Formalising Mathematics
Project 1

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Introduction

For the first project in *Formalising Mathematics*, I decided to formalise a result from measure theory known as Egorov's theorem. Egorov's theorem is a useful result establishing a relation between convergence almost everywhere and uniform convergence. In particular, Egorov's theorem states that a sequence of almost everywhere convergent functions converge uniformly everywhere except on an arbitrarily small set.
Egorov's theorem is used to prove the Vitali convergence theorem (a generalisation of the monotone convergence theorem for uniformly integrable functions) and is also useful to simplify more elementary results such as convergence almost everywhere implies convergence in measure.

Almost Everywhere Filter

In order to formalise Egorov's theorem, we will need to use the almost everywhere filter from `mathlib` in order to state the statement of convergence everywhere. We will in this section quickly introduce this definition.
`mathlib` defines the almost everywhere filter as the set of all co-null sets (set which complement has measure zero), namely it is defined as (omitting the proofs)

```
def measure.ae {α} [measurable_space α] (μ : measure α) : filter α :=
{ sets := {s | μ sᶜ = 0}, ... }
```

With the above filter defined, one may introduce all the standard definitions involving almost everywhere with this filter. In our use case, we would like to state the condition of convergence almost everywhere. To achieve this, we use the definition

```
def eventually (p : α → Prop) (f : filter α) : Prop := {x | p x} ∈ f
```

Unfolding the definition, we observe that `eventually p μ.ae` is the proposition that $\mu(\{x \mid \mathsf{p}(x)\})^c = 0$, namely, the set of elements not satisfying p has measure 0. For ease of use, we denote the statement `eventually p μ.ae` in as `∀ᵐ x ∂μ, p x`. Thus, with

this in mind, we may formulate convergence almost everywhere by taking p the proposition that the sequence of function converges at the point x.
In particular, the proposition that the sequence of functions f_n converges g almost everywhere is represented by the statement

```
∀ᵐ x ∂μ, x ∈ s → tendsto (λ n, f n x) at_top (𝓝 (g x))
```

Formalisation of Egorov's Theorem

We will in this section outline the proof of Egorov's theorem and comment on the formalisation effort. For the remainder of this document, let us assume α is a measure space with measure μ and β is a second-countable metric space.

Theorem 1 (Egorov's Theorem)**.** If $(f_n : \alpha \to \beta)_{n=0}^{\infty}$ is a sequence of measurable functions which converge almost everywhere on a set $s \subseteq \alpha$ of finite measure to $g : \alpha \to \beta$, then, for any $\epsilon > 0$, there exists some $t \subseteq s$ with measure $\mu(t) \leq \epsilon$ and f_n converges uniformly to g on $s \setminus t$.

Proof. Let $\epsilon > 0$ and for $i, j \geq 1$, define

$$C_{ij} := \bigcup_{k=j}^{\infty} \{x \in s \mid |f_k(x) - g(x)| > i^{-1}\}.$$

Since C_{ij} is measurable with finite measure and $C_{i,j+1} \subseteq C_{i,j}$, by the continuity of measures from above, we have

$$\lim_{j \to \infty} \mu(C_{ij}) = \mu\left(\bigcap_{j=1}^{\infty} C_{ij}\right) = 0.$$

Now, since f_k converges to g almost everywhere, by the definition of limits, there exists a subsequence $(C_{i,J(i)})$ such that $\mu(C_{i,J(i)}) < \epsilon 2^{-i}$. Then, by defining

$$t := \bigcup_{i=1}^{\infty} C_{i,J(i)},$$

we have

$$\mu(s \setminus t) = \mu\left(\bigcup_{i=1}^{\infty} C_{i,J(i)}\right) \leq \sum_{i=1}^{\infty} \mu(C_{i,J(i)}) < \sum_{i=1}^{\infty} \epsilon 2^{-i} = \epsilon.$$

Now, by construction, the elements of t are namely the elements x which $|f_k(x) - g(x)|$ converges slower than i^{-1}, the elements in $s \setminus t$ converges at a speed uniform bounded above by i^{-1}, and hence, f converges to g uniformly on $s \setminus t$ as required. □

In Lean, Egorov's theorem is represented by

```
lemma tendsto_uniformly_on_of_ae_tendsto' [is_finite_measure μ]
  (hf : ∀ n, measurable (f n)) (hg : measurable g)
```

```
  (hfg : ∀ᵐ x ∂μ, tendsto (λ n, f n x) at_top (𝓝 (g x)))
  {ε : ℝ} (hε : 0 < ε) :
∃ t, measurable_set t ∧ μ t ≤ ennreal.of_real ε ∧
  tendsto_uniformly_on f g at_top tᶜ
```

To prove this statement several auxiliary definitions were introduced. These are namely,

- `measure_theory.egorov.not_convergent_seq`;
- `measure_theory.egorov.not_convergent_seq_lt_index`;
- `measure_theory.egorov.Union_not_convergent_seq`,

which corresponds to the declarations $C_{ij}, J(i)$ and $\bigcup_{i=1}^{\infty} C_{i,J(i)}$ in the proof above respectively.