The Spectral Theorem for Sesquilinear Forms

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Definition 0.1 (Sesquilinear Form)**.** Let M be a module over the ring R and let σ be an R-antiautomorphism (or equivalently, an isomorphism $\sigma : R \simeq R^{\text{op}}$), then the function $\phi : V \times V \to R$ is a σ-sesquilinear form if and only if for all $x, y, z \in M$, $\alpha \in R$,

- $\phi(x + z, y) = \phi(x, y) + \phi(z, y)$;
- $\phi(x, y + z) = \phi(x, y) + \phi(y, z)$;
- $\phi(\alpha x, y) = \alpha\phi(x, y)$;
- $\phi(x, \alpha y) = \sigma(\alpha)\phi(x, y)$.

A typical example of a sesquilinear form are the inner products of any real or complex vector spaces.

Definition 0.2 (Adjoint Maps)**.** Let ϕ_1, ϕ_2 be sesquilinear forms on the modules M, N respectively, and let $f : N \to M$, $g : M \to N$ be linear maps. Then we say f and g are adjoints if and only if for all $x \in M$, $y \in N$,

$$\phi_1(x, f(x)) = \phi(g(x), y).$$