MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

8.022 **Spring 2011**

Problem Set 6 Solutions

Problem 1: Resistance of Atlantic Ocean The first telegraphic messages crossed the Atlantic Ocean in 1858, by a cable 3000 km long laid between Newfoundland and Ireland. The conductor in this cable consisted of seven copper wires, each of diameter 0.73 mm, bundled together and surrounded by an insulating sheath.

a) Calculate the resistance of the conductor. Use $3\times 10^{-8}\ \Omega\cdot\text{m}$ for the resistivity of copper, which was of somewhat dubious purity.

b) A return path for the current was provided by the ocean itself. Given that the resistivity of seawater is about $0.25\ \Omega\cdot\text{m}$, see if you can show that the resistance of the ocean return would have been much smaller than that of the cable.

Solution:

In an ohmic material, the electric field is proportional to the current density, $\vec{\mathbf{E}} = \rho\vec{\mathbf{J}}$ where the constant of proportionality ρ is called the resistivity. (Note that the symbol ρ does not mean the charge density. A good practice would be to introduce a extra subscript to denote resistivity ρ_r.) The resistance is defined according to

$$R = \frac{|\Delta V|}{|I|} = \frac{\left|\int \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\right|}{\left|\iint_{open} \vec{\mathbf{J}}\cdot d\vec{\mathbf{a}}\right|} = \rho\frac{\left|\int \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\right|}{\left|\iint_{open} \vec{\mathbf{E}}\cdot d\vec{\mathbf{a}}\right|}.$$

For a wire, the electric field is uniform (hence the current density). So the two integrals are easy to calculate

$$R = \rho\frac{\left|\int \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\right|}{\left|\iint_{open} \vec{\mathbf{E}}\cdot d\vec{\mathbf{a}}\right|} = \frac{\rho EL}{EA} = \frac{\rho L}{A}.$$

When current flows in the cable, the ends of each of the seven copper wires are held at the same voltage difference, so the wires are in parallel. Recall that when resistors are in parallel, the equivalent resistance adds inversely:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots.$$

Since resistance is inversely proportional to area, we have that

$$\frac{1}{R_{eq}} = \frac{7}{R} = \frac{7A}{\rho L} = \frac{A_{TOT}}{\rho L}$$

A_{TOT} is the effective area

$$R_{eq} = \frac{\rho L}{A_{TOT}} = \frac{3 \cdot 10^{-8}\,\Omega m \times 3 \cdot 10^{6} m}{7\pi\left(\frac{7.3}{2} \times 10^{-4} m\right)^2} = 3 \times 10^{4}\,\Omega$$

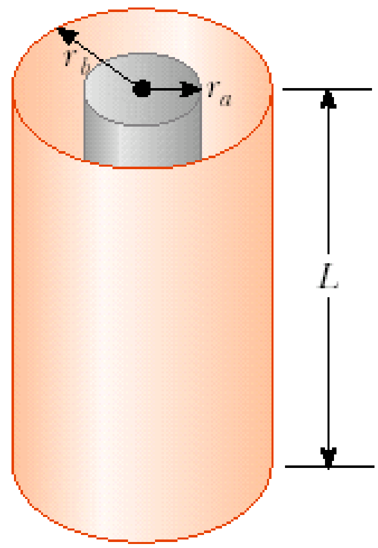
b) In order to calculate the resistance of the return path we must know the area over which the current spreads.

If we assume that the current spreads over an area of $10^4\ m^2$ then

$$R_{ocean} \sim \frac{\rho L}{A_{ocean}} = \frac{0.25\,\Omega m \cdot 3 \cdot 10^{6} m}{10^4\ m^2} = 75\,\Omega$$

that is indeed much smaller than that of the cable.

Problem 2: *Sea Water*



An oceanographer is studying how the ion concentration in sea water depends on depth. She does this by lowering into the water (until completely submerged) a pair of concentric metallic cylinders (see figure) at the end of a cable and taking data to determine the resistance between these electrodes as a function of depth. The water between the two cylinders forms a cylindrical shell of inner radius r_a , outer radius r_b, and length L much larger than r_b. The scientist applies a potential difference ΔV between the inner and outer surfaces, producing an outward radial current I. Let ρ represent the resistivity of the water.

a) Find the resistance of the water between the cylinders in terms of L, ρ, r_a, and r_b.

We will build up the total resistance of the water by considering a number of cylindrical shells of water in series. Consider a thin cylindrical shell of radius r, thickness dr, and length L. Its contribution to the overall resistance of the water is

$$dR = \frac{\rho d\ell}{A} = \frac{\rho dr}{(2\pi r)L} = \left(\frac{\rho}{2\pi L}\right)\frac{dr}{r}$$

The resistance of the whole annulus is the series summation of the contributions of the thin shells:

$$R = \frac{\rho}{2\pi L}\int_{r_a}^{r_b}\frac{dr}{r} = \frac{\rho}{2\pi L}\ln\left(\frac{r_b}{r_a}\right)$$

(b) Express the resistivity of the water in terms of the measured quantities L, r_a, r_b, ΔV, and I.
Since

$$R = \frac{\rho}{2\pi L}\ln\left(\frac{r_b}{r_a}\right) = \frac{\Delta V}{I},$$

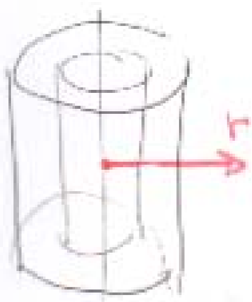
the resistivity can be written as

$$\rho = \frac{2\pi L(\Delta V / I)}{\ln(r_b/r_a)}$$

a) There is a steady radial current I.
Due to the symmetry of the problem

$$\vec{E} = E_r(r)\,\hat{r} \qquad (1)$$

$$I = \int \vec{J}\cdot d\vec{A} \underset{\substack{\text{through a}\\ \text{cylindrical}\\ \text{shell of}\\ \text{radius } r\in(r_a,r_b)}}{=} J\cdot 2\pi r L \underset{J=\sigma E_r}{=} 2\pi r E_r \sigma L \qquad (2)$$



For the current to be steady

$$E_r \sim r^{-1}$$

Therefore we make the ansatz

$$\phi(r) = A \ln r \qquad (3)$$

Boundary Conditions $\Delta V = \phi(r_b) - \phi(r_a) = A \ln\left(\frac{r_b}{r_a}\right)$

$$\Downarrow$$

$$A = \frac{\Delta V}{\ln\left(\frac{r_b}{r_a}\right)}$$

$$E_r(r) = -\frac{\partial\phi}{\partial r} = -\frac{A}{r} = \frac{\Delta V}{r}\frac{1}{\ln(r_b/r_a)} \qquad (4)$$

Substitute (4) in (2)

$$I = 2\pi \cancel{r} \sigma L \frac{\Delta V}{\cancel{r}}\frac{1}{\ln(r_b/r_a)} = \frac{2\pi\sigma L}{\ln(r_b/r_a)}\Delta V \overset{\text{Ohm's law}}{=} \frac{\Delta V}{R}$$

$$R = \frac{\ln(r_b/r_a)\,\rho}{2\pi L}$$

Problem 3: ***Non-uniform Conductivity*** A cylindrical glass rod is heated with a torch until it conducts enough current to cause a light bulb to glow. The rod has a length $L = 2\text{ cm}$, a diameter $d = 0.5\text{ cm}$, and its ends, plated with material of infinite conductivity, are connected to the rest of the circuit. When red hot, the rod's conductivity varies with position x measured from the center of the rod as $\sigma(x) = \sigma_0 L^4 / x^4$, with $\sigma_0 = 4\times10^{-2}(\Omega\cdot\text{m})^{-1}$.

a) What is the resistance of the glass rod? Express your answer both symbolically and as a value in ohms.

We shall consider a small slice of the rod of thickness dx located a distance x measured from the center of the rod . The resistance dR of this small section is given by

$$dR = \frac{\rho_r dx}{A} = \frac{dx}{\sigma_c A} = \frac{4x^4 dx}{\sigma_0 L^4 \pi d^2}.$$

The total resistance of the rod is then the integral over the whole rod since these small sections can be thought of as in series,

$$R = \int_{-L/2}^{L/2} dR = \int_{-L/2}^{L/2} \frac{4x^4 dx}{\sigma_0 L^4 \pi d^2} = \left.\frac{4x^5}{5\sigma_0 L^4 \pi d^2}\right|_{-L/2}^{L/2} = \frac{L}{20\sigma_0 \pi d^2}.$$

$$R = \frac{(2\times10^{-2}\text{ m})}{(20)(4\times10^{-2}(\Omega\cdot\text{m})^{-1})(\pi)(0.5\times10^{-2}\text{ m})^2} = 318\,\Omega$$

b) When a voltage ΔV is applied between the two ends, what is the current density $\vec{\mathbf{J}}(x)$ and what is the steady-state electric field $\vec{\mathbf{E}}(x)$?

The current density is given by

$$\left|\vec{\mathbf{J}}(x)\right| = \frac{I}{A} = \frac{4\Delta V}{R\pi d^2} = \frac{80\sigma_0 \Delta V}{L}$$

This is independent of the position in the rod. The electric field is given by

$$\vec{\mathbf{E}}(x) = \frac{\vec{\mathbf{J}}(x)}{\sigma} = \frac{80\sigma_0 \Delta V / L}{\sigma_0 L^4 / x^4}\hat{\mathbf{i}} = \frac{80\Delta V}{L^5}x^4\hat{\mathbf{i}}.$$

c) In steady-state, what is the volume charge density $\rho(x)$ found within the rod? (Note that you are not asked for the surface charge σ that will accumulate on the surfaces.)

From the equation: $\rho = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{\mathbf{E}}(x)$, and our previous result we have that

$$\rho(x) = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{\mathbf{E}}(x) = \frac{1}{4\pi}\frac{d}{dx}\frac{80\Delta V}{L^5}x^4 = \frac{80\Delta V}{\pi L^5}x^3$$

Problem 4: ***Battery Lifetime***

A standard D cell can supply 10 mA at 1.5 V for about 300 hours. An alkaline D cell can do about the same. Assume that the chemistry of the batteries will produce the same amount of energy and will maintain the emf $\mathcal{E} = 1.5\text{ V}$ regardless of the current drawn from the battery. However the internal resistance of a standard D cell is about $1\,\Omega$ and the internal resistance of an alkaline D cell is about $0.1\,\Omega$, so the amount of energy dissipated internally is different. Suppose that you have a multi-speed winch that is 50% efficient, and that you are trying to lift a mass of 60 kg. The winch acts as load with a variable resistance R_L at different speeds.

a) Suppose the winch is set to super-slow speed. Then the load resistance is much greater than the internal resistance and you can assume that there is no loss of energy to internal resistance. How high can each battery lift the mass before the battery uses all of its energy?

This is just a question of energy. Each battery has an energy storage of

$$U_{emf} = (\Delta V)(I)(\Delta t) = (1.5\text{ V})(10\text{ mA})(300\text{ hours})(\frac{3600\text{ s}}{\text{hour}}) = 1.6\times 10^4\text{ J}.$$

So they can both lift the mass the same distance:

$$U = mgh \Rightarrow \quad h = \frac{U}{mg} = \frac{(1.6\times 10^4\text{ J})\frac{1}{2}}{(60\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})} = 13.8\text{ m}$$

The factor of a half is there because the winch is only 50% efficient.

b) For each battery, what should the resistance of the winch be set at in order to have the battery lift the mass at the fastest rate?

The power delivered to the winch (this is called the load) is

$$P_L = I^2 R_L = \left(\frac{\varepsilon}{R_i + R_L}\right)^2 R_L.$$

We can maximize this by considered the derivative with respect to R_L:

$$\frac{dP_L}{dR_L} = \mathcal{E}^2\left(\left(\frac{1}{R_i + R_L}\right)^2 - 2R_L\left(\frac{1}{R_i + R_L}\right)^3\right) = 0.$$

Solve this equation for R_L:

$$\left(\frac{1}{R_i + R_L}\right)^2 = 2R_L\left(\frac{1}{R_i + R_L}\right)^3,$$

$$R_i + R_L = 2R_L,$$

$$R_L = R_i.$$

The current is then

$$I_{\max} = \frac{\varepsilon}{R_i + R_L} = \frac{\varepsilon}{2R_i}.$$

The power delivered to the load is

$$P_{L,\max} = I_{\max}{}^2 R_L = \left(\frac{\varepsilon}{2R_i}\right)^2 R_i = \frac{1}{4}\frac{\varepsilon^2}{R_i}$$

The maximum speed can be determined by the relation

$$(efficency)(P_{\max}) = Fv_{\max} = mgv_{\max}$$

Note that only half the power is used in lifting the weight, the other (1/2) is wasted in the winch. (This inefficiency is different then the efficiency due to the power wasted as internal heat in the battery.)

Therefore the maximum velocity is

$$v_{\max} = \frac{(efficency)(P_{\max})}{mg} = \frac{(efficency)}{mg}\frac{1}{4}\frac{\varepsilon^2}{R_i}.$$

The energy supplied by the battery when it is delivering maximum current is

$$U_{emf.\max} = \varepsilon I_{\max}\Delta t_{\max} = \varepsilon\frac{\varepsilon}{2R_i}\Delta t_{\max}$$

So the battery will operate for a time interval

$$\Delta t_{\max} = \frac{2R_i}{\varepsilon^2}U_{emf.\max}.$$

During the time the battery is operating, the winch will lift the object a distance,

$$h_{\max} = v_{\max}\Delta t_{\max} = \left(\frac{(efficency)}{mg}\frac{1}{4}\frac{\varepsilon^2}{R_i}\right)\frac{2R_i}{\varepsilon^2}U_{emf.\max} = \frac{1}{2}\left(\frac{(efficency)}{mg}U_{emf.\max}\right).$$

So the maximum height is independent of the internal resistance and hence is the same for both batteries.

c) What is the fastest rate in m/sec that each battery can lift you?

For the standard D-cell: The internal resistance is $R_i = 1\,\Omega$. So the maximum current is

$$I_{\text{standard,max}} = \frac{\varepsilon}{2R_i} = \frac{(1.5\text{ V})}{(2)(1.0\,\Omega)} = 0.75\text{ A}$$

The maximum power is

$$P_{\text{standard,max}} = \frac{1}{4}\frac{\varepsilon^2}{R_i} = \frac{1}{4}\frac{(1.5\text{ V})^2}{(1\,\Omega)} = 0.56\text{ W}.$$

The maximum speed the winch can operate is

$$v_{\text{standard,max}} = \frac{(efficency)(P_{\text{standard,max}})}{mg} = \frac{(0.5)(0.56\text{ W})}{(60\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})} = 4.8\times10^{-4}\text{m}\cdot\text{s}^{-1}$$

For the alkaline D cell with a internal resistance is about $R_i = 0.1\,\Omega$. So the maximum current is

$$I_{\text{alkaline,max}} = \frac{\varepsilon}{2R_i} = \frac{(1.5\text{ V})}{(2)(0.1\,\Omega)} = 7.5\text{ A}$$

The maximum power is

$$P_{\text{alkaline,max}} = \frac{1}{4}\frac{\varepsilon^2}{R_i} = \frac{1}{4}\frac{(1.5\text{ V})^2}{(0.1\,\Omega)} = 5.6\text{ W}.$$

The maximum speed the winch can operate is

$$v_{\text{alkaline,max}} = \frac{(efficency)(P_{\text{alkaline,max}})}{mg} = \frac{(0.5)(5.6\text{ W})}{(60\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2})} = 4.8\times10^{-3}\text{m}\cdot\text{s}^{-1}$$

d) At this fastest lift rate, what is the maximum height that you can lift the mass before the battery dies?

During the time either battery is operating at maximum power output to the load, the winch will lift the object a distance,

$$h_{\max} = \frac{1}{2}\left(\frac{(efficency)}{mg}U_{emf.\max}\right) = \frac{1}{2}\frac{(0.5)(1.6\times10^4\ \text{J})}{(60\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})} = 6.9\ \text{m}\,.$$

The maximum speed height is ½ the super slow speed height because ½ of the energy is wasted in the internal resistance when the winch is operating at maximum speed.

e) How long does it take to reach this height?

For the standard D-cell,

$$\Delta t_{\text{standard, max}} = \frac{2R_i U_{emf.\max}}{\varepsilon^2} = \frac{(2)(1.0\ \Omega)(1.6\times10^4\ \text{J})}{(1.5\ \text{V})^2} = 1.4\times10^4\ \text{s} = 4\ \text{hours}\,.$$

For the alkaline D-cell,

$$\Delta t_{\text{alkaline, max}} = \frac{2R_i U_{emf.\max}}{\varepsilon^2} = \frac{(2)(0.1\ \Omega)(1.6\times10^4\ \text{J})}{(1.5\ \text{V})^2} = 1.4\times10^5\ \text{s} = 0.4\ \text{hours}\,.$$

5) Solution:

The equivalent resistance, R', due to the three resistors on the right is

$$\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_0 + R_1} = \frac{R_0 + 2R_1}{R_1(R_0 + R_1)}$$

or

$$R' = \frac{R_1(R_0 + R_1)}{R_0 + 2R_1}$$

Since R' is in series with the fourth resistor R_1, the equivalent resistance of the entire configuration becomes

$$R_{\mathrm{eq}} = R_1 + \frac{R_1(R_0 + R_1)}{R_0 + 2R_1} = \frac{3R_1^2 + 2R_1R_0}{R_0 + 2R_1}$$

If $R_{\mathrm{eq}} = R_0$, then

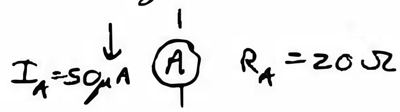
$$R_0(R_0 + 2R_1) = 3R_1^2 + 2R_1R_0 \Longrightarrow R_0^2 = 4R_1^2$$

or

$$R_1 = \frac{R_0}{\sqrt{3}}$$

Problem 6: Design an ohmmeter You own a micro-ammeter that reads $50\ \mu\text{A}$ at full scale deflection, and the coil in the meter movement has a resistance of $20\ \Omega$. By adding two resistors, R_1 and R_2, and a $1.5\ \text{V}$ battery you can convert this into an ohmmeter. When the two outgoing leads of this ohmmeter are connected together, the meter should register 0 ohms by giving full-scale deflection. When the leads are connected across an unknown resistance R_u, the deflection will indicate the resistance value if the scale is appropriately marked. In particular, we want half-scale deflection to indicate $15\ \Omega$. What values of R_1 and R_2, are required, how should the connections be made, and where on the ohm scale will the marks be (with reference to the old micro-ammeter calibration) for $5\ \Omega$ and $50\ \Omega$? Are there several different combinations that can solve this problem?

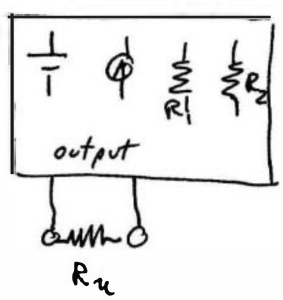
Problem: The ammeter has internal resistance $R_A = 20\,\Omega$. We want $50\,\mu A$ to flow through the ammeter at full scale deflection.



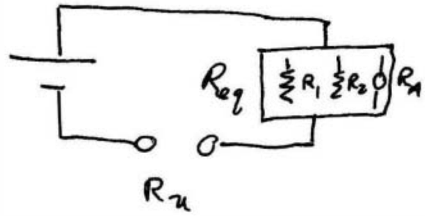
So the voltage difference across the ammeter

$$\Delta V_A = R_A I_A = (20\,\Omega)(50\times10^{-6})A = 10^{-3}\,V$$

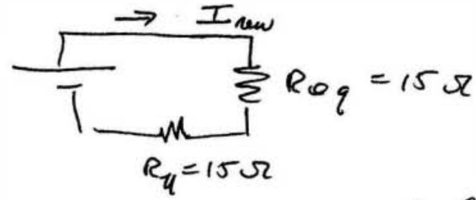
is only a milli-volt. If we introduce a $\mathcal{E} = 1.5$ volt battery and two resistors, we would like to convert current readings to resistance measurements on an unknown resistor R_u. We will calibrate our "ohmmeter" by demanding full scale deflection when $R_u = 0$, and when $R_u = 15\,\Omega$, ½ scale deflection which means $25\,\mu A$ flows through the ammeter.



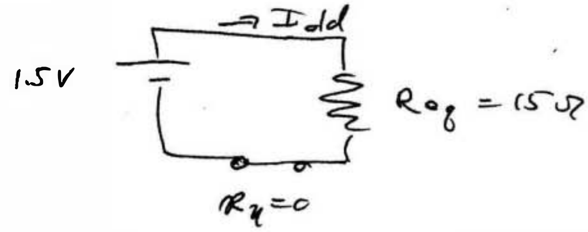
One possibility is putting the resistors and ammeter in some combination such that $R_{eq} = 15\,\Omega$.



Then when

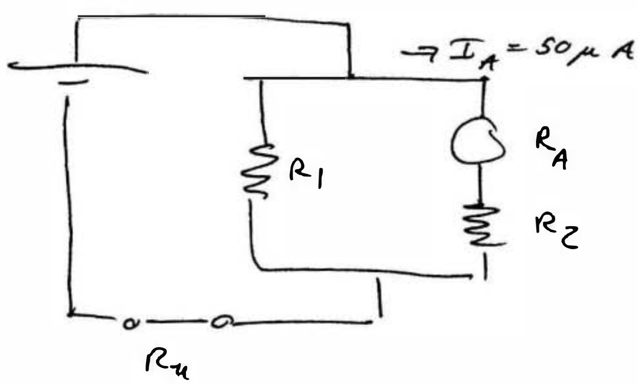


the $R_u = 15\,\Omega$ resister is connected, $I_{new} = \frac{1}{2} I_{old}$

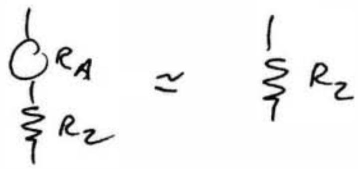


Note that $I_{old} = \dfrac{\mathcal{E}}{R_{eq}} = \dfrac{1.5\text{ V}}{15\,\Omega} = .1\text{ A}$

So this large current must be split



One way to accomplish this is to put a very large resistor R_2 in series with R_A then



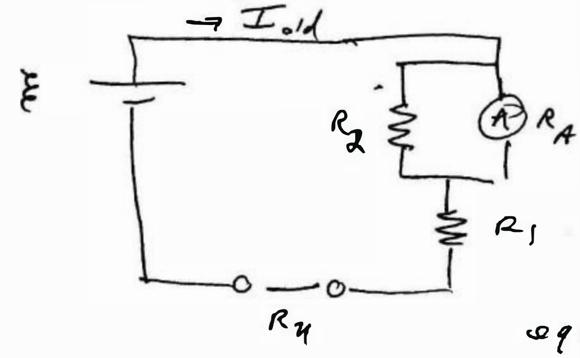
So $\quad I_A = \frac{\mathcal{E}}{R_2} \Rightarrow \quad R_2 = \frac{\mathcal{E}}{I_A} = \frac{1.5\,V}{50\,\mu A} = 3 \times 10^4\,\Omega$

then $\quad R_{eq} \cong \frac{R_1 R_2}{R_1 + R_2} = 15\,\Omega$

requires that R_2 be large compared to R_1.

So $\quad R_{eq} \simeq \frac{R_1 R_2}{R_2} \simeq R_1 = 15\,\Omega$.

A second solution is to shunt the current away from the ammeter with a very small resistor R_2. Then



$R_{eq}' = \frac{R_2 R_A}{R_A + R_2} \simeq R_2$

So the overall equivalent resistance

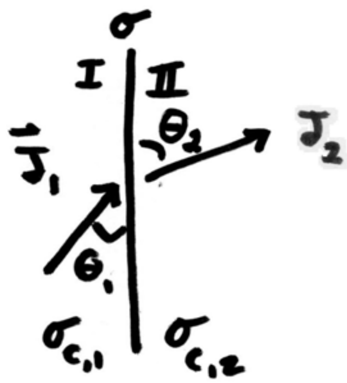
$R_{eq} = R_1 + R_{eq}' \simeq R_1 + R_2 \cong R_1 = 15\,\Omega$

Since we are apply $\Delta V_A = I_A R_A = 10^{-3}\,V$ across the ammeter, and $I_{old} = \frac{\mathcal{E}}{R_{eq}} \simeq \frac{\mathcal{E}}{R_1} = \frac{1.5\,V}{15\,\Omega} = .1\,A$

$R_2 \cong \frac{\Delta V_A}{I_{old}} \cong \frac{10^{-3}\,V}{.1\,A} \cong 10^{-2}\,\Omega$

Problem 7: An infinite conducting medium has two regions with conductivity $\sigma_{c,1}$ and $\sigma_{c,2}$, separated by a plane interface. In region 1 a uniform current density $\vec{\mathbf{J}}_1$ flows up to the interface, making at an angle θ_1 with the interface. $\vec{\mathbf{J}}_2$ flows away from the interface and makes an angle θ_2 with the interface. Assume that the charge density σ is not changing in time.

a) Find the direction and magnitude of the current density $\vec{\mathbf{J}}_2$ in region 2.
b) Find the surface charge density σ on the interface.



The tangential component of the electric field is continuous

$$E_1 \cos\theta_1 = E_2 \cos\theta_2 \,. \tag{0.1}$$

From Ohm's Law,

$$\vec{\mathbf{J}}_1 = \sigma_1 \vec{\mathbf{E}}_1 \,, \tag{0.2}$$

and

$$\vec{\mathbf{J}}_2 = \sigma_2 \vec{\mathbf{E}}_2 \,. \tag{0.3}$$

.

Therefore the tangential boundary condition becomes

$$\frac{J_1}{\sigma_1} \cos\theta_1 = \frac{J_2}{\sigma_2} \cos\theta_2 \,. \tag{0.4}$$

.

This implies that

$$\frac{J_2}{J_1}=\frac{\sigma_2}{\sigma_1}\frac{\cos\theta_1}{\cos\theta_2}. \qquad (0.5)$$

.

Then the normal component of the current density is continuous at the boundary

$$J_2\sin\theta_2 - J_1\sin\theta_1 = 0. \qquad (0.6)$$

which implies that

$$\frac{J_2}{J_1}=\frac{\sin\theta_1}{\sin\theta_2}. \qquad (0.7)$$

.

Setting Eq. (0.7) equal to Eq. (0.5) yields

$$\frac{\sigma_2}{\sigma_1}\frac{\cos\theta_1}{\cos\theta_2}=\frac{\sin\theta_1}{\sin\theta_2}, \qquad (0.8)$$

which simplifies to

$$\tan\theta_2=\frac{\sigma_1}{\sigma_2}\tan\theta_1. \qquad (0.9)$$

.

Thus the current density makes an angle

$$\theta_2=\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\tan\theta_1\right). \qquad (0.10)$$

.

Then we can use Eq. (0.7) and Eq. (0.10) to solve for the current density in region 2

$$J_2=J_1\frac{\sin\theta_1}{\sin\theta_2}=J_1\frac{\sin\theta_1}{\sin\left[\tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\tan\theta_1\right)\right]}. \qquad (0.11)$$

In order to solve for the surface charge density, we use the fact that the normal component of the electric field is discontinuous:

$$E_2 \sin\theta_2 - E_1 \sin\theta_1 = 4\pi\sigma\,. \tag{0.12}$$

Substituting Eq. (0.2) and Eq. (0.3) into Eq. (0.12) yields

$$\frac{J_2}{\sigma_2}\sin\theta_2 - \frac{J_1}{\sigma_1}\sin\theta_1 = 4\pi\sigma\,. \tag{0.13}$$

.

From Eq. (0.6), $J_2 \sin\theta_2 = J_1 \sin\theta_1$, therefore the current density becomes

$$\sigma = \frac{1}{4\pi} J_1 \sin\theta_1 \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right). \tag{0.14}$$

Problem 8: *Multiple Loop Circuits*

Consider the following circuit consisting of a voltage source ε_1, three resistors with resistances R_1, R_2, and R_3, and a capacitor with capacitance C connected together as shown in Figure 9.

This circuit has three branches. Identify the branches in this multiloop circuit and choose positive directions for the flow of currents I_1, I_2, and I_3 in each branch. Draw the direction of your currents in the figure below.

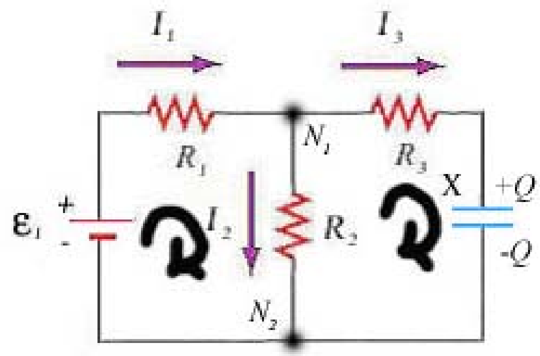


Figure: Multiloop current

Current conservation:

There are two node points in the circuit, N_1 and N_2, where current branches off (point N_1) or recombines (point N_2). (It helps to think of the flow of water in pipe that branches into two pipes and then recombines into one pipe). At each point the current into the node equals the current flowing out of the node,

$$I_{in} = I_{out}.$$

Write down the equation for current conservation.

$$I_1 = I_2 + I_3$$

Loop Rules:

There are three closed loops:

- Loop 1: formed by the voltage source ε_1 and the two resistors R_1 and R_2,
- Loop 2: formed by the capacitor C, and the two resistors R_3 and R_2,
- Loop 3: formed by the voltage source ε_1, the capacitor C, and the two resistors R_1 and R_3.

Loop 1 and Loop 2 are clearly visible. However, the outer perimeter of the circuit also forms Loop 3.

Choose directions of circulations for each loop and use the Loop Law to write down equations for each loop describing the sum of the voltage differences around the closed loop.

Loop 1: $\varepsilon_1 - I_1 R_1 - I_2 R_2 = 0$

Loop 2: $-\dfrac{Q}{C} + I_2 R_2 - I_3 R_3 = 0$

Loop 3: $\varepsilon_1 - I_1 R_1 - I_3 R_3 - \dfrac{Q}{C} = 0$

By either adding (or subtracting) the loop equations for Loop 1 and Loop 2 (depending on your choice of circulation direction), the voltage difference across resistor R_2 for these two loops have opposite (or the same) signs. Hence when we add (or subtract) the two equations, the voltage difference across resistor R_2 cancel, leaving the Loop Rule for Loop 3 as the result. Thus even though there are three loops, there are only two independent equations.

In order to find the differential equations that describe this multi-loop circuit, find a relationship between the charge on the capacitor and the current that flows in that branch.

In branch 3, the current that charges the capacitor is I_3, therefore

$$I_3 = \frac{dQ}{dt}$$

Using your results from above, write down the two Loop equations as differential equations for the charge on the capacitor plate Your equations should include terms that involve Q, dQ/dt, R_1, R_2, and R_3, and $\boldsymbol{\varepsilon}_1$.

Current conservation: $I_1 = I_2 + \dfrac{dQ}{dt}$

Loop 1: $\varepsilon_1 - I_1 R_1 - I_2 R_2 = 0$

Loop 2: $-\dfrac{Q}{C} + I_2 R_2 - \dfrac{dQ}{dt} R_3 = 0$

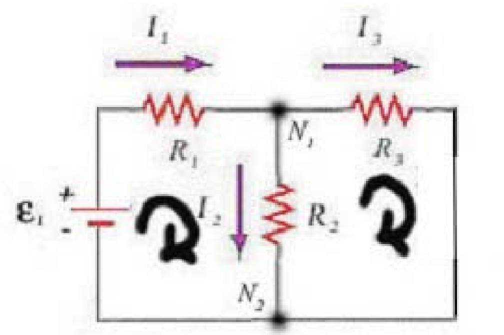
At $t = 0$, what is the voltage difference across the capacitor?

Once again the capacitor is uncharged so the voltage difference across the capacitor is zero.

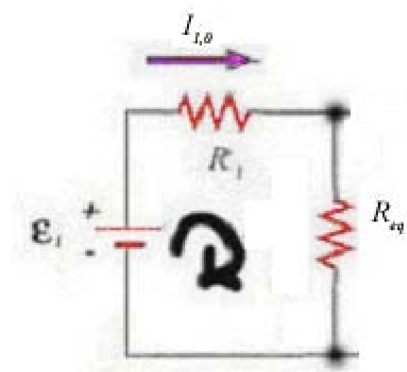
$$V_C = \frac{Q(t=0)}{C} = 0 .$$

Find the current that flows in the branch containing the voltage source at $t=0$.

Since the capacitor acts like a short circuit the circuit looks like



This circuit can be easily solved by reducing the two resistors, R_2, and R_3, that are in parallel, to an equivalent resistor $R_{eq} = R_2R_3/R_2 + R_3$. Then the circuit looks like



The current from the voltage source can now be easily determined,

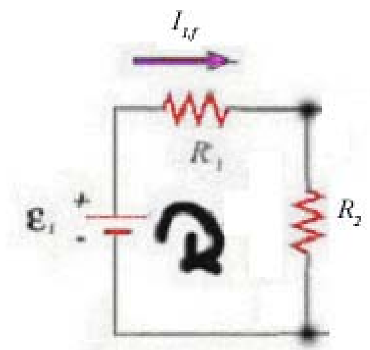
$$I_{1,0} = \frac{\varepsilon_1}{R_1 + R_{eq}} = \frac{\varepsilon_1\left(R_2+R_3\right)}{R_1\left(R_2+R_3\right)+R_2R_3}.$$

When a long time has passed after the switch was closed, what is the current that flows in the branch of the circuit that included the capacitor?

Now the capacitor is fully charge so no current flows in branch 3, hence the capacitor acts like an open circuit and $I_3 = 0$.

When a long time has passed after the switch was closed (based on your result form Question 8), find the current that flows from the voltage source?

The circuit looks like



The current in the circuit is then

$$I_{1,f} = \frac{\varepsilon_1}{R_1 + R_2}$$

Try to combine your two loop equations and current conservation, to find a single differential equation describing the rate of change of the charge on the capacitor plate. Your equations should include terms that involve Q, dQ/dt, R_1, R_2, and R_3, and $\boldsymbol{\varepsilon}_1$.

Using the loop 1 equation from question 5, $\varepsilon_1 - I_1 R_1 - I_2 R_2 = 0$**,** we can solve for the current I_1 in terms of R_1, R_2, I_2, and ε_1,

$$I_1 = \frac{\varepsilon_1 - I_2 R_2}{R_1}.$$

We can use our above result for I_1 in the current conservation equation from question 5, $I_1 = I_2 + \dfrac{dQ}{dt}$, and solve for I_2,

$$I_2 = \frac{(\varepsilon_1/R_1 - dQ/dt)}{(1 + R_2/R_1)}.$$

We can now use loop 2 equation from question 5, $-\dfrac{Q}{C} + I_2 R_2 - \dfrac{dQ}{dt} R_3 = 0$, and our above result for I_2, to derive the differential equation for the charge on the capacitor

$$-\frac{dQ}{dt}\left(R_3 + \frac{R_1 R_2}{R_1 + R_2}\right) - \frac{Q}{C} + \varepsilon_1\left(\frac{R_2}{R_1 + R_2}\right) = 0.$$

Determine the time constant for this circuit? You do not have to solve your equation. See if you can 'read off' the time constant based on the comparison between your equation and the charging equation for a single loop RC circuit.

We can rewrite this equation as

$$\frac{dQ}{dt} = +\mathcal{E}_1\left(\frac{R_2}{R_3R_1 + R_3R_2 + R_1R_2}\right) - \frac{Q\left(R_1 + R_2\right)}{C\left(R_3\left(R_1 + R_2\right) + R_1R_2\right)}.$$

When we compare this to our equation for the simple RC circuit,

$$\frac{dQ}{dt} = \frac{1}{R}\left(\varepsilon - \frac{Q}{C}\right).$$

We can set

$$R \equiv \frac{R_3\left(R_1 + R_2\right) + R_1R_2}{R_1 + R_2}$$

and

$$\frac{\varepsilon}{R} \equiv \mathcal{E}_1\left(\frac{R_2}{R_3R_1 + R_3R_2 + R_1R_2}\right).$$

This implies that

$$\varepsilon = \mathcal{E}_1\left(\frac{R_2}{R_1 + R_2}\right).$$

We can then use our standard solution to the differential equation for the charge on the capacitor, with the above values for ε and R,

$$Q(t) = C\varepsilon\left(1 - e^{-t/RC}\right).$$

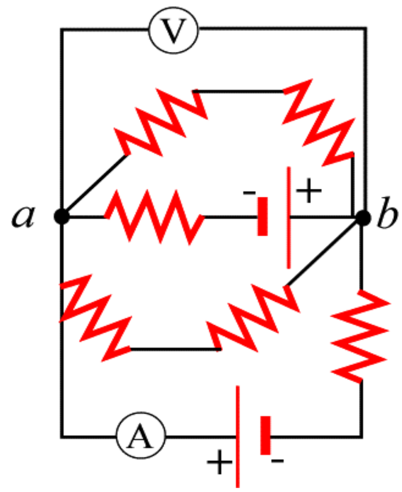
The time constant is then

$$\tau = RC = \left(\frac{R_3\left(R_1 + R_2\right) + R_1R_2}{R_1 + R_2}\right)C.$$

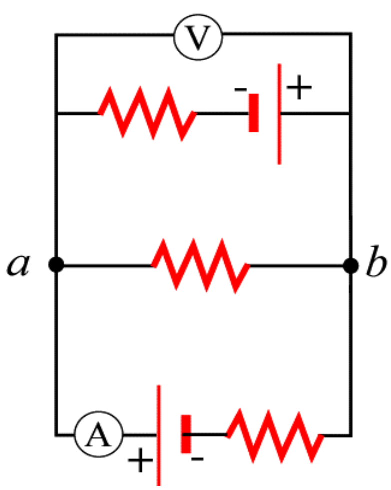
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics:

Problem 9 alternative Resistor Network Solution

Problem: Find the current measured by the ammeter and the voltage (potential difference) read by the voltmeter in the circuit at right, given that all resistors have value R and both batteries have value ε.



Solution:
The first step in solving a circuit problem is to see if anything can be done to simplify the circuit. One common simplification is to collapse any resistors that are in series or in parallel into a single, equivalent resistor. Note that this only works if they really are in series or in parallel – be careful!) Another simplification is to redraw the circuit to straighten it out and make more apparent what is in series and what is in parallel. To be safe, I label the nodes and make sure that everything going in and out stays the same.

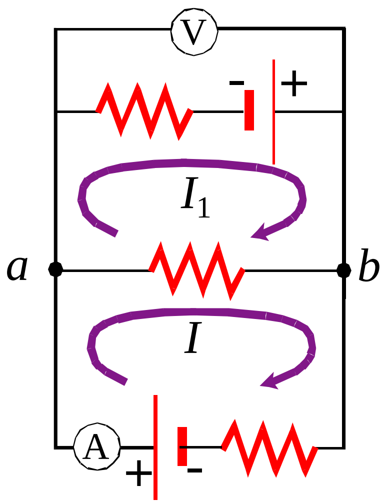


Applying these methods to the above circuit we see that there are two sets of resistors in series (each of equivalent resistance $2R$) and that these equivalent resistors are in parallel, with equivalent resistance

$$\frac{1}{R_{eq}} = \frac{1}{2R} + \frac{1}{2R} = \frac{1}{R} \rightarrow R_{eq} = R$$

The other two resistors are not in parallel or series with any other resistor because they are each in series with batteries (preventing them from being in parallel with another resistor) and have nodes on either side (preventing them from being in series with another resistor). Redrawing gives us the circuit at left.

Now we have a much simpler circuit with two loops (I don't count the loop with the voltmeter since its very high resistance ensures that no current will flow through the voltmeter – or more accurately, so little that it doesn't matter). One methods of solving this is the following.



From the conservation of current (Kirchhoff's first law): $I_1 = I_2 + I_3$ (which is clear at node *b*).

The next step is to write down the loop equations (one for each variable). Starting at node a and moving in the direction of the current loops (its easiest if you keep this convention) we get:

$$-I_1R + \varepsilon - \left(I_1 - I_2\right)R = 0$$
$$-\left(I_2 - I_1\right)R - I_2R + \varepsilon = 0$$

Note that the combination of currents across the central resistor is due to the fact that both currents are going through it. You have a potential drop if the current is going the same direction that you are walking around, and a potential rise for the current in the opposite direction (hence the I_1-I_2 in the first equation, I_2-I_1 in the second).

Combining terms we can rewrite both:
$$\varepsilon - 2I_1R + I_2R = 0$$
$$\varepsilon - 2I_2R + I_1R = 0$$

We want to know I_2 so lets multiply the bottom equation by 2 and add them to eliminate I_1:

$$3\varepsilon - 3I_2R = 0 \rightarrow I_2 = \frac{\varepsilon}{R}$$

This is the reading on the ammeter (with the current going as pictured, from right to left—NOTICE an ideal ammeter has zero resistance). We could get the voltmeter reading from any of the three branches, but lets solve for I_1 and get it from the center branch:

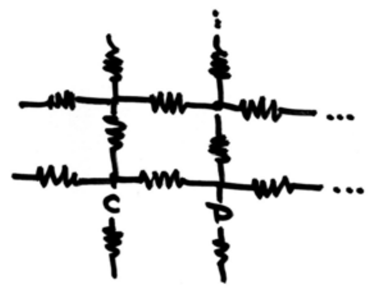
$$\varepsilon - 2I_2R + I_1R = 0 \rightarrow I_1 = \frac{1}{R}\left(2I_2R - \varepsilon\right) = \frac{\varepsilon}{R}$$

So, maybe surprisingly, the currents are equal and opposite and cancel out ($I_3 = 0$), meaning there is no potential difference between nodes *a* and *b* – the voltmeter will read 0.

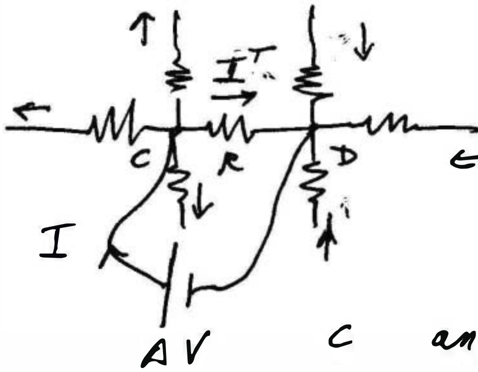
Problem 9: Suppose you have an infinite lattice of resistors, all of the same resistance R, as illustrated in the figure. Suppose you apply a potential difference between two adjacent nodes C and D which causes a current of $I = 1\text{ A}$ to flow into C and a current of $I = 1\text{ A}$ to flow out of D.

a) How much current flows through the resistor connecting C to D?

b) What is the total resistance R_{CD} of the lattice between the two points?

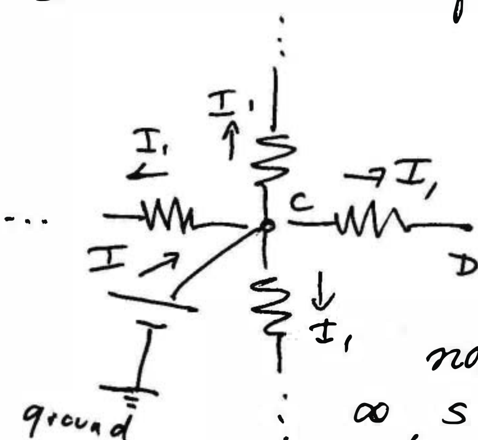


Consider an infinite network of identical resistors with a voltage source ΔV connected between nodes C and D such that 1A flows into C and out of D, with $\Delta V = I^T R$



we shall use superposition to determine I^T

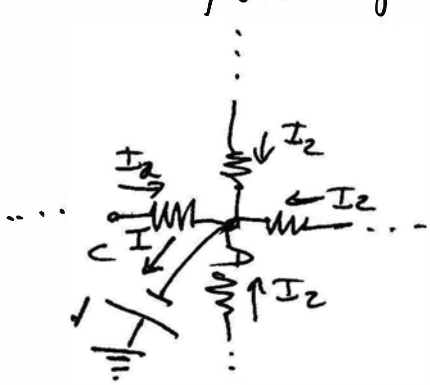
a) Consider an infinite network with a voltage source connected between C and ground ($V=0$) s.t. $I = 1A$ flows into C. By symmetry the current $I_1 = \frac{1}{4} I$.



note: the ground can be taken as ∞ so the current in the network flows to ∞.

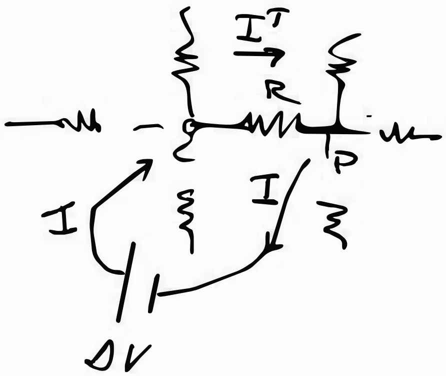
b) Now consider an infinite network with a voltage source connected between D and ground s.t. $I = 1A$ flows from D to ground. Again by symmetry

$$I_2 = \frac{1}{4} I.$$



Now the current flows from ∞ to node D

By superposition if we add these two cases we have $I^T = I_1 + I_2 = \frac{I}{2}$ flowing from C to D. Thus



$$\Delta V = \frac{I}{2} R$$

$\Rightarrow$

We can replace the network with an equivalent resister

$$R_{eq} = \frac{\Delta V}{I} \Rightarrow \frac{I\,R/2}{I} = R/2$$