THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2241 Introduction to Mathematical Analysis

Tutorial 5

Compulsory problems:

1. Prove or disprove: $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is a divergent series.

   (*Hint*: Note that $2^n > n$ for any $n \in \mathbb{N}$.)

2. **(Limit Comparison Test)**
   Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for some $L \geq 0$.

   (a) Prove that if $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{n=1}^{\infty} b_n$ converges.

   (b) Prove that if $L = 0$ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

   (c) Give an example of sequences $\{a_n\}$ and $\{b_n\}$ of positive terms such that

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = 0, \quad \sum_{n=1}^{\infty} a_n \text{ converges but } \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

   (d) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{n^2+n-1}{27n^5 - n^4 + 9n^3 - 2n + 6}$ converges.

3. Prove that the series

   $$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

   converges and find its sum.

   (*Hint*: You may use the fact that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.)

4. Prove each of the following statements:

   (a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \cos a_n$ diverges.

   (b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} \sin a_n$ converges absolutely.

5. Determine whether each of the following series converges absolutely, converges, or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+\sqrt{n}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+\sqrt{n}}$

For self-studying:

6. (a) **(Cauchy Condensation Test)**
   Let $\{a_n\}$ be a decreasing sequence with $a_n \geq 0$ for any $n \in \mathbb{N}$. Prove that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} \text{ converges.}$$

   (b) Let $p > 0$. Use the Cauchy Condensation Test to prove that the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

7. Use the ϵ-δ definition of limit to prove that $\displaystyle\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$ for any $c \in \mathbb{R} \setminus \{0\}$.