Differentiable Equations

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1 Introduction

While we have seen differential equations in year one, we have mostly focused on the different methods of solving specific differential equations. This cannot be expected for general differential equations and in this year, we will focus on existence and uniqueness of solutions to differential equations and develop qualitative tools to help us understand these solutions.

We recall that an algebraic equation is an equation of the form $f(x) = 0$ while a differential equation is an equation of the form $\dot{x} = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$. That is, an algebraic equation has real numbers as solutions while an differential equation has functions as its solution.

As an example, let us consider the simple differentiable equation

$$\dot{x} = ax, \tag{1}$$

for some $a \in \mathbb{R}$. Then, a function $\lambda : I \to \mathbb{R}$ solves 1 if $\dot{\lambda} = a\lambda$ for all $t \in I$ where $I \subseteq \mathbb{R}$ is a interval. These types of differentiable equations occurs often in relation in growth and decay and one can easily see that the family of functions

$$\lambda_b : \mathbb{R} \to \mathbb{R} = t \mapsto be^{at}, \;\; b \in \mathbb{R},$$

are solutions to 1. Of course, we know this already, so an more interesting question would be whether or not this family contains all the solutions to 1. It turns out to be true, and to show this we will assume $\mu : I \to \mathbb{R}$ is a solution to $\dot{x} = ax$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu e^{-at}\right) = \dot{\mu} e^{-at} - a\mu e^{-at} = 0,$$

since $\dot{\mu} = a\mu$ and so, μe^{-at} is constant, i.e. there exists $b \in \mathbb{R}$ such that $\mu e^{-at} = b$ and hence,

$$\mu = be^{at}.$$

This demonstrates that all solutions to 1 are members of the aforementioned solution family and hence, we have found **all** of the solutions to 1.

With the above example, we see that rather than working with solutions that are in finite-dimensional vector spaces, our solution are in function spaces which are typically infinite-dimensional. This is studied in more detail in the next year's *functional analysis* course, and in general, infinite-dimensional spaces are more difficult to grasp. However, for the vast majority of materials in this course, a finite-dimensional thinking suffices while we will also cover some material from functional analysis to understand the differentiable equations as well.

1.1 Ordinary Differential Equations and Initial Value Problems

There are two types of differential equations – *autonomous differential equations* and *nonautonomous differentiable equations*. Autonomous differential equations are differentiable equations of the form $\dot{x} = f(x)$ such as equation 1 while nonautonomous differential equations are equations of the form $\dot{x} = f(t, x)$.

We note that this does not cover higher-order differential equations, but from last year, we recall that one may reduce a higher-order differential equations into a first-order differential equation in vector form and thus, the theories we develop within this course will also apply to higher-order differential equations.

Definition 1.1 (Ordinary Differential Equation)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be open, and a function $f \to D \to \mathbb{R}^d$. Then, an equation of the form

$$\dot{x} = f(t, x)$$

is called a d-dimensional (first-order) ordinary differential equation.

A differentiable function $\lambda : I \to \mathbb{R}^d$ on some interval $I \subseteq \mathbb{R}$ is called a solution of the differential equation if and only if for all $(t, \lambda(t) \in D$, if $t \in I$ then,

$$\dot{\lambda}(t) = f(t, \lambda(t)).$$

We say that an ordinary differential equation is autonomous if f is independent of t and nonautonomous otherwise.

We will only consider ordinary differential equations (ODE) in this course while partial differential equations, that is differential equations which solutions are functions which depends on multiple variables are covered in the second year course **Partial Differential Equations in Action**.

Proposition 1 (Constant solutions to autonomous differential equations)**.** Let $D \subseteq \mathbb{R}^d$ be an open set and $f : D \to \mathbb{R}^d$ be a function where $d \in \mathbb{N}$. Then, there exists a constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ to the autonomous differential

$$\dot{x} = f(x)$$

for some $a \in \mathbb{R}^d$ if and only if $f(a) = 0$.

Proof. ($\implies$) Suppose that $\lambda : I \to \mathbb{R}^d : x \mapsto a$ is a solution the $\dot{x} = f(x)$. Then

$$0 = \dot{\lambda}(t) = f(\lambda(t)) = f(a).$$

($\impliedby$) Suppose there exists some $a \in \mathbb{R}^d$ such that $f(a) = 0$, then verifying, we find $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to the differential equation. $\square$

This proposition allows us to find solutions to many autonomous ODEs as, indeed, if $f : D \to \mathbb{R}^d$ has a root $a \in \mathbb{R}^d$, the above proposition guarantees that $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to $\dot{x} = f(x)$.

Definition 1.2 (Initial Value Problem)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be an open set, and $f : D \to \mathbb{R}^d$ be a function. The system of equations from combining the differential equation

$$\dot{x} = f(t, x),$$

with the initial condition

$$x(t_0) = x_0$$

where $(t_0, x_0) \in D$ is called an initial value problem.

A solution to the above initial value problem is a function $\lambda : I \to \mathbb{R}^d$ that is a solution to the differential equation $\dot{x} = f(t, x)$ and $\lambda(t_0) = x_0$.

While, previously, we have seen a differential equation which always has a solution. This, however, is not always the case.

Example 1. Consider the differential equation $\dot{x} = f(x)$, where

$$f(x) = \begin{cases} 1, & x < 0; \\ -1, & x \geq 0, \end{cases}$$

with the boundary condition $x(0) = 0$. As we will see on the problem sheet, this differential equation indeed does not have a solution.

Example 2. Consider the initial value problem $\dot{x} = f(x) = \sqrt{|x|}$ with the boundary condition $x(0) = 0$.

Since $f(0) = 0$ we have $x(t) = 0$ is a constant solution by proposition 1. Furthermore, by consider the function

$$\lambda_b(t) = \begin{cases} 0, & t \leq b; \\ \frac{1}{4}(t-b)^2, & t > b, \end{cases}$$

we find λ_b to also be a solution for any $b \in \mathbb{R}_0^+$. However, as for all $t \leq b$, $\lambda_b(t) = 0$, we see that x is not unique given values of x at t.

Before moving on, let us quickly recall the *separation of variables* procedure for solving differential equations. Suppose we are to solve a differential equation of the form,

$$\dot{x} = g(t)h(x),$$

with the boundary condition $x(t_0) = x_0$ where $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are continuous functions. Then, we have

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{h(y)} = \int_{t_0}^{t} g(s)\mathrm{d}s.$$

Thus, by evaluating the integral, we can express x in t. We see that this procedure does indeed provide us with a correct solution by simply applying FTC on both sides of the equation.

Example 3. Consider the initial value problem $\dot{x} = tx^2$ with the boundary condition $x(t_0) = x_0$ where $x_0 \neq 0$. By the separation of variables, we find

$$x = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}.$$

However, we see that for this solution, x does not necessarily exist for all t. Indeed, if $t_0 = 0, x_0 = 1$, we find

$$x = \frac{2}{2 - t^2},$$

which does not have a solution for $t = \pm\sqrt{2}$, and so, the solution does not exist globally.

1.2 Visualisations

Visualisations are very important for differential equations as it provides us a mental image of how to think about differential equations. In general, there are two main methods to visualise differential equations:

- nonautonomous differential equations $\dot{x} = f(t, x)$ via the *solution portrait* in the extended phase space;
- autonomous differential equations $\dot{x} = f(x)$ via the *phase portrait* in the phase space.

Suppose we have the nonautonomous differential equations $\dot{x} = f(t, x)$ where $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, have a solution $\lambda : I \to \mathbb{R}^d$, i.e. for all $t \in I$, $\dot{\lambda}(t) = f(t, \lambda(t))$. Thus, we see that the vector $(1, f(t_0, \lambda(t_0))$ for some $t_0 \in I$ is tangential to the solution which passes through $(t_0, \lambda(t_0))$. This can be done for all $p \in \mathbb{R} \times \mathbb{R}^d$ and so, by drawing these vectors at a sufficient number of points, we can have a mental image of what the solution looks like. A plot of these vectors is refereed to as a **vector field**.

A **solution portrait** is given by a visualisation of several solution curves in the (t, x)-space, the so called *extended phase space*. This is called as such since the x-space is normally refereed as the phase space, and so, we are extending it by the time axis.

On the other hand, suppose we have the autonomous differential equation $\dot{x} = f(x)$ where $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ has a solution $\lambda : I \to \mathbb{R}^d$. By considering that f is independent of t, we see that translating λ along t is also a valid solution. Indeed, at $x = \lambda(t_0)$, if λ' is another solution such that $\lambda'(t_1) = \lambda(t_0)$, then $\dot{\lambda}'(t_1) = \dot{\lambda}(t_0)$. This property is refereed to as *translation invariance* and all solutions of autonomous differential equations are translation invariant.

Proposition 2. let $\dot{x} = f(x)$ be an autonomous differential equation. Then, if $\lambda : I \to \mathbb{R}^d$ is a solution to this differential equation, so is

$$\mu : \bar{I} \to \mathbb{R}^d : t \mapsto \lambda(t + \tau),$$

where $\tau \in \mathbb{R}$, $\bar{I} := I + \tau$.

Proof. Follows straight away by chain rule. $\square$

Example 4. Consider the harmonic oscillator $\ddot{x} = -x$. By converting it into a first order differential equation, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

By solving the system, we find

$$\lambda(t) = \begin{bmatrix} \cos(t) \sin(t) \\ -\sin(t) \cos(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is a solution. Indeed, this describes a oscillatory motion in the phase space where the position is circular.

With consideration with the example above, we see that we may project a solution portrait $(t, x) \mapsto (1, f(x))$ onto $x \mapsto f(x)$ resulting in a **phase portrait**.

2 Existence and Uniqueness

As we have seen, differential equations need not have unique solutions given an initial value. Indeed, we have also seen that it is not guaranteed to have a solution at all. We will in this chapter resolve the question on whether or not a solution exists by presenting a theory that guarantees existence and uniqueness for solutions to initial value problems.

2.1 Picard iterates

We observe that often times, to solve an differential equation, we need to reformulate it as an integral equation.

Proposition 3. Consider the initial value problem

$$\dot{x} = f(t, x); \quad x(t_0) = x_0,$$

where $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let $\lambda : I \to \mathbb{R}^d$, λ solves the IVP if and only if λ is continuous and solves the integral equation[1]

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\mathrm{d}s.$$

Proof. Follows from FTC. □

However, as f depends on λ, we in general cannot compute the integral on the right hand side. Nonetheless, this brings us closer to the formulation of the Picard iterates.

Proposition 4. Let f be a continuous function. Then by defining a_0 for some value and $a_{n+1} = f(a_n)$, if $(a_n)_{n=1}^{\infty}$ converges to some value a, then $a = f(a)$.

Proof. We see that if $(a_n)_{n=1}^{\infty}$ converges to some value a, $f(a_n) \to f(a)$ by sequential continuity. However, as $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = a$, by uniqueness of limits on Hausdorff spaces, we have $a = f(a)$. □

With this proposition in mind, we may apply a similar iteration on the integral equation resulting in the Picard iterates.

Definition 2.1 (Picard Iterates)**.** Given an IVP, the Picard iterates is the sequence of functions $(\lambda_n : J \to \mathbb{R}^d)_{n=1}^{\infty}$ be defined such that $\lambda_0(t) = x_0$, and

$$\lambda_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\mathrm{d}s.$$

With connotation to the motivating proposition, we hope that (λ_n) converges to the solution λ_∞ in some notion. As we shall see, the Picard iterates needs to converge to λ_∞ uniformly in order for λ_∞ to be a solution.

Proposition 5. Given an IVP, if the Picard iterates $(\lambda_n)_{n=0}^{\infty}$ converges uniformly to λ_∞, then λ_∞ is a solution to the IVP.

[1] We recall that the integral of a vector valued function is simply the integral of the components.

Proof. Consider the following chain of equalities,

$$\begin{aligned}\lambda_\infty(t) = \lim_{n\to\infty} \lambda_{n+1}(t) = \lim_{n\to\infty} x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\mathrm{d}s \\ = x_0 + \int_{t_0}^{t} f(s, \lim_{n\to\infty} \lambda_n(s))\mathrm{d}s = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\mathrm{d}s,\end{aligned}$$

where the third equality is true as $\lambda_n \to \lambda_\infty$ uniformly (see first year analysis for proof). Thus, by proposition 3, λ_∞ solves the IVP. □

With this proposition in mind, it is *sometimes* possible to show that a particular Picard iterates converges uniformly towards some function resulting in a solution to the corresponding IVP.

2.2 Lipschitz Continuity

We recall from first year analysis the definition of Lipschitz continuity. We are interested in Lipschitz continuity since it is very helpful when showing the existence and uniqueness of solutions of IVPs. Indeed, by reformulating the Picard iterates as an operator P on the Banach space $\mathcal{C}^\circ(J, \mathbb{R}^d)$, we find that P is a contraction (i.e. P has Lipschitz constant < 1) if f satisfies certain Lipschitz condition, and hence, by Banach's fixed point theorem, has a fixed point.

We recall that a normed vector space is a vector space V equipped with a norm $\|\cdot\|$ such that the norm satisfies positive definiteness, absolute homogeneity and the triangle inequality. Indeed, we recall the chain of induced structure: Inner product space $\implies$ Normed vector space $\implies$ (Metric space $\implies$ Topological space) $\vee$ Vector space.

Definition 2.2 (Lipschitz Continuity)**.** Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be two normed vector spaces and suppose $X \subset V$ and $Y \subseteq W$, then $f : X \to Y$ is Lipschitz continuous if there exists some $K > 0$ (a Lipschitz constant) such that for all $x, y \in X$,

$$\|f(x) - f(y)\|_W \leq K\|x - y\|_V.$$

We recall the sufficient conditions for which a function is Lipschitz continuous.

Proposition 6. Let $f : I \to \mathbb{R}$ be differentiable with bounded derivative on some interval I, then f is Lipschitz continuous.

Proof. By MVT, for all $x, y \in I$, there exists some $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

So by taking absolute value on both sides, and by choosing the Lipschitz constant as the supremum of $|f'|$ we have found some $K := \sup_{c\in I} \|f'(c)\|$, so

$$|f(x) - f(y)| \leq K\|x - y\|.$$

□

We see that, if f is continuously differentiable, f' is continuous on the compact set I, and so f' is uniformly continuous, and so is bounded. Thus, if $f : I \to \mathbb{R}$ is continuously differentiable, then it is Lipschitz continuous.

For higher dimensions, we require the mean value inequality for higher dimensions and so an similar result is achieved. To look at the mean value theorem for higher dimensions, let us first introduce the operator norm for linear maps.

Definition 2.3 (Operator Norm)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then the operator norm of f is

$$\|f\|_{\text{op}} := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|f(x)\|}{\|x\|} \equiv \sup_{x \in \mathbb{R}^n, \|x\|=1} \|f(x)\|,$$

where $\|\cdot\|$ are the Euclidean norms.

As we saw in linear algebra last term, the operator norm form norms on the space of linear maps as the name suggests. By considering that the set $S_1 := \{x \in \mathbb{R}^n, \|x\| = 1\}$ is closed and bounded, and hence compact by Heine-Borel, as linear maps are continuous, f attains its maximum as on S_1, so we may in fact write the operator norm as

$$\|f\|_{\text{op}} := \max_{x \in \mathbb{R}^n, \|x\|=1} \|f(x)\| < \infty.$$

Proposition 7. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $x \in \mathbb{R}^n$, then,

$$\|f(x)\| \leq \|f\|_{\text{op}} \|x\|.$$

Proof. If $x = 0$ then both sides are zero so suppose otherwise. If $x \neq 0$, then $\|x\| > 0$ and so it suffices to show

$$\frac{\|f(x)\|}{\|x\|} \leq \|f\|_{\text{op}}.$$

But, this follows directly from the definition so we are done. □

Proposition 8. If $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, then f is Lipschitz continuous with the Lipschitz constant $\|f\|_{\text{op}}$.

Proof. This follows as, for all $x, y \in \mathbb{R}^n$

$$\|f(x) - f(y)\| = \|f(x - y)\| \leq \|f\|_{\text{op}} \|x - y\|.$$

□

Definition 2.4 (Closed Line Segment)**.** For convenience, for all $x, y \in \mathbb{R}^n$, we define the closed line segment

$$[x, y] := \{\alpha x + (1 - \alpha) y \mid \alpha \in [0, 1]\}.$$

Theorem 1 (Mean Value Inequality)**.** Let $D \subseteq \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^m$ be a function which is continuously differentiable. Then, for all $x, y \in D$ with $[x, y] \subseteq D$, there exists some $\xi \in [x, y]$ such that

$$\|f(x) - f(y)\| \leq \|f'(\xi)\|_{\text{op}} \|x - y\|.$$

Proof. Let $g : [0,1] \to \mathbb{R}^m : \alpha \mapsto f(\alpha x + (1-\alpha)y)$ and we see that g is continuously differentiable as it is the composition of two continuous differentiable functions. Consider,

$$\|f(x) - f(y)\| = \|g(1) - g(0)\| = \left\| \int_0^1 g'(\alpha) \mathrm{d}\alpha \right\|.$$

By the chain rule, we have $g'(\alpha) = (x-y)f'(\alpha x + (1-\alpha)y)$ and so,

$$\begin{aligned}
\left\| \int_0^1 g'(\alpha) \mathrm{d}\alpha \right\| &= \left\| \int_0^1 f'(\alpha x + (1-\alpha)y)(x-y) \mathrm{d}\alpha \right\| \\
&= \int_0^1 \|f'(\alpha x + (1-\alpha)y)(x-y)\| \mathrm{d}\alpha \\
&\leq \|x-y\| \int_0^1 \|f'(\alpha x + (1-\alpha)y)\|_{\mathrm{op}} \mathrm{d}\alpha \\
&\leq \|x-y\| \max_{\alpha \in [0,1]} \|f'(\alpha x + (1-\alpha)y)\|_{\mathrm{op}}.
\end{aligned}$$

So, by defining $\xi = \alpha x + (1-\alpha)y$ where α maximises $\|f'(\alpha x + (1-\alpha)y)\|_{\mathrm{op}}$, we have

$$\|f(x) - f(y)\| \leq \|f'(\xi)\|_{\mathrm{op}} \|x-y\|.$$

□

We note that in the proof above, we assumed $\| \int f \| \leq \int \|f\|$. We shall prove this claim now.

Proposition 9. Let $I \subseteq \mathbb{R}$ be a interval and let $f : I \to \mathbb{R}^m$ be continuous. Then,

$$\left\| \int_{t_0}^{t_1} f(s) \mathrm{d}s \right\| \leq \left| \int_{t_0}^{t_1} \|f(s)\| \mathrm{d}s \right|,$$

for all $t_0, t_1 \in I$.

Proof. We use Riemann rather than Darboux sums for this occasion. Since, as we have shown in first year analysis that Riemann and Darboux sums are equivalent, this does not matter.

Wlog. assume $t_0 < t_1$, and let us consider the n-th Riemann sum of $\left\| \int_{t_0}^{t_1} f(s) \mathrm{d}s \right\|$,

$$\left\| \frac{t_1 - t_0}{n} \sum_{i=0}^{n-1} f\left(t_0 + \frac{i}{n}(t_1 - t_0)\right) \right\| \leq \frac{t_1 - t_0}{n} \left\| \sum_{i=0}^{n-1} f\left(t_0 + \frac{i}{n}(t_1 - t_0)\right) \right\|.$$

Thus, by taking $n \to \infty$, the inequality is achieved. □

Now, with the mean value inequality under our belt, we may generalise the result about Lipschitz continuity for higher dimensions.

Corollary 1.1. Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ be continuously differentiable and $C \subseteq U$ be compact and convex. Then, $f\,|_C : C \to \mathbb{R}^m$ is Lipschitz continuous.

Proof. Follows straight away from the mean value inequality. □

We note the restriction to a convex subset C since, if otherwise, for all $x, y \in \mathbb{R}^n$, it is not necessarily true that $[x, y] \subseteq U$, and so, the mean value inequality does not apply.

2.3 Picard-Lindelöf Theorem

We have now come to a very important theorem in this course – the Picard-Lindelöf theorem. The Picard-Lindelöf theorem is a strong statement providing us the existence and uniqueness of solutions to particular IVPs.

As we have seen previously, we shall approach this using the Banach's fixed point theorem. By considering the Banach space $C^\circ(J, \mathbb{R}^d)$ equipped with the supremum norm, and by showing a particular mapping is a contraction, we may apply Banach's fixed point theorem resulting in the existence of a solution to the IVP.

Theorem 2 (Picard-Lindelöf Theorem (global version))**.** Consider a nonautonomous differential equation

$$\dot{x} = f(t, x),$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfy the global Lipschitz condition, that is, there exists some $K \in \mathbb{R}^+$, such that,

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\|,$$

for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$. Then, the IVP

$$\dot{x} = f(t, x); \;\; x(t_0) = x_0,$$

has a unique solution on $[t_0 - h, t_0 + h]$, given by $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$, where $h := 1/2K$.

To prove this theorem, let us first prove the following lemma in which we shall simply use the same notations as established for convenience.

Lemma 2.1. Let X be the Banach space $C^\circ([t_0 - h, t_0 + h], \mathbb{R}^d)$ equipped with the supremum norm $\|\cdot\|_\infty$. Then the function,

$$P : X \to X : \lambda \mapsto \left(t \mapsto x_0 + \int_{t_0}^{t} f(s, \lambda(s))\mathrm{d}s\right)$$

is a contraction on X.

Proof. To show P is a contraction, it suffices to show that P is Lipschitz continuous with some Lipschitz constant $K \in \mathbb{R}^+_{<1}$; in this case, we shall show P has Lipschitz constant $1/2$.

Let $u_1, u_2 \in X$, by the definition of the suprmum norm, it suffices to show that for all $t \in [t_0 - h, t_0 + h]$,

$$\|P(u_1)(t) - P(u_2)(t)\| \leq \frac{1}{2}\|u_1 - u_2\|_\infty.$$

This follows as,

$$\begin{aligned}
\|P(u_1)(t) - P(u_2)(t)\| &= \left\|\int_{t_0}^{t} f(s, u_1(s))\mathrm{d}s - \int_{t_0}^{t} f(s, u_2(s))\mathrm{d}s\right\| \\
&= \left\|\int_{t_0}^{t} f(s, u_1(s)) - f(s, u_2(s))\mathrm{d}s\right\| \\
&\leq \left|\int_{t_0}^{t} \|f(s, u_1(s)) - f(s, u_2(s))\|\mathrm{d}s\right|
\end{aligned}$$

Since f satisfy the Lipschitz condition, $\|f(s, u_1(s)) - f(s, u_2(s))\| \leq K\|u_1(s) - u_2(s)\| \leq K\|u_1 - u_2\|_\infty$ and so,

$$\begin{aligned}\left|\int_{t_0}^{t} \|f(s, u_1(s)) - f(s, u_2(s))\| \mathrm{d}s\right| &\leq K\|u_1 - u_2\|_\infty \left|\int_{t_0}^{t} \mathrm{d}s\right| \\ &\leq Kh\|u_1 - u_2\|_\infty = \frac{1}{2}\|u_1 - u_2\|_\infty.\end{aligned}$$

□

With that, we can apply the Banach fixed point theorem.

Proof. (Picard-Lindelöf theorem). Let P to be the function as before, then by the Banach fixed point theorem, there exists a unique $\lambda \in X$ such that $P(\lambda) = \lambda$. So, λ is the unique local solution to our IVP. □

While the theorem only provides us with a local solution, the solution can be easily extended on the whole space by reapplying the theorem at the end points of the local unique solution (see problem sheet).

We remark that we have proven the global version of the Picard-Lindelöf theorem in contrast to the local version. The global version requires f is globally defined and is also the Lipschitz condition is held globally; these conditions will be relaxed for the local version. Indeed, it is not difficult to see that the global Lipschitz condition is too strong by considering the differential equations such as $\dot{x} = tx^2$.

Let us now differentiate between the functions that are globally and locally Lipschitz continuous.

Definition 2.5 (Globally Lipschitz Continous)**.** $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is said to be globally Lipschitz continuous with respect to x if there exists some $K > 0$ such that for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$,

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\|.$$

Definition 2.6 (Locally Lipschitz continuous)**.** $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is said to be locally Lipschitz continuous with respect to x if for all $(t_0, x_0) \in D$, there exists some neighbourhood U of (t_0, x_0) and some $K > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\|,$$

for all $(t, x), (t, y) \in U$.

Theorem 3 (Picard-Lindelöf Theorem (local version))**.** Let $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be open and $f : D \to \mathbb{R}^d$ be continuous and locally Lipschitz continuous. Then, for all $(t_0, x_0) \in D$, the IVP

$$\dot{x} = f(t, x); \quad x(t_0) = x_0$$

has a unique solution on an interval of the form $[t_0 - h, t_0 + h]$ where $h = h(t_0, x_0)$ (Qualitative version). Furthermore, if we denote U for the neighbourhood of (t_0, x_0) on which the Lipschitz condition is held with the Lipschitz constant $K > 0$, by defining

$$W^{\tau,\delta}(t_0, x_0) := [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)} \subseteq U,$$

for some sufficent $\tau, \delta > 0$, and $M :\geq \|f(t,x)\|$ on $W^{\tau,\delta}(t_0,x_0)$, there exists a unique solution on the interval $[t_0 - h, t_0 + h]$, where $h := \min\{\tau, 1/2K, \delta/M\}$ (Quantitative version).

Proof. See extra meterials. $\square$

Proposition 10. Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \Omega \to \mathbb{R}^d$ be continuously differentiable. Then f is locally Lipschitz continuous.

Proof. For al $x \in \Omega$, let $U \subseteq \Omega$ be a compact set containing x (U exists as Ω is open, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq \Omega$, and so, we can simply take $U = \overline{B_{\delta/2}(x)}$). Then, since f' is continuous, it is bounded by some $M > 0$. Thus, for all $x, y \in U$, by the mean value inequality, there exists some $\xi \in [x, y]$ such that

$$\|f(x) - f(y)\| \leq \|f'(\xi)\|_{\mathrm{op}}\|x - y\| \leq M\|x - y\|.$$

Thus, f is locally Lipschitz at x on U with the Lipschitz constant M. $\square$

Lemma 2.2. Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and locally Lipschitz where D is open. Then, given two solutions $\lambda : I \to \mathbb{R}^d$ and $\mu : J \to \mathbb{R}^d$ of the differential equation $\dot{x} = f(t,x)$, either $\lambda(t) = \mu(t)$ or $\lambda(t) \neq \mu(t)$ for all $t \in I \cap J$.

Proof. Suppose there exists some $t_0, t_1 \in I \cap J$ such that $\lambda(t_0) = \mu(t_0)$ and $\lambda(t_1) \neq \mu(t_1)$, and furthermore, Wlog. assume $t_1 > t_0$. Then, by defining

$$\tilde{t} := \sup\{t > t_0 \mid \lambda(t') = \mu(t'),\ \forall t' \in [t_0, t]\},$$

by the continuity of λ and μ, $\lambda - \mu$ is continuous, and so, by considering $0 = \lim_{n\to\infty}(\lambda - \mu)(\tilde{t} - 1/n) = (\lambda - \mu)(\tilde{t})$, we have $\lambda(\tilde{t}) = \mu(\tilde{t})$. Then, by applying the local version of the Picard-Lindelöf theorem with the initial value $(\tilde{t}, \lambda(\tilde{t}) = \mu(\tilde{t}))$, we see that there exists two different solutions #. $\square$

As we have seen, the local version of Picard-Lindelöf provides us with a unique solution on some interval around the initial value. We would now like to maximise the size of this interval around the initial value and such solutions are called maximal solutions.

Definition 2.7 (Maximal Existence Interval)**.** Given a initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, we define

$$I_+(t_0, x_0) := \sup\{t \geq t_0 \mid \exists \text{ solution on } [t_0, t]\},$$

$$I_-(t_0, x_0) := \inf\{t \leq t_0 \mid \exists \text{ solution on } [t, t_0]\},$$

and the maximal existence interval,

$$I_{\max}(t_0, x_0) := (I_-(t_0, x_0), I_+(t_0, x_0)).$$

Theorem 4. There exists a maximal solution[2]

$$\lambda_{\max} : I_{\max}(t_0, x_0) \to \mathbb{R}^d$$

[2] Any solution is defined on a subset of $I_{\max}(t_0, x_0)$.

of the IVP $\dot{x} = f(t, x)$, $x(t_0) = x_0$ where f is continuous and locally Lipschitz. Furthermore, if $I_+(t_0, x_0) < \infty$, then either the maximal solution is unbounded for $t \supseteq t_0$, that is,

$$\sup_{t \geq t_0} \|\lambda_{\max}(t)\| = \infty,$$

or $\partial D \neq \varnothing$, and

$$\lim_{t \uparrow I_+(t_0, x_0)} \operatorname{dist}((t, \lambda_{\max}(t)), \partial D) = 0,$$

where $\operatorname{dist}(y, A) := \inf\{\|x - y\| \mid x \in A\}$. Analogous case for $I_-(t_0, x_0) > -\infty$.

Proof. Let $\bar{t} \in I_{\max}(t_0, x_0)$, then, by the definition of the maximal existence interval, there exists some solution $\mu : I \to \mathbb{R}^d$ such that $\bar{t}, t_0 \in I$. Now, by considering that the solutions cannot cross each other, we see that all solutions of the IVP on $[t_0, \bar{t}]$ must coincide as they crosses at (t_0, x_0). Thus, the definition

$$\lambda_{\max}(\bar{t}) := \mu(\bar{t})$$

is well-defined. Indeed, $\lambda_{\max}$ is a solution at $\bar{t}$ since

$$\dot{\lambda}_{\max}(\bar{t}) = \dot{\mu}(\bar{t}) = f(\bar{t}, \mu(\bar{t})) = f(\bar{t}, \lambda_{\max}(\bar{t})).$$

Now, by the construction of $I_{\max}$, the only possible case on which the the theorem does not hold is that $I_+(t_0, x_0) \in I_{\max}(t_0, x_0)$. So, for contradiction, assume that there exists some solution $\mu : [t, I_{\max}(t_0, x_0)] \to \mathbb{R}^d$ to the IVP. However, by the local version of Picard-Lindelöf, there exists some open U containing $I_{\max}(t_0, x_0)$ such that there exists a unique solution η of the IVP on the interval $(I_{\max}(t_0, x_0) - h, I_{\max}(t_0, x_0) + h)$, and hence, we may extend μ to be a solution on $[t, I_{\max}(t_0, x_0) + h]$ contradicting the maximum condition #.

For the second part of the theorem, assume $I_+(t_0, x_0) < \infty$. Then, we may find some sequence $(t_n)_{n=1}^{\infty} \subseteq D$ that converges to $I_-(t_0, x_0)$ and assume, for contradiction, there exists some $M > 0$, such that $\|\lambda_{\max}(t_n)\| \leq M$ and $\operatorname{dist}((t_n, \lambda_{\max}(t_n)), \partial D) \geq 1/M$ for all $n \in \mathbb{N}$.

By assumption, we have $(t_n, \lambda_{\max}(t_n))$ is a bounded sequence, and so, by Bolzano-Weierstrass, it has a convergent subsequence $(t_{n_i}, \lambda_{\max}(t_{n_i}))$ which converges to $(t^*, x^*) \in D$. Now, on the problem sheet, we shall see that, the solution is uniform on a neighbourhood of (t^*, x^*). But, this contradicts the maximal condition, so we are done! $\square$

2.4 General Solution and Flows

By the existence of the maximal solution, we find there to be a general solution for all initial pairs.

Proposition 11. Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and locally Lipschitz. Then there exists a maximal solution $\lambda_{\max}$ to the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

for every initial pair $(t_0, x_0) \in D$.

Proof. Follows straight away from the existence of maximal solutions. □

In the case as described in the proposition, we can define $\lambda : D \times \mathbb{R} \to \mathbb{R}^d$ such that $\lambda((t_0, x_0), t)$ is the maximal solution to $\dot{x} = f(t, x)$ and $x(t_0) = x_0$. This λ is called the general solution. Commonly, we shall flatten the arguments of λ such that $\lambda = \lambda(t, t_0, x_0)$.

Proposition 12. Let $\dot{x} = f(t, x)$ be a differential equation with the general solution λ. Then, given $(t_0, x_0) \in D$,

- $I_{\max}(s, \lambda(s, t_0, x_0)) = I_{\max}(t_0, x_0)$;
- $\lambda(t_0, t_0, x_0) = x_0$;
- $\lambda(t, s, \lambda(s, t_0, x_0)) = \lambda(t, t_0, x_0)$.

Proof. The second statement is immediate since the general solution satisfies the IVP so let us consider the two other claims.

Consider the initial pair $(s.\lambda(s, t_0, x_0))$. Then, we see that both $\lambda(\cdot, t_0, x_0)$ and $\lambda(\cdot, s, \lambda(s, t_0, x_0))$ coincides at $(s, \lambda(s, t_0, x_0))$. Now, since the maximal solution is unique, the results follow. □

In the case that the differential equation is autonomous, we find the the general solution contains redundancy due to translational invariant. So, this leads us to the concept of *flow*.

Let $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ be continuous and locally Lipschitz and suppose λ is a general solution to the differential equation $\dot{x} = f(x)$. Then, if $t_0 \in \mathbb{R}$ and $x_0 \in D$ is fixed, then $\lambda(\cdot, t_0, x_0)$ solves the IVP

$$\dot{x} = f(x), \quad x(t_0) = x_0.$$

However, consider $\lambda(\cdot, 0, x_0)$ is a solution to the IVP is initial condition $x(0) = x_0$, we may translate the solution so that

$$\lambda^*(t) = \lambda(t - t_0, 0, x_0),$$

which is also a solution to the IVP with initial condition $x(t_0) = x_0$. So, since the solution is unique, we see that

$$\lambda(t - t_0, 0, x_0) = \lambda(t, t_0, x_0).$$

So, since the left hand equation contains only two variable, we conclude that the general solution has redundant information.

Definition 2.8 (Flow)**.** Let $\dot{x} = f(x)$ with $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$. For all $x_0 \in D$ let us define

$$J_{\max}(x_0) := I_{\max}(0, x_0),$$

and

$$\phi(t, x_0) := \lambda(t, 0, x_0),$$

then ϕ is called the flow of the autonomous differential equation $\cdot x = f(x)$.

Proposition 13. Let ϕ be a flow of $\dot{x} = f(x)$ where $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is continuous and locally Lipschitz. Then for $x \in D$, we have

- $J_{\max}(\phi(t, x)) = J_{\max}(x) - t$;

- $\phi(0, x) = x$;
- $\phi(t, \phi(s, x)) = \phi(t + s, x)$;
- $\phi(-t, \phi(t, x)) = x$.

Proof. Easy as before. $\square$

Definition 2.9 (Orbit)**.** Let ϕ be a flow of $\dot{x} = f(x)$ where $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is continuous and locally Lipschitz. Then, given $x \in D$, the orbit (or trajectory) through x is

$$\sigma(x) := \{\phi(t, x) \in D \mid t \in J_{\max}(x)\}.$$

Furthermore, the positive half-orbit and the negative half-orbit through x are

$$\sigma^+(x) := \{\phi(t, x) \in D \mid t \in J_{\max}(x) \cap \mathbb{R}^+\},$$

$$\sigma^-(x) := \{\phi(t, x) \in D \mid t \in J_{\max}(x) \cap \mathbb{R}^-\},$$

respectively.

We would like to classify the different possible orbits and it turns out there are three possible orbits.

- $\sigma(x)$ is a singleton. This occurs when $f(x) = 0$ and $J_{\max}(x) = \mathbb{R}$ and we call x an equilibrium of the equation;
- $\sigma(x)$ is a closed curve. That is there exists $t > 0$ such that $\phi(t, x) = x$ and $f(x) \neq 0$. In this case $J_{\max}(x) = \mathbb{R}$ and we call x is called a periodic orbit;
- $\sigma(x)$ is not a closed curve nor a singleton. In this case, the solution does not necessarily exist for all time, however, we see that the map $t \mapsto \phi(t, x)$ is injective.

Lastly, let us consider the continuity and differentiability of general solutions and flows. Indeed, by construction, the general solution and flow must be differentiable and so, is continuous, however, we would like to also consider continuity with respect to the other arguments. In this course, we will state without proof that both the general solution and flow are continuous functions under the standard assumptions while this will be further investigated in more advanced modules.