# Computational Higher Inductive Types

## Computing with Custom Equalities

Jason Gross
jgross@mit.edu

MIT CSAIL Student Workshop

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- Leibniz rule: if $x = y$, then $f(x) = f(y)$

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- Equality is defined on an element-by-element basis
  - $[] = []$
  - $[] \neq [a, \ldots]$
  - $[a, \ldots] \neq []$
  - $[x_0, x_1, \ldots, x_n] = [y_0, y_1, \ldots, y_m]$ iff $[x_1, \ldots, x_n] = [y_1, \ldots, y_m]$ and $x_0 = y_0$

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- Fairly easy to prove the properties of equality
  - In Coq, Agda, and Idris, you get all of these properties for free

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- In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)
  - The problem is that either you don't have private fields, or you can't make use of the fact that everything is defined in terms of your public methods.

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- Is this really what we wanted? We asked for unordered sets, and instead made sorted lists.

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- How do we get Leibniz for free?
  - Require proving it each time you define a particular function
  - To define a function that deals with unordered sets, you have to simultaneously prove that your function is invariant under permutations

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- If you look at proofs of equality, and equality of these proofs, and you iterate this process, you get enough math to do topology!
- This is Homotopy Type Theory

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- Very hard to do!
- Can probably be done by way of parametricity (aka "theorems for free"), or a generalization of it
- Parametricity can be given a computational interpretation, but it's very non-trivial to do so