MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — September 9, 2009

Prof. Alan Guth

Lecture Notes 1

THE DOPPLER EFFECT AND SPECIAL RELATIVITY

REQUIRED READING FOR 2009: You are required to read only the following sections from Lecture Notes 1: *Introduction, The Nonrelativistic Doppler Shift, The Doppler Shift for Light Waves, Summary of Special Relativity* (pp. 13–15), and *The Relativistic Doppler Shift* (pp. 17–18). At the beginning of each jump I have added a short paragraph, labeled as *Reading Guide for 2009*, which is intended to help you navigate your thoughts over the skipped sections. You are not required to read any of Lecture Notes 2.

INTRODUCTION:

Probably the centerpiece of modern cosmology is Hubble's law, first proposed by Edwin Hubble in 1929. Hubble's law states that all the distant galaxies are receding from us, with a recession velocity given by

$$v = Hr\ . \tag{1.1}$$

Here

$$\begin{aligned} v &\equiv \text{recession velocity,} \\ H &\equiv \text{Hubble's constant,} \\ \text{and } r &\equiv \text{distance to galaxy.} \end{aligned}$$

Later we will begin to talk about the implications of Hubble's law for cosmology, but for now I just want to discuss how the two ingredients— velocities and distances— are measured. I will consider first the measurement of the velocities, which is done by means of the Doppler shift. I will use this as an excuse to launch a full-scale discussion of special relativity, which I would like to do in any case. Special relativity is fun, and it will also be useful later in the course. Our study of cosmological models will lead to a brief introduction to general relativity, so it will be useful to have a solid foundation in special relativity to use as a starting point. The other ingredient in Hubble's law, the cosmic distance ladder, is described in Chapter 2 of Weinberg's **The First Three Minutes**, in Chapter 3 of Rowan-Robinson's **Cosmology**, and in Sec. 7.4 of Ryden's **Introduction to Cosmology**, and will not be discussed in these notes. (You are expected, however, to learn about it from the reading assignments.)

THE NONRELATIVISTIC DOPPLER SHIFT:

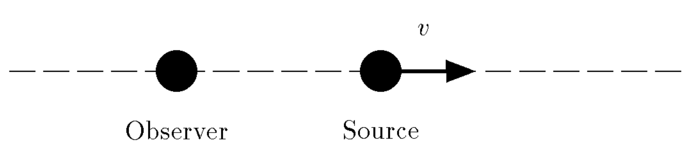
It is a well-known fact that atoms emit and absorb radiation only at certain fixed wavelengths (or equivalently, at certain fixed frequencies). This fact was not understood until the development of quantum theory in the 1920's, but it was known considerably earlier. In 1814-15 the Munich optician Joseph Frauenhofer allowed sunlight to pass through a slit and then a glass prism, and noticed that the spectrum which was formed contained a pattern of hundreds of dark lines, which were always found at the same colors. Today we attribute these dark lines to the selective absorption by the cooler atoms in the atmosphere of the sun. In 1868 Sir William Huggins noticed that a very similar pattern of lines could be seen in the spectra of some bright stars, but that the lines were displaced from their usual positions by a small amount. He realized that this shift was presumably caused by the Doppler effect, and used it as a measurement of the velocity of these distant stars.

As long as the velocities of the stars in question are small compared to that of light, it is sufficient to use a nonrelativistic analysis. We will begin with the nonrelativistic case, which we will then use as a stepping stone in the development of special relativity. To keep the language manifestly nonrelativistic for now, let us consider first the Doppler shift of sound waves. Suppose for now that the source is moving and the observer is standing still, with all motion taking place along a line. We will let

$$u \equiv \text{velocity of sound waves,}$$

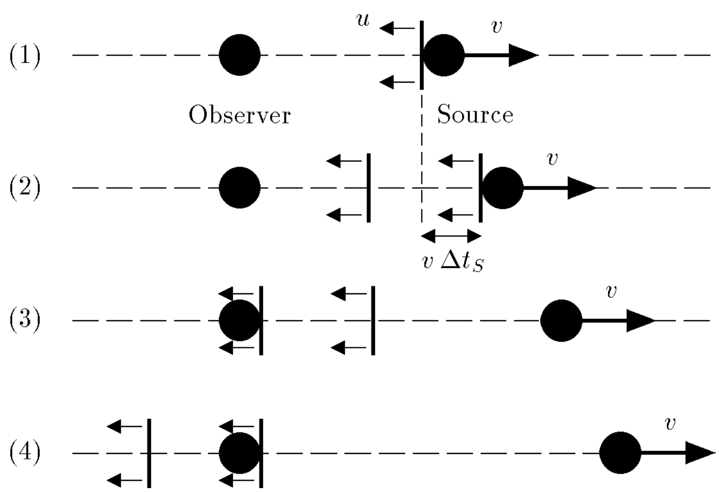
$$v \equiv \text{recession velocity of the source,}$$

$$\Delta t_S \equiv \text{the period of the wave at the source.}$$



Now consider the following sequence, as illustrated below:

(1) The source emits a wave crest.
(2) At a time Δt_S later, the source emits a second wave crest. During this time interval the source has moved a distance $v\Delta t_S$ further away from the observer.
(3) The stationary observer receives the first wave crest.
(4) At some time Δt after (3), the observer receives the second wave crest. Our goal is to find Δt.



The time Δt is then given by

$$\Delta t = \Delta t_S + \frac{v\Delta t_S}{u} \ , \tag{1.2}$$

where Δt_S is the time delay between the emission of the two crests, and $v\Delta t_S/u$ is the extra time it takes for the second pulse to travel over the increased distance between source and observer. Thus,

$$\Delta t = \left(1 + \frac{v}{u}\right) \Delta t_S \ , \tag{1.3}$$

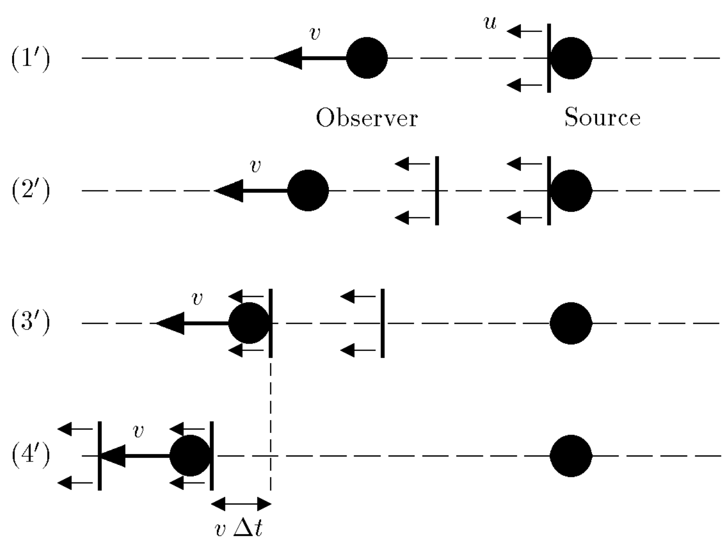
so the observed period (and hence the observed wavelength) of the wave is increased beyond its natural value by a factor of $(1+v/u)$. The factor by which the wavelength is increased is traditionally called $(1 + z)$, so in this case

$$z = v/u \qquad \text{(nonrelativistic, source moving).} \tag{1.4}$$

Suppose now that the source stands still, but the observer is receding at a speed v. In this case, the sequence becomes

(1′) The source emits a wave crest.
(2′) At a time Δt_S later, the source emits a second wave crest. The source is standing still.
(3′) The moving observer receives the first wave crest.

(4′) At a time Δt after (3′), the observer receives the second wave crest. During the time interval between (3′) and (4′), the observer has moved a distance $v\Delta t$ further from the source.



Using the same logic as in the first case, one has

$$\Delta t = \Delta t_S + \frac{v\Delta t}{u} \ , \tag{1.5}$$

and solving for Δt one finds

$$\Delta t = \left(1 - \frac{v}{u}\right)^{-1} \Delta t_S \ , \tag{1.6}$$

and then

$$\begin{aligned} z &= \frac{1}{1-(v/u)} - 1 \\ &= \frac{v/u}{1-(v/u)} \qquad \text{(nonrelativistic, observer moving).} \end{aligned} \tag{1.7}$$

Notice that the difference between the two cases is given by

$$z_{\text{observer moving}} - z_{\text{source moving}} = \frac{(v/u)^2}{1-(v/u)} \ , \tag{1.8}$$

which is proportional to $(v/u)^2$. Thus, the difference is negligible if the speed of recession is much smaller than the wave speed, but can be very significant if the two speeds are comparable.

THE DOPPLER SHIFT FOR LIGHT WAVES:

To derive the Doppler shift for light waves, one must decide which, if either, of the above calculations is applicable.

During the 19th century physicists thought that the situation for light waves was identical to that for sound waves. Sound waves propagate in air, and it was thought that light waves propagate in a medium called the aether which permeates all of space. The aether determines a privileged frame of reference, in which the laws of physics have their simplest form. In particular, Maxwell's equations were believed to have their usual form only in this frame, and it is in this frame that the speed of light was thought to have its standard value of $c = 3.0 \times 10^8$ m/sec in all directions. In a frame of reference which is moving with respect to the aether, the speed of light would be different. Light moving in the same direction as the frame of reference would appear to move more slowly, since the observer would be catching up to it. Light moving in the opposite direction would appear to move faster than normal. Thus, if the source is moving with respect to the aether and the observer is standing still, then the first calculation shown above would apply. If the observer is moving with respect to the aether and the source is standing still, then the second would apply. In either case one would of course replace the sound speed u by the speed of light, c.

In 1905 Albert Einstein published his landmark paper, "On the Electrodynamics of Moving Bodies", in which the theory of special relativity was proposed. The entire concept of the aether, after half a century of development, was removed from our picture of nature. In its place was the principle of relativity: **There exists no privileged frame of reference.** According to this principle, the speed of light will always be measured at the standard value of c, independent of the velocity of the source or the observer. The theory shook the very foundations of physics (which is in general a very risky thing to do), but it has become clear over time that the principle of relativity accurately describes the behavior of nature.

Since special relativity denies the existence of a privileged reference frame, it can make no difference whether it is the source or the observer that is moving. The Doppler shift, and for that matter any physically measurable effect, can depend only on the ***relative*** velocity of source and observer.

READING GUIDE FOR 2009:

The required reading skips to p. 13, *Summary of Special Relativity.* Keep in mind the central premise of special relativity, that the speed of light is always measured at the standard value of c, independent of the velocity of the source or observer. This premise sounds at first to be self-contradictory, but Einstein showed in 1905 that it is part of a consistent system. To make the premise consistent, however, we have to accept the idea that at high velocities (i.e., velocities not negligible compared to that of light), some of our intuitive prejudices about space and time are no longer valid. In particular, we have to accept the notion that measurements of time intervals, measurements of lengths, and judgments about the simultaneity of events can all depend on the velocity of the observer. These requirements are worked out in detail in the next 7 or so pages, but they are not part of this course. You can jump to the summary of these results on p. 13. For now it is only the first of these results — time dilation — that is relevant.

THE DEVELOPMENT OF SPECIAL RELATIVITY:

On the face of it, the principle of relativity appears to be self-contradictory. Suppose that we observe a light pulse which passes us at speed c. Suppose then that a second observer takes off after the light pulse in "super-space-ship" that attains a speed of $0.5c$ relative to us. Surely, one would think, the space ship observer would tend to catch up to the light pulse, and would measure its speed at $0.5c$. How could it possibly be otherwise?

The genius of Albert Einstein is that he was able to figure out how it could be otherwise. The subtlety and the brilliance of the theory lie in the fact that it forces us to change our most fundamental beliefs about the nature of space and time. Special relativity tells us that measurements of length and time are not universal, but instead depend on the motion of the observer. We can, however, maintain our notion about what it means for two events to coincide: if two events appear to occur at the same place ***and*** time to one observer, then they will appear to occur at the same place and time to any observer. (It is standard practice in relativity jargon to use the word "event" to denote a point in spacetime— i.e., an ideal event occurs at a single point in space and at a single instant of time.) Furthermore, in contrast to the 19th century viewpoint, we now believe that the fundamental laws of physics have the same form in any inertial reference frame. While measurements of space and time depend on the observer, the fundamental laws of physics are universal.

To see how special relativity works in detail, let us think more closely about the measurement of the speed of light by the space-ship observer described above. For convenience, we introduce a coordinate system at rest relative to ourselves, and call the coordinates x, y, z, and t. All measurements in the discussion to follow will be made relative to this coordinate system, except when otherwise specified. Note that the synchronization of clocks requires some care, because they must be synchronized by light signals which travel at a finite velocity. They can be synchronized, for example, by sending out a light pulse from the origin at time $t = 0$. Then, when the signal passes a clock with coordinates (x, y, z), the clock is set to the time $\sqrt{x^2 + y^2 + z^2}/c$.

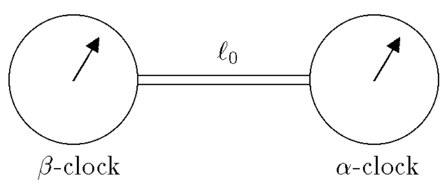
In our coordinate system, as in ***any*** inertial coordinate system, the speed of light will have its usual value of c. This means that light trajectories will be straight lines which satisfy

$$\left|\frac{d\vec{x}}{dt}\right| = c \ . \tag{1.9}$$

We will soon be discussing observations made by a variety of observers, but all these observations will rely on light rays. Regardless of the state of motion of the observer, we will be able to study these observations by computing the trajectories of the light rays in our own coordinate system.

Consider first another observer who is at rest relative to us, but who chooses a coordinate system with an origin displaced relative to ours, both in space and time. Since the recipe given above for synchronizing clocks depended explicitly on the origin of coordinates, it is not quite obvious that clocks which are synchronized by the second observer will also appear synchronized to us. We can find out, however, by using our coordinate system to follow the trajectories of the light pulses that the second observer uses to synchronize his clocks. It is then easy to verify that his time settings will appear synchronous to us— a space-time translation does not alter the concept of simultaneity.

Now consider an observer aboard a "super-space-ship", traveling at a significant fraction of c. The observer will make his own measurement of the speed of light, using a simple device which I will call a SLUG* ("Speed of Light Unraveling Gizmo"):



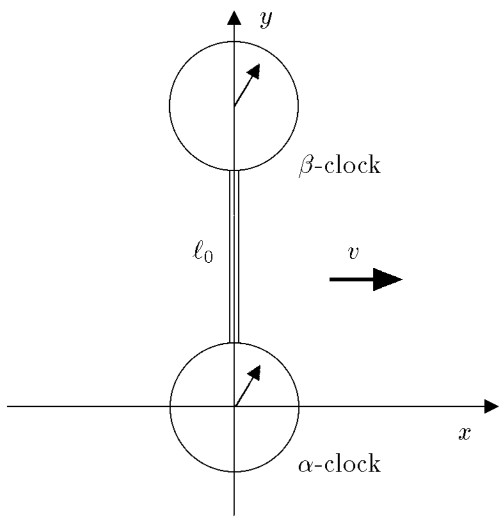
The SLUG consists of two clocks, labeled by the Greek letters α (alpha) and β (beta), joined by a measuring rod. When at rest the rod has a length which will be called ℓ_0; the clocks are synchronized by light signals and run at the normal speed. However, when the SLUG is moving we must not assume that either the length of the rod or the speed of the clocks remain at their original values. Nor can we assume that the moving clocks will appear synchronized. Rather, we will ***assume*** that the speed of light must always be measured at c, and we will ***learn*** how the

* SLUG is an unregistered trademark of Alan Guth. Other suggestions for names are invited.

rod length and the clock speed vary with the velocity of the SLUG. We will also check that the system is consistent, in the sense that if the observer in the rocket decides the use his SLUG to measure the speed of our clocks or the length of our measuring rods, he will find that they differ from their rest values in precisely the same way that we observe for his clocks and measuring rods.

A SLUG Oriented Perpendicular to the Velocity:

Consider first a SLUG oriented with its rod perpendicular to the velocity of the space ship, as shown below:



Our coordinate system is chosen so that at $t = 0$ the α clock is located at the origin, the β clock is located along the y-axis, and the velocity (relative to us) is along the positive x-axis.

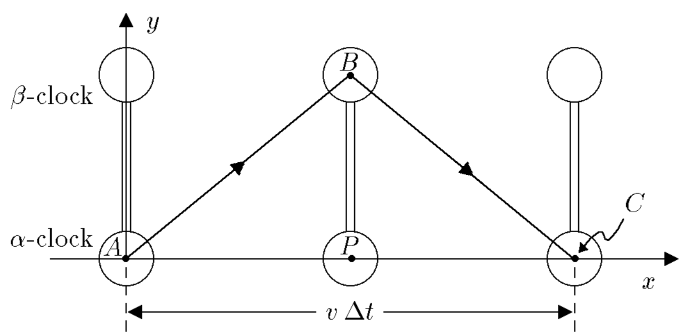
We can now argue that under these circumstances the length of the moving rod, as seen by us, must be equal to its rest length ℓ_0. To see this, imagine introducing a second rod of rest length ℓ_0 which will be at rest in our own coordinate system. Again it can be oriented so that the lower end lies at the origin, and the upper end lies along the y-axis. One can now construct a proof by contradiction. Suppose the length of the moving rod is less than ℓ_0. Now consider the event which occurs when the upper end of the moving rod crosses the stationary rod. This event lies at the upper end of the moving rod, but is in the interior of the stationary rod. The key point is that both we and the moving observer would agree that the event lies in the interior of the stationary rod. (The event, for example, could be an explosion which would sever the stationary rod but not the moving rod.) However, the principle of relativity requires that the two frames must be equivalent, and therefore there can be no observation such as the one just described which indicates an absolute distinction between the two frames. One can similarly prove a contradiction if the length of the moving rod is more than ℓ_0, so one is left with the conclusion that it is equal to ℓ_0.

Consider now the following sequence of events:

(A) At time $t = 0$ (in our frame), a pulse of light is emitted from the α clock. The time on the α clock is set to zero.
(B) The pulse is received at the β clock, and a return signal is sent.
(C) The return signal is received at the α clock.

We are considering the complete round trip in order to postpone worrying about the synchronization of the two clocks.

The following diagram shows the SLUG at three successive times:



Our goal is to find out what behavior is required in order for the speed of light to be measured by the moving SLUG as c. If the total length of time (in our coordinates) between A and C is given by Δt, then the spatial distance between A and C is given by $v\Delta t$. The total lengths of the light trajectory is then given by applying the Pythagorean theorem to the triangle ABP, and then doubling the result:

$$s = 2\sqrt{\ell_0^2 + \left(\frac{1}{2}v\Delta t\right)^2} \; . \tag{1.10}$$

The fact that the speed of light is equal to c in our frame of reference implies that $s = c\Delta t$, and one can then solve for Δt to find

$$\Delta t = \frac{2\ell_0}{c\sqrt{1-\beta^2}} \; , \tag{1.11}$$

where

$$\beta = v/c \; . \tag{1.12}$$

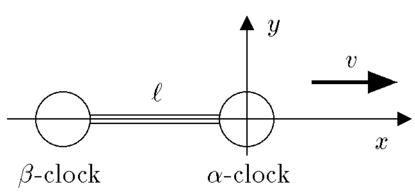
However, the SLUG must also measure the speed of light to be c, and therefore the time interval clicked off by the α clock between events A and C must be $2\ell_0/c$. (Remember that the separation between the clocks, in the rest frame of the SLUG, is by definition equal to ℓ_0.) Since the time interval in our frame is longer than this by a factor of

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}\ ,} \tag{1.13}$$

we are forced to conclude that the moving clock is running slower by a factor of γ. This phenomenon, known as "time dilation", is our first result about the theory of special relativity. Note that the slowing down must occur no matter how the clock is internally constructed, and so it must be thought of as a slowing down of time itself.

A SLUG Oriented Parallel to the Velocity:

Now consider a SLUG oriented with its rod parallel to the velocity of the space ship, as shown below:



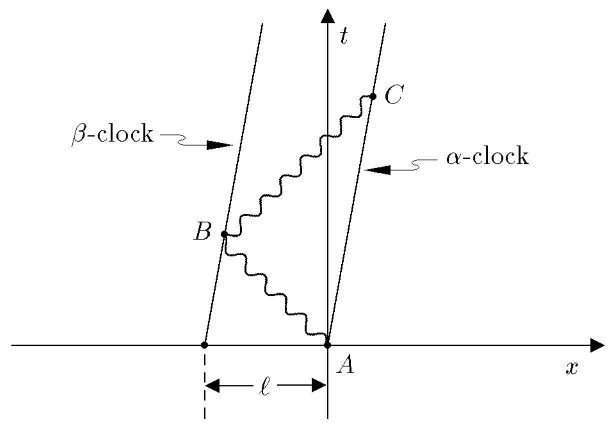
The coordinate system can be chosen so that at $t = 0$ the α clock is located at the origin, and the β clock is located along the negative x-axis. This time there is no argument which says that the length of the moving rod should equal its rest length, so its actual length (in our frame) is a presently unknown quantity that we will denote by ℓ. (The reader might wonder whether a similar argument could be constructed by introducing a second measuring rod at rest in our system. However, to complete the argument one must establish that we and the moving observer would agree on the question of which rod is longer. But such a comparison must involve looking at the two ends at the same time, and the ambiguity in the definition of simultaneity (to be explored further in the next section) will prevent an agreement on this question.) We will assume that the clocks operate in a way which is independent of the orientation of the rod, and they therefore must appear to run slower than normal by a factor of γ.

Again we wish to consider the sequence of events A, B, and C, as described above. Again the speed of light must be measured by the SLUG to be c, and so the time on the α clock at event C must be $2\ell_0/c$. Since the α clock must run more slowly than normal by a factor of γ, it follows that the time in our frame associated with the event C is given by

$$t_C = 2\gamma\ell_0/c \ . \tag{1.14}$$

Our goal is to use this fact to determine the relationship between ℓ and ℓ_0.

The events A, B, and C can be represented on a diagram in the x-t plane:



The units for the x- and t-axes have been chosen so that light pulses travel on lines with unit slope. (For example, one could use the second as the unit of t, and the light-second as the unit of x.) The lines which represent the position of each clock as a function of time are given by

$$x_\alpha(t) = vt \tag{1.15a}$$

$$x_\beta(t) = vt - \ell \ . \tag{1.15b}$$

The light pulse traveling from A to B obeys the trajectory equation

$$x(t) = -ct \ , \tag{1.16}$$

so solving simultaneously with Eq. (1.15b) one finds that the coordinates of event B are given by

$$\begin{aligned} t_B &= \frac{\ell}{c+v} = \frac{\ell/c}{1+\beta} \\ x_B &= -\frac{\ell}{1+\beta} \ . \end{aligned} \tag{1.17}$$

The light pulse from B to C then obeys the equation

$$x(t) = x_B + c(t - t_B). \tag{1.18}$$

Solving this equation simultaneously with Eq. (1.15a), one finds that

$$t_C = \frac{ct_B - x_B}{c - v} = 2\gamma^2 \ell / c \ . \tag{1.19}$$

Comparing the above equation with Eq. (1.14) one learns immediately that

$$\boxed{\ell = \ell_0 / \gamma \ .} \tag{1.20}$$

A rod moving parallel to its length appears to be contracted by a factor of γ. This effect is known as the Lorentz-Fitzgerald contraction.

The Relativity of Simultaneity:

We are now in a position to calculate the synchronization of the two clocks. Consider first the case described directly above, in which the SLUG is oriented parallel to the velocity.

We assume that the observer on the space ship with the SLUG synchronizes his clocks as he always would, without noticing that we are watching him. For example, he might use the light pulse from event A to event B to carry out this synchronization. When the light pulse is emitted at A, the α clock is set to zero. When the pulse arrives as B, the β clock is set to

$$t_\beta(B) = \ell_0 / c \ . \tag{1.21}$$

The notation $t_\beta(B)$ is used here to denote the reading on the β clock at event B. Now we wish to calculate the reading on the α clock at the same time (as seen by us, of course). The α clock was set to zero at $t = 0$, and has since been running slowly by a factor of γ. Thus, at time t_B the reading on the α clock is given by

$$t_\alpha(B) = t_B / \gamma \tag{1.22a}$$

$$= (1 - \beta)\ell_0 / c \ . \tag{1.22b}$$

Here $t_\alpha(B)$ denotes the reading on the α clock at a time which ***to us*** appears simultaneous with event B. Eq. (1.22b) was obtained with the use of Eqs. (1.17), (1.20), and (1.13). Subtracting Eq. (1.22b) from Eq. (1.21),

$$\boxed{t_\beta - t_\alpha = \beta \ell_0 / c \ .} \tag{1.23}$$

We have calculated this time difference at the time of event B, but since both clocks are moving at the same velocity and hence are running at the same rate, the time difference will remain the same for all time.

For the case of a SLUG oriented perpendicular to the velocity, there is simply no criterion by which one could distinguish between the two clocks, to decide which one might be expected to be ahead of the other. The conclusion is that the two clocks must appear synchronized. As a check, we can carry out an analysis similar to the one above. The time of event B is given by one half of the Δt which appears in Eq. (1.11), so

$$t_B = \gamma \ell_0 / c \ . \tag{1.24}$$

Eqs. (1.21) and (1.22a) still hold, so one has

$$t_\alpha(B) = t_B/\gamma = \ell_0/c = t_\beta(B) \ , \tag{1.25}$$

as expected.

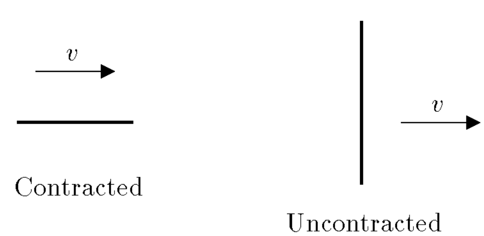
SUMMARY OF SPECIAL RELATIVITY:

At the risk of being redundant, the key results of special relativity can be summarized as follows:

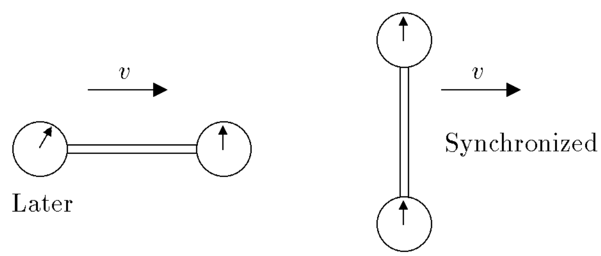
(1) TIME DILATION: Any clock which is moving at speed v relative to a given reference frame will appear (to an observer using that reference frame) to run slower than normal by a factor denoted by the Greek letter γ (gamma), and given by

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \ , \qquad \beta \equiv v/c \ . \tag{1.26}$$

(2) LORENTZ-FITZGERALD CONTRACTION: Any rod which is moving at a speed v along its length relative to a given reference frame will appear (to an observer using that reference frame) to be shorter than its normal length by the same factor γ. A rod which is moving perpendicular to its length does not undergo a change in apparent length.



(3) RELATIVITY OF SIMULTANEITY: Suppose a rod which has rest length ℓ_0 is equipped with a clock at each end. The clocks can be synchronized in the rest frame of the system by using light pulses. If the system moves at speed v along its length, then the trailing clock will appear to read a time which is later than the leading clock by an amount $\beta\ell_0/c$. If, on the other hand, the system moves perpendicular to its length, then the synchronization of the clocks is not disturbed.



There are a few minor qualifications that must be appended to the above statements. First, they hold only for inertial reference frames— they do not hold for rotating or accelerating reference frames. Any reference frame which moves at a uniform velocity relative to an inertial reference frame is also an inertial reference frame. Second, one must define the word "appear" which occurs in each of the three statements. In plain English, the word "appear" normally refers to the perception of the human eyes. However, in these situations the perception of the human eyes would be very complicated. The complication is that one sees with light, and light travels with a less-than-infinite speed. Thus, when an object is moving toward you, the light which you see coming from the front of the object has left the object later than the light which you see coming from the back of the object. Effects of this kind lead to complicated distortions, which are not taken into account in the statements above. For purposes of interpreting these statements, one can imagine that each reference frame is covered by an infinite number of local observers, each of which observes only events so close that the time delay for light travel is negligible. Each observer carries a clock which has been synchronized with the others by light pulses. The "appearance" is then the description which is assembled after the fact by combining the reports of these local observers.

READING GUIDE FOR 2009:

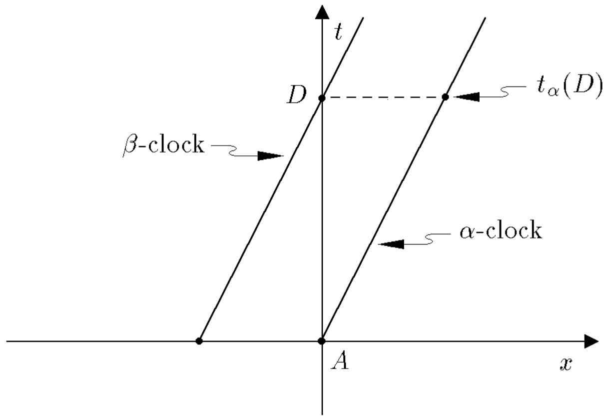
The next section, which you can skip, deals with the consistency of special relativity. For example, suppose that we are at rest, and we see a rocket pass us at high speed. Time dilation means that a clock on the rocket will appear to us to be running slowly. However, to an astronaut riding on the rocket it will appear that our clock is moving, and therefore she would apply the rules of special relativity and conclude that our clock must be running slower than hers. How can both these statements be true at the same time? The answer is complicated, but it works. The answer is rooted in the fact that the measurement of the rate at which a moving clock ticks is a complicated operation. As the moving clock passes us we get one chance to reach out and feel where the hands are when it is at our location, but to measure the rate of the clock we have to also know where the hands were at some earlier time, or where they will be at some later time. This remote measurement has to be made by waiting for a light-signal to reach us from the location of the clock, or perhaps by having a friend at the location of the clock who can make the measurement and send us the results. It is far from obvious, but if we take into account the effects of time dilation, length contraction, and the relativity of simultaneity on each part of the story, then everything is consistent. We would see the clock on the spaceship running slowly, the astronaut on the spaceship would see our clock running slowly, and there would be no contradiction. If you want to see how this works in detail, then feel free to read the next section.

Otherwise, you can skip to p. 17, *The Relativistic Doppler Shift*. There we will repeat the two calculations of the Doppler shift that were done at the beginning of the chapter, but this time thinking in the context of light waves and special relativity. We will find that the only relativistic effect that is relevant is time dilation. Furthermore, we will find that when time dilation is taken into account, we will get the same answer whether the source is moving and the observer is stationary, or vice versa.

CONSISTENCY OF SPECIAL RELATIVITY:

The goal of special relativity is to construct a system in which all inertial reference frames can be considered equivalent. In the above description, however, it is not clear that the space ship reference frame is equivalent to ours. We have shown how the rods and clocks inside the space ship must appear to us to be distorted, so that the space ship observer measures the speed of light at the usual value of c. Suppose, however, that the observer in the space ship uses his rods and clocks to measure the length of our rods and the speed of our clocks. For the theory to be consistent, he must observe the same distortions in our measuring devices that we observe in his.

Consider first the measurement of the velocity itself. The space ship observer can measure our velocity relative to him by using his SLUG oriented parallel to the velocity. He can then observe the time at which the origin of our coordinate system crosses the α clock, and then the time at which it crosses the β clock. In our own spacetime diagram, we can view these two events as follows:



At event A the α clock reads zero. To find the speed measured by the spaceship, we must find $t_\beta(D)$, the reading of the β clock at event D. Using Eqs. (1.15b) and (1.20), one sees that the t-coordinate of D is given by

$$t_D = \ell/v = \ell_0/\gamma v \ . \tag{1.27}$$

The α clock is synchronized with our clocks at $t = 0$ and then runs slowly by a factor of γ, so its reading at the time of event D (as seen by us) is given by

$$t_\alpha(D) = t_D/\gamma = \ell_0/\gamma^2 v \ . \tag{1.28}$$

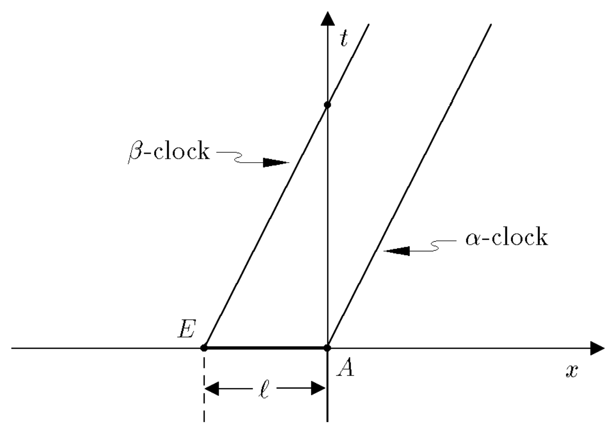
The difference in readings between the two clocks at a fixed time is given by Eq. (1.23), which gives

$$t_\beta(D) = t_\alpha(D) + \frac{\beta\ell_0}{c} = \frac{\ell_0}{v} \ . \tag{1.29}$$

Thus, the space ship observer concludes that the relative velocity between the space ship and our reference frame is v, as expected. If he had found anything other than v, then there would be a distinction between his reference frame and ours, contrary to the basic assumptions of special relativity.

Similarly, the space ship observer can check the apparent speed of our clocks. For example, the clock at our origin reads zero at event A, and according to Eq. (1.27) it reads $\ell_0/\gamma v$ at event D. For the observer in the space ship, the time interval between these two events is measured by $t_\beta(D) - t_\alpha(A) = t_\beta(D)$, as is given by Eq. (1.29). He therefore concludes that ***our*** clocks are running slowly by a factor of γ.

Finally, the space ship observer can measure the apparent length of our measuring rods. For convenience, suppose that we introduce a rod of length $\ell = \ell_0/\gamma$, at rest in our frame, which lies along the x-axis between $x = -\ell$ and $x = 0$. At $t = 0$ the two endpoints of this rod correspond to events E and A, as shown below:



The space ship observer sees these two events occurring at opposite ends of his SLUG, so he measures the distance between these two events as ℓ_0. He does not, however, regard these two events as simultaneous. To him the time of event E is measured by the β clock, which according to Eq. (1.23) reads $\beta\ell_0/c$. Thus, he concludes that, at time zero in his frame, the left end of our rod was located at a distance

$$\ell_0 - \frac{\beta v \ell_0}{c} = \frac{\ell}{\gamma} \tag{1.30}$$

from the right end, and hence he sees its length contracted by a factor of γ from its rest length ℓ. Note that the relativity of simultaneity is essential for the consistency of the theory.

THE RELATIVISTIC DOPPLER SHIFT:

We can now apply these ideas to the Doppler shift for light. We will first consider the case in which the source is moving relative to our reference frame, with the observer stationary. We will then consider the opposite possibility. The derivations will look very different in these two cases, but the principle of relativity guarantees us that the results must be the same— we are simply describing the situation from the point of view of two different reference frames.

For the case of the moving source, one can refer back to the nonrelativistic derivation. The sequence of events is the same, except for step (2). The moving source is a type of moving clock, and it therefore runs slower than normal by a factor of γ. Thus, step (2) would read:

(2) At a time $\gamma \Delta t_S$ later, the source emits a second wave crest. During this time interval the source has moved a distance $v\gamma\Delta t_S$ further away from the observer.

One then has

$$\Delta t = \gamma \left(1 + \frac{v}{c}\right) \Delta t_S = \boxed{\sqrt{\frac{1+\beta}{1-\beta}}\, \Delta t_S \ .} \tag{1.31}$$

Now consider the case in which the observer is moving, with the source stationary. The logic used in deriving Eq. (1.6) remains valid, except that in the relativistic case the clock that the observer is using to time the interval between crests appears to run slower than normal. Thus the time interval Δt which is marked off by this clock is shorter than normal, by a factor of γ. Thus,

$$\Delta t = \gamma^{-1} \left(1 - \frac{v}{c}\right)^{-1} \Delta t_S = \sqrt{\frac{1+\beta}{1-\beta}}\, \Delta t_S \ . \tag{1.32}$$

As expected, the two answers agree. Eqs. (1.31) or (1.32) describe the relationship in special relativity between the Doppler shift and the velocity of recession. Here v denotes the relative speed between source and observer (assumed to lie on the line which joins the source and observer), and it is ***impossible*** to know which of the two is actually in motion. The quantity z is given by

$$\boxed{z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \qquad \text{(relativistic).}} \tag{1.33}$$