Advanced Financial Models

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Continuous Time Models

Consider a marker with $n = 1 + d$ assets $P = (B, S)$ where all assets are numéraires. We assume the prices follow

$$\begin{cases} \mathrm{d}B_t &= B_t r_t \mathrm{d}t \\ \mathrm{d}S_t^i &= S_t^i \left(\mu_t^i \mathrm{d}t + \sum_{j=1}^m \sigma_t^{ij} \mathrm{d}W_t^j\right) \end{cases}$$

for all $i = 1, \cdots, d$ where r, μ, σ are predictable and suitably integrable and W is a m-dimensional standard Brownian motion.

With Itô's formula, we have

$$\begin{cases} B_t &= B_0 \exp\left(\int_0^t r_s \mathrm{d}s\right) \\ S_t^i &= S_0^i \exp\left(\int_0^t \left(\mu_s^i - \frac{1}{2}\sum_{j=1}^m (\sigma_s^{ij})^2\right) \mathrm{d}s + \int_0^t \sum_{j=1}^m \sigma_s^{ij} \mathrm{d}W_s^j\right) \end{cases}$$

If $H = (\theta^0, \theta)$ is a self-financing strategy, i.e. $\mathrm{d}(H_t \cdot P_t) = H_t \cdot \mathrm{d}P_t$, then denoting

$$X_t = H_t \cdot P_t = \theta^0 B_t + \theta \cdot S_t,$$

we have

$$\mathrm{d}X_t = \theta^0 \mathrm{d}B_t + \theta \cdot \mathrm{d}S_t.$$

Now, by definition, $\theta^0 = (X_t - \theta \cdot S_t)/B_t$ and $\mathrm{d}B_t = B_t r_t \mathrm{d}t$, we have

$$\mathrm{d}X_t = r_t(X_t - \theta \cdot S_t)\mathrm{d}t + \theta \cdot \mathrm{d}S_t.$$

By Itô's formula, we have

$$\mathrm{d}(X_t/B_t) = \theta_t \cdot \mathrm{d}(S_t/B_t),$$

and so, if $X_0 = x$, then

$$X_t = B_t\left(x + \int_0^t \theta_s \cdot \mathrm{d}\left(\frac{S_s}{B_s}\right)\right).$$

We can explicitly describe the structure of the local martingale deflators in this market.

Theorem 1. Suppose λ is a m-dimensional predictable process satisfying $\int_0^t \|\lambda\|^2 \mathrm{d}s < \infty$ for all $t \geq 0$ and

$$\sigma_t \lambda_t = \mu_t - r_t 1.$$

Then, for Y given by $Y_0 > 0$ and

$$\mathrm{d}Y_t = Y_t(-r_t \mathrm{d}t - \lambda_t \cdot \mathrm{d}W_t)$$

is a local martingale deflator.

Proposition 0.1. Suppose H is an admissible strategy replicating a European claim with maturity T and payout ξ_T, then if Y is a local martingale deflator,

$$H_t \cdot P_t \geq \frac{1}{Y_t} \mathbb{E}[Y_T \xi_T \mid \mathscr{F}_t].$$

This follows as non-negative local martingales are supermartingales. NB. the above proposition holds in any continuous-time market.

Theorem 2. Suppose $m = \#$ of BM. $= d = \#$ of stocks, σ invertible, $\lambda = \sigma^{-1}(\mu - r1)$ and the filtration is generated by

$$W_t + \int_0^t \lambda_s \mathrm{d}s.$$

Then, if M is a true martingale satisfying

$$\mathrm{d}M = -M\lambda \cdot \mathrm{d}W, M_0 = 1,$$

and for $T > 0$, define $\mathbb{Q}$ such that $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = M_T$, we have for any claim $\xi_T \geq 0$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\xi_T}{B_T}\right] < \infty,$$

there exists an admissible strategy H such that

$$H_t \cdot P_t = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{\xi_T}{B_T} \mid \mathscr{F}_t\right]$$

for all $0 \leq t \leq T$. Namely, H replicates the claim.