Complex Analysis

Kexing Ying

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1 Complex Numbers

We recall some properties about the complex numbers $\mathbb{C}$.

From **Analysis II** we recall the topological properties of $\mathbb{R}^2$. As there exists a natural homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}$, we conclude that the complex numbers also has these properties.

Proposition 1. The set of complex numbers $\mathbb{C}$ forms a metric space with the induced metric from the Pythagorean norm, that is, the metric

$$d : \mathbb{C} \times \mathbb{C} \to \mathbb{R} : (z, w) \mapsto |z - w| .$$

Proof. One can trivially show that the Pythagorean norm is a norm on $\mathbb{C}$, and hence, the induced metric is a metric on $\mathbb{C}$. $\square$

Theorem 1. The complex numbers equipped with the distance as defined above is Lipschitz equivalent to $\mathbb{R}^2$ equipped with Euclidean metric; so, they are also homeomorphic.

Proof. Trivial. $\square$

Corollary 1.1. The complex numbers is complete and a subset of $\mathbb{C}$ is compact if and only if $\mathbb{C}$ is closed and bounded.

Proof. Follows from the Heine-Borel theorem and the fact that $\mathbb{R}^2$ is complete. $\square$

Certain definitions are also induced for the complex numbers by the fact that it is a metric space. We shall define them here again for referencing.

Definition 1.1. An open disk (ball) in $\mathbb{C}$ centred at $z_0 \in \mathbb{C}$ with radius $r > 0$ is the set

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

The boundary of a disk is the set

$$C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r\}.$$

Lastly, we write $\mathbb{D} := D_1(0)$ for shorthand.

Definition 1.2. Let $S \subseteq \mathbb{C}$ and $z_0 \in S$. We call z_0 an interior point of S if and only if there exists some $r > 0$ such that $D_r(z_0) \subseteq S$. We call the set of interior points of S, S^o – the interior of S and we call S open if and only if every element of S is an interior point of S, i.e. $S = S^o$.

We see that the above definition for open is equivalent to that which is induced by the metric space.

Definition 1.3. Let S be a subset of $\mathbb{C}$, then

- S is closed if and only if S^c is open, or, equivalently, S is closed if and only if for all convergent sequences $(x_n) \subseteq S$, (x_n) converges in S.

- the closure of S, $\overline{S}$ is the smallest closed set containing S, or equivalently, the union of S and its limit points.
- the boundary of S is defined to be $\partial S = \overline{S} \setminus S^o$.
- if S is bounded, then the diameter of S is

$$\operatorname{diam}(S) = \sup_{z,w \in S} |z - w| .$$

- S is (path) connected if and only if for all $z, w \in S$, there exists some continuous function $\gamma : [0,1] \to S$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

We remark that there is no confusion regarding the definition of connectedness in $\mathbb{C}$ since path-connectedness is a stronger notion than connectedness in arbitrary topological spaces while in $\mathbb{R}^n$, open sets are path-connected if they are connected, and so, by traversing the homeomorphism, an open set $S \subseteq \mathbb{C}$ is connected if and only if it is path-connected.

As $\mathbb{C}$ is complete the following proposition follows as compact sets in $\mathbb{C}$ are closed and bounded.

Proposition 2. Let (S_n) be a sequence of non-empty decreasing subsets of $\mathbb{C}$ such that $\operatorname{diam}(S_n) \to 0$ as $n \to 0$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \bigcap_n S_n$.

Proof. This result was previously proved for closed and bounded sets in arbitrary complete metric spaces and so, this result follows as an application of that. □

For good measure, let us also recall some lemmas from school regarding algebraic manipulations of the complex numbers.

Theorem 2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Corollary 2.1 (De Moivre's Formula)**.** Let $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

We note that the above implies $\arg z_1 + \arg z_2 = \arg z_1 z_2$ but it is in general **not** true that $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg} z_1 z_2$ where $\operatorname{Arg} z$ denote the principle argument of z.

2 Complex Functions

As with all spaces, we would like to study the properties of mappings between the complex numbers.

Definition 2.1 (Mapping)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$. Then,

$$f : \Omega_1 \to \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$, there exists only one complex number $w = u + iv$ such that $w = f(z)$.

In this case, we denote $w = f(z) = u(x, y) + iv(x, y)$.

We define a special mapping – the Möbius transformation.

Definition 2.2 (The Möbius Transformation)**.** The Möbius transformation is a mapping such that

$$w = f(z) = \frac{az + b}{cz + d},$$

for some $a, b, c, d \in \mathbb{C}$ where $cz + d \neq 0$ on the domain.

As the complex plane is a metric space, we again have the induces notion of continuity.

Definition 2.3 (Continuity)**.** Let $f : \Omega_1 \to \Omega_2$ be some complex mapping and let $z_0 \in \Omega_1$. We say f is continuous at z_0 if for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all $z \in \mathbb{C}$, $|z - z_0| < \delta$, we have $|f(z) - f(z_0)|$.

We say f is continuous on Ω_1 if it is continuous at every point in Ω_1.

Since the complex plane is homeomorphic to the Euclidean space $\mathbb{R}^2$, one might think to establish a notion of derivative on $\mathbb{C}$. This is achieved, however, not through the definition on general Euclidean spaces, but through another definition.

Definition 2.4 (Holomorphic)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. Then we say f is holomorphic (differentiable) at some $z_0 \in \Omega_1$ if the limit

$$\lim_{h \in \mathbb{C} \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. Here we restricts $z_0 + h \in \Omega_1$ which is fine since Ω_1 is open, and hence, there exists some $\delta > 0$ such that $B_\delta(z_0) \subseteq \Omega_1$. If f is holomorphic at z_0 then we call the quotient its derivative and denote it by $f'(z_0)$.

Let $S \subseteq \mathbb{C}$ be some complex set, then, we say f is holomorphic on S if

- S is open and f is holomorphic on every point of S;
- S is closed and f is holomorphic on some open set containing S.

If f is holomorphic on $\mathbb{C}$ itself then we say f is entire.

We note that we are allowed to make this definition as there exists a notion of division on $\mathbb{C}$ while the same cannot be said for general Euclidean spaces.

The function $f(z) = \overline{z}$ is not holomorphic. Indeed, the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

doest not have a limit as $n \to \infty$ and so our claim.

Proposition 3. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = o(h),$$

or equivalently (without the syntactic sugar),

$$f(z_0 + h) - f(z_0) - ah = h\psi(h)$$

where $\psi : D_\epsilon(0) \to \mathbb{C}$ is a function such that $\lim_{h\to 0} \psi(h) = 0$ for some $\epsilon > 0$.

Proof. Straight away, by dividing both side by h, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \to 0$$

as $h \to 0$. □

By taking $h \to 0$ on both sides of the equation, we have $f(z) \to f(z_0)$ as $z \to z_0$ and so, the following corollary.

Corollary 2.2. A holomorphic function f is continuous.

As one might imagine, the normal properties of derivatives hold for this definition as well.

Proposition 4. If f, g are holomorphic in Ω then,

- $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$;
- fg is holomorphic in Ω and $(fg)' = f'g + fg'$;
- if $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and $(f/g)' = \frac{f'g+fg'}{g^2}$;

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are both holomorphic, the chain rule holds, that is

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Proof. Omitted. One can use the above proposition to make life easier. □

2.1 Cauchy-Riemann Equations

Consider the limit

$$f'(z_0) = \lim_{h=h_1+ih_2\to 0} \frac{f(z_0+h)-f(z_0)}{h}.$$

Assuming that $h = h_1$, namely $h_2 = 0$ and by writing,

$$f(z_0) = f(x_0+iy_0) = u(x_0,y_0) + iv(x_0,y_0),$$

we have,

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \\ &= \lim_{h_1\in\mathbb{R}\to 0} \frac{u(x_0+h_1,y_0)+iv(x_0+h_1,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}x_0,y_0 = u'_x(x_0,y_0)+iv'_x(x_0,y_0). \end{aligned}$$

Similarly, if we let $g = ih_2$ by letting $h_1 = 0$, we have

$$f'(z_0) = \frac{1}{i}\frac{\partial u}{\partial v}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = -iu'_y(x_0,y_0)+v'_y(x_0,y_0).$$

So, if f is holomorphic at z_0, then the two limit should agree, and hence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These two equations together are called the *Cauchy-Riemann equations.*

Definition 2.5 (Cauchy-Riemann Equations)**.** Let $f(z) = u(x,y)+iv(x,y)$ be a mapping, then the Cauchy-Riemann equations are the system of equations

$$u'_x = v'_y; \quad u'_y = -v'_x.$$

With the Cauchy-Riemann equations, we have a necessary condition for a function to be holomorphic. As shown above, we have found that the conjugate function $f = z \mapsto \overline{z}$ is not holomorphic, and we see that as well with its Cauchy-Riemann equations since $u'_x = 1 \neq -1 = v'_y$.

The Cauchy-Riemann equations links real and complex analysis in some sense. By defining the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right),$$

we have the following theorem.

Theorem 3. Let $f = u+iv$. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0,$$

and

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Proof. Trivially follows by using the Cauchy-Riemann equations (except perhaps for showing $f'(z_0) = \partial u/\partial z$ which follows since we can write $f'(z_0) = u'_x(z_0) + iv'_x(z_0)$ and so, the result follows by rewriting with the Cauchy-Riemann equations). $\square$

Similar to the necessary and sufficient conditions for the existence of derivatives for general Euclidean spaces, we would like a similar theorem for determining whether or not a complex valued function is holomorphic. This is achieved with the following theorem.

Theorem 4. Suppose $f = u + iv$ is a complex-valued function defined on some open set Ω. If u, v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and

$$f'(z) = \frac{\partial f}{\partial z}(z).$$

The proof of this theorem is similar to the equivalent statement in $\mathbb{R}^n$, namely, a function is differentiable if and only if it's partial derivatives are continuously differentiable and its derivative is the Jacobian.

Proof. Let $h = h_1 + ih_2$, then,

$$u(x + h_1, y + h_2) - u(x, y) = u'_x(x, y)h_1 + u'_y(x, y)h_2 + o(|h|),$$

and

$$v(x + h_1, y + h_2) - v(x, y) = v'_x(x, y)h_1 + v'_y(x, y)h_2 + o(|h|).$$

By the Cauchy-Riemann equations,

$$v'_x = -u'_y; \;\; v'_y = u'_x,$$

we find that,

$$\begin{aligned} f(z+h) - f(z) &= u(x + h_1, y + h_2) + iv(x + h_1, y + h_2) - u(x, y) - iv(x, y) \\ &= (u(x + h_1, y + h_2) - u(x, y)) + i(v(x + h_1, y + h_2) - v(x, y)) \\ &= u'_x(x, y)h_1 + u'_y(x, y)h_2 + o(|h|) + i(v'_x(x, y)h_1 + v'_y(x, y)h_2 + o(|h|)) \\ &= u'_x h_1 + u'_y h_2 + iu'_x h_2 - iu_y h_1 + o(|h|) \\ &= (u'_x - iu'_y)(h_1 + ih_2) + o(|h|). \end{aligned}$$

This gives us

$$f(z + h) - f(z) - (u'_x - iu'_y)h = o(|h|),$$

implying f is differentiable at z with derivative

$$f'(z) = u'_x - iu'_y = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

$\square$

Lastly, by transforming the complex numbers into their polar forms, we have the following Cauchy-Riemann equations,

$$u'_r = \frac{1}{r}v'_\theta; \;\; v'_r = -\frac{1}{r}u'_\theta.$$

The derivation of this is completely mundane so is omitted here (see lecture slides). This form of Cauchy-Riemann equations can be useful when dealing with certain functions where the function is easier to deal with in polar coordinates.

2.2 Complex Exponentials

We recall the definition and properties of a complex power series.

Definition 2.6. A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$.

We call the aforementioned power series convergent at some $z \in \mathbb{C}$ if the partial sum $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has a limit in $\mathbb{C}$, $S(z) = \lim_{N\to\infty} S_N(z)$. If that is the case we write $\sum_{n=0}^{\infty} a_n z^n$.

Furthermore, we define a power series $\sum a_n z^n$ to be absolutely if $sum\, |a_n|\, |z|^n$ converges. As we have seen from last year, absolute convergence implies converges, and furthermore, if $S(z) = \sum a_n z^n$ then $\lim_{N\to\infty} S(z) - S_N(z) = 0$.

Theorem 5. Given $\sum a_n z^n$ be a power series, then there exists some $R \in \mathbb{R}^+$ such that for all $z \in \mathbb{C}$, if $|z| < R$ then the series converges absolutely and if $|z| > R$ then the series diverges. Moreover,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

The number R is called the radius of convergence and the domain D_R is refered as the disc of convergence.

Proof. See last year's notes. □

The power series provide us with an important class of holomorphic functions.

Theorem 6. The power series $f(z) = \sum_{n=0} a_n z^n$ defines a holomorphic function in its disc of convergence. In fact the derivative of f is simply the sum of the derivative of its individual terms, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f has the same radius of convergence as f.

Proof. We note that $\sum_{n=1}^{\infty} a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^n$ have the same radius of convergence by considering the above proposition. Indeed, if $\sum_{n=1}^{\infty} a_n z^{n-1}$ has radius of convergence R, then $R = (\limsup_{n\to\infty} |a_n|^{1/n})^{-1} = (\limsup_{n\to\infty} |na_n|^{1/n})^{-1}$, which is the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^n$. Beware that we used the fact that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\log n}{n}\right) = 1.$$

Thus, it remains to show that $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Let R be the radius of convergence of f, $|z_0| < r < R$ and define $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and,

$$S_N(z) = \sum_{n=0}^{N} a_n z^n; \quad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then, Wlog. assume $|z_0 + h| < r$, so

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = &\left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) \\ &+ (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right),\end{aligned}$$

for arbitrary $N \in \mathbb{N}$. Thus, by taking $N \to \infty, h \to 0$, we see that the first two term of the right hand side vanished while further examination is required for the last term. Indeed, by the triangle inequality,

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where the last term tends to 0 as $N \to \infty$. Hence,

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) \to 0,$$

as $h \to 0$ and the result is complete. □

As the term-wise derivate of a power series is also a power series, the above theorem results in the following corollary.

Corollary 6.1. A power series is infinitely differentiable within its disc of convergence, and its higher derivatives are also power series obtained by term-wise differentiation.

This reason for this revision and seemingly digression from complex functions is because we would like to introduce complex exponentials. We recall from last year that we defined the exponential function as

$$\exp : \mathbb{R} \to \mathbb{R} : x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and we write $e^x := \exp(x)$. We define the complex exponential on top of this definition.

Definition 2.7. Given $z = x + iy \in \mathbb{C}$, we define the complex exponential,

$$e^z := e^x \cos y + i e^x \sin y.$$

As with the real exponential, the complex exponential has several nice properties.

Proposition 5. Let us denote $e^{(\cdot)}$ for the complex exponential, then,

- if $z = x + 0i$, then $e^z = e^x$;

- $e^{(\cdot)}$ is entire (holomorphic for all $z \in \mathbb{C}$);
- $\frac{\partial}{\partial z} e^z = e^z$;
- if $g(z)$ is holomorphic, then $\frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z)$;
- $e^{z_1+z_2} = e^{z_1} e^{z_2}$;
- $|e^z| = e^x$;
- $(e^z)^n = e^{nz}$;
- $\arg e^z = y + 2\pi k$ for $k = 0, \pm 1, \pm 2, \cdots$.

Proof. Straight forward either by definition or by the Cauchy-Riemann equations. $\square$

From the definition of the complex exponential function, we define the complex trigonometric functions.

Definition 2.8 (Complex Trigonometric Functions)**.** For any $z \in \mathbb{C}$, we define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}); \;\; \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

To see why the trigonometric functions are extended in such a way, one can simply consider Euler's identity where we have

$$e^{i\theta} = \cos\theta + i \sin\theta; \;\; e^{-i\theta} = cos\theta - i\sin\theta,$$

for any $\theta \in \mathbb{R}$. Thus, by assuming $\theta \in \mathbb{C}$, and solving the system of equations, we have the definition of the complex trigonometric functions.

As the complex trigonometric functions are simply linear combinations of the complex exponential, we again get many nice properties for free.

Proposition 6. Let $z = x + iy \in \mathbb{C}$, then

- $\sin(\cdot)$ and $\cos(\cdot)$ are entire functions;
- $\frac{\partial}{\partial z} \sin z = \cos z$ and $\frac{\partial}{\partial z} \cos z = -\sin z$;
- $\sin^2 z + \cos^2 z = 1$;
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ and $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$;

Lastly, we define the complex logarithmic functions.

Definition 2.9 (Complex Logarithm)**.** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then

$$\log z = \log|z| + i \arg z = \log r + i(\theta + 2\pi k),$$

for $k = 0, \pm 1, \pm 2, \cdots$.

Clearly, $e^{\log z} = e^{\log r + i(\theta + 2\pi k)} = re^{i(\theta+2\pi k)} = re^{i\theta} = z$ so it behaves as we would expect. However, as log results in an infinite family of values, and so, is not a function. To account for this, we consider the principle logarithmic function.

Definition 2.10 (Principle Logarithm)**.** The principle logarithmic function is

$$\operatorname{Log} : \mathbb{C} \to \mathbb{C} : z \mapsto \log|z| + i \operatorname{Arg} z.$$

Again, we find that the complex logarithms behave as one might expect.

Proposition 7. Let $z \in \mathbb{C}$, then

- $\log(z_1 z_2) = \log z_1 + \log z_2$;
- the principle logarithm is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

With these definitions in place, we may finally define complex powers.

Definition 2.11. For all $\alpha \in \mathbb{C}$, we define

$$z^\alpha = e^{\alpha \log z}$$

as a multi-valued function and we define its principle value to be

$$z^\alpha = e^{\alpha \mathrm{Log} z}.$$

Proposition 8. Let $z, \alpha_1, \alpha_2 \in \mathbb{C}$, then $z^{\alpha_1 + \alpha_2} = z^{\alpha_1} z^{\alpha_2}$.

2.3 Parametrised Curve

As we shall study integration along curves in the next section, let us first look at parametrised curves.

Definition 2.12 (Parametrised Curve)**.** A parametrised curve is a function $z : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$. We say that the parametrised curve is smooth if z' exists and is continuous on $[a, b]$ and $z'(t) \neq 0$ for all $t \in [a, b]$. At the points $t = a$ and $t = b$, the derivative of z at t is defined to be the one-sided limits

$$z'(a) = \lim_{h \to 0, h > 0} \frac{z(a+h) - z(a)}{h}, \quad z'(b) = \lim_{h \to 0, h < 0} \frac{z(b+h) - z(b)}{h}.$$

Similarly, we say that the parameterised curve is piecewise smooth if z is continuous on $[a, b]$ and there exist a finite number of points $a = a_0 < a_1 < \cdots < a_n = b$ such that z is smooth on $[a_k, a_{k+1}]$. In particular, the right-hand and the left-hand derivative of z at a_k need not to agree.

Definition 2.13 (Equivalence of Parametrisations)**.** Let $z : [a, b] \to \mathbb{C}$ and $\hat{z} : [c, d] \to \mathbb{C}$ be two parametrisations: Then, we say z is equivalent to $hatz$ if there exists a continuous bijective $t : [c, d] \to [a, b]$ such that $t' > 0$ on $[c, d]$ and

$$\hat{z}(s) = z(t(s)),$$

for all $s \in [c, d]$.

The requirement for $t' > 0$ says precisely that the orientation is preserved, since as s travels from c to d, $t(s)$ travels from a to b.

Parametrising curves allows us more easily integrate functions along curves.

Definition 2.14 (Complex Integration)**.** Given a smooth curve $\gamma \subseteq \mathbb{C}$ is parameterised by $z : [a, b] \to \mathbb{C}$, and $f : \gamma \to \mathbb{C}$ is a continuous function, we define

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

In order for this definition to be meaningful, we must show that the integration along γ is independent from the parametrisation. That is, if $\hat{z}$ is an equivalent parametrisation to z, then

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t = \int_c^d f(\hat{z}(t))\hat{z}'(t)\mathrm{d}t.$$

However, this is easy to see by the chain rule:

$$\int_c^d f(\hat{z}(s))\hat{z}'(s)\mathrm{d}s = \int_c^d f(z(t(s))z'(t(s))t'(s)\mathrm{d}t = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

As one might expect, if $\gamma \subseteq \mathbb{C}$ is a piecewise smooth curve and if z is a piecewise smooth parametrisation of γ, then

$$\int_\gamma f(z)\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\mathrm{d}t.$$

Intuitively, if we have a curve in space, it is obvious that we can reverse the orientation to obtain a new curve. We formalise this notion by defining γ^- for any curve γ such that γ and γ^- consists of the same points. Then, if $z : [a, b] \to \mathbb{C}$ is any parametrisation of γ, there is a corresponding parametrisation of γ^-1 defined by

$$z^- : [a, b] \to \mathbb{C} : t \mapsto z(b + a - t).$$

Definition 2.15 (Closed)**.** A smooth or piecewise smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations $z : [a, b] \to \mathbb{C}$.

Definition 2.16 (Simple)**.** A smooth or piecewise smooth curve is simple if for all $s, t \in [a, b]$, $s \neq t$ implies $z(s) \neq z(t)$ for any of its parametrisations $z : [a, b] \to \mathbb{C}$.

We see that a curve is simple if and only if all of its parametrisations are injective. Indeed, if $z(x) = z(y)$, then $x = y$ by contrapositive.

Definition 2.17 (Contour)**.** A contour is a simple closed curve which is piecewise continuously differentiable.