Lebesgue Measure & Integration

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1 Motivation

We recall from **Analysis I** the definition of the Darboux integral. While this notion of integration was sufficient for our use case last year, as we shall see, there are some limitations with this notion of integration. These limitations will be addressed by the means of measure theory.

Definition 1.1 (Darboux Integrable)**.** A function $f : [a, b] \to \mathbb{R}$ is called Darboux integrable if for any partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ for some $n \geq 1$ if $[a, b]$, by defining the lower and upper Darboux sums,

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} (t_i - t_{i-1}) \inf_{t \in [t_{i-1}, t_i]} f(t),$$

and

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup_{t \in [t_{i-1}, t_i]} f(t),$$

one has

$$\sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

If this is the case we define the integral of f over $[a, b]$ to be this value, i.e.

$$\int_a^b f := \sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

Many functions are Darboux integrable and in fact, as demonstrated last year, all functions in $C^\circ_{pw}([a, b])$, that is piecewise continuous functions on $[a, b]$ are Darboux integrable. Nonetheless, however, the class of Darboux integrable functions is also rather limited.

Consider the Dirichlet function

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

That is, the indicator function for $\mathbb{Q}$. We see that $\mathbf{1}_{\mathbb{Q}}$ is not Darboux integrable since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ and so, for any partition $\mathcal{P}$ of $[a, b]$, $L(\mathbf{1}_{\mathbb{Q}}, \mathcal{P}) = 0$ while $U(\mathbf{1}_{\mathbb{Q}}, \mathcal{P}) = 1$. This is not ideal, since, as $\mathbb{Q}$ is countable while $\mathbb{R} \setminus \mathbb{Q}$ is not, we intuitively expect that a satisfactory theory of integration would assign $\int_a^b \mathbf{1}_{\mathbb{Q}} = 0$.

Moreover, by defining $P = (q_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}$ be some enumeration of $\mathbb{Q} \cap [a, b]$, we can define the following sequence of functions,

$$f_n(x) := \begin{cases} 1, & x \in \{q_0, \cdots, q_n\}; \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that $\int_a^b f_n = 0$ for all n and $f_n \to \mathbf{1}_{\mathbb{Q}}$ pointwise. However, this implies

$$0 = \lim_{n \to \infty} \int_a^b f_n \neq \int_a^b \lim_{n \to \infty} f_n = \int_a^b \mathbf{1}_{\mathbb{Q}},$$

and in fact, the right hand side is not even defined (as $\mathbf{1}_{\mathbb{Q}}$ is not Darboux integrable)!

To solve this issue we will introduce the notion of the Lebesgue measure and furthermore, its associated Lebesgue integral which extends our Darboux integral such that it has the "nice" properties we desire.

We will in this course also look at L^p spaces. From the perspective of analysis, it is often convenient to work in Banach spaces (complete normed vector spaces) such that we can utilise many existing theorems we have proved in **Analysis II**, e.g. Banach's fixed point theorem. For instance, one can endow $C^{\circ}_{pw}([a,b])$ with the (semi-)norm

$$\|f\|_{L^1} := \int_a^b |f| \,.$$

Then, by considering the aforementioned sequence $(f_n) \subseteq C^{\circ}_{pw}([a,b])$, one can easily show that (f_n) is a Cauchy sequence with respect to $\|\cdot\|_{L^1}$. However, $f_n \to \mathbf{1}_{\mathbb{Q}}$ pointwise. This motivates us to introduce the Banach space $L^1([a,b])$ of integrable functions, and more generally, L^p-spaces later in the course.

Lastly, as we have seen within last term's probability module, measure theory lays below as the foundations for probability theory. As a quick reminder, we recall that a probability space is a special type of measure space and random variables defined on these probability spaces are simply measurable functions to $\mathbb{R}$ (or more exotic fields). This can be interpreted with connotations to real world situations in several ways.

2 Abstract Measure Theory

2.1 Measures and Measure Spaces

As we would like an adequate theory to assign a notion of "size" on sets, we need to construct a function from a set of sets to $\overline{\mathbb{R}_0^+}$. To achieve this, the natural idea is to construct a function with domain being the power set, however, this is not necessarily always possible or meaningful. Thus, instead of assigning every subset of some set a size, we only look at some collection of "nice" sets.

Definition 2.1 (Algebra)**.** Let X be some set and suppose we denote $\mathcal{X}$ for the power set of X, then a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is called an algebra over X if

- $X \in \mathcal{A}$;
- for all $A \in \mathcal{A}$, $A^c = X \setminus A \in \mathcal{A}$;
- if $(A_k)_{i=1}^n$ is a finite sequence of sets in $\mathcal{A}$, then $\bigcup_{k=1}^n A_k \in \mathcal{A}$.

Definition 2.2 (σ-algebra)**.** Let X be a set, then a σ-algebra $\mathcal{A}$ on X is an algebra on X such that $\mathcal{A}$ is closed under countable unions, i.e. if $(A_k)_{k=1}^\infty$ is a sequence of sets in $\mathcal{A}$, then $\bigcup_{k=1}^\infty A_k \in \mathcal{A}$.

As σ-algebras (and algebras) are simply sets of sets, there is an induced order on σ-algebras by $\subseteq$. If there are two σ-algebras $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, then we say $\mathcal{A}$ is coarser than $\mathcal{B}$.

Trivially, we find $\{\varnothing, X\}$ is a σ-algebra. Indeed, this is the coarsest σ-algebra. Furthermore, given a set X and a subset $A \subseteq X$, we have $\{\varnothing, A, A^c, X\}$ is also a σ-algebra.

While every σ-algebra is also an algebra, the converse is not true. An counter-example of this is by consider the algebra

$$\mathcal{A} := \{\varnothing\} \cup \left\{ U \mid \exists \bigcup_{k=1}^{m} (a_k, b_k], m \geq 1, 0 \leq a_k < b_k \leq 1 \right\},$$

on $X = (0, 1]$. We see that $\mathcal{A}$ is an algebra (since $(a, b]^c = (0, a] \cup (b, 0]$) however $\mathcal{A}$ is not a σ-algebra on X since we can define the sequence $A_k = (0, 1 - 1/k] \in \mathcal{A}$ but $\bigcup_k A_k = (0, 1) \notin \mathcal{A}$.

Proposition 1. Let $\mathcal{F}$ be an arbitrary collection of σ-algebra (or algebras) over X. Then the intersections

$$\bigcap \mathcal{F} := \bigcap_{\mathcal{A} \in \mathcal{F}} \mathcal{A},$$

is a σ-algebra (or algebra).

Proof. Straight forward by definition. □

With this, we have a notion of infimum on σ-algebras and hence, we can also define a notion closure.

Definition 2.3 (σ-algebra Generated by a set)**.** Let $\mathcal{C} \subseteq \mathcal{P}(X)$, then

$$\sigma(\mathcal{C}) := \bigcap \{\mathcal{A} \mid \mathcal{C} \subseteq \mathcal{A} \wedge \mathcal{A} \in \mathcal{F}\},$$

where $\mathcal{F}$ is the set of all σ-algebras on X.

As previously shown, $\sigma(\mathcal{C})$ is a intersection of σ-algebras, and so the name suggests, the σ-algebra generated by $\mathcal{C}$ is the smallest σ-algebra containing $\mathcal{C}$. Indeed, we find $\sigma(\varnothing) = \{\varnothing, X\}$ and $\sigma(A) = \{\varnothing, A, A^c, X\}$. Moreover, we see that $\mathcal{C}$ is a σ-algebra if and only if $\sigma(\mathcal{C}) = \mathcal{C}$.

Definition 2.4 (Borel σ-algebra)**.** If $(X, \mathcal{T})$ is a topological space, then the Borel σ-algebra over X is

$$B(X) := \sigma(\mathcal{T}).$$

Unlike topologies, the unions and intersections in a σ-algebra is treated symmetrically.

Proposition 2. If $\mathcal{A}$ is a σ-algebra, then if $(A_k)_{k=1}^{\infty}$ is a sequence of sets in $\mathcal{A}$, then

$$\bigcap_{k=1}^{\infty} A_k \in \mathcal{A}.$$

Proof. By considering de Morgen's identity, we have $\bigcap_{k=1}^{\infty} A_k = (\bigcup_{k=1}^{\infty} A_k^c)^c$. So, since $\bigcup_{k=1}^{\infty} A_k^c \in \mathcal{A}$ as each component is, $\bigcup_{k=1}^{\infty} A_k^c)^c$ and hence $\bigcap_{k=1}^{\infty} A_k$ is also in $\mathcal{A}$. □

Definition 2.5 (Measurable Space)**.** A set X equipped with a σ-algebra $\mathcal{A}$ is called a measurable space and is written as a tuple $(X, \mathcal{A})$. Furthermore, if $A \subseteq X$ is in $\mathcal{A}$, then we say A is a measurable set.

Definition 2.6 (Measure)**.** Let $(X, \mathcal{A})$ is a measurable space. Then a measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

- $\mu(\varnothing) = 0$;
- if $(A_k)_{k=1}^{\infty} \subseteq \mathcal{A}$ is a sequence of pairwise disjoint sets, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

.

We call the second property σ-additivity.

Definition 2.7 (Measure Space)**.** A measurable space $(X, \mathcal{A})$ equipped with the measure μ is called a measure space and is written as a triplet $(X, \mathcal{A}, \mu)$.

A commonly used measure on any arbitrary measurable space $(X, \mathcal{A})$ is the counting measure μ. As the name suggests, for all $A \in \mathcal{A}$, $\mu(A) = |A|$ if A is finite and ∞ otherwise. Another example of a measure is the Dirac measure $\delta_x : \mathcal{A} \to [0, \infty]$ for some $x \in X$ where for all $A \in \mathcal{A}$, $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise. For the last example, let X be uncountable and let $\mathcal{A} := \{A \subseteq X \mid A \text{ or } A^c \text{ is uncountable}\}$. Then, one can show that $\{X, \mathcal{A}\}$ forms a measurable space and we find that the function $\mu : \mathcal{A} \to [0, \infty]$ defined as $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable is a measure on $(X, \mathcal{A})$.

Proposition 3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

- if $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$;
- if $n \geq 1$, $(A_k)_{k=1}^n$ is a sequence of pairwise disjoint sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k);$$

- if $(A_k)_{k=1}^\infty$ is a sequence of monotonically increasing sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

We note that the limit in part 3 exists since the limit in monotonically increasing on the extended reals (so bounded by ∞).

- if $(A_k)_{k=1}^\infty$ is a sequence of monotonically decreasing sets in $\mathcal{A}$, if $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

- if $A \in \mathcal{A}$ and $(A_k)_{k=1}^\infty$ is a sequence in $\mathcal{A}$, then $\mu(A) \leq \sum_{k=1}^\infty \mu(A_k)$.

The last property is refereed to as σ-sub-additivity.

Proof. Part 2 is trivial.

(Part 1) As $B = A \sqcup B \setminus A = A \sqcup (A^c \cap B)$ where $A^c \cap B$ is measurable since both A^c and B are. So, by σ-additivity,

$$\mu(B) = \mu(A \sqcup (A^c \cap B)) = \mu(A) + \mu(A^c \cap B) \geq \mu(A).$$

(Part 3) Define $B_1 = A_1$ and $B_{k+1} = A_{k+1} \setminus A_k$. Then, $(B_k)_{k=1}^\infty$ is a sequence of disjoint subset in $\mathcal{A}$. So, by σ-additivity,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \mu(A_k).$$

(Part 4) Define $B_k = A_1 \setminus A_k$, then $(B_k)_{k=1}^\infty$ is a sequence of monotonically increasing sets in $\mathcal{A}$. So, by part 3,

$$\mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_l).$$

Furthermore, as $A \subseteq B$ implies $\mu(B) - \mu(A) = \mu(B \setminus A)$, we have

$$\mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \mu\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \mu(B_k) = \mu(A_1) - \lim_{k\to\infty} \mu(A_k),$$

hence the result.

(Part 5) Exercise. □