Further Linear Algebra

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1 Introduction

As we have learnt from last year, linear algebra is a important subject regarding matrices, vector spaces, linear maps, and this year we will also take a look at some geometrical interpretations of these concepts.

1.1 Matrices

Definition 1.1 (Similar matrices)**.** Let $A, B \in \mathbb{F}^{n\times n}$ for some field $\mathbb{F}$. We say A is similar to B if and only if there exists some $P \in \mathbb{F}^{n\times n}$ such that,

$$B = P^{-1}AP.$$

We recall that *similar* is an equivalence relation and similar matrices shares many useful properties such as

- same determinant
- same characteristic polynomial
- same Eigenvalues
- same rank

and many more. As similar matrices share so many properties, one major aim in linear algebra is to find a "nice" matrix B given any arbitrary square matrix A such that A and B are similar. We first saw a version of this question last year through the *diagonalisation* of matrices. However, as we have seen, not all matrices are diagonalisable, therefore, in this course, we will take a look at some *weaker* versions that are more general.

A version of our aim is the triangular theorem which states that; given $A \in \mathbb{C}^{n\times n}$ (note that this theorem is not true for arbitrary field), there exists (and not uniquely) some upper triangular matrix $B \in \mathbb{C}^{n\times n}$ such that A is similar to B.

Another version of this aim is the *Jordan Canonical Form* theorem. It turns out if $A \in \mathbb{C}^{n\times n}$, then A is similar to a *unique* matrix in the Jordan canonical form. This theorem is powerful due to the canonical nature of this theorem. One immediate result of this theorem is that we can check whether two matrices are similar to each other by checking where or not they have the same *JCF* (which is computationally easy to do).

However, we see that neither of the above version are theorems over arbitrary fields. The *Rational Canonical Form* attempts to solve this.

Definition 1.2 (Companion matrix)**.** Given an arbitrary field $\mathbb{F}$, $p \in \mathbb{F}[X]$ such that p is monic (i.e. the coefficient of the highest term of p is 1) and $\deg p = k$, the companion matrix of p is the the $k \times k$ matrix

$$C(p) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}$$

where a_i is the coefficient of p of the term X^i in $\mathbb{F}$.

The companion matrix is a nice matrix and it we can in fact show that the characteristic polynomial of the companion matrix of some p is p.

Theorem 1. *Let $A \in \mathbb{F}^{n\times n}$ with characteristic polynomial p. Then, there exists a polynomial factorisation such that $p = \prod_{i=1}^{k} p_i$ and*

$$A \sim \begin{bmatrix} C(p_1) & 0 & \cdots & 0 \\ 0 & C(p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(p_k) \end{bmatrix}.$$

Furthermore, it turns out this factorisation is unique under certain assumptions which we will take a look at in the course.

1.2 Geometry

Recall the dot product on $\mathbb{R}^n$ where given $u, v \in \mathbb{R}^n$, $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$. Furthermore, recall we also took a look at *orthogonal* and *symmetric* matrices last year. All of these, of course, has geometric interpretations and we will in this part of the course generalise and axiomatise these to the theory of *inner product spaces* of V over $\mathbb{R}$. We will also extend this theory to arbitrary fields $\mathbb{F}$ - the *Theory of Bilinear Forms.*

2 Algebraic & Geometric Multiplicities of Eigenvalues

We recall some basic definitions and properties of Eigenvectors.

Definition 2.1. Let V be some vector space, $T : V \to V$ a linear map and λ an Eigenvalue of T. Then the λ-Eigenspace of T is the subspace of V,

$$E_\lambda := \{v \in V \mid (\lambda I_V - T)v = \mathbf{0}\}.$$

We see that this is a subspace as it is the kernel of the linear map $\lambda I_V - T$.

Theorem 2. *Let V be some vector space, $T : V \to V$ a linear map. Suppose that $\{v_1, \cdots, v_k\}$ are Eigenvectors corresponding to distinct Eigenvalues $\lambda_1, \cdots, \lambda_k$, then it is linearly independent.*

Proof. We will prove by contrapositive. Suppose that $\{v_1, \cdots, v_k\}$ are Eigenvectors that are linearly independent. Then by definition, there exists a minimal set of $\{\mu_i \mid i \in I\}$, such that $\sum_{i \in I} \mu_i v_i = 0$ (we see that $\mu_i \neq 0$ for all i as otherwise it is not minimal). Now, let $j \in I$, then by rewriting, we have $v_j = \sum_{i \neq j} \mu_i' v_i$. Thus,

$$\lambda_j \sum_{i \neq j} \mu_i' v_i = \lambda_j v_j = T(v_j) = T\left(\sum_{i \neq j} \mu_i' v_i\right) = \sum_{i \neq j} \mu_i' T(v_i) = \sum_{i \neq j} \mu_i' \lambda_i v_i.$$

So, by rearranging, $0 = \sum_{i \neq j} (\lambda_i - \lambda_j)\mu_i' v_i$. Now, if for all $i \neq j$, $\lambda_i \neq \lambda_j$, we have found a smaller subset of $\{v_1, \cdots, v_k\}$ that is linearly dependent, contradicting our assumption, so there must be some i such that $\lambda_i = \lambda_j$. $\square$

Corollary 2.1. *Let V be a n-dimensional vector space. Then if the characteristic polynomial of the linear map $T : V \to V$ has n distinct roots, then T is diagonalisable.*