

JOHNSON  
Cornell University

# NBA 5420: Investment and Portfolio Management

## Class 2: Passive Investing

Professor Matt Baron  
February 1, 2016





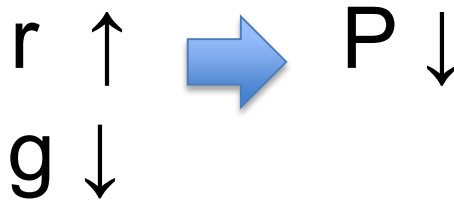
# Class announcements

- New TA
  - Sam Hempel: Thu. 12-1 PM (usually Sage 134)
  - Kate Volkova: Wed. 3-4 PM (Sage 134)
- Problem set due Friday at 4 PM
  - Turn in: box in Sage 304
- Readings for Wed
  - WSJ: “Is there a case for actively managed funds?”
  - Buffett II.A,B,F; III.A; IV.A



# The Method in the Market's Current Madness

$$\begin{aligned} Price &= \sum_{t=1}^{\infty} \frac{(Cash\ Flow)_t}{(1+r)^t} \\ &= \frac{(Cash\ Flow)_{t=0}}{r-g} \end{aligned}$$





# S&P 500 index





# Today's class

1. Discounting & Net Present Value
2. Risk and Return

## Wednesday

1. Portfolio theory



# Why does paper have value?

- Stocks
- Bonds
- Currencies
- Derivatives (options, futures, swaps, etc)
- All assets are priced using the same basic idea:  
**price = net present value (NPV) of future cash flow (profits)**



**PRICE = NET PRESENT VALUE (NPV)**



# Compounding

- Annualized:

$$FV_T = C_0(1 + r)^T$$

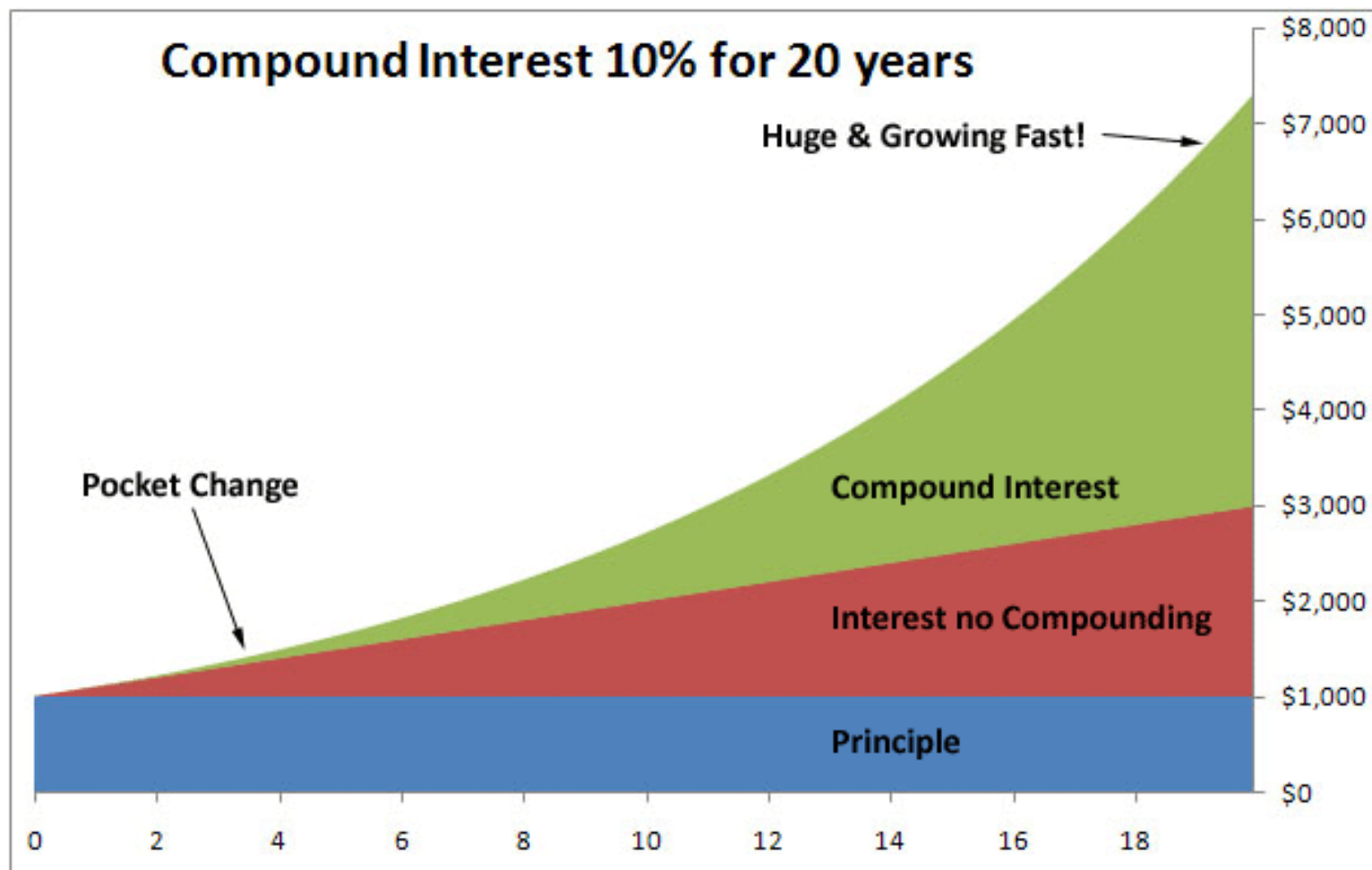
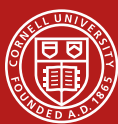
- Compounding frequency (n times a year):

$$FV_T = C_0\left(1 + \frac{r}{n}\right)^{nT}$$

- Continuous compounding:

$$FV_T = C_0e^{rT}$$







# Compounding

- Often, it's easier to work with log returns in the data:

$$FV_T = C_0(1 + r_1)(1 + r_2) \dots (1 + r_T)$$

$$\log FV_T = \log C_0 + \log(1 + r_1) + \dots + \log(1 + r_T)$$



# Discounting

## – General Principle

- \$1 in the future is worth less than \$1 today
- Because, if you get that \$1 in the future, you forgo its use now and can't earn interest on it

## – Present value of \$1 tomorrow = $\frac{1}{(1+r)}$ today

- Because with  $\frac{1}{(1+r)}$  today, you'd invest it to get  $\frac{1}{(1+r)} * (1 + r) = \$1$  tomorrow



# Discounting

– In general:

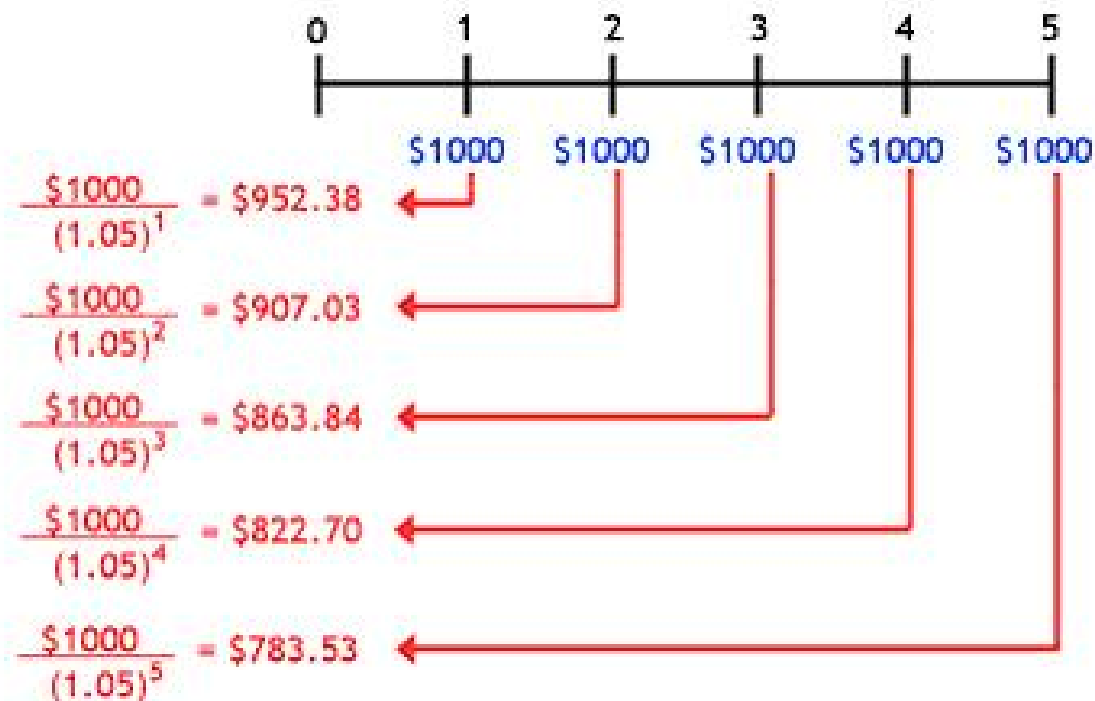
- $X$  dollars in  $T$  days =  $\frac{X}{(1+r)^T}$  dollars today

– Net present value:

- $Price = NPV = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \dots + \frac{C_T}{(1+r)^T}$



# Discounting



Present Value of an  
Ordinary Annuity = \$4329.48



# Discounting

- Infinite cash flow stream:

$$Price = NPV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \dots$$

$$= C \sum_{t=1}^{\infty} \left[ \frac{1}{(1+r)} \right]^t = C \left[ \frac{1}{r} \right]$$

$$= \frac{C}{r}$$



# Infinite summations

$$\sum_{t=0}^{\infty} x^t = \left( \frac{1}{1-x} \right)$$

$$\sum_{t=1}^{\infty} x^t = x \sum_{t=0}^{\infty} x^t = x \left( \frac{1}{1-x} \right)$$

Therefore, letting  $x = 1/(1+r)$ :

$$\sum_{t=1}^{\infty} \left[ \frac{1}{(1+r)} \right]^t = \frac{1}{(1+r)} \left( \frac{1}{1 - \frac{1}{(1+r)}} \right) = \frac{1}{r}$$



# RISK AND RETURN





# Probability

## – Discrete random variable

I states of the world ( $i=1,2,3,\dots I$ )

$x_i$  = payoff in state  $i$

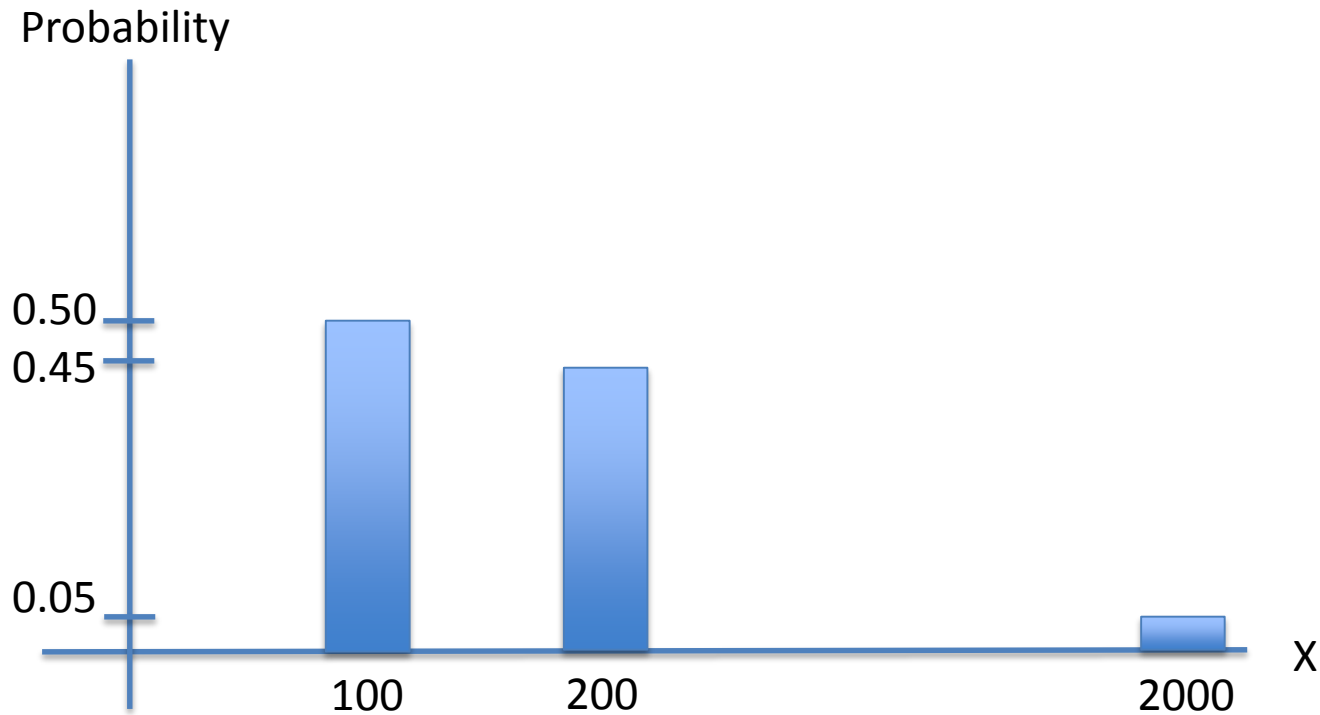
$p_i$  = probability of state  $i$  occurring

Example: Apple stock

- $i = 1$ , no new technology:  $p_1 = 0.50$ ,  $x_1 = 100$
- $i = 2$ , improvement in current tech:  $p_2 = 0.45$ ,  $x_2 = 200$
- $i = 3$ , big technological leap:  $p_3 = 0.05$ ,  $x_3 = 2000$



# Discrete Probability Distribution





# Risk and return

Example: Apple stock

- $i=1$ , no new technology:  $p_1 = 0.50, x_1 = 100$
- $i=2$ , improvement in current tech:  $p_2 = 0.45, x_2 = 200$
- $i=3$ , big technological leap:  $p_3 = 0.05, x_3 = 2000$

Expected value:

$$\mu = E[x] = \sum_{i=1}^I p_i x_i$$

$$\mu = .5 * 100 + .45 * 200 + .05 * 2000 = 240$$

Variance:

$$\sigma^2 = Var(x) = E[(x - \mu)^2] = \sum_{i=1}^I p_i (x_i - \mu)^2$$

$$\sigma^2 = .5(-140)^2 + .45 * (-40)^2 + .05(1760)^2 = 165,400$$



# Risk and return

Useful formula for Variance:

$$\begin{aligned}\sigma^2 &= E[(x - \mu)^2] = E[x^2] - \mu^2 \\ &= \left( \sum_{i=1}^I p_i x_i^2 \right) - \mu^2\end{aligned}$$

Standard deviation (volatility):

$$\sigma = \sqrt{\text{Variance}} = 406.7$$

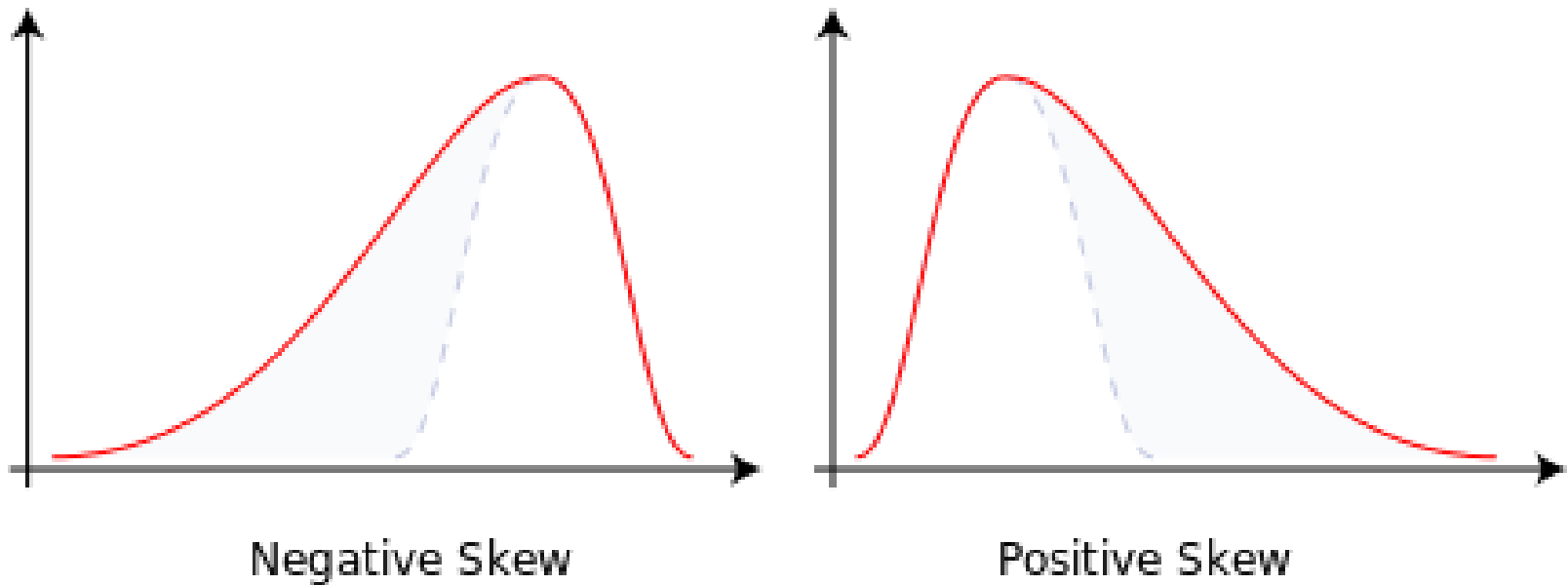
Higher moments:

$$\text{Skewness} = \frac{1}{\sigma^3} E[(x - \mu)^3]$$

$$\text{Kurtosis} = \frac{1}{\sigma^4} E[(x - \mu)^4]$$

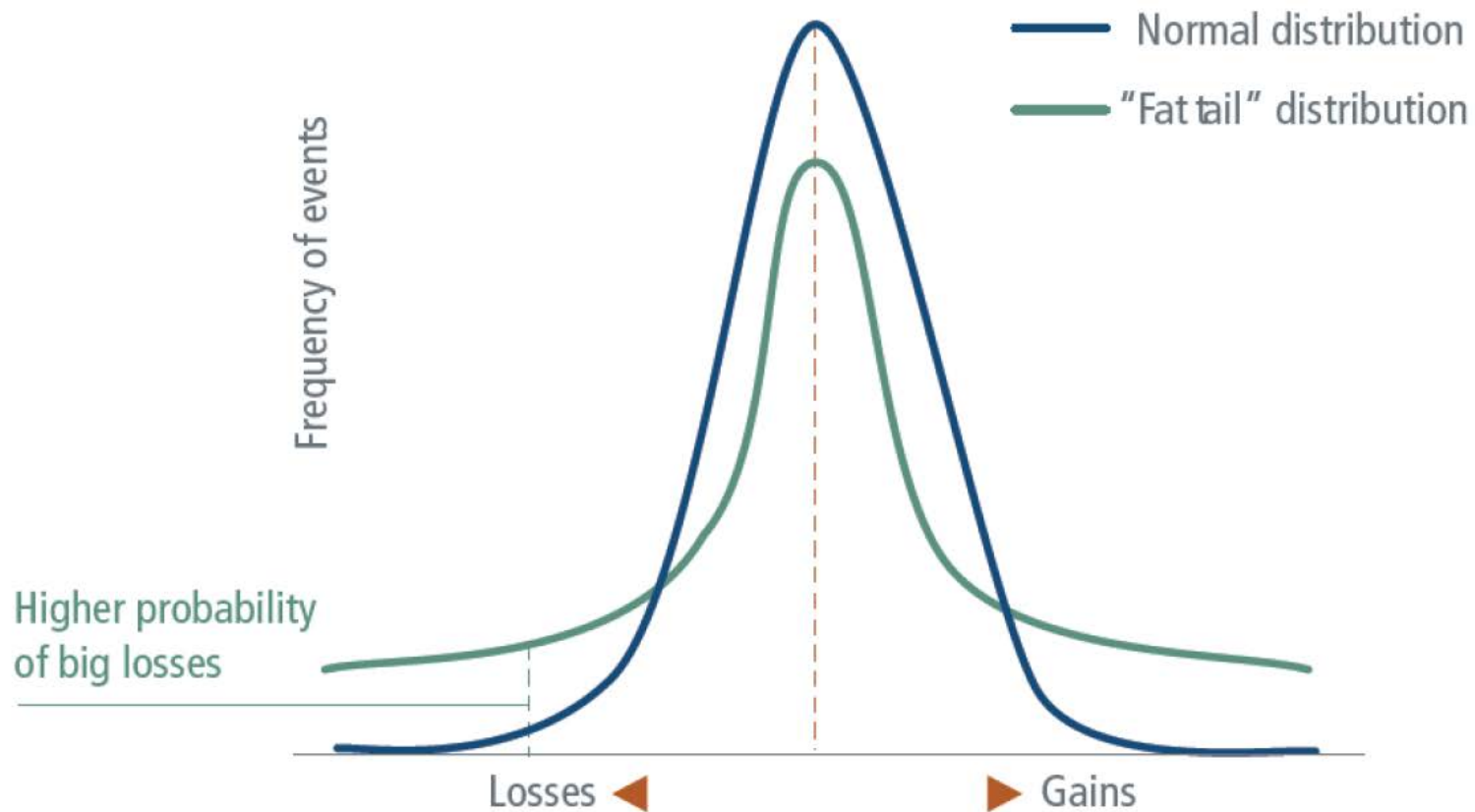


# Skewness





# Kurtosis





# Correlation

Two assets: Apple stock ( $x$ ), Netflix stock ( $y$ )

- $i=1$ , no new technology:  $p_1 = 0.50$ ,  $x_1 = 100$ ,  $y_1 = 50$
- $i=2$ , improvement in current tech:  $p_2 = 0.45$ ,  $x_2 = 200$ ,  $y_2 = 150$
- $i=3$ , big technological leap:  $p_3 = 0.05$ ,  $x_3 = 2000$ ,  $y_3 = 2000$

Covariance:

$$\begin{aligned}\sigma_{XY} &= Cov(x, y) = E[(x - \mu_X)(y - \mu_Y)] \\ &= \sum_{i=1}^I p_i (x_i - \mu_X)(y_i - \mu_Y) \\ &= E[xy] - E[x]E[y] \\ &= \sum_{i=1}^I p_i x_i y_i - \left(\sum_{i=1}^I p_i x_i\right)\left(\sum_{i=1}^I p_i y_i\right)\end{aligned}$$

Correlation coefficient:

$$\rho = corr(x, y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

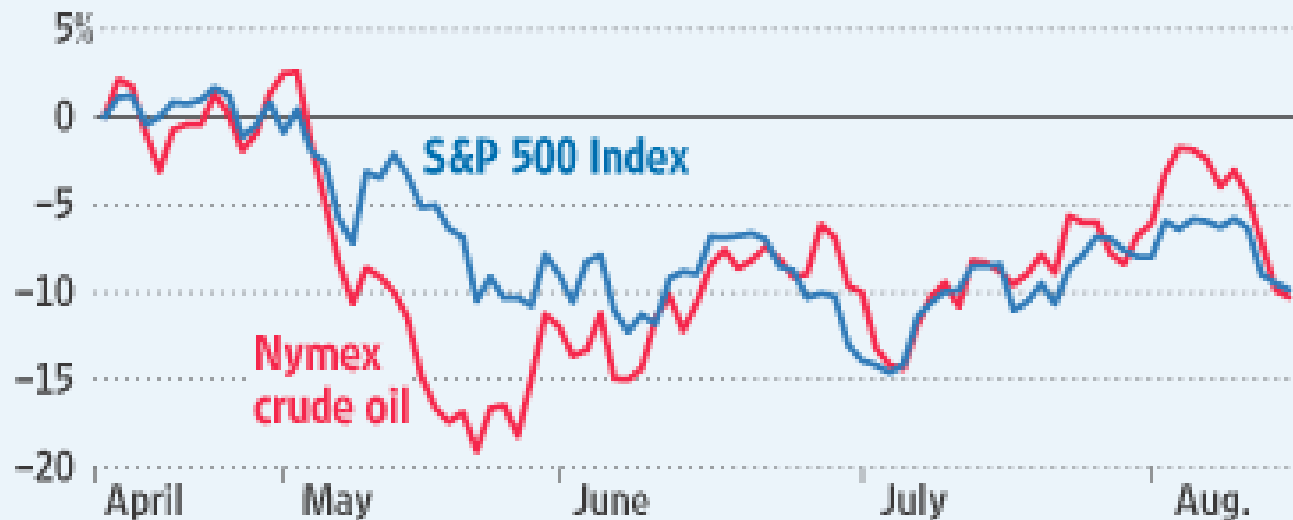
(satisfies:  $-1 \leq \rho \leq 1$ )



# Positive correlation

## A Walk Together on the Wild Side

Prices of stocks and oil have become increasingly correlated.

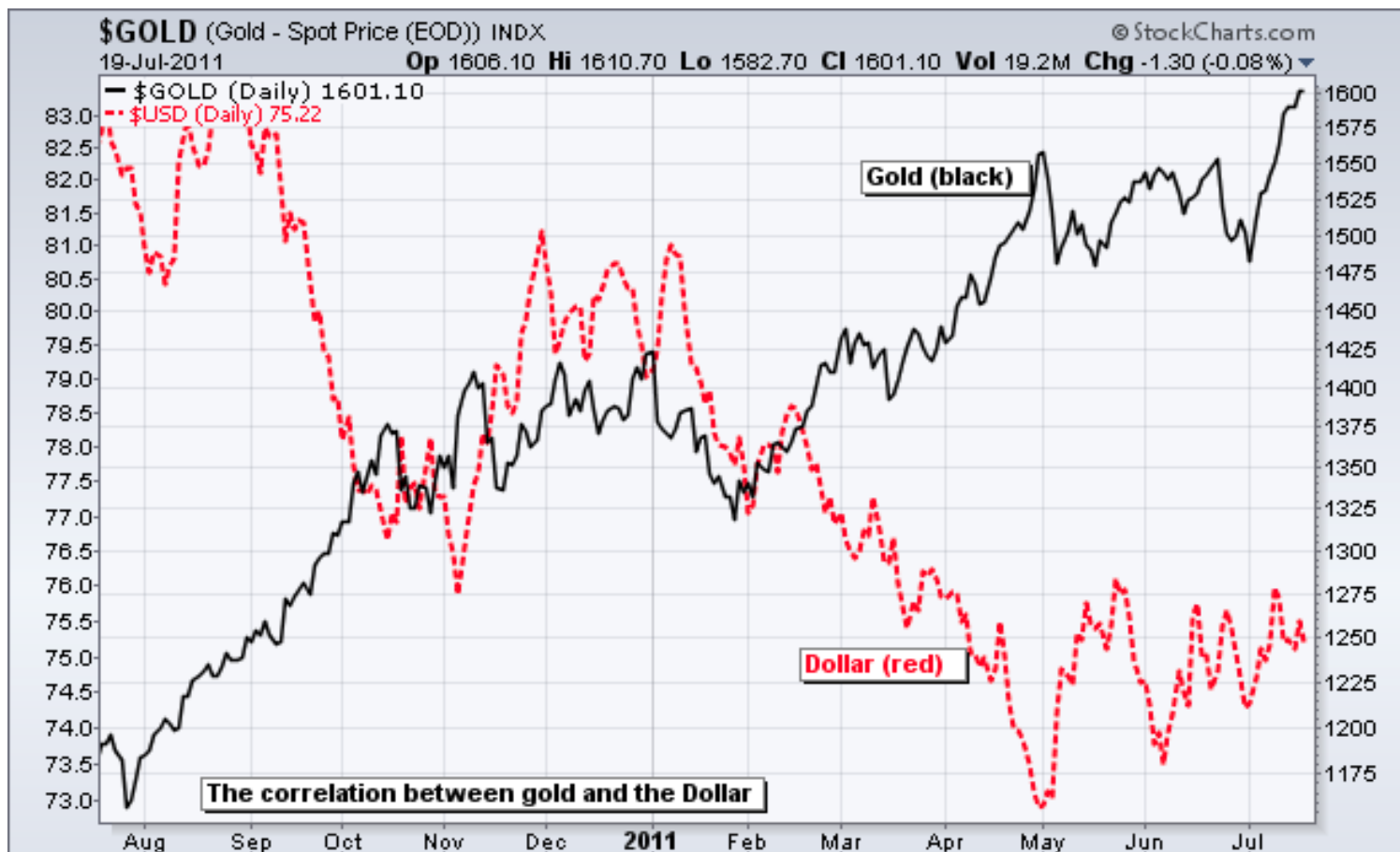


Source: WSJ Market Data Group





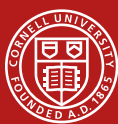
# Negative correlation





# Correlations of Asset Classes

	Dow Jones	S&P 500	Nasdaq	Russell 2000	DJ REIT	Crude Oil	DJ Commodities	MSCI EM	Gold
Dow Jones	100%								
S&P 500	98%	100%							
Nasdaq	89%	92%	100%						
Russell 2000	90%	94%	90%	100%					
DJ REIT	75%	78%	73%	84%	100%				
Crude Oil	38%	43%	38%	41%	31%	100%			
DJ Commodities	44%	49%	43%	47%	35%	94%	100%		
MSCI EM	84%	87%	83%	84%	71%	48%	55%	100%	
Gold	3%	5%	2%	9%	7%	35%	41%	24%	100%



# Correlations

- Stock prices and their volatility tend to be negatively correlated
  - Empirical fact known as the “leverage effect”

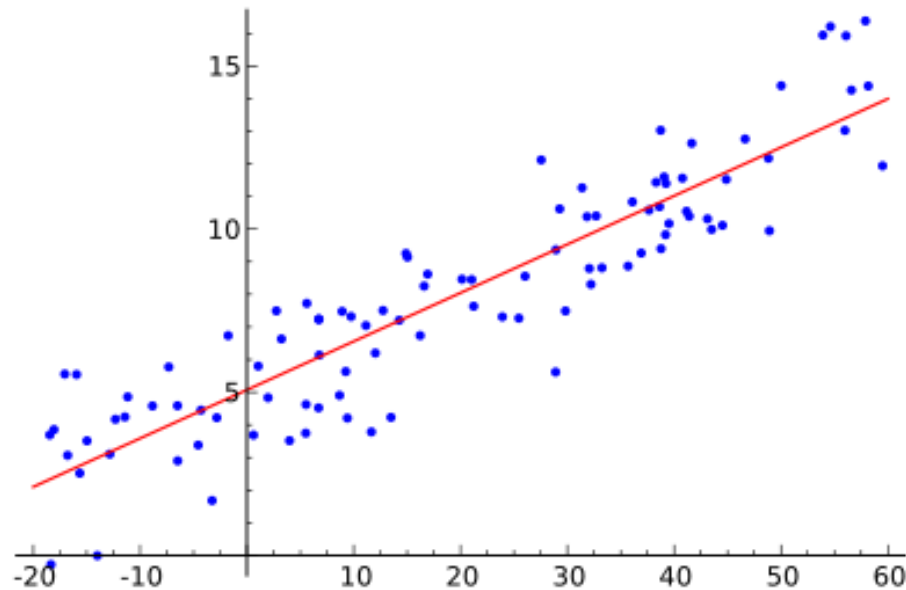
## Smoother Ride

Share prices are moving less wildly as investors grow more confident.





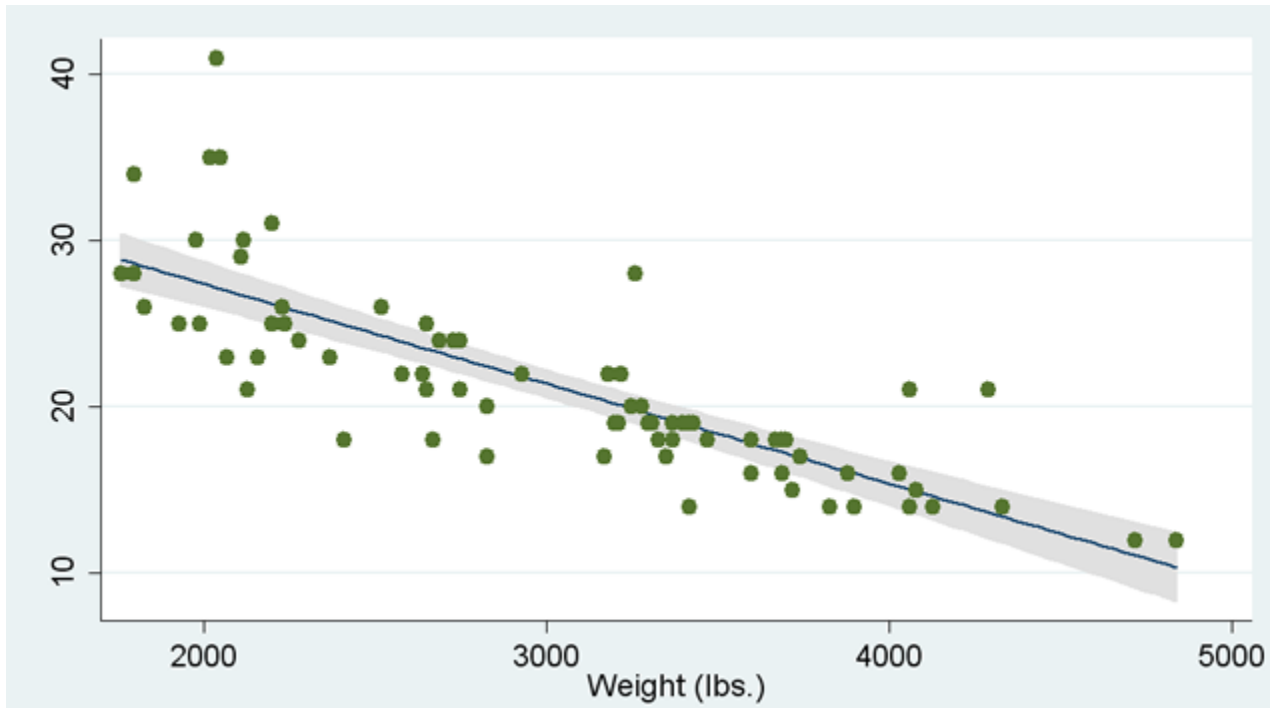
# Regression analysis



$$y_i = \alpha + \beta x_i + \epsilon_i$$



# Estimating $\alpha$ and $\beta$



Estimated values of  $\alpha$  and  $\beta$  have standard errors



# How to read a regression table

Table 1: Regression table

	(1) Price	(2) Price
Weight (lbs.)	1.747** (2.72)	3.465*** (5.49)
Mileage (mpg)	-49.51 (-0.57)	21.85 (0.29)
Car type		3673.1 *** (5.37)
Constant	1946.1 (0.54)	-5853.7 (-1.73)
Observations	74	74

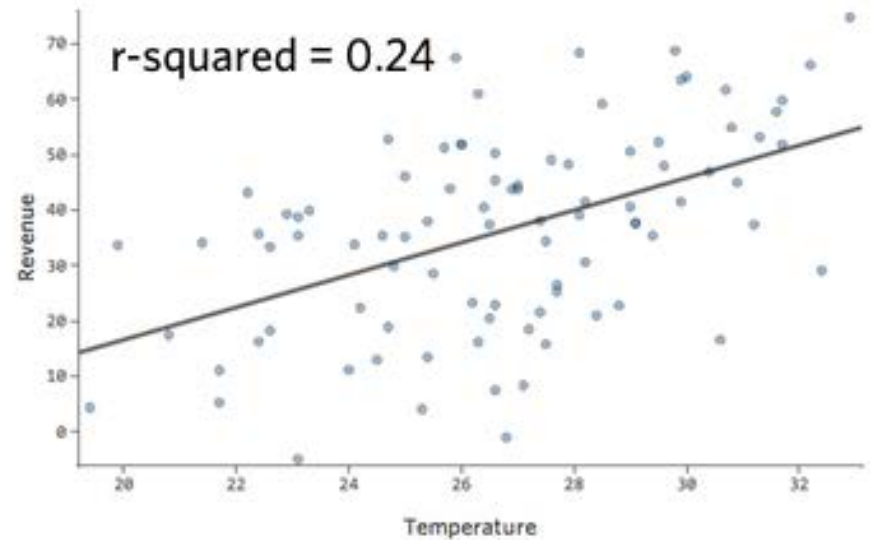
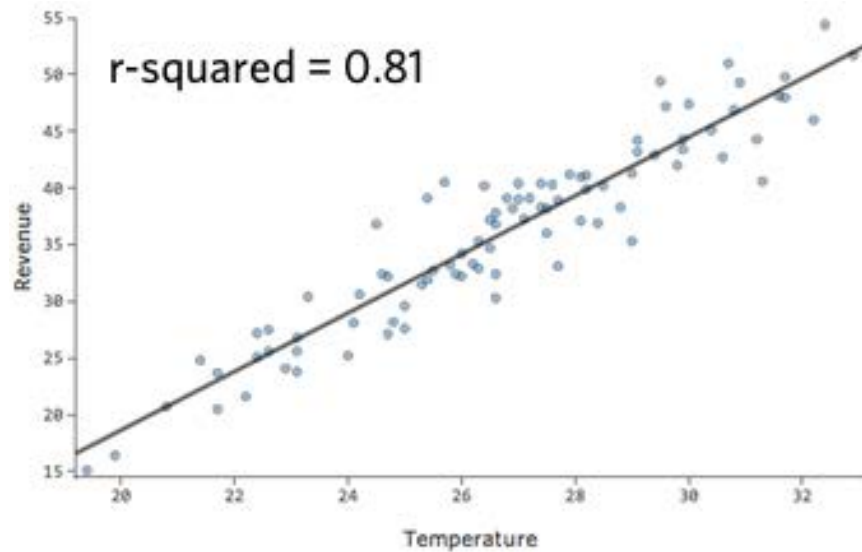
*t* statistics in parentheses

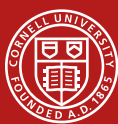
\*  $p < 0.05$ , \*\*  $p < 0.01$ , \*\*\*  $p < 0.001$

$$price = \alpha + \beta_1 Weight + \beta_2 Mileage + \beta_3 CarType + \epsilon_i$$



# $R^2$ = goodness of fit





# Properties of Moments

## – Useful formulas

$$E[aX + b] = aE[X] + b$$

$$E[aX + bY] = aE[X] + bE[Y]$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X]$$

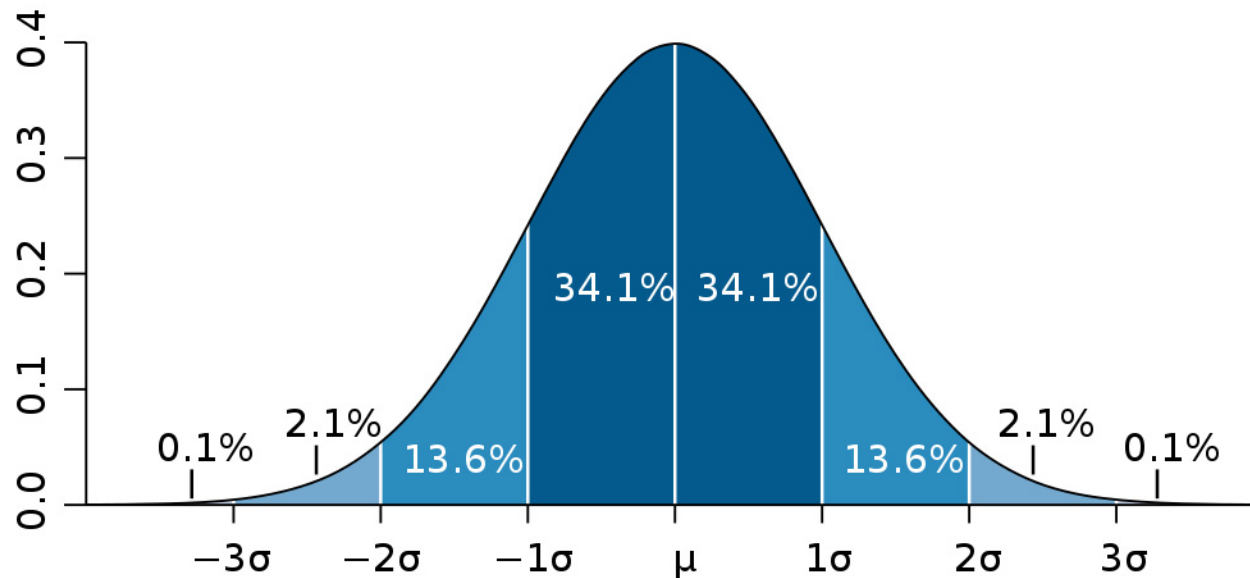
$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] + 2ab \text{Cov}[X, Y]$$

[X,Y random variables; a,b constants]





# Normal or Gaussian Distribution



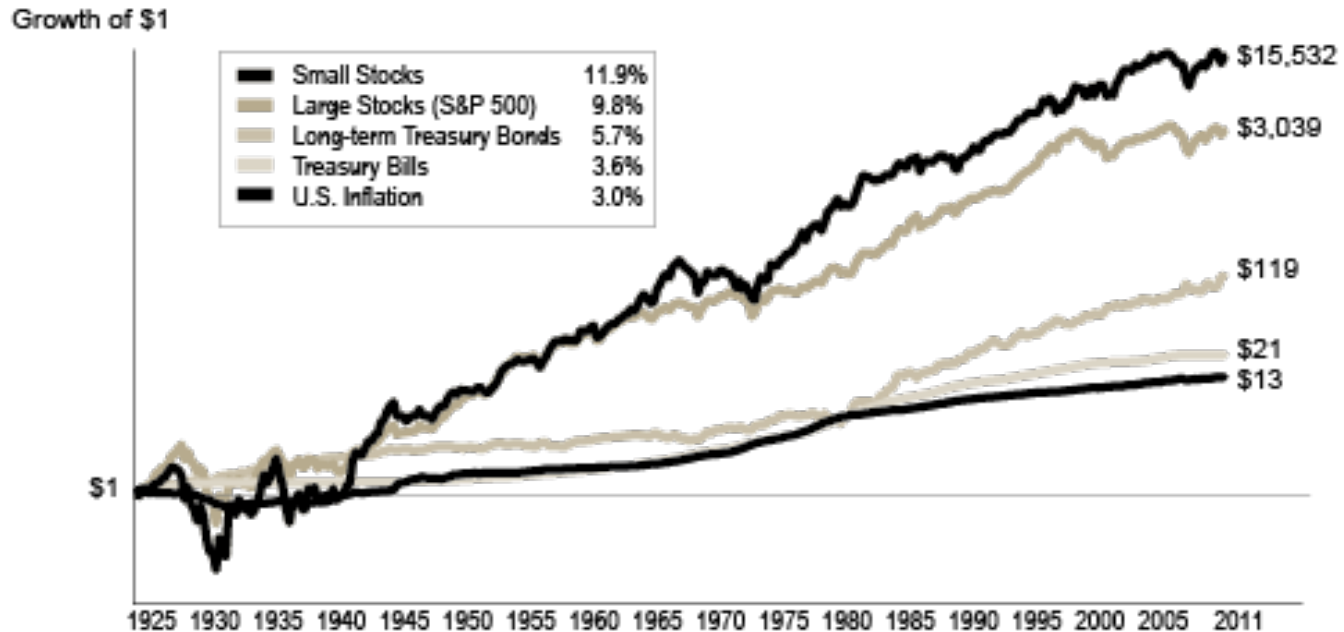
We denote normally distributed variable  $X$ :

$$X \sim N(\mu, \sigma^2)$$

$$E[X] = \mu, \quad Var[X] = \sigma^2, \quad \text{Skew.}=0, \quad \text{Kurt.}=3$$



# Investing for the long-run

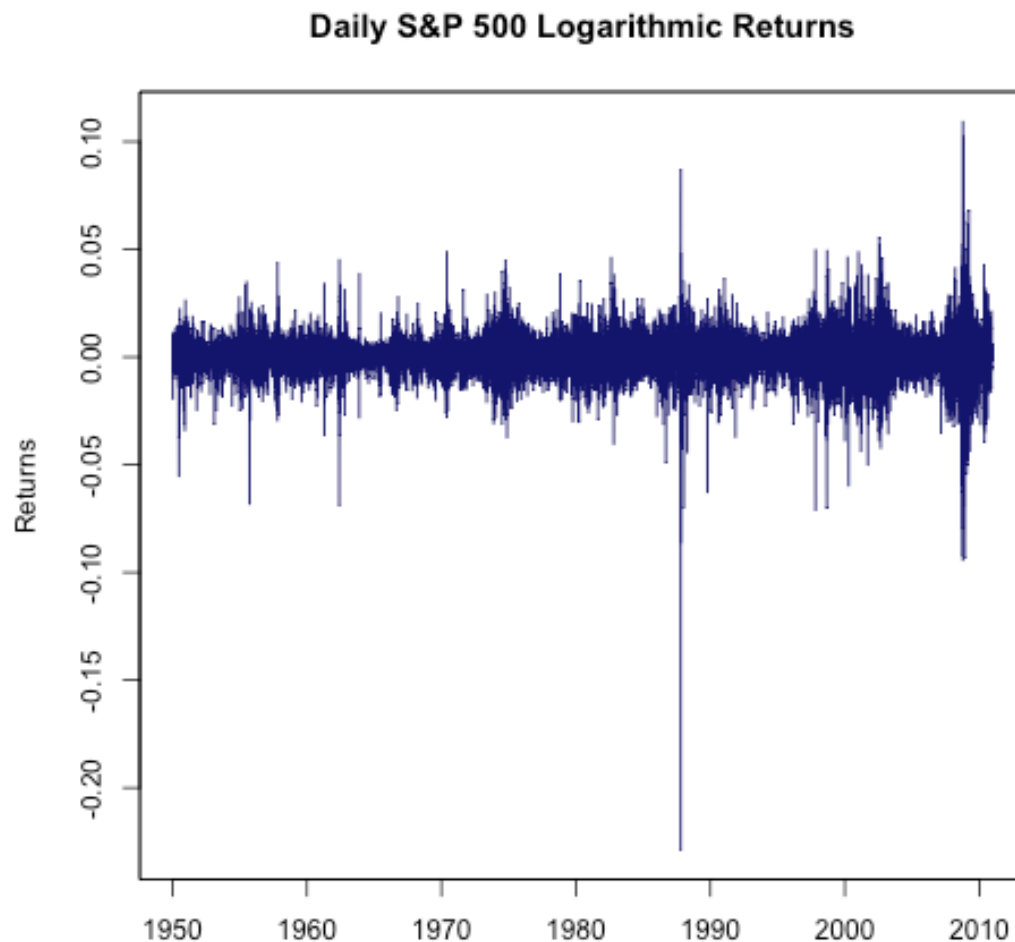


## U.S. long-run returns (last 140 years):

- 7% real returns for stocks (st. dev. = 15%)
- 2% real returns for bonds (st. dev. = 3.5%)



# Normal distribution

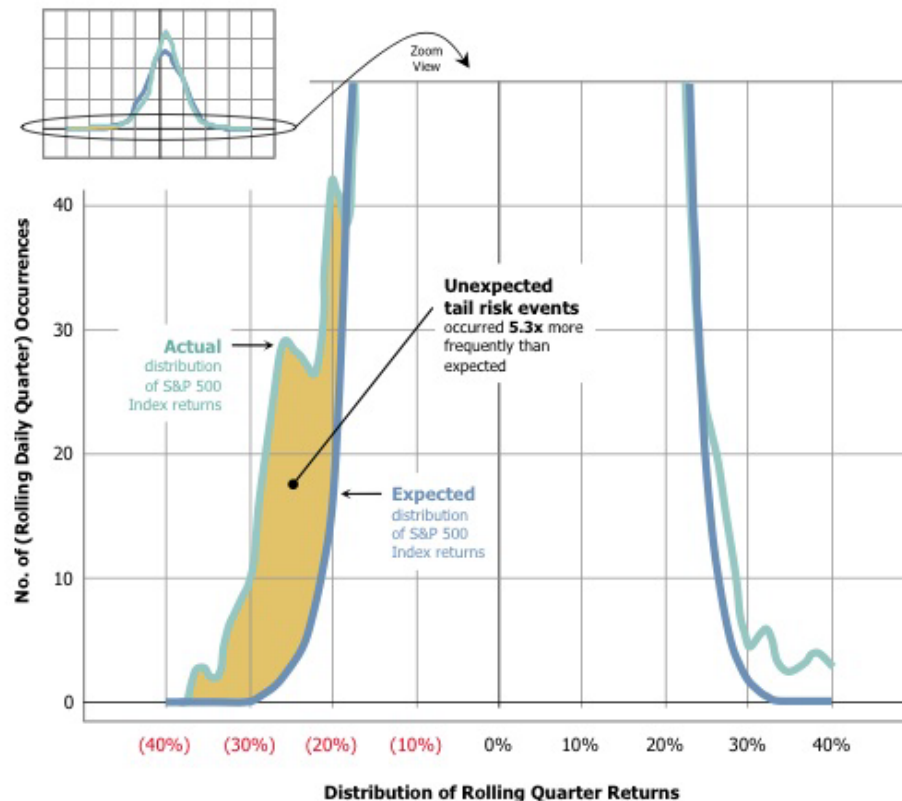




# Normal distribution

- Can be a poor representation of asset returns:  
does not have fat tails

Unexpected Equity Tail Risk Based on Actual vs. Expected  
S&P 500 Index Returns Over the Last 50 Years (1960 to June 2010)





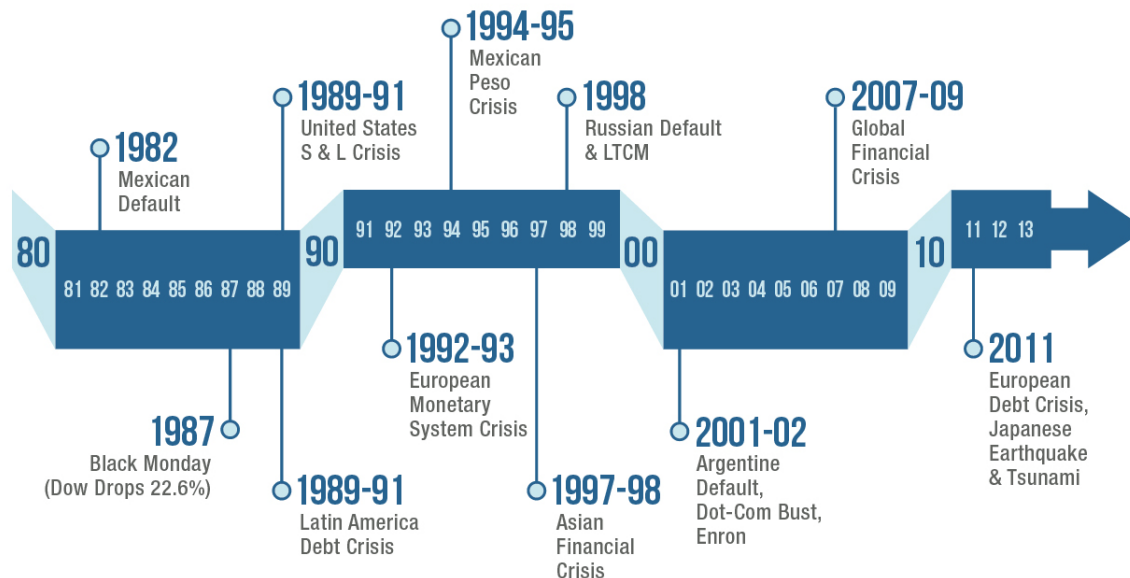
# Tail events

- Under a Normal distribution with historical values for  $\mu$  and  $\sigma$ :
  - Daily returns greater than 7 (in abs. value) should happen once every 300,000 years
  - But there were 48 such days in the 20<sup>th</sup> century!
  - And most of those are left tail events!



# Underlying Causes

- Left tail events mainly driven by mechanisms related to the macroeconomy and banking sector.
  - Cannot be treated from a purely statistical perspective.
  - Must understand the underlying economic ties between the macroeconomy, banking sector, and financial markets





# PORTFOLIO THEORY



# Discounting

## – Net present value:

- $Price = NPV = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \dots + \frac{C_T}{(1+r)^T}$

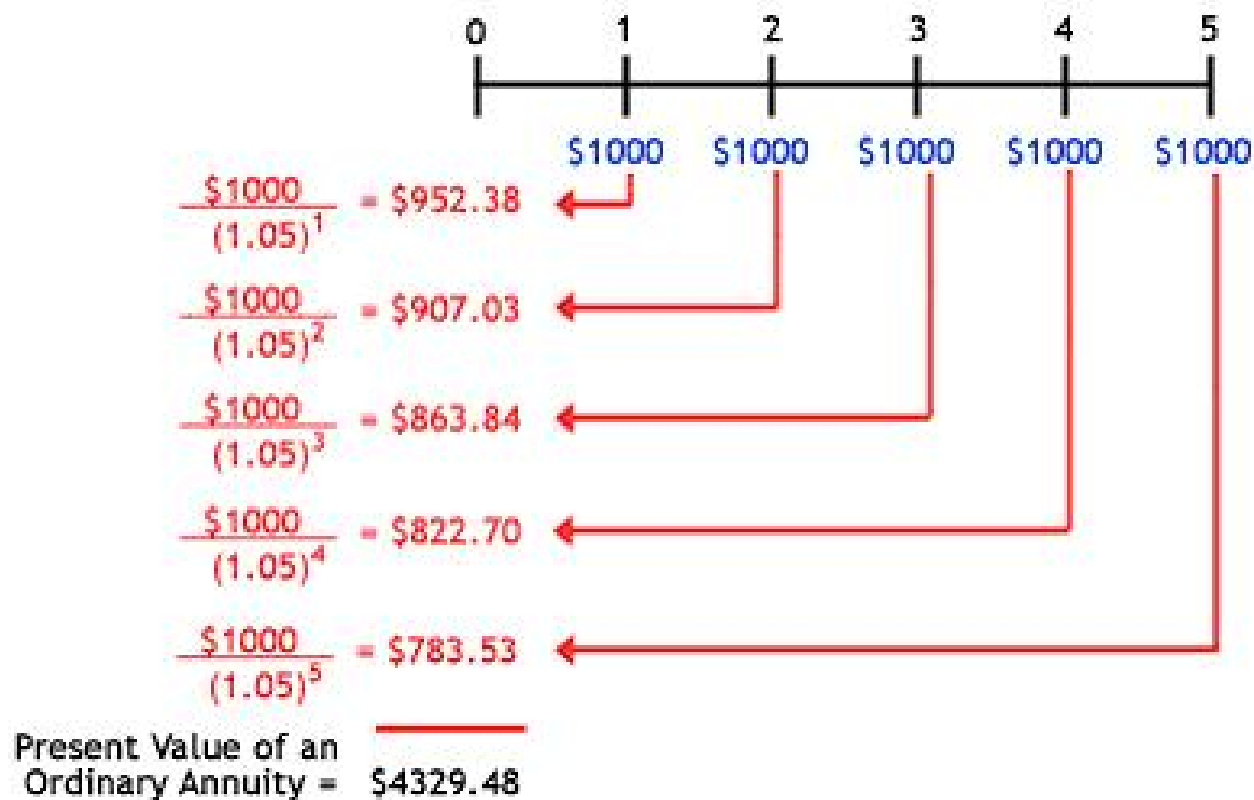
## – Internal Rate of Return (IRR)

- IRR is the interest rate  $r^*$  that makes  $NPV=0$
- In Excel, guess and check:
  - If  $NPV > 0$ , then try a higher  $r$
  - If  $NPV < 0$ , then try a lower  $r$





# Discounting





# Properties of Moments

## – Useful formulas

$$E[aX + b] = aE[X] + b$$

$$E[aX + bY] = aE[X] + bE[Y]$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X]$$

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] + 2ab \text{Cov}[X, Y]$$

[X,Y random variables; a,b constants]



# Optimal (Markowitz) Portfolio



A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies.

— *Harry Markowitz* —

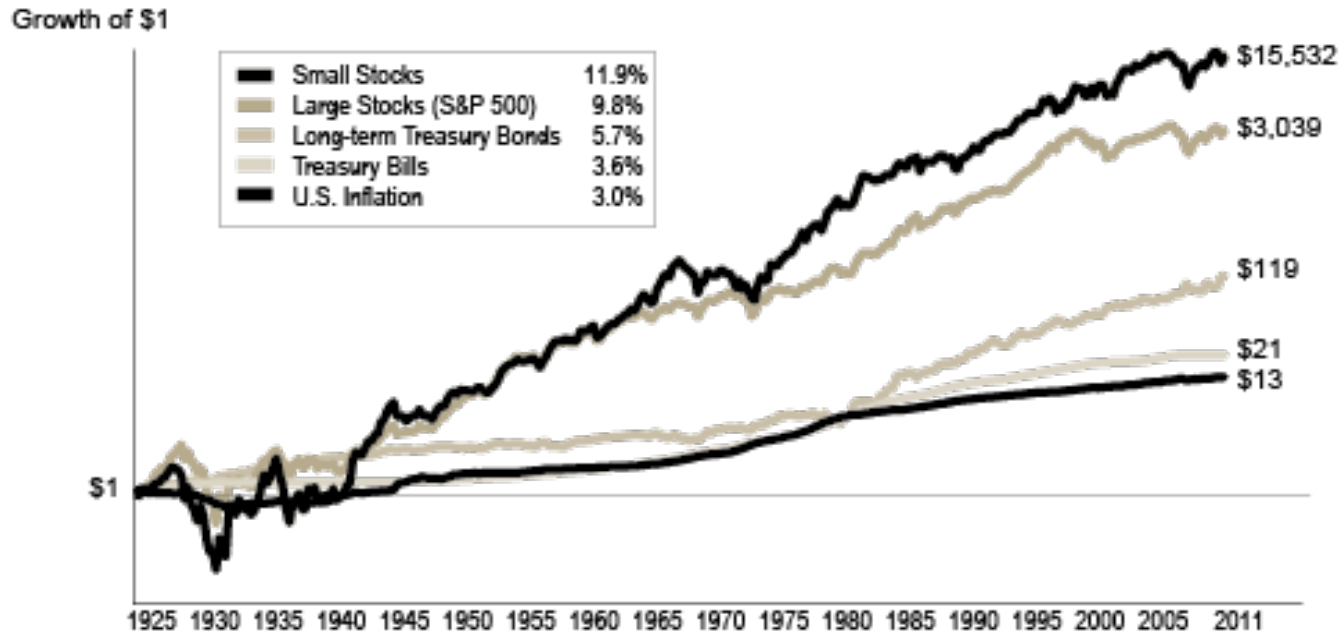


# The goal

1. Higher expected return
  2. Lower risk
- We need to optimally diversify:
    - **Total portfolio risk is less than the sum of the parts**



# Investing for the long-run



## U.S. long-run returns (last 140 years):

- 7% real returns for stocks (st. dev. = 15%)
- 2% real returns for bonds (st. dev. = 3.5%)



# Optimal (Markowitz) Portfolio

– Two risky assets:

$$E[r] = E[\omega r_1 + (1 - \omega)r_2] = \omega E[r_1] + (1 - \omega)E[r_2]$$

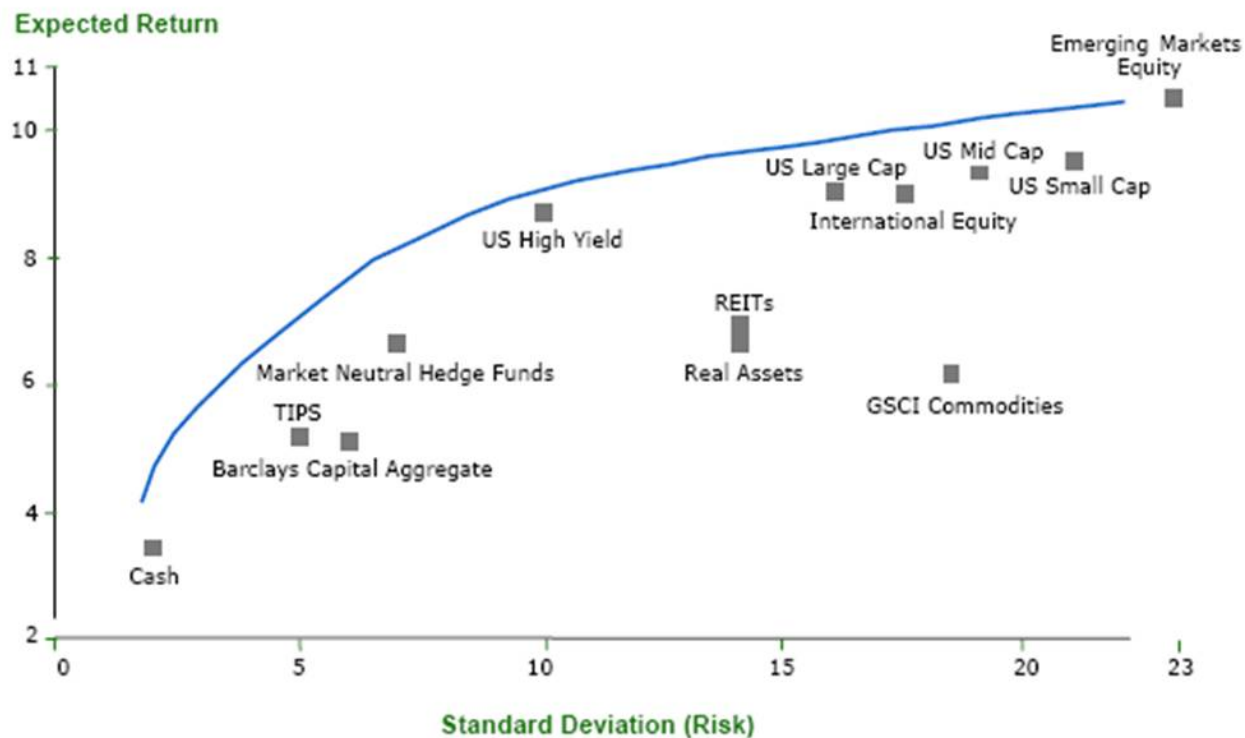
$$\begin{aligned} Var[r] &= Var[\omega r_1 + (1 - \omega)r_2] \\ &= \omega^2 Var[r_1] + (1 - \omega)^2 Var[r_2] \\ &\quad + 2\omega(1 - \omega)cov(r_1, r_2) \end{aligned}$$

Efficient frontier forms ‘parabola’

when  $(E[r], \sigma[r])$  traced out as function of  $\omega$

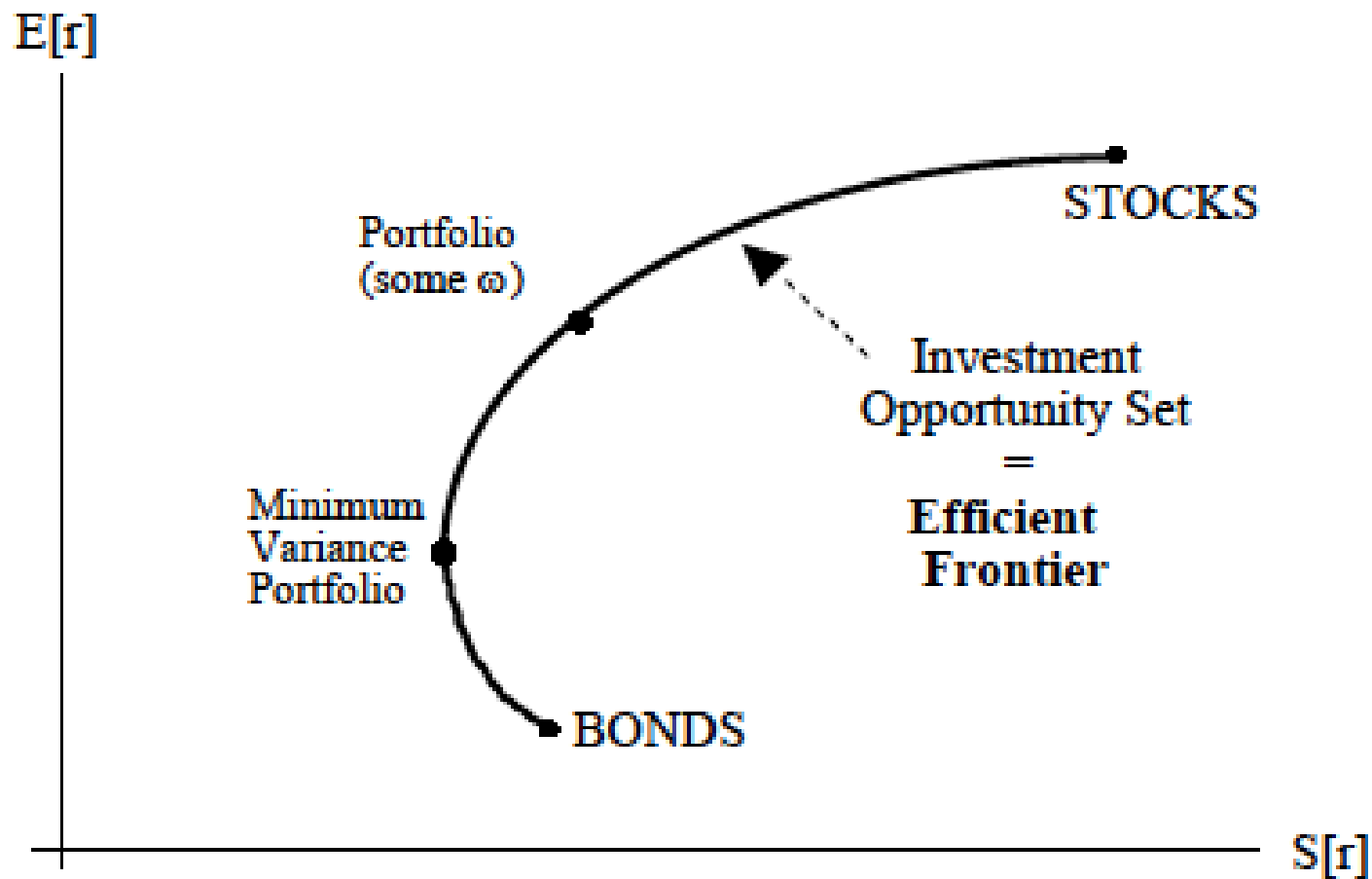


# Asset allocation





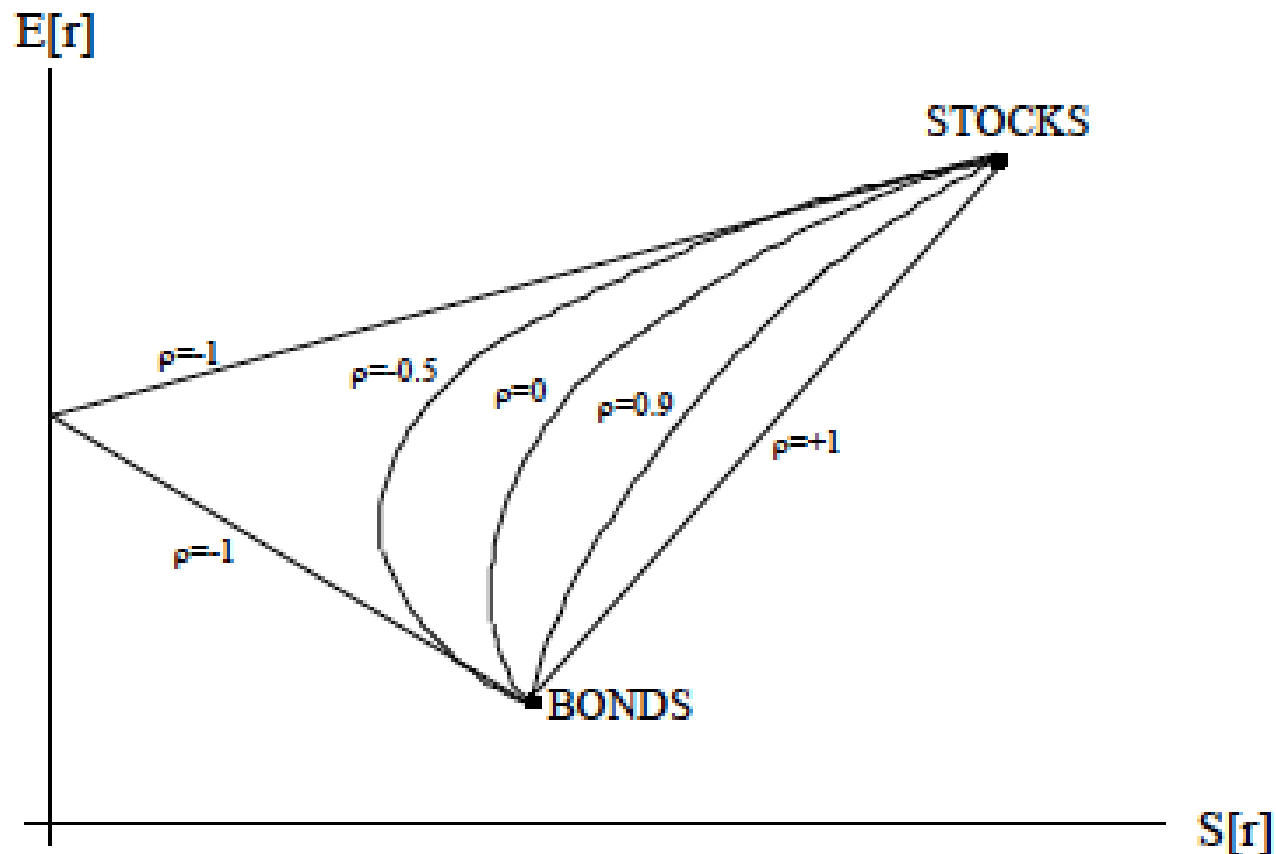
# Efficient frontier





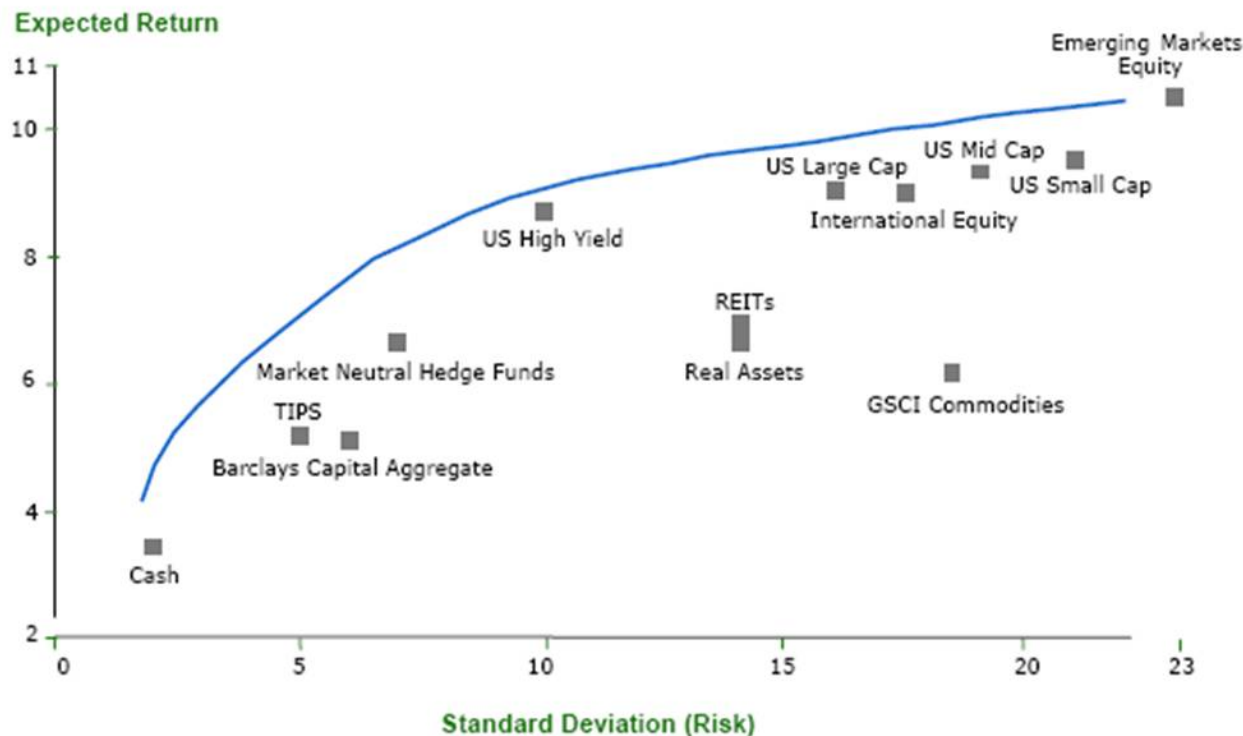


# For various correlation coefficients





# Generalizes to N risky assets



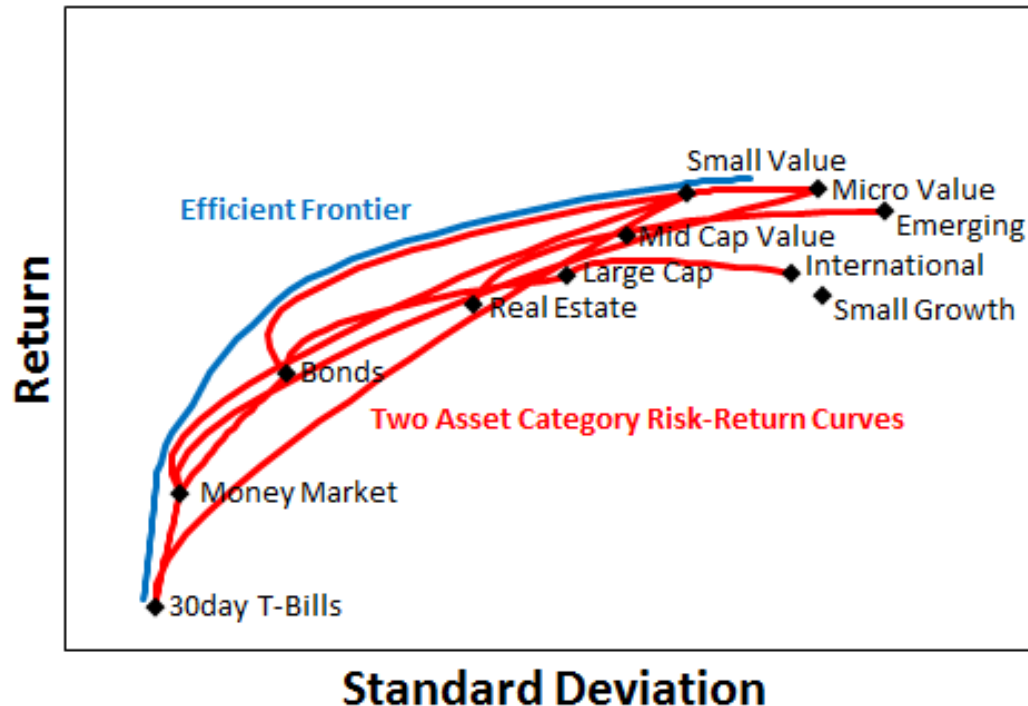
$$E[r] = \sum_{i=1}^N \omega_i \bar{r}_i$$

$$V[r] \equiv \sigma^2 = \sum_{i=1}^N \omega_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j$$



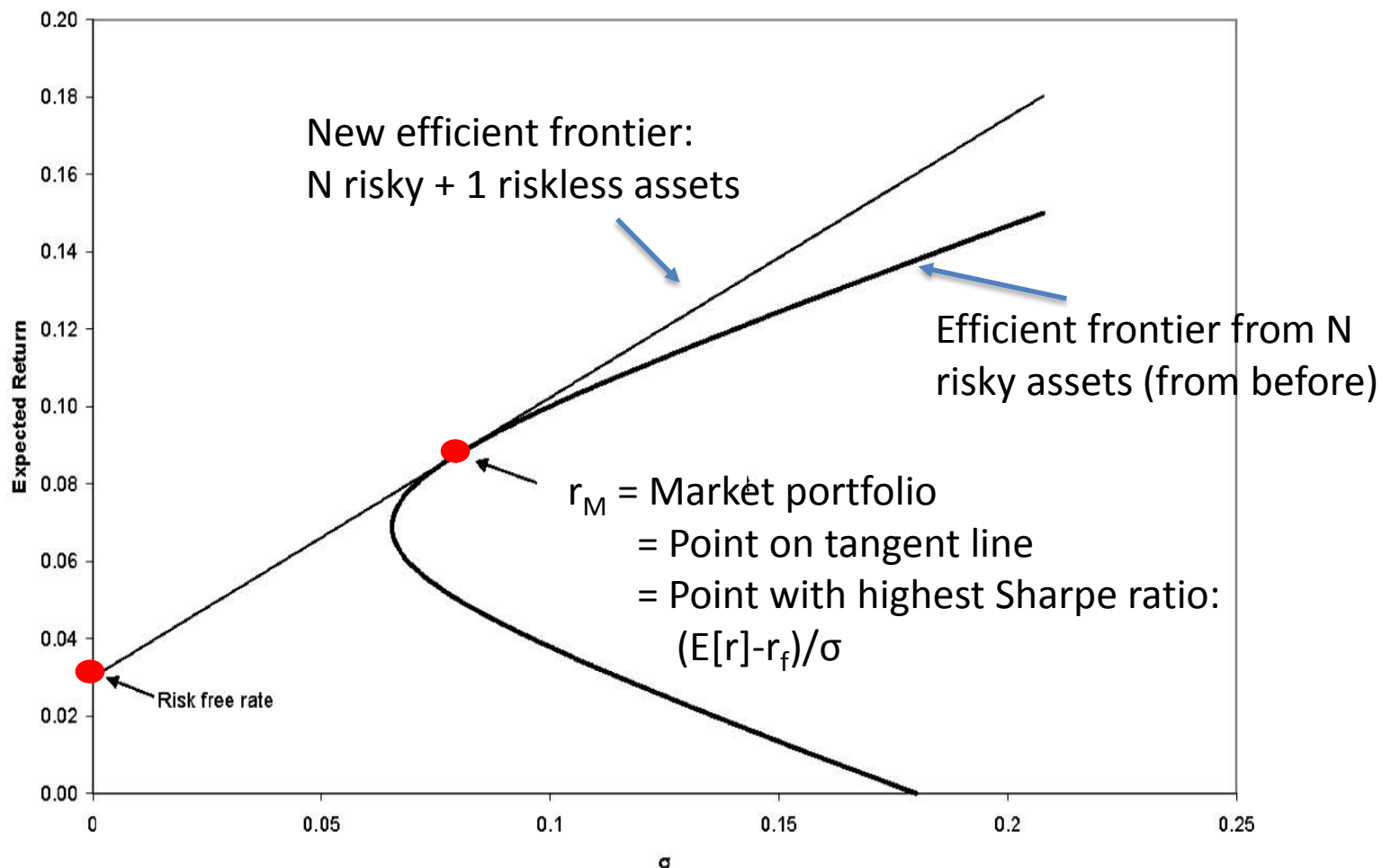
# Each additional asset expands the efficient frontier

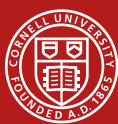
- Diversification benefits





# Optimal risky portfolio + riskless asset





# Optimal risky portfolio + riskless asset

- New efficient frontier is a line:

$$E[r] = E[\omega r_M + (1 - \omega)r_f] = \omega E[r_M] + (1 - \omega)E[r_f]$$

$$\begin{aligned} Var[r] &= Var[\omega r_M + (1 - \omega)r_f] \\ &= \omega^2 Var[r_M] \end{aligned}$$

Where  $r_M$  is 'market portfolio'  
= the point on the tangent line  
= the point with the highest Sharpe



# Two fund separation theorem

- Theorem:
  - The efficient frontier of ( $N$  risky + 1 riskless) assets is the same as finding the tangent line to the efficient frontier of just  $N$  risky assets
- Practical consequence:
  - All investors will hold the same mix of risky assets (they'll all hold the 'market portfolio')
  - Differences in risk preferences will simply put different investors at different points on the tangent line (will determine their mix of the 'market portfolio' versus cash)



# Optimization (two risky assets)

Maximize ( $\gamma$  is your risk-aversion coefficient)

$$U = \mu - \frac{\gamma}{2} \sigma^2$$

where:

$$\mu = \mu_1 \omega + \mu_2 (1 - \omega)$$

$$\sigma^2 = \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\sigma_{12}$$

Plugging in the above two expressions into  $U$ ,  
differentiating with respect to  $\omega$  and setting to 0 yields:

$$\omega = \frac{\frac{1}{\gamma}(\mu_1 - \mu_2) + (\sigma_2^2 - \sigma_{12})}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$



# Minimum variance portfolio

This is not the 'optimal' portfolio, but it is sometimes useful (if you really dislike risk):

In the two asset case, minimize:

$$\sigma^2 = \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\sigma_{12}$$

By differentiating with respect to  $\omega$  and setting to 0.  
This yields:

$$\omega = \frac{(\sigma_2^2 - \sigma_{12})}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$





# PORTFOLIO THEORY IN PRACTICE



# Class announcements

- Problem Set #1 graded: pick up from Annie in 304
- Problem Set #2 due this Friday
  - In general with Problem Sets, don't invest too much with write-ups (also, keep it short)
  - Just need to see evidence that you understand the approach
- Today:
  1. Do an example of portfolio optimization
  2. Finish CAPM
  3. Start with no-arbitrage pricing



# Data for Excel optimizer

Annualized CPI Relative Risk-Return Estimates: 1973-2014

Asset Classes	Return (%)	Std. Deviation (%)
Fixed Income	0.9	1.2
Intermediate Term Bond	3.1	4.5
Long Term Bond	4.9	10.0
High Yield Bonds	6.1	8.3
International Govt Bonds	4.6	10.4
Commodities	3.7	19.8
Large Cap Equity	7.0	15.6
Mid Cap Equity	9.4	17.6
Small Cap Equity	11.1	21.5
International Equity	5.6	17.5
Emerging Market Equity	11.6	23.8
REITs	9.5	19.0



# Data for Excel optimizer

Historical Correlations, Benchmark Relative to CPI: 73-14, Ledoit Estimated

Money Market	Intermediate Term Bond	Long Term Bond	High Yield Bonds	International Govt Bonds	Commodities	Large Cap Equity	Mid Cap Equity	Small Cap Equity	International Equity	Emerging Market Equity	REITs
1	0.42	0.31	0.1	0.18	-0.12	0.14	0.11	0.08	0.13	0.05	0.06
0.42	1	0.87	0.33	0.51	0.07	0.23	0.22	0.12	0.19	0.1	0.2
0.31	0.87	1	0.36	0.44	0.05	0.26	0.25	0.14	0.19	0.11	0.24
0.1	0.33	0.36	1	0.16	0.08	0.61	0.68	0.66	0.51	0.55	0.61
0.18	0.51	0.44	0.16	1	0.3	0.08	0.07	-0.01	0.46	0.15	0.12
-0.12	0.07	0.05	0.08	0.3	1	0.02	0.06	0.04	0.19	0.23	0.06
0.14	0.23	0.26	0.61	0.08	0.02	1	0.93	0.76	0.64	0.67	0.6
0.11	0.22	0.25	0.68	0.07	0.06	0.93	1	0.88	0.63	0.7	0.68
0.08	0.12	0.14	0.66	-0.01	0.04	0.76	0.88	1	0.53	0.65	0.66
0.13	0.19	0.19	0.51	0.46	0.19	0.64	0.63	0.53	1	0.67	0.47
0.05	0.1	0.11	0.55	0.15	0.23	0.67	0.7	0.65	0.67	1	0.49
0.06	0.2	0.24	0.61	0.12	0.06	0.6	0.68	0.66	0.47	0.49	1



# The case for portfolio optimization

## Pros:

1. Suitable for long-term investors looking for optimal diversification
  - Retirement investors, pension funds, univ. endowments
2. In more modern versions, can account for:
  - Tail risk
  - Dynamic correlations
  - Uncertainty about the parameters  $\mu_i$   $\sigma_i$   $\rho_{ij}$  (Black-Litterman)
  - Liquidity needs and payout needs (e.g., for insurance and pension funds)

## Cons:

1. Assumes you know future  $\mu_i$   $\sigma_i$   $\rho_{ij}$  and that these parameters are known and static
2. Suitable for asset allocation ( $N = 10$ ) but not stock selection ( $N = 5000$ )
  - Too much computational complexity
3. Assumes passive investing approach
  - Tells you nothing about whether assets are over-priced or under-priced



# THE CAPM



# Prices and returns are inverses

Price = 100

Corporate profits: 5 5 5 5 5 5 5 5 ....

Long-run returns =  $5/100 = 5\%$

Now, price goes up to 120

Corporate profits: 5 5 5 5 5 5 5 5 ....

Long-run returns =  $5/120 = 4.16\%$



# Prices and returns are inverses

General pricing formula:

$$Price = -C_0 + \frac{C_1}{(1+k)} + \frac{C_2}{(1+k)^2} + \dots + \frac{C_T}{(1+k)^T}$$

Constant cash flows:

$$\begin{aligned} Price &= \frac{C}{(1+k)} + \frac{C}{(1+k)^2} + \dots = C \sum_{t=1}^{\infty} \left[ \frac{1}{(1+k)} \right]^t \\ &= \frac{C}{k} \end{aligned}$$

Stock dividend  $D_0$  grows at rate  $g$ :

*“Gordon growth model” - we’ll derive in future lecture*

$$Price = \frac{D_0}{k - g}$$





# Discount rates

- So if we know the appropriate discount rate
  - Then we can combine it with our cash-flow projections to get the price
- But how do we know the correct discount rate?
  - That's where CAPM comes in
  - In general, higher risk requires a higher expected return



# Discount rates

Other concepts you should be familiar with:

1. The discount rate ( $k$ ) can be decomposed into the risk-free rate + the risk-premium:

$$k = RiskPremium + r_f$$

- The risk-free rate can in turn be decomposed into the “real rate” plus expected inflation.
  - We use the variable “ $k$ ” (rather than “ $r$ ”) when we want to be clear (as with equities) that it is not simply an interest rate. With bonds, we sometimes use “ $y$ ” for yield.
2. We use the terminology “discount rate”, “expected return”, “required return”, “risk premium”, and “cost of equity capital” interchangeably, depending on the context



# Discount rates

- Why do we use “discount rate” and “expected return” interchangeably?

$$P_t = \frac{C_1}{(1+k)} + \frac{C_2}{(1+k)^2} + \dots$$

$$P_{t+1} = \frac{C_2}{(1+k)} + \frac{C_3}{(1+k)^2} + \dots$$

Combining these two equations:

$$P_{t+1} = P_t(1+k) - C_1$$

Therefore:

$$\text{Wealth at time } (t+1) = P_{t+1} + C_1 = P_t(1+k)$$

So your expected return is actually  $(1+k)$ , the discount rate



# Systematic vs. Idiosyncratic Risk

- Amazingly, the resulting formula is very simple:
  - CAPM predicts stocks have a strong common factor (“market factor”):

$$R_i - r_f = \underbrace{\beta_i (R_m - r_f)}_{\text{systematic}} + \underbrace{e_i}_{\text{idiosyncratic}}$$
$$\beta_i = \frac{\text{Cov}(r_i, r_M)}{\text{Var}(r_M)} = \text{correlation of stock } i \text{ with the market}$$

- One problem with the CAPM is that the one-factor model is not quite empirically true (only about 2/3 of the variation in returns explained by the market factor)
  - From the work of Fama and French, we know that there are 3 (or possibly 4 or 5) factors that explain 95% the cross-section of stock returns

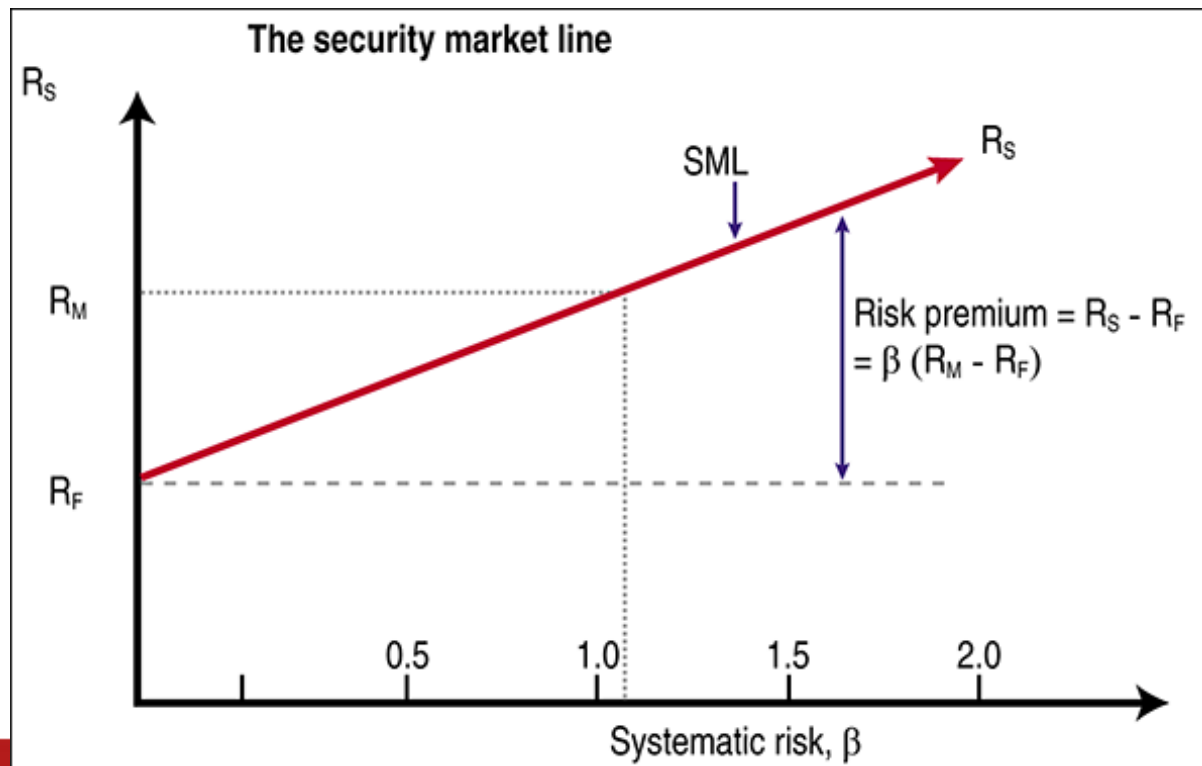


# CAPM

Putting expectations on both sides, CAPM predicts:

- A trade-off between systematic risk and expected return

$$E(R_i - r_f) = \beta_i E(R_m - r_f)$$





# CAPM derivation

- As we'll see: the CAPM one-factor structure emerges from a basic model in which all investors rationally choose optimal Markowitz portfolio



# CAPM derivation

- In the Optimal Portfolio problem:
  - We took expected returns as given and solved for the optimal portfolio.
- Now, we do the opposite:
  - Assume that **investors hold the optimal portfolio and solve for the expected returns.**
  - In aggregate, that optimal portfolio has to be the (fixed) supply of assets.
  - So equate **the supply of assets to the demand for asset** from the optimal portfolio problem and solve:
    - Prices (and thus expected returns) have to adjust to make investors choose to hold exactly the supply of assets out there.



# CAPM derivation

## Step 1: Solve Optimal Portfolio problem

- Taking  $\mathbf{R}=(R_1, \dots, R_I)$ , choose  $\boldsymbol{\omega}=(\omega_1, \dots, \omega_I)$  to maximize:

$$E[U] = \boldsymbol{\omega}' E[\mathbf{R} - r_f \mathbf{1}] - \frac{\gamma}{2} \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega}$$

- Take derivative with respect to  $\omega$  and set to 0:

$$\frac{dE[U]}{d\boldsymbol{\omega}} = E[\mathbf{R} - r_f \mathbf{1}] - \gamma \boldsymbol{\Sigma} \boldsymbol{\omega} = 0$$

- Solve for Optimal Portfolio:

$$\boldsymbol{\omega}^* = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} E[\mathbf{R} - r_f \mathbf{1}] \quad (1)$$





# CAPM derivation

Step 2: Equate supply (market portfolio) with demand (optimal portfolio) of assets

$$R_m = \sum_{j=1}^I w_j^* R_j$$

- Then, plugging the above equation into  $\text{Cov}(R_i, R_m)$ :

$$\begin{aligned} \text{Cov}(R_i, R_m) &= \text{Cov}\left(R_i, \sum_{j=1}^I w_j^* R_j\right) = \sum_{j=1}^I w_j^* \text{Cov}(R_i, R_j) \\ &= \sum_{j=1}^I w_j^* \Sigma_j \end{aligned}$$

- Writing this in vector notation:

$$\Sigma \omega^* = [\text{cov}(R_1, R_m), \dots, \text{cov}(R_I, R_m)]$$

- Combine with Eqn. (1) from previous slide:

$$\text{cov}(R_i, R_m) = \frac{1}{\gamma} E(R_i - r_f) \quad (2)$$



# CAPM derivation

## Step 3: Solve for $\gamma$

- Choosing asset  $i$  to be the market portfolio  
i.e. putting  $R_i = R_m$  into Eqn. 2 gets:

$$\text{cov}(R_m, R_m) = \frac{1}{\gamma} E(R_m - r_f)$$

- Now that we know  $\gamma$ , substituting above formula back into Eqn 2:

$$E(R_i - r_f) = \underbrace{\frac{\text{cov}(R_i, R_m)}{\text{Var}(R_m)}}_{\beta_i} E(R_m - r_f)$$



# The Case for the CAPM

## Pros:

1. First simple model of the risk-return trade-off
2. Good way to estimate the appropriate discount rate
  - Used in corporate finance to calculate cost of equity capital
3. Fund of funds: Easy way to see if a money manager is beating the market through more market risk (beta) vs. skill (alpha)
4. Demonstrates how correlations unrelated to firm fundamentals can arise through investor demand
  - e.g., the “financialization of commodities” in the mid-2000s

## Cons:

1. Evidence of a clear trade-off between systematic risk and return trade-off is **lacking** in the data
2. The cross-section of stock returns seems to have 3-5 important factors (not 1 factor)



# PASSIVE INVESTING IN PRACTICE



# How should you invest your money

- Be boring!
  - Invest in index funds (as low cost as possible, aim for  $< 0.10\%$  fees)
- If  $X$  is your age:
  - $X\%$  government bonds and cash
  - $(1-X)\%$  S&P 500 index
- If you want to get a little fancy, you can choose a bit more diversification:
  - Stock component: 50% large cap, 25% U.S. small-cap, 25% foreign
  - Bond/cash component: cash, US Treasuries and investment-grade corporate bonds.
    - Munis only make sense if you are in the top income tax bracket.
- But the added diversification benefits are a lot less than you think and **may not be worth the higher fees of these asset classes.**



# How should you invest your money

- This strategy is about:
  - Wealth accumulation & preservation; long-term growth
  - Elimination of idiosyncratic risk through diversification
  - Most importantly, eliminating management fees & transaction costs.
- Don't speculate with individual stocks, don't buy high-fee alternative asset classes, etc.
- Don't pay high fees to actively-managed mutual fund managers! (They can't beat the market)
- Don't try to time the market.



# Don't be the person who sold in March 2009





# Institutional Money Management

- Do sensitivity analysis for each asset class:  
expected returns, volatility, & **especially** correlations
  - Be careful of using historical performance
  - Correlations tend to rise substantially in times of stress
- Consider broader use of alternative asset classes
  - Long term horizon is needed to exploit illiquid and less efficient markets in alternative asset classes.
  - David Swensen & Yale Endowment Model
- We will study institutional money manager strategies and performance in more detail later in this course