

# Computational Higher Inductive Types

## Computing with Custom Equalities

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## Properties of Equality

### Warm Up: Linked Lists

#### Example: Unordered Sets

Canonical Inhabitants

Higher Inductive Types

### Computing with Higher Inductive Types

Thank you

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  - ▶  $[] = []$
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  - ▶  $[x_0, x_1, \dots, x_n] = [y_0, y_1, \dots, y_m]$  iff  $[x_1, \dots, x_n] = [y_1, \dots, y_m]$  and  $x_0 = y_0$

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- ▶ Fairly easy to prove the properties of equality
  - ▶ In Coq, Agda, and Idris, you get all of these properties for free

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- ▶ In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)
  - ▶ The problem is that either you don’t have private fields, or you can’t make use of the fact that everything is defined in terms of your public methods.

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- ▶ What if we don’t have an ordering on the elements, only equality?
- ▶ Is this really what we wanted? We asked for unordered sets, and instead made sorted lists.

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  - ▶ Take the reflexive symmetric transitive closure of the given relation
- ▶ How do we get Leibniz for free?
  - ▶ Require proving it each time you define a particular function
  - ▶ To define a function that deals with unordered sets, you have to simultaneously prove that your function is invariant under permutations

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- ▶ This is Homotopy Type Theory

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Questions?

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- ▶ Very hard to do!
- ▶ Can probably be done by way of parametricity (aka “theorems for free”), or a generalization of it
- ▶ Parametricity can be given a computational interpretation, but it’s very non-trivial to do so