Complex Analysis

Kexing Ying

January 11, 2021

Contents

1 Complex Numbers

We recall some properties about the complex numbers $\mathbb{C}$.

From **Analysis II** we recall the topological properties of $\mathbb{R}^2$. As there exists a natural homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}$, we conclude that the complex numbers also has these properties.

Proposition 1. The set of complex numbers $\mathbb{C}$ forms a metric space with the induced metric from the Pythagorean norm, that is, the metric

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R} : (z, w) \mapsto |z - w| \, .$$

Proof. One can trivially show that the Pythagorean norm is a norm on $\mathbb{C}$, and hence, the induced metric is a metric on $\mathbb{C}$. □

Theorem 1. The complex numbers equipped with the distance as defined above is Lipschitz equivalent to $\mathbb{R}^2$ equipped with Euclidean metric; so, they are also homeomorphic.

Proof. Trivial. □

Corollary 1.1. The complex numbers is complete and a subset of $\mathbb{C}$ is compact if and only if $\mathbb{C}$ is closed and bounded.

Proof. Follows from the Heine-Borel theorem and the fact that $\mathbb{R}^2$ is complete. □

Certain definitions are also induced for the complex numbers by the fact that it is a metric space. We shall define them here again for referencing.

Definition 1.1. An open disk (ball) in $\mathbb{C}$ centred at $z_0 \in \mathbb{C}$ with radius $r > 0$ is the set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of a disk is the set

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Lastly, we write $\mathbb{D} := D_1(0)$ for shorthand.

Definition 1.2. Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. We call z_0 an interior point of S if and only if there exists some $r > 0$ such that $D_r(z_0) \subseteq S$. We call the set of interior points of S, S^o – the interior of S and we call S open if and only if every element of S is an interior point of S, i.e. $S = S^o$.

We see that the above definition for open is equivalent to that which is induced by the metric space.

Definition 1.3. Let S be a subset of $\mathbb{C}$, then

- S is closed if and only if S^c is open, or, equivalently, S is closed if and only if for all convergent sequences $(x_n) \subseteq S$, (x_n) converges in S.

- the closure of S, $\overline{S}$ is the smallest closed set containing S, or equivalently, the union of S and its limit points.
- the boundary of S is defined to be $\partial S = \overline{S} \setminus S^o$.
- if S is bounded, then the diameter of S is

$$\operatorname{diam}(S) = \sup_{z,w \in S} |z - w| .$$

- S is (path) connected if and only if for all $z, w \in S$, there exists some continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

We remark that there is no confusion regarding the definition of connectedness in $\mathbb{C}$ since path-connectedness is a stronger notion than connectedness in arbitrary topological spaces while in $\mathbb{R}^n$, open sets are path-connected if they are connected, and so, by traversing the homeomorphism, an open set $S \subseteq \mathbb{C}$ is connected if and only if it is path-connected.

As $\mathbb{C}$ is complete the following proposition follows as compact sets in $\mathbb{C}$ are closed and bounded.

Proposition 2. Let (S_n) be a sequence of non-empty decreasing subsets of $\mathbb{C}$ such that $\operatorname{diam}(S_n) \to 0$ as $n \to 0$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \bigcap_n S_n$.

Proof. This result was previously proved for closed and bounded sets in arbitrary complete metric spaces and so, this result follows as an application of that. $\square$

For good measure, let us also recall some lemmas from school regarding algebraic manipulations of the complex numbers.

Theorem 2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Corollary 2.1 (De Moivre's Formula)**.** Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

We note that the above implies $\arg z_1 + \arg z_2 = \arg z_1 z_2$ but it is in general **not** true that $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} z_1 z_2$ where $\operatorname{Arg} z$ denote the principle argument of z.

2 Complex Functions

As with all spaces, we would like to study the properties of mappings between the complex numbers.

Definition 2.1 (Mapping)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$. Then,

$$f : \Omega_1 \to \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$, there exists only one complex number $w = u + iv$ such that $w = f(z)$.

In this case, we denote $w = f(z) = u(x, y) + iv(x, y)$.

We define a special mapping – the Möbius transformation.

Definition 2.2 (The Möbius Transformation)**.** The Möbius transformation is a mapping such that

$$w = f(z) = \frac{az + b}{cz + d},$$

for some $a, b, c, d \in \mathbb{C}$ where $cz + d \neq 0$ on the domain.

As the complex plane is a metric space, we again have the induces notion of continuity.

Definition 2.3 (Continuity)**.** Let $f : \Omega_1 \to \Omega_2$ be some complex mapping and let $z_0 \in \Omega_1$. We say f is continuous at z_0 if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all $z \in \mathbb{C}$, $|z - z_0| < \delta$, we have $|f(z) - f(z_0)|$.

We say f is continuous on Ω_1 if it is continuous at every point in Ω_1.

Since the complex plane is homeomorphic to the Euclidean space $\mathbb{R}^2$, one might think to establish a notion of derivative on $\mathbb{C}$. This is achieved, however, not through the definition on general Euclidean spaces, but through another definition.

Definition 2.4 (Holomorphic)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. Then we say f is holomorphic (differentiable) at some $z_0 \in \Omega_1$ if the limit

$$\lim_{h \in \mathbb{C} \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Here we restricts $z_0 + h \in \Omega_1$ which is fine since Ω_1 is open, and hence, there exists some $\delta > 0$ such that $B_\delta(z_0) \subseteq \Omega_1$. If f is holomorphic at z_0 then we call the quotient its derivative and denote it by $f'(z_0)$.

Let $S \subseteq \mathbb{C}$ be some complex set, then, we say f is holomorphic on S if

- S is open and f is holomorphic on every point of S;
- S is closed and f is holomorphic on some open set containing S.

If f is holomorphic on $\mathbb{C}$ itself then we say f is entire.

We note that we are allowed to make this definition as there exists a notion of division on $\mathbb{C}$ while the same cannot be said for general Euclidean spaces.

The function $f(z) = \overline{z}$ is not holomorphic. Indeed, the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

doest not have a limit as $n \to \infty$ and so our claim.

Proposition 3. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = o(h),$$

or equivalently (without the syntactic sugar),

$$f(z_0 + h) - f(z_0) - ah = h\psi(h)$$

where $\psi : D_\epsilon(0) \to \mathbb{C}$ is a function such that $\lim_{h \to 0} \psi(h) = 0$ for some $\epsilon > 0$.

Proof. Straight away, by dividing both side by h, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \to 0$$

as $h \to 0$. □

By taking $h \to 0$ on both sides of the equation, we have $f(z) \to f(z_0)$ as $z \to z_0$ and so, the following corollary.

Corollary 2.2. A holomorphic function f is continuous.

As one might imagine, the normal properties of derivatives hold for this definition as well.

Proposition 4. If f, g are holomorphic in Ω then,

- $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$;
- fg is holomorphic in Ω and $(fg)' = f'g + fg'$;
- if $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and $(f/g)' = \frac{f'g+fg'}{g^2}$;

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are both holomorphic, the chain rule holds, that is

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Proof. Omitted. One can use the above proposition to make life easier. □

2.1 Cauchy-Riemann Equations

Consider the limit

$$f'(z_0) = \lim_{h=h_1+ih_2\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Assuming that $h = h_1$, namely $h_2 = 0$ and by writing,

$$f(z_0) = f(x_0+iy_0) = u(x_0,y_0) + iv(x_0,y_0),$$

we have,

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{h_1\in\mathbb{R}\to 0} \frac{u(x_0+h_1,y_0)+iv(x_0+h_1,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}x_0,y_0 = u'_x(x_0,y_0)+iv'_x(x_0,y_0). \end{aligned}$$

Similarly, if we let $g = ih_2$ by letting $h_1 = 0$, we have

$$f'(z_0) = \frac{1}{i}\frac{\partial u}{\partial v}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = -iu'_y(x_0,y_0)+v'_y(x_0,y_0).$$

So, if f is holomorphic at z_0, then the two limit should agree, and hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These two equations together are called the *Cauchy-Riemann equations*.

Definition 2.5 (Cauchy-Riemann Equations)**.** Let $f(z) = u(x,y)+iv(x,y)$ be a mapping, then the Cauchy-Riemann equations are the system of equations

$$u'_x = v'_y; \quad u'_y = -v'_x.$$

With the Cauchy-Riemann equations, we have a necessary condition for a function to be holomorphic. As shown above, we have found that the conjugate function $f = z \mapsto \overline{z}$ is not holomorphic, and we see that as well with its Cauchy-Riemann equations since $u'_x = 1 \neq -1 = v'_y$.

The Cauchy-Riemann equations links real and complex analysis in some sense. By defining the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right),$$

we have the following theorem.

Theorem 3. Let $f = u+iv$. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0,$$

and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Proof. Trivially follows by using the Cauchy-Riemann equations (except perhaps for showing $f'(z_0) = \partial u/\partial z$ which follows since we can write $f'(z_0) = u'_x(z_0) + iv'_x(z_0)$ and so, the result follows by rewriting with the Cauchy-Riemann equations). $\square$

Similar to the necessary and sufficient conditions for the existence of derivatives for general Euclidean spaces, we would like a similar theorem for determining whether or not a complex valued function is holomorphic. This is achieved with the following theorem.

Theorem 4. Suppose $f = u + iv$ is a complex-valued function defined on some open set Ω. If u, v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and

$$f'(z) = \frac{\partial f}{\partial z}(z).$$