Rough paths notes

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Space of rough paths

Let V be a Banach space and $T > 0$. A V-valued α-Hölder rough path is a pair of functions

$$X : [0, T] \to V \text{ and } \mathbb{X} : [0, T]^2 \to V^{\otimes 2}$$

such that

$$\|X\|_\alpha = \sup_{s \neq t \in [0,T]} \frac{\|X_{s,t}\|}{|s-t|^\alpha}, \|\mathbb{X}\|_{2\alpha} = \sup_{s \neq t \in [0,T]} \frac{\|\mathbb{X}_{s,t}\|}{|s-t|^{2\alpha}} < \infty$$

and moreover, satisfy *Chen's relation*:

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t} \tag{1}$$

where we denote $X_{s,t} = X_t - X_s$.
Recall that the tensor norm is defined as

$$\|x\| = \inf\left\{\sum \|v\|\|w\| : x = \sum v \otimes w\right\}.$$

Intuitively, the lift $\mathbb{X}_{s,t}$ is representing $\int_s^t X_{s,r} \otimes \mathrm{d}X_r$ and Chen's relation follows by asserting that

- the map $f :\mapsto \int f_r \mathrm{d}X_r$ is linear;
- $\int_s^t \mathrm{d}X_r = X_t - X_s$ and
- $\int_s^t f_r \mathrm{d}X_r = \int_s^u f_r \mathrm{d}X_r + \int_u^t f_r \mathrm{d}X_r$.

We remark that the lift $\mathbb{X}$ is not unique from Chen's relation. Indeed, if $\mathbb{X}$ is a lift, then so is $\mathbb{X}_{s,t} + F_t - F_s$. In fact, they are all of this form.
We say a rough path $(X, \mathbb{X})$ is weakly geometric if

$$\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} X_{s,t} \otimes X_{s,t}.$$

This relation is determined by integration by parts.
We introduce the following vocabulary:

- the map $X \mapsto (X, \mathbb{X})$ is called a rough path lift.
- $\mathscr{C}^\infty$ denotes the space of smooth rough paths.
- $\mathscr{L}(\mathscr{C}^\infty) \subseteq \mathscr{C}^\infty$ denotes the canonical lift of a smooth paths.
- $\mathscr{C}_g^\alpha$ the space of weakly geometric rough paths.
- $\mathscr{C}_g^{0,\alpha} = \overline{\mathscr{L}(\mathscr{C}^\infty)}^{\mathscr{C}^\alpha}$.

For $\alpha < \beta$, we have the inclusion $\mathscr{C}_g^\beta \subseteq \mathscr{C}_g^{0,\alpha} \subseteq \mathscr{C}_g^\alpha$.

We would like to introduce a topology on the space of rough paths via a norm which makes the space of rough paths a linear subspace of $C^\alpha \oplus C_2^{2\alpha}$. By observing that the norm $\|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}$ does not respect Chen's relation, we remark that this norm does not make the space of rough paths a linear subspace. Instead, define

$$|||(X, \mathbb{X})|||_\alpha = \|X\|_\alpha + \sqrt{\|\mathbb{X}\|_{2\alpha}}.$$

We also record the following (in-homogeneous) α-Hölder metric on $\mathscr{C}^\alpha([0,T], V)$:

$$\rho_\alpha(\mathscr{X}, \mathscr{Y}) = \sup_{s \neq t \in [0,T]} \frac{\|X_{s,t} - Y_{s,t}\|}{|s-t|^\alpha} + \sup_{s \neq t \in [0,T]} \frac{\|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}\|}{|s-t|^{2\alpha}}.$$

The space of rough paths can be alternatively described by a path taking values in a certain additive Lie group. Define

$$T^{(2)}(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2}$$

with the multiplication

$$(a, b, c) \otimes (a', b', c') = (aa', ab' + a'b, ac + a'c' + b \otimes b').$$

$T^{(2)}(V)$ is known as the step-2 truncated tensor algebra over V. Then, considering

$$T_1^{(2)}(V) = \{1\} \oplus V \oplus V^{\otimes 2}$$

with the inherited multiplication from $T^{(2)}(V)$, we have that $T_1^{(2)}(V)$ is a Lie group with the identity $(1, 0, 0)$ and inverse given by

$$(1, b, c)^{-1} = (1, -b, b^{\otimes 2} - c).$$

Then, we consider paths taking values in $T_1^{(2)}(V)$ and in particular, interpret

$$t \mapsto \mathscr{X}_t = (1, X_t, \mathbb{X}_{0,t}) \in T_1^{(2)} V.$$

Chen's relation in this case becomes $\mathscr{X}_{s,u} \otimes \mathscr{X}_{u,t} = \mathscr{X}_{s,t}$ where $\mathscr{X}_{s,t} = \mathscr{X}_s^{-1} \otimes \mathscr{X}_t$.

Denoting $T_0^{(2)}(V) = V \oplus V^{\otimes 2}$, we define its Lie bracket by

$$[(b, c), (b', c')] = b \otimes b' - b' \otimes b.$$

This makes $T_0^{(2)}(V)$ a Lie algebra. Then, denoting $[V, V] = \overline{\langle [v, w] : v, w \in V \rangle}$, we define $g^{(2)}(V)$ to be the subalgebra of $T_0^{(2)}$ defined by

$$g^{(2)}(V) = V \oplus [V, V].$$

We remark that in $\mathbb{R}^d$, $[\mathbb{R}^d, \mathbb{R}^d]$ is the space of symmetric matrices and so is automatically closed. Define the map

$$\exp : T_0^{(2)}(V) \to T_1^{(2)}(V) : (b,c) \mapsto \left(1, b, c + \frac{1}{2} b \otimes b\right).$$

Then, denoting $G^{(2)}(V) = \exp\left(g^{(2)}(V)\right) \leq T_1^{(2)}(V)$, it is easy the check that paths taking values in $G^{(2)}(V)$ corresponds to the weakly geometric rough paths.

Integration of one-forms

Integration of rough paths is motivated by Taylor expansion. Suppose $F : V \to \mathscr{L}(V,W) \in C^{1+}$, and $\mathscr{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, then Taylor expansion gives

$$F(X_r) \approx F(X_s) + DF(X_r)X_{s,r}$$

with $DF(X_t) \in \mathscr{L}(V, \mathscr{L}(V,W)) = \mathscr{L}(V \otimes V, W)$. Then, the Riemann-Stieltjes sum can be interpreted as

$$\int_s^t F(X_r) \mathrm{d}\mathscr{X}_r = \lim_{|\mathscr{P}| \to 0} \sum_{[u,v] \in \mathscr{P}} \left(F(X_u)X_{u,v} + DF(X_r)\mathbb{X}_{s,r}\right) \tag{2}$$

where we hope the higher order terms vanished. This turns out to be the case for $\alpha \in (1/3, 1/2]$.

Lemma 0.1. Let $F : V \to \mathscr{L}(V,W) \in C_b^2$ and $\mathscr{X} = (X, \mathbb{X})$ be an α-Hölder rough path with $\alpha > 1/3$. Then, denoting $Y = F(X)$, $Y' = DF(X)$ and $R_{s,t}^Y = Y_{s,t} - Y_s' X_{s,t}$, then

$$Y, Y' \in C^\alpha \text{ and } R^Y \in C_2^{2\alpha}.$$

Moreover, $\|Y\|_\alpha \leq \|DF\|_\infty \|X\|_\alpha$, $\|Y'\|_\alpha \leq \|D^2F\|_\infty \|X\|_\alpha$ and $\|R^Y\|_{2\alpha} \leq \frac{1}{2}\|D^2F\|_\infty \|X\|_\alpha^2$.

The above lemma allows us to apply the sewing lemma with increments

$$\Xi_{s,t} = Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}$$

resulting in the following theorem.

Theorem 1 (Lyons)**.** Let $F : V \to \mathscr{L}(V,W) \in C_b^2$ and $(X, \mathbb{X})$ be an α-Hölder rough path with $\alpha > 1/3$. Then, the limit (2) exists and we have the bound

$$\left\| \int_s^t F(X_r) \mathrm{d}\mathscr{X}_r - F(X_s)X_{s,t} - DF(X_s)\mathbb{X}_{s,t} \right\| \lesssim_\alpha \|F\|_{C_b^2} (\|X\|_\alpha^3 + \|X\|_\alpha \|\mathbb{X}\|_{2\alpha})|t-s|^{3\alpha}.$$

Since fundamentally, the above theorem relies solely on Lemma 0.1, it is natural to make the following definition for the space of integrands.

Definition 1. Let $X \in C^\alpha([0,T], V)$ and $Y \in C^\alpha([0,T], \bar{W})$. We say Y is controlled by X if there exists some $Y' \in C^\alpha([0,T], \mathscr{L}(V, \bar{W}))$ and $R^Y \in C_2^{2\alpha}$ with $R_{s,t}^Y = Y_{s,t} - Y_s' X_{s,t}$. We write the space of all such pairs of (Y, Y') as $\mathscr{D}_X^{2\alpha}([0,T], \bar{W})$.

We endow $\mathscr{D}_X^{2\alpha}([0,T], \bar{W})$ with the seminorm

$$\|Y, Y'\|_{X,2\alpha} = \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.$$

Theorem 2 (Gubinelli)**.** Let $\mathscr{X} = (X, \mathbb{X}) \in \mathscr{C}^{\alpha}([0.T], V)$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(V,W))$ where $\alpha \in (1/3, 1/2]$. Then, the limit

$$\int_s^t Y_r \mathrm{d}\mathscr{X}_r = \lim_{|\mathscr{P}|\to 0} \sum_{[u,v]\in\mathscr{P}} \left(Y_u X_{u,v} + Y'_u \mathbb{X}_{u,v}\right)$$

exists. Moreover, we have the bound

$$\left\| \int_s^t Y_r \mathrm{d}\mathscr{X}_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right\| \lesssim_\alpha (\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha)|t-s|^{3\alpha}.$$

Finally, the map

$$\Phi : \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(V,W)) \to \mathscr{D}_X^{2\alpha}([0,T], W) : (Y, Y') \mapsto \left(\int_0^{\cdot} Y_r \mathrm{d}\mathscr{X}_r, Y\right)$$

is a continuous linear map between Banach spaces with the bound

$$\|\Phi(Y,Y')\|_{X,2\alpha} \le \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} + C_\alpha T^\alpha (\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha).$$

Finally, if $(X, \mathbb{X}) \in \mathscr{C}^\alpha([0,T], V)$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0,T], \mathscr{L}(\bar{V}, barW))$, $(Z, Z') \in \mathscr{D}_Y^{2\alpha}([0,T], \bar{V})$, then defining

$$\Xi_{s,t} = Y_s Z_{s,t} + Y'_s Z'_s X_{s,t}$$

where we interpret $Y'_s \in \mathscr{L}(V, \mathscr{L}(\bar{V}, \bar{W})) \simeq \mathscr{L}(V \otimes \bar{V}, \bar{W})$, we may apply the sewing lemma in order to make sense of $\int_s^t Y_r \mathrm{d}Z_r$.

Stability

Given $(X, \mathbb{X}), (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$ and $(Y, Y') \in \mathscr{D}_X^{2\alpha}$, $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}_{\tilde{X}}^{2\alpha}$, we define the following notion of "distance":

$$\|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} = \|Y' - Y\|_{2\alpha} + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}.$$

While obviously, this is not a metric as (Y, Y') and $(\tilde{Y}, \tilde{Y}')$ are not necessarily in the same space, even in the case where $X = \tilde{X}$, this distance still does not differentiate between Y and $Y + c$. Nonetheless, by considering the canonical embedding

$$\iota_X : \mathscr{D}_X^{2\alpha} \to C^\alpha \oplus C^{2\alpha} : (Y, Y') \mapsto (Y', R^Y),$$

which is an injection if we fix $Y_0 = \xi$ (since then $Y_t = \xi + R^Y_{0,t} + Y'_0 X_{0,t}$), we have that

$$\|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha} = \|\iota_X(Y, Y') - \iota_{\tilde{X}}(\tilde{Y}, \tilde{Y}')\|_{\alpha,2\alpha}.$$

To this end, we have a bound of the form

$$\|\Phi(Y, Y'); \Phi(\tilde{Y}, \tilde{Y}')\|_{X,\tilde{X},2\alpha} \le C(\rho_\alpha(\mathscr{X}, \tilde{\mathscr{X}}) + |Y'_0 - \tilde{Y}'_0| + T^\alpha \|Y, Y'; \tilde{Y}, \tilde{Y}'\|_{X,\tilde{X},2\alpha})$$