MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Experimental Study Group

Physics 8.022, Spring 2011

Problem Set 3
Gauss's law and electric potential

Due: Monday, February 21

Problem 1: Practice With ∇

(a) Calculate the gradient of each of these scalar fields:

(i) xyz.
(ii) $x^2 + y^2 + z^2$.
(iii) $1/r$ (in spherical coordinates).
(iv) $(\cos\theta)/r^2$ (in spherical coordinates).

(b) Calculate the divergence of each of these vector fields:

(i) $\hat{x}x + \hat{y}y + \hat{z}z$.
(ii) $(\hat{x}y - \hat{y}x)/\sqrt{x^2+y^2}$.
(iii) $\hat{r}/r^2$ (in spherical coordinates).
(iv) $\hat{r}(2\cos\theta)/r^3 + \hat{\theta}(\sin\theta)/r^3$ (in spherical coordinates).

(c) Calculate the curl of each of these vector fields:

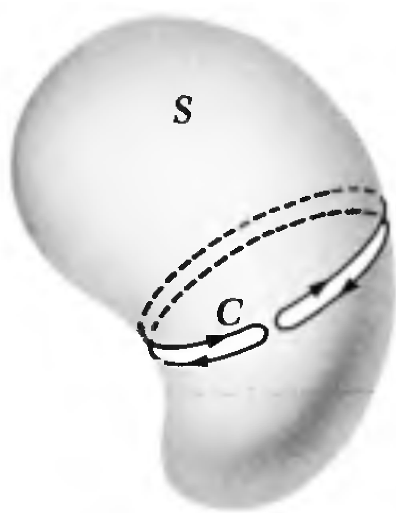
(i) $\hat{x}yz + \hat{y}xz + \hat{z}xy$.
(ii) $\hat{x}xy + \hat{y}y^2 + \hat{z}yz$.
(iii) $(1/r^2)\hat{r}$ (in spherical coordinates).
(iv) $(1/R)\hat{\phi}$ (in cylindrical coordinates).

Problem 2: Purcell 2.16

If $\vec{A}$ is any vector field with continuous derivatives, $\text{div}(\text{curl}\,\vec{A}) = 0$ or, using the "del" notation, $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. We shall need this theorem later. The problem now is to prove it. Here are two different ways in which that can be done:

(a) (Uninspired straightforward calculation in a particular coordinate system): Using the formula for $\vec{\nabla}$ in Cartesian coordinates, work out the string of second partial derivatives that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$ implies.

(b) (With the divergence theorem and Stokes' theorem, no coordinates are needed): Consider the surface S in the figure below, a balloon almost cut in two which is bounded by the closed curve C. Think about the line integral, over a curve like C, of any vector field. Then invoke Stokes and Gauss with suitable arguments.



Problem 3: Stokes's Theorem in Action

Consider the vector field $\vec{F} = \hat{x}z^2 + \hat{y}x^2 - \hat{z}y^2$.

(a) Calculate $\oint \vec{F} \cdot d\vec{r}$ around a square path with corners $(x_0 \pm s/2, \quad y_0 \pm s/2, \quad 0)$. The square has center $(x_0, y_0, 0)$, side length s, and its sides are parallel to the x- and y-axes. The sense of rotation of the path is counter-clockwise as viewed from the $+z$ direction.

(b) Divide your answer to (a) by the area of the square, and take the limit as $s \to 0$.

(c) Calculate $\vec{\nabla} \times \vec{F}$ at the center of the square.

(d) Verify that your answer to (b) is equal to the normal component of $\vec{\nabla} \times \vec{F}$ evaluated at the center of the square.

Problem 4: Gauss's Theorem in Action

Consider a vector field $\vec{F} = r\hat{r}$ (in spherical coordinates), and a closed surface S that is a cube with one corner at the origin and the opposite corner at (b, b, b). Verify Gauss's theorem,

$$\oint \vec{F} \cdot \hat{n}\, dS = \int (\vec{\nabla} \cdot \vec{F})\, dV,$$

for this particular case by performing both the surface integral on the left side, and the volume integral on the right side, and showing that they are equal.

Problem 5: Purcell 2.4 & 2.8

(a) Describe the charge distribution that goes with the following potential:

$$\phi = x^2 + y^2 + z^2 \qquad \text{for } x^2 + y^2 + z^2 < a^2$$

$$\phi = -a^2 + \frac{2a^3}{(x^2 + y^2 + z^2)^{1/2}} \qquad \text{for } a^2 < x^2 + y^2 + z^2$$

Discuss what happens at the boundary $(x^2 + y^2 + z^2 = a^2)$.

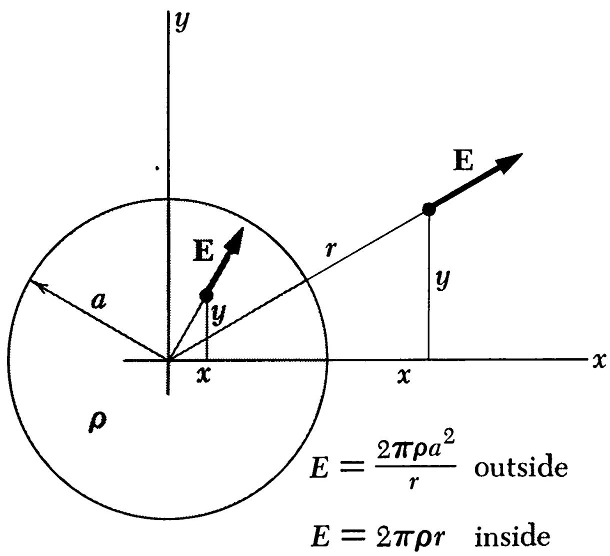
(b) Consider a very long cylinder of radius R that is filled with a uniform charge density ρ (Fig. 2.17). Use the following two different approaches to find the electric field, $\vec{E}$, both inside and outside the cylinder.

 (i) Apply Gauss' law

 (ii) Integrate Poisson's equation: $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$.

Be sure that the $\vec{E}$ field inside and the $\vec{E}$ field outside match at the boundary (i.e., at R).

FIGURE 2.17
The field inside and outside a uniform cylindrical distribution of charge.



Problem 6: Potential of an Electric Dipole

Compute the potential $\varphi(x, y, z)$ of a dipole charge configuration. The dipole consists of a charge $+q$ located at $z = a/2$ and a charge $-q$ located at $z = -q$.

(a) Write down $\varphi(x, y, z)$ (i.e., in Cartesian coordinates).

(b) Expand $\varphi(x, y, z)$ in a Maclaurin series (i.e., a Taylor series about $a = 0$) to first order in a.

(c) Convert your results to spherical coordinates: $\varphi(r, \theta, \phi)$.

(d) Compute $\vec{\nabla}\varphi(r, \theta, \phi)$, in spherical coordinates, to find $\vec{E}$.

Problem 7: Purcell 2.29

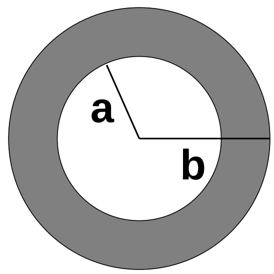
One of two nonconducting spherical shells of radius a carries a charge Q uniformly distributed over its surface, the other a charge $-Q$, also uniformly distributed. The spheres are brought together until they touch. What does the electric field look like, both outside and inside the shells? How much work is needed to move them far apart?

Problem 8: Purcell 2.14 (Laplace's equation)

Does the function $f(x,y) = x^2+y^2$ satisfy the two-dimensional Laplace equation? Does the function $g(x,y) = x^2 - y^2$? Sketch the latter function, calculate the gradient at the points $(x = 0, y = 1)$; $(x = 1, y = 0)$; $(x = 0, y = -1)$; and $(x = -1, y = 0)$ and indicate by little arrows how these gradient vectors point.

Problem 9: Electric field, potential, and flux

A hollow spherical shell carries charge density $\rho = k/r^2$ in the region $a \leq r \leq b$:



(a) Find the electric field $\vec{E}$ everywhere in space

(b) Find the potential ϕ everywhere in space.

(c) Calculate the electric flux through

- (i) A concentric sphere with radius $r_1 > b$
- (ii) A concentric sphere with radius $a \leq r_2 \leq b$
- (iii) A concentric sphere with radius $r_3 < a$
- (iv) A nonconcentric sphere centered on *any* point on the outer surface of the shell, and of radius $r_4 = 2b$.

Problem 10: Purcell 2.20 (Optional) (Potential at the center of a gold nucleus)

As a distribution of electric charge, the gold nucleus can be described as a sphere of radius 6×10^{-13} cm with a charge $Q = 79e$ distributed fairly uniformly through its interior. What is the potential ϕ_0 at the center of the nucleus, expressed in megavolts? (First derive a general formula for ϕ_0 for a sphere of charge Q and radius a. Do this by using Gauss's law to find the internal and external electric field and then integrating to find the potential.)

Ans. $\phi_0 = 3Q/2a = 95\,000$ statvolts $= 28.5$ megavolts.