MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 10 Solutions
Maxwell's equations, waves

Due: Wednesday, May 4th 10 am IN CLASS

Problem 1: Purcell 9.1

Problem

9.1 If the electric field in free space is $\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \sin (2\pi/\lambda)(z + ct)$, with $E_0 = 2$ statvolts/cm, the magnetic field, not including any static magnetic field, must be what?

Figure 1: Purcell 9.1

Solution

We are given the electric field:

$$\vec{E} = E_0(\hat{x} + \hat{y}) \sin\left(\frac{2\pi}{\lambda}(z + ct)\right)$$

The corresponding magnetic field must satisfy Maxwell's equations. Using Faraday's Law, we find:

$$\vec{\nabla} \times \vec{E} = E_0(-\hat{x} + \hat{y}) \left(\frac{2\pi}{\lambda}\right) \cos\left(\frac{2\pi}{\lambda}(z + ct)\right) = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \rightarrow \quad \vec{B} = E_0(\hat{x} - \hat{y}) \sin\left(\frac{2\pi}{\lambda}(z + ct)\right)$$

where we have dropped a constant of integration (static magnetic field).

Figure 2: Solution Purcell 9.1

Problem 2: Purcell 9.5a

Problem

9.5 Here is a particular electromagnetic field in free space:

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$

$$B_x = 0 \qquad B_y = 0 \qquad B_z = -E_0 \sin(kx + \omega t)$$

(a) Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

Figure 3: Purcell 9.5

Solution

Just to be complete, let's test all four of Maxwell's equations on this wave.

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 + 0 + 0 = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 + 0 + 0 = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_y}{\partial x}\hat{z} = 0 + E_0 k \cos(kx + \omega t)\hat{z} \qquad -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = E_0 \frac{\omega}{c}\cos(kx + \omega t)\hat{z} \quad \rightarrow \quad \omega = ck$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial B_z}{\partial y}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y} = 0 + E_0 k \cos(kx + \omega t)\hat{z} \qquad \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = E_0 \frac{\omega}{c}\cos(kx + \omega t)\hat{z} \quad \rightarrow \quad \omega = ck$$

Figure 4: Solution Purcell 9.5a

Problem 3: Discovery of magnetic charge

Problem

You discover magnetic charge. The units of magnetic charge density, μ, are chosen such that $\vec{\nabla} \cdot \vec{B} = 4\pi\mu$.
(a) When this magnetic charge is in motion, there is a "magnetic current density" $\vec{L} = \mu\vec{v}$. In analogy to electric charge density and electric current densities, write down the equation of continuity for magnetic charge.
(b) What do Maxwell's equations become with this new charge?

Solution

(a)

$$\frac{\partial \mu}{\partial t} + \vec{\nabla} \cdot \vec{L} = 0. \tag{1}$$

(b) Take the divergence of the $\vec{\nabla} \times \vec{E}$ equation; you'll find that the new equation of magnetic charge continuity is violated. To fix it, you must add a term that is proportional to $\vec{L}$ to Faraday's law. The resulting Maxwell equations are:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c}\vec{L} \tag{2}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \tag{3}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{4}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi\mu. \tag{5}$$

Problem 4: Magnetic field of a moving charge

Problem

A charge q moving along the x-axis at constant speed $v \ll c$. When it is at $x = -d$, what is the magnetic field at $(x, y, z) = (0, r, 0)$?

(a) Solve this first using Biot-Savart. (Hint: the current from the moving charge isn't particularly well defined. However, B-S only needs the combination $Idl = (dq/dt)dl = dq(dl/dt) \simeq q_{\text{pt charge}}(dl/dt)$. Sloppy physicist calculus in action!)

(b) Now solve this using displacement current. Look at a circle of radius r centered at the origin and passing through the point $(0, r, 0)$. By symmetry, $\vec{B}$ will be constant on this circle and oriented in the tangential direction. Find a surface which has this circle as a boundary and for which $\int \vec{E} \cdot d\vec{a}$ is simple. Evaluate this flux, apply the "generalized" form of Ampere's law (integral formulation) and you're there.

Note, there's a third way: Lorentz transform from the rest frame electric field (which you used on a previous pset). All three answers should agree, at least in the limit $v \ll c$.

Solution

a) Solve this first using Biot-Savart.

For low speed $v \ll c$, we can ignore the relativistic effects. To apply B-S law, we simply treat the moving charge at $x = -d$ as an element current at the same location and $Id\vec{l} = qv\hat{x}$. Then we have

$$\vec{B} = \frac{qv\hat{x} \times \hat{r}_1}{cr_1^2}, \tag{6}$$

where $\vec{r}_1$ is the position vector from the moving charge to the test point $(0, r, 0)$. Using $r_1 = \sqrt{r^2 + d^2}$ and $\hat{x} \times \hat{r}_1 = \hat{z} \sin\theta_1 = \hat{z} r/r_1$, we find

$$\vec{B} = \hat{z}\frac{qvr}{c(r^2 + d^2)^{3/2}}. \tag{7}$$

(b) Now solve this using displacement current.

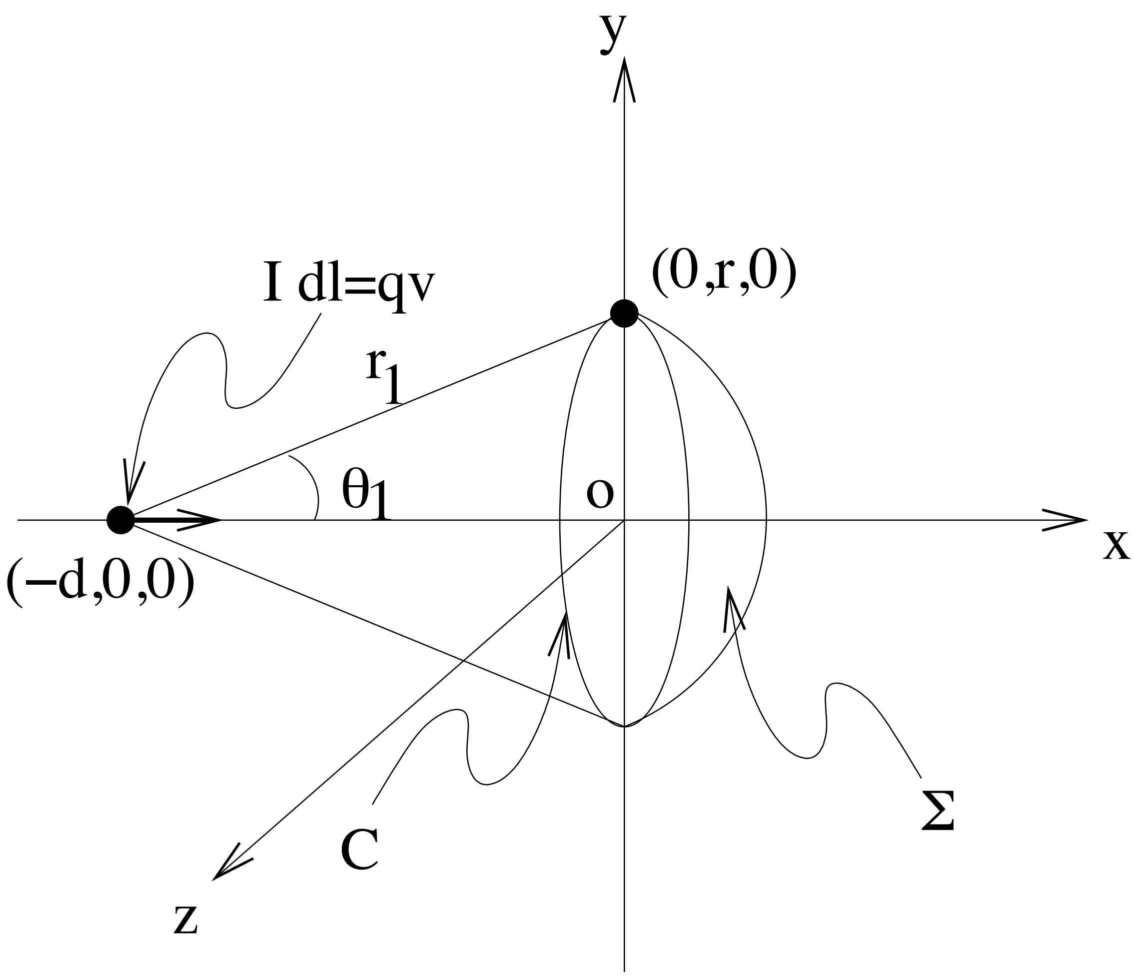


Figure 5: Calculation of magnetic field of a moving charge by "generalized" Ampere's law.

Consider a surface Σ whose boundary is the circle C centered at the origin and in Y-Z plane, and all points on Σ have the same distance r_1 to the moving charge at $(-d, 0, 0)$. Apply Stoke's Theorem,

$$\int_\Sigma \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \int_C \vec{B} \cdot d\vec{l} \tag{8}$$

while

$$\begin{aligned}\int_\Sigma \vec{\nabla} \times \vec{B} \cdot d\vec{a} &= \frac{1}{c} \int_\Sigma \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} \\ &= \frac{1}{c} \frac{\partial}{\partial t} \int_\Sigma \vec{E} \cdot d\vec{a} \end{aligned}\tag{9}$$

so

$$\int_C \vec{B} \cdot d\vec{l} = \frac{1}{c} \frac{\partial}{\partial t} \int_\Sigma \vec{E} \cdot d\vec{a}. \tag{10}$$

On the surface Σ, $E = q/r_1^2$ in radial direction (seen from the moving charge). So

$$\begin{aligned}
\int \vec{E}\cdot d\vec{a} &= \frac{q}{r_1^2}\int da \\
&= \frac{q}{r_1^2} r_1^2 \int_0^{\theta_1} \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= 2\pi q\left[\cos(0) - \cos(\theta_1)\right] \\
&= 2\pi q\left(1 - \frac{d}{\sqrt{d^2+r^2}}\right). \qquad (11)
\end{aligned}$$

Then, since $d(-d)/dt = v$,

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\int_\Sigma \vec{E}\cdot d\vec{a} &= \frac{2\pi q v}{c}\frac{r^2}{(d^2+r^2)^{3/2}} \qquad (12)\\
\int_C \vec{B}\cdot d\vec{l} &= 2\pi r B \qquad (13)\\
B &= \frac{qvr}{c(r^2+d^2)^{3/2}}. \qquad (14)
\end{aligned}$$

Problem 5: The Director's Challenge — Extra credit!!!

Problem

Formulate an interesting problem that relates a topic from 8.022 to your intended major or any other topic about which you are passionate. Give references to help future students to understand the context. Try to give a solution. Any method — theoretical, analytical, numerical, experimental — is acceptable. If you can't give a full solution, outline partial solutions. Enjoy!

Problem 6: Magnetic monopole:experiments

Problem

Magnetic monopoles: experiment. [EXTRA CREDIT, 15 bonus points] One way to search for magnetic monpoles is by monitoring the current through a highly conductive (preferably superconducting) loop. Suppose a monopole with magnetic charge s passes through a perfectly conducting circular loop with self-inductance L. The monopole has a constant speed v, perpendicular to the plane of the loop. It approaches from very far away, and then recedes to infinity. Calculate the current I that flows around the loop as a result of the monopole's passage.

(Note: experiments of this type have been running for decades, and have produced a few candidate events, but there has been no unambiguous detection.)

Figure 6: Magnetic monopole

Solution

We want to calculate the rate of change of the flux passing through a superconducting loop. First, let's calculate the flux of a monopole sitting a distance z below the loop along the axis of the loop. We will orient the loop so that the surface element $d\vec{a}$ is pointing in the positive z direction. The expression for this flux is then:

$$\Phi = \int d\vec{a} \cdot \vec{B} = \int_0^{2\pi} d\phi \int_0^R dr \frac{s}{r^2+z^2} \hat{r}' \cdot \hat{z} = \int_0^{2\pi} d\phi \int_0^R dr \frac{s}{r^2+z^2} \frac{z}{\sqrt{r^2+z^2}}$$

where s is the magnetic charge and $\hat{r}'$ is the radial vector from the monopole. We have take θ to be the angle between the z-axis and the vector $\vec{r}'$. Keep in mind that $\vec{r}'$ points from the monopole to a point on the superconducting ring. Evaluating the integral, we find:

$$\Phi = 2\pi s z \left(-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z} \right) = 2\pi s \left(1 - \frac{vt}{\sqrt{R^2+(vt)^2}} \right)$$

Figure 7: Magnetic monopole

where $z = vt$. A superconductor (no resistance) will actually produce a current that will exactly cancel the flux passing through the loop. The electro-motive force generated is from the self inductance:

$$\varepsilon = L\frac{dI}{dt}$$

We also know that the induced electro-motive force is related to the time derivative of the flux.

$$\varepsilon = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

Comparing the two expressions and integrating, we find that the current is proportional to the flux up to a constant:

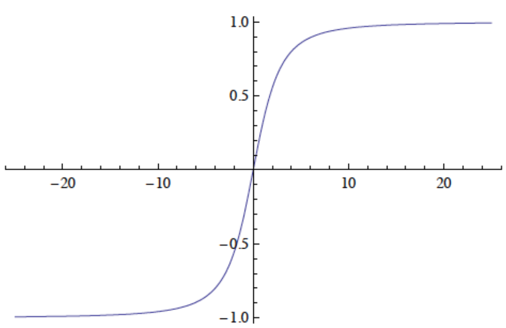
$$I(t) = -\frac{1}{Lc}\Phi(t) + C = -\frac{1}{Lc} 2\pi s \left(1 - \frac{vt}{\sqrt{R^2+(vt)^2}} \right) + C$$

Therefore, the change in current before and after a monopole passes through the loop is:

$$I(t \to \infty) - I(t \to -\infty) = \frac{4\pi s}{Lc}$$

Figure 8: Magnetic monopole

Lc



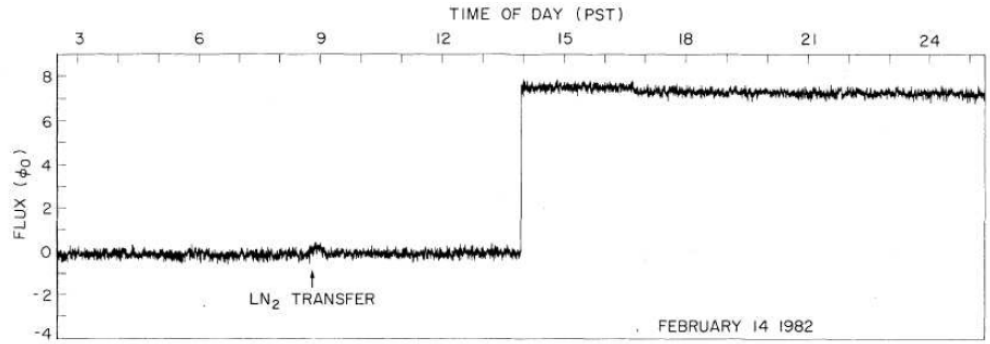
The plot to the left shows the behavior of the current as a function of time (or position, since position is proportional to time). The The plot is made with parameters:

$$\frac{2\pi s}{Lc} = 1 \quad v = 1 \quad R = 3$$

Notice how the most of the current is generated within a distance R of the loop.

An actual experimental candidate for monopole detection in this setup was found about 25 years ago. Please see Cabrera, Blas. *First Results from a Superconductive Detector for Moving Monopoles*, Phys. Rev. Lett. (**48**) 1378 (1982) for more information.

Figure 9: Magnetic monopole



The jump in flux corresponds especially well with the expected value of the magnetic monopole charge from a fairly simple argument made by P.A.M. Dirac. "LN_2 TRANSFER" denotes the time at which liquid nitrogen was added to the container of the superconducting ring to keep it cool (and superconductive). However, no such signal was detected again, leaving us with no firm evidence for the existence of monopoles.

Figure 10: Magnetic monopole