Massachusetts Institute of Technology
Experimental Study Group

Physics 8.022, Spring 2011

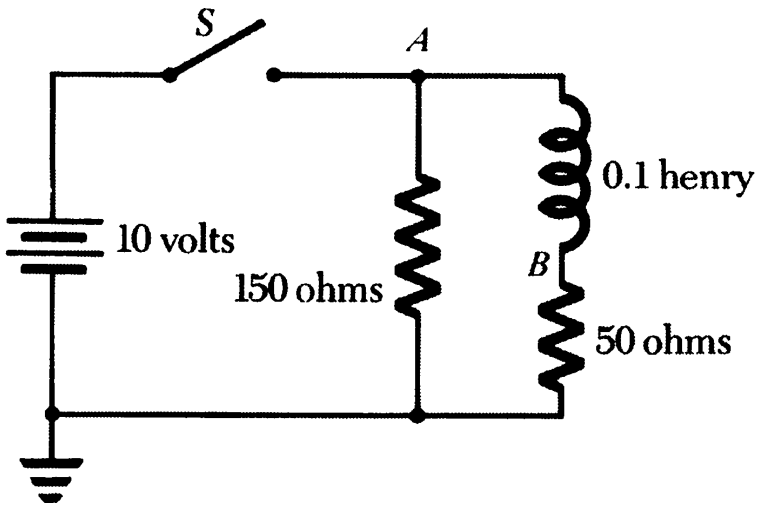
Problem Set 10 Solutions
RLC circuits, AC circuits

Due: Wednesday, April 27th, 10 pm

Problem 1: Purcell 7.17

Problem

In the circuit shown in the diagram the 10-volt battery has negligible internal resistance. The switch S is closed for several seconds, then opened. Make a graph with the potential of point A with respect to ground, just before and then for 10 milliseconds after the opening of switch S. Show also the variation of the potential at point B in the same period of time.



Extra question — By grounding this circuit, we make the switch safer to operate. Describe why a large spark jumps across the switch when it is not grounded, and why the spark does not happen when it is grounded.

Solution

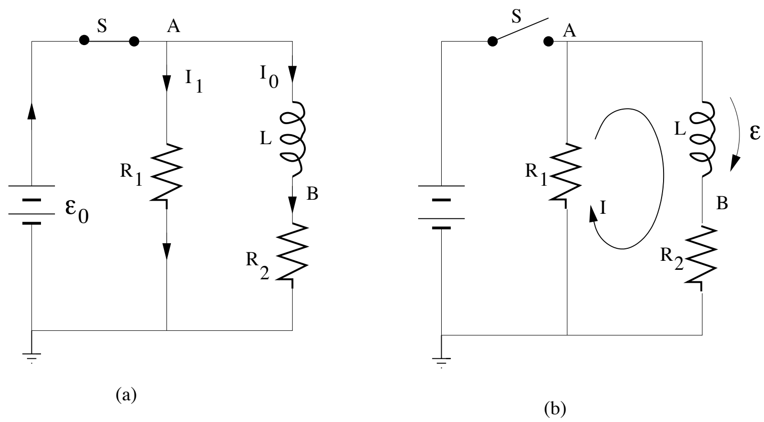


Figure 1: LR circuit: (a) steady state when switch S has been closed for a long time; (b) the current in LR circuit when switch S is opened.

Note: we'll use SI unit in this problem for convenience; and below $\mathcal{E}_0 = 10$ volts, $R_1 = 150$ ohms, $R_2 = 50$ ohms, $L = 0.1$ henry.

Switch S is closed (see Figure 1a) for several seconds, which is long enough that the current in the inductor is steady; hence the EMF due to the inductor $\mathcal{E} = 0$. Right right before S is opened, the current on L is $I_0 = \mathcal{E}_0/R_2 = 0.2\,\text{amp}$. The potential of point A with respect to ground is, for $t < 0$, $V_A = \mathcal{E}_0 = 10$volts; and the potential of point B is, for $t < 0$, $V_B = I_0 R_2 = \mathcal{E}_0 = 10$volts.

After switch S has been opened (Figure 1b), current will decay in the loop containing L, R_2 and R_1, and the change in current induces an electromotive force $\mathcal{E}$ on the inductor. Define the positive electromotive force and positive current as shown in Figure 1b; under this convention, they satisfy $\mathcal{E} = -L\frac{dI}{dt}$. Apply Kirchhoff's rule,

$$-\mathcal{E} + I(R_1 + R_2) = 0. \tag{1}$$

A simple calculation gives an ordinary differential equation for I:

$$L\frac{dI}{dt} = -(R_1 + R_2)I. \tag{2}$$

The solution of Equation 2 with initial condition $I(t = 0) = I_0$ is, for $t > 0$,

$$I(t) = I_0 e^{-t/\tau}, \tag{3}$$

$$\text{where} \qquad \tau = L/(R_1 + R_2) = 0.5\,\text{milliseconds}. \tag{4}$$

The potentials of point A and point B with respect to ground are, for $t > 0$

$$V_A = -I(t)R_1 = -I_0 R_1 e^{-t/\tau} = -30e^{-t/0.5}\ \text{volts}; \tag{5}$$

$$V_B = I(t)R_2 = I_0 R_2 e^{-t/\tau} = 10\ e^{-t/0.5}\ \text{volts}, \tag{6}$$

where t is in milliseconds. The plots of potentials Vs. time are shown in Figure 3.

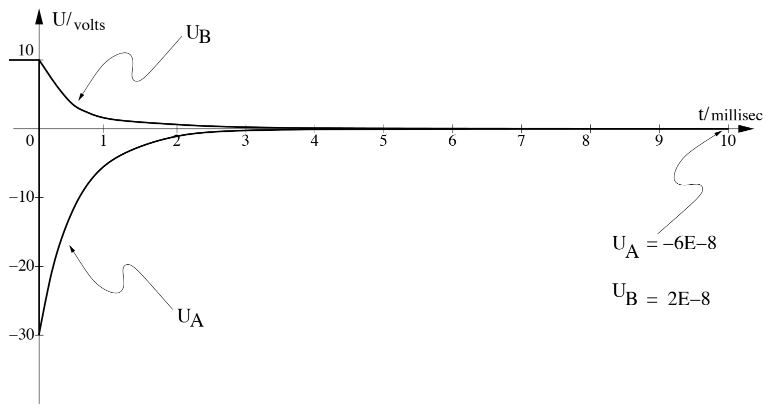


Figure 2: The exponential decay of the potentials in the LR circuit. At $t < 0$, both V_A and V_B are at 10 volts; at $t = 0$, V_A abruptly drops to -30 volts, while V_B is continuous.

Additional question:

Notice that the current in the 150 Ohm resistor has to abruptly switch magnitude and direction at the moment that the switch is opened. If the circuit were not grounded, the only way to suddenly change the direction of the current would be to "suck" the current needed to maintain continuity over from the battery – zap!

Ground fixes this: we can think of ground as an infinite sink or source of charge, and hence of current. By sucking (or dumping) the current needed from ground, the current in the 150 Ohm resistor can switch directions very rapidly – there is no need for a big zap across the switch.

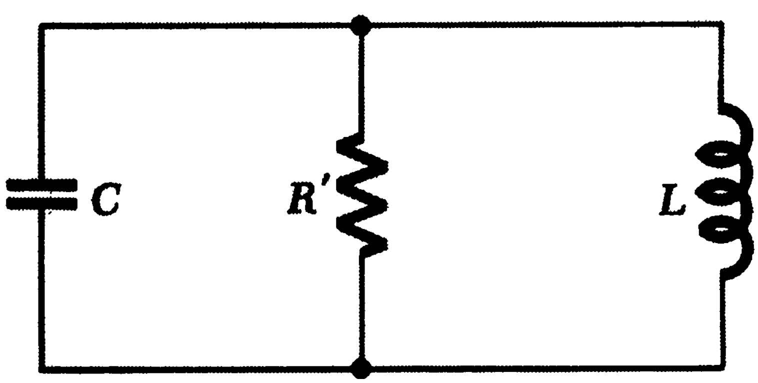
Problem 2: Purcell 8.4

Problem

In the resonant circuit shown in the figure below, the dissipative element is a resistor R' connected in parallel rather than in series, with the LC combination. Work out the equation analogous to Equation 2 in Purcell,

$$\frac{d^2V}{dt^2} + \left(\frac{R}{L}\right)\frac{dV}{dt} + \left(\frac{1}{LC}\right)V = 0,$$

which applies to this circuit. Find also the conditions on the solution analogous to those that hold in the series RLC circuit. If a series RLC and a parallel $R'LC$ circuit have the same L, C, and Q (quality factor), how must R' be related to R?



Solution

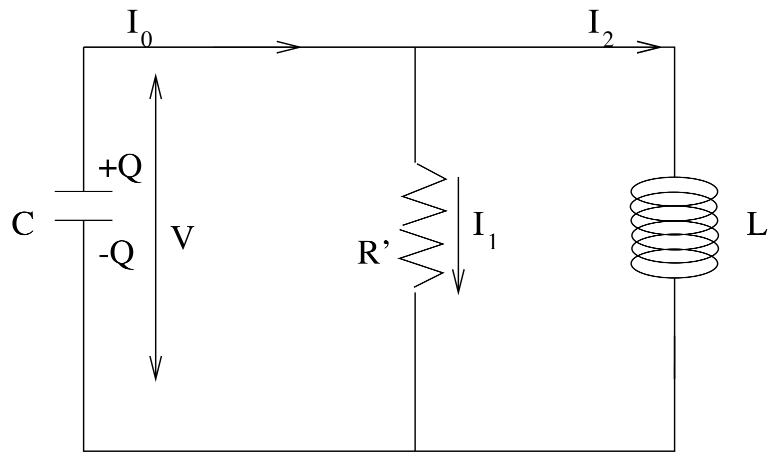


Figure 3: RLC circuit

If $V = Q/C$ is the voltage drop across the capacitor, then Kirchoff's laws for the left and right-hand loops above give:

$$V - I_1 R' = 0 \tag{7}$$

$$-L\frac{dI_2}{dt} + I_1 R' = 0 \tag{8}$$

$$I_0 = I_1 + I_2 \tag{9}$$

We also know that $I_0 = -dQ/dt$ and that $Q = CV$, so we can write

$$I_0 = -C\frac{dV}{dt} \tag{10}$$

The first Kirchoff equation gives us $I_1 = V/R'$. The derivative dI_2/dt in the second equation can be written:

$$\frac{dI_2}{dt} = \frac{d(I_0 - I_1)}{dt} = \frac{dI_0}{dt} - \frac{dI_1}{dt} = -C\frac{d^2V}{dt^2} - \frac{1}{R'}\frac{dV}{dt} \tag{11}$$

Thus we can rewrite the second Kirchoff equation as a differential equation for V:

$$\frac{d^2V}{dt^2} + \left(\frac{1}{CR'}\right)\frac{dV}{dt} + \left(\frac{1}{LC}\right)V = 0 \tag{12}$$

Compare this to the differential equation for V in the case of a serial LRC circuit, Purcell, Ch.8, eq.(2).

$$\frac{d^2V}{dt^2} + \left(\frac{R}{L}\right)\frac{dV}{dt} + \left(\frac{1}{LC}\right)V = 0 \tag{13}$$

Thus the solution to Equation 12 can be read off directly from the solution to Equation 13 (i.e. the discussion following eq.(2) of Purcell Ch.8) by the substitution

$$R \to \qquad \frac{L}{R'C} \tag{14}$$

and in particular $$\alpha = \frac{R}{2L} \to \qquad \alpha' = \frac{1}{2R'C} \tag{15}$$

Analogous to the serial RLC, the oscillation frequency is real and hence the solution oscillates when $\alpha' < \omega_0 = 1/\sqrt{LC}$; this means we must have

$$R' > \frac{1}{2}\sqrt{\frac{L}{C}} \tag{16}$$

When the parallel and serial circuits have the same L, C, and quality factor Q (not to be confused with the charge on the capacitor), we want to find a relation between R' and R. Recall that the quality factor for the serial RLC may be written

$$Q = \frac{\omega}{2\alpha} = \frac{\sqrt{\omega_0^2 - \alpha^2}}{2\alpha} \tag{17}$$

Thus Q for the parallel RLC

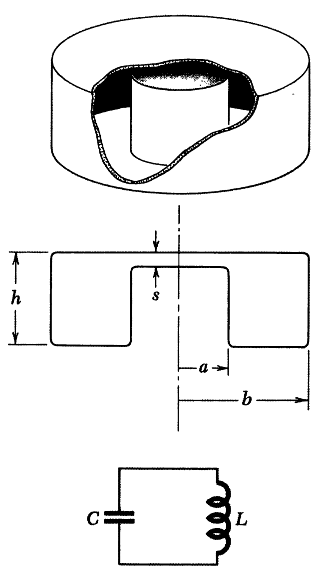
$$Q' = \frac{\sqrt{\omega_0^2 - \alpha'^2}}{2\alpha'} \tag{18}$$

Clearly $Q' = Q$ implies that $\alpha' = \alpha$, or $R' = L/RC$.

Problem 3: Purcell 8.7

Problem

A resonant cavity of the form illustrated is an essential part of many microwave oscillators. It can be regarded as a simple LC circuit. The inductance is that of a toroid with one turn. Find an expression for the resonant frequency of this circuit and show by a sketch the configuration of the magnetic and electric fields. Hint: the capacitor is composed by the upper and lower disks



Solution

The resonant cavity is equivalent to a simple LC circuit. The inductor is a configuration of currents uniformly flowing up and down on the surface of the inner conducting cylinder and the outer one.

The capacitor is a pair of parallel plates of separation s and area $A = \pi a^2$. Quote the result in Problem 8 of pset 8: the inductance *per unit length* is

$$L/l = \frac{2}{c^2} \ln\left(\frac{b}{a}\right).$$

This gives the total inductance

$$L = (h-s)\frac{2}{c^2} \ln\left(\frac{b}{a}\right).$$

[You could also use Purcell (58) of Chapter 7; you would get h in place of $h-s$, which is good enough for small s.]

The capacitor is $C = A/4\pi s = a^2/4s$. So the resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \left(\frac{a^2(h-s)}{2sc^2}\ln\left(\frac{b}{a}\right)\right)^{-1/2} \approx \left(\frac{a^2 h}{2sc^2}\ln\left(\frac{b}{a}\right)\right)^{-1/2}. \tag{19}$$

The configuration of the electric and magnetic field is shown in Figure 4.

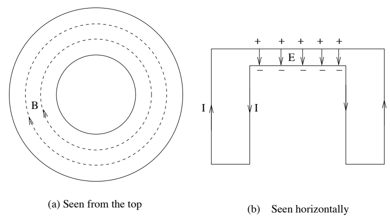


Figure 4: The configuration of $\vec{B}$ (a) and $\vec{E}$ field (b). The magnetic field lines are circles between the inner and the outer cylinders. The electric field lines are parallel straight lines between the top plates.

Problem 4: Purcell 8.9

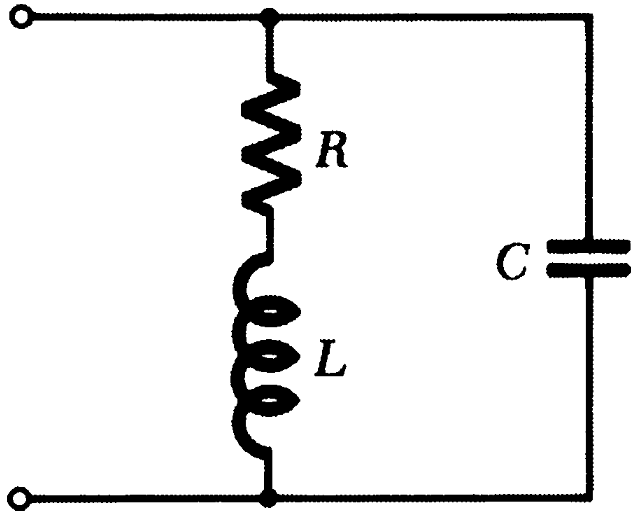
Problem

Using the equations 10 and 13 in Purcell (below), express the effect of damping on the frequency of a series RLC circuit.

$$\omega^2 = \frac{1}{LC} - \alpha\frac{R}{L} + \alpha^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$Q = \omega\frac{\text{energy stored}}{\text{average power dissipated}}$$

Let $\omega_0 = 1/\sqrt{LC}$ be the frequency of the undamped circuit. Suppose enough resistance is added to bring Q from ∞ down to 1000. By what percentage is the frequency ω thereby shifted from ω_0?



Solution

Use eqs.(13) or (14) of Purcell p.301-302.

$$R = \frac{\omega L}{Q} \quad \rightarrow \quad \frac{R}{L} = \frac{\omega}{Q}\,. \tag{20}$$

Plug this into eq.(10) of Purcell p.299.

$$\omega^2 = \omega_0^2 - \frac{\omega^2}{4Q^2}, \tag{21}$$

$$\text{so} \qquad \omega = \omega_0 \left(1 + \frac{1}{4Q^2}\right)^{-1/2} \tag{22}$$

$$= \omega_0 \frac{2Q}{\sqrt{1+4Q^2}}. \tag{23}$$

where $\omega_0 = 1/\sqrt{LC}$. The percentage change is

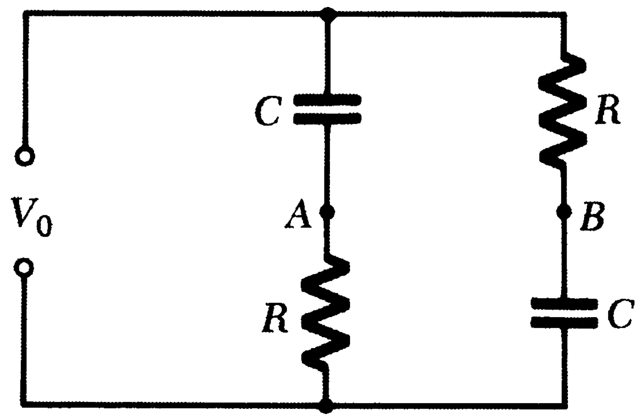
$$\begin{aligned}\text{Percentage change} &= \frac{\omega_0 - \omega}{\omega_0} \times 100\% \\ &= \left[1 - \left(1 + \frac{1}{4Q^2}\right)^{-1/2}\right] \times 100\% \\ &\approx \frac{1}{8Q^2} \times 100\% \\ &= 1.25 \times 10^{-5}\%,\end{aligned} \tag{24}$$

for $Q = 1000$. (The binomial expansion was used to simplify the exact formula.)

Problem 5: Purcell 8.12

Problem

Let $V_{AB} = V_B - V_A$, in this circuit. Show that $|V_{AB}|^2 = V_0^2$ for any frequency ω. Find the frequency for which V_{AB} is 90° out of phase with V_0.



Solution

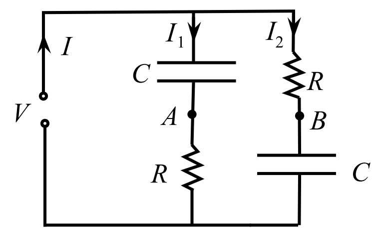


Figure 5: Circuit with resistors and capacitors

Considering the two circuit loops that contain V_0, we can find the following equations relating the currents I_1 and I_2 to the voltage V_0:

$$V_0 = I_1 Z_1 = I_1 \left((\frac{-i}{\omega C}) + R \right) \qquad V_0 = I_2 Z_2 = I_2 \left(R + (\frac{-i}{\omega C}) \right)$$

The voltage $V_{AB} = V_B - V_A$ is then:

$$V_{AB} = -I_2 R + I_1 (\frac{-i}{\omega C})$$

Substituting for the currents I_1 and I_2, we find:

$$V_{AB} = -\frac{V_0 R}{R + (\frac{-i}{\omega C})} + \frac{V_0 (\frac{-i}{\omega C})}{R + (\frac{-i}{\omega C})} = -V_0 \frac{R + \frac{i}{\omega C}}{R - \frac{i}{\omega C}} = V_0 \frac{1 - i\omega RC}{1 + i\omega RC}$$

We can take the absolute value of both side to get:

$$|V_{AB}|^2 = |V_0|^2 \left(\frac{1 - i\omega RC}{1 + i\omega RC} \right) \left(\frac{1 + i\omega RC}{1 - i\omega RC} \right) = |V_0|^2 \frac{1 + (RC\omega)^2}{1 + (RC\omega)^2} = |V_0|^2$$

In order for V_{AB} and V_0 to be out of phase by $\phi = -\pi/2$, we must have

$$V_{AB} = V_0 e^{i\phi} = V_0 e^{-i\frac{\pi}{2}} = -iV_0$$

Therefore ,

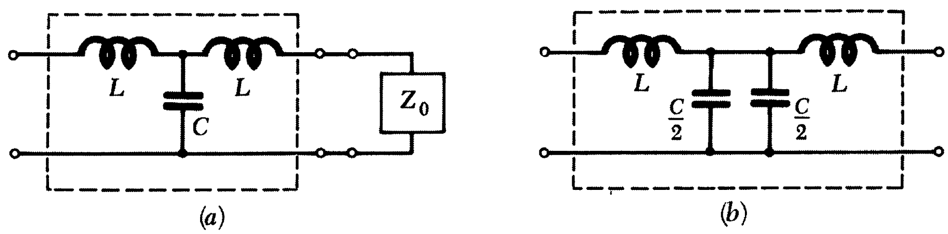
$$\frac{1 - i\omega RC}{1 + i\omega RC} = -i \quad \Longrightarrow \quad \frac{1 - (RC\omega)^2}{1 + (RC\omega)^2} - i \frac{2RC\omega}{1 + (RC\omega)^2} = -i$$

Equating the real parts (the imaginary parts give the same result), we find:

$$1 - (RC\omega)^2 = 0 \quad \Longrightarrow \quad \omega = \frac{1}{RC}$$

Problem 6: Purcell 8.16

Problem



The box (a) with four terminals contains a capacitor C and two inductors of equal inductance L connected as shown. An impedance Z_0 is to be connected to the terminals on the right. For a given frequency ω find the value which Z_0 must have if the resulting impedance across the left terminals is Z_0. You will find that the required Z_0 is a pure resistance R_0 provided $\omega^2 < 2/LC$. What is Z_0 in the special case $\omega = \sqrt{2/LC}$? It helps in understanding that case to note that the contents of the box (a) can be equally well represented by box (b).

Solution

We combine the impedances like resistances so that the total impedance is

$$Z = Z_L + \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L + Z_0}} \quad ,$$

with $Z_L = i\omega L$ and $Z_C = 1/i\omega C$. We set this equal to Z_0 and simplify to obtain

$$Z_0 = \sqrt{-\omega^2 L^2 + 2L/C} \quad .$$

This will be pure real and thus a pure resistance if

$$-\omega^2 L^2 + 2\frac{L}{C} > 0 \quad ,$$

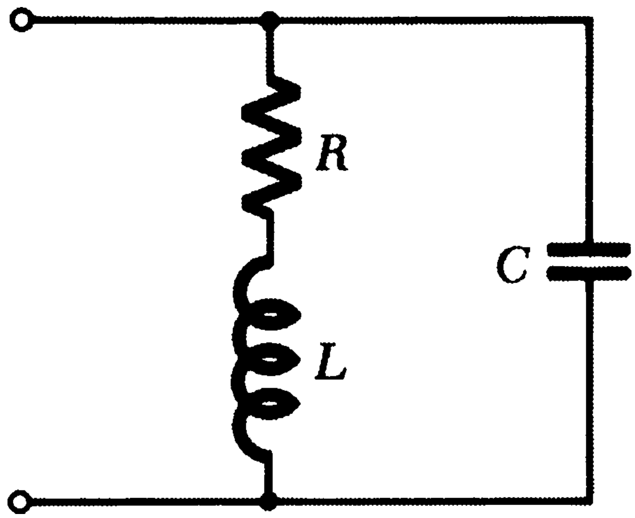
$$\omega^2 < \frac{2}{LC} \quad .$$

In the special case $\omega = \sqrt{2/LC}$, we have $Z_0 = 0$.

Problem 7: Optional Purcell 8.10

Problem

Is it possible to find a frequency at which the impedance at the terminals of the circuit below will be purely real?



Solution

$$
\begin{aligned}
\frac{1}{Z} &= i\omega C + \frac{1}{R + i\omega L} \\
\frac{1}{Z} &= \left[i\omega C + \frac{R - i\omega L}{R^2 + (\omega L)^2} \right] \\
&= \left[i(\omega C - \frac{\omega L}{R^2 + (\omega L)^2}) + (\text{real part}) \right].
\end{aligned} \tag{25}
$$

To make Z purely real, Z^{-1} must be real. So

$$\omega C = \frac{\omega L}{R^2 + (\omega L)^2}, \tag{26}$$

$$\omega = \sqrt{\frac{L - CR^2}{CL^2}}. \tag{27}$$

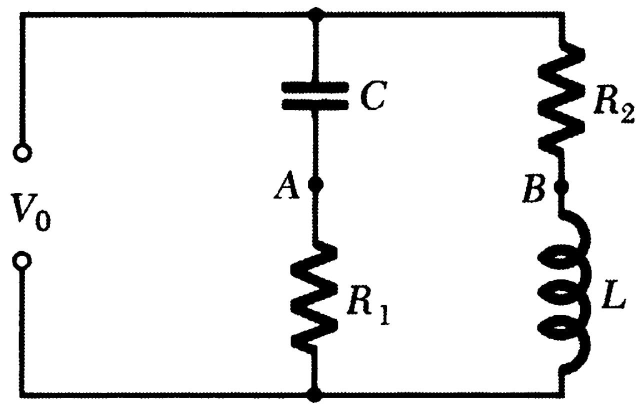
Note that the argument inside a square root must be positive. Therefore the condition under which it is possible to find a frequency so that Z is purely real is

$$L/C > R^2. \tag{28}$$

Problem 8: Optional Purcell 8.13

Problem

Show that, if the condition $R_1 R_2 = L/C$ is satisfied by the components of the circuit below, the difference in voltage between points A and B will be zero at any frequency. Discuss the suitability of this circuit as an AC bridge for measurement of an unknown inductance.



Solution

Refer to the figure for Problem 8.13 of Purcell p.321. The impedance of the left path ($C - R_1$) and the right ($R_2 - L$) are respectively

$$Z_1 = R_1 - \frac{i}{\omega C}, \qquad Z_2 = R_2 + i\omega L.$$

Set the voltage at the bottom of the circuit to be zero. Then the voltages at the points A and B are respectively

$$\begin{aligned} V_A &= \frac{V_0}{Z_1} R_1 = \frac{V_0 R_1}{R_1 - \frac{i}{\omega C}}, \\ V_B &= \frac{V_0}{Z_2}(i\omega L) = \frac{V_0 i\omega L}{R_2 + i\omega L}. \end{aligned} \tag{29}$$

Solve $V_A = V_B$.

$$\begin{aligned} R_1(R_2 + i\omega L) &= i\omega L(R_1 - \frac{i}{\omega C}) \\ \text{or} \qquad R_1 R_2 &= L/C. \end{aligned} \tag{30}$$

Since condition (30) doesn't depend on the value of frequency, we conclude that, if condition (30) is satisfied, the voltage difference between points A and B will be zero at any frequency.

We may employ this condition to measure unknown inductance. The circuit is exactly the same, but we connect points A and B by an AC voltmeter. The values of R_1, R_2 and C are known; at least one of the resistance should be adjustable. We then adjust the resistance so that the reading in the voltmeter vanishes. If necessary, we may adjust the frequency to check that this vanishing doesn't depend on frequency. At the vanishing point the condition (30) is satisfied and the inductance is measured by $L = R_1 R_2 C$.

Problem 9: Optional Purcell 8.14

Problem

In the laboratory, you find an inductor of unknown inductance L and unknown internal resistance R. Using a DC ohm-meter, an AC volt-meter of high impedance, a 1-microfarad capacitor, and a 1000-Hz signal generator, determine L and R as follows: According to the ohm-meter, R is 35 ohms. You connect the capacitor in series with the inductor and the signal generator. The voltage across both is 10.1 volts. The voltage across the capacitor alone is 15.5 volts. You note also, as a check, that the voltage across the inductor alone is 25.4 volts. How large is L? Is the check consistent?

Solution

Consider the RLC in series.

$$Z_{LR} = R + i\omega L\,, \qquad Z_c = -i/\omega C\,, \qquad Z_{\text{total}} = R + i(\omega L - 1/\omega C)\,. \tag{31}$$

From the potential across both and across the capacitor,

$$\frac{V_{\text{total}}}{V_C} = \frac{Z_{\text{total}}}{Z_C} = i\omega CR + (1 - \omega^2 CL)\,, \tag{32}$$

$$\text{so} \qquad \left|\frac{V_{\text{total}}}{V_C}\right| = \sqrt{(\omega CR)^2 + (1 - \omega^2 CL)^2}\,, \tag{33}$$

After doing some math, we find the expression for L,

$$L = \frac{1}{\omega^2 C}\left[1 \pm \sqrt{\left|\frac{V_{\text{total}}}{V_C}\right|^2 - (\omega CR)^2}\right] \tag{34}$$

Plug in $\omega = 2\pi f = 2\pi \times 1000$ Hz, $V_{\text{total}} = 10.1$ volts, $V_C = 15.5$ volts, $C = 10^{-6}$ farad, $R = 35$ ohm, we get

$$L = 0.041 \text{ henry} \qquad \text{or} \qquad 0.0098 \text{ henry}.$$

To check the consistency, calculate the ratio

$$\frac{V_{LR}}{V_C} = \frac{Z_{LR}}{Z_C} = i\omega CR - \omega^2 CL \tag{35}$$

$$\text{so} \qquad \left|\frac{V_{LR}}{V_C}\right| = \sqrt{(\omega CR)^2 + (\omega^2 CL)^2} \tag{36}$$

$$= 1.63 \text{ for L=0.041 henry}$$

$$0.45 \text{ for L=0.0098 henry}. \tag{37}$$

The measurement gives $\left|\frac{V_{LR}}{V_C}\right| = 25.4/15.5 = 1.64$. So the correct value for L is $L = 0.041$ henry and the check is consistent.