THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2241 Introduction to Mathematical Analysis

Tutorial 11

Compulsory problems:

1. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{x^2+1}$. Prove that f is integrable on $[0,1]$.

2. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1], \\ -1, & \text{if } x = 0. \end{cases}$$

   Prove that f is integrable on $[0,1]$.
   (*Hint:* For any $\epsilon > 0$, consider $P = \{0, \epsilon/3, 1\}$)

3. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \in (0,1], \\ 1, & \text{if } x = 0. \end{cases}$$

   Prove that f is integrable on $[0,1]$.
   (*Hint:* For any $\epsilon > 0$, consider $P = \{0, \epsilon/4, 1\}$)

4. If $f > 0$ on $[0,1]$ and $\int_0^1 f$ exists, is it true that $\int_0^1 f > 0$?

5. Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Is f integrable on some closed interval $[a,b]$ with $a < b$? Prove your answer.

   For self-studying:

6. Let $f : [1,3] \to \mathbb{R}$ with $f(x) = 2x^2 - 1$. Define $P_n = \{1 + \frac{2i}{n} \mid 0 \le i \le n\}$.

   (a) Show that
$$U(f, P_n) = \frac{2}{3n^2}(23n^2 + 24n + 4).$$
   (b) Find a similar expression for $L(f, P_n)$.
   (c) Using these, deduce that f is integrable, then find $\int_1^3 f$.