MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 7 Solutions
Magnetic Force, Special Relativity, Review

Due: Monday, March 28 at 10 pm
Review past concepts & problems!!!
Reading on complex numbers: `http://web.mit.edu/sahughes/www/8.022/complex.pdf`

Problem 1:

Problem

Particle A with charge q and mass m_A and particle B with charge $2q$ and mass m_B, are accelerated from rest by a potential difference ΔV, and subsequently deflected by a uniform magnetic field into semicircular paths. The radii of the trajectories by particle A and B are R and $2R$, respectively. The direction of the magnetic field is perpendicular to the velocity of the particle. What is their mass ratio?

Solution

(In SI Units)

The kinetic energy gained by the charges is equal to

$$\frac{1}{2}mv^2 = q\Delta V$$

which yields

$$v = \sqrt{\frac{2q\Delta V}{m}}$$

The charges move in semicircles, since the magnetic force points radially inward and provides the source of the centripetal force:

$$\frac{mv^2}{r} = qvB$$

The radius of the circle can be readily obtained as:

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2q\Delta V}{m}} = \frac{1}{B}\sqrt{\frac{2m\Delta V}{q}}$$

which shows that r is proportional to $\sqrt{m/q}$. The mass ratio can then be obtained from

$$\frac{r_A}{r_B} = \frac{\sqrt{m_A/q_A}}{\sqrt{m_B/q_B}} \implies \frac{R}{2R} = \frac{\sqrt{m_A/q}}{\sqrt{m_B/2q}}$$

which gives

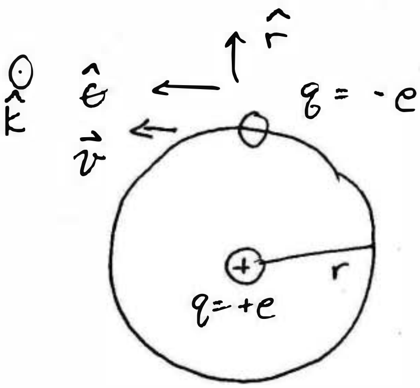
$$\frac{m_A}{m_B} = \frac{1}{8}$$

Problem 2:

Problem

Let us treat the motion of an electron in a hydrogen atom classically. Suppose that an electron follows a circular orbit of radius r around a proton. What is the angular frequency of the orbit ω? Suppose now that a *small* magnetic field perpendicular to the plane of the orbit is switched on. Assuming that the radius of the orbit does not change, calculate the shift in orbital frequency in terms of the magnitude of B. This is known as the "Zeeman effect".

Solution



$\vec{F}$	$= m\vec{a}$
$-\frac{e^2}{r^2}\hat{r} =$	$-m r \omega_0^2 \hat{r}$

$$\Rightarrow \quad \omega_0 = \left(\frac{e^2}{m r^3}\right)^{1/3}$$

Now assume there is a magnetic field

$$\vec{B} = B_z \hat{k}$$

The force becomes

$$\vec{F} = -\frac{e^2}{r^2}\hat{r} + (-e)\,\vec{v} \times \vec{B}$$

$$= -\frac{e^2}{r^2}\hat{r} + (-e)\,v_\theta \hat{\theta} \times B_z \hat{k}$$

$$= -\frac{e^2}{r^2}\hat{r} + -e\, v_\theta B_z \hat{r}$$

$$\vec{F} = \left(\frac{-e^2}{r^2} - e v_\theta B_z\right)\hat{r}$$

$$\vec{F} = m\vec{a}$$

$$\Rightarrow \quad \frac{-e^2}{r^2} - e v_\theta B_z = -m r \omega_f^2$$

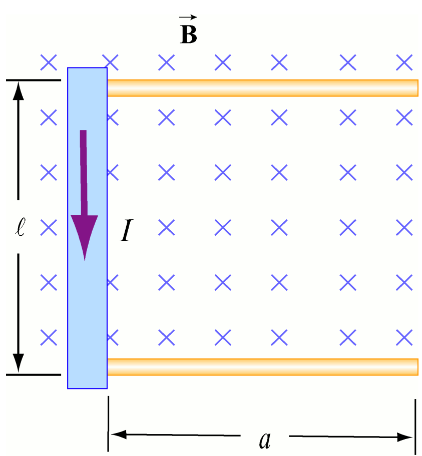
$$\Rightarrow \quad \omega_f^2 = \frac{e^2}{m r^3} + \frac{e v_\theta B_z}{m r}$$

$$\omega_f^2 = \omega_0^2 + \frac{e v_\theta B_z}{m r}$$

Problem 3:

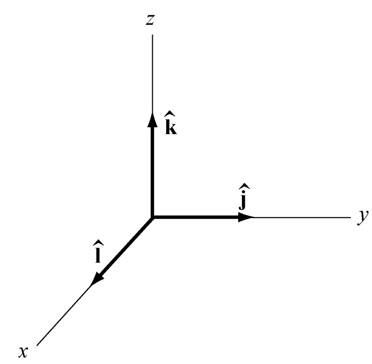
Problem

A rod with a mass m and a radius R is mounted on two parallel rails of length a separated by a distance ℓ, as shown in the figure below. The rod carries a current I and rolls without slipping along the rails which are placed in a uniform magnetic field $\vec{B}$ directed into the page. If the rod is initially at rest, what is its speed as it leaves the rails?



Solution

(In SI Units)



Using the coordinate system shown on the right, the magnetic force acting on the rod is given by

$$\vec{F}_B = \vec{I\ell} \times \vec{B} = I(\ell\hat{i}) \times (-B\hat{k}) = I\ell B\hat{j}$$

The total work done by the magnetic force on the rod as it moves through the region is

$$W = \int \vec{F}_B \cdot d\vec{s} = F_B a = (I\ell B)a$$

By the work-energy theorem, W must be equal to the change in kinetic energy:

$$\Delta K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where both translation and rolling are involved. Since the moment of inertia of the rod is given by $I = mR^2/2$, and the condition of rolling with slipping implies $\omega = v/R$, we have

$$I\ell Ba = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\left(\frac{v}{R}\right)^2 = \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2$$

Thus, the speed of the rod as it leaves the rails is

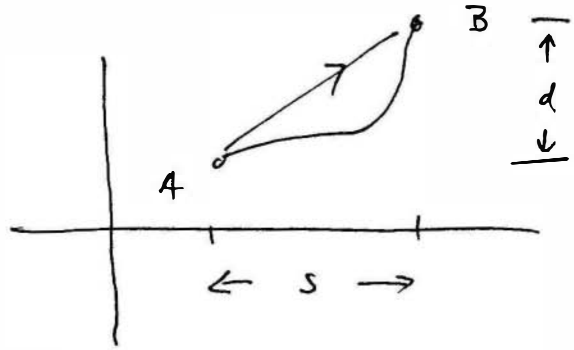
$$v = \sqrt{\frac{4I\ell Ba}{3m}}$$

Problem 4:

Problem

A wire of arbitrary shape which is confined to the $x - y$ plane carries a current I from point A to point B in the plane. Show that if a uniform magnetic field B perpendicular to the $x - y$ plane is present, the force that the wire experiences is the same as that which would be felt by a wire running straight from A to B.

Solution



$\vec{L} = s\,\hat{i} + d\,\hat{j}$

straight line connecting a to b

For arbitrary wire:

$d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$

$$\vec{F}_{wire} = \frac{I}{c}\int (dx\,\hat{i} + dy\,\hat{j}) \times \vec{B}$$

$$= \left(\frac{I}{c}\int dx\,\hat{i} + dy\,\hat{j}\right) \times \vec{B}$$

$\vec{B}$ uniform

$$= \left(\frac{I}{c}\, s\,\hat{i} + y\,\hat{j}\right) \times \vec{B}$$

$$= \frac{I}{c}\,\vec{L} \times \vec{B}$$

force on straight wire from A to B

Problem 5:

Problem

Calculate the divergence of the magnetic field of a straight wire in Cartesian coordinates.

Solution

Put $r = \sqrt{x^2 + y^2}$. With a little trigonometry, you should be able to convince yourself that

$$\begin{aligned}\hat{\phi} &= \hat{y}\cos\phi - \hat{x}\sin\phi \\ &= \frac{x\hat{y}}{\sqrt{x^2+y^2}} - \frac{y\hat{x}}{\sqrt{x^2+y^2}},\end{aligned}$$

so

$$\vec{B} = \frac{2I}{c}\left[\frac{x\hat{y}}{x^2+y^2} - \frac{y\hat{x}}{x^2+y^2}\right].$$

The divergence of this field is

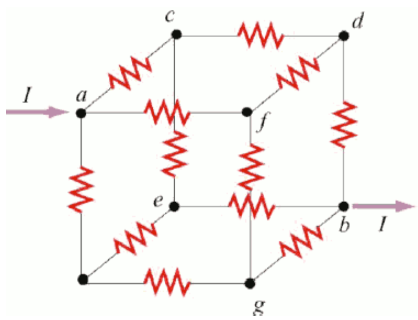
$$\begin{aligned}\vec{\nabla}\cdot\vec{B} &= \frac{2I}{c}\left[\frac{2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right] \\ &= 0.\end{aligned}$$

We could have guessed this without doing any calculation: if we make any small box, there will be just as many field lines entering it as leaving.

Problem 6:

Problem

Consider a cube which has identical resistors with resistance R along each edge, as shown below.



Find the equivalent resistance between points a and b.

Solution

From symmetry arguments, the current which enters a must split evenly, with $I/3$ going to each branch. At the next junction, say c, $I/3$ must further split evenly, with $I/6$ going through the two paths ce and cd. The current going through the resistor in db is the sum of the currents from fd and cd: $I/6 + I/6 = I/3$.

Thus, the potential difference between a and b can be obtained as

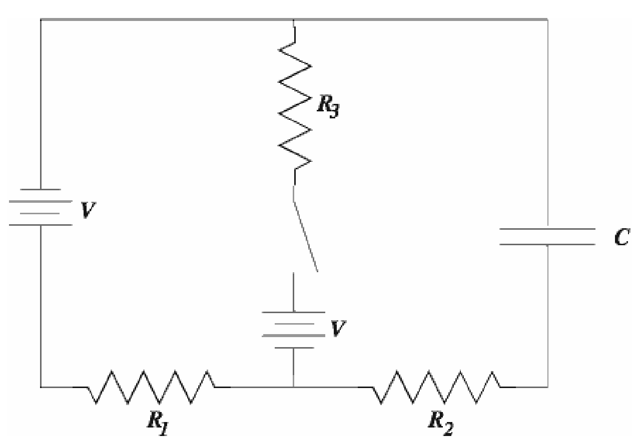
$$V_{ab} = V_{ac} + V_{cd} + V_{db} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR$$

hence the equivalent resistance is

$$R_{\text{eq}} = \frac{5}{6}R.$$

Problem 7:

Problem



Suppose that the capacitor is initially charged. The switch is closed at $t = 0$. Find the current $I_{R_3}(t)$ through resistor R_3 (i.e., in the middle branch) as a function of time, and the charge in the capacitor $Q(t)$. (Hint: this problem can be simplified greatly using Thévenin. Compute $Q(t)$ first.)

Solution

Version 1 takes $Q(0) = 0$. Version 2 takes $Q(0) = CV$, which is more physically reasonable.

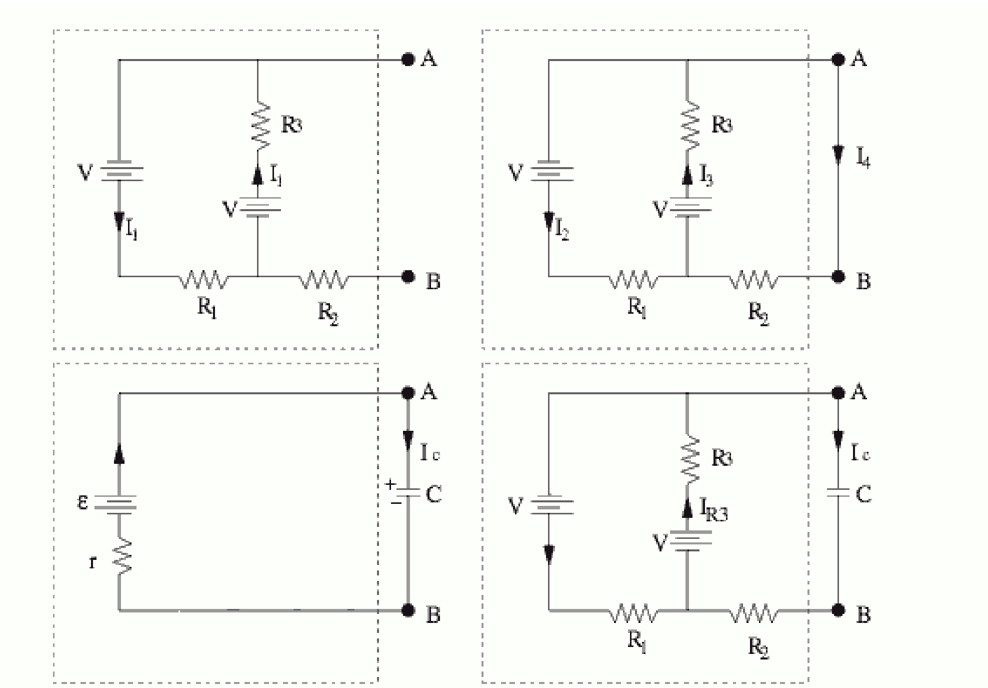


Figure 2: RC circuit network and it's Thévenin equivalence.

VERSION 1:

First calculate the Thévenin equivalent circuit. In the open circuit, $I_1 = 2V/(R_1 + R_3)$, so the equivalent EMF is

$$V_{OC} = V_{AB} = V - I_1 R_3 = \frac{R_1 - R_3}{R_1 + R_3} V \tag{7}$$

In the short circuit,

$$-V + I_2 R_1 = V - I_3 R_3 = I_4 R_2$$
$$I_3 = I_2 + I_4$$

I_4 as drawn here is the short circuit current. Solving these equations, we get

$$I_4 = I_{SC} = \frac{(R_1 - R_3)V}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

Thus the equivalent resistance is

$$R_{TH} = \frac{V_{OC}}{I_{SC}} = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1 + R_3} \tag{8}$$

Connecting A and B with a capacitor C, the equivalent circuit is a simple RC circuit with EMF V_{OC}:

$$V_{OC} - I_c(t)\, R_{TH} = Q(t)/C \tag{9}$$

$$I_c(t) = dQ/dt \tag{10}$$

Equations (9) and (10) yield

$$\frac{dQ}{dt} = \frac{CV_{OC} - Q(t)}{R_{TH}C}$$

Solving it together with the initial condition $Q(t=0) = 0$, we get

$$Q(t) = CV_{OC}\left[1 - e^{-t/(R_{TH}C)}\right] \tag{11}$$

where V_{OC} and R_{TH} are given by (7) and (8). Thus

$$I_c(t) = \frac{V_{OC}}{R_{TH}} e^{-t/(R_{TH}C)}$$

To calculate the current through R_3,

$$V - R_3 I_{R3} = I_C R_2 + Q(t)/C$$

$$\begin{aligned}
I_{R3}(t) &= \frac{1}{R_3}\left[V - V_{OC} + V_{OC}\left(1 - \frac{R_2}{R_{TH}}\right)e^{-t/(R_{TH}C)}\right] && (12)\\
&= \frac{2V}{R_1 + R_3} + \frac{V_{OC}}{R_3}\left(1 - \frac{R_2}{R_{TH}}\right)e^{-t/(R_{TH}C)} && (13)\\
&= \frac{2V}{R_1 + R_3} + \frac{V_{OC}R_1}{R_1R_2 + R_2R_3 + R_3R_1}e^{-t/(R_{TH}C)} && (14)\\
&= \frac{2V}{R_1 + R_3} + \frac{VR_1(R_1 - R_3)}{(R_1 + R_3)(R_1R_2 + R_2R_3 + R_3R_1)}e^{-t/(R_{TH}C)} && (15)
\end{aligned}$$

VERSION 2:

Much of the previous solution carries over. However, we now have a different boundary condition on the capacitor: we start with

$$\frac{dQ}{dt} = \frac{CV_{OC} - Q(t)}{R_{TH}C}$$

Let's solve this differential equation with the boundary condition $Q(t=0) = CV$:

$$\frac{dQ}{Q - CV_{OC}} = -\frac{dt}{R_{TH}C}$$

$$\int_{Q=CV_{OC}}^{Q=Q(t)} \frac{dQ}{Q - CV_{OC}} = -\int_{t=0}^{t} \frac{dt}{R_{TH}C}$$

$$\ln\left[\frac{Q(t)-CV_{OC}}{CV-CV_{OC}}\right] = -\frac{t}{R_{TH}C}$$

Exponentiating and solving for $Q(t)$ yields

$$Q(t) = CVe^{-t/R_{TH}C} + CV_{OC}(1-e^{-t/R_{TH}C}) .$$

Taking the derivative for the current from the capacitor yields

$$I_c(t) = \frac{dQ}{dt} = \frac{V_{OC}-V}{R_{TH}}e^{-t/R_{TH}C} .$$

The current through R_3 is found using the same formula as in version 1,

$$V - R_3 I_{R3} = I_c R_2 + Q(t)/C$$

Again, there are many equivalent ways to write the solution. The simplest thing to do is just leave the answer in terms of V_{OC} and R_{TH}:

$$I_{R3} \;=\; \frac{1}{R_3}\left[(V-V_{OC}) + (V-V_{OC})\left(\frac{R_2}{R_{TH}}-1\right)e^{-t/R_{TH}C}\right]$$

Using $V_{OC} = V(R_1-R_3)/(R_1+R_3)$, we find that $V - V_{OC} = 2VR_3/(R_1+R_3)$, and this can be rewritten

$$I_{R3} \;=\; \frac{2V}{R_1+R_3}\left[1+\left(\frac{R_2}{R_{TH}}-1\right)e^{-t/R_{TH}C}\right]$$

If we instead substitute $R_{TH} = (R_1R_2+R_1R_3+R_2R_3)/(R_1+R_3)$, the factor $(R_2/R_{TH} - 1)$ simplifies:

$$\frac{R_2}{R_{TH}} - 1 = -\frac{R_1R_3}{R_1R_2+R_1R_3+R_2R_3} .$$

The answer can then be written

$$I_{R3} \;=\; \frac{1}{R_3}\left[(V-V_{OC}) - (V-V_{OC})\left(\frac{R_1R_3}{R_1R_2+R_1R_3+R_2R_3}\right)e^{-t/R_{TH}C}\right]$$

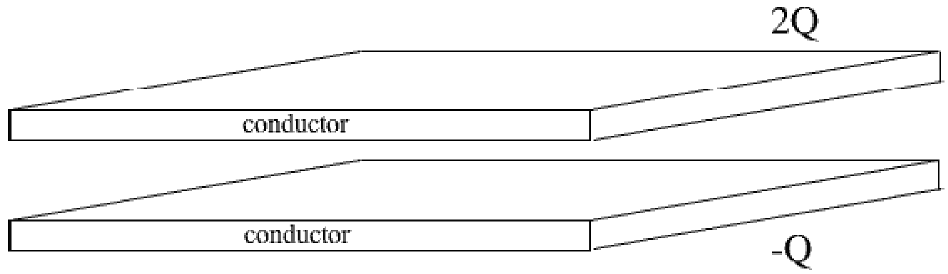
Finally, if we make both substitutions, the answer becomes

$$I_{R3} \;=\; \frac{2V}{R_1+R_3}\left[1-\left(\frac{R_1R_3}{R_1R_2+R_1R_3+R_2R_3}\right)e^{-t/R_{TH}C}\right] .$$

Problem 8:

Problem

Consider two parallel plates as shown in the sketch. They each have a surface area A and are separated by a distance s that is small compared to the dimensions of the plate. The top plate has a charge $+2Q$ deposited on it, while the bottom plate has charge $-Q$ deposited on it. The two plates are not connected in any way. For the purposes of this problem, assume that the charge densities are uniform on both surfaces of both plates, and ignore edge effects.



(a) Find $\vec{E}$ between the two plates.

(b) Find the surface charge densities on the top and bottom faces of each of the two plates.

(c) Find $\vec{E}$ just above the top plate and just below the bottom plate.

(d) Show explicitly that the jump in $\vec{E}$ across each of the two plates is equal to $4\pi\sigma\hat{n}$ where σ is the total surface charge density on that plate.

Solution

(a) Find $\vec{E}$ between the two plates.

Solution. We will use superposition principle to determine the electric field in this problem. The electric field due to the top plate at a point in between the two plates is $2\pi(2Q)/A$ and it acts downwards. The electric field due to the bottom plate at that point is $2\pi Q/A$ and acts downwards. Hence the electric field at a point between the two plates (I) is

$$E_I = E_I^{top} + E_I^{bot} = 6\pi\frac{Q}{A}(\text{downwards}) \tag{47}$$

(b) Find the surface charge densities on the top and bottom faces of each of the two plates.

Solution. The surface charge density on the bottom of the top plate is (using Gauss's law)

$$\sigma_{low}^{top} = \frac{E_I}{4\pi} = \frac{3Q}{2A} \tag{48}$$

Note that we have made use of the fact that electric field vanishes inside a conductor to obtain the above result. Hence, the surface charge density on the top surface of the top plate is

$$\sigma_{low}^{top} = \frac{2Q}{A} - \frac{3Q}{2A} = \frac{Q}{2A} \tag{49}$$

Similarly, the surface charge density on the top surface of the plate at the bottom is

$$\sigma_{up}^{bot} = -\frac{E_I}{4\pi} = -\frac{3Q}{2A} \tag{50}$$

The surface charge density on the bottom surface of the second plate is

$$\sigma_{low}^{bot} = \frac{-Q}{A} + \frac{3Q}{2A} = \frac{Q}{2A} \tag{51}$$

(c) Find $\vec{E}$ just above the top plate and just below the bottom plate. We can use superposition principle again to find these electric fields. The electric field above (II) the top plate is

$$E_{II} = 2\pi\frac{2Q}{A}(\text{upwards}) - 2\pi\frac{Q}{A}(\text{upwards}) = 2\pi\frac{Q}{A}(\text{upwards}) \tag{52}$$

Similarly, the electric field below (III) the bottom plate is

$$E_{III} = 2\pi\frac{2Q}{A}(\text{downwards}) - 2\pi\frac{Q}{A}(\text{downwards}) = 2\pi\frac{Q}{A}(\text{downwards}) \tag{53}$$

(d) Show explicitly that the jump in $\vec{E}$ across each of the two plates is equal to $4\pi\sigma\hat{n}$ where σ is the total surface charge density on that plate.

Solution. It is clear from the results of the previous parts of the problem that

$$\left[\vec{E}_{II}\cdot\hat{n} - \vec{E}_I\cdot\hat{n}\right]_{top} = 2\pi\frac{Q}{A} + 6\pi\frac{Q}{A} = 4\pi\frac{2Q}{A} = 4\pi\sigma_{top} \tag{54}$$

Similarly,

$$\left[\vec{E}_{III}\cdot\hat{n} - \vec{E}_I\cdot\hat{n}\right]_{bot} = 2\pi\frac{Q}{A} - 6\pi\frac{Q}{A} = -4\pi\frac{Q}{A} = 4\pi\sigma_{bot} \tag{55}$$