Multivariable Calculus & Differential Equations

Kexing Ying

May 15, 2020

0.1 Tensor Notation

- $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{A}_i \mathbf{B}_i$
- $\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} \hat{e}_i \mathbf{A}_j \mathbf{B}_k$
- $\operatorname{div} \mathbf{A} = \partial \mathbf{A}_i / \partial x_i$
- $\nabla \phi = \hat{e}_i \partial \phi / \partial x_i$
- $\operatorname{curl} \mathbf{A} = \epsilon_{ijk} \hat{e}_i \partial \mathbf{A}_k / \partial x_j$

0.2 Identities

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
- $\partial \phi / \partial s = \hat{s} \cdot \nabla \phi$

0.3 Finding Equation of a Tangent Plane to $\phi = \phi(P)$

We have $\nabla \phi$ evaluated at P is normal to the surface at P, and so the equation of the tangent plane is

$$(\mathbf{r} - \mathbf{r}_P) \cdot (\nabla \phi)_P = 0,$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}_P = P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}$.

0.4 Results Regarding the Gradient Operator

- $\nabla(\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$
- $\operatorname{div}(\phi \mathbf{A}) = \phi \operatorname{div} \mathbf{A} + \nabla \phi \cdot \mathbf{A}$
- $\operatorname{curl}(\phi \mathbf{A}) = \phi \operatorname{curl} \mathbf{A} + \nabla \phi \times \mathbf{A}$
- $\operatorname{div}(\nabla \phi) = \nabla^2 \phi = \partial^2 \phi / \partial x_i^2$
- $\operatorname{curl}(\nabla \phi) = 0$
- $\operatorname{div}(\operatorname{curl} \mathbf{A}) = 0$

- $\text{curl}(\text{curl}\,\mathbf{A}) = \nabla(\text{div}\,\mathbf{A}) - \nabla^2\mathbf{A}$
- $\text{div}(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl}\,\mathbf{A} - \mathbf{A}\cdot\text{curl}\,\mathbf{B}$
- $\nabla^2(1/r) = 0$

0.5 Integration

Path integrals over some path γ on the field $\mathbf{F}$:

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_\gamma \mathbf{F}\cdot\hat{\mathbf{t}}ds,$$

where $\hat{\mathbf{t}}$ is the path element.

If $\mathbf{F} = \nabla\phi$ for some scalar field ϕ, then if γ is a path that joins points A to B,

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A),$$

and we call $\mathbf{F}$ a conservative field. In this case if γ forms a loop, that is $A = B$, $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$.

In evaluating surface integrals, one can either integrate directly through (perhaps with the help with substitution) or one can use the projection theorem.

Theorem 1 (Projection Theorem)**.** Let S be a surface such that it does not contain a point at which it is orthogonal to $\mathbf{k}$. Then,

$$\int_S f(P)dS = \int_\Sigma f(P)\frac{dx\;dy}{|\hat{\mathbf{n}}\cdot\mathbf{k}|},$$

where f is a function over S and Σ is the projection on to the plane $z = 0$.

The projection theorem can be easily extended where we project onto another plane rather than $z = 0$.

Theorem 2 (Green's Theorem)**.** Let R be a closed plane region bounded by the curve C and let L, M be continuous functions of x, y of type $C^1(R)$, then

$$\oint_C (Ldx + Mdy) = \int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\;dy,$$

where C is integrated positively (counter-clockwise).

Green's theorem can be used to deduce the divergence and Stokes theorem in the 2-dimensional case.

Theorem 3 (Divergence Theorem)**.** If τ is the volume enclosed by a closed surface S with unit outward normal $\hat{\mathbf{n}}$ and $\mathbf{A}$ is a vector field of type $C^1(\tau)$, then,

$$\int_S \mathbf{A}\cdot\hat{\mathbf{n}}dS = \int_\tau \text{div}\,\mathbf{A}d\tau.$$

We in general refers the value of the integral $\int_S \mathbf{A}\cdot\hat{\mathbf{n}}dS$ as the flux of $\mathbf{A}$ across S.

Theorem 4 (Stokes Theorem)**.** Let S be an open surface with the boundary γ and let $\mathbf{A}$ be a vector field with continuous partial derivatives, then

$$\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = \int_S \operatorname{curl} \mathbf{A} \cdot \hat{\mathbf{n}} dS.$$

A result of the Stokes theorem (in combination of considering the properties of a conservative field) is that, a necessary and sufficient condition for $\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = 0$ for any simply closed curve γ is that $\operatorname{curl} A = 0$ within the region bounded by γ.

0.6 Curvilinear Coordinates

Definition 0.1. Let $f_i : \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of functions with continuous second derivatives, then, the coordinate system resulted from the transformation $u_i = f_i(x_j \mid j = 1, \cdots, n)$ is called a curvilinear coordinate system.

Definition 0.2 (Jacobian Matrix)**.** The Jacobian matrix of a given curvilinear coordinate transformation is the matrix $J(x_u)$ with entries $[J]_{ij} = \partial x_i / \partial u_i$ where $\{x_i\}$ is the original coordinates and $\{u_i\}$ is the transformed coordinates. We call the determinant of the Jacobian matrix $|J|$ the Jacobian.

From analysis we recall the inverse function theorem which states that the Jacobian at some point v is non-zero if and only if f_i is locally bijective at v.

Let $u_i = u_i(x_j \mid j)$ be a curvilinear coordinate system, then, by conidering $u_i = c_i$ for some constants, we have a system of families of surfaces. Let $P(x, y, z)$ be some point such that there passes one surface of each family, then, we can define $\hat{\mathbf{a}}_i$ be the unit normal of each surface. Clearly, we have

$$\hat{\mathbf{a}}_i = \frac{\nabla u_i}{|\nabla u_i|}.$$

If each $\hat{\mathbf{a}}_i$ is orthogonal to one another, we say the coordinate system if an orthogonal curvilinear coordinate system.

We find $\partial \mathbf{r} / \partial u_i = \hat{e}_i h_i$, where $h_i = |\partial \mathbf{r} / \partial u_i|$ and we call this quantity the length element for the coordinate system.

For a curvilinear system, we have

- $d\mathbf{r} = \sum \hat{e}_i h_i du_i$
- $d\tau = \prod h_i du_i$
- $dS = |J| \prod du_i$
- $\hat{e}_i = h_i \nabla u_i$
- $\nabla \Phi = \sum \hat{e}_i \frac{1}{h_i} \frac{\partial \Phi}{\partial u_i}$
- $\operatorname{div} \mathbf{A} = 1 / \prod h_i (\partial / \partial u_i (A_i \prod_{j \neq i} h_j))$

0.7 Calculus of Variations

Euler-Lagrange equation with multiple variables:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dx}\frac{\partial L}{\partial x_i'} = 0$$

Euler-Lagrange equation with constraint:

$$\frac{\partial}{\partial x_i}(L + \lambda g) - \frac{d}{dx}\frac{\partial}{\partial x_i'}(L + \lambda g) = 0$$

Suppose we denote the operator $\hat{e}_i \partial/\partial p_i$ by ∇_p for some vector p, the Euler-Lagrange equation in higher dimensions becomes

$$\frac{\partial L}{\partial F} - \operatorname{div}(\nabla_{\nabla F} L) = 0$$