1 Week 1

1.1 Vector properties

Properties
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$
1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2) $\vec{u} + \vec{v} + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3) $\vec{0} + \vec{v} = \vec{v}$
where $\vec{0} = [0 \dots 0]^T$ in $\mathbb{R}^n$
Inverse

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \dots u_n \end{bmatrix} \implies -\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ \dots - u_n \end{bmatrix}$$

$$\vec{u} + (-\vec{u}) = \vec{0}$$

Standard Basis

$$\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

1.2 Dot Product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties
1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3) $(c\vec{u} \cdot \vec{v}) = c(\vec{u} \cdot \vec{v})$
4) $\vec{v} \cdot \vec{v} \geq 0$ with $\vec{v} \cdot \vec{v} = 0 \iff \vec{v} = 0$
Length / Norm / Magnitude

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

$$c \in \mathbb{R}, \vec{v} \in \mathbb{R}^n \implies \|c\vec{v}\| = |c|\|\vec{v}\|$$

Unit Vector

$$\|\vec{v}\| = 1$$

Normalization

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

To find a vector in the same direction apply $\hat{v}$ formula
Find angle between $\vec{v}$ and $\vec{w}$
The angle between them is $0 \le \theta \le \pi$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta, \text{ that is, } \theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

where

$$|\vec{v} \cdot \vec{w}| \le \|\vec{v}\|\vec{w}\| \text{ for all } \vec{v}, \vec{w} \in \mathbb{R}^n.$$

Orthogonal / Perpendicular
It is orthogonal when

$$\vec{u} \cdot \vec{v} = 0$$

Also occurs when $\theta = \dfrac{\pi}{2}$ if applied to the formula above

1.3 Projection, Components, Perpendicular

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v} \cdot \vec{w})}{\|\vec{w}\|^2}\,\vec{w} = \frac{(\vec{v} \cdot \vec{w})}{\vec{w} \cdot \vec{w}}\,\vec{w}.$$

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v}).$$

$$\text{perp}_{\vec{w}}(\vec{v}) \cdot \text{proj}_{\vec{w}}(\vec{v}) = 0.$$

1.4 Cross Product

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$

Properties of Cross Product
Let $\vec{z} = \vec{u} \times \vec{v}$,
1. $\vec{z} \cdot \vec{u} = 0$ & $\vec{z} \cdot \vec{v}$
2. $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$
3. $\vec{u}, \vec{v} \neq 0 \implies \|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$, θ is the angle between $\vec{u}, \vec{v}$

Area

$$\|\vec{u}\|h = \|\vec{u}\|\|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$

More Properties
1. $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

2 Week 2

2.1 Linear Combinations and Span

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} \text{ is a linear combination of } \vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$$
$$\text{Span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\} = \{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} : c_1, c_2, \ldots, c_k \in \mathbb{F}\}.$$

2.2 Lines

Parametric equations of a line in $\mathbb{R}^2$ through the point $(x1, y1)$with slope $\frac{p}{q}$ are

$$\begin{array}{c} x = x_1 + qt \\ y = y_1 + pt \end{array}, \quad t \in \mathbb{R}.$$

$$\vec{l} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} q \\ p \end{bmatrix}, t \in \mathbb{R}$$

Direction: $\begin{bmatrix} q \\ p \end{bmatrix}$

$$\mathcal{L} = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$$

Parametric Equations of a Line in $\mathbb{R}^n$

$$\vec{l} = \vec{u} + t\vec{v}, \quad t \in \mathbb{R}.$$

The parametric equations of the line L in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$ are

$$\begin{array}{c} \ell_1 = u_1 + tv_1 \\ \ell_2 = u_2 + tv_2 \\ \vdots \\ \ell_n = u_n + tv_n \end{array}, \quad t \in \mathbb{R}.$$

2.3 Plane

Plane through Origin

$$\mathcal{P} = \text{Span}\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$
$$\vec{p} = s\vec{v} + t\vec{w}$$

Plane in $\mathbb{R}^n$

$$\mathcal{P} = \{\vec{u} + s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$$
$$\vec{p} = \vec{u} + s\vec{v} + t\vec{w}, \quad s, t \in \mathbb{R}$$

Let $\mathcal{P}$ be a plane **in** $\mathbb{R}^3$ with direction vectors $\vec{v}$ and $\vec{w}$ and a normal (orthogonal) vector (usually $\vec{v} \times \vec{w}$) $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}, \vec{u} \in \mathcal{P}$ and $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{P}$ Normal form of $\mathcal{P}$ is given by

$$\vec{n} \cdot (\vec{p} - \vec{u}) = 0.$$

General form (scalar equation) given by

$$ax + by + cz = d = \vec{n} \cdot \vec{u}$$

Goes through origin if the following:

$$\iff \vec{0} \text{ satisfies the equation}$$

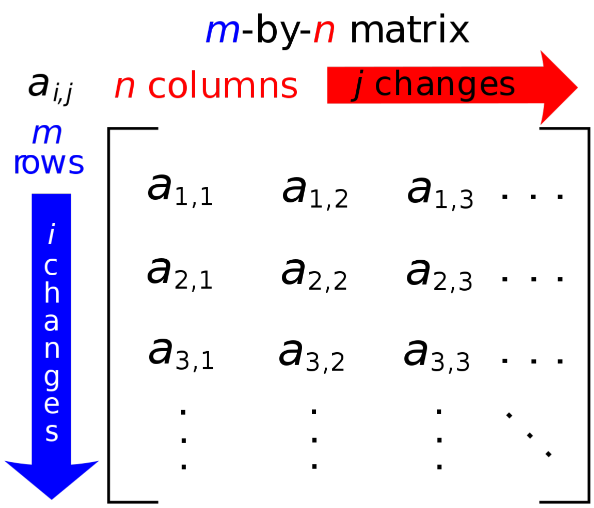
$$\iff (\vec{v} \times \vec{w}) \cdot (\vec{0} - \vec{u}) = 0$$

$$\iff \vec{u} = a\vec{v} + b\vec{w}, a, b \in \mathbb{R}$$

2.4 Systems of Linear Equation

$$\begin{array}{cccccc} a_{1,1}y_1 & +a_{1,2}y_2 & + & \cdots & +a_{1,n}y_n & = & b_1 \\ a_{2,1}y_1 & +a_{2,2}y_2 & + & \cdots & +a_{2,n}y_n & = & b_2 \\ & & & \vdots & & & \\ a_{m,1}y_1 & +a_{m,2}y_2 & + & \cdots & +a_{m,n}y_n & = & b_m \end{array}$$

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a solution to the system



3 Week 3

Given a system:

$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & b_2 \\ & & & \vdots & & & \\ a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & b_m \end{array}$$

3.1 Coefficient Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

3.2 Leading Entry

The leftmost non-zero entry in any non-zero row of a matrix is called the leading entry of that row. If the leading entry is a 1, then it is called a leading one.

3.3 Row Echelon Form (REF)

1. All zero rows occurs as final rows
2. The leading entry in any non-zero row appears in a column to the right of the columns containing the leading entries of any of the rows above it.

3.4 Pivot

If a matrix is in REF, then the leading entries are referred to as pivots and their positions in the matrix are called pivot positions. Any column that contains a pivot position is called a pivot column. Any row that contains a pivot position is called a pivot row.

3.5 Reduced Row Echelon Form (RREF)

1. It is in REF.
2. All its pivots are leading ones.
3. The only non-zero entry in a pivot column is the pivot itself.

3.6 Inconsistent

$$[0 \ldots 0 | b]$$

where $b \neq 0$. Can stop algorithm and conclude the system is inconsistent. Hence, no solutions.

3.7 Augmented Matrix

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

3.8 Basic, Free Variables

Consider a system of linear equations. Let R be a REF of the coefficient matrix of this system. If the i^{th} column of this matrix contains a pivot, then we call x a basic variable. Otherwise, its called a free variable.
e.g.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 are **basic variables** whereas x_3 is a **free variable,** R_3 is also **called a zero row**. This equation also has an infinite amount of solutions if we let $x_3 =$ an arbitrary variable

3.9 Rank

Let $A \in M_{m\times n}(\mathbb{F})$ s.t. $RREF(A)$ has r pivots. So $\text{rank}(A) = r$.

3.10 Rank Bounds

If $A \in M_{m\times n}(\mathbb{F})$ then rank(A) $\leq$ min{m,n}

3.11 Consistent System Test

The system is consistent if and only if rank(A) = $\text{rank}\left(\left[A|\overrightarrow{b}\right]\right)$

3.12 System Rank Theorem

Let $A \in M_{m\times n}(\mathbb{F})$ with rank(A) = r.

1. Let $b \in \mathbb{F}^m$ If the $[A|\overrightarrow{b}]$ is consistent, then the solution set contains $n - r$ parameters
2. $[A|\overrightarrow{b}]$ is consistent for every $\overrightarrow{b} \in \mathbb{F}^n \iff r = m$

3.13 Nullity

Let $A \in M_{m\times n}(\mathbb{F})$ with rank(A) = r. Define nullity of A, written nullity(A) to be integer $n - r$.

4 Week 4

4.1 Homogeneous and Non-homogeneous Systems

We say that a system of linear equations is homogeneous if all the constant terms on the right-hand side of the equations are zero. Otherwise we say the system is non-homogeneous.
The trivial solution is to let all variables $= 0$

4.2 Theorem on homogeneous systems

Let $A\vec{x} = \vec{0}$ be a homogeneous system of linear equation with solution set $\mathcal{S}$. If $\vec{x}, \vec{y} \in \mathcal{S}$ and if $c \in \mathbb{F}$ then $\vec{x} + \vec{y} \in \mathcal{S}$ and $c\vec{x} \in \mathcal{S}$

4.3 Associated Homogeneous system

Let $A\vec{x} = \vec{b}, b \neq 0$ The associated homogeneous system is the system $A\vec{x} = \vec{0}$

4.4 Solutions to a system

Let $A\vec{x} = \vec{b}, \vec{b} \neq 0$ be a **consistent non-homogeneous** system with solution set $\hat{\mathcal{S}}$ Let $A\vec{x} = \vec{0}$ be the associated homogeneous system with solution set S. If $\vec{x}_p \in \hat{\mathcal{S}}$ then

$$\hat{\mathcal{S}} = \{\vec{x}_p + \vec{x} : \vec{x} \in S\}$$

4.5 Corollary to solution of systems

Given two consistent and non-homogeneous systems

$$A\vec{x} = \vec{b} \text{ and } A\vec{x} = \vec{c}$$

where $\vec{b} \neq \vec{c}$ And the solution sets are $\mathcal{S}_b$ and $\mathcal{S}_c$ respectively, with particular solutions $\vec{x}_b$ and $\vec{x}_c$ then,

$$\mathcal{S}_c = \{(\vec{x}_c - \vec{x}_b) + \vec{z} : \vec{z} \in \mathcal{S}_b\}$$

4.6 Nullspace

Solution set to the homogeneous system of linear equations with coefficient matrix A is called **nullspace** denoted as $\text{Null}(A)$.

4.7 Row Vector

Matrix with exactly one row. $A \in M_{m\times n}(\mathbb{F})$, the i-th row of A denoted by $\vec{\text{row}}_i(A)$

4.8 Linearity of Matrix Multiplication

a) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
b) $A(c\vec{x}) = cA\vec{x}$

4.9 Column Space

Let $A \in M_{m\times n}(\mathbb{F})$. A column space is

$$\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$$

If $A\vec{x} = \vec{b}$ is consistent $\iff$ $\vec{b} \in \mathrm{Col}(A)$

4.10 Transpose

Let $A \in M_{m\times n}(\mathbb{F})$. Denoted by A^T where

$$(A^T)_{ij} = (A)_{ji}$$

e.g.

$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix}$

4.11 Row Space

Let $A \in M_{m\times n}(\mathbb{F})$.

$$\mathrm{Row}(A) = \mathrm{Span}\{(\overrightarrow{row_1}(A))^T, \ldots, \overrightarrow{row_m}(A))^T\}$$

e.g.

$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix} \implies \mathrm{Row}(A) = \mathrm{Span}\{\begin{bmatrix} 1 & 2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -5 & 6 \end{bmatrix}\}$

Remark: Row(A) = $\mathrm{Col}(A^T)$

4.12 Row Equivalent

If B is row equivalent to A, then

$$\mathrm{Row}(B) = \mathrm{Row}(A)$$

4.13 Matrix Equality

Matrix $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{p\times q}(\mathbb{F})$ are equal if:

1. A and B have the same size (m = n, p = q)
2. The corresponding entries of A and B are equal. $\forall i, j = 1, 2, \ldots, n, a_{ij} = b_{ij}$

4.14 Column Extraction

Let $A = [\vec{a}_1 \ldots \vec{a}_n] \in M_{m\times n}(\mathbb{F})$ Then $A\vec{e}_i = \vec{a}_i$ where $\vec{e}_i$ is the standard basis

4.15 Matrix Equality

$$A = B \iff A\vec{x} = B\vec{x}, \forall \vec{x} \in \mathbb{F}^n$$

4.16 Matrix Multiplication

$$C = AB = A[\vec{b}_1 \vec{b}_2 \dots \vec{b}_p] = [A\vec{b}_1 A\vec{b}_2 \dots A\vec{b}_p]$$

The j^{th} column is obtained by

$$\vec{c}_j = A\vec{b}_j$$

Properties:

1. $(A + B)C = AC + BC$
2. $A(C + D) = AC + AD$
3. $(AC)E = A(CE) = ACE$

4.17 Matrix Addition

Must be the same size!
Properties

1. $A + B = B + A$
2. $A + B + C = (A + B) + C = A + (B + C)$

4.18 Zero Matrix / Additive Inverse

$\mathcal{O}_{m \times n}$ is a matrix whose entries are all 0
The inverse is just the thing negative
Properties:

1. $A + \mathcal{O} = \mathcal{O} + A = A$
2. $A + (-A) = (-A) + A = \mathcal{O}$

4.19 Scalar Properties

1. $(cA)_{ij} = ca_{ij}$
2. $s(A + B) = sA + sB$
3. $(r + s)A = rA + sA$
4. $r(sA) = (rs)A$
5. $s(AC) = (sA)C = A(sC)$

4.20 Transpose Properties

1. $(A+B)^T = A^T + B^T$
2. $(sA)^T = sA^T$
3. $(AC)^T = C^T A^T$
4. $(A^T)^T = A$

4.21 Square Matrices

It is called a **square matrix** when $A \in M_{n\times n}(\mathbb{F})$
It is called an **upper triangle** if $a_{ij} = 0, i > j$
It is called a **lower triangle** if $a_{ij} = 0, i < j$
Diagonal entries, $\text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$
e.g. Upper Triangle $= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$
Diagonal Matrix $= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, diag(3, 4, 0)

4.22 Identity Matrix

diag(1, 1, ..., 1) denoted by I_n to indicate the matrix is $n \times n$
Remark: Multiplicative Identity
For $A \in M_{m\times n}(\mathbb{F})$,

$$I_m A = A \text{ and } AI_n = A$$

This also holds true for vectors $I_n \vec{x} = \vec{x}$, and $cI_n = diag(c, c, \ldots, c)$

5 Week 5

5.1 Elementary Matrix

Matrix that can be obtained by performing a **single** ERO
e.g.
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ swapping $R_1 \longleftrightarrow R_3$ on I_3

Proposition:
Let $A \in M_{m\times n}(\mathbb{F})$ and suppose a single ERO is performed on it to produce B. We can perform the same ERO on matrix I_m to produce matrix E s.t.

$$B = EA$$

Corollary:
Let $A \in M_{m\times n}(\mathbb{F})$ and suppose that a finite number of EROs $(1 \dots k)$ are performed on A to produce B. Let E_i denote the elementary matrix corresponding to the i^{th} ERO $(i \leq i \leq k)$ applied to I_m

$$B = E_k \dots E_2 E_1 A$$

5.2 Invertible Matrix

A $n \times n$ matrix A is **invertible** if there exists a $n \times n$ matrices B and C s.t.

$$AB = CA = I_n$$

Proposition:
Let $A \in M_{n\times n}(\mathbb{F})$ If there exist matrices B and C in $M_{n\times n}(\mathbb{F})$ s.t. $AB = CA = I_n$ then $B = C$
(If A is invertible, then its left and right inverses are equal)

5.3 Left Right Invertible Theorem

For $A \in m_{n\times n}(\mathbb{F})$, there exists a $n \times n$ matrix B s.t. $AB = I_n \iff$ there exists an $n \times n$ matrix C s.t. $CA = I_n$

5.4 Definition of Inverse

If A is invertible $AB = I_n$, we denote the inverse of A by A^{-1}. The inverse satisfies

$$AA^{-1} = A^{-1}A = I_n$$

(If $AB = I_n$, we don't need to verify $BA = I_n$)

5.5 Criteria for Invertibility

Let $A \in M_{m\times n}(\mathbb{F})$ The following 3 criteria are equivalent

- A is invertible
- $\text{rank}(A) = n$
- $\text{RREF}(A) = I_n$

5.6 Determining Invertibility

1. Construct a super-augmented Matrix $[A|I_n]$
2. Find RREF, $[R|B]$, of $[A|I_n]$
3. If $R \neq I_n$, A is not invertible. If $R = I_n$, A is invertible and that $A^{-1} = B$

5.7 Inverse of 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is invertible iff $ad - bc \neq 0$. Furthermore, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Remark: $ad - bc$ is called the **determinant**)

5.8 Transformations

Let $A \in M_{m\times n}(\mathbb{F})$ The function determined by the matrix A is

$$T_A : \mathbb{F}^n \to \mathbb{F}^m$$

defined by $T_A(\vec{x}) = A\vec{x}$
e.g.
Let $A = \begin{bmatrix} 1 & 4 \\ -2 & -5 \\ 4 & 6 \end{bmatrix}$ If $x \in \mathbb{R}^2$,

$$T_A(\vec{x}) = \begin{bmatrix} 1 & 4 \\ -2 & -5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ -2x_1 - 5x_2 \\ 4x_1 + 6x_2 \end{bmatrix}$$

Note that $T_A(\vec{x}) \in \mathbb{R}^3$ s.t. $T_A\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ -1 \\ 10 \end{bmatrix}$
(Note: takes in input in $\mathbb{R}^2$ and outputs $\mathbb{R}^3$)

5.9 Function Determined by a Matrix is Linear

Let $A \in M_{m \times n}(\mathbb{F})$ and let T_A be the function determined by matrix A. Then T_A is linear for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$:

1. $T_A(\vec{x} + \vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$
2. $T_A(c\vec{x}) = cT_A(\vec{x})$

5.10 Linear Transformations

We say a function $T : \mathbb{F}^n \to \mathbb{F}^m$ is a **linear transformation / mapping** if for $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$, the following properties hold:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2. $T(c\vec{x}) = cT(\vec{x})$

F_n is referred to as the **domain** of T and F^m as the **codomain** of T
Proposition 1:
T is a linear transformation if and only if for $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$

$$T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y})$$

Proposition 2:
Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation then

$$T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$$

6 Week 6

6.1 Range

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We defined the **range** of T to be

$$\text{Range}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$$

The range of T is a subset of $\mathbb{F}^m$, will always have $\vec{0}_{\mathbb{F}^m} \in \text{Range}(A)$, the set is NEVER empty.

6.2 Range of Linear Transformation

Let $A \in M_{m\times n}(\mathbb{F})$, and let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation determined by A

$$\text{Range}(T_A) = \text{Col}(A)$$

Remark:

$$A\vec{x} = \vec{b} \text{is consistent } \iff \vec{b} \in \text{Range}(T_A)$$

6.3 Onto

A transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is **onto/surjective** if $\text{Range}(T) = \mathbb{F}^m$
Following statements are equivalent:

- T_A is onto
- $\text{Col}(A) = \mathbb{F}^m$
- $\text{rank}(A) = m$

6.4 Kernel

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. The **kernel** of T, denoted $\text{Ker}(T)$ to be the set of inputs of T whose output is 0.

$$\text{Ker}(T) = \left\{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\right\}$$

Again it is a subset of $\mathbb{F}^n$ and its never empty as $\vec{0}_{\mathbb{F}^n} \in \text{Ker}(T)$

$$\text{Ker}(T_A) = \text{Null}(A)$$

It is equal to the solution set of the homogeneous system $A\vec{x} = \vec{0}$

6.5 One-to-one

We say the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is one-to-one/injective if, whenether

$$T(\vec{x}) = T(\vec{y}) \implies \vec{x} = \vec{y}$$

Contrapositive:

$$\vec{x} \neq \vec{y} \implies T(\vec{x}) \neq T(\vec{y})$$

Following statements are equivalent:

- T_A is one-to-one
- $\text{Null}(A) = \{\vec{0}_{\mathbb{F}^n}\}$
- $\text{nullity}(A) = 0$
- $\text{rank}(A) = n$

6.6 Invertibility Criteria - Extended

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix and let T_A be the linear transformation by matrix A.

1. A is invertible
2. T_A is one-to-one
3. T_A is onto
4. $\text{Null}(A) = \{\vec{0}\}$ is the only solution to the homogeneous system
5. $\text{Col}(A) = \mathbb{F}^n$
6. $\text{nullity}(A) = 0$
7. $rank(A) = n$
8. $\text{RREF}(A) = I_n$

6.7 Standard Matrix

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. The standard matrix of T denoted by $[T]_\epsilon$, to be $m \times n$ matrix whose columns are the images under T of vectors in the standard basis of F^n

$$[T]_\epsilon = [T(\vec{e}_1)T(\vec{e}_2)\dots T(\vec{e}_n)]$$

6.8 Every Linear Transformation determined by Matrix

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation and let $[T]_\epsilon$ be the standard matrix of T. Then for all $\vec{x} \in \mathbb{F}^n$

$$T(\vec{x}) = [T]_\epsilon \vec{x}$$

$T = T_{[T]_\epsilon}$ is the linear transformation determined by matrix $[T]_\epsilon$
Proposition:
Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Then there exist a real number $m \in \mathbb{R}$ s.t. $T(x) = mx$ for all $x \in \mathbb{R}$

6.9 Properties of Standard Matrix

Let $A \in M_{m\times n}(\mathbb{F}), let T_A : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation determined by A and let $T : \mathbb{F}^n \to F^m$ be a linear transformation. Then,

- $T_{[T]_\epsilon} = T$
- $[T_A]\epsilon = A$
- T is onto iff rank($[T]_\epsilon = m$
- T is one-to-one iff rank($[T]_\epsilon) = n$

6.10 Composition of Linear Transformations

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be a linear transformation. We defined the composite function $T_2 \circ T_1$ as

$$(T_2 \circ T_1)\vec{x} = T_2(T_1(\vec{x}))$$

6.11 Composition of Linear Transformation is Linear

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be a linear transformation. Then $T_2 \circ T_1$ is a linear transformation.

6.12 Standard Matrix of Linear Transformations

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. Then the standard matrix of $T_2 \circ T_1$ is equal to the product of standard matrices of T_2 and T_1 that is

$$[T_2 \circ T_1]_\epsilon = [T_2]_\epsilon [T_1]_\epsilon$$

6.13 Identity Transformation

$$\forall \vec{x} \in \mathbb{F}^n, \mathrm{id}_n(\vec{x}) = \vec{x}$$

6.14 Exponent of transformations

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ and let $p > 1$ be an integer. We define the p-th power of T denoted by T^p

$$T^p = T \circ T^{p-1}$$

Also define $T_0 = \mathrm{id}_n$
Corollary:
Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation and let $p > 1$ be an integer. Then the standard matrix of the p-th power of the standard matrix of T is

$$[T^p]_\epsilon = ([T]_\epsilon)^p$$

7 Week 7

7.1 Definition of Determinant

If $A = \begin{bmatrix} a_{11} \end{bmatrix}$ then,

$$\det(A) = a_{11}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then,

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

7.2 Submatrix

$M_{ij}(A)$ obtained by removing the i-th and j-th row column from A. The determinant of it is also known as the $(i,j)^{th}$ **minor** of A.

7.3 Determinant of a $n \times n$ matrix

$A \in M_{n\times n}(\mathbb{F})$

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{1+j}\det(M_{ij}(A))$$

7.4 Easy Determinants

- If there is a row or column consisting of only zeros, $\det(A) = 0$
- $A = \begin{bmatrix} a_{11} & * & * & \dots & * \\ 0 & a_{22} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$. Then $\det(A) = a_{11}a_{22}\dots a_{nn}$
- $\det(I_n) = 1$
- $\det(A) = \det(A^T)$

7.5 EROS on Determinants

Let B be the matrix obtained from performing an ERO

1. Row Swap $\det(B) = -\det(A)$
2. Row Scale $m \neq 0, \det(B) = m\det(A)$
3. Row Addition, adding a non zero multiple of one row to another $\det(B) = \det(A)$

7.6 Corollary

If there are two identical rows $\det(A) = 0$

7.7 Determianants of Elementary Matrix

1. Row Swap $\det(E) = -1$
2. Row Scale $\det(E) = m$
3. Row Addition $\det(E) = 1$
4. $\det(B) = \det(E_k) \dots \det(E_1) \det(A)$

7.8 Invertibility

A is invertible if and only if $\det(A) \neq 0$

7.9 Determinant of a Product

- $\det(AB) = \det(A) \det(B)$
- $\det(AB) = \det(BA)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- $\det(cA) = c^n \det(A)$
- $\det(A) = \det(A^T)$

7.10 Cofactor

$$C_{ij}(A) = (-1)^{i+j} \det(M_{ij}(A))$$

7.11 Adjugate

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T$$

7.12 Adjugate Properties

1. $A \operatorname{adj}(A) = \operatorname{adj}(A)A = \det(A)I_n$
2. $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

7.13 Cramer's Rule

Consider $A\vec{x} = \vec{b}$ We construct B_j by replacing the j-th column of A by the column vector of $\vec{b}$ then $j \in [1, n]$

$$x_j = \frac{\det(B_j)}{\det(A)}$$

7.14 Area of Parallelogram

Let $\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ and $\vec{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ Area is given by

$$\left| \det(A) = \left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \right) \right|$$

8 Week 8

8.1 Eigenvector/value/pair

A **non-zero** vectr $\vec{x}$ is an **eigenvector** of A over $\mathbb{F}$ if there exists a scalar $\lambda \in \mathbb{F}$ s.t.

$$A\vec{x} = \lambda\vec{x}$$

The scalar is then called the **eigenvalue** of A and the pair $(\lambda, \vec{x})$ is an **eigenpair** of A

8.2 Eigenvalue Equation / Eigenvalue Problem

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}$$

8.3 Characteristic Polynomial

$$C_A(\lambda) = \det(A - \lambda I)$$

8.4 Characteristic Equation

$$C_A(\lambda) = 0$$

8.5 Properties of Eigenvalues

A is invertible if and only if $\lambda = 0$ is **NOT** an eigenvalue

8.6 Trace

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

8.7 Features of Characteristic Polynomial

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{(n-1)} + \cdots + c_1\lambda + c_0$$

1. $c_n = (-1)^n$
2. $c_{n-1} = (-1)^{(n-1)}\operatorname{tr}(A)$
3. $c_0 = \det(A)$

8.8 Characteristic Polynomial over Complex Numbers

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{(n-1)} + \cdots + c_1\lambda + c_0$$

and n eigeinvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ **in** $\mathbb{C}$ (possibly repeated). Then,

1. $c_{n-1} = (-1)^{(n-1)} \sum_{i=1}^{n} \lambda_i \to \operatorname{tr}(A)$ (Over $\mathbb{C}$)
2. $c_0 = \Pi_{i=1}^{n}\lambda_i \to \det(A)$

8.9 Linear Combinations of Eigenvector

Suppose $(\lambda_1, \vec{x})$ and $(\lambda_1, \vec{y})$ are eigenpairs of a matrix with the same value λ_1 then if $c\vec{x}+d\vec{y} \neq 0$, $(\lambda_1, c\vec{x}+d\vec{y})$ is also an eigenpair of A

8.10 Eigenspace

$$E_\lambda(A) = \text{Null}(A - \lambda I)$$

8.11 Similar

A is similar to B if there exists an invertible matrix P s.t.

$$A = PBP^{-1}$$

8.12 Properties of Similar

If A is similar to B then,

- A^k is similar to B^k
- $C_A(\lambda) = C_B(\lambda)$
- A and B have the same eigenvalues
- $\text{tr}(A) = \text{tr}(B)$ and $\det(A) = \det(B)$

8.13 Diagonalizable

A is **diagonalizable** over $\mathbb{F}$ if it is similar to D; that is if there exists an invertible matrix P s.t.

$$P^{-1}AP = D$$

8.14 Distinct Eigenvalues and Diagonizable

If $A \in M_{n\times n}(\mathbb{F})$ and has n **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then A is diagonalizable over $\mathbb{F}$
If we let $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2), \ldots, (\lambda_n, \vec{v}_n)$ be eigenpairs of A over $\mathbb{F}$. then,

1. $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$ is invertible
2. $P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Warning: if A is diagonalizable that does not mean it has n distinct eigenvalues

9 Week 9

9.1 Subspace

A subset $V \in \mathbb{F}^n$ is called a subspace if following properties are met

1. $\vec{0} \in V$
2. $\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$
3. $\forall \vec{x} \in V, c \in \mathbb{F}, c\vec{x} \in V$

9.2 Alternate Definiton

V is a subspace if and only if:

- V is not empty
- $\forall \vec{x}, \vec{y} \in V, c \in \mathbb{F}, c\vec{x} + \vec{y} \in V$

$\mathrm{Col}(A), \mathrm{Range}(T), \mathrm{Ker}(T), E_\lambda$ are all subspaces

9.3 Linear Dependence

A set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent if there exists scalar $c_1, c_2, \ldots, c_k \in \mathbb{F}$ not all zeros such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

9.4 Linear Independence

When the above equation is only satisfied when the only solution is the trivial solution where

$$c_1 = c_2 = \ldots c_k = 0$$

9.5 Propositions

1. If $\vec{0} \in S$, S is linearly dependent
2. If $S = \{\vec{x}\}$ containing only one vector, then S is linearly dependent if and only if $\vec{x} = \vec{0}$
3. If $S = \{\vec{x}, \vec{y}\}$ containing only two vectors, S is linearly dependent if and only if one of the vectors is a multiple of the other.

9.6 Linear Dependence Check

- Let $k \geq 2$ The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly dependent if and only if one of the vectors can be written as a linear combination of some other vector
- The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent if and only if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \implies c_1 = \cdots = c_k = 0$$

9.7 Pivots and Linear Independence

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be the set of k vectors in $\mathbb{F}^n$ Let $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k \end{bmatrix}$ be the $n \times k$ matrix whose columns are the vectors in S
Suppose that the $\operatorname{rank}(A) = r$ and has pivots in columns $q_1, q_2, \dots, q_r$
Let set $U = \{\vec{v}_{q_1}, \dots, \vec{v}_{q_r}\}$

1. S is linearly independent if and only if $r = k$
2. U is linearly independent
3. If $\vec{v} \in S \setminus U$ then the set $\{\vec{v}_{q_1}, \dots, \vec{v}_{q_r}, \vec{v}\}$ is linearly dependent.
4. $\operatorname{Span}(U) = \operatorname{Span}(S)$

9.8 Bound on Number of Linearly Independent Vectors

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be the set of k vectors in $\mathbb{F}^n$. If $n < k$, then S is linearly dependent.
Basically: $\operatorname{rank}(A) \leq n < k$

10 Week 10

10.1 Basis

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a finite set of vectors contained in V. We say $\mathcal{B}$ is a **basis** for V if

- $\mathcal{B}$ is linearly independent
- $V = \text{Span}(\mathcal{B})$

e.g. Standard Basis for $\mathbb{F}^n$, is linearly independent and a spanning set of $\mathbb{F}^n$

10.2 Every Subspace Has a Spanning Set

Let V be a subspace of $\mathbb{F}^n$. Then there exists vector $\vec{v}_1, \ldots, \vec{v}_k \in V$ such that

$$V = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$$

10.3 Every Subspace Has a Basis

Let V be a subspace of $\mathbb{F}^n$. Then V has a basis.

10.4 Span of Subset

Let V be a subspace of $\mathbb{F}^n$ and let $S = \{\vec{v}_1, \ldots \vec{v}_k\} \subseteq V$ Then

$$\text{Span}\, S \subseteq V$$

10.5 Span and Rank

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of k vectors in $\mathbb{F}^n$ and let $A = \begin{bmatrix}\vec{v}_1 \ldots \vec{v}_k\end{bmatrix}$ be the matrix whose columns are the vectors in S Then

$$\text{Span}\, S = \mathbb{F}^n \iff \text{rank}(A) = n$$

10.6 Size of Basis for $\mathbb{F}^n$

Must have exactly n vectors and is linearly independent and spanning.

10.7 Span and linear dependency

Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set of n vectors in $\mathbb{F}^n$.

$$S \text{ is linearly independent} \iff \text{Span}\, S = \mathbb{F}^n$$

10.8 Basis for $\text{Col}(A)$

Let $A = \begin{bmatrix}\vec{a}_1 \dots \vec{a}_n\end{bmatrix} \in M_{m\times n}(\mathbb{F})$ and suppose RREF(A) has pivots in columns $q_1, \dots, q_r$ where $r = \text{rank}(A)$. Then $\{\vec{a}_{q_1}, \dots, \vec{a}_{q_r}\}$ is a basis for Col(A)
e.g.

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ -1 & -1 & -1 & -4 & 2 \end{bmatrix}, RREF(A) = \begin{bmatrix} \mathbf{1} & 0 & 2 & 3 & 0 \\ 0 & \mathbf{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col(A) $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

10.9 Basis for $\text{Null}(A)$

1. Consider equation $A\vec{x} = \vec{0}$
2. Apply Gauss-Jordan and obtain k free parameters so the solution set is given by
$$\text{Null}(A) = \{t_1\vec{x}_1 + \cdots + t_k\vec{x}_k : t_1, \dots, t_k \in \mathbb{F}\}$$
3. The number of parameters obtained by nullity(A) $= n - \text{rank}(A)$
4. Let vectors $\vec{x}_i$ for $1 \le i \le k$ obtained from below
5. $\{\vec{x}_1, \dots, \vec{x}_k\}$ is a basis for Null(A)

Example: Find a basis for nullspace of $A = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ -1 & -1 & -1 & -4 & 2 \end{bmatrix}$

$RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Let $x_3 = s, x_4 = t$

$$\vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, s, t \in \mathbb{F}$$

So Basis $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

10.10 Dimensions Well Defined

Let V be a subspace of $\mathbb{F}^n$. If $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}, \mathcal{C} = \{\vec{w}_1, \ldots, \vec{w}_\ell\}$ are basis for V, then $k = \ell$.
All bases for a given subspace have the same number of vectors.

10.11 Dimensions

The number of elements in a basis for a subspace V of $\mathbb{F}^n$ is called the **dimension** of V denoted by $\dim(V)$

10.12 Bound on Dimensions of Subspace

Let V be a subspace of $\mathbb{F}^n$ then $\dim(V) \leq n$

10.13 Some Properties

If V and W are subspaces of $\mathbb{F}^n$ s.t. $W \subseteq V$

1. $\dim(W) \leq \dim(V)$
2. $\dim(W) = \dim(V) \iff W = V$
3. $\operatorname{rank}(A) = \dim(\operatorname{Col}(A))$
4. $\operatorname{nullity}(A) = \dim(\operatorname{Null}(A))$

10.14 Rank Nullity Theorem

Let $A \in M_{m\times n}(\mathbb{F})$ Then

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A) = \dim(\operatorname{Col}(A)) + \dim(\operatorname{Null}(A))$$

10.15 Unique Representation Theorem

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{F}^n$. Then for every vector $\vec{v} \in \mathbb{F}^n$ there exist unique scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

10.16 Coordinates with Respect to $\mathcal{B}$

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{F}^n$. Let the vector $\vec{v} \in \mathbb{F}^n$ have representation

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \sum_{i=1}^{n} c_i\vec{v}_i$$

We call the scalars $c_1, c_2, \ldots, c_n$ the coordinates or components of $\vec{v}$ with respect to $\mathcal{B}$ or the $\mathcal{B}$ coordinates of $\vec{v}$

10.17 Ordered Basis

An ordered basis for $\mathbb{F}^n$ is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for $\mathbb{F}^n$ together with fixed ordering.
Note: We consider sets $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_2, \vec{v}_1\}$ as different in terms of order though they are the same basis.
Standard Basis: $\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\} \impliedby$ Standard Ordering

10.18 Coordinate Vector

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an ordered basis for $\mathbb{F}^n$. Let $\vec{v} \in \mathbb{F}^n$ have the coordinates $c_1, \dots, c_n$ with respect to $\mathcal{B}$, where the ordering of the scalars c_i matches the ordering in $\mathcal{B}$, that is,

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i$$

Then, the **coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$** is the column vector in $\mathbb{F}^n$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Kinda like standard basis notation

10.19 Linearity of Taking Coordinates

Let $\mathcal{B}$ an ordered basis. The function $[?]_{\mathcal{B}} : \mathbb{F}^n \to \mathbb{F}^n$ defined by sending $\vec{v}$ to $[\vec{v}]_{\mathcal{B}}$ is linear: $\forall \vec{u}, \vec{v} \in \mathbb{F}^n$

1. $[\vec{u} + \vec{v}]_{\mathcal{B}} = [\vec{u}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}}$
2. $[c\vec{v}]_{\mathcal{B}} = c[\vec{v}]_{\mathcal{B}}$

10.20 Change of Basis Matrix

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be ordered bases for $\mathbb{F}^n$.
The **change of basis** or change of coordinate matrix from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates in $n \times n$ matrix

$${}_{\mathcal{C}}[I]_{\mathcal{B}} = [[\vec{v}_1]_{\mathcal{C}} \dots [\vec{v}_n]_{\mathcal{C}}]$$

whose columns are the $\mathcal{C}$ coordinates of the vectors $\vec{v}_i$ in $\mathcal{B}$. Similarly, the change of basis matrix from $\mathcal{C} \to \mathcal{B}$

$${}_{\mathcal{B}}[I]_{\mathcal{C}} = [[\vec{w}_1]_{\mathcal{B}} \dots [\vec{w}_n]_{\mathcal{B}}]$$

10.21 Changing Basis

Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$ be ordered bases for $\mathbb{F}^n$.
Then $[\vec{x}]_\mathcal{C} = {}_\mathcal{C}[I]_\mathcal{B}[\vec{x}]_\mathcal{B}$ and $[\vec{x}]_\mathcal{B} = {}_\mathcal{B}[I]_\mathcal{C}[\vec{x}]_\mathcal{C}$ for all $\vec{x} \in \mathbb{F}^n$
Special case:
when $\mathcal{B} = \mathcal{E}, [\vec{x}]_\mathcal{C} = {}_\mathcal{C}[I]_\mathcal{E}\vec{x}$
when $\mathcal{C} = \mathcal{E}, \vec{x} = {}_\mathcal{E}[I]_\mathcal{B}[\vec{x}]_\mathcal{B}, I = [\vec{v}_1 \dots \vec{v}_n]$

10.22 Inverse of Change of Basis Matrix

Let $\mathcal{B}$ and $\mathcal{C}$ be two ordered bases of $\mathbb{F}^n$ then

$${}_\mathcal{B}[I]_\mathcal{C} \cdot {}_\mathcal{C}[I]_\mathcal{B} = I_n$$

$${}_\mathcal{C}[I]_\mathcal{B} \cdot {}_\mathcal{B}[I]_\mathcal{C} = I_n$$

So ${}_\mathcal{C}[I]_\mathcal{B} = ({}_\mathcal{B}[I]_\mathcal{C})^{-1}$

11 Week 11

11.1 Linear Operator

A transformation T where the domain and the codomain are the same set. $T : \mathbb{F}^n \to \mathbb{F}^n$ is a linear transformation.
Note: it will always be square

11.2 $\mathcal{B}$-Matrix of T

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ be an ordered basis for $\mathbb{F}^n$ We define $\mathcal{B}$ matrix of T

$$[T]_\mathcal{B} = [[T(\vec{v}_1)]_\mathcal{B} \ldots [T(\vec{v}_n)]_\mathcal{B}]$$

Apply the action of T to each member of $\mathcal{B}$
Note: usually choose $\mathcal{B}$ that is simpler (diagonal)

11.3 Proposition

$$[T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B}$$

11.4 Similarity of Matrix Representation

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ and $\mathcal{B}, \mathcal{C}$ be ordered basis for $\mathbb{F}^n$

$$[T]_\mathcal{C} = {}_\mathcal{C}[I]_\mathcal{B}[T]_{\mathcal{B}\mathcal{B}}[I]_\mathcal{C} = ({}_\mathcal{B}[I]_\mathcal{C})^{-1}[T]_{\mathcal{B}\mathcal{B}}[I]_\mathcal{C}$$

11.5 Eigenstuff with Linear Operator

$$T(\vec{x}) = \lambda\vec{x}$$

$\vec{x}$ is an eigenvector, λ is an eigenvalue, $(\lambda, \vec{x})$ is an eigenpair.
It is an eigenpair of T if and only if it is an eigenpair of $[T]_\mathcal{B}$

11.6 Diagonalizable

We say T is diagonalizable if there exists an ordered basis $\mathcal{B}$ such that $[T]_\mathcal{B}$ is diagonalizable.

11.7 Eigenvector basis criterion for diagonalizability

T is diagonalizable if and only if there exists an order basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ consisting the eigenvectors of T

11.8 Eigen Basis criteria for Diagonalizability

A is diagonizable over $\mathbb{F}$ if and only if there exists a basis in $\mathbb{F}$ consisting of the eigeinvectors of A.

12 Week 12

12.1 Eigenvectors corresponding to Distinct Eigenvalues are linearly independent

Let A have eigenpairs $(\lambda_1, \vec{v}_1), \dots, (\lambda_n, \vec{v}_n)$
If the eigenvalues $\lambda_1, \dots, \lambda_n$ are all distinct, the set of eigenvectors are linearly independent.

12.2 Corollary

More specifically, if we let $P = [\vec{v}_1 \dots \vec{v}_n]$

- P is invertible
- $P^{-1}AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$

12.3 Algebraic Multiplicity

$$a_{\lambda_i} = \text{how many times}(\lambda - \lambda_i) \text{ appears}$$

12.4 Geometric Multiplicity

$$g_{\lambda_i} = \dim(E_{\lambda_i})$$

12.5 Geometric and Algebraic Multiplicity

$$1 \leq g_{\lambda_i} \leq a_{\lambda_i}$$

12.6 Proposition

If the corresponding eigenspaces $E_{\lambda_1}, \dots, E_{\lambda_k}$ have basis $\mathcal{B}_1, \dots, \mathcal{B}_k$, then $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$

12.7 Diagonizability Test

Let $\lambda_1, \dots, \lambda_k$ are all distinct eigenvectors of A with algebraic multiplicities

$$a_{\lambda_1}, \dots, a_{\lambda_k}$$

then $h(\lambda)$ is a polynomial in λ that is irreducable over $\mathbb{F}$. Then A is diagonizable if and only if $h(\lambda)$ is a constant polynomial and $a_{\lambda_i} = g_{\lambda_i}$ for each $1, \dots, k$