MATH2101 Linear Algebra I Notes

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Introduction

Linear algebra can be treated as a study of vectors, matrices and systems of linear equations. Linear algebra is not only found useful in mathematics as a tool to solve mathematical problems but has also been broadly impacting areas like physics, engineering, economics and social science. Personally, the quickest way to learn this subject is to read and try some sample questions.

We strongly suggest readers learn the concepts from other materials before reading this "note". A great source is perhaps Prof. Hui's Linear Algebra I Notes.

If you wish to find some questions for exam preparation or wish to explore more interesting examples in linear algebra, this "note" might be what you want, as it is composed of some fascinating examples and some exam-style questions.

Recommended reading: Spence, Insel & Friedberg: *Elementary Linear Algebra – A Matrix Approach* (Pearson, 2014)

Contents

1 Matrices, Systems of Linear Equations, and Linear Independence

Practice Question 1.1. Prove that the sum of two $n \times n$ upper triangular matrices is still an upper triangular matrix.

Solution. Let A $= (a_{ij})$ and $B = (b_{ij})$ be two upper triangular matrices and $C = A + B = c_{ij})$. If $i > j$, then $a_{ij} = b_{ij} = 0$ by definition. Hence, $c_{ij} = a_{ij} + b_{ij} = 0$. Therefore C is also an upper triangular matrix.

Practice Question 1.2 (IMPORTANT!)**.** Write a MATLAB code to solve the following system of linear equations:
$$\begin{aligned} 12x + 34y + 56z &= 2830 \\ 10x - 24y + 37z &= 826 \\ -6x + y + 11z &= 325 \end{aligned}$$

Solution.

```
syms x y z
eqn1 = 12*x + 34*y + 56*z == 2830;
eqn2 = 10*x - 24*y + 37*z == 826;
eqn3 = -6*x + y + 11*z == 325;
sol = solve([eqn1, eqn2, eqn3], [x, y, z]);
xSol = sol.x
ySol = sol.y
zSol = sol.z
```

The output of the programme should be:

```
xSol = 12
ySol = 23
zSol = 34
```

Practice Question 1.3. Consider the following systems of linear equations $\begin{cases} x + y + z = 1 \\ x - y + 7z = 0 \\ 2x + 4y - 4z = 2 \end{cases}$. How many solutions does it have?

Solution. There is no solution, this can be done by Gaussian elimination. A quicker way to see this is to note that $x + y + z = 1$ and $x - y + 7z = 0$, then $2x + 4y - 4z = 3(x + y + z) - (x - y + 7z) = 3 - 0 = 3 \neq 2$.

2 Matrix Multiplication, Invertibility, and LU Decomposition

Practice Question 2.1. If A is a square matrix that satisfies $A^2 + 5A + 6I = O$ where O is the zero matrix.
(a) Show that A is invertible.
(b) Must $A + I$ be invertible? (b) Must $A + 3I$ be invertible?

Solution. (a) Since $A(\frac{A+5A}{-6}) = \frac{A^2+5A}{-6} = \frac{-6I}{-6} = I$, A is invertible.
(b) Yes, since $(A+I)\frac{A+4I}{-2} = \frac{A^2+5A+4I}{-2} = \frac{-2I}{-2} = I$, so $A+I$ is invertible.
(c) No, for instance, if $A = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$,then $A^2+5A+6I = O$ but $A+3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible.

3 Determinants

Practice Question 3.1. (a) Calculate the determinant of the matrix $A = \begin{pmatrix} 6 & 7 & -3 \\ 0 & -13 & 7 \\ -24 & 2 & 1 \end{pmatrix}$ using cofactor expansion along the first row.
(b) The determinant of a 4×4-matrix can also be calculated using cofactor expansion. Calculate the determinant of the matrix:

$$B = \begin{pmatrix} 6 & 4 & 3 & 1 \\ 0 & -1 & 7 & -3 \\ -4 & 2 & 13 & 0 \\ 9 & 1 & 1 & 1 \end{pmatrix}$$

Solution. (a)

$$\begin{aligned} \det A &= 6\begin{vmatrix} -13 & 7 \\ 2 & 1 \end{vmatrix} - 7\begin{vmatrix} 0 & 7 \\ -24 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & -13 \\ -24 & 2 \end{vmatrix} \\ &= 6(-27) - 7(168) - 3(-312) \\ &= -402 \end{aligned}$$

(b) Consider cofactor expansion along the first column, we have

$$\begin{aligned} \det A &= 6\begin{vmatrix} -1 & 7 & -3 \\ 2 & 13 & 0 \\ 1 & 1 & 1 \end{vmatrix} - 0 - 4\begin{vmatrix} 4 & 3 & 1 \\ -1 & 7 & -3 \\ 1 & 1 & 1 \end{vmatrix} - 9\begin{vmatrix} 4 & 3 & 1 \\ -1 & 7 & -3 \\ 2 & 13 & 0 \end{vmatrix} \\ &= 6(6) - 4(26) - 9(111) \\ &= -1067 \end{aligned}$$

Practice Question 3.2. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (y + z, 3x - z, x + y + z)$. What is the volume of the image of the rectangular cube $C = \{(x, y, z) \in \mathbb{R}^3 : -2 \le x \le 2, -3 \le y \le 0, -4 \le z \le 2\}$ under T?

Solution. The linear transformation T on a vector in $\mathbb{R}^3$ can be viewed as multiplying $\begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ to that vector. The volume of $T(C)$ is thus

$$|\begin{vmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix}| \times \text{volume of } C = 1 \times (4 \times 3 \times 6) = 72$$

4 Vector Spaces and Linear Transformations

Practice Question 4.1. Let A be any matrix. Prove that Row $A \cap$ Null $A = \{\mathbf{0}\}$. (Here, we identify row vectors as column vectors in the natural sense.)

Solution. Consider $\mathbf{v} \in \text{Row}A \cap \text{Null}A$, because of $\mathbf{v} \in \text{Row}A$ (we have assumed Row$A$ contains column vectors), there exists a column vector $\mathbf{c}$ such that $\mathbf{v} = A^T\mathbf{c}$. (In fact, if the rows of A are $\mathbf{r}_1, \ldots, \mathbf{r}_m$ and $\mathbf{v} = c_1\mathbf{r}_1 + \cdots + c_m\mathbf{r}_m$, then $\mathbf{c} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}^T$.) Moreover, because of $\mathbf{v} \in \text{Null}\, A$, we have $A\mathbf{v} = \mathbf{0}$. Then

$$\mathbf{v}^T\mathbf{v} = \left(A^T\mathbf{c}\right)^T \mathbf{v} = \mathbf{c}^T A\mathbf{v} = \mathbf{c}^T(A\mathbf{v}) = \mathbf{c}^T\mathbf{0} = \mathbf{0}.$$

If we write $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$, then the equation becomes $v_1^2 + \cdots + v_n^2 = 0$, which implies that $v_1 = \cdots = v_n = 0$. Thus $\mathbf{v} = \mathbf{0}$. Conversely, $\mathbf{0} \in \text{Row}A \cap \text{Null}A$ and the result follows.

Practice Question 4.2. Let $\mathcal{A} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$. It is known that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \ldots, \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\}$ is also a basis for $\mathbb{R}^n$. Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^n$ with $[\mathbf{v}]_{\mathcal{A}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. Find $[\mathbf{v}]_{\mathcal{B}}$.

Solution. From the expression for $[\mathbf{v}]_{\mathcal{A}}$, we have $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n$. We need to determine scalars $c_1, \ldots, c_n$ for which $\mathbf{v} = c_1\mathbf{u}_1 + c_2(\mathbf{u}_1 + \mathbf{u}_2) + \cdots + c_n(\mathbf{u}_1 + \cdots + \mathbf{u}_n)$. This can be rewritten as $\mathbf{v} = (c_1 + \cdots + c_n)\mathbf{u}_1 + (c_2 + \cdots + c_n)\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$. Hence, we have to solve

$$\begin{cases} c_1 + c_2 + \cdots + c_n = a_1 \\ \qquad\; c_2 + \cdots + c_n = a_2 \\ \qquad\qquad\qquad \cdots \\ \qquad\qquad\qquad c_n = a_n. \end{cases}$$

And we get $c_j = a_j - a_{j+1}$ for $j = 1, 2, \cdots, n-1$ and $c_n = a_n$. Hence

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 - a_2 \\ a_2 - a_3 \\ \vdots \\ a_{n-1} - a_n \\ a_n \end{bmatrix}.$$

5 Eigenvalues, Eigenvectors, and Diagonalization

Practice Question 5.1. Consider the matrix $\begin{pmatrix} 5 & \frac{13}{2} & \frac{9}{2} \\ 0 & 6 & 3 \\ 0 & -7 & -4 \end{pmatrix}$

(a) Write a MATLAB programme to find its characteristic polynomial (here we define $p_A(x) = \det(xI - A)$).
(b) Write a MATLAB programme to find its eigenvalues and eigenvectors.

Solution. (a) Method 1:

```
E=[5,6.5,4.5;0,6,3;0,-7,-4]
charpoly(E)
```

The output of the programme should be

```
E=
5.0000 6.5000  4.5000
0      6.0000  3.0000
0     -7.0000 -4.0000

ans=
1 -7 7 15
```

Method 2:

```
syms t
E=[5,6.5,4.5;0,6,3;0,-7,-4]
det(t*eye(3,3)-E)
```

The output of the programme should be

```
E=
5.0000 6.5000  4.5000
0      6.0000  3.0000
0     -7.0000 -4.0000

ans=
t^3 - 7*t^2 + 7*t + 15
```

So the required characteristic polynomial is $t^3 - 7t^2 + 7t + 15$.
(b)

```
E=[5,6.5,4.5;0,6,3;0,-7,-4]
[V,D]=eig(E)
```

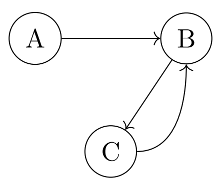
The output of the programme should be

```
E=
5.0000 6.5000  4.5000
0       6.0000  3.0000
0      -7.0000 -4.0000

V=
1.0000 -0.5774 -0.2540
0        0.5774 -0.3810
0       -0.5774  0.8890

D=
5  0  0
0  3  0
0  0 -1
```

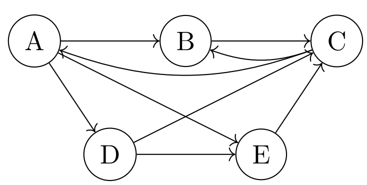
This means the eigenvalues of the matrix are 5,3 and -1 and the corresponding eigenvectors are (1,0,0), (-0.5774,0.5774,-0.5774) (or (1,-1,1)) and (-0.254,-0.381,0.889) (or (2,3,-7)) respectively.

Example 5.1. We now present an interesting application of eigenvalues - the PageRank Algorithm. It is an algorithm Google Search uses to rank web pages in their search engine results.

It is natural to model the linkage between webs by a graph (i.e. a diagram with nodes and edges). We draw an arrow from node a to node b if there is a link from page a to page b. For instance, the following diagram implies that there is a link from page A to page B, a link from page B to page C and a link from page C to page B.



Consider the following network of web pages. Which web page do you think is the most important or popular and which one is the least important?



You probably think C is the most important because it is linked to the greatest number of pages. Likewise, you might think D or A is the least important because both are linked to the least number of pages. If we compare further, D might be of lesser importance than A is, as the webpage linked to A (C) is more important than the webpage linked to D (A). This is a reasonable judgement, and let's now use linear algebra to create an "importance score" for the web pages.

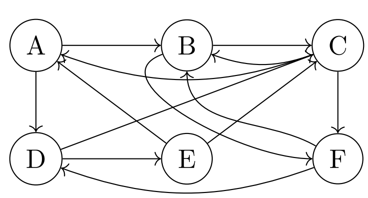
We will construct a 5×5 matrix G. We may call pages A, B, C, D and E in the previous examples nodes 1, 2, 3, 4 and 5 respectively. The (i, j)-th entry of G will be $\frac{1}{n_j}$, where n_j is the number of arrows pointing towards node j. For instance, the second column of G is $\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$ because arrows are pointing from node 1 and node 3 to node 2 but not from other nodes.

Similarly, one can deduce $G = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}$

Note that the sum of entries of each column of G is 1. It can be proved that any matrix whose sum of entries in each column is $k > 0$ will have k being one of its eigenvalues. Moreover, there exists an k-eigenvector of the matrix with all entries being non-negative and sum to 1. Thus, there is a 1-eigenvector of G with nonzero entries that sum to 1. The "importance score" of node i will be the i-th entry of this 1-eigenvector. For example, note that $\begin{pmatrix} \frac{5}{18} \\ \frac{1}{9} \\ \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{9} \end{pmatrix}$ is the required 1-eigenvector in this example, so the "importance scores" of pages A to E are $\frac{5}{18}$, $\frac{1}{9}$, $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{9}$ respectively. The higher the "importance score", the more important the web page should be.

The above algorithm is the PageRank algorithm. It turns out that under the PageRank algorithm, C has the greatest importance while B and E are of the least importance in the above case! No searching algorithm is perfect. Interested readers might read more about the limitations of the PageRank algorithm somewhere else.

Practice Question 5.2. For the following web pages, calculate their importance score using the PageRank algorithm.



Solution. We consider the following matrix $\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$ Its 1-eigenvector with entries sum to 1 is $\begin{pmatrix} \frac{24}{115} \\ \frac{3}{23} \\ \frac{24}{115} \\ \frac{18}{115} \\ \frac{4}{23} \\ \frac{2}{15} \end{pmatrix}$. So the importance scores of pages A to F are $\frac{24}{115}$, $\frac{3}{23}$, $\frac{24}{115}$, $\frac{18}{115}$, $\frac{4}{23}$ and $\frac{2}{15}$ respectively.

6 Dot Product, Orthogonality, and Orthogonal Matrices

Practice Question 6.1. An inconsistent system of linear equation $A\mathbf{x} = \mathbf{b}$ is given with

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -2 \\ -2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Find the matrix P_W where W represents the column space of A.
(b) Use the method of least squares to find the vectors $\mathbf{z}$ that minimize $\|A\mathbf{z}-\mathbf{b}\|$.
(c) Find the solution of least norm to the equation $A\mathbf{x} = P_W\mathbf{b}$.

Solution. (a) Notice that the first two columns of A form a basis for $\operatorname{Col} A$.
Let $C = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -2 & -3 \end{bmatrix}$. It follows that

$$P_W = C\left(C^T C\right)^{-1} C^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

(b) The vectors $\mathbf{z}$ that minimize $\|A\mathbf{z} - \mathbf{b}\|$ are the solutions to
$A\mathbf{x} = P_W\mathbf{b} = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}^T$. The general solution of this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

These are the vectors desired. Remark. Also note that for these vectors, we have

$$\|A\mathbf{z} - \mathbf{b}\| = \|P_W\mathbf{b} - \mathbf{b}\| = \left\| \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\| = \sqrt{3}.$$

(c) Based on the vector form of the solution we obtained in part (b), a vector $\mathbf{v}$ is a solution if and only if $\mathbf{v} = \mathbf{v}_0 + \mathbf{z}$, where

$$\mathbf{v}_0 = \begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{z} \in Z = \operatorname{Null} A = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Setting $C' = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, we can get the orthogonal projection matrix as

$$P_Z = C' \left(C^{TT} C' \right)^{-1} C'^T = \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Hence, $\mathbf{v}_0 - P_Z \mathbf{v}_0 = (I_3 - P_Z)\, \mathbf{v}_0 = \frac{1}{3} \begin{bmatrix} -8 \\ 2 \\ -10 \end{bmatrix}$ is the solution of least norm.

Practice Question 6.2. If V is a subspace of $\mathbb{R}^n$, prove that $(V^\perp)^\perp = V$.

Solution. Let $W = V^\perp$. We first show that $V \subseteq W^1$. For any $\mathbf{v} \in V$, it follows from the definition of W that $\mathbf{v} \cdot \mathbf{w} = 0$ for any $\mathbf{w} \in W$. Since $\mathbf{v}$ is orthogonal to every vector in $W, \mathbf{v} \in W^\perp$ by definition. This implies $V \subseteq W^\perp$. Note that V and W^1 are subspaces of $\mathbb{R}^n$. Moreover,

$$\dim W^1 = n - \dim W = n - (n - \dim V) = \dim V.$$

It follows from $V \subseteq W^\perp$ and $\dim W^\perp = \dim V$ that $V = W^\perp = \left(V^\perp\right)^\perp$.

Exam Questions

For students who care how to answer questions in the examination, we use the 2019 May Exam as a demonstration.

1. Give answers only to the following questions. Explanation is not required.

(a) If $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, compute the reduced row echelon form of A.

(b) Continuing the previous part, find a non-trivial solution to the system $A\mathrm{x} = 0$.

(c) With the same matrix A as above, find an eigenvector of A.

(d) If the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 8 \\ x \end{bmatrix}$ are linearly independent, find all possible values of x.

(e) Compute the determinant of the matrix $\begin{bmatrix} 2 & 4 & 6 & 9 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

(f) Let $B = \begin{bmatrix} * & * & * \\ 1 & 2 & 3 \\ * & * & * \end{bmatrix}$, where the $*$'s denote (possibly the same or different) real numbers. If $\operatorname{adj} B = \begin{bmatrix} * & 5 & * \\ 4 & 3 & 2 \\ * & 1 & * \end{bmatrix}$, find $\det B$.

(g) Consider the subspace $V = \operatorname{Span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ of $\mathbb{R}^4$. Find $\dim V$.

(h) What is the projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

Solution. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(d) $x \neq 9$

(e) -6

(f) 14

(g) 2

(h) $\begin{bmatrix} \frac{15}{7} \\ \frac{10}{7} \\ \frac{5}{7} \end{bmatrix}$

2. In each multiple-choice question below, some numbered choices are given and some are correct (wordings like is and are do not indicate the singularity/plurality of the number of correct choices). Answer the question by adding up the numbers of the correct choices. In case all choices are wrong, answer 0. Explanation is not required.

(a) Let A and B be invertible square matrices of the same size. Which of the following must be true?

(1) $(2A)(6B) = (3B)(4A)$

(2) $(2AB)^{-1} = (2B)^{-1}A^{-1}$

(4) $(AB)^3 = \left(A^2B\right)\left(AB^2\right)$

(8) $(A+B)(A-B) = (A-B)(A+B)$

(b) Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. If $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in two different ways, which of the following must be true?

(1) The system $A\mathbf{x} = \mathbf{b}$ is consistent.
(2) The system $A\mathrm{x} = \mathrm{b}$ has infinitely many solutions.
(4) The rank of A is less than n.
(8) a_n can be expressed as a linear combination of $\mathbf{a}_1, \mathrm{a}_2, \ldots, \mathbf{a}_{n-1}$ and b.

(c) In which of the following cases is T a linear transformation?
(1) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y$
(2) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2xy$
(4) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x \end{bmatrix}$
(8) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} |x| \\ 0 \end{bmatrix}$

(d) If $\{\mathbf{x}, \mathbf{y}, \mathrm{z}\}$ is a basis for a subspace V of $\mathbb{R}^n$, so is
(1) $\{2\mathbf{x}, 2\mathbf{y}, 2\mathbf{z}\}$
(2) $\{2x, 3y, 4z\}$
(4) $\{\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{x}+\mathrm{y}+\mathrm{z}\}$
(8) $\{\mathrm{x}+\mathrm{y}, \mathrm{y}+\mathrm{z}, \mathrm{z}+\mathrm{x}\}$

(e) A 4×4 matrix whose eigenvalues are all real must be diagonalisable if
(1) it has 4 distinct eigenvalues
(2) its eigenvalues are all positive
(4) all its eigenvalues have algebraic multiplicity 1
(8) all its eigenvalues have geometric multiplicity 1

(f) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Which of the following must be true?
(1) $u \cdot v \geq 0$
(2) $\|\mathrm{u}\| \cdot \|\mathrm{v}\| \geq \mathrm{u} \cdot \mathrm{v}$
(4) $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$
(8) $\|\mathrm{u}\|^2 + \|\mathrm{v}\|^2 \geq \|\mathrm{u} + \mathrm{v}\|^2$

(g) Let S be an orthonormal subset of $\mathbb{R}^n$. Which of the following must be true?
(1) $\left(S^\perp\right)^\perp = S$
(2) S is a basis for $\mathbb{R}^n$
(4) S is an orthogonal set
(8) $S^\perp$ is a subspace of $\mathbb{R}^n$

(h) Let S be a subset of $\mathbb{R}^n$ and T be the orthogonal complement of S. Which of the following must be true?
(1) T is a subspace of $\mathbb{R}^n$
(2) $\dim T = n - |S|$
(4) $S \cap T = \{0\}$
(8) $S \cup T = \mathbb{R}^n$

Solution. (a) 2 (b) 7 (c) 5 (d) 11 (e) 5 (f) 6 (g) 12 (h) 1

3. For each of the following statements, write (T) if it is true and (F) if it is false, and then give a very brief (say, one-line) explanation.

(a) Let A be a 3×4 reduced row echelon matrix. Then A has at least 6 zero entries.
(b) There exists matrix A with no zero entries such that $A^3 = 8A$ but $A^2 \neq 8I$.
(c) There exists a 4×4 matrix A such that adj $A = O$ and no two entries of A are the same.
(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation by an angle θ about the origin. If θ is not an integer multiple of $180°$, then T has no real eigenvalue.
(e) An elementary row operation will not change the column space of a matrix.
(f) If A has two distinct eigenvalues 3 and 4, then the intersection of the 3-eigenspace and the 4-eigenspace of A is a subspace of $\mathbb{R}^n$.
(g) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. If $\mathbf{u} \cdot \mathbf{v} = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
(h) A set of n orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Solution. (a) T, this can be proved by checking all possible forms of 3×4 RREFs: $\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ all have at least 6 zeros. Other 3×4 RREFs are in the form $\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ which contains at least 6 zeros.

(b) T, for instance, $A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

(c) T, for instance, $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 10 & 15 & 20 \\ 6 & 12 & 18 & 24 \\ 7 & 14 & 21 & 28 \end{bmatrix}$

(d) T, if T has a real eigenvalue r and $\mathbf{v}$ is a r-eigenvector of T then the rotation of $\mathbf{v}$ by θ is parallel to $\mathbf{v}$, so θ must be an integer multiple of $180°$.

(e) F, for instance, changing $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ by type III ERO, the column space changes from span$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ to span$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$.

(f) T, the intersection of the 3-eigenspace and the 4-eigenspace of A is the zero subspace of $\mathbb{R}^n$.

(g) F, for instance, in $\mathbb{R}^2$, $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ but $\mathbf{e}_1, \mathbf{e}_2 \neq 0$.

(h) F, for instance, the set of n zero vectors in $\mathbb{R}^n$ is a set of n orthogonal vectors in $\mathbb{R}^n$, but it is not a basis for $\mathbb{R}^n$.

4. Let

$$A = \begin{bmatrix} 1 & 4 & * & -1 \\ 2 & 3 & * & -1 \\ 2 & 4 & * & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & * & * & * \end{bmatrix}$$

Here the $*$'s denote (possibly the same or different) real numbers. It is known

that R is the reduced row echelon form of A.
(a) Show that $\operatorname{rank} A = 3$.
(b) Find, with careful explanation, all the unknown entries in A and R.
(c) Which columns of A form a basis for $\operatorname{Col} A$? Using the Gram-Schmidt process, turn this basis into an orthonormal basis. Can you find another (much simpler) orthonormal basis?

Solution. (a) Since A has three rows, $\operatorname{rank} A \leq 3$. Next, note that $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ are linearly independent columns of A, so $\operatorname{rank} A \geq 3$ and hence $\operatorname{rank} A = 3$.
(b) Since we have shown $\operatorname{rank} A = 3$ in (a), R has three leading ones. This forces $R = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Note that
$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we deduce that the third column of A is $3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -2 \end{bmatrix}$.
(c) The first, second and fourth columns of A form a basis for $\operatorname{Col} A$. Take $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.
Now take $e_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$e_2 = v_2 - \frac{v_2 \cdot e_1}{e_1 \cdot e_1} e_1 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} - \frac{18}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$e_3 = v_3 - \frac{v_3 \cdot e_2}{e_2 \cdot e_2} e_2 - \frac{v_3 \cdot e_1}{e_1 \cdot e_1} e_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \frac{-5}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{-1}{5} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-2}{45} \\ \frac{-4}{45} \\ \frac{1}{9} \end{bmatrix}$$

we get an orthogonal basis $\{e_1, e_2, e_3\}$ of $\operatorname{Col} A$. By normalisation, $\{\frac{e_1}{3}, \frac{e_2}{\sqrt{5}}, 3\sqrt{5} e_3\}$ is an orthonormal basis for $\operatorname{Col} A$. A much simpler orthonormal basis for $\operatorname{Col} A$ is the first, second and fourth columns of R, which is the standard basis for $\mathbb{R}^3$.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation which preserves norm, i.e.

$$\|T(\mathrm{x})\| = \|\mathrm{x}\| \quad \text{for all } \mathrm{x} \in \mathbb{R}^n$$

Let also A be the standard matrix of T.
(a) Show that the columns of A form an orthonormal basis for $\mathbb{R}^n$. Hence deduce that $A^TA = I$.
(b) If $\det A = d$ and λ is an eigenvalue of A, find all possible values of d and λ.
(c) Must T also preserve dot product? That is, must it be true that

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n?$$

Solution. (a) The columns of A are Ae_1, Ae_2, $\cdots$, Ae_n, where $\{e_1, \cdots, e_n\}$ is the standard basis for $\mathbb{R}^n$.
First, $||Ae_i||^2 = ||T(e_i)||^2 = ||e_i||^2 = 1$ for $1 \leq i \leq n$.
Next, for $1 \leq i \neq j \leq n$, e_i is orthogonal to e_j, so

$$||Ae_i||^2 + ||Ae_j||^2 + 2e_iA^TAe_j = ||A(e_i + e_j)||^2 = ||T(e_i + e_j)||^2 = ||e_i||^2 + ||e_j||^2$$

and hence $e_iA^TAe_j = 0$ for $1 \leq i \neq j \leq n$. This means the columns of A are orthogonal and hence they form an orthonormal basis for $\mathbb{R}^n$. Consequently, the (i, j)-th entry of A^TA is $e_iA^TAe_j$ which equals δ_{ij} for $1 \leq i, j \leq n$, so $A^TA = I$.
(b) By (a), we have

$$d^2 = \det A = \det A^T \det A = \det(A^TA) = \det I = 1$$

so the possible values of d are ± 1.
It is not hard to see that $||A(x + iy)|| = ||x + iy||$ for $x, y \in \mathbb{R}^n$, which means $||Av|| = ||v||$ for all $v \in \mathbb{C}^n$. If $\lambda \in \mathbb{C}$ is an eigenvalue of A, take $v \in \mathbb{C}^n$ to be a λ-eigenvector of A, one has $|\lambda| = 1$.
(c) Yes, T must preserve dot product as
$T(x) \cdot T(y) = (Ax)^T(Ay) = x^TA^TAy = x^Ty = x \cdot y$ for all $x, y \in \mathbb{R}^n$.

6. Let A and B be similar square matrices, with $A = P^{-1}BP$. Suppose that λ is an eigenvalue of A with algebraic multiplicity a and geometric multiplicity g.
(a) Explain why λ is also an eigenvalue of B with algebraic multiplicity a.
(b) If $\mathbf{x}$ is a λ-eigenvector of A, show that Px is a λ-eigenvector of B.
(c) Prove or disprove the following statements.
(i) A and B must be row equivalent.
(ii) λ must be an eigenvalue of B with geometric multiplicity g.

Solution. (a) This follows directly from the fact that similar matrices have the same characteristic polynomial: $p_A(t) = \det(A - tI) - \det(P^{-1}BP - tI) = \det(P^{-1}(B - tI)P) = \det P^{-1} \det(B - tI) \det P = \det(B - tI) = p_B(t)$. By definition of algebraic multiplicity, $p_B(t) = p_A(t) = (t - \lambda)^a q_A(t)$ for some $q_A(t)$ not divisible by $t - \lambda$.
(b) $BPx = PP^{-1}BPx = PAx = P(\lambda x) = \lambda Px$, so Px is a λ-eigenvector of B.
(c)(i) The statement is incorrect. For instance, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and
$B = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ are similar matrices as $A = P^{-1}BP$ where $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

However, they are not row equivalent because Row $A = \text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ while Row $B = \text{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$

(ii) The statement is correct. Let $g_A (= g)$ and g_B denote the dimension of the λ-eigenspace of A and B respectively. If $\{v_1, v_2, \cdots, v_{g_A}\}$ is a basis for the λ-eigenspace of A, then one can use (b) to show that $\{Pv_1, Pv_2, \cdots, Pv_{g_A}\}$ is a linearly independent set of the λ-eigenspace of B. Thus, $g_B \geq g_A$. Conversely, if $\{w_1, w_2, \cdots, w_{g_B}\}$ is a basis for the λ-eigenspace of B, then one similarly show that $\{P^{-1}w_1, P^{-1}w_2, \cdots, P^{-1}w_{g_B}\}$ is a linearly independent set of the λ-eigenspace of A. Thus, $g_A \geq g_B$ and hence they are equal.