Invariant measure of SDEs
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Semigroup theory

Let X be a Banach space and let $\{T_t : X \to X\}_{t \geq 0}$ be a family of bounded linear operators. We say that $\{T_t\}$ is a semigroup if $T_{s+t} = T_s \circ T_t$ and $T_0 = \text{id}$. If futhermore, the map $(x, t) \mapsto T_t x$ is continuous, then we say $\{T_t\}$ is a C_0-semigroup.

Lemma 1. A semigroup $\{T_t\}$ is C_0 if and only if for any $x \in X$, the map $t \mapsto T_t x$ is continuous at 0 and there exists some constants M, a such that $\|T_t\| \leq M e^{at}$.

Proof. The forward direction is clear by decomposing $\|T_t\| \leq (\sup_{0 \leq s \leq 1} \|T_s\|) \|T_1\|^{\lfloor t \rfloor}$.
For the converse, we first realize that the map $t \mapsto T_t x$ is continuous for any $x \in X$. Indeed, for $t, h > 0$, we have

$$\|T_{t+h} x - T_t x\| = \|T_t (T_h x - x)\| \to 0$$

as $h \downarrow 0$. Moreover,

$$\|T_t x - T_{t-h} x\| = \|T_{t-h}(T_h x - x)\| \leq M e^{(at \vee 0)} \|T_h x - x\| \to 0$$

as $h \downarrow 0$. Thus, the map is continuous in t everywhere. With this in mind, it is easy to show that the semigroup is C_0 by the triangle inequality. $\square$

By the Banach-Steinhaus theorem, the condition $\|T_t\| \leq M e^{at}$ can be dropped.
For a C_0-semigroup T, we will be interested in its *generator* defined as

$$\mathscr{L} : \mathscr{D}(\mathscr{L}) \subseteq X \to X : \mathscr{L} f = \lim_{t \downarrow 0} \frac{T_t f - f}{t}.$$

Moreover, we define its adjoint as operator $\mathscr{L}^*$ such that

$$\mathscr{L}^* : \mathscr{D}(\mathscr{L}^*) \supseteq X^* \to X^* : \mathscr{L}^* l = l \circ L.$$

This is well-defined as $\mathscr{D}(\mathscr{L})$ is dense in X.
The resolvent of the generator is defined as

$$\rho(\mathscr{L}) = \{\lambda \in \mathscr{C} : (\lambda - \mathscr{L}) : \mathscr{D}(\mathscr{L}) \to X \text{ is invertible}\}.$$

For each $\lambda \in \rho(\mathscr{L})$, we define $R_\lambda = (\lambda - \mathscr{L})^{-1}$. The spectrum of $\mathscr{L}$ is $\sigma(\mathscr{L}) = \rho(\mathscr{L})^c$. An important identity for the resolvent is

$$R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda$$

for any $\lambda, \mu \in \rho(\mathscr{L})$.

Lemma 2. $\mathscr{D}(\mathscr{L})$ is dense in X, invariant under T_t and

$$\partial_t T_t x = \mathscr{L} T_t x = T_t \mathscr{L} x$$

for any $x \in \mathscr{D}(\mathscr{L})$. Moreover, for any $l \in \mathscr{D}(\mathscr{L}^*)$, we have that

$$\partial_t \langle l, T_t x \rangle = \langle L^* l, T_t x \rangle.$$

Proof. Defining $x_t = \int_0^t T_s x \mathrm{d}s$ for any $x \in X$, we have that $\mathscr{L} x_t = T_t x - x$ so $x_t \in \mathscr{D}(\mathscr{L})$ and $t^{-1} x_t \to x$ in X as $t \downarrow 0$. For the second statement, integrate both side and write $T_t x$ as $\mathscr{L} x_t + x$ □

Lemma 3. Let T be a C_0-semigroup satisfying $\|T_t\| \leq M e^{at}$ and let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) > a$. Then $\lambda \in \rho(\mathscr{L})$ and

$$R_\lambda = \int_0^\infty e^{-\lambda t} T_t \mathrm{d}t.$$

Invariant measures

Let (B_t) be a standard Brownian motion in $\mathbb{R}^d$, we look at an SDE of the form

$$\mathrm{d}x_t = b(x_t)\mathrm{d}t + \sigma(x_t)\mathrm{d}B_t \tag{1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are sufficiently regular to guarantee certain well-posedness. We are interested in finding a/the invariant measure of the SDE.

It is well known that the solution of the SDE (1) is a Markov process and we will first look at its infinitesimal generator. To find this, we recall that we say the semigroup T_t corresponds to the Markov process (x_t) if for all $s, t \geq 0$, we have that $T_t f(x_s) = \mathbb{E}[f(x_{s+t}) \mid \mathscr{F}_s]$ for any bounded measurable function f. Thus, denoting T the transition semigroup of (x_t), by Itô's formula (and adopting Einstein's notation), we have that for any $f \in C_b^2$

$$\begin{aligned}
T_t f(x) &= \mathbb{E}[f(x_t) \mid x_0 = x] \\
&= \mathbb{E}\Big[f(x) + \partial_i f(x)\mathrm{d}x_t^i + \partial_{ij}^2 f(x)\mathrm{d}\langle x^i, x^j\rangle_t\Big] \\
&= f(x) + \mathbb{E}\left[\partial_i f(x) b_i(x_t)\mathrm{d}t + \partial_i f(x)\sigma_i(x_t)\mathrm{d}B_t + \frac{1}{2}\partial_{ij}^2 f(x)\sigma_i(x_t)\sigma_j(x_t)\mathrm{d}t\right] \\
&= f(x) + \partial_i f(x)\int_0^t b_i(x_s)\mathrm{d}s + \frac{1}{2}\partial_{ij}^2 f(x)\int_0^t \sigma_i(x_s)\sigma_j(x_s)\mathrm{d}s.
\end{aligned}$$

Hence, taking derivatives in t at 0, we have that

$$\mathscr{L} f(x) = \partial_i f(x) b_i(x) + \frac{1}{2}\partial_{ij}^2 f(x)\sigma_i(x)\sigma_j(x). \tag{2}$$

The generator is useful in determining the invariant measure of the SDE via the following argument. Suppose that $\mathrm{d}\pi = \rho(x)\mathrm{d}x$ is an invariant measure of the SDE, then for any $f \in D(\mathscr{L})$, we have that

$$\int T_t f \,\mathrm{d}\pi - \int f \,\mathrm{d}\pi = 0$$

for any $t \geq 0$, and thus,

$$\int \mathscr{L} f \,\mathrm{d}\pi = \lim_{h \downarrow 0} \int T_h f \,\mathrm{d}\pi - \int f \,\mathrm{d}\pi = 0.$$

On the other hand, denoting $\mathscr{L}^*$ the adjoint of $\mathscr{L}$, we have

$$\int \mathscr{L} f \,\mathrm{d}\pi = \int (\mathscr{L} f(x))\rho(x)\mathrm{d}x = \int f(x)\mathscr{L}^*\rho(x)\mathrm{d}x.$$

Hence, we have that $\int f(x)\mathscr{L}^*\rho(x)\mathrm{d}x = 0$ for any $f \in D(\mathscr{L})$. Thus, as $D(\mathscr{L})$ is dense in $L^2(\pi)$, we have that $\mathscr{L}^*\rho = 0$. With this in mind, by finding the adjoint $\mathscr{L}^*$ and solving the PDE $\mathscr{L}^*\rho = 0$, we obtain a candidate for the invariant measure of the SDE (1). This is a special case of the Fokker-Planck equation where the system is autonomous.

To compute $\mathscr{L}^*$ we find

$$\begin{aligned}\mathscr{L}^* g(x) = {} & -\partial_i g(x) b_i(x) + \frac{1}{2}\partial_{ij}^2 g(x)\sigma_i(x)\sigma_j(x) \\ & + \partial_i g(x)\sigma_i(x)\partial_j\sigma_j(x) + \frac{1}{2}\partial_{ij}^2(\sigma_i\sigma_j)(x) - g(x)\nabla \cdot b(x).\end{aligned}$$

In the case of additive noise, i.e. $\sigma = I$, we thusly have

$$\mathscr{L}^* g(x) = -\partial_i g(x) b_i(x) + \frac{1}{2}\partial_{ij}^2 g(x) - g(x)\nabla \cdot b(x).$$