# Functional Analysis

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**Abstract**

This note contains parts of the course *Functional Analysis* taught by András Zsák for Part III students at the University of Cambridge. I will omit the initial parts of the course reviewing linear operator theory and numerous Hahn-Banach theorems. I will also omit the proof of the Radon-Nikodym theorem.

## Contents

# 1 The Dual of $L_p$ and $C(K)$

We will always work in the measure space $(\Omega, \mathscr{F}, \mu)$. We recall the Radon-Nikodym theorem and its related results.

**Theorem 1** (Hahn decomposition)**.** Given a signed measure $\nu : \mathscr{F} \to \mathbb{R}$, there exists a disjoint partition $A, B \in \mathscr{F}$, $A \sqcup B = \Omega$ such that for all $S \subseteq A$, $\nu(S) \geq 0$ and for all $S \subseteq B$, $\nu(S) \leq 0$.

**Corollary 1.1** (Hahn-Jordan decomposition of a signed measure)**.** Given a signed measure $\nu$, there exists unique measures $\nu^+, \nu^-$ such that for all $S \in \mathscr{F}$, $\nu(S) = \nu^+(S) - \nu^-(S)$.

**Theorem 2** (Radon-Nikodym)**.** Given $\mu$ is $\sigma$-finite, $\nu : \mathscr{F} \to \mathbb{C}$ is a complex measure such that $\nu \ll \mu$, there exists a unique $f \in L_1(\mu)$ such that for all $S \in \mathscr{F}$,

$$\nu(A) = \int_A f \, \mathrm{d}\mu.$$

**Remark.** This $f$ is said to be the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ and we denote it by $\mathrm{d}\nu/\mathrm{d}\mu$.

**Remark.** In the case $\nu$ is not necessarily absolutely continuous with respect to $\mu$, we can decompose $\nu = \nu_1 + \nu_2$ where $\nu_1, \nu_2$ are complex measures such that $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$ (i.e. there exists some $S \in \mathscr{F}$ such that $\nu_2(S) = 0 = \mu(S^c)$).

## 1.1 Dual of $L_p$

Utilizing the Radon-Nikodym theorem, we in this section show that for all $p \in [1, \infty)$, $L_p^*$ is isometrically isomorphic to $L_q$ for $p, q$ Hölder conjugates.

The map we consider is

$$\phi : L_q \to L_p^* : g \mapsto \phi_g$$

where we define $\phi_g(f) := \int f g \mathrm{d}\mu$. This map is well defined since $|\phi_g(f)| \leq \|g\|_q \|f\|_p$ and so $\|\phi_g\| \leq \|g\|_q < \infty$. As $\phi$ is clearly linear, this furthermore shows that $\phi$ is bounded.

**Theorem 3.** For $p \in (1, \infty)$, $\phi$ is a isometric isomorphism between $L_q$ and $L_p^*$. Furthermore, in the case that $\mu$ is $\sigma$-finite, the same remains to hold for $p = 1$.

*Proof.* We first consider the case $p \in (1, \infty)$ and we show that $\phi$ is isometric.

Let $g \in L_q$, we have already shown that $\|\phi_g\| \leq \|g\|_q$. We now show the converse inequality. Define

$$f = \begin{cases} \frac{|g|^q}{g}, & g \neq 0, \\ 0, & g = 0. \end{cases}$$

It suffices to show $|\phi_g(f)|/\|f\|_p$ achieves $\|g\|_q$. Indeed,

$$\int |f^p| \mathrm{d}\mu = \int |g|^{(q-1)p} \mathrm{d}\mu = \int |g|^q \mathrm{d}\mu < \infty$$

and so $f \in L_p$ and $\|f\|_p^p = \|g\|_q^q$. Thus,

$$|\phi_g(f)| = \int |g|^q \mathrm{d}\mu = \|g\|_q^q = \|f\|_p^p,$$

implying

$$\frac{|\phi_g(f)|}{\|f\|_p} = \|f\|_p^{p-1} = \|g\|_q^{\frac{q(p-1)}{p}} = \|g\|_q$$

as claimed.

We now show that $\phi$ is surjective. We first consider the case that $\mu$ is finite.

Fix $\psi \in L_p^*$. Define

$$\nu(A) = \psi(1_A).$$

I claim that $\nu$ is a complex measure. Indeed,

- $\nu(\varnothing) = \psi(0) = 0$, and
- for $(A_n) \subseteq \mathscr{F}$ disjoint,

$$\begin{aligned}\left|\nu\left(\bigcup_n A_n\right) - \sum_{n=1}^N \nu(A_n)\right| &= \left|\psi\left(1_{\bigcup_n A_n} - \sum_{n=1}^N 1_{A_n}\right)\right| \\ &\leq \|\psi\| \left\|1_{\bigcup_n A_n} - \sum_{n=1}^N 1_{A_n}\right\|_p = \|\psi\| \mu\left(\bigcup_{n=N}^\infty A_n\right)^{1/p}\end{aligned}$$

  which converges to 0 as $N \to \infty$ implying $\sigma$-additivity.

Furthermore, it is clear that $\nu \ll \mu$ and so by the Radon-Nikodym theorem, there exists a unique $g \in L_1(\mu)$ such that for all $S \in \mathscr{F}$,

$$\psi(1_S) = \nu(S) = \int_S g \,\mathrm{d}\mu.$$

Thus, it follows that for any simple function $f$, $\int f g \,\mathrm{d}\mu = \psi(f)$.

Now approximating $f \in L_\infty$ by simple functions $f_n \uparrow f$, we have

$$\psi(f) = \lim_{n\to\infty} \psi(f_n) = \lim_{n\to\infty} \int f_n g \,\mathrm{d}\mu \overset{\text{MCT}}{=} \int f g \,\mathrm{d}\mu.$$

With this, we would like to conclude by the density of $L_\infty$ in $L_p$ (this is what requires $\mu$ to be finite). However to do so, we need to first check that $g \in L_q$ and so $\phi_g$ is in fact in $L_p^\infty$. Let us check this now:

For $n \in \mathbb{N}$, let $A_n = \{0 < |g| < n\}$ and $f = 1_{A_n}|g|^q/g \in L_\infty \subseteq L_p$. Then,

$$\int |g|^q \mathrm{d}\mu = \int f g \,\mathrm{d}\mu = \psi(f) \leq \|\psi\| \|f\|_p = \|\psi\| \left(\int_{A_n} |g|^q \mathrm{d}\mu\right)^{1/p}.$$

Thus, taking $n \to \infty$, we have by the monotone convergence theorem

$$\|\psi\| \geq \left(\int_{A_n} |g|^q \mathrm{d}\mu\right)^{1-1/p} = \|g\|_q$$

and so $g \in L_q$ as required. With this, by the previous remark, we conclude that $\phi_g = \psi$ for $\mu$ finite case by leveraging the density of $L_\infty$ in $L_p$.

Before proving the general case, let us first introduce the following notations: given $B \in \mathscr{F}$, we denote $\mathscr{F}_B = \{A \in \mathscr{F} \mid A \subseteq B\}$ and $\mu_B = \mu|_B$ so $(\Omega, \mathscr{F}_B, \mu_B)$ is a measure space and $L_p(\mu_B) \subseteq L_p(\mu)$. Furthermore, given $\psi \in L_p(\mu)^*$, we denote $\psi_B = \psi|_{L_p(\mu_B)}$ so $\psi_B \in L_p(\mu_B)^*$ and $\|\psi_B\| \leq \|\psi\|$. We note the following claim:

*Given $B, C \in \mathscr{F}$ are disjoint, $\|\psi_{B\cup C}\|^q = \|\psi_B\|^q + \|\psi_C\|^q$.*

*Proof of claim.* Let $f \in L_p(\mu_{B\cup C})$. Then,

$$\begin{aligned}
|\psi_{B\cup C}(f)| &= |\psi_B(f|_B) + \psi_C(f|_C)| \\
&\leq |\psi_B(f|_B)| + |\psi_C(f|_C)| \\
&\leq \|\psi_B\|\|f|_B\|_p + \|\psi_C\|\|f|_C\|_p \\
&\overset{\text{Hölder}}{\leq} (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q}(\|f|_B\|_p^p + \|f|_C\|_p^p)^{1/p} \\
&= (\|\psi_B\|^q + \|\psi_C\|^q)^{1/q}\|f\|_p
\end{aligned}$$

implying $\|\psi_{B\cup C}\|^q \leq \|\psi_B\|^q + \|\psi_C\|^q$.

On the other hand, for the reverse direction, fix $a, b \geq 0$ such that $a^p + b^q = 1$ and

$$a\|\psi_B\| + b\|\psi_C\| = (\|\psi_B\|^a + \|\psi_C\|^b)^{1/q}.$$

Then, given $f \in L_p(\mu_B), g \in L_p(\mu_C)$, with $\|f\|_p, \|g\|_p \leq 1$, $\alpha, \beta$ scalars such that $|\alpha| = |\beta| = 1$ and

$$\alpha\psi_B(f) = |\psi_B(f)| \text{ and } \beta\psi_C(g) = |\psi_C(g)|,$$

we observe

$$a|\psi_B(f)| + b|\psi_C(g)| = \psi_{B\cup C}(a\alpha f + b\beta g) \leq \|\psi_{B\cup C}\|\|a\alpha f + b\beta g\|_p \leq \|\psi_{B\cup C}\|$$

implying $\|\psi_{B\cup C}\|^q \geq \|\psi_B\|^q + \|\psi_C\|^q$ as required. □

Let us now consider the case $\mu$ is $\sigma$-finite. In this case, by definition, there exists a countable measurable partition of $\Omega$: $(A_n)$ such that $\mu(A_n) < \infty$ for all $n$. So, for $\psi \in L_p(\mu)^*$, we can restrict $\psi$ onto $A_n$ and apply the previous case. Namely, for all $n$, there exists some $g_n \in L_q(\mu_{A_n})$ such that

$$\psi_{A_n}(f) = \int f g_n \mathrm{d}\mu_{A_n} = \int_{A_n} f g_n \mathrm{d}\mu.$$

Observe that for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \|g_n\|_q^q = \sum_{n}^{N} \|\psi_{A_n}\| \overset{(*)}{=} \|\psi_{\bigcup_{n=1}^{N} A_n}\| \leq \|\psi\| < \infty.$$

where $(*)$ follows by the claim.

So, by defining $g = g_n$ on $A_n$, we have by the monotone convergence theorem $g \in L_q(\mu)$ and $\phi_g = \psi$ on $L_p(\mu_{A_n})$ for all $n$. Hence, as $\bigcup L_p(\mu_{A_n})$ has dense linear span, $\psi = \phi_g$ as required.

Finally, for the general case, take $\psi \in L_p(\mu)^*$ and choose $(f_n)$ to be a sequence in $L_p(\mu)$ such that $\|f_n\| \leq 1$ for all $n$ and

$$\psi(f_n) \to \|\psi\| \text{ as } n \to \infty.$$

Recall that for $f \in L_p(\mu)$,

$$\{f \neq 0\} = \bigcup_n \{|f| > n^{-1}\}$$

which is $\sigma$-finite as by Markov's inequality,

$$\mu(\{|f| > n^{-1}\}) \leq n^p \|f\|_p^p < \infty$$

for all $n$. Thus, defining $B = \bigcup_n \{f_n \neq 0\}$, $B$ is $\sigma$-finite and by the $\sigma$-finite case, there exists some $g \in L_q(\mu_B)$ such that $\phi_g = \psi_B$. Now, by the claim,

$$\|\psi\|^q = \|\psi_B\|^q + \|\psi_{\Omega \setminus B}\|^q$$

while by construction, $\|\psi\|^q = \|\psi_B\|^q$. Thus, $\psi_{\Omega\setminus B} = 0$ and $\psi = \phi_g$.

We now start proving the case $p = \infty$ and $\mu$ is $\sigma$-finite. We first show $\phi$ is isometric. Let $g \in L_\infty(\mu)$. We've already shown $\|\phi_g\| \leq \|g\|_\infty$ so it suffices to show the reverse inequality. WLOG. assume that $g \neq 0$ and fix $0 < s < \|g\|_\infty$ and define $A = \{|g| > s\}$. Straightaway, we note $\mu(A) > 0$ and so, as $\mu$ is $\sigma$-finite, there exists some $B \subseteq A$, $0 < \mu(B) < \infty$. Defining $f = 1_B |g|/g$, we have $f \in L_1$ and

$$s \leq \int_B |g| \mathrm{d}\mu = \phi_g(f) \leq \|\phi_g\| \|f\|_1 = \|\phi_g\| \mu(B).$$

Hence, $s \leq \|\phi_g\|$ and as $s < \|g\|_\infty$ was arbitrary, we have $\|\phi_g\| \geq \|g\|_\infty$ as required.

For subjectivity, we proceed similarly to the first case. Given $\psi \in L_1^*$, define

$$\nu(A) = \psi(1_A), \text{ for all } A \in \mathscr{F}.$$

$\nu$ is a complex measure and by Radon-Nikodym, there exists some $g \in L_1$, $\nu(A) = \int_A g \mathrm{d}\mu$ for all $A \in \mathscr{F}$. Then, by approximating with simple functions, it is clear that $\psi(f) = \int f g \mathrm{d}\mu$ for all $f \in L_\infty$.

We now show $g \in L_\infty$. Fix

$$t > \|\psi\|, A = \{|g| > t\}, f = 1_A \frac{|g|}{g}.$$

Then $f \in L_\infty$ and thus,

$$t\mu(A) \leq \int_A |g| \mathrm{d}\mu = \int f g \mathrm{d}\mu = \psi(f) \leq \|\psi\| \|f\|_1 = \|\psi\| \mu(A).$$

However, as $t > \|\psi\|$ by definition, we have $\mu(A) = 0$ implying $g \in L_\infty$.

So far we've shown $\psi = \phi_g$ on $L_\infty$. To show $\psi = \phi_g$ on $L_1$, we use the fact that $L_\infty \subseteq L_1$ is dense for all *finite* measures $\mu$. As $\mu$ is $\sigma$-finite, let $(A_n)$ be a measurable partition of $\Omega$ of finite measures. Then For all $\psi \in L_1(\mu)^*$, as $\mu_n = \mu|_{A_n}$ is finite, there exists some $g_n \in L_\infty(\mu_n)$ such that

$$\psi_n(f) = \psi|_{A_n}(f) = \int_{A_n} f g_n \mathrm{d}\mu_n = \int g_n f \mathrm{d}\mu.$$

Now, as $\phi$ is isometric, $\|g_n\|_\infty = \|\psi_n\| \leq \|\psi\|$. Hence, taking $g = g_n$ on $A_n$, $g \in L_\infty$ and $\phi_g = \psi$ as required. $\square$

**Corollary 3.1.** For all $1 < p < \infty$, $L_p(\mu)$ is reflexive.

*Proof.* The previous theorem provides the isometric isomorphism

$$\phi : L_q \to L_p^*, \langle f, \phi(g)\rangle = \int f g \mathrm{d}\mu.$$

Then, its dual (see example sheet 1) $\phi^* : (L_p^*)^* \to L_q^*$ is also an isometric isomorphism. Now, denoting $\psi : L_p \to L_q^*$ the isometric isomorphism from $L_p$ to $L_q^*$ (constructed the same way as $\phi$), It suffices to show that $(\phi^*)^{-1} \circ \psi : L_p \to (L_p^*)^*$ is the canonical embedding. Indeed, for all $f \in L_p, g \in L_q$,

$$\langle g, \phi^*(\hat{f})\rangle = \langle \phi(g), \hat{f}\rangle = \langle f, \phi(g)\rangle = \int f g \mathrm{d}\mu = \langle g, \psi(f)\rangle,$$

so $\phi^*(\hat{f}) = \psi(f)$ as claimed. □

## 1.2 Dual of $C(K)$

### 1.2.1 Preliminary definitions

For this section, we take $K$ to be a compact Hausdorff space and introduce the following notations:

- $C(K) = \{f : K \to \mathbb{C} \mid f \text{ continuous}\}$ equipped with the sup-norm;
- $C^{\mathbb{R}}(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$;
- $C^+(K) = \{f \in C^{\mathbb{R}}(K) \mid f \geq 0\}$;
- $M(K) = C(K)^*$;
- $M^{\mathbb{R}}(K) = \{\phi \in M(K) \mid \forall \phi \in C^{\mathbb{R}}(K), \phi(f) \in \mathbb{R}\}$;
- $M^+(K) = \{\phi : C(K) \to \mathbb{C} \mid \phi \text{ linear and } \forall f \in C^+(K), \phi(f) \geq 0\}$.

We call elements of $M^+(K)$ positive linear functionals.

**Lemma 1.1.** Given $\phi \in M(K)$, there exists unique $\phi_1, \phi_2 \in M^{\mathbb{R}}(K)$ such that $\phi = \phi_1 + i\phi_2$.

*Proof. Uniqueness*: We observe for $f \in C^{\mathbb{R}}(K)$, if $\phi = \phi_1 + i\phi_2$,

$$\phi(f) = \phi_1(f) + i\phi_2(f) \text{ and } \overline{\phi(f)} = \phi_1(f) - i\phi_2(f).$$

Thus,

$$\begin{cases} \phi_1(f) = \mathrm{Re}(\phi(f)) = \frac{\phi(f)+\overline{\phi(f)}}{2}, \\ \phi_2(f) = \mathrm{Im}(\phi(f)) = \frac{\phi(f)-\overline{\phi(f)}}{2i}, \end{cases}$$

so $\phi_1$ and $\phi_2$ are uniquely determined by $\phi$ on $C^{\mathbb{R}}(K)$ and hence also on $C(K) = C^{\mathbb{R}}(K) + iC^{\mathbb{R}}(K)$.

*Existence*: This works:

$$\begin{cases} \phi_1(f) = \frac{\phi(f)+\overline{\phi(\bar{f})}}{2}, \\ \phi_2(f) = \frac{\phi(f)-\overline{\phi(\bar{f})}}{2i}. \end{cases}$$

□

**Lemma 1.2.** The map

$$\phi \mapsto \phi|_{C^{\mathbb{R}}(K)} : M^{\mathbb{R}}(K) \to C^{\mathbb{R}}(K)^*$$

is an isometric isomorphism.

*Proof.* Take $\phi \in M^{\mathbb{R}}(K)$. It is clear that $\|\phi|_{C^{\mathbb{R}}(K)}\| \leq \|\phi\|$. On the other hand, for $f \in C(K)$, take $\lambda \in S^1 \subseteq \mathbb{C}$ such that $\lambda\phi(f) = |\phi(f)|$. Then,

$$|\phi(f)| = \phi(\lambda f) = \phi(\mathrm{Re}(\lambda f)) + i\phi(\mathrm{Im}(\lambda f)).$$

However, as the left hand side is real, $\phi(\mathrm{Im}(\lambda f)) = 0$ and so

$$|\phi(f)| = \phi(\mathrm{Re}(\lambda f)) \leq \|\phi|_{C^{\mathbb{R}}(K)}\|\|\mathrm{Re}(\lambda f)\| = \|\phi(f)\|\|f\|$$

proving isometry.
To prove subjectivity, take $\psi \in C^{\mathbb{R}}(K)$. Then, defining

$$\phi(f) = \phi(\mathrm{Re}(f)) + i\psi(\mathrm{Im}(f))$$

for all $f \in C(K)$. It is clear $\phi \in M(K)$ and $\phi|_{C^{\mathbb{R}}(K)} = \psi$ as required. □

**Lemma 1.3.** $M^+(K) \subseteq M(K)$ (and in particular are continuous) and

$$M^+(K) = \{\phi \in M(K) \mid \|\phi\| = \phi(1_K)\}.$$

*Proof.* Let $\phi \in M^+(K)$ and $f \in C^{\mathbb{R}}(K), \|f\|_\infty \leq 1$ so that $1_K \pm f \geq 0$. Then,

$$0 \leq \phi(1_K \pm f) = \phi(1_K) \pm \phi(f)$$

implying $\phi(1_K) \geq |\phi(f)|$ and hence $\|\phi|_{C^{\mathbb{R}}(K)}\| = \phi(1_K)$. Thus, by the previous lemma, $\phi \in M^{\mathbb{R}}(K)$ with $\|\phi\| = \phi(1_K)$, i.e. we've shown

$$M^+(K) \subseteq \{\phi \in M(K) \mid \|\phi\| = \phi(1_K)\}$$

Now, suppose $\phi \in M(K)$ is such that $\|\phi\| = \phi(1_K)$, we want to show $\phi \in M^+(K)$. WLOG. assume $\|\phi\| = 1$. Then, taking $f \in C^{\mathbb{R}}(K)$, $\|f\|_\infty \leq 1$, let us denote $\phi(f) = a + ib$ for some $a, b \in \mathbb{R}$. Observe, for $t \in \mathbb{R}$,

$$|\phi(f + it1_K)|^2 = |a + (b+t)i|^2 = a^2 + b^2 + 2bt + t^2$$

while on the other hand,

$$|\phi(f + it1_K)|^2 \leq \|\phi\|^2\|f + it1_K\|^2 \leq 1 + t^2$$

and so $a^2 + b^2 + 2bt \leq 1$ for all $t$ which is only possible if $b = 0$. Thus, $\phi$ takes value in $\mathbb{R}$.

Now taking $f \in C^+(K)$, $\|f\|_\infty \leq 1$, we have $0 \leq f \leq 1_K$ and so

$$-1_K \leq 1_K - 2f \leq 1_K$$

implying $\|1_K - 2f\|_\infty \leq 1$. Hence

$$1 - 2\phi(f) = \phi(1_K - 2f) \leq 1$$

implying $\phi(f) \geq 0$ and so $\phi \in M^+(K)$ as claimed. □

**Lemma 1.4.** For all $\phi \in M^{\mathbb{R}}(K)$, there exists unique $\phi^+, \phi^- \in M^+(K)$ such that $\phi = \phi^+ - \phi^-$ and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$.

*Proof. Existence*: Define $\phi^+$ on $C^+(K)$ as follows: for all $f \in C^+(K)$, take

$$\phi^+(f) = \sup\{\phi(g) \mid g \in C^+(K), g \leq f\}.$$

It is clear that $\phi^+(f) \geq \phi(0) = 0$ and $\phi^+(f) \geq \phi(f)$. Furthermore, $\phi^+$ is additive since for all $f_1, f_2 \in C^+(K)$, $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, we have

$$\phi^+(f_1 + f_2) \geq \phi(g_1 + g_2) = \phi(g_1) + \phi(g_2).$$

Hence, taking the supremum over $g_1$ and $g_2$ provides

$$\phi^+(f_1 + f_2) \geq \phi^+(f_1) + \phi^+(f_2).$$

One the other hance, given $0 \leq g \leq f_1 + f_2$,

$$\phi(g) = \phi(g \wedge f_1) + \phi(g - (g \wedge f_1)) \leq \phi^+(f_1) + \phi^+(f_2)$$

since $g \wedge f_1 \leq f_1$ and $g - (g \wedge f_1) \leq g - f_1 \leq f_2$.
Now, we define $\phi^+$ on $C^{\mathbb{R}}(K)$. such that for all $f \in C^{\mathbb{R}}(K)$, by writing $f = f^+ - f^-, f^\pm \in C^+(K)$, we take

$$\phi^+(f) = \phi^+(f^+) - \phi^+(f^-).$$

Finally, to define $\phi^+$ on $C(K)$, for all $f \in C(K)$, we take

$$\phi^+(f) = \phi^+(f_1) + i\phi^+(f_2)$$

where $f_1, f_2 \in C^{\mathbb{R}}(K)$ are such that $f = f_1 + if_2$.
Of course, now we've defined $\phi^+ \in M^+(K)$, we take $\phi^- = \phi^+ - \phi$. $\phi^- \in M^+(K)$ also, since for all $f \in C^+(K)$,

$$\phi^-(f) = \phi^+(f) - \phi(f) \geq 0.$$

It remains to show $\|\phi\| = \|\phi^+\| + \|\phi^-\|$. Indeed, by considering

$$\|\phi\| \leq \|\phi\|^+ + \|\phi^-\| = \phi^+(1_K) + \phi^-(1_K) = 2\phi^+(1_K) - \phi(1_K). \tag{1}$$

On the other hand, for all $0 \leq f \leq 1_K$, we have

$$-1_K \leq 2f - 1_K \leq 1_f$$

and so $\|2f - 1_K\|_\infty \leq 1$. Hence,

$$2\phi(f) - \phi(1_K) = \phi(2f - 1_K) \leq \|\phi\|.$$

Thus, taking the supremum over $f$, the right hand side of equaton (1) is less equal to the operator norm of $\phi$ implying

$$\|\phi\| = \|\phi\|^+ + \|\phi^-\|$$

as required.

*Uniqueness*: Suppose $\phi = \psi_1 - \psi_2$ and $\|\phi\| = \|\psi_1\| + \|\psi_2\|$ for some $\psi_1, \psi_2 \in M^+(K)$. Then, for all $0 \leq g \leq f$,

$$\phi(g) = \psi_1(g) - \psi_2(g) \leq \psi_1(g) \leq \psi_1(f),$$

implying $\psi_1 \geq \phi^+$ and $\psi_1 - \phi^+ \in M^+(K)$. Thus, we also have $\psi_2 - \phi^- = \psi_1 - \phi^+ \in M^+(K)$. Then,

$$\begin{aligned}
\|\psi_1 - \phi^+\| + \|\psi_2 - \phi^-\| &= \psi_1(1_K) - \phi^+(1_K) + \psi_2(1_K) - \phi^-(1_K) \\
&= (\psi_1(1_K) + \psi_2(1_K)) - (\phi^+(1_K) + \phi^-(1_K)) \\
&= (\|\psi_1\| + \|\psi_2\|) - (\|\phi^+\| + \|\phi^-\|) \\
&= \|\phi\| - \|\phi\| = 0
\end{aligned}$$

providing uniqueness. □

### 1.2.2 Topological preliminaries

We recall the following facts:

- $K$ is said to be normal if for all disjoint closed subsets $E_1, E_2 \subseteq K$, there exists disjoint open sets $U_1, U_2 \subseteq K$ such that $E_1 \subseteq U_1$ and $E_2 \subseteq U_2$.

  Equivalently, if $E \subseteq U \subseteq K$ are such that $E$ is closed and $U$ is open, then there exists a open $V$ such that $E \subseteq V \subseteq \overline{V} \subseteq U$.

- *Urysohn's lemma*: Given disjoint closed subsets $E_1, E_2 \subseteq K$, there exists a continuous function $f : K \to [0,1]$ such that $f = 0$ on $E_1$ and $f = 1$ on $E_2$.

*Notations*: $f \prec U$ denotes the fact that

- $U$ is open,
- $f : K \to [0,1]$ is continuous,
- and $\mathrm{supp}(f) = \overline{\{x \mid f(x) \neq 0\}} \subseteq U$.

On the other hand, $E \prec f$ denotes

- $E$ is closed;
- $f : K \to [0,1]$ is continuous,
- and $f = 1$ on $E$.

Using this notation, Urysohn's lemma provides the existence of a $f$ such that $E \prec f \prec U$.

**Lemma 1.5.** Let $E \subseteq K$ be closed and let $U_j \subseteq K$ be open sets such that $E \subseteq \bigcup_{j=1}^n U_k$. Then,

- there exists open $V_j$ such that $\overline{V_j} \subseteq U_j$ and $E \subseteq \bigcup_{j=1}^n V_j$.
- there exists $f_j \prec U_j$ such that $\sum_{j=1}^n f_j \leq 1$ on $K$ and $\sum_{j=1}^n f_j = 1$ on $E$.

*Proof.* For the first part we induct on $n$.
Since $E \setminus U_n \subseteq \bigcup_{j=1}^{n-1} U_j$ and $E \setminus U_n$ is closed, we can apply the inductive hypothesis to obtain $V_j$ for $j = 1, \cdots, n-1$ such that $\overline{V_j} \subseteq U_j$ and $E \setminus U_n \subseteq \bigcup_{j=1}^{n-1} V_j$. Then,

$$E \setminus \bigcup_{j=1}^{n-1} V_j \subseteq U_n,$$

and thus, by the normality of $K$, there exists some open set $V$, such that

$$E \setminus \bigcup_{j=1}^{n-1} V_j \subseteq V \subseteq \overline{V} \subseteq U_n.$$

Hence, we have $E \subseteq \bigcup_{j=1}^n V_j$ where $\overline{V_j} \subseteq U_j$ for all $j$.
For the second part, choose $V_j$ as in the first part. Then, by Urysohn's lemma, there exists $g_j$ such that $\overline{V_j} \prec g_j \prec U_j$ and $K \setminus \bigcup V_j \prec g_0 \prec K \setminus E$. So, defining $g = \sum_{j=0}^n g_j$, $g$ is continuous and satisfies $g \geq 1$ on $K$. Thus, setting $f_j = g_j/g$ for $j = 1, \cdots, n$, we have $f_j : K \to [0,1]$ is continuous and satisfies

$$\sum_{j=1}^n f_j = \sum_{j=1}^n \frac{g_j}{g} \leq 1$$

on $K$, and by noting $g_0 = 0$ on $E$,

$$\sum_{j=1}^n f_j = \sum_{j=1}^n \frac{g_j}{g} = 1$$

on $E$. $\square$

**Definition 1.1** (Regular)**.** A Borel measure $\mu$ on the Borel space $X$ is said to be regular if

- for all compact $E \subseteq X$, $\mu(E) < \infty$,
- for all $A \in \mathscr{B}(X)$,
$$\mu(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathscr{G}\}$$
where $\mathscr{G}$ is the collection of all open sets in $X$.
- for all $U \in \mathscr{G}$,
$$\mu(U) = \sup\{\mu(E) \mid E \subseteq U, E \text{ compact}\}.$$

A compact measure $\nu$ is said to be regular if $|\nu|$ is.

**Proposition 1.1.** If $X$ is compact Hausdorff, then TFAE:

- The Borel measure $\mu$ is regular;
- $\mu(X) < \infty$ and for all $A \in \mathscr{B}(X)$,

$$\mu(A) = \inf\{\mu(U) \mid A \subseteq U \in \mathscr{G}\};$$

- $\mu(X) < \infty$ and for all $A \in \mathscr{B}(X)$,

$$\mu(A) = \sup\{\mu(E) \mid E \subseteq A, E \text{ closed}\}.$$

### 1.2.3 Riesz-Markov representation theorem

If $\nu$ is a complex Borel measure on $K$, for any $f \in C(K)$, we have $f$ is Borel-measurable and by observing

$$\int |f| \mathrm{d}|\nu| \leq \|f\|_\infty |\nu|(K) < \infty,$$

$f$ is also $\nu$-integrable. Thus, we may define the bounded linear functional

$$\phi : C(K) \to \mathbb{C} : f \mapsto \int f \,\mathrm{d}\nu.$$

$\phi$ is clearly linear and it is bounded since

$$|\phi(f)| \leq \int |f| \mathrm{d}|\nu| \leq \|f\|_\infty |\nu|(K)$$

so $\phi \in M(K) = C(K)^*$ and $\|\phi\| \leq \|\nu\|_1$. If $\nu$ is a signed measure, then $\phi \in M^{\mathbb{R}}(K) \simeq C^{\mathbb{R}}(K)^*$ and if $\nu$ is a positive measure, then $\phi \in M^+(K)$. It turns out that the converse is also true, namely elements of $M(K)$ can also be represented by complex measures.

**Theorem 4** (Riesz-Markov representation)**.** Given $\phi \in M^+(K)$, there exists a unique regular Borel measure $\mu$ on $K$ such that for all $f$ $\mu$-integrable, $\phi(f) = \int f \,\mathrm{d}\mu$. Furthermore, $\|\phi\| = \mu(K) = \|\mu\|_1$.

*Proof. Uniqueness*: Suppose we have two regular Borel measures $\mu_1, \mu_2$ both representing $\phi$ in the sense as above. Then for all $E \subseteq U \subseteq K$ with $E$ closed and $U$ open, by Urysohn's lemma, there exists some $f : K \to [0,1]$ such that $E \prec f \prec U$. Hence,

$$\mu_1(E) \leq \int f \,\mathrm{d}\mu_1 = \phi(f) = \int f \,\mathrm{d}\mu_2 \leq \mu_2(U).$$

So, as both $\mu_1$ and $\mu_2$ are regular, this implies $\mu_1 \leq \mu_2$. By symmetry, we also have $\mu_2 \leq \mu_1$ providing the uniqueness.

*Existence*: We would like to define a measure akin to $\mu(A) = \phi(1_A)$. However, as $1_A$ is not continuous, we will approximate this construction by defining an outer measure $\mu^*$.

Given $U \in \mathscr{G}$ (recall that $\mathscr{G}$ is the set of all open sets in $K$), we define

$$\mu^*(U) := \sup\{\phi(f) \mid f \prec U\}.$$

Observe straightaway that $\mu^*(\varnothing) = 0$ and $\mu^*(K) = \phi(1_K) = \|\phi\|$.

We will now show $\mu^*$ satisfies sub-$\sigma$-additivity. Suppose we have $U \subseteq \bigcup_{k=1}^\infty U_k$ for some $U, U_k \in \mathscr{G}$. Then, given $f \prec U$, by compactness, there exists some $n$ such that

$$\operatorname{supp}(f) \subseteq \bigcup_{k=1}^n U_k.$$

By the partition of unity, for each $k = 1, \cdots, n$, there exists some $h_k \prec U_k$ such that $\sum h_k \leq 1$ on $K$ and $\sum h_k = 1$ on $\operatorname{supp}(f)$. Thus,

$$\phi(f) = \phi\left(\sum_{k=1}^n h_k f\right) = \sum_{k=1}^n \phi(h_k f) \leq \sum_{k=1}^n \mu^*(U_k) \leq \sum_{k=1}^\infty \mu^*(U_k).$$

Hence, as this inequality holds for all $f \prec U$, we have $\mu^*(U) \leq \sum_{k=1}^\infty \mu^*(U_k)$. Furthermore, it follows that given $U, V \in \mathscr{G}$, $U \subseteq V$, we have $\mu^*(U) \leq \mu^*(V)$ and so,

$$\mu^*(U) = \inf\{\mu^*(V) \mid U \subseteq V \in \mathscr{G}\}.$$

With this in mind, we extend $\mu^*$ to all of $2^K$ by defining

$$\mu^*(A) = \inf\{\mu^*(V) \mid A \subseteq V \in \mathscr{G}\}$$

for any $A \subseteq K$.

Again, it is clear that $\mu^*(\varnothing) = 0$ and $\mu^*(K) = \|\phi\|$. For sub-$\sigma$-additivity, let $A \subseteq \bigcup_{k=1}^\infty A_n$. Then, for any $\epsilon > 0$, for each $n$, we may choose $U_n \in \mathscr{G}$ such that $A_n \subseteq U_n$ and

$$\mu^*(U_n) < \mu^*(A_n) + \epsilon 2^{-n}.$$

Hence, $A \subseteq \bigcup_{k=1}^\infty U_k$ and so,

$$\mu^*(A) \leq \mu^*\left(\bigcup_{k=1}^\infty U_k\right) \leq \sum_{k=1}^\infty \mu^*(U_k) \leq \sum_{k=1}^\infty \mu^*(A_k) + \epsilon.$$

Thus, as $\epsilon > 0$ was arbitrary, it follows $\mu^*(A) \leq \sum_{k=1}^\infty \mu^*(A_k)$ and $\mu^*$ is an outer measure on $K$.

Now, by Carathéodory extension, $\mu^*$ restricts to a measure on the set of sets which are $\mu^*$-measurable. Thus, by showing all open sets of $K$ are $\mu^*$-measurable, we may restrict $\mu^*$ on to $\mathscr{B}(K)$ to obtain the desired Borel measure. Take $U \in \mathscr{G}$ and $A \subseteq K$, we need to show

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U).$$

First, let us consider the cas that $A = V \in \mathscr{G}$. Then, taking $f \prec U \cap V$ and $g \prec V \setminus \operatorname{supp}(f)$ so that $f, g$ are disjointedly supported on $V$, we have $f + g \prec V$ and so,

$$\mu^*(V) \geq \phi(f + g) = \phi(f) + \phi(g).$$

Taking the supremum over $g$, we have

$$\mu^*(V) \geq \phi(f) + \mu^*(V \setminus \operatorname{supp}(f)) \geq \phi(f) + \mu^*(V \setminus U).$$

Now, taking the supremum over $f$,

$$\mu^*(V) \geq \mu^*(U \cap V) + \mu^*(V \setminus U)$$

as required.

For general $A$, let $V \in \mathscr{G}$ such that $A \subseteq V$. Then,

$$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(A \cap U) + \mu^*(A \setminus U).$$

Hence, taking the infimum over $V$, it follows

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \setminus U)$$

as required.

Thus, $\mu := \mu^*|_{\mathscr{B}}$ is a Borel measure on $K$ and it is regular by construction. It remains to show that $\mu$ represents $\phi$. It is sufficient to show $\phi(f) \leq \int f \,\mathrm{d}\mu$ for all $f \in C^{\mathbb{R}}(K)$ since if this holds, then

$$-\phi(f) = \phi(-f) \leq \int -f \,\mathrm{d}\mu = -\int f \,\mathrm{d}\mu$$

providing the reverse inequality.

Let $f \in C^{\mathbb{R}}(K)$ and choose $a < b \in \mathbb{R}$ so that $f(K) \subseteq [a, b]$. WLOG. assume $a > 0$ and fix $\epsilon > 0$ and choose

$$0 < y_0 < a < y_1 < \cdots < y_n = b$$

such that $y_j - y_{j-1} < \epsilon$. Let $A_j := f^{-1}((y_{j-1}, y_j])$ so $K = \bigcup_{j=1}^{n}$ is a Borel partition of $K$. For each $j$, choose $U_j \in \mathscr{G}$ such that $A_j \subseteq U_j \subseteq f^{-1}((y_{j-1}, y_j + \epsilon))$ and

$$\mu(U_j) < \mu(A_j) + \frac{\epsilon}{n}.$$

Then by the partition of unity, there exists $h_j \prec U_j$ such that $\sum_{j=1}^{n} h_j = 1_K$ so

$$\begin{aligned}
\phi(f) = \sum_{j=1}^{n} \phi(h_j f) &\leq \sum \phi((y_j + \epsilon)h_j) = \sum (y_j + \epsilon)\phi(h_j) \\
&\leq \sum (y_j + \epsilon)\mu(U_j) \leq \sum (y_j + \epsilon)\left(\mu(A_j) + \frac{\epsilon}{n}\right) \\
&= \int \sum y_j 1_{A_j} \,\mathrm{d}\mu + 2\epsilon\mu(K) + (b + 2\epsilon)\epsilon \\
&\leq \int f \,\mathrm{d}\mu + C\epsilon.
\end{aligned}$$

Hence, as $\epsilon$ was arbitrary, $\phi(f) \leq \int f \,\mathrm{d}\mu$ are required. □

**Corollary 4.1.** For all $\phi \in M(K)$, there exists a unique regular Borel complex measure $\nu$ such that for all $f \in C(K)$, $\phi(f) = \int f \,\mathrm{d}\nu$ and $\|\phi\| = \|\nu\|_1$. Furthermore, if $\phi \in M^{\mathbb{R}}(K)$, then $\nu$ is a signed measure.

*Proof.* Existence follows by Jordan decomposition while uniqueness follows from $\|\phi\| = \|\nu\|_1$. We will show $\|\phi\| = \|\nu\|_1$. We've seen that $\|\phi\| \le \|\nu\|_1$ so it remains to show the reverse. Recall that

$$\|\nu\| = |\nu|(K) = \sup\left\{\sum_{j=1}^{n} |\nu(A_j)| \mid (A_j)_{j=1}^{n} \text{ is a Borel partition of } K\right\}.$$

So, taking $(A_j)$ a Borel partition of $K$, for each $j$ let us choose $E_j$ closed such that $E_j \subseteq A_j$ and

$$|\nu|(A_j \setminus E_j) < \frac{\epsilon}{n}$$

which exists by regularity. Noting that $E_j \subseteq K \setminus \bigcup_{i\neq j} E_i$ which is open, there exists some open $U_j$ such that $E_j \subseteq U_j \subseteq K \setminus \bigcup_{i\neq j} E_i$ and

$$|\nu|(U_j \setminus E_j) < \frac{\epsilon}{n}.$$

Then, $E := \bigcup_{j=1}^{n} E_j \subseteq \bigcup_{j=1}^{n} U_j$ and by the partition of unity, there exists $h_j \prec U_j$ such that $\sum h_j \le 1$ on K and $\sum h_j = 1$ on $E$. Now, as $E_j$ are disjoint, $h_j = 1$ on $E_j$. Thus, choosing $\lambda_j \in \mathbb{C}$, $|\lambda| = 1$ such that $|\nu|(E_j) = \lambda_j \nu(E_j)$, we have

$$\begin{aligned}\left|\sum |\nu(E_j)| - \phi\left(\sum \lambda h_j\right)\right| &= \left|\sum \lambda_j \int (1_{E_j} - h_j)\mathrm{d}\nu\right| \\ &\le \sum \int |1_{E_j} - h_j|\mathrm{d}\nu \le \sum |\nu|(U_j \setminus E_j) < \epsilon.\end{aligned}$$

Hence,

$$\begin{aligned}\sum |\nu(A_j)| &\le \sum |\nu(E_j)| + \epsilon \\ &\le \left|\phi\left(\sum \lambda_j 1_{E_j}\right)\right| \le \|\phi\| \left\|\sum \lambda_j h_j\right\|_\infty + 2\epsilon \le \|\phi\| + 2\epsilon,\end{aligned}$$

implying $\|\nu\|_1 = \|\phi\|$ as required. □

**Corollary 4.2.** The space of regular complex Borel measures is a complex Banach space with the total variation norm and it is isometrically isomorphic to $M(K) = C(K)^*$.

# 2 Weak Topology

## 2.1 General weak topology

Let $X$ be a set and $\mathscr{F}$ be a collection of functions such that for each $f \in \mathscr{F}$, $f : X \to Y_f$ where $Y_f$ is a topological space. Then the weak topology $\sigma(X, \mathscr{F})$ is the smallest topological space such that for all $f \in \mathscr{F}$, $f$ is continuous. We have the following straight forward properties about the weak topology.

**Proposition 2.1.** Taking $X, \mathscr{F}$ as above,

- $S := \{f^{-1}(U) \mid f \in \mathscr{F}, U \text{ open in } Y_f\}$ generates $\sigma(X, \mathscr{F})$.
- $V \subseteq X$ is open in $\sigma(X, \mathscr{F})$ iff for all $x \in V$, there exists $f_1, \cdots, f_n \in \mathscr{F}$ and open sets $U_1, \cdots, U_n$ such that $U_i \subseteq Y_{f_i}$ and
$$x \in \bigcap_{j=1}^n f_j^{-1}(U_j) \subseteq V.$$
- If $S_f$ generates the topology of $Y_f$ for all $f \in \mathscr{F}$, then $\{f^{-1}(U) \mid U \in S_f, f \in \mathscr{F}\}$ generates $\sigma(X, \mathscr{F})$.
- If $Y_f$ is Hausdorff for all $f \in \mathscr{F}$ and $\mathscr{F}$ separates points, then, so is $\sigma(X, \mathscr{F})$ Hausdorff.
- If $Y \subseteq X$, then $\sigma(X, \mathscr{F})|_Y = \sigma(Y, \mathscr{F}|_Y)$.
- (universal property) Given $Z$ a topological space and $g : Z \to X$, then $g$ is continuous with respect to $\sigma(X, \mathscr{F})$ iff for all $f \in \mathscr{F}$, $f \circ g : Z \to Y_f$ is continuous.

The weak topology generalizes the subspace topology by considering $\sigma(Y, \{\iota\})$ for $\iota : Y \hookrightarrow X$ the inclusion map and the product topology which has the topology
$$\sigma\left(\prod_{\gamma \in \Gamma} X_\gamma, \{\pi_\gamma \mid \gamma \in \Gamma\}\right),$$
where $\pi_\gamma : \prod_{\gamma \in \Gamma} X_\gamma \to X_\gamma$ is the projection map.

**Proposition 2.2.** Let $X$ be a set and for each $n \in \mathbb{N}$, let $(Y_n, d_n)$ be metric spaces. Then, if $\mathscr{F} := \{f_n : X \to Y_n \mid n \in \mathbb{N}\}$ separates points, then $\sigma(X, \mathscr{F})$ is metrizable.

*Proof.* WLOG. by replacing $d_n$ by $d_n \wedge 1$ which is equivalent, we may assume that $d_n \leq 1$. Then, it is easy to check that
$$d(x, y) := \sum_{n=1}^\infty 2^{-n} d_n(f_n(x), f_n(y)),$$
form a metric on $X$.

By noting that any $f_n$ in the above proposition is Lipschitz with respect to the topology generated by $d$, $f_n$ is $\mathscr{T}_d$-continuous and so $\sigma(X, \mathscr{F}) \subseteq \mathscr{T}_d$. Conversely, as each $f_n$ is $\sigma(X, \mathscr{F})$-continuous, the map
$$(x, y) \mapsto d_n(f_n(x), f_n(y))$$

is also $\sigma(X,\mathscr{F})$-continuous. Hence, by the Weierstass-M-test, it follows $d$ is also $\sigma(X,\mathscr{F})$-continuous implying

$$\mathscr{T}_d = \sigma(X,\mathscr{F})$$

as required. $\square$

**Theorem 5** (Tychonov)**.** The product of compact spaces is compact in the product topology.

*Proof.* Let $\Gamma$ be the index set and for each $\gamma \in \Gamma$, let $X_\gamma$ be a compact space and denote $X = \prod_{\gamma\in\Gamma} X_\gamma$. We will show $X$ is compact by showing: for any non-empty family of closed sets $\mathscr{A}$ with the finite intersection property (fip.), that is, for all $A_1, \cdots, A_n \in \mathscr{A}$, we have $\bigcap_{i=1}^n A_i \neq \varnothing$, then $\bigcap_{A\in\mathscr{A}} A \neq \varnothing$.

By Zorn's lemma, there exists a maximal family $\mathscr{B}$ of (not necessarily closed) subsets of $X$ with fip. and satisfies $\mathscr{A} \subseteq \mathscr{B}$. Then,

$$\bigcap_{A\in\mathscr{A}} A \supseteq \bigcap_{B\in\mathscr{B}} B$$

and so, it suffices to show $\bigcap_{B\in\mathscr{B}} B \neq \varnothing$.

By the maximality of $\mathscr{B}$, we observe that if $A \subseteq X$ satisfies $A \cap B \neq \varnothing$ for all $B \in \mathscr{B}$, then $A \in B$.

Fix $\gamma \in \Gamma$, $\{\pi_\gamma(B) \mid B \in \mathscr{B}\}$ has fip. and hence, as $X_\gamma$ is compact, it follows $\bigcap_{B\in\mathscr{B}} \overline{\pi_\gamma^{-1}(B)} \neq \varnothing$. Choose $x_\gamma \in \bigcap_{B\in\mathscr{B}} \overline{\pi_\gamma^{-1}(B)}$, we will show $x = (x_\gamma)_{\gamma\in\Gamma} \in \bigcap_{B\in\mathscr{B}} \overline{B}$. Let $V$ be an open neighborhood of $x$, we need to show $V \cap B \neq \varnothing$ for all $B \in \mathscr{B}$. WLOG. write

$$V = \bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i)$$

for some $\gamma_1, \cdots, \gamma_n \in \Gamma$ and $U_1, \cdots, U_n$ open neighborhoods of $x_{\gamma_i}$.

Since $x_{\gamma_i} \in \bigcap_{B\in\mathscr{B}} \pi_{\gamma_i}^{-1}(B)$, we have $U_{\gamma_i} \cap \pi_{\gamma_i}(B) \neq \varnothing$ for all $B \in \mathscr{B}$. Thus, by maximality, we have $\pi_{\gamma_i}^{-1}(U_i) \in \mathscr{B}$ and so

$$V = \bigcap_{i=1}^n \pi_{\gamma_i}^{-1}(U_i) \in \mathscr{B}$$

implying $V \cap B \neq \varnothing$ for all $B \in \mathscr{B}$. Hence, as $V$ was chosen arbitrarily, $x \in B$ for all $B \in \mathscr{B}$ as required. $\square$

## 2.2 Weak topology on vector spaces

Let $E$ be a real or complex vector space and $F$ a subspace of the space of all linear functionals on $E$ that separates points of $E$. We will in this section consider $\sigma(E,F)$. We recall that $U \subseteq E$ is weakly open iff for all $x \in U$, there exists $f_1, \cdots, f_n \in F$, $\epsilon > 0$ such that

$$\{y \in E \mid |f_i(y - x)| < \epsilon, i = 1, \cdots, n\} \subseteq U.$$

For $f \in F$, define $p_f : E \to \mathbb{R}$ by $p_f(x) = |f(x)|$. Then,

$$\mathscr{P} := \{p_f \mid f \in F\}$$

is a family of semi-norms which separates points of $E$. Thus, the weak topology on $E$ generated by $F$ is the same as the LCS topology generated by $\mathscr{P}$.

**Lemma 2.1.** Let $E$ be a real or complex vector space and $f, g_1, \cdots, g_n$ linear functionals on $E$ such that

$$\bigcap_{i=1}^{n} \ker g_i \subseteq \ker f_i.$$

Then, $f \in \langle g_1, \cdots, g_n \rangle$.

*Proof.* Define $T : E \to \mathbb{F}^n$ by $Tx = (g_i(x))_{i=1}^n$. Then, $\ker T \subseteq \ker f$. Thus, there exists some linear $h : \mathrm{Im}(T) \to \mathbb{F}$ such that $f = h \circ T$. Thus, by Hahn-Banach, extending $h$ to $\mathbb{F}^n \to \mathbb{F}$, we can write $h(y) = \sum_{i=1}^n a_i y_i$ for all $y = (y_i)_{i=1}^n \in \mathbb{R}^n$. Hence, for all $x \in E$,

$$f(x) = h(Tx) = \sum_{i=1}^{n} a_i g_i(x),$$

implying $f \in \langle g_1, \cdots, g_n \rangle$ as required. □

**Proposition 2.3.** Let $E, F$ as above. A linear functional $f$ on $E$ is weakly continuous iff $f \in F$. Namely, $(E, \sigma(E,F))^* = F$.

*Proof.* The converse is true by definition. For the other direction, let $f$ be a weakly continuous linear functional. Then, $V := f^{-1}(B_1(0))$ is an open neighborhood of 0 in $(E, \sigma(E,F))$. Thus, there exists $g_1, \cdots, g_n \in F$ and $\epsilon > 0$ such that

$$U := \{x \in E \mid |g_i(x)| < \epsilon, i = 1, \cdots, n\} \subseteq V.$$

Then, for all $x \in \bigcap_{i=1}^n \ker g_i$, for all $\lambda \in \mathbb{F}$ such that $\lambda x \in U \subseteq V$ and so, $|f(\lambda x)| = |\lambda||f(x)| < 1$ implying $x \in \ker f$. Thus, by the previous lemma $f \in \langle g_1, \cdots, g_n \rangle$ and so $f \in F$. □

If $X$ is a normed space, the weak topology on $X$ is $w := \sigma(X, X^*)$. By Hahn-Banach, $X^*$ separates points of $X$ and so the weak topology is Hausdorff. As before, a subset $U \subseteq X$ is weakly open iff for all $x \in U$, there exists $f_1, \cdots, f_n \in X^*$ and $\epsilon > 0$ such that

$$\{y \in X \mid |f_i(y - x)| < \epsilon, i = 1, \cdots, n\} \subseteq U.$$

Now, by identifying $X$ in $(X^*)^*$ by the canonical embedding, we define the weak-* topology on $X^*$ by $w^* := \sigma(X^*, X)$. A subset $U \subseteq X^*$ is weak-* open iff for all $f \in U$, there exists $x_1, \cdots, x_n \in X$ and $\epsilon > 0$ such that

$$\{g \in X^* \mid |g(x_i) - f(x_i)| < \epsilon, i = 1, \cdots, n\} \subseteq U.$$

**Proposition 2.4.** $(X, w), (X^*, w^*)$ are locally convex spaces. In particular, they are Hausdorff and their vector space operations are continuous. Furthermore,

- $w \subseteq \|\cdot\|-$topology with equality iff $X$ is finite dimensional.
- $w^* \subseteq \sigma(X^*, (X^*)^*) \subseteq \|\cdot\|-$topology with the first inclusion becoming an equality iff $X$ is reflexive and the second inclusion becoming an equality iff $X$ is finite dimensional.
- if $Y \leq X$, then
$$\sigma(X, X^*)|_Y = \sigma(Y, \{f|_Y \mid f \in X^*\}) = \sigma(Y, Y^*)$$
where the last equality follows by Hahn-Banach.

- The canonical embedding $X \to (X^*)^*$ is a w-to-w* homeomorphism between $X$ and $\hat{X}$.

**Proposition 2.5.** Let $X$ be a normed space. Then,

- a linear functional $f$ on $X$ is weakly continuous iff $f \in X^*$.
- a linear functional $\phi$ on $X^*$ is weak-* continuous iff $\phi \in \hat{X}$.
- $\sigma(X^*, X) = \sigma(X^*, (X^*)^*)$ iff $X$ is reflexive.

*Proof.* The only slightly non-trivial part is the forward direction of the third statement. But this is also straight forward. Let $\phi \in (X^*)^*$, we need to show $\phi \in \hat{X}$. Since $\sigma(X^*, X) = \sigma(X^*, (X^*)^*)$, $f$ is weak-* continuous and the result follows by the second claim. $\square$

**Definition 2.1.** Let $X$ be a normed space, $A \subseteq X$ is said to be weakly bounded if $\{f(x) \mid x \in A\}$ is bounded for all $f \in X^*$.

Clearly, as all $f \in X^*$ are bounded, bounded in $\|\cdot\|$ implies weakly bounded.

We recall the principle of uniformly boundedness (PUB).

**Theorem 6.** Let $X$ be a Banach space, Y a normed space and $\mathscr{T} \subseteq \mathscr{B}(X, Y)$. Then, if $\mathscr{T}$ is point-wise bounded, i.e.

$$\sup_{T \in \mathscr{T}} \|Tx\| < \infty,$$

then $\sup_{T \in \mathscr{T}} \|T\| < \infty$.

**Proposition 2.6.** If $X$ is a normed space,

- if $A \subseteq X$ is weakly bounded, then $A$ is $\|\cdot\|$-bounded.
- if $X$ is in addition complete, then if $B \subseteq X^*$ is w*-bounded, then $B$ is $\|\cdot\|$-bounded.

*Proof.* Firstly, defining $\hat{A} = \{\hat{x} \mid x \in A\}$, as $A$ is weakly bounded, for all $f \in X^*$,

$$\sup_{\hat{x} \in \hat{X}} \|\hat{x}(f)\| = \sup_{x \in A} \|f(x)\| < \infty.$$

Thus, by PUB (note that we are using the fact $(X^*)^*$ is complete), $\sup_{x \in A} \|x\| = \sup_{\hat{x} \in \hat{A}} \|\hat{x}\| < \infty$ as required.

On the other hand, if $X$ i complete and $B \subseteq X^*$ is w*-bounded, then we may directly apply PUB to obtain the bound in $\|\cdot\|$ as required. $\square$

# 3 Banach Algebra

## 3.1 Definitions

**Definition 3.1** (Algebra)**.** A real or complex algebra is a real or resp. complex vector space $A$ with a multiplication

$$A \times A \to A : (a, b) \mapsto ab,$$

such that

- $(ab)c = a(bc)$,
- $a(b+c) = ab + ac$,
- $(a+b)c = ac + bc$,
- for all $\lambda \in \mathbb{R}$ or resp. $\mathbb{C}$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$.

**Definition 3.2** (Unital)**.** An algebra is said to be *unital* if there exists a $1 \neq 0 \in A$ such that for all $a \in A$,

$$a1 = 1a = a.$$

Such an element is unique and is called the unit of $A$.

**Definition 3.3** (Algebra norm)**.** An algebra norm on an algebra $A$ is a vector norm $\|\cdot\|$ such that for all $a, b \in A$,

$$\|ab\| \leq \|a\|\|b\|.$$

This property implies that multiplication is continuous wrt. the topology induced by the norm.

**Definition 3.4** (Normed algebra)**.** A normed algebra is an algebra with an algebra norm.

**Definition 3.5** (Banach algebra)**.** A Banach algebra is a complete normed algebra.

**Definition 3.6** (Unital normed algebra)**.** A unital normed algebra is a unital algebra with a algebra norm such that $\|1\| = 1$.

If $A$ is a unital algebra with an algebra norm $\|\cdot\|$, then defining another norm

$$\|a\|' := \sup\{\|ab\| \mid \|b\| \leq 1\}.$$

$\|\cdot\|$ and $\|\cdot\|'$ are equivalent and $\|1\|' = 1$. Thus, we can always make a unital algebra with an algebra norm into a unital normed algebra with the same topology.

**Definition 3.7** (Algebra homomorphism)**.** Let $A, B$ be algebras. A homomorphism from $A$ to $B$ is a linear map $\theta : A \to B$ such that

$$\theta(xy) = \theta(x)\theta(y)$$

for all $x, y \in A$. If $A, B$ are in addition unital, then we also require $\theta(1_A) = 1_B$.
If $\theta$ is bijective, then we say it is an isomorphism.

We note that for $A, B$ normed algebras, a homomorphism is *not* assumed to be continuous while isomorphism is assumed to be continuous with a continuous inverse.
As our focus is on spectral theory, from this point forward, we will assume the scalar field is $\mathbb{C}$.

**Example 3.1.** Let $K$ be a compact Hausdorff space. Then $C(K)$ is a commutative unital Banach algebra under pointwise multiplication.

Furthermore, a uniform algebra on $K$ is a closed subalgebra of $C(K)$ which separates points of $K$ and contain the constant functions. In the real case, Stone-Weierstrass implies that it must be all of $C(K)$. In our case however (with complex scalar field), Stone-Weierstrass in addition requires the subalgebra to be closed under conjugation.

An example of this is

$$A(\Delta) = \{f \in C(\Delta) \mid f \text{ holomorphic on } \Delta^\circ\}$$

where $\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

More generally, let $K \subseteq \mathbb{C}$ be a non-empty compact subset. Then, we have the following uniform algebras on $K$:

$$\mathscr{P}(K) \subseteq \mathscr{R}(K) \subseteq \mathscr{O}(K) \subseteq A(K) \subseteq C(K),$$

where $\mathscr{P}(K), \mathscr{R}(K), \mathscr{O}(K)$ are the closures of resp. polynomials, rational functions without poles in $K$ and holomorphic functions on some open neighborhood of $K$. We shall see later that $\mathscr{R}(K) = \mathscr{O}(K)$ (always) and

$$\mathscr{P}(K) = \mathscr{R}(K) \iff \mathbb{C} \setminus K \text{ is connected.}$$

On the other han, $\mathscr{R}(K) \neq A(K)$ and

$$A(K) = C(K) \iff K^\circ = \varnothing.$$

**Example 3.2.** $L_1(\mathbb{R})$ with the $L_1$-norm and convolution as multiplication is a commutative Banach algebra without a unit (Riemann-Lebesgue lemma).

**Example 3.3.** Let $X$ be a Banach space. Then $\mathscr{B}(X)$ (bounded linear operators from $X$ to itself) with the operator norm and composition as multiplication is a unital Banach algebra. It is not commutative if $\dim X \geq 2$.

In the special case that $X$ is a Hilbert space, then $\mathscr{B}(X)$ is what is known as a $C^*$-algebra.

## 3.2 Constructions

*Subalgebra*: Let $A$ be an algebra and $B$ a subalgebra of $A$. If $A$ is unital with unit 1, then $B$ is unital if $1 \in B$. If $A$ is a normed algebra, then $\overline{B}$ is also a subalgebra.

*Unitization*: If $A$ is a normed algebra. The unitization of $A$ is the vector space $A_+ = A \oplus \mathbb{C}$ with multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda a + b\mu, \lambda\mu).$$

Then, $A_+$ is a unital algebra with the unit $(0, 1)$. The set $\{(a, 0) \mid a \in A\}$ is an ideal of $A_+$ and is isomorphic as an algebra to $A$. We write

$$A_+ = \{a + \lambda 1 \mid a \in A, \lambda \in \mathbb{C}\}.$$

If $A$ is a normed algebra, then so is $A_+$ with the norm

$$\|a + \lambda 1\| = \|a\| + |\lambda|$$

and in this case, $A$ is a closed ideal of $A_+$. Furthermore, if $A$ is a Banach algebra, so is $A_+$.

*Ideals*: Let $A$ be a normed algebra. If $J \trianglelefteq A$, then also $\overline{J} \trianglelefteq A$. If $J$ is a closed ideal of $A$, then we can define $A/J$ which is a normed algebra with the quotient norm.

If $A$ is in addition unital, and $J$ is a proper ideal, then $A/J$ is a unital normed algebra with the unit $1+J$.

*Completion*: Let $A$ be a normed algebra and $\tilde{A}$ be its completion. For $a, b \in \tilde{A}$, by construction, we may choose sequences $(a_n), (b_n) \subseteq A$ such that $a_n \to a$ and $b_n \to b$. Then, defining

$$ab = \lim_{n\to\infty} a_n b_n$$

where the right hand side exists and it's Cauchy, $\tilde{A}$ is a Banach algebra which contains $A$ as a dense subalgebra.

*Operator algebra*: Let $A$ be a unital Banach algebra. For each $a \in A$, we define

$$L_a : A \to A : x \mapsto ax.$$

$L_a$ is clearly linear, and is bounded as $\|ax\| \leq \|a\|\|x\|$. The map $a \mapsto L_a : A \to \mathscr{B}(A)$ is an isometric homomorphism. Thus, every Banach algebra is a closed subalgebra of $\mathscr{B}(X)$ for some $X$.

**Lemma 3.1.** Let $A$ be a unital Banach algebra and let $a \in A$. Then, $a$ is invertible if $\|a-1\| < 1$. Furthermore,

$$\|a^{-1}\| \leq \frac{1}{1-\|1-a\|}.$$

*Proof.* Let $h = 1-a$ so $a = 1-h$, $\|h\| < 1$ and $\|h^n\| \leq \|h\|^n$. Thus, $\sum_{n=0}^{\infty} \|h^n\|$ converges in $\mathbb{R}$ and so $b := \sum_{n=0}^{\infty} h^n$ converges in $A$ (as $A$ is a Banach space). With this in mind, we observe

$$ab = (1-h)\sum_{n=0}^{\infty} h^n = \sum h^n - \sum h^{n+1} = 1.$$

Similarly $ba = 1$ so $a$ is invertible. Moreover,

$$\|a^{-1}\| \leq \sum_{n=0}^{\infty} \|h\|^n = \frac{1}{1-\|h\|} = \frac{1}{1-\|1-a\|}$$

as required. □

We introduce the notation

$$G(A) = \{a \in A \mid a \text{ invertible}\}.$$

**Corollary 6.1.** Let $A$ be a unital Banach algebra, then

1. $G(A)$ is open.
2. $x \mapsto x^{-1} : G(A) \to G(A)$ is continuous.
3. If $(x_n) \subseteq G(A)$ converges to $x \notin G(A)$, then $\|x_n^{-1}\| \to \infty$.

4. If $x \in \partial G(A)$, then there exists a sequence $(z_n)$ with $\|z_n\| = 1$ for all $n$ such that

$$z_n x \to 0 \text{ and } x z_n \to 0.$$

It follows that $x$ has no left or right inverse (even in any unital Banach algebra containing $A$ isometrically).

*Proof.*

1. Let $x \in G(A), y \in A$. If $\|y - x\| < \|x^{-1}\|^{-1}$, then

$$\|1 - x^{-1} y\| = \|x^{-1}(x - y)\| \leq \|x^{-1}\| \|x - y\| < 1$$

and so, $x^{-1} y \in G(A)$ and hence also $y \in G(A)$.

2. Fix $x, y \in G(A)$, then

$$\|y^{-1} - x^{-1}\| = \|y^{-1}(x - y)x^{-1}\| \leq \|y^{-1}\| \|x^{-1}\| \|x - y\|.$$

Then, if $\|x - y\| < (2\|x^{-1}\|)^{-1}$, we have

$$\|y^{-1}\| - \|x^{-1}\| \leq \|y^{-1} - x^{-1}\| \leq \frac{1}{2}\|y^{-1}\|$$

implying $\|y^{-1}\| \leq 2\|x^{-1}\|$. Thus,

$$\|y^{-1} - x^{-1}\| \leq \|y^{-1}\| \|x^{-1}\| \|x - y\| \leq 2\|x^{-1}\|^2 \|x - y\|$$

which converges to 0 as $y \to x$.

3. From 1, for all $y \in A$ and $\|y - x_n\| < \|x_n^{-1}\|^{-1}$, we have $y \in G(A)$. Hence, $\|x_n - x\| \geq \|x_n^{-1}\|^{-1}$ implying $\|x_n^{-1}\| \to \infty$ as claimed.

4. Choose $(x_n)$ in $G(A)$ such that $x_n \to x$. Then, defining

$$z_n := \frac{x_n^{-1}}{\|x_n^{-1}\|},$$

we have $\|z_n\| = 1$ and

$$\|z_n x\| = \|z_n x + z_n(x - x_n)\| \leq \frac{1}{\|x_n^{-1}\|} + \|z_n\| \|x - x_n\|.$$

Now, as $\|x_n^{-1}\|^{-1}$ converges to 0 by 3, the right hand side converges to 0 as $n \to \infty$ allowing us to conclude.

□

### 3.3 Spectrum

**Definition 3.8** (Spectrum)**.** Let $A$ be an algebra and let $x \in A$. We define the spectrum $\sigma_A(x)$ of $x$ to be

$$\sigma_A(x) := \{\lambda \in \mathbb{C} \mid \lambda 1 - x \notin G(A)\}$$

if $A$ is unital and

$$\sigma_A(x) := \sigma_{A_+}(x)$$

if $A$ is not unital.

**Example 3.4.** If $A = M_n(\mathbb{C})$, then $\sigma_A(x)$ is the set of eigenvalues of $x$.

**Example 3.5.** If $A = C(K)$ for a compact Hausdorff $K$, $f \in A$, then $\sigma_A(f) = f(K)$ since $g \in A$ is invertible if and only if $0 \notin g(K)$.

**Example 3.6.** If $X$ is a Banach space, $A = \mathscr{B}(X)$, $T \in A$. Then,

$$\sigma_A(T) = \{\lambda \in \mathbb{C} \mid \lambda \mathrm{id} - T \text{ is not an isomorphism}\}.$$

**Theorem 7.** Let $A$ be a Banach algebra and $x \in A$. Then $\sigma_A(x)$ is non-empty compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|x\|\}$.

*Proof.* By unitization, we may assume $A$ is unital. Consider that the map

$$\lambda \mapsto \lambda 1 - x : \mathbb{C} \to A$$

is continuous and $\sigma_A(x)$ is the inverse image of $A \setminus G(A)$ under this map, $\sigma_A(x)$ must be closed.
Now, if $|\lambda| > \|x\|$, then $\|x/\lambda\| < 1$ and so by the previous theorem, $1 - x/\lambda \in G(A)$. Thus, as $\lambda \neq 0$, $\lambda(1 - x/\lambda) = \lambda 1 - x \in G(A)$ and hence, $\lambda \notin \sigma_A(x)$. As we've shown that $\sigma_A(x) \subseteq \mathbb{C}$ is closed and bounded, it is thusly compact.
Finally, we will show it is non-empty. Suppose otherwise, then we can define the (resolvent) map

$$R : \mathbb{C} \to G(A) : \lambda \mapsto (\lambda 1 - x)^{-1}$$

which in particular is holomorphic since

$$R(\lambda) - R(\mu) = R(\lambda)((\mu 1 - x) - (\lambda 1 - x))R(\mu) = (\mu - \lambda)R(\lambda) - R(\mu).$$

Thus,

$$\frac{R(\lambda) - R(\mu)}{\lambda - \mu} = -R(\lambda)R(\mu) \to -R(\mu)^2$$

as $\lambda \to \mu$ since $R$ is continuous.
Now, for $|\lambda| > \|x\|$, $R(\lambda) = \lambda^{-1}(1 - x/\lambda)^{-1}$ and so,

$$\|R(\lambda)\| \leq \frac{1}{|\lambda|} \frac{1}{1 - \|x/\lambda\|} = \frac{1}{|\lambda| - \|x\|}$$

which converges to 0 as $|\lambda| \to \infty$. Hence, $R = 0$ by the vector valued Liouville's theorem which is a contradiction. ☐

**Corollary 7.1** (Gelfand-Mazur)**.** A complex unital normed division algebra (i.e. $G(A) = A \setminus \{0\}$) is isometrically isomorphic to $C$.

*Proof.* The map we want is

$$\theta : \mathbb{C} \to A : \lambda \mapsto \lambda 1.$$

It is clear that $\theta$ is an isometric homomorphism.

For surjectivity, let $B$ be a completion of $A$, so $B$ is a unital Banach algebra. Given $x \in A$, by the previous theorem $\sigma_B(x)$ is non-empty and so we may choose $\lambda \in \sigma_B(x)$. Then, $\lambda 1 - x \notin G(B)$ and so $\lambda 1 - x \notin G(A)$. However, as $A$ is a division algebra, this means $\lambda 1 - x = 0$ and so $\theta(\lambda) = x$ as required. □

**Definition 3.9** (Spectral radius)**.** Let $A$ be a Banach algebra and $x \in A$. The spectral radius of $x$ is defined to be

$$r_A(x) := \sup\{|\lambda| \mid \lambda \in \sigma_A(x)\} \leq \|x\|.$$

**Lemma 3.2.** If $A$ is a unital algebra, $x, y \in A$ and $xy = yx$, then $x, y \in G(A)$ if and only if $xy \in G(A)$.

*Proof.* Let $b = (xy)^{-1}$, then, $(by)x = b(yx) = b(xy) = 1 = (xy)b = x(yb)$. □

**Lemma 3.3** (Polynomial spectral mapping theorem)**.** Let $A$ be a unital Banach algebra and let $x \in A$. Then, for any complex polynomial $p(x) = \sum_{k=0}^{n} a_k z^k$, we have

$$\sigma_A(p(x)) = \{p(\lambda) \mid \lambda \in \sigma_A(x)\}.$$

*Proof.* The lemma is clear for constant polynomials as $\sigma_A(\lambda 1) = \{\lambda\}$.

Assume now $n \geq 1$ and $a_n \neq 0$. Then, fixing $\mu \in \mathbb{C}$, we write

$$\mu - p(z) = c \prod_{\gamma=1}^{n} (\lambda_\gamma - z)$$

for some $c \neq 0$, $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Then, by the above lemma,

$$\mu 1 - p(x) = c \prod_{\gamma=1}^{n} (\lambda_\gamma 1 - x)$$

is invertible if and only if $\lambda_\gamma 1 - x$ is invertible for all $\gamma$. Thus, $\mu \in \sigma_A(p(x))$ if and only if one of the $\lambda_\gamma \in \sigma_A(x)$ which occurs if and only if $p(\lambda_\gamma) = \mu$. □

**Theorem 8** (Beurling-Gelfand spectral radius formula)**.** Let $A$ be a Banach algebra and let $x \in A$. Then,

$$r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_n \|x^n\|^{1/n}.$$

*Proof.* By unitization, we may assume $A$ is unital.

Observe that for $\lambda \in \sigma_A(x)$, $\lambda^n \in \sigma_A(x^n)$ (by polynomial spectral mapping) and so $|\lambda^n| \leq \|x^n\|$. Thus, $|\lambda| \leq \|x^n\|^{1/n}$ and it follows that $r_A(x) \leq \inf_n \|x^n\|^{1/n}$.

Consider again the resolvent operator

$$R : \{\lambda \in \mathbb{C} \mid |\lambda| > r_A(x)\} \to G(A) : \lambda \to (\lambda 1 - x)^{-1}.$$

We've previously shown $R$ is holomorphic and hence, for any $\phi \in A^*$, $\phi \circ R$ has a Laurent expansion. In particular, for $|\lambda| > \|x\| (\geq r_A(x))$, we have

$$R(\lambda) = \frac{1}{\lambda}\left(1 - \frac{x}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{x^n}{\lambda^n}.$$

Hence,

$$\phi \circ R(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\phi\left(\frac{x^n}{\lambda^n}\right) = \sum_{n=0}^{\infty}\frac{\phi(x^n)}{\lambda^{n+1}}$$

implying $\sum_{n=0}^{\infty}\frac{\phi(x^n)}{\lambda^{n+1}}$ is the Laurent expansion of $\phi \circ R$. Thus, for all $\lambda \in \mathbb{R}$ with $|\lambda| > r_A(x)$, $\phi(x^n/\lambda^n) \to 0$ for any $\phi \in A^*$. With this, $\{x^n/\lambda^n \mid n \in \mathbb{N}\}$ is weakly bounded and hence is bounded in norm by some constant $M$. Then, for all $n$, $\|x^n/\lambda^n\| \leq M$ and so,

$$\|x^n\|^{1/n} \leq M^{1/n}|\lambda| \text{ implying } \limsup \|x^n\|^{1/n} \leq |\lambda|$$

for every $\lambda$ satisfying $|\lambda| > r_A(x)$. Thus, we have

$$r_A(x) \leq \inf \|x^n\|^{1/n} \leq \liminf \|x^n\|^{1/n} \leq \limsup \|x^n\|^{1/n} \leq r_A(x).$$

□

**Theorem 9.** Let $A$ be a unital Banach algebra and $B$ a unital subalgebra of $A$. Then, given $x \in B$,

$$\sigma_B(x) \supseteq \sigma_A(x) \text{ and } \partial\sigma_B(x) \subseteq \partial\sigma_A(x).$$

It follows that $\sigma_B(x)$ is the union of $\sigma_A(x)$ with some of the bounded components of $\mathbb{C} \setminus \sigma_A(x)$.

*Proof.* If $\lambda \notin \sigma_B(x)$, then $\lambda 1 - x \in G(B)$ and so, $\lambda 1 - x \in G(A)$ implying $\lambda \notin \sigma_A(x)$.
On the other hand, let us take $\lambda \in \partial\sigma_B(x)$ ($\lambda \in \sigma_B(x)$ as $\sigma_B(x)$ is compact and hence closed). So, choosing $(\lambda_n) \subseteq \mathbb{C} \setminus \sigma_B(x) \subseteq \mathbb{C} \setminus \sigma_A(x)$ such that $\lambda_n \to \lambda$, it suffices to show that $\lambda \in \sigma_A(x)$.
Observe that $\lambda_n 1 - x \in G(B) \subseteq G(A)$ for all $n$ and $\lambda_n 1 - x \to \lambda 1 - x \notin G(B)$. Namely, $\lambda 1 - x \in \partial G(B)$. Thus, if $\lambda 1 - x \in G(A)$, by the continuity of the inverse,

$$(\lambda_n 1 - x)^{-1} \to (\lambda 1 - x)^{-1}.$$

However, as $(\lambda_n 1 - x)^{-1} \in B$, and $B$ is closed, it follows $(\lambda 1 - x)^{-1} \in B$ contradicting $\lambda 1 - x \notin G(B)$. Hence, $\lambda 1 - x \notin G(A)$ implying $\lambda \in \sigma_A(x)$ as required. □

**Proposition 3.1.** Let $A$ be a unital Banach algebra and $C$ a maximal commutative subalgebra of $A$. Then $C$ is closed, unital and for all $x \in C$, we have $\sigma_A(x) = \sigma_C(x)$.

*Proof.* As multiplication is continuous, it follows $\overline{C}$ is also a commutative subalgebra. Thus, for $C$ to be maximal, $C = \overline{C}$ implying $C$ is closed. $C$ is unital as 1 commutes with all elements of $C$ and so can always be added in to create a larger commutative subalgebra.
Fix $x \in C$. We already know $\sigma_C(x) \supseteq \sigma_A(x)$. Now, for $\lambda \notin \sigma_A(x)$, there exists some $y \in A$,

$$y(\lambda 1 - x) = (\lambda 1 - x)y = 1.$$

On the other hand, as $\lambda 1 - x \in C$, it commutes with any $z \in C$. Thus,

$$yz = yz(\lambda 1 - x)y = y(\lambda 1 - x)zy = zy$$

implying $y \in C$ by maximality. Thus $\lambda \notin \sigma_C(x)$ as required. □

## 3.4 Commutative Banach algebra

**Definition 3.10** (Character)**.** A character on an algebra $A$ is a non-zero homomorphism $\phi : A \to \mathbb{C}$. We denote the set of all characters on $A$ by $\Phi_A$ and we call it the spectrum of $A$ (when it is equipped with the Gelfand topology, see below).

In the case $A$ is unital, then for all $\phi \in \Phi_A$, $\phi(1) = 1$.

**Lemma 3.4.** Let $A$ is a Banach algebra, $\phi \in \Phi_A$, then $\phi$ is bounded and $\|\phi\| \leq 1$. Moreover, if $A$ is unital, then $\|\phi\| = 1$.

*Proof.* By defining $\phi_+ : A_+ \to \mathbb{C}$, $\phi_+(x + \lambda 1) = \phi(x) + \lambda$, we have $\phi_+ \in \Phi_+$ with $\phi_+|_A = \phi$. Thus, it suffices to show $\|\phi_+\| \leq 1$ allowing us to assume $A$ is unital.

Let $x \in A$ and suppose $\phi(x) > \|x\|$. Then, $\phi(x)1 - x \in G(A)$ (since for all $\lambda \in \sigma_A(x)$, $|\lambda| \leq \|x\|$). Thus, there exists some $y \in A$ such that $(\phi(x)1 - x)y = 1$ and applying $\phi$ on both sides results in

$$1 = \phi(1) = (\phi(\phi(x)1) - \phi(x))\phi(y) = (\phi(x) - \phi(x))\phi(y) = 0$$

which is a contradiction. Thus, $\phi(x) \leq \|x\|$. On the other hand, as $\|\phi(1)\| = 1$, it follows $\|\phi\| = 1$. □

**Lemma 3.5.** Let $A$ be a unital Banach algebra. If $J$ is a proper ideal of $A$, then so is $\overline{J}$. Hence, maximal ideals are always closed.

*Proof.* Since $J$ is proper, $J \cap G(A) = \varnothing$. Thus, as $G(A)$ is open, we also have $\overline{J} \cap G(A) = \varnothing$. Hence, $\overline{J}$ is a proper ideal of $A$ as required. □

We introduce the notation $\mathscr{M}_A$ for the set of all maximal ideals of $A$.

**Theorem 10.** Let $A$ be a commutative unital Banach algebra. Then the map

$$\phi \mapsto \ker\phi : \Phi_A \to \mathscr{M}_A$$

is a bijection.

*Proof.* Firstly, the map is well-defined as it is clear $\ker\phi$ is an ideal of $A$ while it is maximal since $\operatorname{codim}(\phi) = 1$.

*Injectivity*: Let $\phi, \psi \in \Phi_A$ with $\ker\phi = \ker\psi$. Then, for all $x \in A$, $\phi(x)1 - x \in \ker\psi$ and thus, $\phi(x) - \psi(x) = 0$ as required.

*Surjectivity*: Let $M \in \mathscr{M}_A$ so $A/M$ is a field and a unital Banach algebra. By Gelfand-Mazur, $A/M$ is isometrically isomorphic to $\mathbb{C}$ and thus the quotient map is a character with kernel $M$ □

**Corollary 10.1.** Let $A$ be a commutative unital Banach algebra with $x \in A$. Then,

- $x \in G(A)$ if and only if for all $\phi \in \Phi_A$, $\phi(x) \neq 0$.
- $\sigma_A(x) = \{\phi(x) \mid \phi \in \Phi_A\}$.
- $r_A(x) = \sup\{|\phi(x)| \mid \phi \in \Phi_A\}$.

*Proof.*

- If $x \in G(A)$, then for all $\phi \in \Phi_A$, $1 = \phi(1) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = 0$ if $\phi(x) = 0$.

  On the other hand, if $x \notin G(A)$, we can define $M$ to be a maximal ideal containing $x$. Thus, by the above theorem, there exists some $\phi \in \Phi_A$ such that $\ker\phi = M \ni x$.

- By the first part, $\lambda \in \sigma_A(x)$ if and only if there exists some $\phi \in \Phi_A$ such that $\phi(\lambda 1 - x) = 0$. Namely $\lambda \in \sigma_A(x)$ if and only if there exists some $\phi \in \Phi_A$ such that $\phi(x) = \lambda$.

- Clear by the second part.

□

**Corollary 10.2.** Let $A$ be a Banach algebra and $x, y \in A$ such that $xy = yx$. Then,

$$r_A(x+y) \leq r_A(x) + r_A(y) \text{ and } r_A(xy) \leq r_A(x)r_A(y).$$

*Proof.* By unitization we may assume $A$ is unital. By consider the subalgebra generated by $x, y$, we can also assume $A$ is commutative. Thus, the conclusion is clear by the third part of the above corollary. □

**Example 3.7.** Let $A = C(K)$ where $K$ is a compact Hausdorff space. Then

$$\Phi_A = \{\delta_k \mid k \in K\}$$

where $\delta_k(f) = f(k)$.

Clearly $\delta_k \in \Phi_A$ for any $k$ so we will only consider the reverse inclusion. Let $M \in \mathscr{M}_A$ and we need to show that there is some $k \in K$ such that

$$M = \{f \mid f(k) = 0\}.$$

Suppose otherwise, then for all $k \in K$, there exists some $f_k \in M$ such that $f_k(k) \neq 0$. Then, for each $k$, $f_k$ is not zero on a open neighborhood of $k$. Let us denote this neighborhood by $U_k$ and it is clear that $\{U_k\}_{k \in K}$ form an open cover of $K$. Thus, by compactness, there exists $k_1, \cdots, k_n$ such that $\{U_{k_i}\}_{i=1}^n$ covers $K$. Now, defining

$$g := \sum_{i=1}^n |f_{k_i}|^2,$$

by construction, $g$ does not have any 0 in $K$ and in particular, $g \in G(A)$. However, since $g = \sum_{i=1}^n \overline{f_{k_i}} f_{k_i} \in M$, it follows that $M = A$ which is a contradiction.

**Example 3.8.** Let $K \subseteq \mathbb{C}$ be a non-empty compact set. Then $\Phi_{C(K)} = \{\delta_\omega \mid \omega \in K\}$.

**Example 3.9.** For $A(\Delta)$ the disk algebra, $\Phi_{A(\Delta)} = \{\delta_n \mid n \in \Delta\}$.

**Example 3.10.** Denote the Wiener algebra

$$W = \left\{ f \in C(\mathbb{T}) \mid \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty \right\}$$

where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta})e^{-in\theta} d\theta$. Then, $W$ is a commutative unital Banach algebra with pointwise operations and norm

$$\|f\|_1 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|.$$

This is isometrically isomorphic to the commutative unital Banach algebra $l_1(\mathbb{Z})$ with pointwise vector operations, the $l_1$-norm and convolution as multiplication:

$$(a * b)_n = \sum_{j+k=n} a_j b_k.$$

In this case, $\Phi_W = \{\delta_\omega \mid \omega \in \mathbb{T}\}$ and so $f \in W$ is invertible if and only if $f$ is nowhere 0 (Wiener's theorem).

Let $A$ be a commutative unital Banach algebra. Then, we may write

$$\begin{aligned}\Phi_A &= \{\phi \in B_{A^*} \mid \phi(1) = 1, \phi(xy) = \phi(x)\phi(y),\ \forall x, y \in A\} \\ &= B_{A^*} \cap \hat{1}^{-1}(1) \cap \bigcap_{x,y \in A} (\widehat{xy} - \hat{x}\hat{y})^{-1}(0).\end{aligned}$$

which is a w*-closed subset of $A$. So, $\Phi_A$ is a *compact* Hausdorff space in the w*-topology and this topology is known as the Gelfand topology. We call $\Phi_A$ equipped with the Gelfand topology the spectrum of $A$.

For $x \in A$, we define its Gelfand transform to be

$$\hat{x} : \Phi_A \to \mathbb{C} : \phi \mapsto \phi(x).$$

The map $x \mapsto \hat{x} : A \to C(\Phi_A)$ is called the Gelfand map.

**Theorem 11** (Gelfand representation theorem). The Gelfand map $A \to C(\Phi_A)$ is a continuous, unital homomorphism. Moreover, for $x \in A$,

- $\|\hat{x}\|_\infty = r_A(x) \le \|x\|$.
- $\sigma_{C(\Phi_A)}(\hat{x}) = \sigma_A(x)$.
- $x \in G(A)$ if and only if $\hat{x} \in G(C(\Phi_A))$.

*Proof.* Continuity and the first part of the theorem follows as

$$\|\hat{x}\| = \sup\{|\hat{x}(\phi)| \mid \phi \in \Phi_A\} = \sup\{\phi(x) \mid \phi \in \Phi_A\} = r_A(x) \le \|x\|.$$

The second part of the theorem follows as

$$\sigma_{C(\Phi_A)}(\hat{x}) = \{\phi(x) \mid \phi \in \Phi_A\} = \sigma_A(x)$$

where the first equality holds since by the above example, $\Phi_{C(\Phi_A)} = \{\delta_\phi \mid \phi \in \Phi_A\}$.
Finally, the third part follows directly from the second. □

We remark that the Gelfand map is in general *not* injective nor surjective. The Gelfand map has kernel

$$\{x \in A \mid r_A(x) = 0\} = \{x \mid \liminf_{n\to\infty} \|x^n\|^{1/n} = 0\} = \bigcap_{\phi \in \Phi_A} \ker \phi = \bigcap_{M \in \mathscr{M}_A} M.$$

We call $x \in A$ quasi-nilpotent if $\liminf_{n\to\infty} \|x^n\|^{1/n} = 0$. The ideal $J(A) := \bigcap_{M \in \mathscr{M}_A} M$ is known as the Jacobson radical and we say $A$ is semi-simple if $J(A) = 0$.

# 4 Holomorphic Functional Calculus

Let $U \subseteq \mathbb{C}$ be non-empty and open. Recall that

$$\mathcal{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$$

which is a locally convex space with seminorms

$$\|f\|_K = \sup_{x \in K} |f(x)|$$

for all $K \subseteq U$ non-empty and compact.

$\mathcal{O}(U)$ is an algebra with pointwise multiplication.

We introduce the notation $e, u \in \mathcal{O}(U)$ where $e(z) = 1, u(z) = z$ for all $z \in U$. $\mathcal{O}(U)$ is then unital with the unit $e$.

The main theorem of the chapter is the following.

**Theorem 12** (Holomorphic functional calculus, HFC)**.** Let $A$ be a commutative Banach algebra with $x \in A$. Let $U \subseteq \mathbb{C}$ be a non-empty open set with $\sigma_A(x) \subseteq U$. Then, there exists a unique, continuous, unital homomorphism

$$\Theta_x : \mathcal{O}(U) \to A, \text{ such that } \Theta_x(u) = x.$$

Moreover, for all $\phi \in \Phi_A, f \in \mathcal{O}(U)$ we have $\phi(\Theta_x(f)) = f(\phi(x))$ and

$$\sigma_A(\Theta_x(f)) = \{f(\lambda) \mid \lambda \in \sigma_A(x)\}.$$

Heuristically, we can think of $\Theta_x$ as the evaluation at $x$ and we write $f(x)$ for $\Theta_x(f)$.

Since $e(x) = \Theta_x(e) = 1$, $u(x) = \Theta_x(u) = x$ and $\Theta_x$ is a homomorphism, it follows that if $p(z) = \sum_{k=0}^n a_k z^k$ is a complex polynomial, then $p(x) = \Theta_x(p) = \sum_{k=0}^n a_k x^k$.

To prove this theorem, we will need Runge's approximation theorem which allows us to approximate holomorphic functions by rational functions.

**Theorem 13** (Runge's approximation theorem)**.** Let $K \subseteq \mathbb{C}$ be a non-empty compact set. Then, $\mathcal{O}(K) = \mathcal{R}(K)$, i.e. if $f$ is holomorphic on some open set containing $K$, then for all $\epsilon > 0$, there exists a rational function $r$ without poles in $K$ such that $\|f - r\|_K < \epsilon$.

More precisely, given a set $\Lambda$ which contains a point from each bounded component of $\mathbb{C} \setminus K$. For any $\epsilon > 0$ and $f$ holomorphic on some open set containing $K$, there exists a rational function $r$ with poles in $\Lambda$ such that $\|f - r\|_K < \epsilon$.

We remark that if $\mathbb{C} \setminus K$ is connected, then taking $\Lambda = \emptyset$, we have $\mathcal{O}(K) = \mathcal{P}(K)$.

## 4.1 Vector-valued integration

Let $a < b$ be in $\mathbb{R}$, $X$ a Banach space, $f : [a, b] \to X$ a continuous function. We define the integral of $f$ over $[a, b]$ as follows:

Take sequences $\mathcal{D}_n$: $a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n}^{(n)} = b$ such that $|\mathcal{D}_n| = \max_{1 \le j \le k_n} |t_j^{(n)} - t_{j-1}^{(n)}| \to 0$ as $n \to \infty$.

Since $f$ is continuous on a compact set, it is uniformly continuous and so the limit

$$\lim_{n\to\infty}\sum_{j=1}^{k_n} f(t_j^{(n)})(t_j^{(n)}-t_{j-1}^{(n)})$$

exists and is independent of the choice of $\mathscr{D}_n$. We denote this limit by $\int_a^b f(t)\mathrm{d}t$.
We note that for $\phi \in X^*$,

$$\phi\left(\int_a^b f(t)\mathrm{d}t\right)=\int_a^b \phi(f(t))\mathrm{d}t.$$

If $\phi$ is the norming functional of $\int_a^b f(t)\mathrm{d}t$, then

$$\left\|\int_a^b f(t)\mathrm{d}t\right\| = \phi\left(\int_a^b f(t)\mathrm{d}t\right)=\int_a^b \phi(f(t))\mathrm{d}t \le \int_a^b \|\phi\|\|f(t)\|\mathrm{d}t = \int \|f(t)\|\mathrm{d}t.$$

Next, let $\gamma : [a,b] \to \mathbb{C}$ be a path (i.e. $\gamma$ continuously differentiable) and $f : [\gamma] \to X$ a continuous function ($[\gamma]=\gamma([a,b])$). Define

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t.$$

For a chain $\Gamma=(\gamma_1,\cdots,\gamma_n)$ and $f : [\Gamma] \to X$ (where $[\Gamma]=\bigcup_{i=1}^n[\gamma_i]$), we define

$$\int_\Gamma f(z)\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\mathrm{d}z.$$

From this, we observe

$$\left\|\int_\Gamma f(z)\mathrm{d}z\right\| \le \sum_{i=1}^n \left\|\int_{\gamma_i} f(z)\mathrm{d}z\right\| \le l(\Gamma)\sup_{z\in\Gamma}\|f(z)\|$$

where $l(\Gamma)$ denotes the length of $\Gamma$, i.e. $l(\Gamma)=\sum_{i=1}^n l(\gamma_i)$ where $l(\gamma)=\int_a^b|\gamma'(t)|\mathrm{d}t$ for any path $\gamma$.

## 4.2 Proof of HFC

**Theorem 14** (Vector valued Cauchy theorem)**.** Let $U \subseteq \mathbb{C}$ be an open set, $\Gamma$ a cycle in $U$ such that $n(\Gamma,\omega)=0$ for all $\omega \notin U$. Then, for a holomorphic function $f : U \to X$, we have

$$\int_\Gamma f(z)\mathrm{d}z = 0.$$

*Proof.* Indeed, for all $\phi \in X^*$,

$$\phi\left(\int_\Gamma f(z)\mathrm{d}z\right)=\int_\Gamma \phi(f(z))\mathrm{d}z = 0$$

as $\phi\circ f$ is holomorphic. Hence, the result follows by Hahn-Banach. □

**Lemma 4.1.** Let $A$ be a unital Banach algebra and $x \in U \subseteq \mathbb{C}$ for some non-empty open set $U$. Furthermore, denote $K = \sigma_A(x)$. Then, for a cycle $\Gamma$ in $U \setminus K$, with

$$n(\Gamma, \omega) = \begin{cases} 1, & \omega \in K, \\ 0, & \omega \notin K, \end{cases}$$

defining

$$\Theta_x : \mathcal{O}(U) \to A,\ f \mapsto \frac{1}{2\pi i} \int_\Gamma f(z)(z1 - x)^{-1} \mathrm{d}z,$$

then $\Theta_x$ is well-defined, unital, linear and continuous. Furthermore,

- For a rational function $r$ without poles in $U$, we have $\Theta_x(r) = r(x)$.
- For all $\phi \in \Phi_A$ and $f \in \mathcal{O}(Y)$,

$$\phi(\Theta_x(f)) = f(\phi(x)) \text{ and } \sigma_A(\Theta_x(f)) = f(\sigma_A(x)).$$

We remark that the above lemma is not quite HFS. Indeed, it is missing the condition that $\Theta_x$ is multiplicative.

*Proof. Well-defined*: By construction, as $\Gamma \cap K = \emptyset$, for all $z \in [\Gamma]$, $z1 - x \in G(A)$ and so $f(z)(z1 - x)^{-1}$ is defined on $[\Gamma]$ and furthermore is continuous.
*Linearity*: follows as the integral is linear.
*Continuity*:

$$\|\Theta_x(f)\| \leq \frac{1}{2\pi} l(\Gamma) \sup_{z \in [\Gamma]} |f(z)| \|(z1 - x)^{-1}\| = M \sup_{z \in [\Gamma]} |f(z)| = M \|f\|_{[\Gamma]}$$

with $\|\cdot\|_{[\Gamma]}$ being one of the semi-norms used to define the LCS topology on $\mathcal{O}(U)$ implying $\Theta_x$ is continuous.
*Unital*:

$$\Theta_x(e) = \frac{1}{2\pi i} \int_\Gamma (z1 - x)^{-1} \mathrm{d}z = \frac{1}{2\pi i} \int_{|z|=R} (z1 - x)^{-1} \mathrm{d}x = \frac{1}{2\pi i} z^{-1} \int_{|z|=R} (1 - x/z)^{-1} \mathrm{d}z$$

where the second equality holds as $\Gamma$ and $\{|z| = R\}$ are homologous in $\mathbb{C} \setminus K$ for sufficiently large $R$. Thus, taking $R > \|x\|$, we have

$$\Theta_x(e) = \frac{1}{2\pi i} \int_{|z|=R} \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} \mathrm{d}z = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{|z|=R} \frac{\mathrm{d}z}{z^{n+1}} \right) x^n = 1$$

Now, given rational function $r$ without poles in $U$, we can write $r = p/q \in \mathcal{O}(U)$ where $p, q$ are complex polynomials where $q$ does not have any zeros in $U$. By the polynomial spectral mapping theorem, we know that

$$\sigma_A(q(x)) = \{q(\lambda) \mid \lambda \in \sigma_A(x)\}.$$

Thus, $0 \notin \sigma_A(q(x))$ implying $q(x) \in G(A)$ and we may define $r(x) = p(x)q(x)^{-1}$. Denote $s$ the complex polynomial in two variables such that for all $z, w \in \mathbb{C}$, we have

$$p(z)q(w) - q(z)p(w) = (z - w)s(z, w).$$

Hence,

$$p(z)q(x) - q(z)p(x) = (z1 - x)s(z,x)$$

implying

$$r(z)1(z1-x)^{-1} - r(x)(z1-x)^{-1} = s(z,x)q(z)^{-1}q(x)^{-1}.$$

Thus,

$$\Theta_x(r) = \frac{1}{2\pi i}\int_\Gamma r(z)(z1-x)^{-1}\mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma (z1-x)^{-1}\mathrm{d}z r(x) + \frac{1}{2\pi i}\int_\Gamma s(z,x)q(z)^{-1}\mathrm{d}z q(x)^{-1}$$

where $\frac{1}{2\pi i}\int_\Gamma (z1-x)^{-1}\mathrm{d}z = \Theta_x(e) = 1$ and the second term vanishes by the scalar Cauchy integral formula resulting in $\Theta_x(r) = r(x)$ as required.

Finally, fixing $\phi \in \Phi_A$ and $f \in \mathscr{O}(U)$, we have

$$\begin{aligned}\phi(\Theta_x(f)) = \phi\left(\frac{1}{2\pi i}\int_\Gamma f(z)(z1-x)^{-1}\mathrm{d}z\right) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\phi(x)}\mathrm{d}z \\ &= n(\Gamma,\phi(x))f(\phi(x)) = f(\phi(x))\end{aligned}$$

where $n(\Gamma,\phi(x)) = 1$ since $\phi(x) \in \sigma_A(x) = K$. With this, we have

$$\sigma_A(\Theta_x(f)) = \{\phi(\Theta_x(f)) \mid \phi \in \Phi_A\} = \{f(\phi(x)) \mid \phi \in \Phi_A\} = \{f(\lambda) \mid \lambda \in \sigma_A(x)\}$$

as required. □

*Proof of Rangu's theorem.* Let $U \subseteq \mathbb{C}$ be open such that $U \supseteq K$. Let $A = \mathscr{R}(K)$, $x, \in A$ be such that $x(z) = z$ for all $z \in K$. Then,

$$\sigma_A(x) = \{\phi(x) \mid \phi \in \Phi_A\} = \{\delta_z(x) \mid z \in K\} = K.$$

Then, for all $f \in \mathscr{O}(U), z \in K$, we have $\Theta_x(f) = \delta_z(\Theta_x(f)) = f(\delta_z(x)) = f(z)$, namely $\Theta_x(f) = f|_K$. Next, taking $B$ to be the closed subalgebra of $A$ generated by $1, x, (\lambda 1 - x)^{-1}$, for all $\lambda \in \Lambda$, i.e. $B$ is the closure in $C(K)$ of rational functions whose poles are in $\Lambda$ (recall that $\Lambda \subseteq \mathbb{C}$ is a given set which contains a point from each bounded components of $\mathbb{C} \setminus K$). It is clear that $B$ is a closed unital subalgebra of $A$.

If $V$ is a bounded component of $\mathbb{C} \setminus K = \mathbb{C} \setminus \sigma_A(x)$, then there exists some $\lambda \in \Lambda \cap V$ such that $\lambda 1 - x$ is invertible in $B$, namely $\lambda \notin \sigma_B(x)$. It follows then $\sigma_B(x) = \sigma_A(x) = K$. Hence, $\Theta_x$ takes value in $B$. □

**Corollary 14.1.** Let $U \subseteq \mathbb{C}$ be a non-empty open set. Then, the subalgebra rational functions of $\mathscr{O}(U)$ is dense, i.e. $\mathscr{R}(U) = \mathscr{O}(U)$.

*Proof.* Let $K \subseteq U$ be a non-empty compact set. Let $\hat{K} = K \cup$ bounded components of $\mathbb{C} \setminus K$ in $U$. Then, $\hat{K}$ is compact and $\hat{K} \subseteq U$. For every bounded component $V$ of $\mathbb{C} \setminus \hat{K}$, we know $V \setminus U \neq \varnothing$, so we can pick $\lambda_V \in V \setminus U$. Let $\Lambda$ be the set of all such $\lambda_v$s. By Rangu's theorem, given $f \in \mathscr{O}(U), \epsilon > 0$, there exists a rational function $r$ whose poles lie in $\Lambda$ and $\|f - r\|_{\hat{K}} < \epsilon$. Thus, $r \in \mathscr{R}(U)$ and $\|f - r\|_K < \epsilon$ as required. □

*Proof of HFC.* Let $A, x, U$ be as in the theorem.

*Existence*: Since, we have already defined $\Theta_x$ and shown it is unital, linear and continuous, we just need to check $\Theta_x(fg) = \Theta_x(f)\Theta_x(g)$ for all $f, g \in \mathcal{O}(U)$. However, we have already shown this for rational functions, and thus, we may conclude by using the density of rational functions in $\mathcal{O}(U)$ and the continuity of $\Theta_x$.

*Uniqueness*: If $\Psi : \mathcal{O}(U) \to A$ is another continuous unital homomorphism such that $\Psi(u) = x$, then, as $\Psi$ is a homomorphism, $\Psi(r) = r(x)$ for all rational functions $r$. Thus, $\Theta_x = \Psi$ by continuity and density. $\square$

# 5 $C^*$-algebra