Massachusetts Institute of Technology
Experimental Study Group
8.022 Spring 2011

Optional problems: Special relativity

1. Lenght contraction. Measure the length of a train car in the train reference frame and in the station reference frame. How are they related? Hint: Place a light source at the back of a train and a mirror at the front. The train moves with constant velocity $\vec{v} = v\hat{x}$. We flash the light in the forward direction where it reflects and returns to the back of the train. What are the travel times in the two reference frames in consideration?

2. Invariant interval (8 pts): A quantity that is left unchanged by Lorentz transformations is called a "Lorentz invariant". Consider two events described in the laboratory frame by (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2). Show that

$$\Delta s^2 \equiv -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

is a Lorentz invariant.

3. Galilean tranformations (10 pts): Prior to special relativity, people related coordinates between different frames with the "Galilean transformation" — clocks in different reference frame tick at the same rate, spatial positions are shifted by a term that depends on the relative velocity just as you would expect. For example, for frames that are moving with respect to each other in the x direction, we would have

$$\begin{aligned} t' &= t \\ x' &= x - vt\ . \end{aligned}$$

Using the binomial expansion on γ, show that for small v/c the Lorentz tranformations reduce to the Galilean tranformations. At what value of v does the next term in the expansion change the x transformation by 1%?

4. Transforming velocities (15 pts): A bullet is fired with velocity $\vec{u}'$ in the (x', y') plane of a moving frame F'. Frame F' moves with speed v in the $+x$ direction with respect to the laboratory frame F.

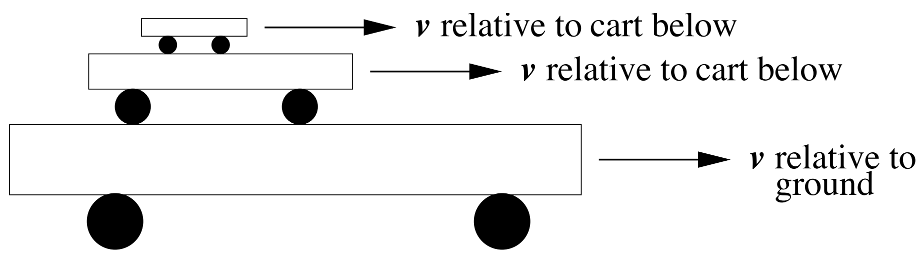
   (a) [6 pts] Find the angle that the velocity vector makes with x axis of the lab frame.

   (b) [3 pts] What is this angle in the limit $|\vec{u}'| = c$? Does anything wierd happen?

   (c) [6 pts] Show that when $|\vec{u}'| = c$, $|\vec{u}| = c$ — the speed of light is the same in both frames.

5. "Beating the speed of light" (20 pts): Your crazy uncle is fed up with all this fancy book learnin' you're getting at MIT. He's particularly unhappy that you're studying special relativity. He claims that he can build a device that will easily disprove the idea that nature has a speed limit (i.e., that nothing can go faster than the speed of light).

   The idea is as follows. We make a cart roll across the floor with speed v. We put a smaller cart on top of that cart, and roll it with speed v *with respect to the first cart*, and in the same direction as the first cart. We put a third cart on this second cart; it rolls with speed v *with respect to the second cart.* We put a fourth cart ... you get the idea. Your uncle claims that there is some n at which the cart must be going faster than the speed of light.



   (a) [10 pts] Prove him wrong. Using mathematical induction, prove that if $v < c$, then $v_n < c$, where v_n is the velocity of the nth cart. Show that this holds even for extremely large n.

   (b) [10 pts] Calculate the value of v_n given v and n. To facilitate this, note the following very useful identity:

$$\tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}$$

   Define $v_n = c \tanh x_n$. Using the above formula and comparing to the special relativity rule for adding velocities, first work out how x_n adds, then deduce v_n. What is v_∞?

6. Energy-momentum identity (7 pts): Show that

$$E^2 = p^2c^2 + m^2c^4$$

   where $p = |\vec{p}|$.

7. Transformation of fields (15 pts): A very large sheet of charge lies in the $x - y$ plane of the frame F. The charge per unit area of this sheet is σ. In the frame F', this sheet moves to the right with speed v.

   (a) [3 pts] What is the electric field in the rest frame (above and below the sheet)?

   (b) [4 pts] What is the electric field in the frame F' (above and below the sheet)?

   (c) [5 pts] What is the magnetic field in the frame F' (above and below the sheet)? (To do this, first think of what kind of current must come from a moving sheet of charge. Dimensional analysis should help you here.)

   (d) [3 pts] Show that the results of (b) and (c) are consistent with the general Lorentz transformations for electric and magnetic fields, Eq. (60) of Purcell Chapter 6.

8.

A proton has a mass $m_p = 938\,\mathrm{MeV}/c^2$. (Note: It is common when dealing with elementary particles to use units of "electron volts," abbreviated "eV." 1 eV is the energy imparted to a unit of electronic charge when it is accelerated through a potential difference of 1 volt, i.e. $1\,\mathrm{eV} = 1.6 \times 10^{-19}\,\mathrm{J}$. Common variations are: kilo-electron volt [keV], 10^3 eV; Mega-electron volt [MeV], 10^6 eV; Giga-electron volt [GeV], 10^9 eV; and Tera-electron volt [TeV], 10^{12} eV. The Large Hadron Collider at CERN, just beginning large-scale science operations, works at energies measured in TeV.) Very rarely one measures a cosmic ray particle (a proton that has been accelerated to high energies through some astrophysical process) that has energy $E \simeq 3 \times 10^{20}\,\mathrm{eV}$.

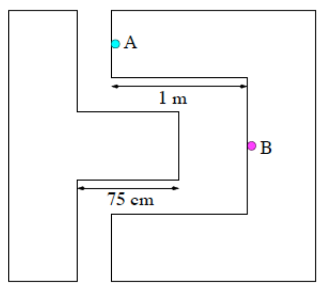
(a) [5 pts] If this energy ($3 \times 10^{20}\,\mathrm{eV}$) were used to throw a baseball ($m_{\mathrm{BB}} = 140\,\mathrm{gm}$), how fast would it go?

(b) [5 pts] In the rest frame of the $3 \times 10^{20}\,\mathrm{eV}$ cosmic ray particle, how wide is the Milky Way galaxy? (It is about 10^5 light years across in our rest frame.) How much time, in seconds, does it take the cosmic ray to cross the galaxy as measured in the ray's rest frame?

9.

A U-shaped structure made of ultrastrong steel contains a detonator. The depth of the U is 1 meter in its own rest frame. If the activation switch (at point B) is hit while the detonator is armed, a tremendous explosion goes off. However, there is a disarming switch at point A. When the switch at A is hit, a signal is sent (at the speed of light) to B, disarming the detonator. If point B is hit after the disarming signal has been received, then the explosion does not happen. Bear in mind that the disarming signal cannot arrive sooner than the light travel time from A to B. If point B doesn't "know" that point A has been hit, the explosion clearly must occur.

A T-shaped structure made of the same steel fits inside the U; the long arm of the T is 75 cm in its rest frame. When both structures are at rest in our laboratory, the T cannot reach far enough to hit the activation switch; instead, the disarmament switch is pressed.



The detonator is armed. The T structure is moved far to the left and accelerated to a speed $v = \sqrt{3}c/2$. It zooms in and smashes into the U.

(a) [3 pts] Compute the length of the T's arm in the rest frame of the U. Which switch is hit first, A or B?

(b) [3 pts] Compute the depth of the U in the rest frame of the T. Which switch is hit first, A or B?

(c) [9 pts] *Does the explosion happen or not?* Clearly, it cannot happen in one rest frame and not the other! Resolve the apparent contradiction in the two viewpoints.

Hint: Bodies cannot be perfectly rigid in special relativity. Information can only travel through the body at some finite speed, the speed of sound. This speed cannot exceed that of light (and is typically far, far smaller).