

# Complex Analysis

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# 1 Complex Numbers

We recall some properties about the complex numbers  $\mathbb{C}$ .

From **Analysis II** we recall the topological properties of  $\mathbb{R}^2$ . As there exists a natural homeomorphism from  $\mathbb{R}^2$  to  $\mathbb{C}$ , we conclude that the complex numbers also has these properties.

**Proposition 1.** The set of complex numbers  $\mathbb{C}$  forms a metric space with the induced metric from the Pythagorean norm, that is, the metric

$$d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R} : (z, w) \mapsto |z - w|.$$

*Proof.* One can trivially show that the Pythagorean norm is a norm on  $\mathbb{C}$ , and hence, the induced metric is a metric on  $\mathbb{C}$ .  $\square$

**Theorem 1.** The complex numbers equipped with the distance as defined above is Lipschitz equivalent to  $\mathbb{R}^2$  equipped with Euclidean metric; so, they are also homeomorphic.

*Proof.* Trivial.  $\square$

**Corollary 1.1.** The complex numbers is complete and a subset of  $\mathbb{C}$  is compact if and only if  $\mathbb{C}$  is closed and bounded.

*Proof.* Follows from the Heine-Borel theorem and the fact that  $\mathbb{R}^2$  is complete.  $\square$

Certain definitions are also induced for the complex numbers by the fact that it is a metric space. We shall define them here again for referencing.

**Definition 1.1.** An open disk (ball) in  $\mathbb{C}$  centred at  $z_0 \in \mathbb{C}$  with radius  $r > 0$  is the set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of a disk is the set

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Lastly, we write  $\mathbb{D} := D_1(0)$  for shorthand.

**Definition 1.2.** Let  $S \subseteq \mathbb{C}$  and  $z_0 \in S$ . We call  $z_0$  an interior point of  $S$  if and only if there exists some  $r > 0$  such that  $D_r(z_0) \subseteq S$ . We call the set of interior points of  $S$ ,  $S^\circ$  – the interior of  $S$  and we call  $S$  open if and only if every element of  $S$  is an interior point of  $S$ , i.e.  $S = S^\circ$ .

We see that the above definition for open is equivalent to that which is induced by the metric space.

**Definition 1.3.** Let  $S$  be a subset of  $\mathbb{C}$ , then

- $S$  is closed if and only if  $S^c$  is open, or, equivalently,  $S$  is closed if and only if for all convergent sequences  $(x_n) \subseteq S$ ,  $(x_n)$  converges in  $S$ .

- the closure of  $S$ ,  $\overline{S}$  is the smallest closed set containing  $S$ , or equivalently, the union of  $S$  and its limit points.
- the boundary of  $S$  is defined to be  $\partial S = \overline{S} \setminus S^\circ$ .
- if  $S$  is bounded, then the diameter of  $S$  is

$$\text{diam}(S) = \sup_{z, w \in S} |z - w|.$$

- $S$  is (path) connected if and only if for all  $z, w \in S$ , there exists some continuous function  $\gamma : [0, 1] \rightarrow S$  such that  $\gamma(0) = z$  and  $\gamma(1) = w$ .

We remark that there is no confusion regarding the definition of connectedness in  $\mathbb{C}$  since path-connectedness is a stronger notion than connectedness in arbitrary topological spaces while in  $\mathbb{R}^n$ , open sets are path-connected if they are connected, and so, by traversing the homeomorphism, an open set  $S \subseteq \mathbb{C}$  is connected if and only if it is path-connected.

As  $\mathbb{C}$  is complete the following proposition follows as compact sets in  $\mathbb{C}$  are closed and bounded.

**Proposition 2.** Let  $(S_n)$  be a sequence of non-empty decreasing subsets of  $\mathbb{C}$  such that  $\text{diam}(S_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a unique point  $w \in \mathbb{C}$  such that  $w \in \bigcap_n S_n$ .

*Proof.* This result was previously proved for closed and bounded sets in arbitrary complete metric spaces and so, this result follows as an application of that.  $\square$

For good measure, let us also recall some lemmas from school regarding algebraic manipulations of the complex numbers.

**Theorem 2.** Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and let  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ , then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

**Corollary 2.1** (De Moivre's Formula). Let  $z = r(\cos \theta + i \sin \theta)$ , then

$$z^n = r^n (\cos n\theta + i \sin n\theta).$$

We note that the above implies  $\arg z_1 + \arg z_2 = \arg z_1 z_2$  but it is in general **not** true that  $\text{Arg } z_1 + \text{Arg } z_2 = \text{Arg } z_1 z_2$  where  $\text{Arg } z$  denote the principle argument of  $z$ .

## 2 Complex Functions