

Multiscale notes

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Preliminaries

Let $T > 0$ and suppose $f : [0, T] \rightarrow \mathbb{R}^d$ is an α -Hölder continuous function for some α . Then, defining $\hat{f} : [0, T] \rightarrow \mathbb{R}^d$ by $\hat{f}(t) = \int_0^t f(r)dr$, we observe that

$$\|\hat{f}(t) - \hat{f}(s) - f(s)|t - s|\| = \left\| \int_s^t (f(r) - f(s))dr \right\| \leq \|f\|_\alpha |t - s|^{1+\alpha}$$

for all $s, t \in [0, T]$. Thus, we have that

$$\frac{\|\hat{f}(t) - \hat{f}(s)\|}{|t - s|^{1+\alpha}} \leq \|f\|_\alpha + \frac{\|f(s)\|}{|t - s|^\alpha} \quad (1)$$

Hence, if α is somehow negative, the second term in the above equation vanishes as $|t - s| \rightarrow 0$ and this motivates the following definition for functions with negative Hölder continuity.

Definition 1. For $f : [0, T] \rightarrow \mathbb{R}^d$ and $\kappa \in (0, 1)$, we define

$$\|f\|_{-\kappa} = \sup_{s \neq t \in [0, T]} \frac{1}{|t - s|^{1-\kappa}} \int_s^t f(r)dr.$$

We denote $C^{-\kappa} = \{f : [0, T] \rightarrow \mathbb{R}^d : \|f\|_{-\kappa} < \infty\}$.

We observe that, denoting $\hat{f}(t) = \int_0^t f(r)dr$ as above, $\|f\|_{-\kappa} = \|\hat{f}\|_{1-\kappa}$.

Definition 2. For $f : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$ and $\kappa, \gamma \in (0, 1)$, we define

$$\|f\|_{-\kappa, \gamma} = \sup_{x \neq y \in \mathbb{R}^d} \frac{\|f(x, \cdot) - f(y, \cdot)\|_{-\kappa}}{\|x - y\|^\gamma}.$$

Moreover, denote $C_{-\kappa, \gamma} = \{f : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d : \|f\|_{-\kappa, \gamma} < \infty\}$.

Theorem 1 (Multidimensional Kolmogorov's continuity criterion). Let $(X_t)_{t \in [0,1]^d}$ be a stochastic process with values in \mathbb{R}^d and suppose there exists $\alpha, C > 0$ such that for all $s \neq t \in [0,1]^d$,

$$\|X_t - X_s\|_p \leq C \|t - s\|^\alpha.$$

Then, for all $\gamma < \alpha - \frac{d}{p}$, there exists a continuous modification of $(X_t)_{t \in [0,1]^d}$ such that

$$\left\| \sup_{s \neq t \in [0,1]^d} \frac{\|X_t - X_s\|}{\|t - s\|^\gamma} \right\|_p = \| \|X\|_\gamma \|_p \leq C \tilde{C}$$

where $\tilde{C} = \sum_{m \in \mathbb{N}} 2^{m(\gamma p - \alpha p + d)}$.

Theorem 2 (Sewing Lemma). Let W be a Banach space and for any two parameter process $A : [0, T]_{\leq}^2 \rightarrow W$, defining $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t}$, we denote

$$\|A\|_\eta = \sup_{s < t \in [0,T]} \frac{\|A_{s,t}\|}{|t - s|^\eta} \text{ and } \|\delta A\|_{\bar{\eta}} = \sup_{s < u < t \in [0,T]} \frac{\|\delta A_{s,u,t}\|}{|t - s|^{\bar{\eta}}}$$

for some $\eta, \bar{\eta} > 0$. Then, we take

$$C^{\eta, \bar{\eta}}([0, T], W) = \{A : [0, T]_{\leq}^2 \rightarrow W : \|A\|_\eta < \infty, \|\delta A\|_{\bar{\eta}} < \infty\}.$$

For $\eta > 0$ and $\bar{\eta} > 1$, there exists a unique linear map

$$I : C^{\eta, \bar{\eta}}([0, T], W) \rightarrow C^\eta([0, T], W)$$

such that $I(A)_0 = 0$ and for all $s < t \in [0, T]$,

$$\| \underbrace{I(A)_t - I(A)_s}_{=: I(A)_{s,t}} - A_{s,t} \| \leq C |t - s|^{\bar{\eta}} \|\delta A\|_{\bar{\eta}}.$$

Moreover, for any partition \mathcal{P} of the interval $[s, t] \subseteq [0, T]$, we have that

$$\left\| I(A)_{s,t} - \sum_{[u,v] \in \mathcal{P}} A_{u,v} \right\| \leq C |t - s| \|\delta A\|_{\bar{\eta}} |\mathcal{P}|^{\bar{\eta}-1}.$$

Theorem 3 (Young integral). Let $f \in C^\alpha$, $g \in C^\beta$ such that $\alpha + \beta > 1$. Then, defining $A_{s,t} = f_s g_{s,t}$, the sewing lemma applies and we denote the resulting map by

$$\int_s^t f_r dg_r := I(A)_{s,t}.$$

The sewing lemma provides the following bounded which we will use frequently in the next section.

Lemma 1. Let $\alpha, \kappa, \gamma \in (0, 1)$ such that $\alpha\gamma > \kappa$. Then, there exists a continuous linear map

$$\Phi : C_{-\kappa, \gamma} \times C^\alpha([0, T], \mathbb{R}^d) \rightarrow C^{-\kappa}([0, T], \mathbb{R}^d) : (f, x) \mapsto (t \mapsto f(t, x_t)).$$

Moreover, $\|\Phi(f, x)\|_{-\kappa} \lesssim \|f\|_{-\kappa, \gamma} (1 + \|x\|_\alpha^\gamma T^{\alpha\gamma})$.

Proof. In order to show $\Phi(f, x) \in C^{-\kappa}$, we need to show that $\int_0^t f(r, x_r) dr \in C^{1-\kappa}$ to which one simply applies the sewing lemma to $A_{s,t} = \int_s^t f(r, x_s) dr$. \square

Definition 3. For two random variables X, Y , we define their degree of mixing coefficient to be

$$\alpha(X, Y) = \sup_{A \in \sigma(X), B \in \sigma(Y)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

We see straightaway that two independent random variables have mixing coefficient 0 and so intuitively, the mixing coefficient describes how close two random variables are to being independent. Moreover, we recall that in setting of measure preserving systems, the m.p.s. (X, \mathcal{A}, μ, T) is said to be strong mixing if for all $A, B \in \mathcal{A}$,

$$\lim_{n \rightarrow \infty} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| = 0.$$

Hence, the m.p.s. is strong mixing if and only if $\alpha(T^0, T^n) \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 1. Let X, Y be two random variables. Then,

$$4\alpha(X, Y) = \sup_{\|F\|_\infty, \|G\|_\infty \leq 1} |\mathbb{E}[F(X)G(Y)] - \mathbb{E}[F(X)]\mathbb{E}[G(Y)]|.$$

Proposition 2. If there exists some $\delta > 0$ such that $\|X\|_{2+\delta}, \|Y\|_{2+\delta} \leq c$, then

$$\text{Cov}(X, Y) \leq 8c^{\frac{2}{2+\delta}} (\alpha(X, Y))^{\frac{2}{2+\delta}}.$$

Multiscale analysis

We mainly consider the following two standing assumptions.

Assumption 1. Let (y_t) be a stationary process taking value in Y such that $\alpha(y_0, y_t) \lesssim t^{-\delta}$ for some $\delta \in (0, 1)$. We denote $\mu = \mathcal{L}(y_0)$ the stationary measure of the process.

Assumption 2. Let $F : \mathbb{R}^d \times Y \rightarrow \mathbb{R}$ be uniformly (in both arguments) bounded and Lipschitz in \mathbb{R}^d (with Lipschitz constant uniform over Y).

Lemma 2. Assuming Assumption 1 and let $F : \mathbb{R}^d \times Y \rightarrow \mathbb{R}$ be a bounded measurable function in Y and define

$$f_n(x, r) = F(x, y_{nr}) \text{ and } \bar{f}(x) = \int_Y F(x, y) \mu(dy).$$

Then, for all $p \geq 2$ and $x \in \mathbb{R}^d$, we have that

$$\left\| \int_s^t f_n(x, r) - \bar{f}(x)(t-s) \right\|_p \lesssim \|F(x, \cdot)\|_\infty n^{-\frac{\delta}{p}} |t-s|^{1-\frac{\delta}{p}}$$

Proof. As fundamentally x plays no role in the above statement, we will omit it from our notations. Moreover, by scaling and translating, we may assume without loss of generality that $\bar{f} = 0$ and $s = 0$. Then, we observe

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t f_n(r) dr \right)^2 \right] &= \int_0^t \int_0^t \mathbb{E}[f_n(r)f_n(s)] dr ds \\ &\leq 4\|F\|_\infty \int_0^t \int_0^t \alpha(y_{nr}, y_{ns}) dr ds \leq 4\|F\|_\infty n^\delta t^{2-\delta}. \end{aligned}$$

Hence, for all $p \geq 2$, we have that

$$\mathbb{E} \left[\left(\int_0^t f_n(r) dr \right)^p \right] \leq t^{p-2} \|F\|_\infty^{p-2} \mathbb{E} \left[\left(\int_0^t f_n(r) dr \right)^2 \right] \lesssim \|F\|_\infty^p n^\delta t^{p-\delta}.$$

□

Lemma 3. Assuming now Assumption 1 and 2 and denote f and \tilde{f} as in the previous lemma. Then, for all $p \geq 2$ and $x, z \in \mathbb{R}^d$, we have that

$$\left\| \int_s^t (f_n(x, r) - f_n(z, r)) - (\tilde{f}(x) - \tilde{f}(z)) dr \right\|_p \lesssim n^{-\frac{\delta}{p}} |t-s|^{1-\frac{\delta}{p}} \|x-z\|.$$

Proof. Fixing $z \in \mathbb{R}^d$ and defining $\bar{F}(x, y) = F(x, y) - F(z, y)$, the bound follows by observing that $\|\bar{F}(x, \cdot)\|_\infty = \|F(x, \cdot) - F(z, \cdot)\|_\infty \leq \|F\|_1 \|x-z\|$. □

Lemma 4. Let $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be bounded and Lipschitz and suppose $G, \tilde{G} \in C^\alpha([0, T], \mathbb{R}^d)$ for some $\alpha \in (0, 1)$ with $G_0 = \tilde{G}_0$. Then, if

$$z_t = G_t + \int_0^t F(z_s) ds \text{ and } \tilde{z}_t = \tilde{G}_t + \int_0^t F(\tilde{z}_s) ds,$$

we have that

$$\|z - \tilde{z}\|_\alpha \lesssim_{\|F\|_\infty} \|G - \tilde{G}\|_\alpha.$$

Proof. It is easy to see from Grönwall's inequality that

$$\|z - \tilde{z}\|_\infty \leq \|G - \tilde{G}\|_\infty \exp(\|F\|_1 T).$$

On the other hand, since for any $t \in [0, T]$, we have that

$$\|G_t - \tilde{G}_t\| \leq \|(G_t - \tilde{G}_t) - (G_0 - \tilde{G}_0)\| \leq \|G - \tilde{G}\|_\alpha T^\alpha,$$

so, $\|G - \tilde{G}\|_\infty \leq \|G - \tilde{G}\|_\alpha T^\alpha$. Hence, combining the two estimates provides the desired bound. □

Lemma 5. Let $f_n, f : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be functions in $C_{-\kappa, \gamma}$ such that the ODEs

$$x_t^n = x_0 + \int_0^t f_n(s, x_s^n) ds, x_t = x_0 + \int_0^t f(s, x_s) ds$$

have unique solutions $x^n, x \in C^\alpha$ for some $\alpha + \gamma \leq 1$ and $\alpha\gamma > \kappa$. Then, if $\|f_n - f\|_{-\kappa, \gamma} \rightarrow 0$ as $n \rightarrow \infty$, we have that $\|x^n - x\|_\alpha \rightarrow 0$.

Proof. Follows by applying the previous lemma in which we take

$$G_t = x_0 + \int_0^t (f_n(s, x_s^n) - f(s, x_s^n)) ds.$$

□