# Advanced Probability Revision

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## Martingale

Regarding the optional stopping theorem, it is useful to write the stopped value of a process as the following:

$$X_T = X_0 + \sum_{k=0}^{\infty}(X_k - X_{k-1})\mathbb{1}_{\{T \geq k\}}.$$

The stopped process is useful to bound the stopped value should one be able to find an applicable rule such that

$$\mathbb{E}[X_T] = \mathbb{E}\left[\lim_{n\to\infty} X_{T\wedge n}\right] = \lim_{n\to\infty} \mathbb{E}[X_{T\wedge n}].$$

## Brownian motion

**Definition** (Brownian motion). A continuous stochastic process $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^n$ if

- $B_0 = x$ a.e. for some constant $x \in \mathbb{R}^n$;
- $B_t - B_s \sim \mathcal{N}(0, (t-s)\mathrm{id}_n)$ for all $s < t$;
- $(B_t)_t$ has independent increments which are also independent of $B_0$.

if $x = 0$ we say the Brownian motion is standard.

**Proposition.** The standard Brownian motion in $\mathbb{R}^n$ is the unique Gaussian process $(B_t)_t$ (i.e. $(B_{t_1}, \cdots, B_{t_k})$ is a Gaussian random vector) satisfying $\mathbb{E}[B_t] = 0$ and $\mathrm{Cov}(B_s, B_t) = (s \wedge t)\mathrm{id}_n$ for all $s, t \geq 0$.

**Proposition.** A Brownian motion does not reach its maximum almost surely, i.e. for any $t > 0$, the event $\{B_t = \sup_{0\leq s\leq t} B_s\}$ is a null-set.

*Proof.* Fix $t > 0$ and we observe

$$B_t = \sup_{0\leq s\leq t} B_s \iff \inf_{0\leq s\leq t}(B_t - B_s) \geq 0 \iff \inf_{0\leq s\leq t}(B_t - B_{t-s}) \geq 0.$$

Now, it is clear that $(B_t - B_{t-s})_{s\in[0,1]}$ is a standard Brownian motion and thus, the right hand side occurs with probability 0 proving the claim. □

To show a random time is *not* a stopping time, one may invoke the strong Markov property of Brownian motions. Namely, if we want to show $T$ is not a stopping time it suffices to show $(B_{T+t} - B_T)_t$ is not a Brownian motion.

**Proposition.** A Brownian motion is $\alpha$-Hölder continuous for all $\alpha < 1/2$.

*Proof.* By continuity, it suffices to show Hölder continuity on the Dyadic numbers for which we invoke the Kolmogorov's continuity criterion. Namely, if for all $s, t \in \mathscr{D}$,

$$\mathbb{E}[|B_s - B_t|^p] \leq c|s - t|^{1+\epsilon}$$

for some constant $c$, then $(B_t)$ is $\alpha$-Hölder continuous on $\mathscr{D}$ for all $\alpha \in (0, \epsilon/p)$. To show this we observe that (WLOG. $s < t$) $B_t - B_s \sim \mathcal{N}(0, (t-s)\mathrm{id}_n)$ and so,

$$\mathbb{E}[|B_t - B_s|^p] \leq \mathbb{E}[|Z|^p]|t-s|^{p/2},$$

where $Z \sim \mathcal{N}(0, 1)$. Hence, as $\mathbb{E}[|Z|^p] < \infty$ for all $p$, we have $(B_t)$ is $\alpha$-Hölder continuous for all $\alpha \in \left(0, \frac{p/2-1}{p}\right) = \left(0, \frac{1}{2} - \frac{1}{p}\right)$. Finally, for all $\alpha < 2$, there exists some $p$ such that $\frac{1}{2} - \frac{1}{p} > \alpha$ and so, $(B_t)$ is $\alpha$-Hölder continuous. $\square$