THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2241 Introduction to Mathematical Analysis

Tutorial 3

1. Use the Monotone Convergence Theorem to show that the following sequence converges. Find also the limit. (*Hint:* prove by induction.)

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = 1 + \sqrt{x_n} \quad \text{for any } n \in \mathbb{N}.$$

2. Let $a > 1$. Define a sequence $\{x_n\}$ by

$$x_1 = a, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad \text{for any } n \in \mathbb{N}.$$

   (a) Prove by induction that $x_n > \sqrt{a}$ for any $n \in \mathbb{N}$.

   (b) Prove that $\{x_n\}$ is decreasing. Hence show that $\{x_n\}$ converges. Find also the limit.

3. (a) Let $x \geq 1$. Prove that the sequence $\{x^{1/n}\}$ is bounded below and decreasing.

   (b) Show that $\lim\limits_{n\to\infty} x^{1/n} = 1$.

   (c) Show also that $\lim\limits_{n\to\infty} x^{1/n} = 1$ if $0 < x < 1$ by making a change of variable $x = \frac{1}{y}$ for some $y > 1$.

4. Give an example of each of the following, or state that such a situation is impossible by referring to the theorem(s) from the lecture note:

   (a) a sequence $\{a_n\}$ which diverges to positive infinity but has a subsequence which does not diverge to positive infinity.

   (b) an unbounded sequence $\{a_n\}$ which has no convergent subsequence.

   (c) an unbounded sequence $\{a_n\}$ which has a convergent subsequence.

5. Prove or disprove: The sequence $\left\{\cos\dfrac{n\pi}{128}\right\}$ is divergent.

6. (a) Write down the meaning of "a sequence $\{a_n\}$ is NOT a Cauchy sequence".

   (b) Suppose $\{a_n\}$ is sequence satisfying

$$|a_{2n} - a_n| \geq \frac{1}{2} \quad \text{for any } n \in \mathbb{N}.$$

   Use (a) to show that $\{a_n\}$ is NOT a Cauchy sequence.