MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 10 Solutions
Maxwell's equations, waves

Due: Wednesday, May 4, 10 AM IN CLASS

Problem 1: Discovery of magnetic charge

Problem

You discover magnetic charge. The units of magnetic charge density, μ, are chosen such that $\vec{\nabla} \cdot \vec{B} = 4\pi\mu$.

(a) When this magnetic charge is in motion, there is a "magnetic current density" $\vec{L} = \mu\vec{v}$. In analogy to electric charge density and electric current densities, write down the equation of continuity for magnetic charge.

(b) What do Maxwell's equations become with this new charge?

Hint: The following vector identity may be useful: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ for any $\vec{F}$.

Solution

(a)

$$\frac{\partial \mu}{\partial t} + \vec{\nabla} \cdot \vec{L} = 0. \tag{1}$$

(b) Take the divergence of the $\vec{\nabla} \times \vec{E}$ equation; you'll find that the new equation of magnetic charge continuity is violated. To fix it, you must add a term that is proportional to $\vec{L}$ to Faraday's law. The resulting Maxwell equations are:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c}\vec{L} \tag{2}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \tag{3}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{4}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi\mu. \tag{5}$$

Problem 2: Magnetic field of a moving charge

Problem

A charge q moving along the x-axis at constant speed $v \ll c$. When it is at $x = -d$, what is the magnetic field at $(x, y, z) = (0, r, 0)$?

(a) Solve this first using Biot-Savart. (Hint: the current from the moving charge isn't particularly well defined. However, B-S only needs the combination $I\,dl = (dq/dt)\,dl = dq\,(dl/dt) \simeq q_{\text{pt charge}}(dl/dt)$. Sloppy physicist calculus in action!)

(b) Now solve this using displacement current. Look at a circle of radius r centered at the origin and passing through the point $(0, r, 0)$. By symmetry, $\vec{B}$ will be constant on this circle and oriented in the tangential direction. Find a surface which has this circle as a boundary and for which $\int \vec{E} \cdot d\vec{a}$ is simple. Evaluate this flux, apply the "generalized" form of Ampere's law (integral formulation) and you're there.

Note, there's a third way: Lorentz transform from the rest frame electric field. All three answers should agree, at least in the limit $v \ll c$.

Solution

(a) Solve this first using Biot-Savart.

For low speed $v \ll c$, we can ignore the relativistic effects. To apply B-S law, we simply treat the moving charge at $x = -d$ as an element current at the same location and $I d\vec{l} = qv\hat{x}$. Then we have

$$\vec{B} = \frac{qv\hat{x} \times \hat{r}_1}{cr_1^2}, \tag{6}$$

where $\vec{r}_1$ is the position vector from the moving charge to the test point $(0, r, 0)$. Using $r_1 = \sqrt{r^2 + d^2}$ and $\hat{x} \times \hat{r}_1 = \hat{z} \sin\theta_1 = \hat{z} r/r_1$, we find

$$\vec{B} = \hat{z} \frac{qvr}{c(r^2 + d^2)^{3/2}}. \tag{7}$$

(b) Now solve this using displacement current.

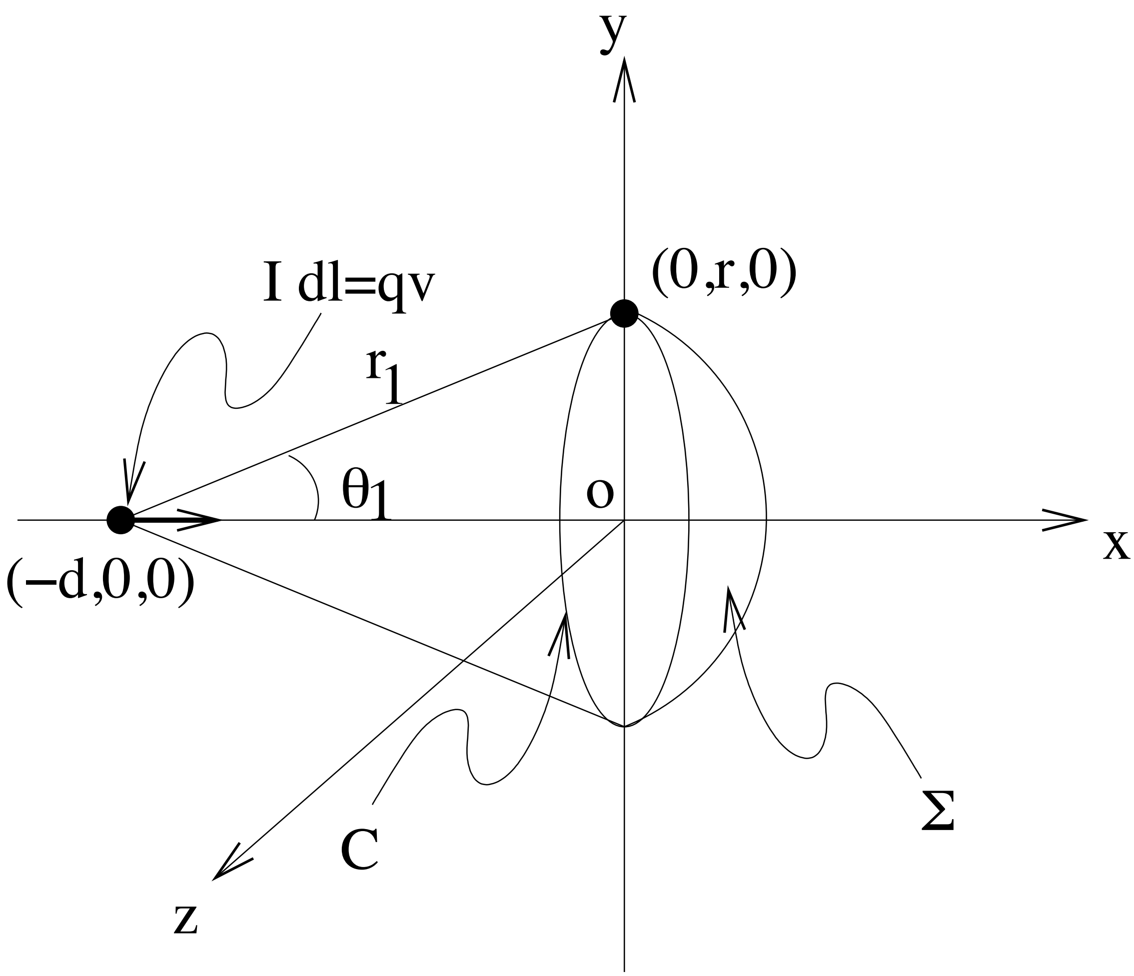


Figure 1: Calculation of magnetic field of a moving charge by "generalized" Ampere's law.

Consider a surface Σ whose boundary is the circle C centered at the origin and in Y-Z plane, and all points on Σ have the same distance r_1 to the moving charge at $(-d, 0, 0)$. Apply Stoke's Theorem,

$$\int_\Sigma \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \int_C \vec{B} \cdot d\vec{l} \tag{8}$$

while
$$\int_\Sigma \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \frac{1}{c} \int_\Sigma \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$
$$= \frac{1}{c} \frac{\partial}{\partial t} \int_\Sigma \vec{E} \cdot d\vec{a} \tag{9}$$

so
$$\int_C \vec{B} \cdot d\vec{l} = \frac{1}{c} \frac{\partial}{\partial t} \int_\Sigma \vec{E} \cdot d\vec{a}. \tag{10}$$

On the surface Σ, $E = q/r_1^2$ in radial direction (seen from the moving charge). So

$$\begin{aligned}
\int \vec{E} \cdot d\vec{a} &= \frac{q}{r_1^2} \int da \\
&= \frac{q}{r_1^2} r_1^2 \int_0^{\theta_1} \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= 2\pi q \left[\cos(0) - \cos(\theta_1)\right] \\
&= 2\pi q \Big(1 - \frac{d}{\sqrt{d^2 + r^2}}\Big). \qquad (11)
\end{aligned}$$

Then, since $d(-d)/dt = v$,

$$\frac{1}{c}\frac{\partial}{\partial t}\int_\Sigma \vec{E} \cdot d\vec{a} = \frac{2\pi q v}{c}\frac{r^2}{(d^2 + r^2)^{3/2}} \qquad (12)$$

$$\int_C \vec{B} \cdot d\vec{l} = 2\pi r B \qquad (13)$$

$$B = \frac{qvr}{c(r^2 + d^2)^{3/2}}. \qquad (14)$$

Problem 3: General questions

Problem

I. $\displaystyle \underset{\substack{\text{closed}\\\text{surface}}}{\oiint} \vec{E} \cdot d\vec{a} = 4\pi q_{\text{enclosed}}$ II. $\displaystyle \underset{\substack{\text{closed}\\\text{surface}}}{\oiint} \vec{B} \cdot d\vec{a} = 0$ III. $\displaystyle \underset{\substack{\text{closed}\\\text{loop}}}{\oint} \vec{E} \cdot d\vec{s} = -\frac{1}{c}\frac{d}{dt} \underset{\substack{\text{open}\\\text{surface}}}{\iint} \vec{B} \cdot d\vec{a}$

IV. $\displaystyle \underset{\substack{\text{closed}\\\text{loop}}}{\oint} \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{\text{enclosed}} + \frac{1}{c}\frac{d}{dt} \underset{\substack{\text{open}\\\text{surface}}}{\iint} \vec{E} \cdot d\vec{a}$

Lorentz Force Equation:

V. $\vec{F}_q = q\vec{E} + q\dfrac{\vec{v}}{c} \times \vec{B}$

Indicate the number(s) of the Maxwell equation(s) or the Lorentz Force Equation (V.) that can be used to explain the given phenomena:

(a) A coil with a sinusoidal current flowing can levitate above a conducting plate.

(b) The electric field of an isolated point charge drops off like $1/r^2$.

(c) There are no magnetic monopoles.

(d) A conducting disc falls more slowly between the poles of a magnet than does a disc which is an insulator.

(e) The lines of $\vec{B}$ never end.

(f) Iron struck by lightning often becomes magnetized.

(g) There is no magnetic equivalent of a Faraday cage.

(h) All unbalanced charge in a metal is found at the surface under static conditions.

(i) Moving a coil through a magnet generates an electric current in the coil.

(j) Radios can tune in to different frequencies.

(k) A transformer can step up or step down voltage.

P.S. You can skip explaining completely part F for now (we did not discuss magnetization yet!).

Solution

A. A coil with a sinusoidal current flowing can levitate above a conducting plate.

III. and V. Faraday's Law implies that the changing magnetic field from the coil will induce Eddy currents in the conducting plane. Due to a slight phase shift from the resistance of the plane, the Lorentz force between the induced current in the plane and the coil will always be repulsive.

B. The electric field of an isolated point charge drops off like $1/r^2$.

I. Gauss's Law only works for inverse square fields

C. There are no magnetic monopoles.

II. No magnetic monopoles means that magnetic fields lines have no objects in which they begin and end on i.e. there are no magnetic sources or sinks, hence magnetic flux through any closed surface is zero.

D. A conducting disc falls more slowly between the poles of a magnet than does a disc which is an insulator.

III. and V. Since the disc is falling through a non-uniform magnetic field, eddy currents will be induced in the conducting disc and no eddy currents in the insulated disc. Hence the Lorentz force on the induced current in the disc will slow its motion.

E. The lines of $\vec{B}$ never end.

II. Similar argument to no magnetic monopoles.

F. Iron struck by lightning often becomes magnetized.

I. , III, IV, V. Lightning involves the ionization of air due to the strong electric field (Gauss' Law) generated by electric sources in the ground and the clouds. When iron is struck by lightning, a current flows through the iron due to the conductivity of iron. There is a potential difference (Faraday's law for slowly varying fields) hence an electric field. Thus there is an electric force on the charges (Lorentz force Law) and a current. The current creates a magnetic field in the iron (Ampere's Law), which torques the iron magnetic moments to line up (Lorentz Force Law) , hence increasing the magnetic field (Ampere's Law).

G. There is no magnetic equivalent of a Faraday cage.

II. No magnetic monopoles that are free to move to shield the B field.

H. All unbalanced charge in a metal is found at the surface under static conditions.

V, I. Definition of static equilibrium implies that no electric force (Lorentz force law) exists inside the conductor. Therefore the electric field in the conductor is zero. Gauss's Law implies that the conducting charges move to the boundaries in order to cancel the electric field in the interior of the conductor.

Moving a coil through a magnet generates an electric current in the coil

III. When a coil is moved through a non-uniform magnetic field generated by a magnet, there is changing magnetic flux through the coil. Faradays's law implies that an induced electric field drives the current in the coil when the magnetic flux through the coil is changing.

I. Radios can tune in to different frequencies.

III. and I-IV. A radio circuit is a driven RLC circuit that we can solve for resonance using Faraday's Law. We tune to this resonance by adjusting the capacitance. The electromagnetic waves associated with the signal involves all of Maxwell's Eqs. The signal is picked up by an antenna that using the electric force (Lorentz force law) from the electric field of the signal acting on the charges in the antenna

J. A transformer can step up or step down voltage.

III. IV Transformers involve calculating the electromotive force (Faraday's Law) produced by changing the magnetic flux through a secondary due to the magnetic flux generated by the current in the primary. Ampere's Law states that a current in the primary produces a magnetic field.

Problem 4: Purcell 9.1

Problem

If the electric field in free space is $\vec{E} = E_0(\hat{x} + \hat{y})\sin[(2\pi/\lambda)(z + ct)]$, with $E_0 = 2$ statvolts/cm, the magnetic field, not including any static magnetic field, must be what?

Solution

We are given the electric field:

$$\vec{E} = E_0(\hat{x} + \hat{y}) \sin\left(\frac{2\pi}{\lambda}(z + ct)\right)$$

The corresponding magnetic field must satisfy Maxwell's equations. Using Faraday's Law, we find:

$$\vec{\nabla} \times \vec{E} = E_0(-\hat{x} + \hat{y}) \left(\frac{2\pi}{\lambda} \cos\left(\frac{2\pi}{\lambda}(z + ct)\right)\right) = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \longrightarrow \vec{B} = E_0(\hat{x} - \hat{y}) \sin\left(\frac{2\pi}{\lambda}(z + ct)\right)$$

where we have dropped a constant of integration (static magnetic field).

Problem 5: Purcell 9.5a

Problem

Here is a particular electromagnetic field in free space:

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$
$$B_x = 0 \qquad B_y = 0 \qquad B_z = -E_0 \sin(kx + \omega t)$$

Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

Solution

Just to be complete, let's test all four of Maxwell's equations on this wave.

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 + 0 + 0 = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 + 0 + 0 = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_y}{\partial x}\hat{z} = 0 + E_0 k \cos(kx + \omega t)\hat{z} \qquad -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = E_0 \frac{\omega}{c} \cos(kx + \omega t)\hat{z} \quad \rightarrow \quad \omega = ck$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial B_z}{\partial y}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y} = 0 + E_0 k \cos(kx + \omega t)\hat{z} \qquad \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = E_0 \frac{\omega}{c} \cos(kx + \omega t)\hat{z} \quad \rightarrow \quad \omega = ck$$

Figure 2: Solution Purcell 9.5a

Problem 6: Electromagnetic Plane Waves

Problem

Suppose that in the absence of any charges (free space) an electric field exists in the form

$$\vec{E} = E_0 \sin(kz + \omega t)\hat{\imath} + E_0 \cos(kz + \omega t)\hat{\jmath}.$$

Show that $\vec{E}$ satisfies Maxwell's equations provided that a certain magnetic field $\vec{B}(x, y, z, t)$ also exists, and a relation between ω and k is satisfied.

(a) What is the relation between ω and k?

(b) What is $\vec{B}(x, y, z, t)$?

(c) Describe what the electric and magnetic fields look like at the origin as a function of time.

Solution

a) $\dfrac{\partial E_x}{\partial z^2} = \dfrac{1}{c^2}\dfrac{\partial^2 E_x}{\partial t^2} \Rightarrow -k^2 E_x = -\dfrac{\omega^2}{c^2} E_x \Rightarrow$

$\underline{\omega = ck}$

b) $\vec{B} = -\int (\vec{\nabla} \times \vec{E})\, dt$

$$\vec{B} = -\int \frac{\partial}{\partial z} E_x \, dt \, \hat{j} - \int \left(-\frac{\partial E_y}{\partial z}\right) dt \, \hat{i}$$

$$= -\int k E_0 \cos(kz + \omega t)\, dt \, \hat{j} + \int - E_0 k \sin(kz + \omega t)\, \hat{i}$$

$$= -\frac{k E_0}{\omega} \sin(kz + \omega t)\, \hat{j} + \frac{E_0 k}{\omega} \cos(kz + \omega t)\, \hat{i}$$

$$= -\frac{E_0}{c} \sin(kz + \omega t)\, \hat{j} + \frac{E_0}{c} \cos(kz + \omega t)\, \hat{i}$$

c) at $z = 0$: $\vec{E} = E_0 \sin \omega t \, \hat{i} + E_0 \cos \omega t \, \hat{j}$

$\vec{B} = -\dfrac{E_0}{c} \sin \omega t \, \hat{j} + \dfrac{E_0}{c} \cos \omega t \, \hat{i}$

Figure 3: Waves

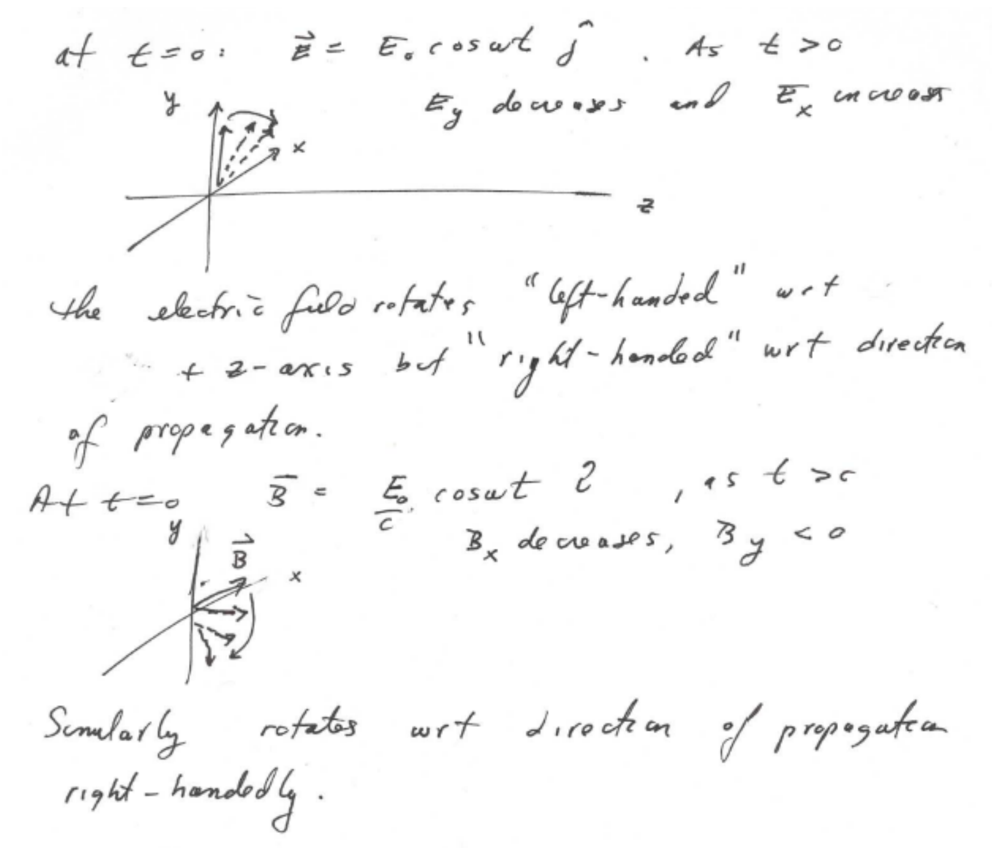


Figure 4: Waves

Problem 7: Purcell 9.8

Problem

Show that the electromagnetic field described by

$$\vec{E} = E_0 \hat{z} \cos kx \cos ky \cos \omega t$$
$$\vec{B} = B_0 (\hat{x} \cos kx \sin ky - \hat{y} \sin kx \cos ky) \sin \omega t$$

will satisfy

$$\operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \div \vec{E} = 0$$
$$\operatorname{curl} \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \qquad \div \vec{B} = 0$$

if $E_0 = \sqrt{2} B_0$ and $\omega = \sqrt{2} ck$. This field can exist inside a square metal box, of dimension π/k in the x and y directions and arbitrary height. What does the magnetic field look like?

Solution

For the EM field

$$\vec{E} = \hat{z}E_0 \cos kx \cos ky \cos \omega t \tag{15}$$
$$\vec{B} = B_0(\hat{x} \cos kx \sin ky - \hat{y} \sin kx \cos ky) \sin \omega t, \tag{16}$$

the two divergencelessness equations of Purcell eqs.(16)

$$\vec{\nabla} \cdot \vec{E} = 0\,;$$

$$\vec{\nabla} \cdot \vec{B} = 0\,,$$

can be easily verified. The other two equations give

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = (\frac{\omega B_0}{c} - E_0 k)(\hat{x} \cos kx \sin ky - \hat{y} \sin kx \cos ky) \cos \omega t = 0$$
$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = (\frac{E_0 \omega}{c} - 2B_0 k)\hat{z} \cos kx \cos ky \sin \omega t = 0$$
$$or \qquad E_0 = \frac{\omega}{kc}B_0 \tag{17}$$
$$E_0 = 2\frac{kc}{\omega}B_0. \tag{18}$$

So the condition is that

$$\omega = \sqrt{2}ck \tag{19}$$
$$E_0 = \sqrt{2}B_0. \tag{20}$$

The magnetic field inside a square metal box of size π/k is shown in Figure 5. Note that the box must run from $-\pi/(2k)$ to $\pi/(2k)$ in both x and y in order to satisfy the boundary condition that $E = 0$ on the box (which follows from the fact that electric fields vanish on conductors).

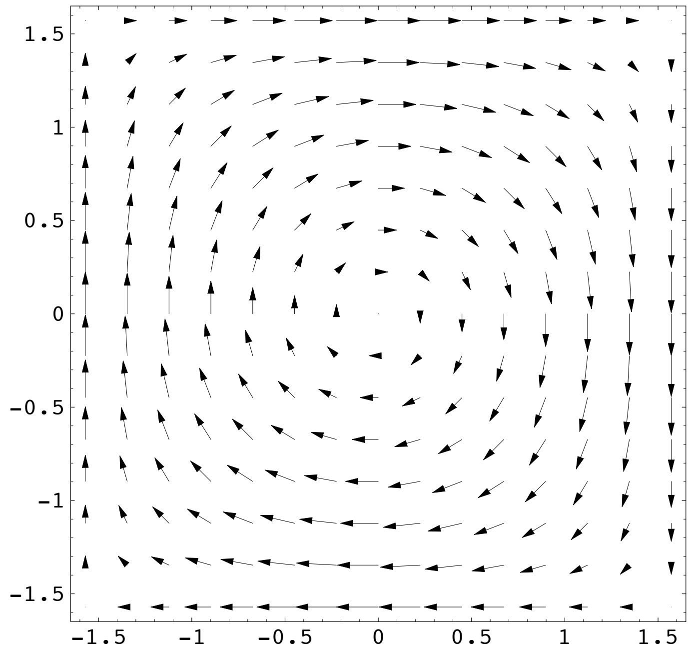


Figure 5: The magnetic field inside a squre metal box at $t = \pi/2\omega$. The horizontal axis is kx and vertical axis ky.

Problem 8: Galilean Transformation of Maxwell's Wave Equation

Problem

Observers in frame F take 8.022 and derive Maxwell's wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

For simplicity and specificity, assume that $\vec{E} = E(x,t)\,\hat{y}$, and therefore, the wave equation reduces to:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

The goal of this problem is to understand what form the wave equation would have for observers in another inertial frame F' frame moving along the x axis with speed v. The Galilean transformation of coordinates between the two frames is:

$$x' = x - vt$$
$$t' = t$$

(a) Use the chain rule to show that

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}$$

(b) Use the chain rule to show that

$$\frac{\partial E}{\partial t} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'}$$

(c) Use the results of parts (a) and (b) to show that the original wave equation in F transforms to

$$\frac{\partial^2 E}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} = -\frac{2v}{c^2}\frac{\partial^2 E}{\partial x'\partial t'} + \frac{v^2}{c^2}\frac{\partial^2 E}{\partial x'^2}$$

in frame F'.

(d) Show that in frame F' a person computing the speed of waves, V, governed by the modified Maxwell wave equation, would find $V = v \pm c$. You may simply assume that the waves are of the form

$$E(x' \pm Vt')$$

where E is an arbitrary function.

Solution

(a) Using chain rule we get,

$$\frac{\partial E}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial E}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial E}{\partial t'}$$

Note that

$$\frac{\partial x'}{\partial x} = 1 \quad \text{and} \quad \frac{\partial t'}{\partial x} = 0$$

Hence,

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}$$

(b) In this case

$$\frac{\partial E}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial E}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial E}{\partial t'}$$

We also know that

$$\frac{\partial x'}{\partial t} = -v \quad \text{and} \quad \frac{\partial t'}{\partial t} = 1$$

Therefore,

$$\frac{\partial E}{\partial t} = -v\frac{\partial E}{\partial x'} + \frac{\partial E}{\partial t'}$$

(c) Using the result of part (a) we get,

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial E}{\partial x'}) = \frac{\partial}{\partial x'}(\frac{\partial E}{\partial x'})$$

Similarly replacing in the result of part (b) E by $\partial E/\partial t$ we get

$$\frac{\partial^2 E}{\partial t^2} = (-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'})(\frac{\partial E}{\partial t})$$

Using the result of part (b) again we get (after some algebra)

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} - \frac{2v}{c^2}\frac{\partial^2 E}{\partial x'\partial t'} + \frac{v^2}{c^2}\frac{\partial^2 E}{\partial x'^2}$$

Hence, the wave equation in fame F' is given by,

$$\frac{\partial^2 E}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2} = -\frac{2v}{c^2}\frac{\partial^2 E}{\partial x'\partial t'} + \frac{v^2}{c^2}\frac{\partial^2 E}{\partial x'^2}$$

(d) Substituting the given ansatz into the wave equation of F' we get,

$$(1 - \frac{V^2}{c^2}) = \mp\frac{2vV}{c^2} - \frac{v^2}{c^2}$$

Hence, the assumed anastz is a solution if

$$1 = \frac{1}{c^2}(V \mp v)^2$$

This implies the speed of the wave in frame F' is $c \pm v$. It is known from experiments that the speed of light in vacuum is frame independent. But just now we had shown that EM waves in frame F' (according to Galilean relativity) is $c \pm v$. Hence, it is clear that "Galilean/Newtonian relativity" is not consistent with Maxwell's equation. This is why we need Einstein's theory of special relativity.

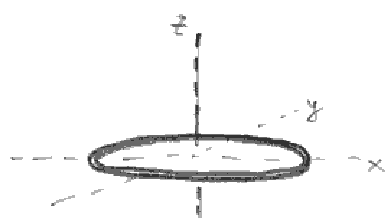
Problem 9: Optional: loop antenna — in SI units

Problem

An electromagnetic wave propagating in air has a magnetic field given by

$$B_x = 0 \qquad\qquad B_y = 0 \qquad\qquad B_z = B_0 \cos(\omega t - kx)$$

It encounters a circular loop antenna of radius a centered at the origin $(x, y, z) = (0, 0, 0)$ and lying in the $x - y$ plane. The radius of the antenna $a \ll \lambda$ where λ is the wavelength of the wave. So you can assume that at any time t the magnetic field inside the loop is approximately equal to its value at the center of the loop.



(a) What is the magnetic flux, $\Phi_{\text{mag}}(t) = \iint\limits_{\text{disk}} \vec{B} \cdot d\vec{a}$, through the plane of the loop of the antenna?

The loop has a self-inductance L and a resistance R. Faraday's law for the circuit

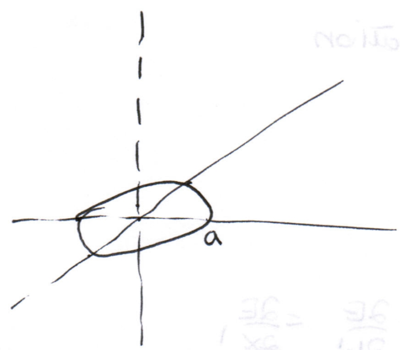
$$IR = -\frac{d\Phi_{\text{mag}}}{dt} - L\frac{dI}{dt}.$$

(b) Assume a solution for the current of the form $I(t) = I_0 \sin(\omega t - \phi)$ where ω is the angular frequency of the electromagnetic wave, I_0 is the amplitude of the current, and ϕ is a phase shift between the changing magnetic flux and the current. Find expressions for the constants ϕ and I_0.

(c) What is the magnetic field created at the center of the loop by this current $I(t)$?

Solution

9



$B_x = 0$

$B_y = 0$

$B_z = B_0 \cos(\omega t - kx)$

$a << \lambda$

a) $\phi_B = \int_S \vec{B} \cdot d\vec{a} \simeq B_0 \cos(\omega t - kx)\pi a^2$

b) $$IR = -\frac{d\phi_B}{dt} - L\frac{dI}{dt}$$

$$-B_0 \omega \sin(\omega t - kx)\pi a^2 = -IR - L\frac{dI}{dt}$$

$I(t) = I_0 \sin(\omega t - \phi) \quad \rightarrow \tilde{I} = I_0 e^{i\omega t} e^{-i\phi}$

$V(t) \equiv \underbrace{B_0 \omega \pi a^2}_{} \sin(\omega t - kx)$

$= V_0 \sin(\omega t - kx) \simeq V_0 \sin(\omega t) \rightarrow \tilde{V} = V_0 e^{i\omega t}$

$$V_0 \sin(\omega t) = IR + L\frac{dI}{dt}$$

Complex

$$\tilde{V} = \tilde{I}\underbrace{(R + i\omega L)} \qquad I_0 = \frac{V_0}{|Z_{TOT}|} = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}}$$

$V_0 e^{i\omega t} = I_0 e^{i\omega t} \boxed{Z_{TOT}} Z_{TOT} e^{-i\phi}$

$= I_0 e^{i\omega t} \boxed{|Z_{TOT}| e^{i\delta}} e^{-i\phi}$

$\delta = \phi = \tan^{-1}\frac{\omega L}{R} \rightarrow I = \frac{B_0 \omega \pi a^2}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t - \phi)$

$\doteq \frac{Im(z)}{Re(z)}$

c) $$\vec{B}_{center} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{a\, d\theta\, \hat{\theta}\, (-a\hat{r})}{a^3} = \frac{\mu_0 I \cancel{\hat{z}}}{2a}\ \frac{\mu_0}{2a} I_0 \sin(\omega t - \phi)\hat{z}$$

Figure 6: Antenna

Problem 10: Optional — Magnetic monopole: experiments

Problem

One way to search for magnetic monopoles is by monitoring the current through a highly conductive (preferably superconducting) loop. Suppose a monopole with magnetic charge s passes through a perfectly conducting circular loop with self-inductance L. The monopole has a constant speed v, perpendicular to the plane of the loop. It approaches from very far away, and then recedes to infinity. Calculate the current I that flows around the loop as a result of the monopole's passage.

(Note: experiments of this type have been running for decades, and have produced a few candidate events, but there has been no unambiguous detection.)

Solution

We want to calculate the rate of change of the flux passing through a superconducting loop. First, let's calculate the flux of a monopole sitting a distance z below the loop along the axis of the loop. We will orient the loop so that the surface element $d\vec{a}$ is pointing in the positive z direction. The expression for this flux is then:

$$\Phi = \int d\vec{a} \cdot \vec{B} = \int_0^{2\pi} d\phi \int_0^R dr \frac{s}{r^2 + z^2} \hat{r}' \cdot \hat{z} = \int_0^{2\pi} d\phi \int_0^R dr \frac{s}{r^2 + z^2} \frac{z}{\sqrt{r^2 + z^2}}$$

where s is the magnetic charge and $\hat{r}'$ is the radial vector from the monopole. We have take θ to be the angle between the z-axis and the vector $\vec{r}'$. Keep in mind that $\vec{r}'$ points from the monopole to a point on the superconducting ring. Evaluating the integral, we find:

$$\Phi = 2\pi s z \left(-\frac{1}{\sqrt{R^2 + z^2}} + \frac{1}{z} \right) = 2\pi s \left(1 - \frac{vt}{\sqrt{R^2 + (vt)^2}} \right)$$

Figure 7: Magnetic monopole

where $z = vt$. A superconductor (no resistance) will actually produce a current that will exactly cancel the flux passing through the loop. The electro-motive force generated is from the self inductance:

$$\varepsilon = L \frac{dI}{dt}$$

We also know that the induced electro-motive force is related to the time derivative of the flux.

$$\varepsilon = -\frac{1}{c} \frac{\partial \Phi}{\partial t}$$

Comparing the two expressions and integrating, we find that the current is proportional to the flux up to a constant:

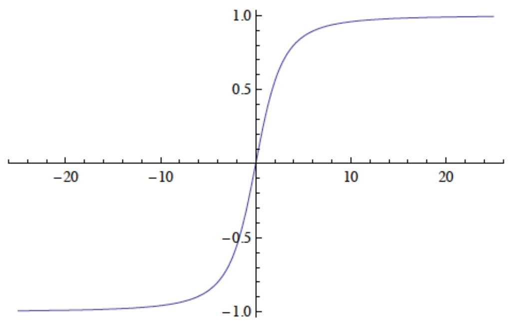
$$I(t) = -\frac{1}{Lc} \Phi(t) + C = -\frac{1}{Lc} 2\pi s \left(1 - \frac{vt}{\sqrt{R^2 + (vt)^2}} \right) + C$$

Therefore, the change in current before and after a monopole passes through the loop is:

$$I(t \to \infty) - I(t \to -\infty) = \frac{4\pi s}{Lc}$$

Figure 8: Magnetic monopole

Lc



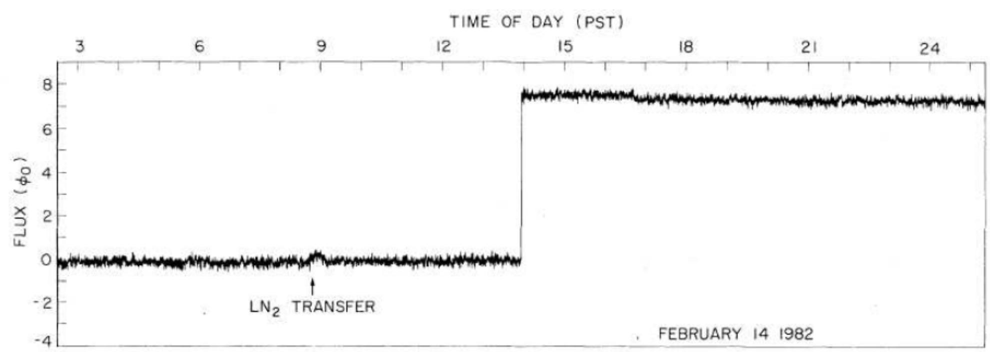
The plot to the left shows the behavior of the current as a function of time (or position, since position is proportional to time). The The plot is made with parameters:

$$\frac{2\pi s}{Lc} = 1 \quad v = 1 \quad R = 3$$

Notice how the most of the current is generated within a distance R of the loop.

An actual experimental candidate for monopole detection in this setup was found about 25 years ago. Please see Cabrera, Blas. *First Results from a Superconductive Detector for Moving Monopoles*, Phys. Rev. Lett. (**48**) 1378 (1982) for more information.

Figure 9: Magnetic monopole



The jump in flux corresponds especially well with the expected value of the magnetic monopole charge from a fairly simple argument made by P.A.M. Dirac. "LN_2 TRANSFER" denotes the time at which liquid nitrogen was added to the container of the superconducting ring to keep it cool (and superconductive). However, no such signal was detected again, leaving us with no firm evidence for the existence of monopoles.

Figure 10: Magnetic monopole

Problem 11: The Director's Challenge — Extra credit!!!

Problem

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!