Multiscale notes

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March 12, 2024

This note is based on the lectures given by Prof. Xue-Mei Li at EPFL during the spring term of 2024.

Preliminaries

Let $T > 0$ and suppose $f : [0, T] \to \mathbb{R}^d$ is an α-Hölder continuous function for some α. Then, defining $\hat{f} : [0, T] \to \mathbb{R}^d$ by $\hat{f}(t) = \int_0^t f(r)\mathrm{d}r$, we observe that

$$\|\hat{f}(t) - \hat{f}(s) - f(s)|t-s|\| = \left\| \int_s^t (f(r) - f(s))\mathrm{d}r \right\| \leq \|f\|_\alpha |t-s|^{1+\alpha}$$

for all $s, t \in [0, T]$. Thus, we have that

$$\frac{\|\hat{f}(t) - \hat{f}(s)\|}{|t-s|^{1+\alpha}} \leq \|f\|_\alpha + \frac{\|f(s)\|}{|t-s|^\alpha} \tag{1}$$

Hence, if α is somehow negative, the second term in the above equation vanishes as $|t-s| \to 0$ and this motivates the following definition for functions with negative Hölder continuity.

Definition 1. For $f : [0, T] \to \mathbb{R}^d$ and $\kappa \in (0, 1)$, we define

$$\|f\|_{-\kappa} = \sup_{s \neq t \in [0,T]} \frac{1}{|t-s|^{1-\kappa}} \int_s^t f(r)\mathrm{d}r.$$

We denote $C^{-\kappa} = \{f : [0, T] \to \mathbb{R}^d : \|f\|_{-\kappa} < \infty\}$.

We observe that, denoting $\hat{f}(t) = \int_0^t f(r)\mathrm{d}r$ as above, $\|f\|_{-\kappa} = \|\hat{f}\|_{1-\kappa}$.

Definition 2. For $f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $\kappa, \gamma \in (0, 1)$, we define

$$\|f\|_{-\kappa,\gamma} = \sup_{x \neq y \in \mathbb{R}^d} \frac{\|f(x, \cdot) - f(y, \cdot)\|_{-\kappa}}{\|x-y\|^\gamma}.$$

Moreover, denote $C_{-\kappa,\gamma} = \{f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d : \|f\|_{-\kappa,\gamma} < \infty\}$.

Theorem 1 (Multidimensional Kolmogorov's continuity criterion)**.** Let $(X_t)_{t\in[0,1]^d}$ be a stochastic process with values in $\mathbb{R}^d$ and suppose there exists $\alpha, C > 0$ such that for all $s \neq t \in [0,1]^d$,

$$\|X_t - X_s\|_p \le C\|t-s\|^\alpha.$$

Then, for all $\gamma < \alpha - \frac{d}{p}$, there exists a continuous modification of $(X_t)_{t\in[0,1]^d}$ such that

$$\left\| \sup_{s\neq t\in[0,1]^d} \frac{\|X_t - X_s\|}{\|t-s\|^\gamma} \right\|_p = \|\|X\|_\gamma\|_p \le C\tilde{C}$$

where $\tilde{C} = \sum_{m\in\mathbb{N}} 2^{m(\gamma p - \alpha p + d)}$.

Theorem 2 (Sewing Lemma)**.** Let W be a Banach space and for any two parameter process $A : [0,T]^2_\le \to W$, defining $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t}$, we denote

$$\|A\|_\eta = \sup_{s<t\in[0,T]} \frac{\|A_{s,t}\|}{|t-s|^\eta} \text{ and } \|\delta A\|_{\bar{\eta}} = \sup_{s<u<t\in[0,T]} \frac{\|\delta A_{s,u,t}\|}{|t-s|^{\bar{\eta}}}$$

for some $\eta, \bar{\eta} > 0$. Then, we take

$$C^{\eta,\bar{\eta}}([0,T],W) = \{A : [0,T]^2_\le \to W : \|A\|_\eta < \infty, \|\delta A\|_{\bar{\eta}} < \infty\}.$$

For $\eta > 0$ and $\bar{\eta} > 1$, there exists a unique linear map

$$I : C^{\eta,\bar{\eta}}([0,T],W) \to C^\eta([0,T],W)$$

such that $I(A)_0 = 0$ and for all $s < t \in [0,T]$,

$$\|\underbrace{I(A)_t - I(A)_s}_{=:I(A)_{s,t}} - A_{s,t}\| \le C|t-s|^{\bar{\eta}}\|\delta A\|_{\bar{\eta}}.$$

Moreover, for any partition $\mathscr{P}$ of the interval $[s,t] \subseteq [0,T]$, we have that

$$\left\| I(A)_{s,t} - \sum_{[u,v]\in\mathscr{P}} A_{u,v} \right\| \le C|t-s|\|\delta A\|_{\bar{\eta}}|\mathscr{P}|^{\bar{\eta}-1}.$$

Theorem 3 (Young integral)**.** Let $f \in C^\alpha$, $g \in C^\beta$ such that $\alpha + \beta > 1$. Then, defining $A_{s,t} = f_s g_{s,t}$, the sewing lemma applies and we denote the resulting map by

$$\int_s^t f_r \mathrm{d}g_r := I(A)_{s,t}.$$

The sewing lemma provides the following bounded which we will use frequently in the next section.

Lemma 1. Let $\alpha, \kappa, \gamma \in (0,1)$ such that $\alpha\gamma > \kappa$. Then, there exists a continuous linear map

$$\Phi : C_{-\kappa,\gamma} \times C^\alpha([0,T],\mathbb{R}^d) \to C^{-\kappa}([0,T],\mathbb{R}^d) : (f,x) \mapsto (t \mapsto f(t,x_t)).$$

Moreover, $\|\Phi(f,x)\|_{-\kappa} \lesssim \|f\|_{-\kappa,\gamma}(1 + \|x\|_\alpha^\gamma T^{\alpha\gamma})$.

Proof. In order to show $\Phi(f,x) \in C^{-\kappa}$, we need to show that $\int_0^{\cdot} f(r,x_r)\mathrm{d}r \in C^{1-\kappa}$ to which one simply applies the sewing lemma to $A_{s,t} = \int_s^t f(r,x_s)\mathrm{d}r$. $\square$

Definition 3. For two random variables X, Y, we define their degree of mixing coefficient to be

$$\alpha(X,Y) = \sup_{A\in\sigma(X), B\in\sigma(Y)} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

We see straightaway that two independent random variables have mixing coefficient 0 and so intuitively, the mixing coefficient describes how close two random variables are to being independent. Moreover, we recall that in setting of measure preserving systems, the m.p.s. $(X, \mathscr{A}, \mu, T)$ is said to be strong mixing if for all $A, B \in \mathscr{A}$,

$$\lim_{n\to\infty} |\mu(T^{-n}A\cap B) - \mu(A)\mu(B)| = 0.$$

Hence, the m.p.s. is strong mixing if and only if $\alpha(T^0, T^n) \to 0$ as $n \to \infty$.

Proposition 1. Let X, Y be two random variables. Then,

$$4\alpha(X,Y) = \sup_{\|F\|_\infty, \|G\|_\infty \le 1} |\mathbb{E}[F(X)G(Y)] - \mathbb{E}[F(X)]\mathbb{E}[G(Y)]|.$$

Proposition 2. If there exists some $\delta > 0$ such that $\|X\|_{2+\delta}, \|Y\|_{2+\delta} \le c$, then

$$\mathrm{Cov}(X,Y) \le 8c^{\frac{2}{2+\delta}}(\alpha(X,Y))^{\frac{2}{2+\delta}}.$$

Multiscale analysis

We mainly consider the following two standing assumptions.

Assumption 1. Let (y_t) be a stationary process taking value in Y such that $\alpha(y_0, y_t) \lesssim t^{-\delta}$ for some $\delta \in (0,1)$. We denote $\mu = \mathscr{L}(y_0)$ the stationary measure of the process.

Assumption 2. Let $F : \mathbb{R}^d \times Y \to \mathbb{R}$ be uniformly (in both arguments) bounded and Lipschitz in $\mathbb{R}^d$ (with Lipschitz constant uniform over Y).

Lemma 2. Assuming Assumption 1 and let $F : \mathbb{R}^d \times Y \to \mathbb{R}$ be a bounded measurable function in Y and define

$$f_n(x,r) = F(x, y_{nr}) \text{ and } \bar{f}(x) = \int_Y F(x,y)\mu(\mathrm{d}y).$$

Then, for all $p \ge 2$ and $x \in \mathbb{R}^d$, we have that

$$\left\| \int_s^t f_n(x,r) - \bar{f}(x)(t-s) \right\|_p \lesssim \|F(x,\cdot)\|_\infty n^{-\frac{\delta}{p}} |t-s|^{1-\frac{\delta}{p}}$$

Proof. As fundamentally x plays no role in the above statement, we will omit it from our notations. Moreover, by scaling and translating, we may assume without loss of generality that $\bar{f} = 0$ and $s = 0$. Then, we observe

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t f_n(r)\mathrm{d}r\right)^2\right] &= \int_0^t \int_0^t \mathbb{E}[f_n(r)f_n(s)]\mathrm{d}r\mathrm{d}s \\ &\le 4\|F\|_\infty \int_0^t \int_0^t \alpha(y_{nr}, y_{ns})\mathrm{d}r\mathrm{d}s \le 4\|F\|_\infty n^\delta t^{2-\delta}.\end{aligned}$$

Hence, for all $p \geq 2$, we have that

$$\mathbb{E}\left[\left(\int_0^t f_n(r)\mathrm{d}r\right)^p\right] \leq t^{p-2}\|F\|_\infty^{p-2}\mathbb{E}\left[\left(\int_0^t f_n(r)\mathrm{d}r\right)^2\right] \lesssim \|F\|_\infty^p n^\delta t^{p-\delta}.$$

□

Lemma 3. Assuming now Assumption 1 and 2 and denote f and $\bar{f}$ as in the previous lemma. Then, for all $p \geq 2$ and $x, z \in \mathbb{R}^d$, we have that

$$\left\|\int_s^t (f_n(x,r) - f_n(z,r)) - (\bar{f}(x) - \bar{f}(z))\mathrm{d}r\right\|_p \lesssim n^{-\frac{\delta}{p}}|t-s|^{1-\frac{\delta}{p}}\|x-z\|.$$

Proof. Fixing $z \in \mathbb{R}^d$ and defining $\bar{F}(x,y) = F(x,y) - F(z,y)$, the bound follows by observing that $\|\bar{F}(x,\cdot)\|_\infty = \|F(x,\cdot) - F(z,\cdot)\|_\infty \leq \|F\|_1\|x-z\|$. □

Lemma 4. Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be bounded and Lipschitz and suppose $G, \tilde{G} \in C^\alpha([0,T],\mathbb{R}^d)$ for some $\alpha \in (0,1)$ with $G_0 = \tilde{G}_0$. Then, if

$$z_t = G_t + \int_0^t F(z_s)\mathrm{d}s \text{ and } \tilde{z}_t = \tilde{G}_t + \int_0^t F(\tilde{z}_s)\mathrm{d}s,$$

we have that

$$\|z - \tilde{z}\|_\alpha \lesssim_{\|F\|_\infty} \|G - \tilde{G}\|_\alpha.$$

Proof. It is easy to see from Grönwall's inequality that

$$\|z - \tilde{z}\|_\infty \leq \|G - \tilde{G}\|_\infty \exp(\|F\|_1 T).$$

On the other hand, since for any $t \in [0,T]$, we have that

$$\|G_t - \tilde{G}_t\| \leq \|(G_t - \tilde{G}_t) - (G_0 - \tilde{G}_0)\| \leq \|G - \tilde{G}\|_\alpha T^\alpha,$$

so, $\|G - \tilde{G}\|_\infty \leq \|G - \tilde{G}\|_\alpha T^\alpha$. Hence, combining the two estimates provides the desired bound. □

Lemma 5. Let $f_n, f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ be functions in $C_{-\kappa,\gamma}$ such that the ODEs

$$x_t^n = x_0 + \int_0^t f_n(s, x_s^n)\mathrm{d}s, x_t = x_0 + \int_0^t f(s, x_s)\mathrm{d}s$$

have unique solutions $x^n, x \in C^\alpha$ for some $\alpha + \gamma \leq 1$ and $\alpha\gamma > \kappa$. Then, if $\|f_n - f\|_{-\kappa,\gamma} \to 0$ as $n \to \infty$, we have that $\|x^n - x\|_\alpha \to 0$.

Proof. Follows by applying the previous lemma in which we take

$$G_t = x_0 + \int_0^t (f_n(s, x_s^n) - f(s, x_s^n))\mathrm{d}s.$$

□