MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

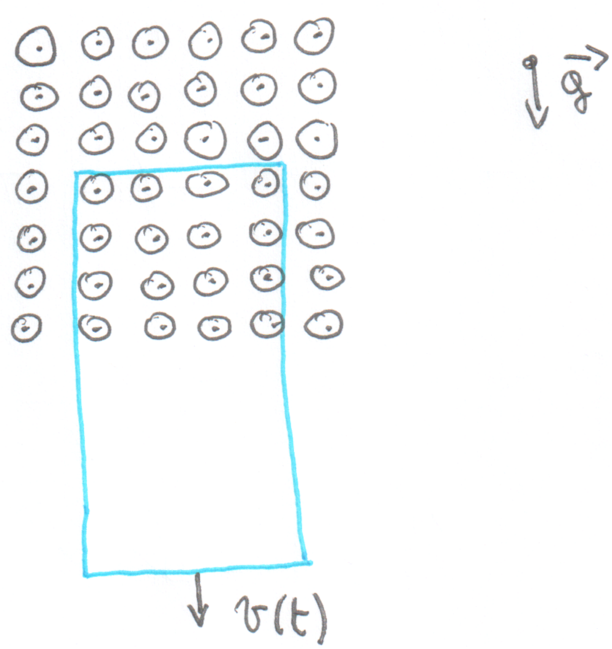
Problem Set 9 Solutions
Lenz's law and Faraday's law

Due: Sunday, April 10th at 10:00 pm

Problem 1: Falling rectangular loop

Problem

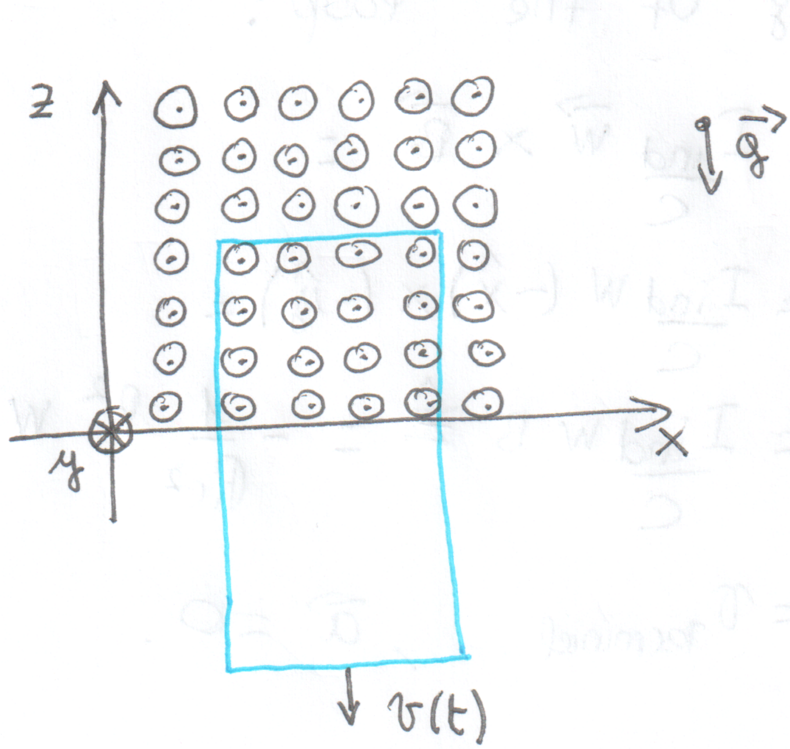
A rectangular loop of wire with mass m, width w, vertical length l, and resistance R falls out of a magnetic field under the influence of gravity. The magnetic field is uniform with magnitude $\vec{B}$ and out of the paper within the area shown in the sketch and zero outside that area. At the time t, the loop is exiting the magnetic field at speed, $v(t)$. What is the terminal velocity of the loop?



Solution

Solution 1

1)



Faraday's law $\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial}$

- Ohm's law: $\mathcal{E} = I_{ind} R$ (2)

- $\Phi_B = \int_S \vec{B} \cdot d\vec{a} = B\int_S da = BzW$ (3)

$\vec{B} = B\hat{y}$
$d\vec{a} = da\,\hat{y}$

$\frac{\partial \Phi_B}{\partial t} = BW\frac{dz}{dt} = BW\,v(t) < 0$ As
$v(t) < 0$ falls
fl

- Substitute (2) + (3) in (1):

$I_{ind} = -\frac{1}{R_0} BWv(t) > 0$

$$\vec{F}_{ind} = \frac{I_{ind}}{c} \vec{W} \times \vec{B} =$$

$$= \frac{I_{ind}}{c} W (-\hat{x}) \times (-\hat{y}) =$$

$$= \frac{I_{ind}}{c} W B \hat{z} = -\frac{1}{Rc^2} B^2 W^2 v($$

- When $v(t) = v_{terminal}$, $\vec{a} = 0$.

Therefore

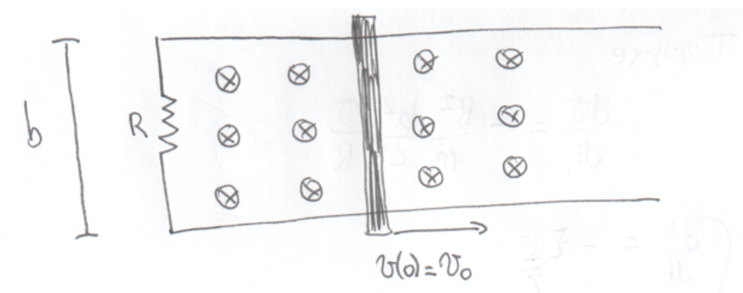
$$0 = \vec{F} = -\frac{1}{Rc^2} B^2 W^2 v_{terminal} - mg$$

$$v_{terminal} = -\frac{mgRc^2}{B^2 W^2} < 0$$

Problem 2: Crossbar sliding in magnetic field

Problem

A metal crossbar of mass m slides without friction on two long parallel conducting rails a distance b apart. A resistor R is connected across the rails at one end. The resistance of the bar and the rails is negligible. There is a uniform magnetic field $\vec{B}$ perpendicular to the page. At time $t = 0$ the crossbar is given a velocity v_0 towards the right.



(a) Write down a differential equation of the form $dv/dt =$ something that governs the motion of the sliding crossbar.

(b) Integrate this to find the velocity $v(t)$ for all $t > 0$.

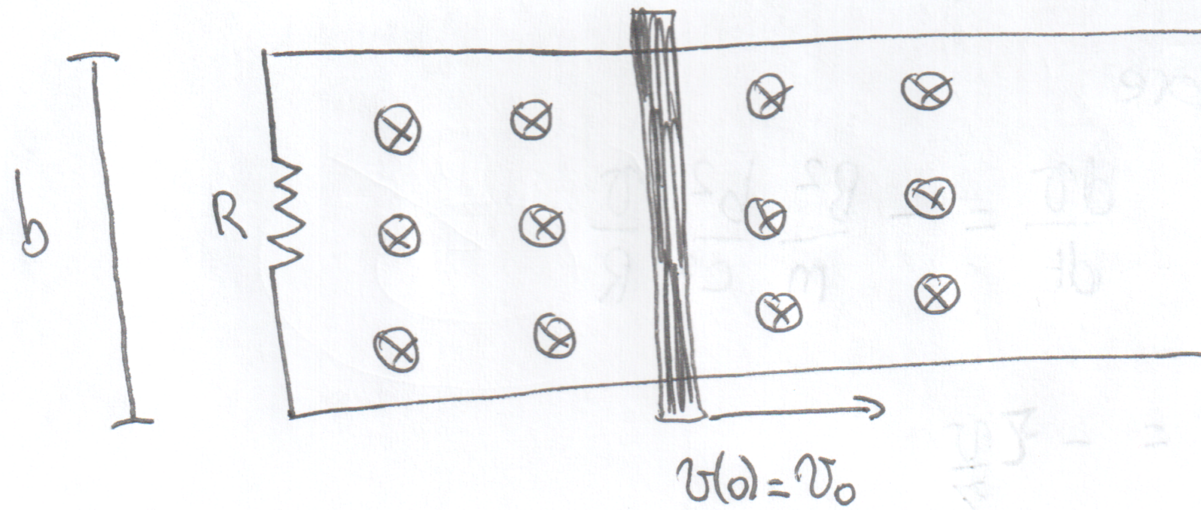
(c) Compute the total distance that the cross bar moves.

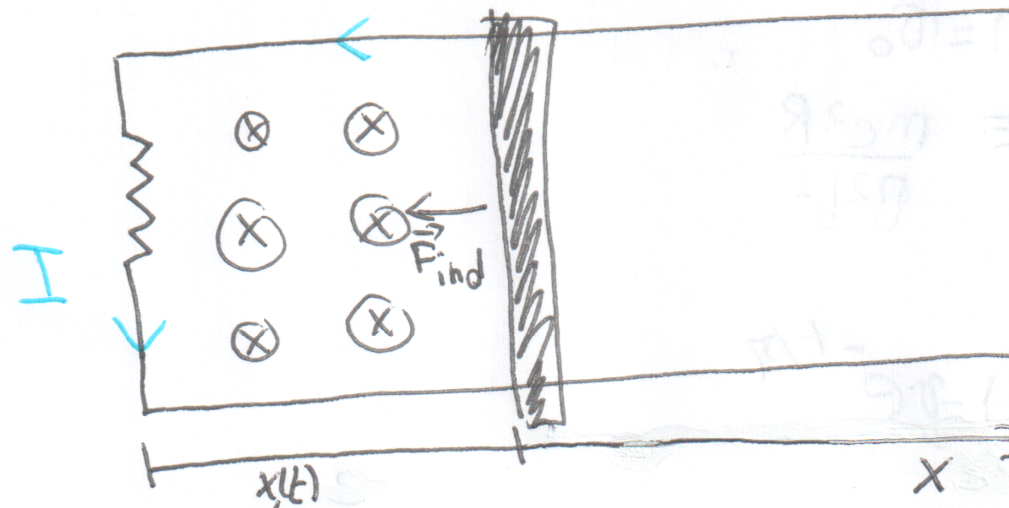
(d) Show that the *total* energy dissipated in the resistor makes sense given the initial velocity of the crossbar.

Solution

Solution

2)





a) • Faraday's law

$$\begin{cases} \mathcal{E} = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t} \\ \Phi_B(t) = B b x_L(t) \end{cases}$$

$$\mathcal{E} = -\frac{1}{c} B b \frac{dx_L}{dt} = -\frac{1}{c} B b v$$

• Lenz's law : the induced current ... the direction that carries ...

Therefore

$$\frac{dv}{dt} = -\frac{B^2}{m}\frac{b^2}{c^2}\frac{v}{R}$$

b)

$$\begin{cases} \dfrac{dv}{dt} = -\dfrac{v}{\tau} \\ v(0) = v_0 \\ \tau \equiv \dfrac{mc^2 R}{B^2 b^2} \end{cases}$$

$$v(t) = v_0 e^{-t/\tau}$$

c)

$$\Delta x_{TOT} = \int_0^\infty v(t)\,dt = \left[-v_0 \tau e^{-t/\tau}\right]_0^\infty = v_0 \tau$$

d) The energy dissipated in the resistor

$$W = \int_0^\infty I^2(t) R\,dt = \int_0^\infty \frac{B^2 b^2}{c^2 R} v^2(t)\,dt =$$

$$= \frac{B^2 b^2}{c^2 R}\left[-v_0^2 \frac{\tau}{2} e^{-2t/\tau}\right]_0^\infty = \frac{B^2 b^2}{c^2 R} v_0$$

Problem 3: Generate electricity at the gym

Problem

In class we discussed that, given the definition of magnetic flux, there are essentially three ways that we can make it vary and thereby create an EMF: we can
a) make the area vary;
b) change the relative orientation of the magnetic field and the area;
c) make the magnetic field vary.

Think how to use gym exercise machines to produce electricity in each of the three ways. Be creative! Try to make some estimates of the power you can produce. Extra credit if you suggest one detailed design.

Problem 4: Build a simple generator

Problem

Group exercise: build a simple generator and use it to light a led or small bulb. Use whatever you want to provide the mechanical work.

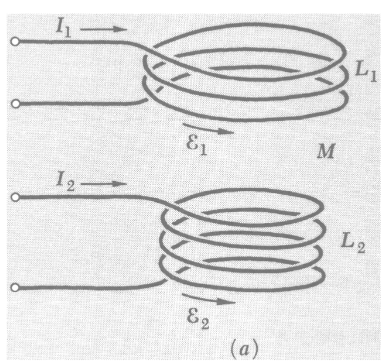
Problem 5: Purcell 7.11 — Inductance of two coils

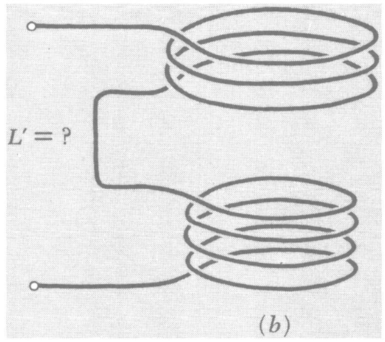
Problem

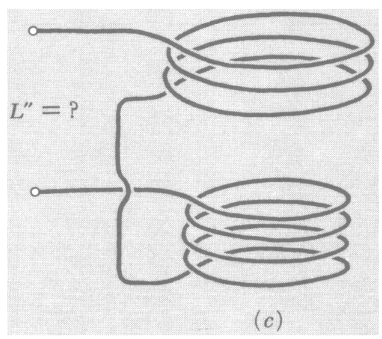
Two coils with self-inductances L_1 and L_2 and mutual inductance M are shown with the positive direction for current and electromotive force indicated. The equations relating currents and emf's are

$$\mathcal{E}_1 = -L_1\frac{dI_1}{dt} \pm M\frac{dI_2}{dt}\ \mathcal{E}_2 = -L_2\frac{dI_2}{dt} \pm M\frac{dI_1}{dt} \tag{1}$$

Given that M is always to be taken as positive, how must the signs be chosen in these equations? What if we had chosen the other direction for positive current and emf in the lower coil? Now connect the two coils together as in b. What is the inductance L' of this circuit? What is the inductance L'' of the circuit formed as shown in c? Which circuit has the greater self-inductance? Considering that the self-inductance of any circuit must be a positive quantity, see if you can deduce anything concerning the relative magnitudes of L_1, L_2, and M.







Solution

Solution

5)

a). • Imagine that $I_2 > 0$ and incr

Then the magnetic field due

coil 2 points up.

As I_2 increases, the flux

coil 1 increases

By Lenz's law the indu

current must be negative (

way the magnetic field pro

coil 1 points downward).

Therefore $\mathcal{E}_1$ is negative as we

$$\mathcal{E}_1 = -L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt}$$

The same argument applies

second equation (imagine $I_1 > 0$

$$\mathcal{E}_2 = -L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt}$$

• If $I_2 < 0$,

signs would be positive:

b) $$\begin{cases} I_1 = I_2 = I \\ \mathcal{E}' = \underbrace{\mathcal{E}_1 + \mathcal{E}_2}_{\text{same sign}} \underset{\text{(1) and (2)}}{=} -L_1 \frac{dI_1}{dt} - M\frac{dI_2}{dt} - L_2 \frac{dI_2}{dt} \end{cases}$$

$$\mathcal{E}' = -(L_1 + 2M + L_2)\frac{dI}{dt}$$

This is equivalent to a single coil wit

$$L' = L_1 + L_2 + 2M$$

c) $I_1 = I_2 = -I$

$$\mathcal{E}'' = \mathcal{E}_1 - \mathcal{E}_2 = -L_1 \frac{dI_1}{dt} - M\frac{dI_2}{dt} + L_2\frac{dI_2}{dt}$$

$$= -(L_1 - L_2 - 2M)\frac{dI}{dt}$$

$$L'' = L_1 - L_2 - 2M$$

The self-inductance must be p

(otherwise any charge in I

result in more current in

Same direction... against der

Problem 6: Purcell 7.18 — Total charge flow from induced EMF

Problem

A circular coil of wire, with N turns of radius a, is located in the field of an electromagnet. The magnetic field is perpendicular to the coil, and its strength has the constant value B_0 over that area. The coil is connected by a pair of twisted leads to an external resistance. The total resistance of this closed circuit is R. Suppose the electromagnet is turned off, its field dropping more or less rapidly to zero. Derive the formula for the total charge which passes through the resistor and explain why it does not depend on the rapidity with which the field drops to zero.

Solution

$\mathcal{E}(t) = -(1/c)d\Phi/dt$ and $I(t) = \mathcal{E}(t)/R$. Therefore the charge flowing through the closed circuit is

$$Q = \int I(t)dt = \int_{t_i}^{t_f} -\frac{d\Phi}{cR} = \frac{(\Phi(t_i) - \Phi(t_i))}{cR} = \frac{NB_0\pi a^2}{cR}. \tag{2}$$

Since it only depends on the initial and final flux (and hence the initial and final field, the final field being zero), it doesn't matter how quickly we shut the field off — we can do it in a microsecond, or we can ramp it down over a decade.

Problem 7: Purcell 7.20 — Magnetic field due to a loop far from the loop

Problem

Can you devise a way to use the theorem $\Phi_{21} = \Phi_{12}$ to find the magnetic field strength due to a ring current at points in the plane of the ring at a distance from the ring much greater than the ring radius? Hint: consider two concentric coplanar rings of radius R_2 and R_1, $R_1 \gg R_2$ and evaluate the effect of a small change of the radius of the outer ring on Φ_{21} and Φ_{12}.

More detailed hint: we wish to calculate the magnetic field of the small loop at the location of the large loop. Since the large loop is of a radius $R_1 \gg R_2$, this will tell us the magnetic field of a small loop at any radius $r \gg R_2$, at least in the plane of the small loop. The trick here is to calculate the *change* in flux, $\Delta\Phi_B$ that occurs if we adjust the radius R_1 by some amount ΔR_1. Do this with the reciprocity theorem: first, work out $\delta\Phi_B$ by running the current through the big loop. Reciprocity says we must get the same $\delta\Phi_B$ when you run the current through the small loop.

Solution

Solution

Problem 8: Purcell 7.22 — Spinning a charged ring

Problem

A thin ring of radius a carries a static charge q. This ring is in a magnetic field of strength B_0, parallel to the ring's axis, and is supported so that it is free to rotate about that axis. If the field is switched off, how much angular momentum will be added to the ring? If the ring has mass m, how much angular velocity will it acquire? Hint: the variation of magnetic field induces an electric field along the ring, which accelerates it. The electric force creates a torque about the ring's axis.

Solution

Apply Faraday's law of induction:

$$\begin{aligned} \Phi_B(t) &= B(t)\pi a^2, \\ \mathcal{E}(t) &= \int_C \vec{E}(t)\cdot d\vec{s} = E(t)\times 2\pi a \\ &= -\frac{1}{c}\frac{d\Phi_B(t)}{dt} = \frac{\pi a^2}{c}\left(-\frac{dB}{dt}\right) && (3) \\ \text{or} \qquad E(t) &= \frac{a}{2c}\left(-\frac{dB}{dt}\right). && (4) \end{aligned}$$

The force on the charge is $F(t) = qE(t)$, along the loop direction; so its torque about the ring's axis is $N(t) = F(t)a = (qa^2/2c)(-dB)/dt$. Therefore the total angular momentum added to the ring is

$$\Delta J = \int_{t_i}^{t_f} N(t)dt = -\frac{qa^2}{2c}\int_{B(t_i)}^{B(t_i)} dB = qa^2B_0/2c, \tag{5}$$

where B_0 is the initial magnetic field and $B(t_f) = 0$. The momentum of inertia of the ring is $I = ma^2$. If the ring is initially at rest, the final angular velocity is

$$\omega = \frac{\Delta J}{I} = \frac{qB_0}{2mc}. \tag{6}$$

Note that eq.(5) only depends on the initial and final values of the magnetic field, not the variation rate.

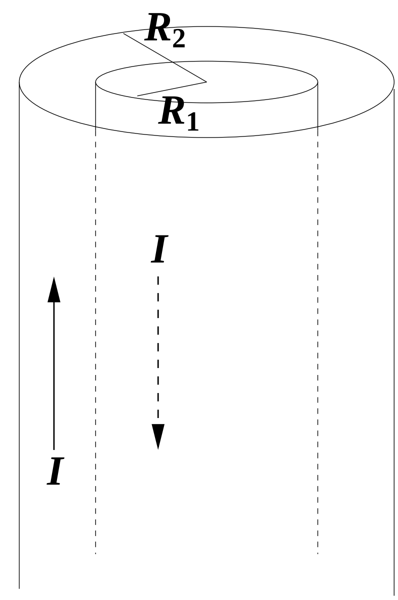
Additional question: If you combine this calculation with the notion of angular momentum conservation, what can you infer about electromagnetic fields?

This simply indicates that electromagnetic field has angular momentum of its own — this angular momentum is transferred to the ring as the magnetic field switches off.

Problem 9: Self inductance per unit length of coaxial conductors

Problem

A transmission line consists of a pair of nested, long cylindrical tubes with radii R_1 and R_2:



The current I flows up the outer tube and down in the inner tube; these currents are uniformly distributed over their respective surface. Compute the self inductance per unit length of this configuration. Hint: make life easy for yourself and do so using magnetic energy.

Solution

The magnetic field energy stored in an inductor is:

$$U = \int_{Everywhere} \frac{B^2}{8\pi} dV = \frac{1}{2} L I^2. \tag{7}$$

The magnetic field around a cylindrical current flow of current I and radius R is given by applying Ampere's law,

$$\begin{aligned} B \times 2\pi r &= \begin{cases} (4\pi/c)I & , \quad r > R \\ 0 & , \quad r < R \end{cases} \\ or \qquad \vec{B} &= \begin{cases} (2I/cr)\hat{\theta} & , \quad r > R \\ 0 & , \quad r < R \end{cases} \end{aligned}$$

where $\hat{\theta}$ is the unit vector in "circumferential" direction and its circulation obeys the right-hand-rule.

For our problem, adding the contributions from the two cylindrical current flows of opposite directions, we have

$$\vec{B} = \begin{cases} -(2I/cr)\hat{\theta} & , \quad R_1 < r < R_2 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{8}$$

So the total magnetic field energy *per unit length* is

$$\begin{aligned} U_l &= \int_{Everywhere} \frac{dV}{l}\frac{B^2}{8\pi} = \int_{R_1}^{R_2} 2\pi r dr \frac{(2I/cr)^2}{8\pi} \\ &= \frac{I^2}{c^2}\ln\left(\frac{R_2}{R_1}\right), \qquad (9) \\ \text{so} \qquad L_l &= \frac{2U_l}{I^2} = \frac{2}{c^2}\ln\left(\frac{R_2}{R_1}\right), \qquad (10) \end{aligned}$$

where L_l is the self-inductance *per unit length.*

Problem 10: The Director's Challenge — Extra credit!!!

Problem

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!