Notes on Lévy processes

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An $\mathbb{R}^d$-valued stochastic process $(Z_t)_{t\in[0,T]}$ is said to be a Lévy process if

1. it has independent increments, i.e. for all $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$, the random variables $(Z_{t_{i+1}} - Z_{t_i})_{i=0}^{n-1}$ are independent;
2. it has stationary increments, i.e. for all $0 \le s < t \le T$ and $h > 0$, we have
$$Z_t - Z_s \sim Z_{t-s};$$
3. it is continuous in probability, i.e. for all $\epsilon > 0$ and $t \in [0,T]$, we have
$$\lim_{h\to 0} \mathbb{P}(|Z_{t+h} - Z_t| > \epsilon) = 0.$$

In the case we are working on a filtered probability space with filtration $\mathscr{F}$, we say Z is a $\mathscr{F}$-Lévy process if it is adapted to $\mathscr{F}$ and independent increments condition is replaced by $Z_{t+s} - Z_s \perp \mathscr{F}_s$. If the filtration is natural then the two definitions are equivalent.

It is easy to see that, if (Z_t) is an L^1-Lévy process is such that $\mathbb{E}[Z_t] = mt$, then $M_t := Z_t - mt$ is a martingale with respect to its natural filtration. Moreover, if (Z_t) is an L^2-Lévy process such that $\mathbb{E}[Z_T^{\otimes 2}] = Vt = (v_{ij})t$, then (M_t) is a square-integrable martingale such that $\langle M^i, M^j\rangle_t = v_{ij}t$.

Generally, it turns out that any Lévy process is a semimartingale and all continuous Lévy processes are Gaussian processes. Thus, combining with the Lévy characterisation of the Brownian motion, we have that a continuous L^2-Lévy process is automatically a Brownian motion if $v_{ij} = \delta_{ij}$. In contrast to the typical definition of Brownian motion, we note that in Kunita, any continuous Lévy process is a called a Brownian motion.

We take a long detour here to discuss the theory of the Poisson random measure.

Let $(\mathscr{X}, \mathscr{B})$ be a measurable space and μ a σ-finite measure on $\mathscr{X}$. From AP, we defined a Poisson random measure N with intensity μ as a random variable taking values in the space of counting measures on $\mathscr{X}$ such that for all disjoint $(A_i)_{i=1}^\infty \subseteq \mathscr{B}$, we have that $N(A_i)$ are *independent* Poisson random variables with intensity $\mu(A_i)$. Recall that this means that, for all $k \in \mathbb{N}$,
$$\mathbb{P}(N(A_i) = k) = e^{-\mu(A_i)}\frac{(\mu(A_i))^k}{k!}.$$

We observe that this definition automatically requires that
$$\sum_{i=1}^\infty N(A_i) = N\left(\bigcup_{i=1}^\infty A_i\right) \sim \mathrm{Poi}\left(\mu\left(\bigcup_{i=1}^\infty A_i\right)\right) = \mathrm{Poi}\left(\sum_{i=1}^\infty \mu(A_i)\right)$$

which is the additive property of the Poisson law.

Kunita's definition for the Poisson random measure is a more explicit as he defines it as the induced random measure of a Poisson point process.

- Let $\mathbb{D}_p \subseteq (0,T]$ be a countable set and $p : \mathbb{D}_p \to \mathscr{X}$. Define the measure N_p on $[0,T] \times \mathscr{X}$ such that
$$N_p((s,t] \times U) = \left|\{r \in (s,t] \cap \mathbb{D}_p : p(r) \in U\}\right|$$
for all $s < t \in [0,T]$ and $U \in \mathscr{B}$.

- A Poisson point process p is a random variable taking values in the space of functions $\mathbb{D}_p \to \mathscr{X}$ such that for any $E_1, \dots, E_n \in \mathscr{B}$ such that for almost every ω, $N_{p(\omega)}((0,T] \times E_i) < \infty$ for all $i = 1, \dots, n$, defining
$$X_t = (N_p((0,t], E_i))_{i=1}^n,$$
$(X_t)_{t\in[0,T]}$ is a n-dimensional Lévy process (note that it is automatically a pure jump process).

- The random measure induced by a Poisson point process p: $\omega \mapsto N_{p(\omega)}$ is called a Poisson random measure.

One can show that a Poisson random measure in the Kunita sense is a Poisson random measure in the AP sense with intensity $\mu(A) = \mathbb{E}[N_p((0,1] \times A)] =: \mathbb{E}[N_1(A)]$. We sketch the proof for $N_t(A) \sim \mathrm{Poi}(t\mu(A))$:

It suffices to show that the characteristic function of $N_t(A)$ is $\phi_t(\alpha) = \exp\big((e^{i\alpha} - 1)t\mu(A)\big)$. Since $N_t(A)$ is a Lévy process, in particular, it is stationary and infinitely divisible ($N_t(A) = \sum_{k=0}^{n-1}(N_{\frac{k+1}{n}t}(A) - N_{\frac{k}{n}t}(A))$), we have by the Lévy-Khintchine formula that $\phi_t(\alpha) = e^{t\psi(\alpha)}$ for some function ψ. Thus, defining
$$M_t^\alpha = \exp(i\alpha N_t(A) - t\psi(\alpha)),$$
we observe $M_t^\alpha (M_s^\alpha)^{-1} \perp \mathscr{F}_s$ for all $s < t$ and so, M_t^α is a martingale. Hence, defining
$$Y_t^\alpha = \int_0^t \frac{1}{M_{s-}^\alpha} \mathrm{d}M_s^\alpha,$$
we have by the Itô formula applied to $\log\big(M_t^\alpha\big)$ that
$$Y_t^\alpha = (e^{i\alpha} - 1)N_t(A) - t\psi(\alpha)$$
(this step requires us to observe that $\sum_{s\le t}(e^{i\alpha\Delta X_s} - 1) = (e^{i\alpha} - 1)N_t(A)$). Consequently, taking expectations, we have that $\mu(A) = \mathbb{E}[N_1(A)] = -\frac{\psi(\pi)}{2}$ which implies that
$$\mathbb{E}[N_t(A)] = -\frac{\psi(\pi)}{2} = t\mu(A),$$
and moreover,
$$\psi(\alpha) = (e^{i\alpha} - 1)\mu(A).$$
Thus, $\phi_t(\alpha) = \exp\big((e^{i\alpha} - 1)t\mu(A)\big)$ as required.

Taking $\mu(A) = \mathbb{E}[N_1(A)]$ as above, we define
$$\hat{N}(\mathrm{d}t\mathrm{d}z) = \mathrm{d}t\mu(\mathrm{d}z) \text{ and } \tilde{N}(\mathrm{d}t\mathrm{d}z) = N(\mathrm{d}t\mathrm{d}z) - \mathrm{d}t\mu(\mathrm{d}z) = N(\mathrm{d}t\mathrm{d}z) - \hat{N}(\mathrm{d}t\mathrm{d}z).$$

We call $\tilde{N}$ the compensated Poisson random measure and $\hat{N}$ the compensator of the Poisson random measure of N.

Similarly defining $\tilde{N}_t(A) = \tilde{N}((0,t] \times A)$, since $\mathbb{E}[N_t(A)] = t\mu(A)$, we have by the previous remarks that $\tilde{N}_t(A)$ is a square integrable martingale. Moreover, $\langle \tilde{N}(A) \rangle_t = t\mu(A)$.

We would now like to integrate with respect to the Poisson random measure. To this end, we define integration with respect to the compensated Poisson random measure $\tilde{N}$ while integration with respect to the compensator $\hat{N}$ is classical as it is a deterministic measure.

We make the following observation: Let $t \in (0, T]$ and $A, B \in \mathscr{B}$. Then, as disjoint sets evaluated under $\tilde{N}_t$ are independent, we have that

$$\begin{aligned} \mathbb{E}[\tilde{N}_t(A)\tilde{N}_t(B)] &= \mathbb{E}[(\tilde{N}_t(A \cap B))^2] + \overbrace{\mathbb{E}[\tilde{N}_t(A \setminus B)]\mathbb{E}[\tilde{N}_t(B)]}^{=0} + \overbrace{\mathbb{E}[\tilde{N}_t(A \cap B)]\mathbb{E}[\tilde{N}_t(B \setminus A)]}^{=0} \\ &= \mathbb{E}[(N_t(A \cap B) - t\mu(A \cap B))^2] \\ &= t\mu(A \cap B) \end{aligned}$$

where the last equality follows as $N_t(A \cap B) \sim \mathrm{Poi}(t\mu(A \cap B))$ and so, $\mathbb{E}[(N_t(A \cap B))^2] = t\mu(A \cap B) + t^2\mu(A \cap B)^2$. Consequently, as $\tilde{N}_{s,t}$ is independent of $\mathscr{F}_s$

$$\mathbb{E}[\tilde{N}_{s,t}(A)\tilde{N}_{s,t}(B) \mid \mathscr{F}_s] = \mathbb{E}[\tilde{N}_t(A)\tilde{N}_t(B)] - \mathbb{E}[\tilde{N}_s(A)\tilde{N}_s(B)] = (t-s)\mu(A \cap B).$$

Thus, we have that

$$\langle \tilde{N}(A), \tilde{N}(B) \rangle_t = t\mu(A \cap B)$$

and we have the Itô isometry for the compensated Poisson random measure:

$$\mathbb{E}\left[\left(\int \psi(z)\tilde{N}_{s,t}(\mathrm{d}z)\right)^2\right] = (t-s)\int \psi(z)^2\mu(\mathrm{d}z)$$

for any $\mathscr{F}_s$ measurable function ψ such that $\int |\psi|^2 \mathrm{d}\mu < \infty$.

As the notation $\tilde{N}(\mathrm{d}t\mathrm{d}z)$ suggest, we would also like to integrate with respect to time. To this end, we first define the stochastic integral with respect to the Poisson random measure for simple functions of the form

$$g(t,z) = \sum_{i=1}^{n} \psi_i(z)\mathbb{1}_{(t_i,t_{i+1}]}(t)$$

where ψ_i is a random variable measurable with respect to $\mathscr{F}_{s_i} \times \mathscr{B}$. We define

$$\int g(t,z)\tilde{N}_{\mathrm{d}t}(\mathrm{d}z) = \sum_{i=1}^{n} \int \psi(z)\tilde{N}_{t_i,t_{i+1}}(\mathrm{d}z).$$

We have the following Itô type isometry for the stochastic integral with respect to the compensated Poisson random measure:

$$\left\langle \int g(t,z)\tilde{N}_{\mathrm{d}t}(\mathrm{d}z) \right\rangle_t = \int g(t,z)^2\hat{N}_{\mathrm{d}t}(\mathrm{d}z) = \int g(t,z)^2\mathrm{d}t\mu(\mathrm{d}z)$$

and

$$\mathbb{E}\left[\left|\int g(t,z)\tilde{N}_{\mathrm{d}t}(\mathrm{d}z)\right|^2\right] = \mathbb{E}\left[\int g(t,z)^2\hat{N}_{\mathrm{d}t}\mathrm{d}z\right] = \mathbb{E}\left[\int g(t,z)^2\mathrm{d}t\mu(\mathrm{d}z)\right].$$

With this, denoting $L^2(\hat{N})$ for the set of all predictable random functionals $g : [0,T]\times\mathscr{Z}\times\Omega\to\mathbb{R}$, i.e. g is measurable with respect to $\mathscr{P}\times\mathscr{B}$ where $\mathscr{P}$ is the predictable σ-algebra on $[0,T]\times\Omega$, which satisfy

$$\int g(t,z)^2\hat{N}_{\mathrm{d}t}(\mathrm{d}z) = \int g(t,z)^2\mathrm{d}t\mu(\mathrm{d}z) < \infty,$$

as the set of simple functions is dense in $L^2(\hat{N})$, we have defined the stochastic integral with respect to the compensated Poisson random measure for all $g \in L^2(\hat{N})$. We remark that the stochastic integral with respect to the compensated Poisson random measure is a martingale in time.

Finally, for any $g \in L^2(\hat{N})$, we define the stochastic integral with respect to the Poisson random measure N as

$$\int g(t,z)N_{\mathrm{d}t}(\mathrm{d}z) = \int g(t,z)\tilde{N}_{\mathrm{d}t}(\mathrm{d}z) + \int g(t,z)\hat{N}_{\mathrm{d}t}(\mathrm{d}z).$$

I think the above construction might work for any martingale random measures. However, as we used the independent increments property a couple of times, the argument will be slightly different.

We also have a version of the Itô formula for this stochastic integral due to Kunita-Watanabe. I state the one-dimensional result here: Let $g,h \in L^2(\hat{N})$ satisfy $gh = 0$ (so that $N_t(h)$ and $\tilde{N}_t(g)$ do not jump simultaneously) and take

$$X_t = X_0 + \int_0^t\int_{\mathscr{Z}} g(s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z) + \int_0^t\int_{\mathscr{Z}} h(s,z)N_{\mathrm{d}s}(\mathrm{d}z).$$

Then, for all $F \in C^2$, we have that

$$\begin{aligned}
F(X_t) &= F(X_0) \\
&\quad + \int_0^t\int_{\mathscr{Z}} [F(X_{s-} + g(s,z)) - F(X_{s-})]N_{\mathrm{d}s}(\mathrm{d}z) \\
&\quad + \int_0^t\int_{\mathscr{Z}} [F(X_s + h(s,z)) - F(X_{s-})]N_{\mathrm{d}s}(\mathrm{d}z) \\
&\quad - \int_0^t\int_{\mathscr{Z}} g(s,z)F'(X_{s-})\hat{N}_{\mathrm{d}s}(\mathrm{d}z)
\end{aligned}$$

Going back to Lévy processes, we have that for any $A \in \mathscr{B}$, by definition $(N_t(A))$ is a Lévy process. On the other hand, starting from a Lévy process Z_t on $\mathbb{R}^n$, we can define a Poisson random measure N by

$$N_{s,t}(A) = |\{r \in (s,t] : \Delta Z_r \in A\}|$$

for any $A \in \mathscr{B}(\mathbb{R}^n \setminus \{0\})$. We remove 0 since ΔZ is zero on all but countably many points. We observe that, if for some $\epsilon > 0$, $A \cap B_\epsilon(0) = \varnothing$, then as all by finitely many jumps of Z_t are in $B_\epsilon(0)$, we have that $N_{s,t}(A) < \infty$. Consequently, we have that $\mu(B_\epsilon^c) = \mathbb{E}[N_1(B_\epsilon^c)] < \infty$.

This construction allows us to represent any Lévy process using its corresponding Poisson random measure. In particular, we have the *unique* Lévy-Itô decomposition[1]:

$$Z_t = \sqrt{a}W_t + bt + \int_0^t\int_{B_1^c} zN_{\mathrm{d}s}(\mathrm{d}z) + \int_0^t\int_{B_1\setminus\{0\}} z\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)$$

[1] Possible typo in Kunita: z is replaced by an x in Kunita.

where W is a standard Brownian motion, N a Poisson random measure on $[0,T]\times\mathbb{R}^n\setminus\{0\}$ with intensity $\mathrm{d}t\mu(\mathrm{d}z)$ which is independent of W. Moreover, μ satisfies

$$\int \frac{|z|^2}{1+|z|^2}\mu(\mathrm{d}z) < \infty \tag{1}$$

and we call it the Lévy measure of Z. We see that $\frac{|z|^2}{1+|z|^2} \geq \frac{1}{2}(1\wedge|z|^2)$ and so, Equation (1) implies that $|z|\mathbb{1}_{B_1\setminus\{0\}} \in L^2(\hat{N})$ which is required to make sense of the integral $\int_0^t\int_{B_1\setminus\{0\}} z\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)$. Thus, as μ determines the Poisson random measure N, we have that the triplet (a,b,μ) uniquely determines the Lévy process Z and we call it the Lévy triple of Z.

In the case where the norm of the jumps of Z_t are bounded below by some positive constant, we have that the Lévy measure is finite and the decomposition can be instead written as

$$Z_t = W_t + b't + \int_0^t\int_{\mathbb{R}^d\setminus\{0\}} zN_{\mathrm{d}s}(\mathrm{d}z).$$

The proof of this is uses a similar idea to proving that $N_t(A) \sim \mathrm{Poi}(t\mu(A))$. However, before sketching the proof, we first observe the following identity regarding integration with respect to a Poisson random measure generated by the Lévy process Z. Denoting $g \in L^2(\hat{N})$, we have that

$$\int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} g(s,z)N_{\mathrm{d}s}(\mathrm{d}z) = \sum_{s\leq t} g(s,\Delta Z_s).$$

To see this, one makes the heuristic observation that the measure $N_t(\mathbb{R}\setminus\{0\})$ gains by 1 precisely if Z jumps at time t. Thus, we have the informal formula that $N_{\delta t}(U) = \mathbb{1}_U(\Delta Z_t)$ which can be made rigorous by considering simple functions. With this in mind, denoting $M_t^\alpha = \exp(i\alpha Z_t - t\psi(\alpha))$ and $Y_t^\alpha = \int_0^t \frac{1}{M_{s-}^\alpha}\mathrm{d}M_s^\alpha$ as before, we find[2]

$$\Delta Y_t^\alpha = \frac{1}{M_{t-}^\alpha}\Delta M_t^\alpha = e^{i\alpha\Delta Z_t} - 1.$$

Hence, denoting $\tilde{Y}_t^\alpha$ for the pure jump part of Y_t^α so that $\tilde{Y}_t^\alpha = \sum_{s\leq t}\Delta Y_s^\alpha$ and $\hat{Y}_t^\alpha = Y_t^\alpha - \tilde{Y}_t^\alpha$ for the continuous part, we have that[3]

$$\tilde{Y}_t^\alpha = \sum_{s\leq t}(e^{i\alpha\Delta Z_t} - 1) = \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}}(e^{i\alpha z}-1)N_{\mathrm{d}s}(\mathrm{d}z),$$

and thus, as $t \mapsto \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}}(e^{i\alpha z}-1)\hat{N}_{\mathrm{d}s}(\mathrm{d}z) = t\mathbb{E}[\exp(i\alpha N_1(\mathbb{R}\setminus\{0\}))-1]$ has finite variation, we have that

$$\infty > \langle\tilde{Y}^\alpha\rangle_t = \left\langle \int_{(0,\cdot]}\int_{\mathbb{R}\setminus\{0\}}(e^{i\alpha z}-1)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)\right\rangle_t = t\int_{\mathbb{R}\setminus\{0\}}\left|e^{i\alpha z}-1\right|^2\mu(\mathrm{d}z)$$

which implies that $\int\frac{|z|^2}{1+|z|^2}\mu(\mathrm{d}z) < \infty$. (?)

[2]Possible typo in Kunita: Z is replaced by Y in Kunita.

[3]I don't see how Kunita got $\tilde{N}$ instead of N here.

Now to determine the shape of Z, we observe that M satisfy the linear SDE

$$\mathrm{d}M_t^\alpha = M_{t-}^\alpha \mathrm{d}Y_t^\alpha.$$

Thus, using the above decomposition, explicitly write out M_t^α in terms of $\hat{Y}_t^\alpha$ and $N, \tilde{N}$.

In general, if (M_t) is a L^2 martingale which is adapted to the natural filtration $\mathscr{F}^Z$ generated by a Lévy process Z_t, then we have the unique representation

$$M_t = h + \int_0^t f(s)\mathrm{d}W_s^i + \int_0^t \int_{\mathbb{R}\setminus\{0\}} g(s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)$$

with $\int_0^T f^2(s)\mathrm{d}s < \infty$ and $g \in L^2(\hat{N})$. This shows that the space of martingales adapted to $\mathscr{F}^Z$ can be decomposed into a direct sum $\mathscr{M}_c \oplus \mathscr{M}_d$ where

$$\mathscr{M}_c = \left\{ \int_0^t f(s)\mathrm{d}W_s : \int_0^T f^2(s)\nu\mathrm{d}s < \infty \right\}$$

with $\nu = \mathbb{E}[W_1^2]$ and

$$\mathscr{M}_d = \left\{ \int_0^t \int_{\mathbb{R}\setminus\{0\}} g(s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z) : g \in L^2(\hat{N}) \right\}.$$

We remark that this sum is indeed a direct sum as the quadratic variation of the Brownian motion and the Poisson random measure is zero since the latter is a process of finite variation.

We state a generalized BDG inequality for Lévy processes: Let X be a semimartingale of the form

$$X_t = x + \int_0^t b(r)\mathrm{d}r + \int_0^t f(r)\mathrm{d}W_r + \int_0^t \int_{\mathscr{Z}} g(r,z)\tilde{N}_{\mathrm{d}r}(\mathrm{d}z).$$

Then,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\le t}|X_s|^p\right] \lesssim_p |x|^p + \mathbb{E}\left[\left(\int_0^t |b(r)|\mathrm{d}r\right)^p\right] + \mathbb{E}\left[\left(\int_0^t |f(r)|^2\mathrm{d}r\right)^{p/2}\right] \\ + \mathbb{E}\left[\left(\int_0^t \int_{\mathscr{Z}} |g(r,z)|^2\mathrm{d}s\mu(\mathrm{d}z)\right)^{p/2}\right] + \mathbb{E}\left[\int_0^t \int_{\mathscr{Z}} |g(r,z)|^p\mathrm{d}s\mu(\mathrm{d}z)\right].\end{aligned}$$

We remark that this coincides with the traditional BDG inequality when $b, g = 0$.

Applying the triangle inequality to $\sup_{s\le t}|X_s|^p$, the inequality involving all but the last term follows from the classical BDG inequality so, we will sketch the proof for the inequality only involving the last term. Let $Y_t = \int_0^t \int_{\mathscr{Z}} g(s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)$, applying Itô's formula to $|Y_t|^p$, we have that

$$|Y_t|^p = M_t + \int_0^t \int_{\mathscr{Z}} \left(|Y_{s-} + g|^p - |Y_{s-}| - p|Y_{s-}|^{p-2}Y_{s-}g\right)\mathrm{d}s\mu(\mathrm{d}z) \tag{2}$$

where

$$M_t = \int_0^t \int_{\mathscr{Z}} (|Y_{s-} + g|^p - |Y_{s-}|)\,\tilde{N}_{\mathrm{d}s}(\mathrm{d}z)$$

is a (local) martingale.

We observe that, by applying MVT twice, there exists some $\theta \in (0,1)$ such that

$$|Y_{s-}+g|^p - |Y_{s-}| - p|Y_{s-}|^{p-2}Y_{s-}g = p(p-1)|Y_{s-}+\theta g|^{p-2}g^2.$$

Hence, using the fact that $(a+b)^n \le 2^n(a^n+b^n)$, we have that

$$|Y_{s-}+g|^p - |Y_{s-}| - p|Y_{s-}|^{p-2}Y_{s-}g \le c_3|Y_{s-}|^{p-2}g^2 + c_4|g|^p.$$

Hence, taking expectations on both sides of Equation (2), we have that

$$\mathbb{E}[|Y_t|^p] \le c_3\mathbb{E}\left[\int_0^t\int_{\mathscr{Z}}|Y_{s-}|^{p-2}g^2\mathrm{d}s\mu(\mathrm{d}z)\right] + c_4\mathbb{E}\left[\int_0^t\int_{\mathscr{Z}}|g|^p\mathrm{d}s\mu(\mathrm{d}z)\right].$$

Applying Hölder's and Young's inequality to the first term provides

$$\mathbb{E}\left[\int_0^t\int_{\mathscr{Z}}|Y_{s-}|^{p-2}g^2\mathrm{d}s\mu(\mathrm{d}z)\right] \le \left(1-\frac{2}{p}\right)\mathbb{E}\left[\sup_{s\le t}|Y_{s-}|^p\right] + \frac{2}{p}\mathbb{E}\left[\left(\int_0^t\int_{\mathscr{Z}}g^2\mathrm{d}s\mu(\mathrm{d}z)\right)^{p/2}\right].$$

Thus, as Y is a martingale, by applying Doob's inequality while substituting the above inequalities, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\le t}|Y_{s-}|^p\right]\\ &\le q^p\left(c_3'\mathbb{E}\left[\sup_{s\le t}|Y_{s-}|^p\right] + c_3''\mathbb{E}\left[\left(\int_0^t\int_{\mathscr{Z}}g^2\mathrm{d}s\mu(\mathrm{d}z)\right)^{p/2}\right] + c_4\mathbb{E}\left[\int_0^t\int_{\mathscr{Z}}|g|^p\mathrm{d}s\mu(\mathrm{d}z)\right]\right)\end{aligned}$$

where we denote q for the Hölder conjugate of p. At this point, Kunita shrinks c_3' so that $q^pc_3' < 1$ and moves the corresponding term to the left hand side to obtain the desired inequality. I'm not sure why we can shrink c_3' in this way as it seems to be fixed by the choice of p.

The above lemma allows us to conclude the following approximation for semimartingales of the form

$$X_t^x = a(x) + \int_0^t b(x,s)\mathrm{d}s + \int_0^t f(x,s)\mathrm{d}W_s + \int_0^t\int_{\mathscr{Z}}g(x,s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z),$$

by semimartingales

$$X_t^{n,x} = a^n(x) + \int_0^t b^n(x,s)\mathrm{d}s + \int_0^t f^n(x,s)\mathrm{d}W_s + \int_0^t\int_{\mathscr{Z}}g^n(x,s,z)\tilde{N}_{\mathrm{d}s}(\mathrm{d}z).$$

In particular, if a^n, b^n and f^n converges respectively to a, b and f uniformly in time and with respect to the p-th Lipschitz norm, and moreover

$$\lim_{n\to\infty}\sup_{s\le T}\left\|\left(\int_{\mathscr{Z}}|g^n(\cdot,s,z)-g(\cdot,s,z)|^{p'}\mu(\mathrm{d}z)\right)^{1/p'}\right\|_p = 0$$

for $p'=2$ and $p'=p>d$. Then, for any $R>0$, we have that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{|x|\le R}\sup_{s\le T}|X_n(x,s)-X(x,s)|^p\right] = 0.$$

The proof of this approximation is a straightforward application of Kolmogorov's continuity criterion and the above BDG inequality.

As a result of this approximation, by a standard Banach fixed point argument, Kunita then concludes the existence of a unique solution to an SDE of the form

$$\mathrm{d}x_t = b(t,x_t)\mathrm{d}t + f(t,x_t)\mathrm{d}W_t + \int_{\mathscr{Z}} g(t,x,z)\tilde{N}_{\mathrm{d}t}(\mathrm{d}z)$$

where we assume linear growth and Lipschitz conditions on b, f, and moreover, g satisfies

$$\frac{g(t,x,z)}{1+|x|} \leq K(z), |g(t,x,z) - g(t,y,z)| \leq L(z)|x-y|$$

where

$$\int_{\mathscr{Z}} (K(z)^p + L(z)^p)\mu(\mathrm{d}z) < \infty \tag{3}$$

for all $p \geq 2$.

Furthermore, under the same conditions, by the BDG inequality, we obtain continuity with respect to the initial conditions via Kolmogorov's continuity criterion.

In the case where our SDE is not just driven by a $\mathscr{F}^Z$-martingale (recall the decomposition of $\mathscr{F}^Z$-martingales into $\mathscr{M}_c \oplus \mathscr{M}_d$) but by Z itself, the above result remains to hold. In particular, suppose we consider the SDE with additive Lévy noise

$$\mathrm{d}x_t = b(t,x_t)\mathrm{d}t + \mathrm{d}Z_t \tag{4}$$

where Z is a Lévy process. Then, by the Itô-Lévy decomposition, we can write the above SDE as

$$\mathrm{d}x_t = b(t,x_t)\mathrm{d}t + \int_{B_1 \setminus \{0\}} z\tilde{N}_{\mathrm{d}t}(\mathrm{d}z) + \int_{B_1^c} zN_{\mathrm{d}t}(\mathrm{d}z)$$

where $\int_0^t \int_{B_1^c} zN_{\mathrm{d}t}(\mathrm{d}z) = \sum_{0 \leq s \leq t} \Delta Z_s \mathbb{1}_{B_1^c}(\Delta Z_s)$. Thus, assuming the growth and Lipchitz conditions, if ϕ is the solution flow of the SDE

$$\mathrm{d}x_t = b(t,x_t)\mathrm{d}t + \int_{B_1 \setminus \{0\}} z\tilde{N}_{\mathrm{d}t}(\mathrm{d}z) \tag{5}$$

it is possible to construct the solution flow of Equation (4) from ϕ. In particular, define the random times $(\sigma_n)_{n=1}^\infty$ where σ_k denotes the k-th jump time for Z where $|Z_{\sigma_k}| \geq 1$. Then, fixing $\omega \in \Omega$, we recursively define the solution flow ψ so that for $t \in [0, \sigma_1(\omega))$, $\psi_t(x) = \phi_t(x)$ and for $t \in (\sigma_k(\omega), \sigma_{k+1}(\omega)]$, we have

$$\psi_t(x) = \phi_{\sigma_k(\omega),t}(\psi_k(x) + \Delta Z_{\sigma_k(\omega)}).$$

Thus, inductively, as each term is continuous in x almost surely, we have that ψ is continuous in x. With this argument in mind, it suffices to only consider SDEs of the form (5).

In summary, for a Lévy process Z with intensity μ, we recall that

$$\int \frac{|z|^2}{1+|z|^2}\mu(\mathrm{d}z) < \infty.$$

Thus, as $\frac{|z|^2}{1+|z|^2} \geq \frac{1}{2}|z|^p$ for all $|z| \leq 1$ and $p \geq 2$, condition (3) is satisfied (as $\mathscr{Z} = B_1 \setminus \{0\}, K(z) = |z|, L = 0$) in this case. Hence, (5) has a unique solution flow which is continuous in the initial condition which in turn implies that (4) has a unique solution flow by the above argument. (c.f. stackexchange post).

Lemma (Strong Markov Property for Lévy Processes)**.** For Z a $\mathscr{F}$-Lévy process and τ a $\mathscr{F}$-stopping time, defining the process $Z'_t = Z_{\tau+t} - Z_\tau$, we have that Z' is a process independent of $\mathscr{F}^+_\tau$ and has the same distribution as Z.

Consequently, taking τ to be the deterministic time s, the above shows $Z_{t+s} - Z_s \perp \mathscr{F}^+_s$ and so Z is a $\mathscr{F}^+$-Lévy process. Thus, if τ is a $\mathscr{F}^+$-stopping time, by viewing Z as a $\mathscr{F}^+$ stopping time, we have that $Z_{t+\tau} - Z_\tau$ is independent of $\mathscr{F}^+_\tau$ and has the same distribution as Z.

We say that a Lévy process Z is α-stable if for all $t > 0$, $Z_t \sim t^{1/\alpha} Z_1$. It seems to me that, for $\alpha \neq 2$, the Lévy measure of Z cannot have finite second moment.

Suppose otherwise $\mathbb{E}[Z_1^2] = \sigma^2 > 0$, then by stationary and independent increments, we have by CLT

$$\frac{Z_{\lfloor t \rfloor}}{\sqrt{\lfloor t \rfloor}} = \frac{1}{\sqrt{\lfloor t \rfloor}} \sum_{k=0}^{\lfloor t \rfloor - 1} (Z_{k+1} - Z_k) \xrightarrow{\mathscr{D}} \mathscr{N}(0, \sigma^2).$$

Thus, if $\alpha < 2$,

$$\frac{Z_{\lfloor t \rfloor}}{\lfloor t \rfloor^{\frac{1}{\alpha}}} = \lfloor t \rfloor^{\frac{1}{2} - \frac{1}{\alpha}} \frac{Z_{\lfloor t \rfloor}}{\sqrt{\lfloor t \rfloor}} \xrightarrow{\mathscr{D}} 0.$$

However, this contradicts the fact that by stability $\lfloor t \rfloor^{-\frac{1}{\alpha}} Z_{\lfloor t \rfloor} \sim Z_1$ for all t. Similarly, in the case where $\alpha > 2$, then

$$\mathscr{N}(0, \sigma^2) \xleftarrow{\mathscr{D}} \frac{Z_{\lfloor t \rfloor}}{\sqrt{\lfloor t \rfloor}} = \lfloor t \rfloor^{\frac{1}{\alpha} - \frac{1}{2}} \frac{Z_{\lfloor t \rfloor}}{\lfloor t \rfloor^{\frac{1}{\alpha}}} \xrightarrow{\mathscr{D}} 0.$$

α cannot be greater than 2 by looking at the characteristic function.