Further Analysis

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Contents

1 Real Analysis in Higher Dimensions

We continue on first-year analysis in higher dimensions.

1.1 Euclidean Spaces

For $n \geq 1$, the n-dimensional *Euclidean space* denoted by $\mathbb{R}^n$, is the set of ordered n-tuples $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ for $x_i \in \mathbb{R}$. Recall that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$, we can use the usual vector space operations, i.e. vector addition and scalar multiplication. Furthermore, $\mathbb{R}^n$ forms a inner product space with the operation,

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \sum_{i=1}^{n} x_i y_i.$$

Thus, as a inner product space induces a normed vector space, we find a natural norm defined for $\mathbb{R}^n$ by,

$$\| \cdot \| : \mathbb{R}^n \to \mathbb{R} : \mathbf{x} \mapsto \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

By manually checking, we find that this norm satisfy the norm axioms, i.e. it satisfy the *triangle inequality*, is *absolutely scalable*, and *positive definite* (In fact, we do not need the norm to be non-negative as it can deduced from the other axioms).

1.1.1 Preliminary Concepts in $\mathbb{R}^n$

Sequences in $\mathbb{R}^n$ can be defined similarly to that of $\mathbb{R}$, and we carry over all notations in all suitable places.

Definition 1.1 (Convergence in $\mathbb{R}^n$)**.** We say a sequence $(\mathbf{x}_i)_{i=1}^{\infty} \subseteq \mathbb{R}^n$ converges to $\mathbf{x} \in \mathbb{R}^n$ if and only if for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $i \geq N$, $\|\mathbf{x}_i - \mathbf{x}\| < \epsilon$.

Proposition 1. *A sequence $(\mathbf{x}_i)_{i=1}^{\infty} \in \mathbb{R}^n$ converges to $\mathbf{x} \in \mathbb{R}^n$ if and only if each component of $\mathbf{x}_i$ converges to the corresponding component of $\mathbf{x}$.*

In the first dimension, we've considered the topology of $\mathbb{R}$ including the examination of open and closed sets. We extend this idea for higher dimensions. The most basic examples we have of an open set (or closed set for that matter) in $\mathbb{R}$ are the open and closed intervals respectively. This is extended in $\mathbb{R}^n$ to be sets of the form

$$\prod_{i=1}^{n}(a_i, b_i) := \{\mathbf{x} \mid a_i < \mathbf{x}_i < b_i, \forall 1 \leq i \leq n\},$$

and similarly for closed intervals. However, while this is nice to look at, it is not very useful. So for this reason, we again will extend the notion of open and closed sets for $\mathbb{R}^n$.

Definition 1.2 (Open Ball)**.** Let $\mathbf{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$, we define the open ball of radius r about $\mathbf{x}$ as the set

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\| < r\}$$

.

Definition 1.3 (Open)**.** A set $U \subseteq \mathbb{R}^n$ is open in $\mathbb{R}^n$ if and only if for all $\mathbf{x} \in U$, there is some $r \in \mathbb{R}^+$ such that $B_r(\mathbf{x}) \subseteq U$.

Definition 1.4 (Closed)**.** A set $U \subseteq \mathbb{R}^n$ is closed if and only if its complement is open.

Straight away from the definition, we can see that every open ball is open (see here). Furthermore, we find the union and intersection of two open sets is open. In fact, the union and any collection of open sets is also open, however, this is not necessarily true for closed sets.

Definition 1.5 (Continuity at a Point)**.** Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f : A \to \mathbb{R}^m$. We say f is continuous at $p \in A$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A \cap B_\delta(p)$, $\|f(x) - f(p)\| < \epsilon$.

If the function f is continuous at every point of A, then we say f is continuous on A.

Definition 1.6. Let $A \in \mathbb{R}^n$ be open in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. For $p \in A$, we say that the limit of f as $\mathbf{x}$ tends to $\mathbf{p}$ in A is equal to $\mathbf{q} \in \mathbb{R}^m$ if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A \cap B_\delta(p), x \neq p$, $\|f(x) - \mathbf{q}\| < \epsilon$.

This is the same notion we used for continuity in the first dimension to say that f is continuous at p if and only if $\lim_{x \to p} f(x) = q$.

Proposition 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then f is continuous if and only if for all open subsets U of $\mathbb{R}^m$, $f^{-1}(U)$ is open in $\mathbb{R}^n$.*

Proof. See here for the proof. $\square$