Differentiable Equations

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1 Introduction

While we have seen differential equations in year one, we have mostly focused on the different methods of solving specific differential equations. This cannot be expected for general differential equations and in this year, we will focus on existence and uniqueness of solutions to differential equations and develop qualitative tools to help us understand these solutions.

We recall that an algebraic equation is an equation of the form $f(x) = 0$ while a differential equation is an equation of the form $\dot{x} = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$. That is, an algebraic equation has real numbers as solutions while an differential equation has functions as its solution.

As an example, let us consider the simple differentiable equation

$$\dot{x} = ax, \tag{1}$$

for some $a \in \mathbb{R}$. Then, a function $\lambda : I \to \mathbb{R}$ solves 1 if $\dot{\lambda} = a\lambda$ for all $t \in I$ where $I \subseteq \mathbb{R}$ is a interval. These types of differentiable equations occurs often in relation in growth and decay and one can easily see that the family of functions

$$\lambda_b : \mathbb{R} \to \mathbb{R} = t \mapsto be^{at}, \;\; b \in \mathbb{R},$$

are solutions to 1. Of course, we know this already, so an more interesting question would be whether or not this family contains all the solutions to 1. It turns out to be true, and to show this we will assume $\mu : I \to \mathbb{R}$ is a solution to $\dot{x} = ax$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu e^{-at}\right) = \dot{\mu} e^{-at} - a\mu e^{-at} = 0,$$

since $\dot{\mu} = a\mu$ and so, μe^{-at} is constant, i.e. there exists $b \in \mathbb{R}$ such that $\mu e^{-at} = b$ and hence,

$$\mu = be^{at}.$$

This demonstrates that all solutions to 1 are members of the aforementioned solution family and hence, we have found **all** of the solutions to 1.

With the above example, we see that rather than working with solutions that are in finite-dimensional vector spaces, our solution are in function spaces which are typically infinite-dimensional. This is studied in more detail in the next year's *functional analysis* course, and in general, infinite-dimensional spaces are more difficult to grasp. However, for the vast majority of materials in this course, a finite-dimensional thinking suffices while we will also cover some material from functional analysis to understand the differentiable equations as well.

1.1 Ordinary Differential Equations and Initial Value Problems

There are two types of differential equations – *autonomous differential equations* and *nonautonomous differentiable equations*. Autonomous differential equations are differentiable equations of the form $\dot{x} = f(x)$ such as equation 1 while nonautonomous differential equations are equations of the form $\dot{x} = f(t, x)$.

We note that this does not cover higher-order differential equations, but from last year, we recall that one may reduce a higher-order differential equations into a first-order differential equation in vector form and thus, the theories we develop within this course will also apply to higher-order differential equations.

Definition 1.1 (Ordinary Differential Equation)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be open, and a function $f \to D \to \mathbb{R}^d$. Then, an equation of the form

$$\dot{x} = f(t, x)$$

is called a d-dimensional (first-order) ordinary differential equation.

A differentiable function $\lambda : I \to \mathbb{R}^d$ on some interval $I \subseteq \mathbb{R}$ is called a solution of the differential equation if and only if for all $(t, \lambda(t) \in D$, if $t \in I$ then,

$$\dot{\lambda}(t) = f(t, \lambda(t)).$$

We say that an ordinary differential equation is autonomous if f is independent of t and nonautonomous otherwise.

We will only consider ordinary differential equations (ODE) in this course while partial differential equations, that is differential equations which solutions are functions which depends on multiple variables are covered in the second year course **Partial Differential Equations in Action**.

Proposition 1 (Constant solutions to autonomous differential euqations)**.** Let $D \subseteq \mathbb{R}^d$ be an open set and $f : D \to \mathbb{R}^d$ be a function where $d \in \mathbb{N}$. Then, there exists a constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ to the autonomous differential

$$\dot{x} = f(x)$$

for some $a \in \mathbb{R}^d$ if and only if $f(a) = 0$.

Proof. ($\implies$) Suppose that $\lambda : I \to \mathbb{R}^d : x \mapsto a$ is a solution the $\dot{x} = f(x)$. Then

$$0 = \dot{\lambda}(t) = f(\lambda(t)) = f(a).$$

($\impliedby$) Suppose there exists some $a \in \mathbb{R}^d$ such that $f(a) = 0$, then verifying, we find $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to the differential equation. $\square$

This proposition allows us to find solutions to many autonomous ODEs as, indeed, if $f : D \to \mathbb{R}^d$ has a root $a \in \mathbb{R}^d$, the above proposition guarantees that $\lambda : \mathbb{R} \to \mathbb{R}^d : x \mapsto a$ is a solution to $\dot{x} = f(x)$.

Definition 1.2 (Initial Value Problem)**.** Let $d \in \mathbb{N}$, $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be an open set, and $f : D \to \mathbb{R}^d$ be a function. The system of equations from combining the differential equation

$$\dot{x} = f(t, x),$$

with the initial condition

$$x(t_0) = x_0$$

where $(t_0, x_0) \in D$ is called an initial value problem.

A solution to the above initial value problem is a function $\lambda : I \to \mathbb{R}^d$ that is a solution to the differential equation $\dot{x} = f(t, x$ and $\lambda(t_0) = x_0$.