Measure Theory

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1 Motivation

We recall from **Analysis I** the definition of the Darboux integral. While this notion of integration was sufficient for our use case last year, as we shall see, there are some limitations with this notion of integration. These limitations will be addressed by the means of measure theory.

Definition 1.1 (Darboux Integraable)**.** A function $f : [a, b] \to \mathbb{R}$ is called Darboux integrable if for any partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ for some $n \geq 1$ if $[a, b]$, by defining the lower and upper Darboux sums,

$$L(f, \mathcal{P}) = \sum_{i=1}^{n} (t_i - t_{i-1}) \inf_{t \in [t_{i-1}, t_i]} f(t),$$

and

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup_{t \in [t_{i-1}, t_i]} f(t),$$

one has

$$\sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

If this is the case we define the integral of f over $[a, b]$ to be this value, i.e.

$$\int_a^b f := \sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P}).$$

Many functions are Darboux integrable and in fact, as demonstrated last year, all functions in $C^{\circ}_{pw}([a, b])$, that is piecewise continuous functions on $[a, b]$ are Darboux integrable. Nonetheless, however, the class of Darboux integrable functions is also rather limited.

Consider the Dirichlet function

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

That is, the indicator function for $\mathbb{Q}$. We see that $\mathbf{1}_{\mathbb{Q}}$ is not Darboux integrable since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ and so, for any partition $\mathcal{P}$ of $[a, b]$, $L(\mathbf{1}_{\mathbb{Q}}, \mathcal{P}) = 0$ while $U(\mathbf{1}_{\mathbb{Q}}, \mathcal{P}) = 1$. This is not ideal, since, as $\mathbb{Q}$ is countable while $\mathbb{R} \setminus \mathbb{Q}$ is not, we intuitively expect that a satisfactory theory of integrable would assign $\int_a^b \mathbf{1}_{\mathbb{Q}} = 0$.

Moreover, by defining $P = (q_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}$ be some enumeration of $\mathbb{Q} \cap [a, b]$, we can define the following sequence of functions,

$$f_n(x) := \begin{cases} 1, & x \in \{q_0, \cdots, q_n\}; \\ 0, & \text{otherwise}. \end{cases}$$

It is not difficult to see that $\int_a^b f_n = 0$ for all n and $f_n \to \mathbf{1}_{\mathbb{Q}}$ pointwise. However, this implies

$$0 = \lim_{n \to \infty} \int_a^b f_n \neq \int_a^b \lim_{n \to \infty} f_n = \int_a^b \mathbf{1}_{\mathbb{Q}},$$

and in fact, the right hand side is not even defined (as $\mathbf{1}_{\mathbb{Q}}$ is not Darboux integrable)!

To solve this issue we will introduce the notion of the Lebesgue measure and furthermore, its associated Lebesgue integral which extends our Darboux integral such that it has the "nice" properties we desire.

We will in this course also look at L^p spaces. From the perspective of analysis, it is often convenient to work in Banach spaces (complete normed vector spaces) such that we can utilise many existing theorems we have proved in **Analysis II**, e.g. Banach's fixed point theorem. For instance, one can endow $C^{\circ}_{pw}([a,b])$ with the (semi-)norm

$$\|f\|_{L^1} := \int_a^b |f| \, .$$

Then, by considering the aforementioned sequence $(f_n) \subseteq C^{\circ}_{pw}([a,b])$, one can easily show that (f_n) is a Cauchy sequence with respect to $\|\cdot\|_{L^1}$. However, $f_n \to \mathbf{1}_{\mathbb{Q}}$ pointwise. This motivates us to introduce the Banach space $L^1([a,b])$ of integrable functions, and more generally, L^p-spaces later in the course.

Lastly, as we have seen within last term's probability module, measure theory lays below as the foundations for probability theory. As a quick reminder, we recall that a probability space is a special type of measure space and random variables defined on these probability spaces are simply measurable functions to $\mathbb{R}$ (or more exotic fields). This can be interpreted with connotations to real world situations in several ways.