MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 8 Solutions
Ampère's law, Biot-Savart law

Due: Sunday, April 3rd at 10 PM

Problem 1: Long flat conductor

Problem

A long flat conductor of width a carries a sheet of current i (see Figure 1). You are asked to find the magnetic field (direction and magnitude) near the center of its flat side and and very close to the surface, such that the distance R from the sheet is $R \ll a$.

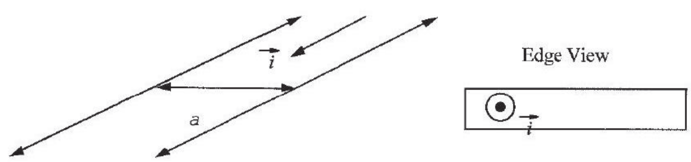
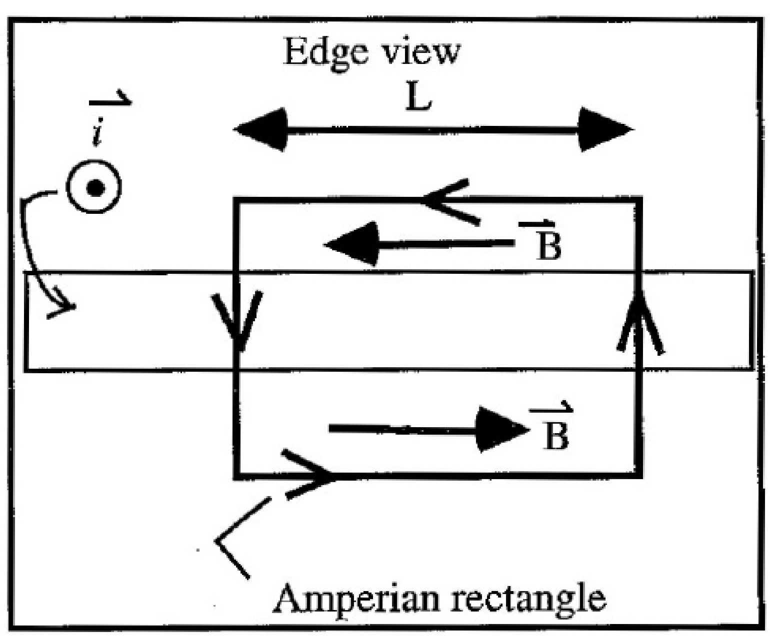


Figure 1: Flat conductor

Solution



As shown in the figure, above the conductor, $\vec{B}$ points to the left, below the conductor, $\vec{B}$ points to the right (the current is pointing out of the paper). Choose an amperian path shown in the plot, since the distance R is much smaller than a, we have:

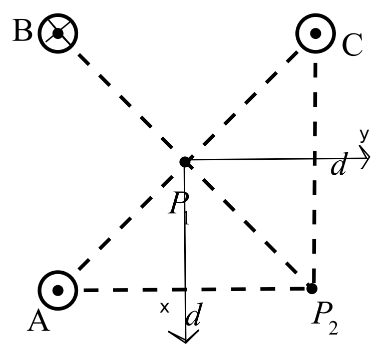
$$2B \times L = \frac{4\pi}{c} I_{\text{enc}} = \frac{4\pi}{c} \frac{Li}{a} \tag{1}$$

Hence $B = \frac{2\pi i}{ac}$.

Problem 2: Magnetic field of three wires — Purcell 6.5

Problem

Three long straight parallel wires are located as shown in the diagram. One wire (B) carries current $2I$ into the paper; each of the others (A and C) carries current I in the opposite direction. What is the strength if the magnetic field at P_1 and P_2?



Solution

The field generated by the wires A and C cancel at the point P_1. Hence, the magnetic field at the point P_1 is only due to the wire B which is,

$$\vec{B}_1 = -\frac{2(2I)}{cd/\sqrt{2}} \left(\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right) = -\frac{4I}{cd} (\hat{x} + \hat{y})$$

This field is perpendicular to the line joining B and P_1 and it points towards A.
The magnetic field due to the wire A at the point P_2 is

$$\vec{B}_{2A} = \frac{2I}{cd} \hat{y}$$

Similarly, the field due to the wire C is

$$\vec{B}_{2C} = \frac{2I}{cd} \hat{x}$$

The contribution of the wire B to the field at point P_2 is half of its contribution to the field at P_1:

$$\vec{B}_{2B} = -\frac{2I}{cd} (\hat{x} + \hat{y})$$

Hence, the net magnetic field at the point P_2 is zero.

Problem 3: Bent wire revisited

Problem

In class we found the magnetic field at the center of a wire bent through 180°. Solve it instead for the wire bent through some arbitrary angle.

Solution

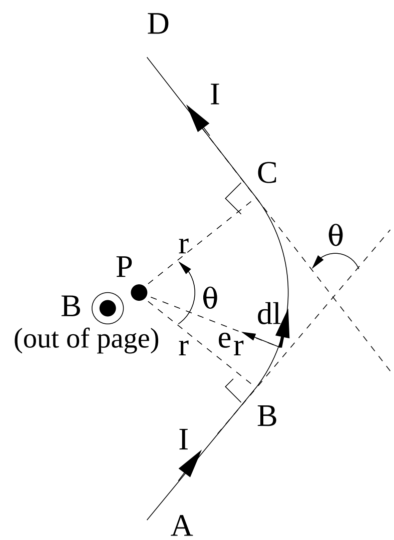


Figure 2: Bent wire

Refer to Figure 2. To calculate the magnetic field, we decompose the wire into two semi-infinite long wires AB and CD and an arc BC of angle θ. Each part of the wire contributes to $\vec{B}$ in the same direction, that is normal to and out of the page. By Ampère's law we can calculate the magnetic field of a *whole* infinite long wire $B \times 2\pi r = (4\pi/c)I$, or $B = 2I/cr$. $\vec{B}$ given by the *half*-infinite long wire AB is just $B_{AB} = (1/2)(2I/cr) = I/cr$. The arc CD contributes the same amount. For the arc BC,

$$B_{BC} = \int \frac{Idl}{cr^2} = \frac{I \times \theta r}{cr^2} = \frac{\theta I}{cr} \quad (\theta \text{ in radians}) \tag{2}$$

So the total magnetic field at point P is

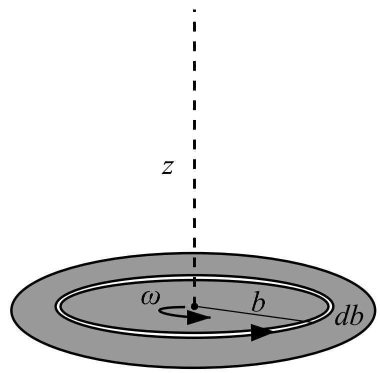
$$\vec{B}_P = \frac{(2+\theta)I}{cr}\hat{z} \ , \tag{3}$$

where $\hat{z}$ points out of the page.

Problem 4: Magnetic field due to a spinning disk

Problem

A flat circular disk with radius R carries a uniform surface charge density σ. It rotates with an angular velocity ω about the z-axis. Find the magnetic field $\vec{B}(z)$ at any point z along the rotation axis.



Solution

We treat the spinning disk as a series of infinitesimal loops for which we already know the field. The field generated by a loop of radius b is

$$B_z = \frac{2\pi b^2 I}{c(b^2+z^2)^{\frac{3}{2}}} \quad \rightarrow \quad dB_z = \frac{2\pi b^2 dI}{c(b^2+z^2)^{\frac{3}{2}}} \quad dI = \sigma\omega b db$$

Integrating over the whole disk we find,

$$B_z = \int_0^R \frac{2\pi b^2 \sigma\omega b}{c(b^2+z^2)^{\frac{3}{2}}}\, db = \frac{2\pi\sigma\omega}{c}\left(\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2|z|\right)$$

Problem 5: Coaxial cable

Problem

A long coaxial cable consists of two concentric conductors, as shown in figure Figure 3 below. The inner conductor is a cylinder with radius a, and it carries a current I uniformly distributed over its cross section. The outer conductor is a cylindrical shell with inner radius b and outer radius c. It carries a current I that is also uniformly distributed over its cross section, and that is opposite in direction to the current of the inner conductor. Calculate the magnetic field $\vec{B}$ and plot the field strength as a function of the distance from the axis.

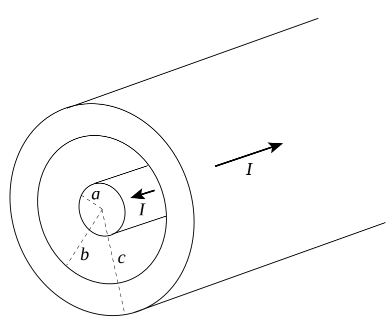


Figure 3: Cross-section of a long coaxial cable.

Solution

We will make use of the cylindrical symmetry and Ampère's law to evaluate the magnetic field everywhere. Let R be the distance from the axis.

- In the region $R < a$,
$$2\pi RB = \frac{4\pi}{c} J_1 \pi R^2 \implies \vec{B} = \frac{2IR}{ca^2}\hat{\theta}$$
where J_1 is the current density in this region and it is given by,
$$J_1 = \frac{I}{\pi a^2}$$
- In the region $(b > R > a)$
$$2\pi RB = \frac{4\pi}{c} J_1 \pi a^2 \implies \vec{B} = \frac{2I}{cR}\hat{\theta}$$
- In the region $(c > R > b)$, the current density is $J_3 = I/\pi(c^2 - b^2)$. Hence,
$$2\pi RB = \frac{4\pi}{c}(I - J_3\pi(R^2 - b^2)) \implies \vec{B} = \frac{2I}{cR}\left(\frac{c^2 - R^2}{c^2 - b^2}\right)\hat{\theta}$$
- In the region $(d > R > c)$
$$B \cdot 2\pi R = \frac{4\pi}{c}(I - I) \implies \vec{B} = 0$$

Problem 6: Toroidal Solenoid — Purcell 6.14

Problem

What is the magnetic field inside and outside of the solenoid in Figure 4?

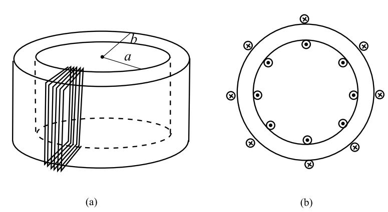


Figure 4: A Toroidal Solenoid

Solution

Cylindrical symmetry demands that the magnetic field must be azimuthal everywhere and has only radial dependence. Note that the current enclosed by a circular Amperian loop with radius less than the inner radius encloses zero current, i.e., $I_{\text{enc}} = 0$. A circular Amperian loop with radius greater than the outer radius encloses zero net current ($I_{\text{enc}} = NI - NI = 0$). Therefore, the field is zero everywhere outside the toroid.

The field inside can be calculated using Ampère's law as follows.

$$\oint \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} I_{\text{enc}} \implies \quad B2\pi R = \frac{4\pi}{c} NI \implies \quad \vec{B} = \frac{2NI}{cR}\hat{\theta}$$

Problem 7: Vector potential of a solenoid

Problem

Find the vector potential $\vec{A}$ inside and outside of an infinite solenoid of radius R with n turns per centimeter, each carrying current I. Find the solution for $\vec{A}$ which is symmetric about the axis of the solenoid.

HINT: You can come up with a *very* simple way to compute $\vec{A}$ by putting together

- The magnetic flux, $\Phi_B = \int \vec{B} \cdot d\vec{a}$
- The definition $\vec{B} = \vec{\nabla} \times \vec{A}$
- Stoke's theorem.

Solution

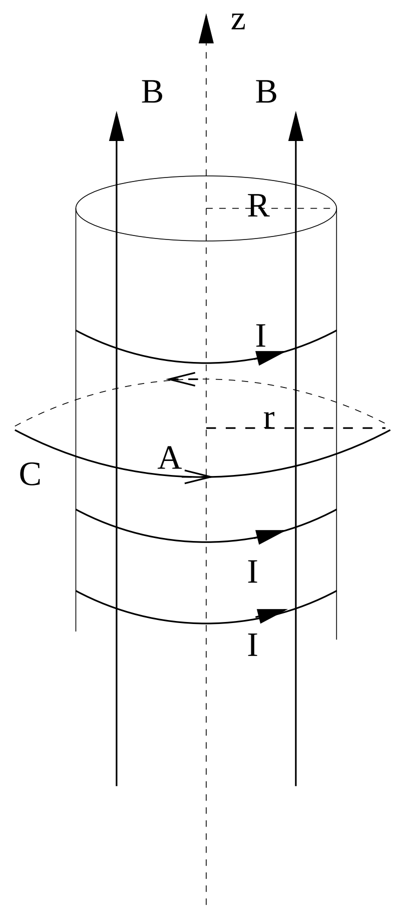


Figure 5: Vector potential of an infinite solenoid

What is the magnetic field inside and outside of the solenoid in figure Figure 4? The magnetic field of a infinite solenoid is uniform inside and zero outside it. Let r be the distance from the axis. By

Ampère's law, $Bl = (4\pi/c)nlI$ for $r < R$ where l is some axial length. So

$$\vec{B} = \begin{cases} (4\pi/c)nI\hat{z} & r < R \\ 0 & r > R \end{cases} \tag{4}$$

The magnetic flux through a surface area bounded by a circle C of radius r (see Figure 5) is:

$$\Phi_B = \begin{cases} (4\pi^2/c)nIr^2 & r < R \\ (4\pi^2/c)nIR^2 & r > R \end{cases} \tag{5}$$

By Stoke's Theorem,

$$\begin{aligned} \Phi_B &= \int \vec{B} \cdot d\vec{a} = \int \nabla \times \vec{A} \cdot d\vec{a} \\ &= \oint \vec{A} \cdot d\vec{l}. \end{aligned}$$

We seek a solution that is symmetric about the z-axis. The simplest one is $\vec{A} = A\hat{\phi}$, i.e., along the "circumferential" direction.

$$\oint \vec{A} \cdot d\vec{l} = A \times 2\pi r.$$

Therefore,

$$A = \begin{cases} (2\pi/c)nIr & r < R \\ (2\pi/c)nIR^2/r & r > R \end{cases} \tag{6}$$

You can check that $\nabla \times \vec{A} = \vec{B}$ in cylindrical coordinates both inside and outside. Note that in the exterior $\vec{A}$ is non-zero, even though $\vec{B}$ is zero there.

Problem 8: The Director's Challenge — Extra credit!!!

Problem

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!