MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 3 Solutions
Gauss's law and electric potential

Due: Monday, February 21

Problem 1: Practice With ∇

Problem

(a) Calculate the gradient of each of these scalar fields:

(i) xyz.
(ii) $x^2 + y^2 + z^2$.
(iii) $1/r$ (in spherical coordinates).
(iv) $(\cos\theta)/r^2$ (in spherical coordinates).

(b) Calculate the divergence of each of these vector fields:

(i) $\hat{x}x + \hat{y}y + \hat{z}z$.
(ii) $(\hat{x}y - \hat{y}x)/\sqrt{x^2+y^2}$.
(iii) $\hat{r}/r^2$ (in spherical coordinates).
(iv) $\hat{r}(2\cos\theta)/r^3 + \hat{\theta}(\sin\theta)/r^3$ (in spherical coordinates).

(c) Calculate the curl of each of these vector fields:

(i) $\hat{x}yz + \hat{y}xz + \hat{z}xy$.
(ii) $\hat{x}xy + \hat{y}y^2 + \hat{z}yz$.
(iii) $(1/r^2)\hat{r}$ (in spherical coordinates).
(iv) $(1/R)\hat{\phi}$ (in cylindrical coordinates).

Solution

(a) (i)

$$\vec{\nabla}(xyz) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(xyz) = \hat{x}yz + \hat{y}zx + \hat{z}xy$$

(ii)

$$\vec{\nabla}\left(x^2+y^2+z^2\right) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(x^2+y^2+z^2\right) = 2\left(\hat{x}x + \hat{y}y + \hat{z}z\right)$$

(iii)

$$\vec{\nabla}\left(\frac{1}{r}\right) = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

(iv)

$$\vec{\nabla}\left(\frac{\cos\theta}{r^2}\right) = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\frac{\cos\theta}{r^2}\right) = -\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}$$

(b) (i)

$$\vec{\nabla}\cdot(\hat{x}x+\hat{y}y+\hat{z}z) = \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right) = 3$$

(ii)

$$\vec{\nabla}\cdot\left(\hat{x}\frac{y}{\sqrt{x^2+y^2}} - \hat{y}\frac{x}{\sqrt{x^2+y^2}}\right) =$$

$$\left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial z}(0)\right) = 0$$

(iii)

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{1}{r^2}\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0) = 0, \text{ for } r\neq 0$$

We will now examine the behavior at the origin by using Gauss's law for an infinitesimal surface enclosing the origin. The surface integral of $\hat{r}/r^2$ over an infinitesimal sphere of radius r enclosing the origin is

$$\int_S \frac{\hat{r}}{r^2}\cdot d\vec{A} = \int_0^{\pi}\int_0^{2\pi}\frac{1}{r^2}r^2\sin\theta\, d\theta\, d\phi = 4\pi$$

We also know that (Divergence theorem)

$$\int \vec{\nabla}\cdot\left(\frac{\hat{r}}{r'^2}\right) dV = \int_S \frac{\hat{r}}{r^2}\cdot d\vec{A} = 4\pi \tag{1}$$

Define $\delta^3(\vec{r})$ to be the three dimensional Dirac delta function defined as follows:

$$\delta^3(\vec{r}) = \begin{cases} 0 & \text{for } r\neq 0 \\ \infty & \text{when } r=0 \end{cases}$$

and the integral of the Dirac delta function over any region ($\mathcal{V}$) enclosing the origin is 1 i.e.,

$$\int_{\mathcal{V}} \delta^3(\vec{r})\, dV = 1.$$

At $r=0$, $\vec{v}\cdot\hat{r}/r^2$ must be different from zero. It is clear from Equation 1 that

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

(iv)

$$\vec{\nabla}\cdot\left(\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right) =$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{2\cos\theta}{r^3}\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\frac{\sin\theta}{r^3}\right)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0)$$

$$= -\frac{2\cos\theta}{r^4} + \frac{2\cos\theta}{r^4} = 0, \text{ for } r \neq 0$$

Note that the vector field given in this problem is the electric field due to a dipole. Hence, the surface integral of this field over a surface enclosing the origin is zero as the total charge enclosed by this surface is zero! Hence, by divergence theorem the divergence vanishes at the origin. Therefore,

$$\vec{\nabla} \cdot \left(\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right) = 0, \text{ everywhere}$$

Note that in the above argument we have assumed that the size of the dipole is so small that every infinitesimal surface enclosing the origin also encloses the dipole. In other words, we have assumed that the dipole is a point dipole located at the origin.

(c) (i)

$$\vec{\nabla} \times (\hat{x}yz + \hat{y}xz + \hat{z}xy) = \hat{x}\left(\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(xy)\right) + \text{cyclic perm.} = 0$$

Note that the above result is consistent with the fact that $\hat{x}yz + \hat{y}xz + \hat{z}xy = \vec{\nabla}(xyz)$.

(ii)

$$\vec{\nabla} \times \left(\hat{x}xy + \hat{y}y^2 + \hat{z}yz\right) =$$
$$\hat{x}\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(yz) + \hat{z}\left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(xy)\right)\right) = \hat{x}z - \hat{z}x$$

(iii)

$$\vec{\nabla} \times \left(\frac{\hat{r}}{r^2}\right) = \frac{\hat{r}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(0) - \frac{\partial}{\partial\phi}(0)\right) + \frac{\hat{\theta}}{r}\left(\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\phi}\left(\frac{\hat{r}}{r^2}\right)\right) - \frac{\partial}{\partial r}(0)\right)$$
$$+\frac{\hat{\phi}}{r}\left(\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial\theta}\left(\frac{\hat{r}}{r^2}\right)\right) = 0 \text{ for } r \neq 0$$

The line integral of $\left(\hat{r}/r^2\right)$ around an infinitesimal closed curve enclosing the origin is zero. Hence, by Stokes' theorem $\vec{\nabla} \times \left(\hat{r}/r^2\right) = 0$ near the origin as well. Hence, $\vec{\nabla} \times \left(\hat{r}/r^2\right) = 0$ everywhere.

(iv)

$$\vec{\nabla} \times \left(\frac{\hat{\phi}}{R}\right) = \hat{R}(0-0) + \hat{\phi}(0-0) + \hat{z}\left(\frac{1}{R}\frac{\partial}{\partial r}\left(\frac{1}{R}\right) - 0\right) = 0 \text{ for } R \neq 0$$

The line integral of $\hat{\phi}/R$ around an infinitesimal closed curve enclosing the origin is 2π. Hence, $\vec{\nabla} \times \left(\hat{\phi}/R\right)$ diverges at the origin. Therefore,

$$\vec{\nabla} \times \left(\frac{\hat{\phi}}{R}\right) = 2\pi\delta^2(\vec{R})$$

where, $\delta^2(\vec{R})$ is the 2-D Dirac delta function defined as follows:

$$\delta^2(\vec{R}) = \begin{cases} 0 & \text{for } R \neq 0 \\ \infty & \text{when } R = 0 \end{cases}$$

and the integral of the Dirac delta function over any region ($\mathcal{A}$) enclosing the origin is 1 i.e.,

$$\int_{\mathcal{A}} \delta^2(\vec{R})\, dA = 1.$$

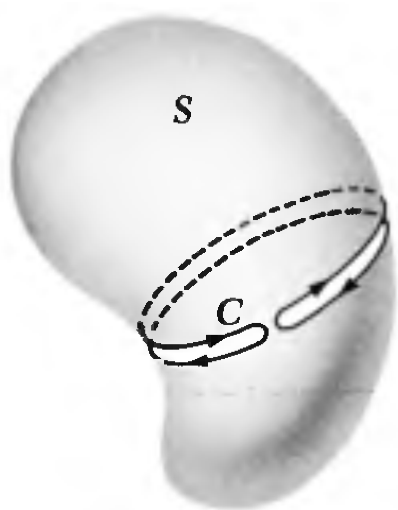
Problem 2: Purcell 2.16

Problem

If $\vec{A}$ is any vector field with continuous derivatives, $\text{div}(\text{curl}\,\vec{A}) = 0$ or, using the "del" notation, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. We shall need this theorem later. The problem now is to prove it. Here are two different ways in which that can be done:

(a) (Uninspired straightforward calculation in a particular coordinate system): Using the formula for $\vec{\nabla}$ in Cartesian coordinates, work out the string of second partial derivatives that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$ implies.

(b) (With the divergence theorem and Stokes' theorem, no coordinates are needed): Consider the surface S in the figure below, a balloon almost cut in two which is bounded by the closed curve C. Think about the line integral, over a curve like C, of any vector field. Then invoke Stokes and Gauss with suitable arguments.



Solution

1.
$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \left(\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right) = 0$$

2. Using Divergence theorem we get,

$$\int \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) dV = \oint \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S}$$

where (for this problem) the surface area element is $d\vec{S}$ instead of the usual $d\vec{A}$ since the vector $\vec{A}$ has already been assigned a different meaning. We can evaluate the surface integral by summing up the surface integral over several infinitesimal cuts (see Figure 1). Hence,

$$\oint \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS = \sum_i \iint_{S_i} \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS$$

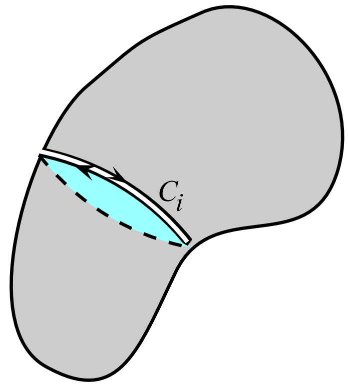


Figure 1: Balloon with a cut

where S_i denotes the region enclosed by C_i on the curved surface (white area). Using Stoke's law we get,

$$\sum_i \oint_{S_i} \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS = \sum_i \oint_{S_i} \vec{A} \cdot d\vec{r}$$

Note that the line integral around the infinitesimal "cut" (C_i) must vanish, as it is a sum of a line integral evaluated in the clockwise direction and the same integral evaluated in the counter-clockwise direction. Hence,

$$\sum_i \oint_{S_i} \vec{A} \cdot d\vec{r} = 0 \implies \oint \left(\vec{\nabla} \times \vec{A}\right) \cdot \hat{n}\, dS = 0 \implies \int \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) dV = 0$$

Since we evaluated the volume integral of $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right)$ for any arbitrary volume, $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$.

Problem 3: Stokes's Theorem in Action

Problem

Consider the vector field $\vec{F} = \hat{x}z^2 + \hat{y}x^2 - \hat{z}y^2$.

(a) Calculate $\oint \vec{F} \cdot d\vec{r}$ around a square path with corners $(x_0 \pm s/2, \quad y_0 \pm s/2, \quad 0)$. The square has center $(x_0, y_0, 0)$, side length s, and its sides are parallel to the x- and y-axes. The sense of rotation of the path is counter-clockwise as viewed from the $+z$ direction.

(b) Divide your answer to (a) by the area of the square, and take the limit as $s \to 0$.

(c) Calculate $\vec{\nabla} \times \vec{F}$ at the center of the square.

(d) Verify that your answer to (b) is equal to the normal component of $\vec{\nabla} \times \vec{F}$ evaluated at the center of the square.

Solution

(a)

$$\oint \vec{F}\cdot d\vec{s} = \int_{x_0-s/2}^{x_0+s/2} F_x\left((x, y_0-\frac{s}{2}, 0\right)dx + \int_{y_0-s/2}^{y_0+s/2} F_x\left((x_0+\frac{s}{2}, y, 0\right)\,dy$$
$$+\int_{x_0+s/2}^{x_0-s/2} F_x\left((x, y_0+\frac{s}{2}, 0\right)\,dx + \int_{y_0+s/2}^{y_0-s/2} F_x\left((x_0-\frac{s}{2}, y, 0\right)\,dy$$

Hence,

$$\oint \vec{F}\cdot d\vec{s} = s\left(x_0+\frac{s}{2}\right)^2 - s\left(x_0-\frac{s}{2}\right)^2 = 2s^2x_0$$

(b)

$$\lim_{s\to 0}\frac{1}{s^2}\oint \vec{F}\cdot d\vec{s} = 2x_0$$

(c) $\vec{\nabla}\times\vec{F}(x_0, y_0, 0) = -2y_0\hat{x} + 2x_0\hat{z}$

(d) The normal to the square points along the $\hat{z}$ direction. hence, the normal component of $\vec{\nabla}\times\vec{F}(x_0, y_0, 0)$ is $2x_0$. This is the answer we got in part (b)!

Problem 4: Gauss's Theorem in Action

Problem

Consider a vector field $\vec{F} = r\hat{r}$ (in spherical coordinates), and a closed surface S that is a cube with one corner at the origin and the opposite corner at (b, b, b). Verify Gauss's theorem,

$$\oint \vec{F}\cdot\hat{n}\,dS = \int (\vec{\nabla}\cdot\vec{F})\,dV,$$

for this particular case by performing both the surface integral on the left side, and the volume integral on the right side, and showing that they are equal.

Solution

A sketch of the cube is shown in Figure 2. The surface integral of $\vec{F}$ over the cube is

$$\begin{aligned}\oint_{\text{cube}} \vec{F}\cdot\hat{n}\,dA = &-\int_0^b\int_0^b \vec{F}(0, y, z)\cdot\hat{x}\,dy\,dz + \int_0^b\int_0^b \vec{F}(b, y, z)\cdot\hat{x}\,dy\,dz\\ &-\int_0^b\int_0^b \vec{F}(x, 0, z)\cdot\hat{y}\,dz\,dx + \int_0^b\int_0^b \vec{F}(x, b, z)\cdot\hat{y}\,dx\,dz\\ &-\int_0^b\int_0^b \vec{F}(x, y, 0)\cdot\hat{z}\,dx\,dy + \int_0^b\int_0^b \vec{F}(x, y, b)\cdot\hat{z}\,dx\,dy\\ &= 0 + b^3 - 0 + b^3 - 0 + b^3 = 3b^3\end{aligned}$$

Now, $\vec{\nabla}\cdot\vec{F} = 3$. Hence,

$$\int \vec{\nabla}\cdot\vec{F}\,dV = 3\int_{\text{cube}} dV = 3b^3$$

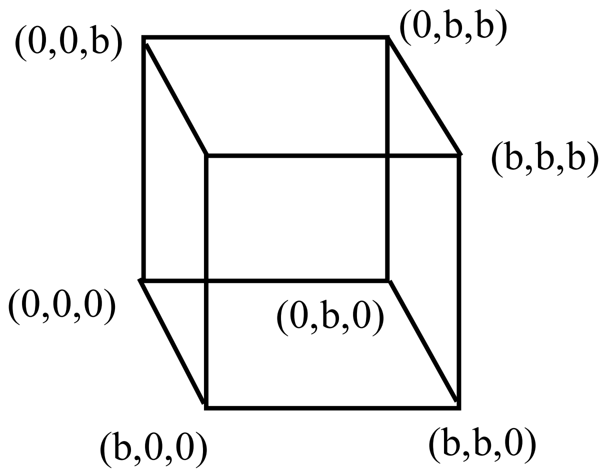


Figure 2: Cube of side b

Thus we have verified that,

$$\oint \vec{F} \cdot \hat{n}\, dA = \int (\vec{\nabla} \cdot \vec{F}) dV$$

Problem 5: Purcell 2.4 & 2.8 — One more time

Problem

(a) Describe the charge distribution that goes with the following potential:

$$\phi = x^2 + y^2 + z^2 \qquad \text{for } x^2 + y^2 + z^2 < a^2$$

$$\phi = -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}} \qquad \text{for } a^2 < x^2 + y^2 + z^2$$

Discuss what happens at the boundary ($x^2 + y^2 + z^2 = a^2$).

(b) Consider a very long cylinder of radius R that is filled with a uniform charge density ρ (Fig. 2.17). Again, find the electric field, $\vec{E}$, both inside and outside the cylinder, by integrating Poisson's equation: $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$.

Be sure that the $\vec{E}$ field inside and the $\vec{E}$ field outside match at the boundary (i.e., at R).

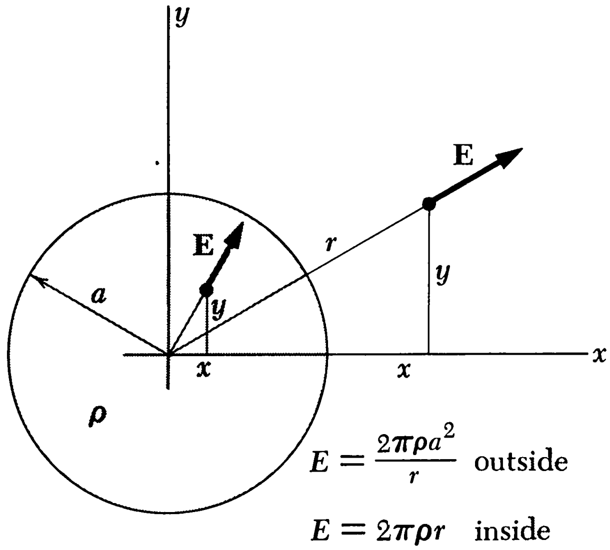


FIGURE 2.17
The field inside and outside a uniform cylindrical distribution of charge.

Solution

(a) For $x^2 + y^2 + z^2 < a^2$, $\phi = x^2 + y^2 + z^2$, recall from Problem Set 2 that

$$\begin{aligned}
\vec{E} &= -\nabla\phi \\
&= -\frac{\partial\phi}{\partial x}\hat{x} - \frac{\partial\phi}{\partial y}\hat{y} - \frac{\partial\phi}{\partial z}\hat{z} \\
&= -2x\hat{x} - 2y\hat{y} - 2z\hat{z},
\end{aligned}$$

or $\vec{E} = (E_x, E_y, E_z) = (-2x, -2y, -2z)$. Then

$$\begin{aligned}
\rho &= \frac{1}{4\pi}\nabla\cdot\vec{E} \\
&= \frac{1}{4\pi}(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) \\
&= -\frac{3}{2\pi}.
\end{aligned}$$

For $x^2 + y^2 + z^2 > a^2$, $\phi = -a^2 + \frac{2a^3}{(x^2+y^2+z^2)^{1/2}}$, recall from Problem Set 2 that

$$\vec{E} = \left(\frac{2a^3 x}{(x^2+y^2+z^2)^{3/2}}\right)\hat{x} + \left(\frac{2a^3 y}{(x^2+y^2+z^2)^{3/2}}\right)\hat{y} + \left(\frac{2a^3 z}{(x^2+y^2+z^2)^{3/2}}\right)\hat{z}.$$

Then

$$\rho = 0.$$

That's not the end of the story: we should be careful of the boundary! Take a limit of $x^2 + y^2 + z^2 \to a^2$ from both inside and outside; the fields are not the same. This indicates there are some surface charge density σ on the boundary $x^2 + y^2 + z^2 = a^2$.

Let $a^- \equiv \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}(a-\varepsilon)$ and $a^+ \equiv \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}(a+\varepsilon)$.

$$\begin{aligned}
\vec{E}_1 &= \vec{E}\left(x^2+y^2+z^2=(a^-)^2\right) = -2x\hat{x}-2y\hat{y}-2z\hat{\mathbf{z}}\\
\vec{E}_2 &= \vec{E}\left(x^2+y^2+z^2=(a^+)^2\right) = 2x\hat{x}+2y\hat{y}+2z\hat{\mathbf{z}}\\
|\vec{E}_2-\vec{E}_1| &= |4x\hat{x}+4y\hat{y}+4z\hat{z}|\\
&= 4\sqrt{x^2+y^2+z^2} = 4a\\
\sigma &= |\vec{E}_2-\vec{E}_1|/4\pi = +a/\pi.
\end{aligned}$$

(b) Outside the cylinder the differential form of Gauss's law takes the following form (in cylindrical coordinates)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r^{\text{out}}\right) = 0$$

Inside the cylinder,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r^{\text{in}}\right) = 4\pi\rho$$

Solving the above equations we find,

$$E_r^{\text{out}} = \frac{A}{r} \text{ and } E_r^{\text{in}} = 2\pi\rho r + \frac{B}{r}$$

Note that B must vanish for the electric field to be well defined at the origin. Equating, the electric field inside and outside we get,

$$E_r^{\text{out}}(R) = E_r^{\text{in}}(R) \implies A = 2\pi R^2\rho$$

Hence,

$$E_r^{\text{out}} = \frac{2\pi R^2\rho}{r} \text{ and } E_r^{\text{in}} = 2\pi r\rho$$

Problem 6: Potential of an Electric Dipole

Problem

Compute the potential $\varphi(x,y,z)$ of a dipole charge configuration. The dipole consists of a charge $+q$ located at $z=a/2$ and a charge $-q$ located at $z=-q$.

(a) Write down $\varphi(x,y,z)$ (i.e., in Cartesian coordinates).

(b) Expand $\varphi(x,y,z)$ in a Maclaurin series (i.e., a Taylor series about $a=0$) to first order in a.

(c) Convert your results to spherical coordinates: $\varphi(r,\theta,\phi)$.

(d) Compute $\vec{\nabla}\varphi(r,\theta,\phi)$, in spherical coordinates, to find $\vec{E}$.

Solution

(a) The potential at a point (x, y, z) due to the dipole is given by

$$\varphi(x,y,z) = \frac{q}{\left(x^2+y^2+(z-a/2)^2\right)^{1/2}} - \frac{q}{\left(x^2+y^2+(z+a/2)^2\right)^{1/2}} \tag{2}$$

(b) Expanding the potential in Equation 2 to first order in a we get

$$\varphi(x,y,z) = \frac{q}{(x^2+y^2+z^2)^{1/2}}\left(\left(1+\frac{1}{2}\frac{az}{x^2+y^2+z^2}\right)-\left(1-\frac{1}{2}\frac{az}{x^2+y^2+z^2}\right)\right)+\mathcal{O}(a^2)$$

After some simplification we get

$$\varphi(x,y,z) = \frac{qaz}{(x^2+y^2+z^2)^{3/2}} \tag{3}$$

(c) The expression for the dipole potential in Equation 3 can be written in spherical polar coordinates as follows,

$$\varphi(r,\theta,\phi) = \frac{qa\cos\theta}{r^2}$$

(d) The electric field is given by

$$\begin{aligned}\vec{E} = -\vec{\nabla}\varphi(r,\theta,\phi) &= -\hat{r}\frac{\partial}{\partial r}\left(\frac{qa\cos\theta}{r^2}\right) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{qa\cos\theta}{r^2}\right)\\ &= \hat{r}\frac{2qa\cos\theta}{r^3}+\hat{\theta}\frac{qa\sin\theta}{r^3}\end{aligned}$$

Problem 7: Purcell 2.29

Problem

One of two nonconducting spherical shells of radius a carries a charge Q uniformly distributed over its surface, the other a charge $-Q$, also uniformly distributed. The spheres are brought together until they touch. What does the electric field look like, both outside and inside the shells? How much work is needed to move them far apart?

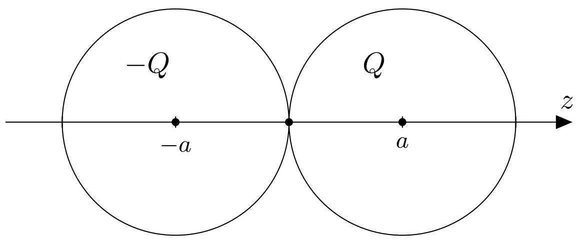
Solution

The electric field due to a single spherical shell of radius a and charge Q is given by

$$\vec{E}(r) = \begin{cases} 0 & \text{for } |\vec{r}-\vec{r}_0| < a \\ \frac{Q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3} & \text{for } |\vec{r}-\vec{r}_0| > a \end{cases}$$

where $\vec{r}$ denotes the position vector of the point at which electric field is evaluated and $\vec{r}_0$ denotes the position vector of the center of the spherical shell. In class, we usually put $\vec{r}_0 = \vec{0}$.

We will now evaluate the electric field due to two non-conducting spherical shells touching each other. Without loss of generality, we can assume that the centers are located at $(0, 0, \pm a)$.



The electric field is given by the superposition of the electric field of each spherical shell. Hence,

$$\vec{E}(\vec{r}) = \begin{cases} -\frac{Q(\vec{r}+a\hat{z})}{|\vec{r}+a\hat{z}|^3} & \text{for } |\vec{r}-a\hat{z}| < a \\ \frac{Q(\vec{r}-a\hat{z})}{|\vec{r}-a\hat{z}|^3} & \text{for } |\vec{r}+a\hat{z}| < a \\ \frac{Q(\vec{r}-a\hat{z})}{|\vec{r}-a\hat{z}|^3} - \frac{Q(\vec{r}+a\hat{z})}{|\vec{r}+a\hat{z}|^3} & \text{otherwise} \end{cases}$$

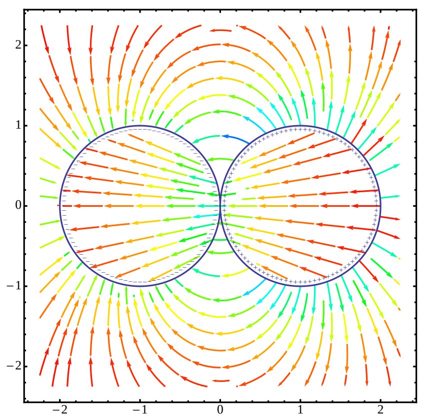


Figure 3: Electric Field Lines

Figure 3 shows a sketch of the electric field lines. Note that $\vec{E}$ depends on r and θ but not ϕ.[1]

We will treat the spherical shells as point charges to calculate the work done to move these shells far apart. This is justified because the average potential of one sphere in presence of the other is the potential at its center. Hence, the work done to move these two shells far apart is

$$W = \frac{Q^2}{2a}.$$

Problem 8: Purcell 2.14 (Laplace's equation)

Problem

Does the function $f(x, y) = x^2+y^2$ satisfy the two-dimensional Laplace equation? Does the function $g(x, y) = x^2 - y^2$? Sketch the latter function, calculate the gradient at the points $(x = 0, y = 1)$; $(x = 1, y = 0)$; $(x = 0, y = -1)$; and $(x = -1, y = 0)$ and indicate by little arrows how these gradient vectors point.

[1] $|r \pm a\hat{z}| = \left(r^2 + a^2 \pm 2ra\cos\theta\right)^{1/2}$

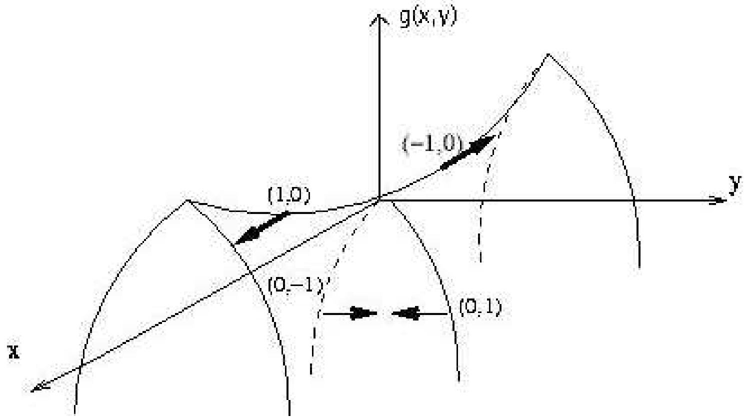
Solution

Since $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2+2 = 4$, $f(x,y)$ does not satisfy the 2-D Laplace equation.

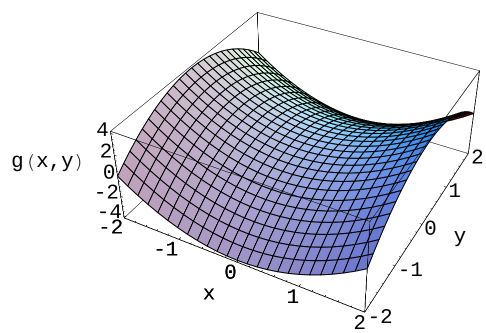
However, $g(x,y)$ does, since $\nabla^2 g(x,y) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 2-2 = 0$.

$$\nabla g(x,y) = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} = 2x\hat{x} - 2y\hat{y}.$$

$$\nabla g|_{(0,1)} = -2\hat{y}, \qquad \nabla g|_{(1,0)} = 2\hat{x},$$
$$\nabla g|_{(0,-1)} = 2\hat{y}, \qquad \nabla g|_{(-1,0)} = -2\hat{x}.$$



(a)



(b)

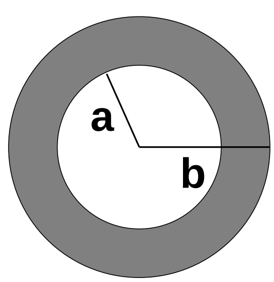
Figure 4: saddle shape

NOTE: $\nabla g|_{(x=0,y=1)}$ anti-parallel to y-axis, $\nabla g|_{(x=1,y=0)}$ parallel to x-axis, $\nabla g|_{(x=0,y=-1)}$ parallel to y-axis, $\nabla g|_{(x=-1,y=0)}$ anti-parallel to x-axis. The gradient vectors are all in xy-plane.

Problem 9: Electric field, potential, and flux

Problem

A hollow spherical shell carries charge density $\rho = k/r^2$ in the region $a \le r \le b$:



(a) Find the electric field $\vec{E}$ everywhere in space

(b) Find the potential ϕ everywhere in space.

(c) Calculate the electric flux through

(i) A concentric sphere with radius $r_1 > b$

(ii) A concentric sphere with radius $a \leq r_2 \leq b$

(iii) A concentric sphere with radius $r_3 < a$

(iv) A nonconcentric sphere centered on *any* point on the outer surface of the shell, and of radius $r_4 = 2b$.

Solution

(a) From Gauss's Law:

$$
\begin{aligned}
r < a, & \vec{E} = 0 \\
a < r < b, & 4\pi r^2 E = 4\pi Q_{\text{enclosed}} = 4\pi \int_a^r \frac{k}{r^2} 4\pi r^2 \, dr = (4\pi)^2 k(r-a) \\
& \Rightarrow E = 4\pi \frac{k(r-a)}{r^2} \\
r > b, & 4\pi r^2 E = 4\pi Q_{\text{enclosed}} = 4\pi \int_a^b \frac{k}{r^2} 4\pi r^2 \, dr = (4\pi)^2 k(b-a) \\
& \Rightarrow E = 4\pi \frac{k(b-a)}{r^2}
\end{aligned}
$$

(b) Assume $\phi(r \to \infty) = 0$

$$
\begin{aligned}
r > b, & \phi(r) - \phi(\infty) = -\int_\infty^r \vec{E}(r') \cdot d\vec{r'} \\
& \Rightarrow \phi(r) = -\int_\infty^r 4\pi \frac{k(b-a)}{r'^2} dr' = 4\pi \frac{k(b-a)}{r}. \\
a < r < b, & \phi(r) - \phi(b) = -\int_b^r \vec{E}(r') \cdot d\vec{r'}, \\
& \Rightarrow \phi(r) = \phi(b) - \int_b^r \vec{E}(r') \cdot d\vec{r'} = 4\pi k[1 - a/r - \ln(r/b)]. \\
r < a, & \phi(r) = 4\pi k \ln(\frac{b}{a}).
\end{aligned}
$$

(c) Flux

(i) For concentric sphere with $r_1 > b$, flux $\phi_1 = (4\pi)^2 k(b-a)$.

(ii) For $a < r_2 < b$ flux $\phi_2 = (4\pi)^2 k(r_2 - a)$.

(iii) For $r < a$ flux ϕ_3 is zero since the electric field is zero.

(iv) From Gauss's law, flux $\phi_4 = \oint \vec{E} \cdot ds = 4\pi Q_{\text{total}} = (4\pi)^2 k(b-a)$

Problem 10: Purcell 2.20 (Potential at the center of a gold nucleus)

Problem

As a distribution of electric charge, the gold nucleus can be described as a sphere of radius 6×10^{-13} cm with a charge $Q = 79e$ distributed fairly uniformly through its interior. What is the potential ϕ_0 at the center of the nucleus, expressed in megavolts? (First derive a general formula for ϕ_0 for a sphere of charge Q and radius a. Do this by using Gauss's law to find the internal and external electric field and then integrating to find the potential.)

Ans. $\phi_0 = 3Q/2a = 95\,000$ statvolts $= 28.5$ megavolts.

Solution

For a uniformly charged sphere of radius a,

$$\vec{E} = \begin{cases} \frac{Q}{r^2}\hat{r} & r > a \\ \frac{Qr}{a^3}\hat{r} & r \le a \end{cases}$$

Since the potential is zero at infinity, the potential at any point P is

$$\phi(P) = -\int_{\infty}^{P} \vec{E} \cdot d\vec{s}.$$

We can make the path of integration come radial straight in. If the point P has $r < a$,

$$\phi(r) = -\int_{\infty}^{r} E_{r'}\, dr' = -\int_{\infty}^{a} \frac{Q}{r'^2}\, dr' - \int_{a}^{r} \frac{Qr'}{a^2}\, dr' = \frac{Q}{a} - \frac{Qr^2}{2a^3} + \frac{Q}{2a}$$

In SI units,

$$\phi(0) = \frac{1}{4\pi\epsilon_0}\frac{3Q}{2a} = \frac{3 \cdot 79(1.6 \times 10^{-19}\,\mathrm{C})}{4\pi(8.85 \times 10^{-12}\,\mathrm{C}^2/(\mathrm{N\,m}^2))2(6 \times 10^{-15}\,\mathrm{m})} = 28.4\,\mathrm{MV}.$$