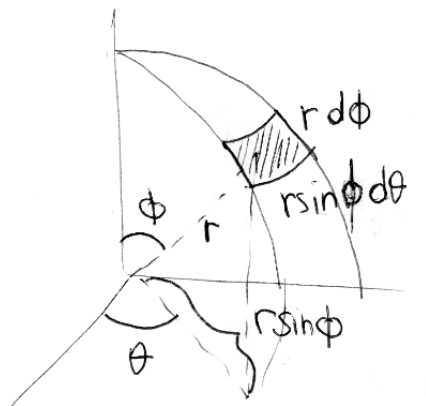
Ans. $Q^2/8R^2$.

Solution

In class, we derived that the pressure exerted by the electric field in a spherical shell is equal to

$$P = 2\pi\sigma^2 = 2\pi \left(\frac{Q}{4\pi R^2} \right)^2 = \frac{Q^2}{8\pi R^4}.$$

The area of a small area element is $R^2 \sin \theta d\theta d\phi$. (See diagram below.)



For a small area element dA , the force is given by

$$d\vec{F} = P(dA)\hat{r} = \frac{Q^2}{8\pi R^4}(R^2 \sin \theta d\theta d\phi)\hat{r}.$$

Due to the symmetry of the problem, the net force on the hemisphere is directed along \hat{z} .

$$\begin{aligned}\vec{F} &= \sum \delta\vec{F}_z \\ &= \sum \delta\vec{F} \cos \theta \hat{z} \\ &= \hat{z} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \frac{Q^2}{8\pi R^2} \\ &= 2\pi \frac{1}{2} \frac{Q^2}{8\pi R^2} \\ &= \frac{Q^2}{8R^2}\end{aligned}$$

Problem 4: Purcell 2.1

Problem

The vector function which follows represents a possible electrostatic field:

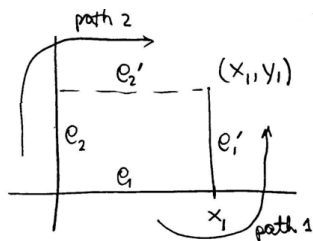
$$E_x = 6xy$$

$$E_y = 3x^2 - 3y^2$$

$$E_z = 0$$

Calculate the line integral of \vec{E} from the point $(0,0,0)$ to the point $(x_1, y_1, 0)$ along the path which runs straight from $(0,0,0)$ to $(x_1, 0, 0)$ and thence to $(x_1, y_1, 0)$. Make a similar calculation for the path which runs along the other two sides of the rectangle, via the point $(0, y_1, 0)$. You ought to get the same answer if the assertion above is true. Now you have the potential function $\phi(x, y, z)$. Take the gradient of this function and see that you get back the components of the given field.

Solution



Compute first along path 1:

$$\begin{aligned}
 \phi(x_1, y_1, 0) &= - \int_{\text{path 1}} \vec{E} \cdot d\vec{s} \\
 &= - \int_0^{x_1} E_x(x, 0, 0) dx - \int_0^{y_1} E_y(x_1, y, 0) dy \\
 &= - \int_0^{x_1} 0 dx - \int_0^{y_1} (3x_1^2 - 3y^2) dy \\
 &= -3x_1^2 y_1 + y_1^3
 \end{aligned}$$

Compute now along path 2:

$$\begin{aligned}
 \phi(x_1, y_1, 0) &= - \int_{\text{path 2}} \vec{E} \cdot d\vec{s} \\
 &= - \int_0^{y_1} E_y(0, y, 0) dy - \int_0^{x_1} E_x(x, y_1, 0) dx \\
 &= - \int_0^{y_1} 3y^2 dy - \int_0^{x_1} 6xy_1 dx \\
 &= y_1^3 - 3x_1^2 y_1
 \end{aligned}$$

The two results agree! Since $E_z = 0$, this gives us, taking the origin as the reference point,

$$\phi(x, y, z) = y^3 - 3x^2 y.$$

Checking our answer,

$$\begin{aligned}
 E_x &= -\frac{\partial \phi}{\partial x} = 6xy \\
 E_y &= -\frac{\partial \phi}{\partial y} = -3y^2 + 3x^2 \\
 E_z &= -\frac{\partial \phi}{\partial z} = 0
 \end{aligned}$$

Problem 5: Purcell 2.4

Problem

Describe the electric field that goes with the following potential:

$$\begin{aligned}
 \phi &= x^2 + y^2 + z^2 && \text{for } x^2 + y^2 + z^2 < a^2 \\
 \phi &= -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}} && \text{for } a^2 < x^2 + y^2 + z^2
 \end{aligned}$$

Discuss what happens at the boundary $(x^2 + y^2 + z^2 = a^2)$.

Solution

For $x^2 + y^2 + z^2 < a^2$, $\phi = x^2 + y^2 + z^2$,

$$\begin{aligned}\vec{E} &= -\nabla\phi \\ &= -\frac{\partial\phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial\phi}{\partial z}\hat{\mathbf{z}} \\ &= -2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}},\end{aligned}$$

or $\vec{E} = (E_x, E_y, E_z) = (-2x, -2y, -2z)$. For $x^2 + y^2 + z^2 > a^2$, $\phi = -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}}$,

$$\vec{E} = \left(\frac{2a^3x}{(x^2 + y^2 + z^2)^{3/2}} \right) \hat{\mathbf{x}} + \left(\frac{2a^3y}{(x^2 + y^2 + z^2)^{3/2}} \right) \hat{\mathbf{y}} + \left(\frac{2a^3z}{(x^2 + y^2 + z^2)^{3/2}} \right) \hat{\mathbf{z}}.$$

That's not the end of the story: we should be careful of the boundary! Take a limit of $x^2 + y^2 + z^2 \rightarrow a^2$ from both inside and outside; the fields are not the same. This indicates there are some surface charge density σ on the boundary $x^2 + y^2 + z^2 = a^2$. Let $a^- \equiv \lim_{\varepsilon \rightarrow 0, \varepsilon > 0}(a - \varepsilon)$ and $a^+ \equiv \lim_{\varepsilon \rightarrow 0, \varepsilon > 0}(a + \varepsilon)$.

$$\begin{aligned}\vec{E}_1 &= \vec{E}(x^2 + y^2 + z^2 = (a^-)^2) = -2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \\ \vec{E}_2 &= \vec{E}(x^2 + y^2 + z^2 = (a^+)^2) = 2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}} \\ |\vec{E}_2 - \vec{E}_1| &= |4x\hat{\mathbf{x}} + 4y\hat{\mathbf{y}} + 4z\hat{\mathbf{z}}| \\ &= 4\sqrt{x^2 + y^2 + z^2} = 4a \\ \sigma &= |\vec{E}_2 - \vec{E}_1|/4\pi = +a/\pi.\end{aligned}$$

Problem 6: Purcell 2.8

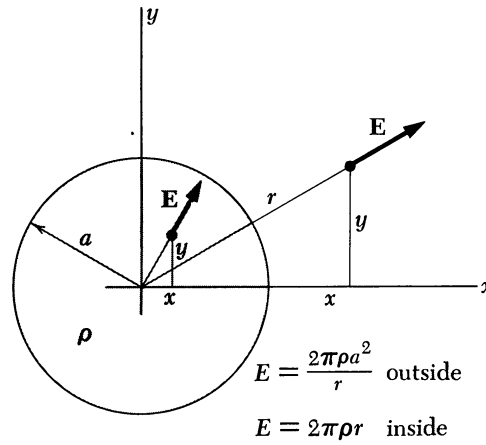
Problem

For the cylinder of uniform charge density in Fig. 2.17:

- Show that the expression there given for the field inside the cylinder follows from Gauss's law.
- Find the potential ϕ as a function of r , both inside and outside the cylinder, taking $\phi = 0$ at $r = 0$.

FIGURE 2.17

The field inside and outside a uniform cylindrical distribution of charge.



Solution

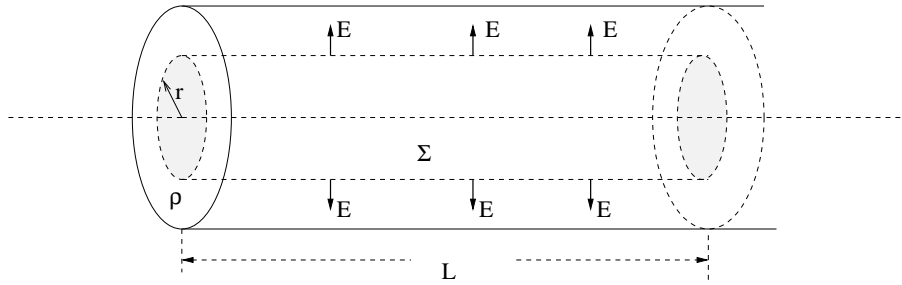


Figure 3: A cylinder inside which charges are uniformly distributed. The fields are along radial direction.

- (a) The symmetry of a cylinder contains the direction of the field \vec{E} to be radial. Refer to figure 3. We consider a cylindrical closed surface Σ whose radius is r and length is L . The flux on the two cross sections are zero. So

$$\oint \vec{E} \cdot d\vec{A} = 4\pi Q_{\text{inside}}$$

$$E \times (2\pi r L) = 4\pi \rho \pi r^2 L$$

or

$$E = 2\pi\rho r,$$

for $r < a$. Similarly, we can work out the field outside the cylinder by Gauss's law,

$$E = 2\pi\rho a^2/r,$$

for $r > a$.

(b) For $r < a$,

$$\begin{aligned}\phi(r) &= - \int_0^r \vec{E} \cdot d\vec{r} + \phi(0) \\ &= - \int_0^r 2\pi\rho r \, dr + 0 \\ &= -\pi\rho r^2.\end{aligned}$$

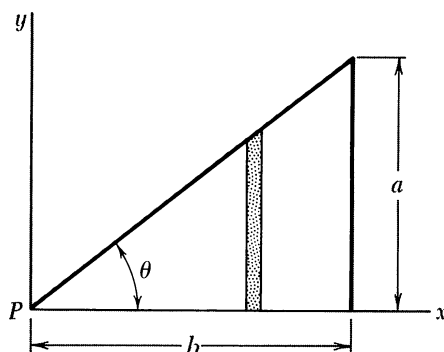
For $r > a$,

$$\begin{aligned}\phi(r) &= - \int_a^r \vec{E} \cdot d\vec{r} + \phi(a) \\ &= - \int_a^r \frac{2\pi\rho a^2}{r} dr - \pi\rho a^2 \\ &= -2\pi\rho a^2 \ln(r/a) - \pi\rho a^2.\end{aligned}$$

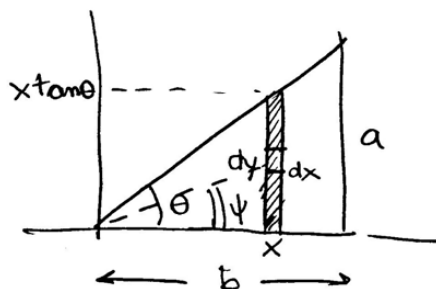
Problem 7: Purcell 2.12

Problem

The right triangle with vertex P at the origin, base b , and altitude a has a uniform density of surface charge σ . Determine the potential at the vertex P . First find the contribution of the vertical strip of width dx at x . Show that the potential at P can be written as $\phi_P = \sigma b \ln[(1 + \sin \theta)/\cos \theta]$.



Solution



To get the potential due to the little vertical strip:

$$d\phi_{\text{strip}} = \frac{\sigma dA}{r}$$

where $dA = dx dy$

$$\begin{aligned}\phi_{\text{strip}} &= \int_{y=0}^{y=x \tan \theta} \frac{(\sigma dx) dy}{\sqrt{x^2 + y^2}} \\ &= \sigma dx \int_{y=0}^{y=x \tan \theta} \frac{dy}{\sqrt{x^2 + y^2}}\end{aligned}$$

It is actually good to change variables to ψ :

Let $y = x \tan \psi$ so that $dy = \frac{x}{\cos^2 \psi} d\psi$. Then

$$\phi_{\text{strip}} = \sigma dx \int_0^\theta \left(\frac{x d\psi}{\cos^2 \psi} \right) \frac{1}{x \sqrt{1 + \tan^2 \psi}}$$

Since $1 + \tan^2 \psi = \sec^2 \psi = 1/\cos^2 \psi$,

$$\begin{aligned}&= \sigma dx \int_0^\theta \frac{d\psi}{\cos \psi} \\ &= \sigma dx \ln |\sec \theta + \tan \theta| \\ &= \sigma dx \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right|\end{aligned}$$

Now:

$$\begin{aligned}\phi_{\text{tot}} &= \int_{x=0}^{x=b} \phi_{\text{strip}} \\ &= \int_0^b \sigma dx \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right| \\ &= \sigma b \ln \left| \frac{1 + \sin \theta}{\cos \theta} \right|\end{aligned}$$

Check: As $\theta \rightarrow 0$, $\phi_{\text{tot}} \rightarrow \sigma b \ln 1 = 0$.

Problem 8: Purcell 2.30

Problem

Consider a charge distribution which has the constant density ρ everywhere inside a cube of edge b and is zero everywhere outside that cube. Letting the electric potential ϕ be zero at infinite distance from the cube of charge, denote by ϕ_0 the potential at the center of the cube and ϕ_1 the potential at a corner of the cube. Determine the ratio ϕ_0/ϕ_1 . The answer can be found with very little calculation by combining a dimensional argument with superposition. (Think about the potential at the center of a cube with the same charge density and with twice the edge length.)

Solution

Let's do dimensional analysis. The potential ϕ_0 at the center is proportional to the total charge of the cube and inversely proportional to the characteristic length b , i.e.

$$\phi_0 = c \frac{\rho b^3}{b} = c \rho b^2 \propto b^2,$$

where c is a constant depending only on the shape of the object. If we assemble eight identical cubes together, each of which has the same charge density ρ and edge length b , the potential ϕ'_0 at the *new* center is four times the previous ϕ_0 . Note that the center of the *bigger* cube is a corner of the previous small cube; so $\phi'_0 = 8\phi_1$. There we get $4\phi_0 = \phi'_0 = 8\phi_1$, or

$$\frac{\phi_0}{\phi_1} = 2.$$

Graphically, treating ϕ_0 as a function of charge distribution, this is

$$4 \cdot \phi_0 \left(\text{cube} \right) = \phi_0 \left(\text{big cube} \right) = \phi_0 \left(\text{8 small cubes} \right) = 8 \cdot \phi_0 \left(\text{corner cube} \right)$$