

# Computational Higher Inductive Types

## Computing with Custom Equalities

Jason Gross  
`jgross@mit.edu`

MIT CSAIL Student Workshop

June 1, 2025

## Properties of Equality

## Warm Up: Linked Lists

Example: Unordered Sets  
Canonical Inhabitants  
Higher Inductive Types

## Computing with Higher Inductive Types

Thank you

# Properties of Equality

# Properties of Equality

- ▶ Reflexivity:  $x = x$

# Properties of Equality

- ▶ Reflexivity:  $x = x$
- ▶ Symmetry: if  $x = y$  then  $y = x$

# Properties of Equality

- ▶ Reflexivity:  $x = x$
- ▶ Symmetry: if  $x = y$  then  $y = x$
- ▶ Transitivity: if  $x = y$  and  $y = z$ , then  $x = z$

# Properties of Equality

- ▶ Reflexivity:  $x = x$
- ▶ Symmetry: if  $x = y$  then  $y = x$
- ▶ Transitivity: if  $x = y$  and  $y = z$ , then  $x = z$
- ▶ Leibniz rule: if  $x = y$ , then  $f(x) = f(y)$

# Warm Up: Linked Lists

- ▶ Two constructors:



## Warm Up: Linked Lists

- ▶ Two constructors: `nil`, or `[]`, and `cons`

## Warm Up: Linked Lists

- ▶ Two constructors: `nil`, or `[]`, and `cons`
- ▶ Two accessors on non-nil lists:

## Warm Up: Linked Lists

- ▶ Two constructors: `nil`, or `[]`, and `cons`
- ▶ Two accessors on non-`nil` lists: `head` and `tail`

## Warm Up: Linked Lists

- ▶ Two constructors: `nil`, or `[]`, and `cons`
- ▶ Two accessors on non-nil lists: `head` and `tail`
- ▶ Equality is defined on an element-by-element basis
  - ▶  $[] = []$
  - ▶  $[] \neq [a, \dots]$
  - ▶  $[a, \dots] \neq []$
  - ▶  $[x_0, x_1, \dots, x_n] = [y_0, y_1, \dots, y_m]$  iff  $[x_1, \dots, x_n] = [y_1, \dots, y_m]$  and  $x_0 = y_0$

# Warm Up: Linked Lists

- ▶ Two constructors: `nil`, or `[]`, and `cons`
- ▶ Two accessors on non-`nil` lists: `head` and `tail`
- ▶ Equality is defined on an element-by-element basis
  - ▶  $[] = []$
  - ▶  $[] \neq [a, \dots]$
  - ▶  $[a, \dots] \neq []$
  - ▶  $[x_0, x_1, \dots, x_n] = [y_0, y_1, \dots, y_m]$  iff  $[x_1, \dots, x_n] = [y_1, \dots, y_m]$  and  $x_0 = y_0$
- ▶ Fairly easy to prove the properties of equality
  - ▶ In Coq, Agda, and Idris, you get all of these properties for free

# Example: Unordered Sets

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`



## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree
- ▶ Equality is then implemented as “is one a permutation of the other?”

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree
- ▶ Equality is then implemented as “is one a permutation of the other?”
- ▶ Fairly easy to prove that it’s an equivalence relation

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree
- ▶ Equality is then implemented as “is one a permutation of the other?”
- ▶ Fairly easy to prove that it’s an equivalence relation
- ▶ Leibniz rule (if  $x = y$ , then  $f(x) = f(y)$ ) is harder
- ▶ In Haskell, Agda, Coq, and Idris, the Leibniz rule is false!

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree
- ▶ Equality is then implemented as “is one a permutation of the other?”
- ▶ Fairly easy to prove that it’s an equivalence relation
- ▶ Leibniz rule (if  $x = y$ , then  $f(x) = f(y)$ ) is harder
- ▶ In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)

## Example: Unordered Sets

- ▶ `nil`, or  $\emptyset$
- ▶ `add`
- ▶ `remove`
- ▶ `contains`
- ▶ Often implemented internally as a list or a tree
- ▶ Equality is then implemented as “is one a permutation of the other?”
- ▶ Fairly easy to prove that it’s an equivalence relation
- ▶ Leibniz rule (if  $x = y$ , then  $f(x) = f(y)$ ) is harder
- ▶ In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)
  - ▶ The problem is that either you don’t have private fields, or you can’t make use of the fact that everything is defined in terms of your public methods.



# Example: Unordered Sets

## Solution 1: Canonical Inhabitants

- ▶ Give up private fields, but use element-wise equality

# Example: Unordered Sets

## Solution 1: Canonical Inhabitants

- ▶ Give up private fields, but use element-wise equality
- ▶ Define a type of “sorted lists without duplication”, and call them sets

# Example: Unordered Sets

## Solution 1: Canonical Inhabitants

- ▶ Give up private fields, but use element-wise equality
- ▶ Define a type of “sorted lists without duplication”, and call them sets
- ▶ Now we can use element-wise equality, and get Leibniz (and other properties) for free

# Example: Unordered Sets

## Solution 1: Canonical Inhabitants

- ▶ Give up private fields, but use element-wise equality
- ▶ Define a type of “sorted lists without duplication”, and call them sets
- ▶ Now we can use element-wise equality, and get Leibniz (and other properties) for free
- ▶ What if we don't have an ordering on the elements, only equality?

# Example: Unordered Sets

## Solution 1: Canonical Inhabitants

- ▶ Give up private fields, but use element-wise equality
- ▶ Define a type of “sorted lists without duplication”, and call them sets
- ▶ Now we can use element-wise equality, and get Leibniz (and other properties) for free
- ▶ What if we don't have an ordering on the elements, only equality?
- ▶ Is this really what we wanted? We asked for unordered sets, and instead made sorted lists.

# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types

# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ▶ Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation

# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ▶ Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- ▶ How do we get that it's an equivalence relation for free?



# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ▶ Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- ▶ How do we get that it's an equivalence relation for free?
  - ▶ Take the reflexive symmetric transitive closure of the given relation

# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ▶ Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- ▶ How do we get that it's an equivalence relation for free?
  - ▶ Take the reflexive symmetric transitive closure of the given relation
- ▶ How do we get Leibniz for free?

# Example: Unordered Sets

## Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ▶ Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- ▶ How do we get that it's an equivalence relation for free?
  - ▶ Take the reflexive symmetric transitive closure of the given relation
- ▶ How do we get Leibniz for free?
  - ▶ Require proving it each time you define a particular function
  - ▶ To define a function that deals with unordered sets, you have to simultaneously prove that your function is invariant under permutations

# Computing with Higher Inductive Types

- ▶ It seems simple enough, so what's the problem?

# Computing with Higher Inductive Types

- ▶ It seems simple enough, so what's the problem?
- ▶ Having higher inductive types gives you functional extensionality (if  $f(x) = g(x)$  for all  $x$ , then  $f = g$ ), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris

# Computing with Higher Inductive Types

- ▶ It seems simple enough, so what's the problem?
- ▶ Having higher inductive types gives you functional extensionality (if  $f(x) = g(x)$  for all  $x$ , then  $f = g$ ), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris
- ▶ Equality in Coq and Agda (`--without-K`) actually has a rich structure

# Computing with Higher Inductive Types

- ▶ It seems simple enough, so what's the problem?
- ▶ Having higher inductive types gives you functional extensionality (if  $f(x) = g(x)$  for all  $x$ , then  $f = g$ ), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris
- ▶ Equality in Coq and Agda (`--without-K`) actually has a rich structure
- ▶ If you look at proofs of equality, and equality of these proofs, and you iterate this process, you get enough math to do topology!

# Computing with Higher Inductive Types

- ▶ It seems simple enough, so what's the problem?
- ▶ Having higher inductive types gives you functional extensionality (if  $f(x) = g(x)$  for all  $x$ , then  $f = g$ ), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris
- ▶ Equality in Coq and Agda (`--without-K`) actually has a rich structure
- ▶ If you look at proofs of equality, and equality of these proofs, and you iterate this process, you get enough math to do topology!
- ▶ This is Homotopy Type Theory



Thank you

Thanks!

Thank you

Thanks!

Questions?

# Example: Unordered Sets

## Solution 3: Parametricity

- ▶ Make use of the fact that private fields are private

# Example: Unordered Sets

## Solution 3: Parametricity

- ▶ Make use of the fact that private fields are private
- ▶ Very hard to do!

# Example: Unordered Sets

## Solution 3: Parametricity

- ▶ Make use of the fact that private fields are private
- ▶ Very hard to do!
- ▶ Can probably be done by way of parametricity (aka “theorems for free”), or a generalization of it

# Example: Unordered Sets

## Solution 3: Parametricity

- ▶ Make use of the fact that private fields are private
- ▶ Very hard to do!
- ▶ Can probably be done by way of parametricity (aka “theorems for free”), or a generalization of it
- ▶ Parametricity can be given a computational interpretation, but it’s very non-trivial to do so