Complex Analysis

Kexing Ying

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1 Complex Numbers

We recall some properties about the complex numbers $\mathbb{C}$.

From **Analysis II** we recall the topological properties of $\mathbb{R}^2$. As there exists a natural homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}$, we conclude that the complex numbers also has these properties.

Proposition 1. The set of complex numbers $\mathbb{C}$ forms a metric space with the induced metric from the Pythagorean norm, that is, the metric

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R} : (z, w) \mapsto |z - w|.$$

Proof. One can trivially show that the Pythagorean norm is a norm on $\mathbb{C}$, and hence, the induced metric is a metric on $\mathbb{C}$. □

Theorem 1. The complex numbers equipped with the distance as defined above is Lipschitz equivalent to $\mathbb{R}^2$ equipped with Euclidean metric; so, they are also homeomorphic.

Proof. Trivial. □

Corollary 1.1. The complex numbers is complete and a subset of $\mathbb{C}$ is compact if and only if $\mathbb{C}$ is closed and bounded.

Proof. Follows from the Heine-Borel theorem and the fact that $\mathbb{R}^2$ is complete. □

Certain definitions are also induced for the complex numbers by the fact that it is a metric space. We shall define them here again for referencing.

Definition 1.1. An open disk (ball) in $\mathbb{C}$ centred at $z_0 \in \mathbb{C}$ with radius $r > 0$ is the set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of a disk is the set

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Lastly, we write $\mathbb{D} := D_1(0)$ for shorthand.

Definition 1.2. Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. We call z_0 an interior point of S if and only if there exists some $r > 0$ such that $D_r(z_0) \subseteq S$. We call the set of interior points of S, S^o – the interior of S and we call S open if and only if every element of S is an interior point of S, i.e. $S = S^o$.

We see that the above definition for open is equivalent to that which is induced by the metric space.

Definition 1.3. Let S be a subset of $\mathbb{C}$, then

- S is closed if and only if S^c is open, or, equivalently, S is closed if and only if for all convergent sequences $(x_n) \subseteq S$, (x_n) converges in S.

- the closure of S, $\overline{S}$ is the smallest closed set containing S, or equivalently, the union of S and its limit points.
- the boundary of S is defined to be $\partial S = \overline{S} \setminus S^o$.
- if S is bounded, then the diameter of S is

$$\operatorname{diam}(S) = \sup_{z,w \in S} |z - w| .$$

- S is (path) connected if and only if for all $z, w \in S$, there exists some continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

We remark that there is no confusion regarding the definition of connectedness in $\mathbb{C}$ since path-connectedness is a stronger notion than connectedness in arbitrary topological spaces while in $\mathbb{R}^n$, open sets are path-connected if they are connected, and so, by traversing the homeomorphism, an open set $S \subseteq \mathbb{C}$ is connected if and only if it is path-connected.

As $\mathbb{C}$ is complete the following proposition follows as compact sets in $\mathbb{C}$ are closed and bounded.

Proposition 2. Let (S_n) be a sequence of non-empty decreasing subsets of $\mathbb{C}$ such that $\operatorname{diam}(S_n) \to 0$ as $n \to 0$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \bigcap_n S_n$.

Proof. This result was previously proved for closed and bounded sets in arbitrary complete metric spaces and so, this result follows as an application of that. $\square$

For good measure, let us also recall some lemmas from school regarding algebraic manipulations of the complex numbers.

Theorem 2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Corollary 2.1 (De Moivre's Formula)**.** Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

We note that the above implies $\arg z_1 + \arg z_2 = \arg z_1 z_2$ but it is in general **not** true that $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} z_1 z_2$ where $\operatorname{Arg} z$ denote the principle argument of z.

2 Complex Functions

As with all spaces, we would like to study the properties of mappings between the complex numbers.

Definition 2.1 (Mapping)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$. Then,

$$f : \Omega_1 \to \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$, there exists only one complex number $w = u + iv$ such that $w = f(z)$.

In this case, we denote $w = f(z) = u(x, y) + iv(x, y)$.

We define a special mapping – the Möbius transformation.

Definition 2.2 (The Möbius Transformation)**.** The Möbius transformation is a mapping such that

$$w = f(z) = \frac{az + b}{cz + d},$$

for some $a, b, c, d \in \mathbb{C}$ where $cz + d \neq 0$ on the domain.

As the complex plane is a metric space, we again have the induces notion of continuity.

Definition 2.3 (Continuity)**.** Let $f : \Omega_1 \to \Omega_2$ be some complex mapping and let $z_0 \in \Omega_1$. We say f is continuous at z_0 if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all $z \in \mathbb{C}$, $|z - z_0| < \delta$, we have $|f(z) - f(z_0)|$.

We say f is continuous on Ω_1 if it is continuous at every point in Ω_1.

Since the complex plane is homeomorphic to the Euclidean space $\mathbb{R}^2$, one might think to establish a notion of derivative on $\mathbb{C}$. This is achieved, however, not through the definition on general Euclidean spaces, but through another definition.

Definition 2.4 (Holomorphic)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. Then we say f is holomorphic (differentiable) at some $z_0 \in \Omega_1$ if the limit

$$\lim_{h \in \mathbb{C} \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Here we restricts $z_0 + h \in \Omega_1$ which is fine since Ω_1 is open, and hence, there exists some $\delta > 0$ such that $B_\delta(z_0) \subseteq \Omega_1$. If f is holomorphic at z_0 then we call the quotient its derivative and denote it by $f'(z_0)$.

Let $S \subseteq \mathbb{C}$ be some complex set, then, we say f is holomorphic on S if

- S is open and f is holomorphic on every point of S;
- S is closed and f is holomorphic on some open set containing S.

If f is holomorphic on $\mathbb{C}$ itself then we say f is entire.

We note that we are allowed to make this definition as there exists a notion of division on $\mathbb{C}$ while the same cannot be said for general Euclidean spaces.

The function $f(z) = \overline{z}$ is not holomorphic. Indeed, the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

doest not have a limit as $n \to \infty$ and so our claim.

Proposition 3. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = o(h),$$

or equivalently (without the syntactic sugar),

$$f(z_0 + h) - f(z_0) - ah = h\psi(h)$$

where $\psi : D_\epsilon(0) \to \mathbb{C}$ is a function such that $\lim_{h \to 0} \psi(h) = 0$ for some $\epsilon > 0$.

Proof. Straight away, by dividing both side by h, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \to 0$$

as $h \to 0$. □

By taking $h \to 0$ on both sides of the equation, we have $f(z) \to f(z_0)$ as $z \to z_0$ and so, the following corollary.

Corollary 2.2. A holomorphic function f is continuous.

As one might imagine, the normal properties of derivatives hold for this definition as well.

Proposition 4. If f, g are holomorphic in Ω then,

- $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$;
- fg is holomorphic in Ω and $(fg)' = f'g + fg'$;
- if $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and $(f/g)' = \frac{f'g+fg'}{g^2}$;

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are both holomorphic, the chain rule holds, that is

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Proof. Omitted. One can use the above proposition to make life easier. □

2.1 Cauchy-Riemann Equations

Consider the limit

$$f'(z_0) = \lim_{h=h_1+ih_2\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Assuming that $h = h_1$, namely $h_2 = 0$ and by writing,

$$f(z_0) = f(x_0+iy_0) = u(x_0,y_0) + iv(x_0,y_0),$$

we have,

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{h_1\in\mathbb{R}\to 0} \frac{u(x_0+h_1,y_0)+iv(x_0+h_1,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}x_0,y_0 = u'_x(x_0,y_0)+iv'_x(x_0,y_0). \end{aligned}$$

Similarly, if we let $g = ih_2$ by letting $h_1 = 0$, we have

$$f'(z_0) = \frac{1}{i}\frac{\partial u}{\partial v}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = -iu'_y(x_0,y_0)+v'_y(x_0,y_0).$$

So, if f is holomorphic at z_0, then the two limit should agree, and hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These two equations together are called the *Cauchy-Riemann equations.*

Definition 2.5 (Cauchy-Riemann Equations)**.** Let $f(z) = u(x,y)+iv(x,y)$ be a mapping, then the Cauchy-Riemann equations are the system of equations

$$u'_x = v'_y; \quad u'_y = -v'_x.$$

With the Cauchy-Riemann equations, we have a necessary condition for a function to be holomorphic. As shown above, we have found that the conjugate function $f = z \mapsto \overline{z}$ is not holomorphic, and we see that as well with its Cauchy-Riemann equations since $u'_x = 1 \neq -1 = v'_y$.

The Cauchy-Riemann equations links real and complex analysis in some sense. By defining the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right),$$

we have the following theorem.

Theorem 3. Let $f = u + iv$. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0,$$

and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Proof. Trivially follows by using the Cauchy-Riemann equations (except perhaps for showing $f'(z_0) = \partial u/\partial z$ which follows since we can write $f'(z_0) = u'_x(z_0) + iv'_x(z_0)$ and so, the result follows by rewriting with the Cauchy-Riemann equations). $\square$

Similar to the necessary and sufficient conditions for the existence of derivatives for general Euclidean spaces, we would like a similar theorem for determining whether or not a complex valued function is holomorphic. This is achieved with the following theorem.

Theorem 4. Suppose $f = u + iv$ is a complex-valued function defined on some open set Ω. If u, v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and

$$f'(z) = \frac{\partial f}{\partial z}(z).$$

The proof of this theorem is similar to the equivalent statement in $\mathbb{R}^n$, namely, a function is differentiable if and only if it's partial derivatives are continuously differentiable and its derivative is the Jacobian.

Proof. Let $h = h_1 + ih_2$, then,

$$u(x + h_1, y + h_2) - u(x, y) = u'_x(x, y)h_1 + u'_y(x, y)h_2 + o(|h|),$$

and

$$v(x + h_1, y + h_2) - v(x, y) = v'_x(x, y)h_1 + v'_y(x, y)h_2 + o(|h|).$$

By the Cauchy-Riemann equations,

$$v'_x = -u'_y; \;\; v'_y = u'_x,$$

we find that,

$$\begin{aligned}
f(z + h) - f(z) &= u(x + h_1, y + h_2) + iv(x + h_1, y + h_2) - u(x, y) - iv(x, y) \\
&= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\
&= u'_x(x, y)h_1 + u'_y(x, y)h_2 + o(|h|) + i(v'_x(x, y)h_1 + v'_y(x, y)h_2 + o(|h|)) \\
&= u'_x h_1 + u'_y h_2 + iu'_x h_2 - iu_y h_1 + o(|h|) \\
&= (u'_x - iu'_y)(h_1 + ih_2) + o(|h|).
\end{aligned}$$

This gives us

$$f(z + h) - f(z) - (u'_x - iu'_y)h = o(|h|),$$

implying f is differentiable at z with derivative

$$f'(z) = u'_x - iu'_y = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

$\square$

Lastly, by transforming the complex numbers into their polar forms, we have the following Cauchy-Riemann equations,

$$u'_r = \frac{1}{r}v'_\theta; \;\; v'_r = -\frac{1}{r}u'_\theta.$$

The derivation of this is completely mundane so is omitted here (see lecture slides). This form of Cauchy-Riemann equations can be useful when dealing with certain functions where the function is easier to deal with in polar coordinates.

2.2 Complex Exponentials

We recall the definition and properties of a complex power series.

Definition 2.6. A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$.

We call the aforementioned power series convergent at some $z \in \mathbb{C}$ if the partial sum $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has a limit in $\mathbb{C}$, $S(z) = \lim_{N\to\infty} S_N(z)$. If that is the case we write $\sum_{n=0}^{\infty} a_n z^n$.

Furthermore, we define a power series $\sum a_n z^n$ to be absolutely if $sum\, |a_n|\, |z|^n$ converges. As we have seen from last year, absolute convergence implies converges, and furthermore, if $S(z) = \sum a_n z^n$ then $\lim_{N\to\infty} S(z) - S_N(z) = 0$.

Theorem 5. Given $\sum a_n z^n$ be a power series, then there exists some $R \in \mathbb{R}^+$ such that for all $z \in \mathbb{C}$, if $|z| < R$ then the series converges absolutely and if $|z| > R$ then the series diverges. Moreover,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

The number R is called the radius of convergence and the domain D_R is refered as the disc of convergence.

Proof. See last year's notes. □

The power series provide us with an important class of holomorphic functions.

Theorem 6. The power series $f(z) = \sum_{n=0} a_n z^n$ defines a holomorphic function in its disc of convergence. In fact the derivative of f is simply the sum of the derivative of its individual terms, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f has the same radius of convergence as f.

Proof. We note that $\sum_{n=1}^{\infty} a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^n$ have the same radius of convergence by considering the above proposition. Indeed, if $\sum_{n=1}^{\infty} a_n z^{n-1}$ has radius of convergence R, then $R = (\limsup_{n\to\infty} |a_n|^{1/n})^{-1} = (\limsup_{n\to\infty} |n a_n|^{1/n})^{-1}$, which is the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^n$. Beware that we used the fact that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\log n}{n}\right) = 1.$$

Thus, it remains to show that $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Let R be the radius of convergence of f, $|z_0| < r < R$ and define $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$ and,

$$S_N(z) = \sum_{n=0}^{N} a_n z^n; \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then, Wlog. assume $|z_0 + h| < r$, so

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = &\left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right)\\ &+ (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right),\end{aligned}$$

for arbitrary $N \in \mathbb{N}$. Thus, by taking $N \to \infty, h \to 0$, we see that the first two term of the right hand side vanished while further examination is required for the last term. Indeed, by the triangle inequality,

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \, nr^{n-1},$$

where the last term tends to 0 as $N \to \infty$. Hence,

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \to 0,$$

as $h \to 0$ and the result is complete. □

As the term-wise derivate of a power series is also a power series, the above theorem results in the following corollary.

Corollary 6.1. A power series is infinitely differentiable within its disc of convergence, and its higher derivatives are also power series obtained by term-wise differentiation.

This reason for this revision and seemingly digression from complex functions is because we would like to introduce complex exponentials. We recall from last year that we defined the exponential function as

$$\exp : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and we write $e^x := \exp(x)$. We define the complex exponential on top of this definition.

Definition 2.7. Given $z = x + iy \in \mathbb{C}$, we define the complex exponential,

$$e^z := e^x \cos y + ie^x \sin y.$$

As with the real exponential, the complex exponential has several nice properties.

Proposition 5. Let us denote $e^{(\cdot)}$ for the complex exponential, then,

- if $z = x + 0i$, then $e^z = e^x$;

- $e^{(\cdot)}$ is entire (holomorphic for all $z \in \mathbb{C}$);
- $\frac{\partial}{\partial z} e^z = e^z$;
- if $g(z)$ is holomorphic, then $\frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z)$;
- $e^{z_1 + z_2} = e^{z_1} e^{z_2}$;
- $|e^z| = e^x$;
- $(e^z)^n = e^{nz}$;
- $\arg e^z = y + 2\pi k$ for $k = 0, \pm 1, \pm 2, \cdots$.

Proof. Straight forward either by definition or by the Cauchy-Riemann equations. □

From the definition of the complex exponential function, we define the complex trigonometric functions.

Definition 2.8 (Complex Trigonometric Functions)**.** For any $z \in \mathbb{C}$, we define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \;\; \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

To see why the trigonometric functions are extended in such a way, one can simply consider Euler's identity where we have

$$e^{i\theta} = \cos\theta + i\sin\theta; \;\; e^{-i\theta} = cos\theta - i\sin\theta,$$

for any $\theta \in \mathbb{R}$. Thus, by assuming $\theta \in \mathbb{C}$, and solving the system of equations, we have the definition of the complex trigonometric functions.

As the complex trigonometric functions are simply linear combinations of the complex exponential, we again get many nice properties for free.

Proposition 6. Let $z = x + iy \in \mathbb{C}$, then

- $\sin(\cdot)$ and $\cos(\cdot)$ are entire functions;
- $\frac{\partial}{\partial z} \sin z = \cos z$ and $\frac{\partial}{\partial z} \cos z = -\sin z$;
- $\sin^2 z + \cos^2 z = 1$;
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ and $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$;

Lastly, we define the complex logarithmic functions.

Definition 2.9 (Complex Logarithm)**.** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then

$$\log z = \log|z| + i\arg z = \log r + i(\theta + 2\pi k),$$

for $k = 0, \pm 1, \pm 2, \cdots$.

Clearly, $e^{\log z} = e^{\log r + i(\theta + 2\pi k)} = re^{i(\theta + 2\pi k)} = re^{i\theta} = z$ so it behaves as we would expect. However, as log results in an infinite family of values, and so, is not a function. To account for this, we consider the principle logarithmic function.

Definition 2.10 (Principle Logarithm)**.** The principle logarithmic function is

$$\mathrm{Log} : \mathbb{C} \to \mathbb{C} : z \mapsto \log|z| + i \,\mathrm{Arg}\, z.$$

Again, we find that the complex logarithms behave as one might expect.

Proposition 7. Let $z \in \mathbb{C}$, then

- $\log(z_1 z_2) = \log z_1 + \log z_2$;
- the principle logarithm is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

With these definitions in place, we may finally define complex powers.

Definition 2.11. For all $\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha \log z}$$

as a multi-valued function and we define its principle value to be

$$z^\alpha = e^{\alpha \mathrm{Log} z}.$$

Proposition 8. Let $z, \alpha_1, \alpha_2 \in \mathbb{C}$, then $z^{\alpha_1 + \alpha_2} = z^{\alpha_1} z^{\alpha_2}$.

Proof. Easy. $\square$

We remark that the extension of the principle logarithm is rather arbitrary as we strict $-\pi < \mathrm{Arg}\, z \leq \pi$ such that we have restricted the domain of Log to $\mathbb{C} \setminus (-\infty, 0]$. Indeed, this is simply one of a family of "branches" of the logarithm, and we shall discuss this family later on.

3 Integration Along Curves

3.1 Parametrised Curves

As we shall study integration along curves in the next section, let us first look at parametrised curves.

Definition 3.1 (Parametrised Curve)**.** A parametrised curve is a function $z : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$. We say that the parametrised curve is smooth if z' exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for all $t \in [a, b]$. At the points $t = a$ and $t = b$, the derivative of z at t is defined to be the one-sided limits

$$z'(a) = \lim_{h\to 0, h>0} \frac{z(a+h) - z(a)}{h}, \quad z'(b) = \lim_{h\to 0, h<0} \frac{z(b+h) - z(b)}{h}.$$

Similarly, we say that the parameterised curve is piecewise smooth if z is continuous on $[a, b]$ and there exist a finite number of points $a = a_0 < a_1 < \cdots < a_n = b$ such that z is smooth on $[a_k, a_{k+1}]$. In particular, the right-hand and the left-hand derivative of z at a_k need not to agree.

Definition 3.2 (Equivalence of Parametrisations)**.** Let $z : [a, b] \to \mathbb{C}$ and $\hat{z} : [c, d] \to \mathbb{C}$ be two parametrisations: Then, we say z is equivalent to $hatz$ if there exists a continuous bijective $t : [c, d] \to [a, b]$ such that $t' > 0$ on $[c, d]$ and

$$\hat{z}(s) = z(t(s)),$$

for all $s \in [c, d]$.

The requirement for $t' > 0$ says precisely that the orientation is preserved, since as s travels from c to d, $t(s)$ travels from a to b.

Parametrising curves allows us more easily integrate functions along curves.

Definition 3.3 (Complex Integration)**.** Given a smooth curve $\gamma \subseteq \mathbb{C}$ is parameterised by $z : [a, b] \to \mathbb{C}$, and $f : \gamma \to \mathbb{C}$ is a continuous function, we define

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

In order for this definition to be meaningful, we must show that the integration along γ is independent from the parametrisation. That is, if $\hat{z}$ is an equivalent parametrisation to z, then

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t = \int_c^d f(\hat{z}(t))\hat{z}'(t)\mathrm{d}t.$$

However, this is easy to see by the chain rule:

$$\int_c^d f(\hat{z}(s))\hat{z}'(s)\mathrm{d}s = \int_c^d f(z(t(s))z'(t(s))t'(s)\mathrm{d}t = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

As one might expect, if $\gamma \subseteq \mathbb{C}$ is a piecewise smooth curve and if z is a piecewise smooth parametrisation of γ, then

$$\int_\gamma f(z)\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\mathrm{d}t.$$

Intuitively, if we have a curve in space, it is obvious that we can reverse the orientation to obtain a new curve. We formalise this notion by defining γ^- for any curve γ such that γ and γ^- consists of the same points. Then, if $z : [a, b] \to \mathbb{C}$ is any parametrisation of γ, there is a corresponding parametrisation of γ^-1 defined by

$$z^- : [a, b] \to \mathbb{C} : t \mapsto z(b + a - t).$$

Definition 3.4 (Closed)**.** A smooth or piecewise smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations $z : [a, b] \to \mathbb{C}$.

Definition 3.5 (Simple)**.** A smooth or piecewise smooth curve is simple if for all $s, t \in [a, b]$, $s \neq t$ implies $z(s) \neq z(t)$ for any of its parametrisations $z : [a, b] \to \mathbb{C}$.

We see that a curve is simple if and only if all of its parametrisations are injective. Indeed, if $z(x) = z(y)$, then $x = y$ by contrapositive.

Definition 3.6 (Contour)**.** A contour is a simple closed curve which is piecewise continuously differentiable.

Integration along curves satisfy some intuitive properties.

Theorem 7. Let $f, g : \Omega \to \mathbb{C}$ be continuous, $\alpha, \beta \in \mathbb{C}$, and γ a curve in $\mathbb{C}$ such that $\gamma \subseteq \Omega$, then

- $$\int_\gamma (\alpha f(z) + \beta g(z))\mathrm{d}z = \alpha \int_\gamma f(z)\mathrm{d}z + \beta \int_\gamma g(z)\mathrm{d}z;$$
- if γ^- is γ in the reverse orientation, then
$$\int_\gamma f(z)\mathrm{d}z = -\int_{\gamma^-} f(z)\mathrm{d}z;$$
- (ML-inequality)
$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \mathrm{length}(\gamma),$$
where $\mathrm{length}(\gamma) = \int_a^b |z'(t)|\mathrm{d}t$ for some parametrisation of γ, $z : [a, b] \to \mathbb{C}$.

Remark. For continuous f, we have f achieves the maximum, and so the supremum can be replaced by the maximum.

Proof. (Part 1) Linearity follows by the definition and results from last year.

(Part 2) Say we parametrise γ as $z : [0,1] \to \mathbb{C}$, then γ^- can be parametrised as $z^- : [0,1] \to \mathbb{C} : x \mapsto z(1-x)$, and so

$$\int_{\gamma^-} f(z)\mathrm{d}z = \int_0^1 f(z^-(t))\dot{z}^-(t)\mathrm{d}t = -\int_0^1 f(z(1-t))z'(1-t)\mathrm{d}t.$$

By substituting $u = 1-t$,

$$-\int_0^1 f(z(1-t))z'(1-t)\mathrm{d}t = \int_1^0 f(z(u))z'(u)\mathrm{d}u = -\int_0^1 f(z(u))z'(u)\mathrm{d}u = -\int_\gamma f(z)\mathrm{d}z.$$

(Part 3) Follows readily as, if $z : [a,b] \to \mathbb{C}$ is a parametrisation of γ, then,

$$\left|\int_\gamma f(z)\mathrm{d}z\right| = \left|\int_a^b f(z(t))z'(t)\mathrm{d}t\right| \leq \int_a^b |f(z(t))||z'(t)|\mathrm{d}t \leq \sup_{z\in\gamma}|f(z)|\int_a^b |z'(t)|\mathrm{d}t.$$

□

3.2 Primitive Functions and Cauchy's Theorem

Definition 3.7 (Primitive)**.** Let $f : \Omega \to \mathbb{C}$ be some function, then the primitive of f on Ω is a holomorphic function $F : \Omega \to \mathbb{C}$ such that for all $z \in \Omega$, $F'(z) = f(z)$.

We see that the definition of primitive is essentially that of the antiderivative for Euclidean function, and so, similar versions of the fundamental theorem of calculus applies.

Theorem 8. If a continuous function $f : \Omega \to \mathbb{C}$ has a primitive $F : \Omega \to \mathbb{C}$ on Ω and γ is a curve in Ω that begins at w_1 and ends as w_2, then

$$\int_\gamma f(z)\mathrm{d}z = F(w_1) - F(w_2).$$

Proof. If γ is smooth, then let $z : [a,b] \to \mathbb{C}$ be a parametrisation of γ. Then

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t = \int_a^b F'(z(t))z'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))\mathrm{d}t = F(z(b)) - F(z(a)).$$

Since z is a parametrisation of γ, we have $z(a) = w_1$ and $z(b) = w_2$ and hence the result.

On the other hand, if γ is only piecewise smooth, we can parametrised the individual smooth curves resulting in telescoping sum, achieving the same result. □

While this theorem seems intuitive at first, upon closer examination, we see that while the left hand side is dependent on the curve γ, the right hand side is only dependent on the start an end point. Furthermore, if γ, is closed, the above theorem will imply that the integral evaluates to zero.

Corollary 8.1. If γ is a closed curve in an open set $\Omega \subseteq \mathbb{C}$, and $f : \Omega \to \mathbb{C}$ is continuous and has a primitive on Ω, then

$$\oint_\gamma f(z)\mathrm{d}z = 0.$$

This is not necessarily true for functions that does not have a primitive. Consider $f(z) = 1/z$ over $\gamma = C_1(0)$ the unit circle in the positive direction. We see that f does not have a primitive on $\mathbb{C} \setminus \{0\}$ and indeed, if we parametrise γ by $z : [0, 2\pi] \to \mathbb{C} : t \mapsto e^{it}$, we have

$$\oint_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\mathrm{d}t = 2\pi i \neq 0.$$

Corollary 8.2. If $f : \Omega \to \mathbb{C}$ is holomorphic on the open connected set Ω with $f' = 0$, then f is a constant.

Proof. Suppose otherwise, then there exists $w_1, w_2 \in \Omega$ such that $f(w_1) \neq f(w_2)$. So, as Ω is (path)-connected, there exists some γ connecting from w_1 to w_2. Now, as f is the primitive function to $f' = 0$, we have

$$0 = \int_\gamma 0 \ \mathrm{d}z = \int_\gamma f'(z)\mathrm{d}z = f(w_2) - f(w_1) \neq 0,$$

a contraction. # ☐

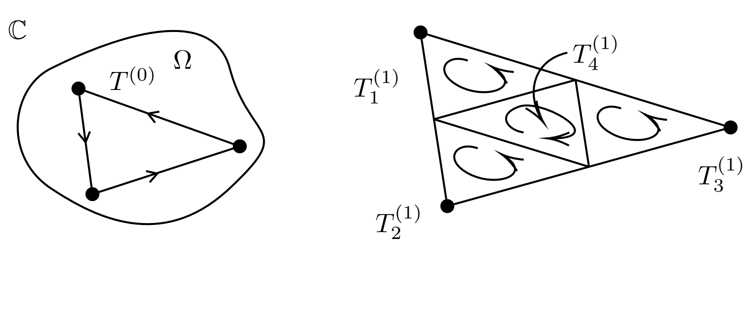
Let us now consider a similar but different result for holomorphic functions.

Theorem 9 (Cauchy's Theorem for Triangles)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set and let $T \subseteq \Omega$ be a triangle whose interior is also contained in Ω, then,

$$\oint_T f(z)\mathrm{d}z = 0,$$

whenever f is holomorphic on Ω.

Proof. Let $T^{(0)}$ be our original triangle with some fixed orientation which we define to be positive and let $d^{(0)}$ and $p^{(0)}$ be the diameter and perimeter of $T^{(0)}$ respectively. By dividing $T^{(0)}$ as the diagram below demonstrates, we can introduce four triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}$ and $T_4^{(1)}$ all of which are similar to $T^{(0)}$.



Then,

$$\oint_{T^{(0)}} f(z)\mathrm{d}z = \oint_{T_1^{(1)}} f(z)\mathrm{d}z + \oint_{T_2^{(1)}} f(z)\mathrm{d}z + \oint_{T_3^{(1)}} f(z)\mathrm{d}z + \oint_{T_4^{(1)}} f(z)\mathrm{d}z.$$

By choosing j such that $\left|\oint_{T_j^{(1)}} f(z)\mathrm{d}z\right|$ is the maximum among the four integrals, we have

$$\left|\oint_{T^{(0)}} f(z)\mathrm{d}z\right| \leq 4\left|\oint_{T_j^{(1)}} f(z)\mathrm{d}z\right|.$$

Now, by renaming $T_j^{(1)} = T^{(1)}$, we observe that

$$d^{(1)} = \frac{1}{2}d^{(0)}; \;\; p^{(1)} = \frac{1}{2}p^{(0)}.$$

So, by repeating this process, we obtain a sequence of triangles $T^{(0)}, T^{(1)}, T^{(2)}, \cdots, T^{(n)}, \cdots$ such that

$$\left|\oint_{T^{(0)}} f(z)\mathrm{d}z\right| \leq 4^n\left|\oint_{T^{(n)}} f(z)\mathrm{d}z\right|,$$

and

$$d^{(n)} = 2^{-n}d^{(0)}; \;\; p^{(n)} = 2^{-n}p^{(0)},$$

where $d^{(n)}, p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$ respectively.

Now, by denoting $\Omega^{(n)}$ be the closed region enclosed by T^n so that $\partial\Omega^{(n)} = T^{(n)}$, we have a sequence of nested nonempty compact sets $\Omega^{(0)} \supseteq \Omega^{(1)} \supseteq \cdots$ whose diameter tends to 0. So, due to results from metric spaces, there exists a unique point z_0 belonging to $\bigcap \Omega^{(n)}$. Since f is holomorphic,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\psi(z)$$

where ψ is a function such that $\psi(z) \to 0$ as $z \to z_0$.

Since $f(z_0) + f'(z_0)(z - z_0)$ is simply a linear function, and thus has a primitive, integrating it over a closed curve will result in 0, and so,

$$\oint_{T^{(n)}} f(z)\mathrm{d}z = \oint_{T^{(n)}} \psi(z)(z - z_0)\mathrm{d}z.$$

Now as $z_0 \in \bigcap \Omega^{(n)}$, $|z - z_0| \leq d^{(n)}$, and so, by ML-inequality,

$$\left|\oint_{T^{(n)}} f(z)\mathrm{d}z\right| \leq \epsilon_n d^{(n)}p^{(n)},$$

where $\epsilon := \sup_{z\in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$. Therefore

$$\left|\oint_{T^{(0)}} f(z)\mathrm{d}z\right| \leq 4^n\left|\oint_{T^{(n)}} f(z)\mathrm{d}z\right| \leq \epsilon_n 4^n 2^{-n}d^{(0)}2^{-n}p^{(0)} \to 0,$$

as $n \to 0$. $\square$

From this theorem, we see that all curves whose shapes are polygons also satisfy Cauchy's theorem. That is, if $\Omega \subseteq \mathbb{C}$ is a open set, $\gamma \subseteq \Omega$ is a curve describing a polygon, and Ω contains the interior of γ, then, if f is holomorphic on Ω,

$$\oint_\gamma f(z)\mathrm{d}z = 0.$$

Indeed, this is true since we can partition any polygon into smaller triangles such that the values from the inner edges cancel, resulting in the total integral to become zero. In fact, a stronger statement is true – the integral of a holomorphic function over any piecewise C^1 curve on a simply connected domain (which we will discuss in the next section) vanishes.

Theorem 10 (Local Existence of Primitives)**.** A holomorphic function in an open disc has a primitive in that disc.

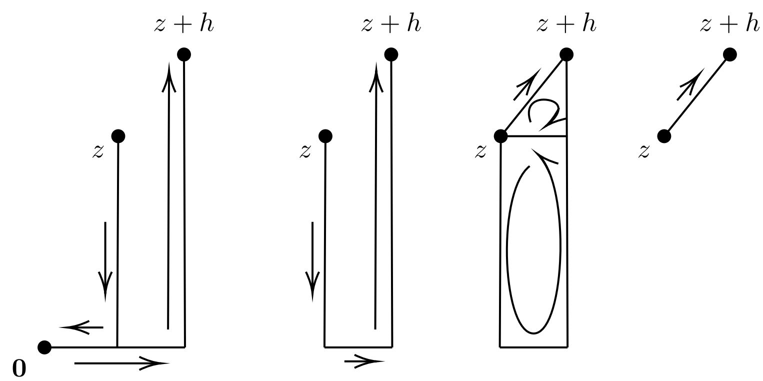
Proof. Wlog. let us assume that the disc D is centred at the origin. Then, for all $z \in D$, we consider the curve $\gamma_z := [0, \mathrm{Re}(z)] \cup \{\mathrm{Re}(z) + ti \mid t \in [0, \mathrm{Im}(z)]\}$, that is the right-angled curve connecting the origin with z following the real axis. Now, as γ_z is piecewise continuous, we may define

$$F(z) := \int_{\gamma_z} f(w)\mathrm{d}w.$$

Consider the difference,

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\mathrm{d}w - \int_{\gamma_z} f(w)\mathrm{d}w,$$

by the diagram below, we find that the difference is in fact the integral along the straight line from z to $z+h$.



By denoting the straight line segment by η, we have

$$F(z+h) - F(z) = \int_\eta f(w)\mathrm{d}w.$$

Now, as f is holomorphic at z, it is also continuous at z, and so $f(w) = f(z) + \psi(w)$ for some function ψ where $\psi(w) \to 0$ as $w \to z$. So,

$$F(z+h) - F(z) = \int_\eta f(z)\mathrm{d}w + \int_\eta \psi(w)\mathrm{d}w = f(z)h + \int_\eta \psi(w)\mathrm{d}w.$$

Thus, by using the ML-inequality,

$$\left| \int_\eta \psi(w) \mathrm{d}w \right| \leq |h| \sup_{w \in \eta} |\psi(w)|,$$

where the right hand side tends to 0 as $w \to z$. So,

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{f(z)h}{h} = f(z),$$

and hence, F is the primitive of f. $\square$

A direct corollary of this is the following.

Corollary 10.1 (Cauchy-Goursat Theorem for a Disk)**.** If f is holomorphic on a disc D, then

$$\oint_\gamma f(z) \mathrm{d}z = 0,$$

for all closed curve $\gamma \subseteq D$.

Similarly, the same can be said about a holomorphic function whose domain contains a disc. That is, if f is holomorphic on the open set Ω which contains the circle C and its interior, then

$$\oint_C f(z) \mathrm{d}z = 0.$$

To see why this is true, we see that, as Ω is open, there exists a larger disk containing C which f is holomorphic on, and so, the result follows.

3.3 Homotopies

Definition 3.8 (Homotopic)**.** Let γ_0, γ_1 be curves in $\Omega \subseteq \mathbb{C}$ with common end-points, that is if γ_0 and γ_1 are two parametrisations defined on $[a, b]$, then $\gamma_0(a) = \gamma_1(a) = \alpha$ and $\gamma_0(b) = \gamma_0(b) = \beta$. Then, we say γ_0 and γ_1 are homotopic in Ω, if for all $s \in [0, 1]$ there exists some curve $\gamma_s \subseteq \Omega$ with parametrisation $\gamma_s : [a, b] \to \Omega$ such that

$$\gamma_s(a) = \alpha; \quad \gamma_s(b) = \beta,$$

and for all $t \in [a, b]$,

$$\gamma_s(t) \mid_{s=0} = \gamma_0(t); \quad \gamma_s(t) \mid_{s=1} = \gamma_1(t).$$

Moreover, $\gamma_s(t)$ is jointly continuous with respect to s and t.

Homotopic curves forms an equivalence relations on the set of curves and processes some very "nice" properties.

Proposition 9. If γ_0 and γ_1 are two homotopic curves, and $f : \Omega \to \mathbb{C}$ is holomorphic on the open set Ω, then

$$\int_{\gamma_0} f(z) \mathrm{d}z = \int_{\gamma_1} f(z) \mathrm{d}z.$$

Proof. If $\gamma_s(t)$ is the homotopy between γ_0 and γ_1, then we can define the continuous function

$$F(s,t) : [0,1] \times [a,b] \to \Omega : (s,t) \mapsto \gamma_s(t).$$

Since F is continuous, and by Heine-Borel $[0,1] \times [a,b]$ is compact, the image of F is also compact and we shall denote it by $\mathcal{K} \subseteq \Omega$.

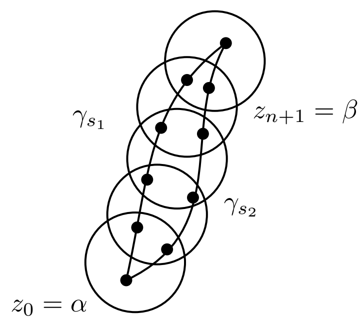
Now, since Ω is open, for all $p \in \mathcal{K}$, there exists some $\epsilon_p > 0$ such that $D_{\epsilon_p}(p) \subseteq \Omega$. Thus, we may construct the open cover $\mathcal{C} := \{D_{\epsilon_p}(p) \mid p \in \mathcal{K}\}$, and so, by compactness, we may find a finite subcover of $\mathcal{C}$ allowing us to choose $\epsilon > 0$ such that for all $p \in \mathcal{K}$, $D_{3\epsilon}(p) \subseteq \Omega$.

Furthermore, as F is uniformly continuous, there exists some δ, such that for all $s_1, s_2 \in \Omega$, $|s_1 - s_2| < \delta$, then

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon.$$

So, we may choose discs $\{D_0, \cdots, D_n\}$ of radius 2ϵ and points $\{z_0, \cdots, z_{n+1}\} \subseteq \gamma_{s_1}$ and $\{w_0, \cdots, w_{n+1}\} \subseteq \gamma_{s_2}$ such that $\gamma_{s_1} \cup \gamma_{s_2} \subseteq \bigcup D_i$ and

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i.$$



Now, by the local existence of primitives, for each disc D_i, there exists some F_i such that F_i is the primitive of f on D_i. By observing that on $D_i \cap D_{i+1}$, F_i and F_{i+1} are both primitives to f, they can only differ by a constant (on $D_i \cap D_{i+1}$). So,

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}),$$

and hence,

$$\begin{aligned}
\int_{\gamma_{s_1}} f(z)\mathrm{d}z - \int_{\gamma_{s_2}} f(z)\mathrm{d}z &= \sum_{i=0}^{n}(F_i(z_{i+1}) - F_i(z_i)) - \sum_{i=0}^{n}(F_i(w_{i+1}) - F_i(w_i)) \\
&= \sum_{i=0}^{n}(F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i))) \\
&= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) \\
&= F_n(\beta) - F_n(\beta) - (F_0(\alpha) - F_0(\alpha)) = 0
\end{aligned}$$

Finally, by dividing the interval $[0,1]$ into subintervals with size less than δ, we see that

$$\int_{\gamma_0} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z.$$

□

Definition 3.9 (Simply Connected)**.** An open set $\Omega \subseteq \mathbb{C}$ is simply connected if any two pairs of curves with the same end points in Ω are homotopic.

Straight away, we see that a disc is connected as given two curves in a disc γ_1, γ_2, we have to homotopy $\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t)$. Furthermore, by the same argument we have any convex sets are also simply connected. On the other hand, we see that any sets which contains a hole is not simply connected.

As mentioned previously, the integral of a holomorphic function over any closed curve on a simply connected set vanishes. This follows straight way from the following theorem.

Theorem 11. Any holomorphic function in a simply connected domain has a primitive.

Proof. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic with the simply connected domain Ω. Then, let us fix a point $z_0 \in \Omega$ and define

$$F(z) = \int_\gamma f(w)\mathrm{d}w,$$

where γ is any curve in Ω joining z_0 to z. We note that this definition is independent of the specific choice of γ since Ω is simply connected. Consider,

$$F(z+h) - F(z) = \int_\eta f(w)\mathrm{d}w,$$

where η is the straight line joining z and $z+h$. By the same argument as our proof for local existence of primitives, we find

$$\lim_{h \to 0} \frac{F(z+h) - F(h)}{h} = f(z),$$

implying F is the primitive of f. $\square$

Corollary 11.1 (Cauchy-Goursat Theorem)**.** Let Ω be a simply connected domain and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then, if γ is a closed, piecewise-smooth curve in Ω,

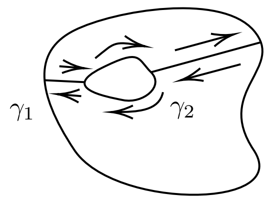
$$\oint_\gamma f(z)\mathrm{d}z = 0.$$

Proof. Indeed, without using the existence of primitives, we see that we may partition γ into two curves with the same end points, and so, as Ω is simply connected, they are also homotopic. Thus, the integral over these two curves result in the same value. Now, by reversing the orientation of one of these curves, we obtain our result. $\square$

Theorem 12 (Deformation Theorem)**.** Let $\gamma_1, \gamma_2 \subseteq \Omega$ be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1. If $f : \Omega \to \mathbb{C}$ is holomorphic in the domain containing the region between γ_1 and γ_2, then

$$\oint_{\gamma_1} f(z)\mathrm{d}z = \oint_{\gamma_2} f(z)\mathrm{d}z.$$

Proof. We note that we cannot directly apply the Cauchy-Goursat theorem as f is not necessarily holomorphic in the region enclosed by γ_2. Instead, we construct a bridge between γ_1 and γ_2.



By considering the slices, we see that the sum of the integrals of f over the top and bottom closed curves is simply

$$\oint_{\gamma_1} f(z)\mathrm{d}z - \oint_{\gamma_2} f(z)\mathrm{d}z.$$

However, as the region covered by the two curves are the same region between γ_1 and γ_2, f is holomorphic on the region enclosed by the two curves, and so, the integrals evaluates to zero by Cauchy-Goursat's theorem and hence,

$$\oint_{\gamma_1} f(z)\mathrm{d}z - \oint_{\gamma_2} f(z)\mathrm{d}z = 0.$$

$\square$

The deformation theorem is very useful whenever we are able to restrict a difficult curve onto a simpler curve (such as a circle) and thus, have a easier time to parametrise the integral. As an example consider the following integral over the curve $\gamma = \{z \in \mathbb{C} \mid |z-1| = 2\}$.

$$\oint_\gamma \frac{\mathrm{d}z}{z^2-4} = \oint_\gamma \frac{\mathrm{d}z}{(z-2)(z+2)} = \frac{1}{4}\oint_\gamma \frac{1}{z-2} - \frac{1}{z+2}\mathrm{d}z.$$

Since $\frac{1}{z+2}$ is holom orphic except at -2 which is outside of γ, we have

$$\oint_\gamma \frac{\mathrm{d}z}{z+2} = 0.$$

On the other hand, to evaluate $\frac{1}{z-2}$, by the deformation theorem, we have

$$\oint_\gamma \frac{\mathrm{d}z}{z-2} = \oint_{\{z \mid |z-2|=1\}} \frac{\mathrm{d}z}{z-2} = \oint_{C_1(0)} \frac{\mathrm{d}w}{w} = 2\pi i.$$

Thus,

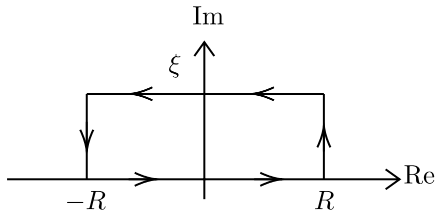
$$\oint_\gamma \frac{\mathrm{d}z}{z^2-4} = i\frac{\pi}{2}.$$

As another example, consider that we would like to show for all $\xi \in \mathbb{R}$,

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi^2}\mathrm{d}x.$$

Suppose $\xi > 0$ (the argument for $\xi < 0$ is similar) and let γ_R be the following curve as described in the diagram below for some $R \in \mathbb{R}^+$. As $f(z) = e^{-\pi z^2}$ is entire, by the Cauchy-Goursat theorem, we have

$$\oint_{\gamma_R} f(z)\mathrm{d}z = 0.$$



On the other hand, we have

$$\oint_{\gamma_R} f(z)\mathrm{d}z = \int_{-R}^{R} f(x)\mathrm{d}x + \int_0^{\xi} f(R+iy)\mathrm{d}y + \int_R^{-R} f(x+i\xi)\mathrm{d}x + \int_{\xi}^{0} f(-R+iy)\mathrm{d}y.$$

Consider

$$\begin{aligned}\left|\int_0^{\xi} f(R+iy)\mathrm{d}y\right| &= \left|\int_0^{\xi} e^{-\pi(R+iy)^2}\mathrm{d}y\right| = \left|\int_0^{\xi} e^{-\pi(R^2-y^2+2iRy)}\mathrm{d}y\right| \\ &= |e^{-\pi R^2}|\left|\int_0^{\xi} e^{\pi y^2}e^{-2i\pi Ry}\mathrm{d}y\right| \leq e^{-\pi R^2}\int_0^{\xi}|e^{\pi y^2}e^{-2i\pi Ry}|\mathrm{d}y \\ &= e^{-\pi R^2}\int_0^{\xi}|e^{\pi y^2}|\mathrm{d}y \leq e^{-\pi R^2}\xi e^{\pi\xi^2}\end{aligned}$$

where the last inequality is due to the ML-indequality. So, the integral tends to 0 as $R \to \infty$. Similarly, $\left|\int_{\xi}^{0} f(R+iy)\mathrm{d}y\right| \to 0$ and so since $\lim_{R\to\infty}\int_{-R}^{R} f(x)\mathrm{d}x = 1$ (the standard Gaussian integral),

$$\begin{aligned}0 &= \lim_{R\to\infty}\oint_{\gamma_R} f(z)\mathrm{d}z = 1 + \lim_{R\to\infty}\int_R^{-R} f(x+i\xi)\mathrm{d}x = 1 - \lim_{R\to\infty}\int_{-R}^{R} f(x+i\xi)\mathrm{d}x \\ &= 1 - \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2}\mathrm{d}x = 1 - \int_{-\infty}^{\infty} e^{-\pi(x^2+2ix\xi-\xi^2)}\mathrm{d}x = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ix\xi}\mathrm{d}x.\end{aligned}$$

So,

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ix\xi}\mathrm{d}x,$$

as required.

4 Cauchy's Integral Formula

Cauchy's integral formulae is an amazing theorem that relates the value of a holomorphic function with a special formula. Indeed, with the Cauchy's integral formula, it is possible to find the value of f as some point z_0 just from the value of f on some neighbourhood. We shall in this chapter investigate Cauchy's integral formula and some of its applications.

Theorem 13 (Cauchy's Integral Formula)**.** Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any z_0 interior to γ, we have

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z.$$

Proof. As $z_0 \in \gamma^\circ$ where γ° is open, there exists some $r' > 0$ such that $B_{r'}(z_0) \subseteq \gamma^\circ$. Furthermroe, as f is holomorphic as z_0, we have f is continuous as z_0. So by fixing $\epsilon > 0$, there exists some $\delta > 0$, such that the continuity criterion applies. So, by letting $r :< \min\{r', \delta\}$, $C := C_r(z_0)$ is a closed curve fully inside of γ such that for all $z \in D_r(z_0)$,

$$|f(z) - f(z_0)| < \epsilon.$$

So, by the deformation theorem, we have

$$\oint_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z = \oint_C \frac{f(z)}{z - z_0}\mathrm{d}z.$$

Then,

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z &= \frac{f(z_0)}{2\pi i}\oint_C \frac{1}{z - z_0}\mathrm{d}z + \frac{1}{2\pi i}\oint_C \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z\\ &= f(z_0) + \frac{1}{2\pi i}\oint_C \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z.\end{aligned}$$

Now, consider, by the ML-inequality,

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z\right| \le \frac{1}{2\pi}\oint_C \left|\frac{f(z) - f(z_0)}{z - z_0}\right|\mathrm{d}z \le \frac{1}{2\pi}\oint_C \frac{\epsilon}{r}\mathrm{d}z = \frac{1}{2\pi}\frac{\epsilon}{r}2\pi r = \epsilon.$$

Since ϵ was arbitrary,

$$\frac{1}{2\pi i}\oint_C \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z = 0$$

and hence the result. ☐

Theorem 14 (Generalised Cauchy's Integral Formula)**.** Let f be holomorphic in an open set Ω, then f has infinitely differentiable in Ω. Moreover, for a simple closed piecewise-smooth curve $\gamma \subseteq \Omega$, and for any $z \in \gamma^\circ$, we have

$$\frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n} = \frac{n!}{2\pi i}\oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\mathrm{d}\eta.$$

Proof. Follows by induction on n. ☐

A corollary of the generalised Cauchy's integral formula is that, if f is holomorphic on Ω, then so is $f', f'', \cdots$.

4.1 Applications of the Cauchy's Integral Formula

The Cauchy's integral formula is used in many places in complex analysis, we shall take a look at some of its applications.

Theorem 15 (Liouville's Theorem)**.** If an entire function is bounded, then it is constant.

Proof. Suppose f is bounded by $M \in \mathbb{R}$, i.e. for all $z \in \mathbb{C}$, $|f(z)| \leq M$. So, for all $z_0 \in \mathbb{C}$, by the generalised Cauchy's integral formula,

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(z)}{(z - z_0)^2} \mathrm{d}z \right| \leq \frac{M}{r} \to 0,$$

as $r \to \infty$. So, for all $z_0 \in \mathbb{C}$, $f'(z_0) = 0$, hence f is constant. $\square$

Unexpectedly, Liouville's theorem results in the fundamental theorem of algebra.

Theorem 16 (Fundamental Theorem of Algebra)**.** Every complex polynomial of degree greater than zero has at least one root.

Proof. Suppose there exists some $f \in \mathbb{C}[z]$ such that for all $z \in \mathbb{C}$, $f(z) \neq 0$. Since f is a polynomial, it is entire, and so, as $f(z) \neq 0$ for all z, $1/f$ is also entire. Furthermore, $1/f$ is bounded since $|1/f(z)| \to 0$ as $|z| \to \infty$, and so, there exists some $R_1 > 0$ such that for all $|z| > R_1$, $|1/f(z)| < M_1$ for some $M_1 \in \mathbb{R}$. Moreover, since $1/f$ is entire, it is continuous, and so, it is bounded on the compact set $\{z \in \mathbb{C} \mid |z| \leq R_1\}$ by some $M_2 \in \mathbb{R}$. So, $1/f$ is bounded by $\max\{M_1, M_2\}$ and hence, by Liouville's theorem, $1/f$ is a constant, and so is f # since $\deg f \neq 0$. $\square$

Corollary 16.1. Every polynomial $f \in \mathbb{C}[z]$ of degree n has precisely n roots $w_1, \cdots, w_n$ such that

$$p(z) = \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{n} (z - w_i).$$

Proof. By FTA, we know f has at least one root w_1, and so, we may factor $f(z) = (z - w_1)g(z)$, where $g \in \mathbb{C}[z]$ and $\deg g = n - 1$. Thus, the result follows by induction. $\square$

Theorem 17 (Morera's Theorem)**.** Suppose f is a continuous function in the open disc D such that for any triangle $T \subseteq D$,

$$\oint_T f(z) \mathrm{d}z = 0,$$

then f is holomorphic.

Straight away, we see that the theorem may be extended so its an if and only if statement since the reverse statement follows by Cauchy-Goursat's theorem.

Proof. We recall that, since f is continuous on a disk, it has a primitive on D, F such that $F' = f$. Now, by the generalised Cauchy's integral formula, we have F is infinitely differentiable, and so f is holomorphic on D. $\square$

Theorem 18 (Taylor Expansion Theorem)**.** Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots,$$

for all $z \in B_r(z_0)$ for some $r > 0$ such that $B_r(z_0) \subseteq \Omega$.

We remark that we do not require f to be infinitely differentiable since this follows from the fact that f is holomorphic.

Proof. Let $\gamma := \{\eta \mid |\eta - z_0| = r\} \subseteq \Omega$ and suppose $z \in \gamma^\circ$. Then, by Cauchy's integral formula,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{\eta - z}\mathrm{d}\eta = \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{(\eta - z_0) - (z - z_0)}\mathrm{d}\eta \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{\eta - z_0}\frac{1}{1 - \frac{z - z_0}{\eta - z_0}}\mathrm{d}\eta \\
&= \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{\eta - z}\left(\sum_{i=0}^{n-1}\left(\frac{z - z_0}{\eta - z_0}\right)^i + \frac{\left(\frac{z - z_0}{\eta - z_0}\right)^n}{1 - \frac{z - z_0}{\eta - z_0}}\right)\mathrm{d}\eta \\
&= \frac{1}{2\pi i}\sum_{i=0}^{n-1}\oint_\gamma \frac{f(\eta)}{\eta - z}\left(\frac{z - z_0}{\eta - z_0}\right)^i\mathrm{d}\eta + \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{\eta - z}\frac{\left(\frac{z - z_0}{\eta - z_0}\right)^n}{1 - \frac{z - z_0}{\eta - z_0}}\mathrm{d}\eta
\end{aligned}$$

So, by recalling the generalised Cauchy's integral formula, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0) + R_n,$$

where

$$R_n = \frac{(z - z_0)^n}{2\pi i}\oint_\gamma \frac{f(\eta)}{(\eta - z)(\eta - z_0)^n}\mathrm{d}\eta.$$

Now, by defining $M = \sup_{\eta \in \overline{\gamma^\circ}} |f(\eta)|$, by the ML-inequality,

$$|R_n| \leq \frac{|z - z_0|^n}{2\pi}\frac{M}{(r - |z - z_0|)r^n(2\pi r)} = \frac{rM}{r - |z - z_0|}\left(\frac{|z - z_0|}{r}\right)^n.$$

Since $|z - z_0| < r$, we have $R_n \to 0$ as $n \to \infty$. □

As with real analysis, we call such an expansion of a function f the Taylor expansion of f and in the specific case that $z_0 = 0$, we call such an expansion the Maclaurin expansion of f.

4.2 Sequences of Holomorphic Functions

Theorem 19. If $(f_n)_{n=1}^\infty$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic on Ω.

We note that this is not necessarily true for real variables, that is, the uniform limit of continuously differentiable functions need not be differentiable. Indeed, by the Weierstrass approximation theorem, every continuous function is the uniform limit of some sequences of polynomials, however, we can easily construct a non-differentiable continuous function.

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in D. Then, since each f_n is holomorphic, Cauchy-Goursat's theorem implies

$$\oint_T f_n(z)\mathrm{d}z = 0,$$

for all n. Furthermore, as $f_n \to f$ uniformly in the closure D, so f is continuous and

$$\oint_T f_n(z)\mathrm{d}z = \oint_T f(z)\mathrm{d}z,$$

and so

$$\oint_T f(z)\mathrm{d}z = 0,$$

hence, by Morera's theorem, we find that f is holomorphic in D. Since D was chosen arbitrarily, f is holomorphic on Ω. $\square$

Corollary 19.1. If $(f_n)_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω such that

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly for all compact subsets of Ω, then F is also holomorphic on Ω.

Proposition 10. Let $(f_n)_{n=1}^{\infty}$ be a sequence of holomorphic functions that converges uniformly to f in every compact subset of Ω. Then the sequence $(f_n')_{n=1}^{\infty}$ converges to f' on every compact subset of Ω.

Proof. Let $U \subseteq \Omega$ such that $\overline{U} \subseteq \Omega$ and for all $\delta > 0$, we define U_δ such that

$$U_\delta := \{z \in U \mid \overline{B_\delta}(z) \subseteq U\}.$$

Then, by the previous theorem, it suffices to show $(f_n')_{n=1}^{\infty}$ converges uniformly to f' on $\overline{U_\delta}$. For any holomorphic function F on U_δ we have,

$$|F'(z)| = \left|\frac{1}{2\pi i}\oint_{|\eta - z| = \delta} \frac{F(z)}{(\eta - z)^2}\mathrm{d}\eta\right| \leq \frac{1}{2\pi}\max_{\eta\in U}|F(\eta)|\frac{1}{\delta^2}2\pi\delta \leq \frac{1}{\delta}\max_{\eta\in U}|F(\eta)|,$$

by the generalised Cauchy's integral formula. Applying this inequality to $F = f_n - f$ completes the proof. $\square$

Using corollary 19.1, we may consider the integral of a holomorphic function as the limit of its Riemann sums, and so show that the integral of a holomorphic function is holomorphic.

Theorem 20. Let $F(z,s)$ be a function defined for $(z,s) \in \Omega \times [0,1]$ where $\Omega \subseteq \mathbb{C}$ is open. Suppose F is holomorphic in Ω with respect to s and is continuous on $\Omega \times [0,1]$. Then, the function f defined on Ω by

$$f(z) = \int_0^1 F(z,s)\mathrm{d}s,$$

is holomorphic.

Proof. Define f_n for the n-th Riemann sum of f, that is

$$f_n(z) := \frac{1}{n}\sum_{k=1}^{n} F(z, k/n).$$

By the construction f_n is holomorphic in Ω and so, it suffices to show that $f_n \to f$ uniformly on any compact $D \subseteq \Omega$. Since $F(z,s)$ is continuous, it is uniformly continuous on D and so, f_n is uniformly continuous on D. Thus, for all $\epsilon > 0$, there exists some $\delta > 0$, for all $a, b \in D$, if $|a - b| < \delta$ then

$$\sup_{z \in D} |F(z,a) - F(z,b)| < \epsilon.$$

So, choosing $n > 1/\delta$, for all $z \in D$, we have

$$\begin{aligned} |f_n(z) - f(z)| &= \left|\sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} F(z,k/n) - F(z,s)\mathrm{d}s\right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z,k/n) - F(z,s)|\mathrm{d}s < \sum_{k=1}^{n} \frac{\epsilon}{n} = \epsilon. \end{aligned}$$

Thus, $f_n \to f$ uniformly on D and so, f is holomorphic. □

4.3 Schwarz Reflection Principle

Before moving on, we will in this short section deal with a simple extension problem for holomorphic functions. From this, we discover the Schwarz Reflection principle which allows one to extend a holomorphic function to a larger domain.

Definition 4.1 (Symmetric Domain)**.** A complex set $\Omega \subseteq \mathbb{C}$ is a symmetric domain if

$$z \in \Omega \iff \bar{z} \in \Omega,$$

and we denote $\Omega^+ := \{z \in \Omega \mid \mathrm{Im} z > 0\}$, $\Omega^- := \{z \in \Omega \mid \mathrm{Im} z < 0\}$ and $I := \{z \in \Omega \mid \mathrm{Im} z = 0\} = \Omega \cap \mathbb{R}$ so that

$$\Omega = \Omega^+ \sqcup \Omega^- \sqcup I.$$

Theorem 21 (Symmetry Principle)**.** If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively and both functions extends continuously to I such that

$$f^+(x) = f^-(x),$$

for all $x \in I$, then the function defined as

$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+ \sqcup I, \\ f^-(z), & z \in \Omega^-, \end{cases}$$

is holomorphic in Ω.

Proof. It suffices to show that f is holomorphic on I since, by construction, it is holomorphic on $\Omega^+ \cup \Omega^-$. For any $x \in I$, let D be a disc centred at x contained in Ω, then, by Morera's theorem, it suffices to show

$$\oint_T f(z)\mathrm{d}z = 0,$$

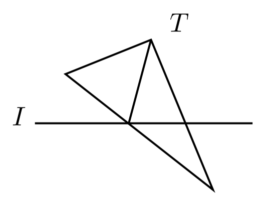
for any triangle T contained in D. If $T \cap I = \varnothing$, then the result is trivial, and so, suppose first that one side of T is contained in I while the rest is contained in (Wlog.) Ω^+. Now, by defining $T_\epsilon := T + \epsilon$ for some $\epsilon > 0$, we have $T_\epsilon \subseteq \Omega^+$, and so,

$$\oint_{T_\epsilon} f(z)\mathrm{d}z = 0.$$

Thus, by taking $\epsilon \to 0$, by continuity, we have

$$\oint_T f(z)\mathrm{d}z = 0.$$

By consulting the following diagram,



we see that every triangle T which $T \cap I \neq \varnothing$ can be decomposed as the three smaller triangles which is covered by the previous cases. We conclude that f is holomorphic in D by Morera's theorem. $\square$

Theorem 22 (Schwarz Reflection Principle)**.** Suppose that f is a holomorphic function in Ω^+ that extends continuously to I such that f is real-valued on I. Then, there exists a function F holomorphic on Ω such that $F|_{\Omega^+} = f$.

Proof. Define F on Ω^- by $F(z) = \overline{f(\bar{z})}$ for all $z \in \Omega^-$. Then to show that F is holomorphic in Ω^-, we note that for all $z, z_0 \in \Omega^-$, the $\bar{z}, \bar{z}_0 \in \Omega^+$, and hence, by Taylor expansion, we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n.$$

Thus,

$$F(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \bar{a}_n(z - z_0)^n,$$

so F is holomorphic on Ω^-. With that, the result follows by the symmetry principle. $\square$

4.4 Complex Logarithm

We return to consider the logarithm as a single-valued function. As we have seen, to define the complex logarithm, we had to restrict the set which it is defined. Indeed, the definition we have seen thus far – the principle logarithm, restricted the domain to $\mathbb{C} \setminus (-\infty, 0]$. This is choice of restriction, however, is rather arbitrary, and as we shall see now, the principle logarithm is simply one of many choices of a *branch* of the logarithm.

Theorem 23. Suppose Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then, in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that,

- F is holomorphic in Ω;
- $e^{F(z)} = z$ for all $z \in \Omega$;
- $F(r) = \log r$ whenever $r \in \mathbb{R}$ and sufficiently close to 1.

In other words, each branch $\log_\Omega$ is an extension of the standard logarithm defined for positive numbers.

Proof. We construct F as a primitive of the function $z \mapsto \frac{1}{z}$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω and so, we may define

$$\log_\Omega(z) = F(z) = \int_\gamma f(z)\mathrm{d}z,$$

where γ is any curve connecting 1 to z. Since Ω is simply connected, this definition does not depend on the path chosen and thus, this definition is well-defined. Now, F is holomorphic on Ω as it is the integral of a holomorphic function, and so, part 1 is established.

For part 2, by rearranging, we see that it suffices to show $ze^{-F(z)} = 1$ for all $z \in \Omega$. Consider,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - z1/z)e^{-F(z)} = 0,$$

and so $ze^{-F(z)}$ is a constant. Thus, as $F(1) = 0$, we have $ze^{-F(z)} \mid_{z=1} = 1$, and so $ze^{-F(z)} = 1$ for all z. $\square$

5 Meromorphic Functions

A meromorphic function on some domain Ω is a holomorphic function except at some isolated points in Ω. Meromorphic functions are useful in that they are nicely behaved as, it turns out, they can be written as a ratio between two holomorphic functions.

5.1 Laurent Series

Let us first consider an important expansion which shall aid us throughout this chapter.

Definition 5.1 (Laurent Series)**.** A series

$$f(z) := \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

is called a Laurent series for f at z_0 whenever the series converges.

Theorem 24 (Laurent Expansion Theorem)**.** Let f be holomorphic in the annulus $D := \{z \mid r < |z - z_0| < R\}$ for some $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}^+$. Then, f can be expressed as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n := \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}} \mathrm{d}\eta,$$

and γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

We note that we consider a annulus rather a disk since as $z \to z_0$, the negative terms diverges to $+\infty$. This allow us to consider the Laurent expansion despite the presence of singularities since we can simply centre the expansion such that the singularity is contained in $\{z \mid |z - z_0| < r\}$.

Proof. Wlog. assume $z_0 = 0$ and define

$$\gamma_1 := \{z \mid |z| < R' < R\} \text{ and } \gamma_2 := \{z \mid |z| > r' > r\}$$

such that $z \in D' := \{z \mid r' < |z| < R\}$. Then,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z} \mathrm{d}\eta - \oint_{\gamma_2} \frac{f(\eta)}{\eta - z} \mathrm{d}\eta := I_1 - I_2.$$

(This is the case since we may appropriate cuts between γ_1 and γ_2) Furthermore, if $\eta \in \gamma_1$, then $|\eta| > |z|$ and so $|z/\eta| < 1$, hence,

$$I_1 = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z} \mathrm{d}\eta = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta(1 - z/\eta)} \mathrm{d}\eta = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}} \mathrm{d}\eta z^n$$

where the last equality is due to the geometric series.

On the other hand, for all $\eta \in \gamma_2$, $|\eta| < |z|$ and so

$$-I_2 = -\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{\eta - z}\mathrm{d}\eta = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{z(1-\eta/z)}\mathrm{d}\eta = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{z^n}\oint_{\gamma_2} f(\eta)\eta^n \mathrm{d}\eta.$$

Thus, by replacing parameters with $-k := n+1$, we have

$$-I_2 = \frac{1}{2\pi i}\sum_{k=-\infty}^{-1}\oint_{\gamma_2} \frac{f(\eta)}{\eta^{k+1}}\mathrm{d}\eta z^k.$$

With that, putting these two integrals we have

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n,$$

with

$$a_n = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}}\mathrm{d}\eta,$$

for non-negative n and

$$a_n = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\eta)}{\eta^{n+1}}\mathrm{d}\eta,$$

for negative n. Hence, the result follows by the deformation theorem. □

5.2 Poles and Zeros

Definition 5.2 (Zeros)**.** We say a holomorphic function $f : \Omega \to \mathbb{C}$ has a zero of order m at $z_0 \in \mathbb{C}$ is

$$f^{(k)}(z_0) = 0,$$

for all $k = 0, 1, \cdots, m-1$, and $f^{(m)}(z_0) \neq 0$.

Proposition 11. A holomorphic function f has a zero of order m at z_0 if and only if it can be written in the form

$$f(z) = (z - z_0)^m g(z),$$

for some g holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. We see that the reverse direction is true straight away, so let us consider the forward direction. Suppose f has a zero of order m at z_0, then, since f is holomorphic, we may write f as a Taylor expansion centred at z_0. Indeed, by the definition of zeros, we have the first m Taylor expansion terms vanishing while the remaining terms all contain $(z - z_0)^m$. By factoring this outside the sum, we have the required decomposition. □

While this proposition seems to be rather obvious, it has some striking consequences.

Corollary 24.1. The zeros of a non-constant holomorphic function are isolated, i.e. every zero has a neighbourhood inside of which it is the only zero.

Proof. If z_0 is a zero of order m of f, then $f(z) = (z - z_0)^m g(z)$ where g is holomorphic at z_0 and $g(z_0) \neq 0$. Then, g is continuous at z_0 and therefore, there exists a neighbourhood around z_0 in which $g(z) \neq 0$. Thus, on this neighbourhood $f(z) = (z - z_0)^m g(z) \neq 0$ except at z_0. $\square$

Definition 5.3 (Singularity)**.** A point z_0 is called a singularity of a complex function f if f is not holomorphic at z_0 but for every neighbourhood of z_0 – U, there exists some $z \in U$ such that f is holomorphic at z.

Definition 5.4 (Isolated)**.** A singularity z_0 of f is said to be isolated if there exists a neighbourhood of z_0 on which z_0 is the only singularity of f.

Definition 5.5. Let z_0 be an isolated singularity of the holomorphic function f on some domain. Then, if

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of f which is valid in some annulus $0 < |z - z_0| < R$,

- if $a_n = 0$ for all $n < 0$, then z_0 is called a removable singularity;
- if $a_n = 0$ for all $n < -m$ where $m \in \mathbb{Z}^+$ is fixed with $a_{-m} \neq 0$, then z_0 is called a pole of order m;
- if $a_n \neq 0$ for infinitely many $n \in \mathbb{Z}^-$, z_0 is called an essential singularity.

Proposition 12. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. If g is holomorphic at z_0 and $g(z_0) \neq 0$, then for some $R > 0$,

$$g(z) = a_0 + a_1(z_1 - z_0) + \cdots,$$

for all $z \in B_R(z_0)$ by Taylor expansion where $a_0 = g(z_0) \neq 0$. Then

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \cdots,$$

for all $z \in B_R(z_0) \setminus \{z_0\}$ and so z_0 is a pole of order m.

Conversely, if f has a pole of order m at z_0, then the Laurent expansion of f is

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots = \frac{1}{(z - z_0)^m}(a_{-m} + a_{-m+1}(z - z_0) + \cdots),$$

which is a power series. Thus by defining

$$g(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots,$$

we have the required function. $\square$

5.3 Residue Theory

Definition 5.6 (Residue)**.** Let f be some complex function with the Laurent expansion at z_0

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n,$$

for $z \in B_R(z_0) \setminus \{z_0\}$. Then, the residue of f at z_0 is

$$\text{Res}[f, z_0] = a_{-1}.$$

Proposition 13. Let $\gamma \subseteq \{z \mid 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve such that $z_0 \in \gamma^\circ$. Then

$$\text{Res}[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z)\mathrm{d}z.$$

Proof. Let $0 < r < R$. Then, by the deformation theorem, we obtain

$$\begin{aligned}\frac{1}{2\pi i} \oint_\gamma f(z)\mathrm{d}z = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\mathrm{d}z = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \sum_{-\infty}^{\infty} a_n (z - z_0)^n \mathrm{d}z \\ = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{in\theta} i r e^{i\theta} \mathrm{d}\theta = a_{-1},\end{aligned}$$

where we parametrised $z = re^{i\theta} + z_0$. □

Proposition 14. Let f be a holomorphic function inside and on a simple, closed piecewise-smooth curve γ except at the singularities $z_1, \cdots z_n$ in its interior. Then

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \text{Res}[f, z_j].$$

Proof. By the deformation theorem, the integral equals to the sum of curves surrounded each of the singularities. Thus,

$$\oint_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\mathrm{d}z,$$

for appropriate γ_j. Hence, the result follows by the previous proposition. □

This result allows us to evaluate complicated integrals. As an example, consider the following integral,

$$\oint_{|z|=1} e^{1/z}\mathrm{d}z.$$

Clearly,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

and so, $\text{Res}[e^{1/z}, 0] = 1$ and hence, $2\pi i = \oint_{|z|=1} e^{1/z}\mathrm{d}z$.

While this result is powerful, it is not always easy to compute the Laurent coefficients explicitly. In this case, we may employ the following method. Suppose f has a pole of order m at z_0 and let $g(z) = (z - z_0)^m f(z)$. Then if $m = 1$, we have

$$g(z) = a_{-1} + a_0(z - z_0) + \cdots ,$$

and so,

$$\mathrm{Res}[f, z_0] = a_{-1} = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Similarly, if $m > 1$, we may simple derivate g, $m - 1$-times resulting in a_{-1} being the first coefficient, allowing us to apply the same argument, i.e.

$$\mathrm{Res}[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} ((z - z_0)^m f(z)).$$

Theorem 25 (Principle of the Argument)**.** Let f be holomorphic in an open set Ω for a finite number of poles and let γ be simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f. Then

$$\oint_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi i(N - P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ respectively.

The name of this theorem follows from the fact that

$$\frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \log(f(z)),$$

and so,

$$\oint_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \oint_\gamma \frac{\mathrm{d}}{\mathrm{d}z} \log(f(z)) \mathrm{d}z = i\Delta \arg f(z).$$