Massachusetts Institute of Technology
Experimental Study Group

Physics 8.022, Spring 2011

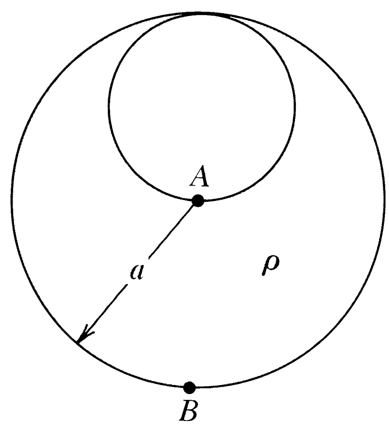
Problem Set 2 Solutions

Due: Sunday, February 13

Problem 1: Purcell 1.16

Problem

The sphere of radius a was filled with positive charge at uniform density ρ. Then a smaller sphere of radius $a/2$ was carved out, as shown in the figure, and left empty. What are the direction and magnitude of the electric field at A? At B?



Solution

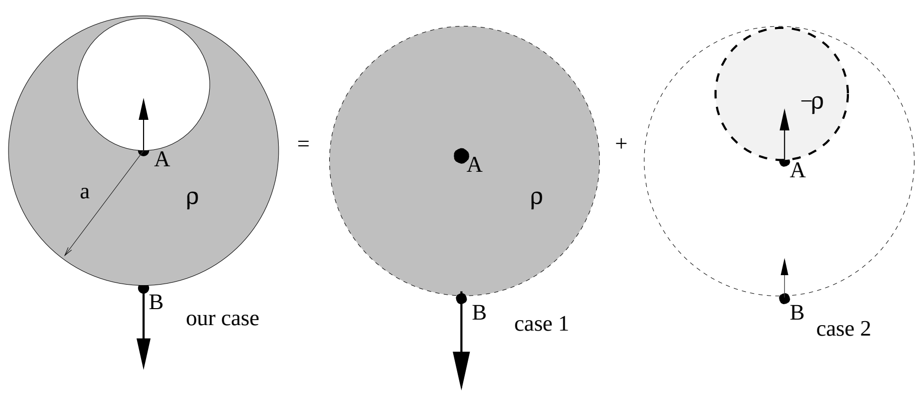


Figure 1: A hollow sphere and the equivalent "decomposition".

Refer to Figure 1. The hollow sphere (our case) is equivalent to the combination of a solidly charged sphere of uniform density ρ (case 1) and a half-radius sphere of uniform density $-\rho$ (case 2). Then we can read off the electric field at point A and point B from the figure.

$$\begin{aligned} E_A &= \frac{\rho \frac{4}{3}\pi (a/2)^3}{(a/2)^2} \\ &= \frac{2}{3}\pi \rho a. \end{aligned}$$

The direction of $\vec{E}_A$ is upward.

$$\begin{aligned} E_B &= \frac{\rho \frac{4}{3}\pi a^3}{a^2} - \frac{\rho \frac{4}{3}\pi (a/2)^3}{(3a/2)^2} \\ &= \frac{34}{27}\pi \rho a. \end{aligned}$$

The direction of $\vec{E_B}$ is downward.

Problem 2: Purcell 1.17

Problem

(a) A point charge q is located at the center of a cube of edge length d. What is the value of $\int \vec{E} \cdot d\vec{a}$ over one face of the cube?

(b) The charge q is moved to one corner of the cube. What is now the value of the flux of $\vec{E}$ through each of the faces of the cube?

Solution

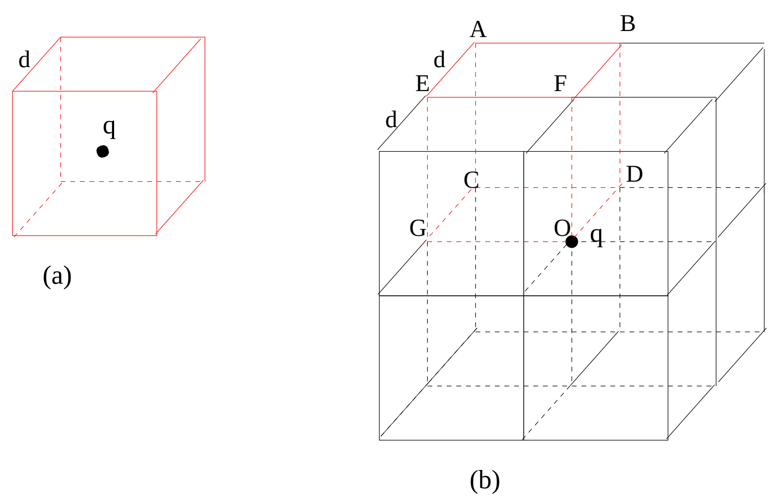


Figure 2: A point charge q at the center (a) and at a corner (b) of a cube (the upper-left-back octant of the larger cube; red in color version).

(a) Since the point charge q is located at the center of a cube, this problem respects the rotational symmetry. Thus the flux $\int \vec{E} \cdot d\vec{a}$ is the same for all six faces of the cube. Gauss's law therefore gives

$$\begin{aligned}\int_{oneface} \vec{E} \cdot d\vec{a} &= \frac{1}{6} \oint \vec{E} \cdot d\vec{a} \\ &= \frac{1}{6} \times 4\pi q \\ &= \frac{2}{3}\pi q.\end{aligned}$$

(b) q is now at a corner of the cube. Imagine we assemble seven other identical cubes together as in fig.2(b). In the new big cube of edge length $2d$, the charge q is again located in its center. Due to the rotational symmetry, we conclude the fluxes through three faces ABCD, AEGC, AEFB are the same and each is equal to one-quarter of the flux through one face of the big cube. Therefore

$$\begin{aligned}\int_{ABCD} \vec{E} \cdot d\vec{a} = \int_{AEGC} = \int_{AEFB} &= \frac{1}{4} \times \frac{2}{3}\pi q \\ &= \frac{1}{6}\pi q.\end{aligned}$$

The fluxes through the other faces, OFEG, OFBD, ODCD are zero since the electric field is parallel to these faces, so $\vec{E} \cdot d\vec{a} = 0$.

Problem 3: Purcell 1.31

Problem

Like the charged rubber balloon described on page 31, a charged soap bubble experiences an outward electrical force on every bit of its surface. Given the total charge Q on a bubble of radius R, what is the magnitude of the resultant force tending to pull any hemispherical half of the bubble away from the other half? (Should this force divided by $2\pi R$ exceed the surface tension of the soap film interesting behavior might be expected!)

Ans. $Q^2/8R^2$.

Solution

Refer to subsection 3.2 in which we want to calculate the total force on the upper hemisphere. For any piece of area element dA, the force is pointing outward from the origin; applying eq.(35) of Purcell ch.1,

$$F = \frac{1}{2}(E_1 + E_2)\sigma$$

(note that "F" in eq.(35) is the force per unit area), the magnitude is determined by

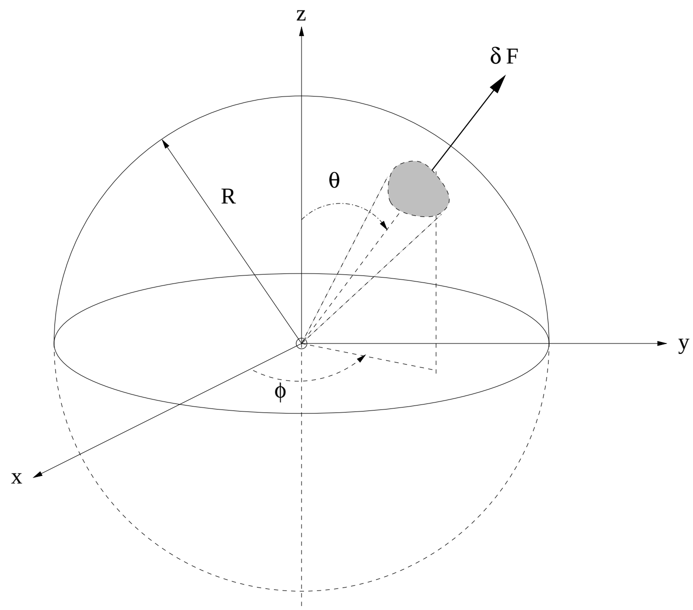


Figure 3: A spherical bubble on which charges are uniformly distributed.

$$\begin{aligned}
\delta F &= \frac{1}{2}(E_{inner} + E_{outer})\,\sigma\,dA \\
&= \frac{Q^2}{8\pi R^2}\sin\theta d\theta d\phi, \\
\text{where,} \qquad E_{\text{inner}} &= 0, \\
E_{\text{outer}} &= 4\pi\sigma, \qquad (1) \\
dA &= R^2 \sin\theta d\theta d\phi \\
\sigma &= Q/(4\pi R^2).
\end{aligned}$$

Equation 1 is easily seen by eq.(31) of Purcell ch.1,

$$E_2 - E_1 = 4\pi\sigma.$$

The projected component of δF on X-Y plane should be completely canceled throughout the hemisphere as σ is constant. So the net force on the hemisphere is the sum of the z-component of δF, i.e.

$$\begin{aligned}
F &= \sum \delta F \cos\theta \\
&= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \frac{Q^2}{8\pi R^2}\sin\theta\cos\theta \\
&= \frac{Q^2}{8R^2}.
\end{aligned}$$

Problem 4: Purcell 2.1

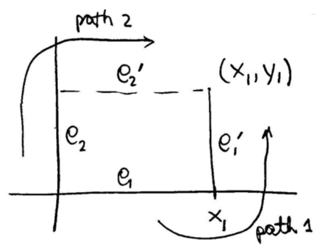
Problem

The vector function which follows represents a possible electrostatic field:

$$E_x = 6xy \qquad E_y = 3x^2 - 3y^2 \qquad E_z = 0$$

Calculate the line integral of $\vec{E}$ from the point $(0,0,0)$ to the point $(x_1, y_1, 0)$ along the path which runs straight from $(0,0,0)$ to $(x_1,0,0)$ and thence to $(x_1,y_1,0)$. Make a similar calculation for the path which runs along the other two sides of the rectangle, via the point $(0,y_1,0)$. You ought to get the same answer if the assertion above is true. Now you have the potential function $\phi(x,y,z)$. Take the gradient of this function and see that you get back the components of the given field.

Solution



Compute first along path 1:

$$\begin{aligned}
\phi(x_1, y_1, 0) &= -\int_{\text{path 1}} \vec{E} \cdot d\vec{s} \\
&= -\int_0^{x_1} E_x(x,0,0)\, dx - \int_0^{y_1} E_y(x_1, y, 0)\, dy \\
&= -\int_0^{x_1} 0\, dx - \int_0^{y_1} (3x_1^2 - 3y^2), dy \\
&= -3x_1^2 y_1 + y_1^3
\end{aligned}$$

Compute now along path 2:

$$\begin{aligned}
\phi(x_1, y_1, 0) &= -\int_{\text{path 2}} \vec{E} \cdot d\vec{s} \\
&= -\int_0^{y_1} E_y(0,y,0)\, dy - \int_0^{x_1} E_x(x, y_1, 0)\, dx \\
&= \int_0^{y_1} 3y^2\, dy - \int_0^{x_1} 6xy_1\, dx \\
&= y_1^3 - 3x_1^2 y_1
\end{aligned}$$

The two results agree! Since $E_z = 0$, this gives us, taking the origin as the reference point,

$$\phi(x,y,z) = y^3 - 3x^2 y.$$

Checking our answer,

$$E_x = -\frac{\partial \phi}{\partial x} = 6xy$$
$$E_y = -\frac{\partial \phi}{\partial y} = -3y^2 + 3x^2$$
$$E_z = -\frac{\partial \phi}{\partial z} = 0$$

Problem 5: Purcell 2.4

Problem

Describe the electric field and the charge distribution that go with the following potential:

$$\phi = x^2 + y^2 + z^2 \qquad \text{for } x^2 + y^2 + z^2 < a^2$$
$$\phi = -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}} \qquad \text{for } a^2 < x^2 + y^2 + z^2$$

Solution

For $x^2 + y^2 + z^2 < a^2$, $\phi = x^2 + y^2 + z^2$,

$$\begin{aligned}\vec{E} &= -\nabla\phi \\ &= -\frac{\partial \phi}{\partial x}\hat{\mathbf{x}} - \frac{\partial \phi}{\partial y}\hat{\mathbf{y}} - \frac{\partial \phi}{\partial z}\hat{\mathbf{z}} \\ &= -2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}},\end{aligned}$$

or $\vec{E} = (E_x, E_y, E_z) = (-2x, -2y, -2z)$.

$$\begin{aligned}\rho &= \frac{1}{4\pi}\nabla \cdot \vec{E} \\ &= \frac{1}{4\pi}(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) \\ &= -\frac{3}{2\pi}.\end{aligned}$$

For $x^2 + y^2 + z^2 > a^2$, $\phi = -a^2 + \frac{2a^3}{(x^2+y^2+z^2)^{1/2}}$,

$$\vec{E} = \left(\frac{2a^3 x}{(x^2 + y^2 + z^2)^{3/2}}\right)\hat{\mathbf{x}} + \left(\frac{2a^3 y}{(x^2 + y^2 + z^2)^{3/2}}\right)\hat{\mathbf{y}} + \left(\frac{2a^3 z}{(x^2 + y^2 + z^2)^{3/2}}\right)\hat{\mathbf{z}}.$$

$$\rho = 0.$$

That's not the end of the story: we should be careful of the boundary! Take a limit of $x^2 + y^2 + z^2 \to a^2$ from both inside and outside; the fields are not the same. This indicates there are some surface charge density σ on the boundary $x^2 + y^2 + z^2 = a^2$. Let $a^- \equiv \lim_{\varepsilon \to 0, \varepsilon > 0}(a - \varepsilon)$ and

$a^+ \equiv \lim_{\varepsilon\to 0, \varepsilon>0}(a+\varepsilon)$.

$$\begin{aligned}
\vec{E}_1 &= \vec{E}\left(x^2+y^2+z^2=(a^-)^2\right) = -2x\hat{\mathbf{x}} - 2y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \\
\vec{E}_2 &= \vec{E}\left(x^2+y^2+z^2=(a^+)^2\right) = 2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}} \\
|\vec{E}_2 - \vec{E}_1| &= |4x\hat{\mathbf{x}} + 4y\hat{\mathbf{y}} + 4z\hat{\mathbf{z}}| \\
&= 4\sqrt{x^2+y^2+z^2} = 4a \\
\sigma &= |\vec{E}_2 - \vec{E}_1|/4\pi = +a/\pi.
\end{aligned}$$

Problem 6: Purcell 2.8

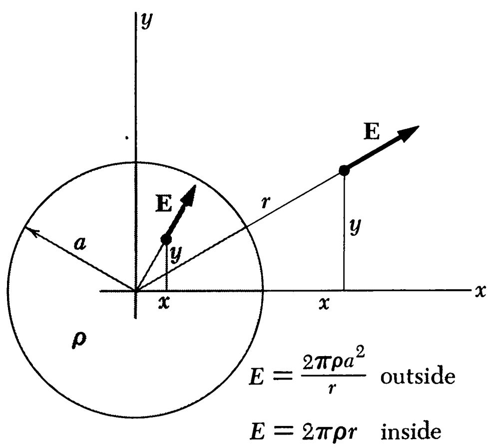
Problem

For the cylinder of uniform charge density in Fig. 2.17:

(a) Show that the expression there given for the field inside the cylinder follows from Gauss's law.

(b) Find the potential ϕ as a function of r, both inside and outside the cylinder, taking $\phi = 0$ at $r = 0$.

FIGURE 2.17
The field inside and outside a uniform cylindrical distribution of charge.



Solution

(a) The symmetry of a cylinder contains the direction of the field $\vec{E}$ to be radial. Refer to figure 4. we consider a cylindrical closed surface Σ whose radius is r and length is L. The flux on the two cross sections are zero. So

$$\oint \vec{E}\cdot d\vec{A} = 4\pi Q_{inside}$$
$$E \times (2\pi r L) = 4\pi\rho\pi r^2 L$$

or

$$E = 2\pi\rho r,$$

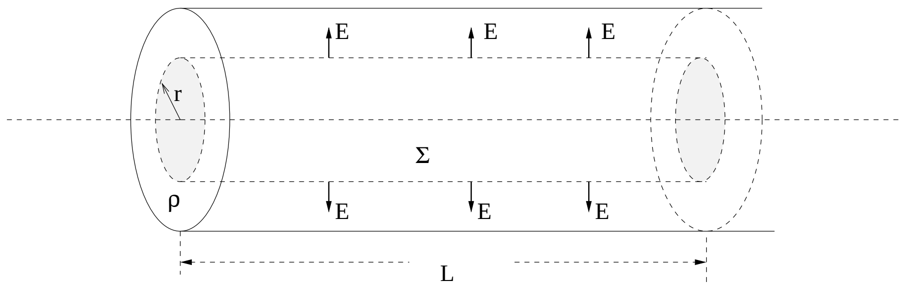


Figure 4: A cylinder inside which charges are uniformly distributed. The fields are along radial direction.

for $r < a$. Similarly, we can work out the field outside the cylinder by Gauss's law,

$$E = 2\pi\rho a^2/r,$$

for $r > a$.

(b) For $r < a$,

$$\begin{aligned}\phi(r) &= -\int_0^r \vec{E}\cdot d\vec{r} + \phi(0)\\ &= -\int_0^r 2\pi\rho r\, dr + 0\\ &= -\pi\rho r^2.\end{aligned}$$

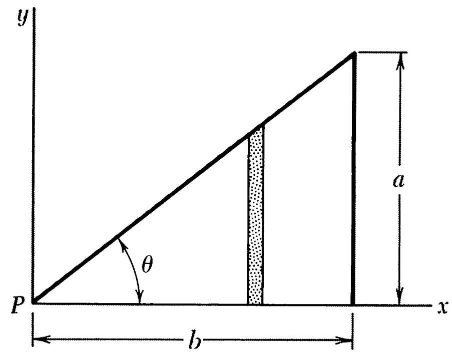
For $r > a$,

$$\begin{aligned}\phi(r) &= -\int_a^r \vec{E}\cdot d\vec{r} + \phi(a)\\ &= -\int_a^r \frac{2\pi\rho a^2}{r} dr - \pi\rho a^2\\ &= -2\pi\rho a^2 \ln(r/a) - \pi\rho a^2.\end{aligned}$$

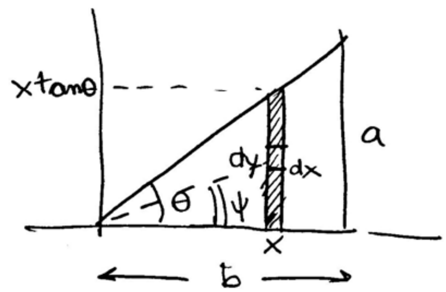
Problem 7: Purcell 2.12

Problem

The right triangle with vertex P at the origin, base b, and altitude a has a uniform density of surface charge σ. Determine the potential at the vertex P. First find the contribution of the vertical strip of width dx at x. Show that the potential at P can be written as $\phi_P = \sigma b \ln[(1+\sin\theta)/\cos\theta]$.



Solution



To get the potential due to the little vertical strip:

$$\begin{aligned}\phi_{\text{strip}} &= \int_{y=0}^{y=x\tan\theta} \frac{(\sigma\,dx)\,dy}{\sqrt{x^2+y^2}} \\ &= \sigma\,dx \int_{y=0}^{y=x\tan\theta} \frac{dy}{\sqrt{x^2+y^2}}\end{aligned}$$

It is actually good to change variables to ψ:

Let $y = x\tan\psi$ so that $dy = \frac{x}{\cos^2\psi}\,d\psi$. Then

$$\phi_{\text{strip}} = \sigma\,dx \int_0^\theta \left(\frac{x\,d\psi}{\cos^2\psi}\right) \frac{1}{x\sqrt{1+\tan^2\psi}}$$

Since $1+\tan^2\psi = \sec^2\psi = 1/\cos^2\psi$,

$$\begin{aligned}&= \sigma\,dx \int_0^\theta \frac{d\psi}{\cos\psi} \\ &= \sigma\,dx \ln|\sec\theta + \tan\theta| \\ &= \sigma\,dx \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|\end{aligned}$$

Now:

$$\begin{aligned}
\phi_{\text{tot}} &= \int_{x=0}^{x=b} \phi_{\text{strip}} \\
&= \int_0^b \sigma\, dx \ln\left|\frac{1+\sin\theta}{\cos\theta}\right| \\
&= \sigma b \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|
\end{aligned}$$

Check: As $\theta \to 0$, $\phi_{\text{tot}} \to \sigma b \ln 1 = 0$.

Problem 8: Purcell 2.30

Problem

Consider a charge distribution which has the constant density ρ everywhere inside a cube of edge b and is zero everywhere outside that cube. Letting the electric potential ϕ be zero at infinite distance from the cube of charge, denote by ϕ_0 the potential at the center of the cube and ϕ_1 the potential at a corner of the cube. Determine the ratio ϕ_0/ϕ_1. The answer can be found with very little calculation by combining a dimensional argument with superposition. (Think about the potential at the center of a cube with the same charge density and with twice the edge length.)

Solution

Let's do dimensional analysis. The potential ϕ_0 at the center is proportional to the total charge of the cube and inversely proportional to the characteristic length b, i.e.

$$\phi_0 = c\frac{\rho b^3}{b} = c\rho b^2 \propto b^2,$$

where c is a constant depending only on the shape of the object. If we assemble eight identical cubes together, each of which has the same charge density ρ and edge length b, the potential ϕ_0' at the *new* center is four times the previous ϕ_0. Note that the center of the *bigger* cube is a corner of the previous small cube; so $\phi_0' = 8\phi_1$. There we get $4\phi_0 = \phi_0' = 8\phi_1$, or

$$\frac{\phi_0}{\phi_1} = 2.$$

Graphically, treating ϕ_0 as a function of charge distribution, this is