Rough paths notes

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This notes is based on *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications* by Peter Friz and Nicolas Victoir.

Basics

Take (E, d) to be a complete metric space.

Definition 1. A map $\omega : \Delta_T \to [0, \infty)$ is a control (denoting $\Delta = \Delta_T = \{(s,t) \in [0,T]^2 : 0 \leq s \leq t \leq T\}$) if it is

- super-additive: $\omega(s,t) + \omega(t,u) \leq \omega(s,u)$ for all $s \leq t \leq u$ in $[0,T]$
- zero on the diagonal: $\omega(s,s) = 0$ for all $s \in [0,T]$.

Given $x \in C^{1-var}([0,T], E)$, we say it is dominated by a control ω if for all $s < t$, $d(x_s, x_t) \lesssim \omega(s,t)$. In particular, if x is dominated by ω, then $\|x\|_{1-var;[s,t]} \leq C\omega(s,t)$ for some constant C. Moreover, the map

$$(s,t) \mapsto \|x\|_{1-var;[s,t]}$$

is a (additive) control dominating x. Denote this control by ω_x.

Denote $C^{0,1-var}([0,T], \mathbb{R}^d) = \overline{C^\infty([0,T], \mathbb{R}^d)}$ where the closure is taken with respect to the C^{1-var} norm. It is a Banach space and moreover,

$$C^{0,1-var}([0,T], \mathbb{R}^d) = \{x : [0,T] \to \mathbb{R}^d \text{ absolutely continuous}\}.$$

Young integration

In stochastic calculus, we defined the Ito integral in which we are allowed to integrate with respect martingales, for which the construction relied on the fact that martingales has a quadratic variation process. The Young integral generalizes this by allowing us to construct an integral of the form $\int_0^\cdot y \, dx$ for $x \in C^{p-var}([0,T], \mathbb{R}^d)$ and $y \in C^{q-var}([0,T], L(\mathbb{R}^d, \mathbb{R}^e))$ where $1/p + 1/q > 1$.