# Differentiable Equations

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# 1 Introduction

While we have seen differential equations in year one, we have mostly focused on the different methods of solving specific differential equations. This cannot be expected for general differential equations and in this year, we will focus on existence and uniqueness of solutions to differential equations and develop qualitative tools to help us understand these solutions.

We recall that an algebraic equation is an equation of the form $f(x) = 0$ while a differential equation is an equation of the form $\dot{x} = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$. That is, an algebraic equation has real numbers as solutions while an differential equation has functions as its solution.

As an example, let us consider the simple differentiable equation

$$\dot{x} = ax, \tag{1}$$

for some $a \in \mathbb{R}$. Then, a function $\lambda : I \to \mathbb{R}$ solves 1 if $\dot{\lambda} = a\lambda$ for all $t \in I$ where $I \subseteq \mathbb{R}$ is a interval. These types of differentiable equations occurs often in relation in growth and decay and one can easily see that the family of functions

$$\lambda_b : \mathbb{R} \to \mathbb{R} = t \mapsto be^{at}, \;\; b \in \mathbb{R},$$

are solutions to 1. Of course, we know this already, so an more interesting question would be whether or not this family contains all the solutions to 1. It turns out to be true, and to show this we will assume $\mu : I \to \mathbb{R}$ is a solution to $\dot{x} = ax$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu e^{-at}\right) = \dot{\mu} e^{-at} - a\mu e^{-at} = 0,$$

since $\dot{\mu} = a\mu$ and so, $\mu e^{-at}$ is constant, i.e. there exists $b \in \mathbb{R}$ such that $\mu e^{-at} = b$ and hence,

$$\mu = be^{at}.$$

This demonstrates that all solutions to 1 are members of the aforementioned solution family and hence, we have found **all** of the solutions to 1.

With the above example, we see that rather than working with solutions that are in finite-dimensional vector spaces, our solution are in function spaces which are typically infinite-dimensional. This is studied in more detail in the next year's *functional analysis* course, and in general, infinite-dimensional spaces are more difficult to grasp. However, for the vast majority of materials in this course, a finite-dimensional thinking suffices while we will also cover some material from functional analysis to understand the differentiable equations as well.

## 1.1 Ordinary Differential Equations and Initial Value Problems

There are two types of differential equations – *autonomous differential equations* and *nonautonomous differentiable equations*. Autonomous differential equations are differentiable equations of the form $\dot{x} = f(x)$ such as equation 1 while nonautonomous differential equations are equations of the form $\dot{x} = f(t, x)$.

We note that this does not cover higher-order differential equations, but from last year, we recall that one may reduce a higher-order differential equations into a first-order differential equation in vector form and thus, the theories we develop within this course will also apply to higher-order differential equations.

**Definition 1.1** (Ordinary Differential Equation)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be open, and a function $f \to D \to \mathbb{R}^d$. Then, an equation of the form

$$\dot{x} = f(t, x)$$

is called a $d$-dimensional (first-order) ordinary differential equation.

A differentiable function $\lambda : I \to \mathbb{R}^d$ on some interval $I \subseteq \mathbb{R}$ is called a solution of the differential equation if and only if for all $(t, \lambda(t) \in D$, if $t \in I$ then,

$$\dot{\lambda}(t) = f(t, \lambda(t)).$$

We say that an ordinary differential equation is autonomous if $f$ is independent of $t$ and nonautonomous otherwise.

We will only consider ordinary differential equations (ODE) in this course while partial differential equations, that is differential equations which solutions are functions which depends on multiple variables are covered in the second year course **Partial Differential Equations in Action**.

**Proposition 1** (Constant solutions to autonomous differential equations)**.** Let $D \subseteq \mathbb{R}^d$ be an open set and $f : D \to \mathbb{R}^d$ be a function where $d \in \mathbb{N}$. Then, there exists a constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ to the autonomous differential

$$\dot{x} = f(x)$$

for some $a \in \mathbb{R}^d$ if and only if $f(a) = 0$.

*Proof.* ( $\Longrightarrow$ ) Suppose that $\lambda : I \to \mathbb{R}^d : x \mapsto a$ is a solution the $\dot{x} = f(x)$. Then

$$0 = \dot{\lambda}(t) = f(\lambda(t)) = f(a).$$

( $\Longleftarrow$ ) Suppose there exists some $a \in \mathbb{R}^d$ such that $f(a) = 0$, then verifying, we find $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to the differential equation. $\square$

This proposition allows us to find solutions to many autonomous ODEs as, indeed, if $f : D \to \mathbb{R}^d$ has a root $a \in \mathbb{R}^d$, the above proposition guarantees that $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to $\dot{x} = f(x)$.

**Definition 1.2** (Initial Value Problem)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be an open set, and $f : D \to \mathbb{R}^d$ be a function. The system of equations from combining the differential equation

$$\dot{x} = f(t, x),$$

with the initial condition

$$x(t_0) = x_0$$

where $(t_0, x_0) \in D$ is called an initial value problem.

A solution to the above initial value problem is a function $\lambda : I \to \mathbb{R}^d$ that is a solution to the differential equation $\dot{x} = f(t, x)$ and $\lambda(t_0) = x_0$.

While, previously, we have seen a differential equation which always has a solution. This, however, is not always the case.

**Example 1.** Consider the differential equation $\dot{x} = f(x)$, where

$$f(x) = \begin{cases} 1, & x < 0; \\ -1, & x \geq 0, \end{cases}$$

with the boundary condition $x(0) = 0$. As we will see on the problem sheet, this differential equation indeed does not have a solution.

**Example 2.** Consider the initial value problem $\dot{x} = f(x) = \sqrt{|x|}$ with the boundary condition $x(0) = 0$.

Since $f(0) = 0$ we have $x(t) = 0$ is a constant solution by proposition 1. Furthermore, by consider the function

$$\lambda_b(t) = \begin{cases} 0, & t \leq b; \\ \frac{1}{4}(t-b)^2, & t > b, \end{cases}$$

we find $\lambda_b$ to also be a solution for any $b \in \mathbb{R}_0^+$. However, as for all $t \leq b$, $\lambda_b(t) = 0$, we see that $x$ is not unique given values of $x$ at $t$.

Before moving on, let us quickly recall the *separation of variables* procedure for solving differential equations. Suppose we are to solve a differential equation of the form,

$$\dot{x} = g(t)h(x),$$

with the boundary condition $x(t_0) = x_0$ where $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are continuous functions. Then, we have

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{h(y)} = \int_{t_0}^{t} g(s)\mathrm{d}s.$$

Thus, by evaluating the integral, we can express $x$ in $t$. We see that this procedure does indeed provide us with a correct solution by simply applying FTC on both sides of the equation.

**Example 3.** Consider the initial value problem $\dot{x} = tx^2$ with the boundary condition $x(t_0) = x_0$ where $x_0 \neq 0$. By the separation of variables, we find

$$x = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}.$$

However, we see that for this solution, $x$ does not necessarily exist for all $t$. Indeed, if $t_0 = 0, x_0 = 1$, we find

$$x = \frac{2}{2 - t^2},$$

which does not have a solution for $t = \pm\sqrt{2}$, and so, the solution does not exist globally.

## 1.2 Visualisations

Visualisations are very important for differential equations as it provides us a mental image of how to think about differential equations. In general, there are two main methods to visualise differential equations:

- nonautonomous differential equations $\dot{x} = f(t, x)$ via the *solution portrait* in the extended phase space;
- autonomous differential equations $\dot{x} = f(x)$ via the *phase portrait* in the phase space.

Suppose we have the nonautonomous differential equations $\dot{x} = f(t, x)$ where $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, have a solution $\lambda : I \to \mathbb{R}^d$, i.e. for all $t \in I$, $\dot{\lambda}(t) = f(t, \lambda(t))$. Thus, we see that the vector $(1, f(t_0, \lambda(t_0))$ for some $t_0 \in I$ is tangential to the solution which passes through $(t_0, \lambda(t_0))$. This can be done for all $p \in \mathbb{R} \times \mathbb{R}^d$ and so, by drawing these vectors at a sufficient number of points, we can have a mental image of what the solution looks like. A plot of these vectors is refereed to as a **vector field**.

A **solution portrait** is given by a visualisation of several solution curves in the $(t, x)$-space, the so called *extended phase space*. This is called as such since the $x$-space is normally refereed as the phase space, and so, we are extending it by the time axis.

On the other hand, suppose we have the autonomous differential equation $\dot{x} = f(x)$ where $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ has a solution $\lambda : I \to \mathbb{R}^d$. By considering that $f$ is independent of $t$, we see that translating $\lambda$ along $t$ is also a valid solution. Indeed, at $x = \lambda(t_0)$, if $\lambda'$ is another solution such that $\lambda'(t_1) = \lambda(t_0)$, then $\dot{\lambda}'(t_1) = \dot{\lambda}(t_0)$. This property is refereed to as *translation invariance* and all solutions of autonomous differential equations are translation invariant.

**Proposition 2.** let $\dot{x} = f(x)$ be an autonomous differential equation. Then, if $\lambda : I \to \mathbb{R}^d$ is a solution to this differential equation, so is

$$\mu : \bar{I} \to \mathbb{R}^d : t \mapsto \lambda(t + \tau),$$

where $\tau \in \mathbb{R}$, $\bar{I} := I + \tau$.

*Proof.* Follows straight away by chain rule. $\square$

**Example 4.** Consider the harmonic oscillator $\ddot{x} = -x$. By converting it into a first order differential equation, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}.$$

By solving the system, we find

$$\lambda(t) = \begin{bmatrix} \cos(t) \sin(t) \\ -\sin(t) \cos(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is a solution. Indeed, this describes a oscillatory motion in the phase space where the position is circular.

With consideration with the example above, we see that we may project a solution portrait $(t, x) \mapsto (1, f(x))$ onto $x \mapsto f(x)$ resulting in a **phase portrait**.

# 2 Existence and Uniqueness

As we have seen, differential equations need not have unique solutions given an initial value. Indeed, we have also seen that it is not guaranteed to have a solution at all. We will in this chapter resolve the question on whether or not a solution exists by presenting a theory that guarantees existence and uniqueness for solutions to initial value problems.

## 2.1 Picard iterates

We observe that often times, to solve an differential equation, we need to reformulate it as an integral equation.

**Proposition 3.** Consider the initial value problem

$$\dot{x} = f(t, x); \quad x(t_0) = x_0,$$

where $f : D \subseteq \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let $\lambda : I \to \mathbb{R}^d$, $\lambda$ solves the IVP if and only if $\lambda$ is continuous and solves the integral equation[1]

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\mathrm{d}s.$$

*Proof.* Follows from FTC. □

However, as $f$ depends on $\lambda$, we in general cannot compute the integral on the right hand side. Nonetheless, this brings us closer to the formulation of the Picard iterates.

**Proposition 4.** Let $f$ be a continuous function. Then by defining $a_0$ for some value and $a_{n+1} = f(a_n)$, if $(a_n)_{n=1}^{\infty}$ converges to some value $a$, then $a = f(a)$.

*Proof.* We see that if $(a_n)_{n=1}^{\infty}$ converges to some value $a$, $f(a_n) \to f(a)$ by sequential continuity. However, as $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = a$, by uniqueness of limits on Hausdorff spaces, we have $a = f(a)$. □

With this proposition in mind, we may apply a similar iteration on the integral equation resulting in the Picard iterates.

**Definition 2.1** (Picard Iterates)**.** Given an IVP, the Picard iterates is the sequence of functions $(\lambda_n : J \to \mathbb{R}^d)_{n=1}^{\infty}$ be defined such that $\lambda_0(t) = x_0$, and

$$\lambda_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\mathrm{d}s.$$

With connotation to the motivating proposition, we hope that $(\lambda_n)$ converges to the solution $\lambda_\infty$ in some notion. As we shall see, the Picard iterates needs to converge to $\lambda_\infty$ uniformly in order for $\lambda_\infty$ to be a solution.

**Proposition 5.** Given an IVP, if the Picard iterates $(\lambda_n)_{n=0}^{\infty}$ converges uniformly to $\lambda_\infty$, then $\lambda_\infty$ is a solution to the IVP.

[1] We recall that the integral of a vector valued function is simply the integral of the components.

*Proof.* Consider the following chain of equalities,

$$\begin{aligned}\lambda_\infty(t) = \lim_{n\to\infty} \lambda_{n+1}(t) = \lim_{n\to\infty} x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\mathrm{d}s \\ = x_0 + \int_{t_0}^{t} f(s, \lim_{n\to\infty} \lambda_n(s))\mathrm{d}s = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\mathrm{d}s,\end{aligned}$$

where the third equality is true as $\lambda_n \to \lambda_\infty$ uniformly (see first year analysis for proof). Thus, by proposition 3, $\lambda_\infty$ solves the IVP. $\square$

With this proposition in mind, it is *sometimes* possible to show that a particular Picard iterates converges uniformly towards some function resulting in a solution to the corresponding IVP.

## 2.2 Lipschitz Continuity

We recall from first year analysis the definition of Lipschitz continuity. We are interested in Lipschitz continuity since it is very helpful when showing the existence and uniqueness of solutions of IVPs. Indeed, by reformulating the Picard iterates as an operator $P$ on the Banach space $\mathcal{C}^\circ(J, \mathbb{R}^d)$, we find that $P$ is a contraction (i.e. $P$ has Lipschitz constant $< 1$) if $f$ satisfies certain Lipschitz condition, and hence, by Banach's fixed point theorem, has a fixed point.

We recall that a normed vector space is a vector space $V$ equipped with a norm $\|\cdot\|$ such that the norm satisfies positive definiteness, absolute homogeneity and the triangle inequality. Indeed, we recall the chain of induced structure: Inner product space $\implies$ Normed vector space $\implies$ (Metric space $\implies$ Topological space) $\vee$ Vector space.

**Definition 2.2** (Lipschitz Continuity)**.** Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be two normed vector spaces and suppose $X \subset V$ and $Y \subseteq W$, then $f : X \to Y$ is Lipschitz continuous if there exists some $K > 0$ (a Lipschitz constant) such that for all $x, y \in X$,

$$\|f(x) - f(y)\|_W \leq K\|x - y\|_V.$$

We recall the sufficient conditions for which a function is Lipschitz continuous.

**Proposition 6.** Let $f : I \to \mathbb{R}$ be differentiable with bounded derivative on some interval $I$, then $f$ is Lipschitz continuous.

*Proof.* By MVT, for all $x, y \in I$, there exists some $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

So by taking absolute value on both sides, and by choosing the Lipschitz constant as the supremum of $|f'|$ we have found some $K := \sup_{c\in I} \|f'(c)\|$, so

$$|f(x) - f(y)| \leq K\|x - y\|.$$

$\square$

We see that, if $f$ is continuously differentiable, $f'$ is continuous on the compact set $I$, and so $f'$ is uniformly continuous, and so is bounded. Thus, if $f : I \to \mathbb{R}$ is continuously differentiable, then it is Lipschitz continuous.

For higher dimensions, we require the mean value inequality and so an similar result is achieved.