Computational Higher Inductive Types Computing with Custom Equalities

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Properties of Equality

Warm Up: Linked Lists

Example: Unordered Sets Canonical Inhabitants Higher Inductive Types

Computing with Higher Inductive Types

Thank you

Properties of Equality

- ightharpoonup Reflexivity: x = x
- ightharpoonup Symmetry: if x = y then y = x
- ▶ Transitivity: if x = y and y = z, then x = z
- ▶ Leibniz rule: if x = y, then f(x) = f(y)

Warm Up: Linked Lists

- Two constructors: nil, or [], and cons
- Two accessors on non-nil lists: head and tail
- Equality is defined on an element-by-element basis
 - **▶** [] = []
 - ightharpoonup $[]
 eq [a, \ldots]$
 - $\blacktriangleright [a,\ldots] \neq []$
 - $[x_0, x_1, \dots, x_n] = [y_0, y_1, \dots, y_m] \text{ iff } [x_1, \dots, x_n] = [y_1, \dots, y_m]$ and $x_0 = y_0$
- Fairly easy to prove the properties of equality
 - ▶ In Coq, Agda, and Idris, you get all of these properties for free

- ▶ nil, or ∅
- add
- remove
- contains
- Often implemented internally as a list or a tree
- Equality is then implemented as "is one a permutation of the other?"
- Fairly easy to prove that it's an equivalence relation
- Leibniz rule (if x = y, then f(x) = f(y)) is harder
- In Haskell, Agda, Coq, and Idris, the Leibniz rule is false! (or at least not internally provable)
 - The problem is that either you don't have private fields, or you can't make use of the fact that everything is defined in terms of your public methods.

Solution 1: Canonical Inhabitants

- Give up private fields, but use element-wise equality
- Define a type of "sorted lists without duplication", and call them sets
- Now we can use element-wise equality, and get Leibniz (and other properties) for free
- What if we don't have an ordering on the elements, only equality?
- ▶ Is this really what we wanted? We asked for unordered sets, and instead made sorted lists.

Solution 2: Higher Inductive Types

- ▶ Higher Inductive Types
- ► Keep the built-in equality (so we get the properties for free), but turn it into equality up to permutation
- ▶ How do we get that it's an equivalence relation for free?
 - Take the reflexive symmetric transitive closure of the given relation
- ► How do we get Leibniz for free?
 - ▶ Require proving it each time you define a particular function
 - To define a function that deals with unordered sets, you have to simultaneously prove that your function is invariant under permutations

Computing with Higher Inductive Types

- It seems simple enough, so what's the problem?
- ▶ Having higher inductive types gives you functional extensionality (if f(x) = g(x) for all x, then f = g), which doesn't yet have a good computational interpretation in Coq nor Agda nor Idris
- Equality in Coq and Agda (--without-K) actually has a rich structure
- If you look at proofs of equality, and equality of these proofs, and you iterate this process, you get enough math to do topology!
- ► This is Homotopy Type Theory

Thank you

Thanks!

Questions?

Solution 3: Parametricity

- ▶ Make use of the fact that private fields are private
- Very hard to do!
- ► Can probably be done by way of parametricity (aka "theorems for free"), or a generalization of it
- Parametricity can be given a computational interpretation, but it's very non-trivial to do so