Comple Analysis

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1 Complex Numbers

We recall some properties about the complex numbers $\mathbb{C}$.

From **Analysis II** we recall the topological properties of $\mathbb{R}^2$. As there exists a natural homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}$, we conclude that the complex numbers also has these properties.

Proposition 1. The set of complex numbers $\mathbb{C}$ forms a metric space with the induced metric from the Pythagorean norm, that is, the metric

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R} : (z, w) \mapsto |z - w| .$$

Proof. One can trivially show that the Pythagorean norm is a norm on $\mathbb{C}$, and hence, the induced metric is a metric on $\mathbb{C}$. $\square$

Theorem 1. The complex numbers equipped with the distance as defined above is Lipschitz equivalent to $\mathbb{R}^2$ equipped with Euclidean metric; so, they are also homeomorphic.

Proof. Trivial. $\square$

Corollary 1.1. The complex numbers is complete and a subset of $\mathbb{C}$ is compact if and only if $\mathbb{C}$ is closed and bounded.

Proof. Follows from the Heine-Borel theorem and the fact that $\mathbb{R}^2$ is complete. $\square$

Certain definitions are also induced for the complex numbers by the fact that it is a metric space. We shall define them here again for referencing.

Definition 1.1. An open disk (ball) in $\mathbb{C}$ centred at $z_0 \in \mathbb{C}$ with radius $r > 0$ is the set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of a disk is the set

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Lastly, we write $\mathbb{D} := D_1(0)$ for shorthand.

Definition 1.2. Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. We call z_0 an interior point of S if and only if there exists some $r > 0$ such that $D_r(z_0) \subseteq S$. We call the set of interior points of S, S^o – the interior of S and we call S open if and only if every element of S is an interior point of S, i.e. $S = S^o$.

We see that the above definition for open is equivalent to that which is induced by the metric space.

Definition 1.3. Let S be a subset of $\mathbb{C}$, then

- S is closed if and only if S^c is open, or, equivalently, S is closed if and only if for all convergent sequences $(x_n) \subseteq S$, (x_n) converges in S.

- the closure of S, $\overline{S}$ is the smallest closed set containing S, or equivalently, the union of S and its limit points.
- the boundary of S is defined to be $\partial S = \overline{S} \setminus S^o$.
- if S is bounded, then the diameter of S is

$$\operatorname{diam}(S) = \sup_{z,w \in S} |z - w| \,.$$

- S is (path) connected if and only if for all $z, w \in S$, there exists some continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

We remark that there is no confusion regarding the definition of connectedness in $\mathbb{C}$ since path-connectedness is a stronger notion than connectedness in arbitrary topological spaces while in $\mathbb{R}^n$, open sets are path-connected if they are connected, and so, by traversing the homeomorphism, an open set $S \subseteq \mathbb{C}$ is connected if and only if it is path-connected.

As $\mathbb{C}$ is complete the following proposition follows as compact sets in $\mathbb{C}$ are closed and bounded.

Proposition 2. Let (S_n) be a sequence of non-empty decreasing subsets of $\mathbb{C}$ such that $\operatorname{diam}(S_n) \to 0$ as $n \to 0$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \bigcap_n S_n$.

Proof. This result was previously proved for closed and bounded sets in arbitrary complete metric spaces and so, this result follows as an application of that. □

For good measure, let us also recall some lemmas from school regarding algebraic manipulations of the complex numbers.

Theorem 2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Corollary 2.1 (De Moivre's Formula)**.** Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

We note that the above implies $\arg z_1 + \arg z_2 = \arg z_1 z_2$ but it is in general **not** true that $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} z_1 z_2$ where $\operatorname{Arg} z$ denote the principle argument of z.

2 Complex Functions