The Stone-Weierstrass Theorem and its Formalisation

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Variables and Definitions

Throughout this poster, we will let X be a compact metric space, M to be the set of all bounded and continuous functions from X to $\mathbb{R}$, M_0 a subset of M, $\bar{M}_0$ the closure of M_0 under lattice operations and uniform convergence to the limit and unless otherwise specified, we let all functions from X to $\mathbb{R}$ be bounded and continuous.

We say M_0 *separates points* if and only if for all distinct $x, y \in X$, there exists some $f \in M_0$ such that $f(x) \neq f(y)$.

Lattice Operations

While we do not form a complete lattice on the set of bounded continuous functions, we define two lattice operations $\vee, \wedge : (X \to \mathbb{R})^2 \to (X \to \mathbb{R})$ such that, for all $f, g : X \to \mathbb{R}$, where f, g are bounded continuous functions

$$f \vee g = \max\{f, g\},$$

and

$$f \wedge g = \min\{f, g\}.$$

R-algebra and Subalgebra

While in many literature algebras over a field are used to examine the Stone-Weierstrass theorem [2], it is in fact not necessary to examine algebras over field. As we will demonstrate, algebras over a commutative ring (or R-algebras) will suffice.

An *R-algebra* is a mathematical object consisting of a commutative ring R and a semi-ring A such that there exists scalar multiplication $\cdot : R \times A \to A$ and a homomorphism from R to A, $\phi : R \to A$ such that

$$r \cdot a = \phi(r) \times a,$$

and

$$\phi(r) \times a = a \times \phi(r)$$

are satisfied for all $r \in R$ and $a \in A$ [3].

A *subalgebra* S of an R-algebra A is a subset of A that's closed under the induced operations carried from A [4].

It was shown that both M and $\mathbb{R}^2$ form a R-algebra over $\mathbb{R}$ (see `http://bit.ly/3eL7LEC`).

The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem states that, given a subalgebra of M, M_0 that is closed under lattice operations and separates points, $\bar{M}_0 = M$, and we say M_0 is dense in M [2].

Outline of the proof

The theorem relies on two crucial lemmas:

Lemma 1. For all $f \in M$, $f \in \bar{M}_0$ if and only if for all $x, y \in X$, $\epsilon > 0$, there exists $g \in M_0$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$, i.e. there M_0 has a function arbitrarily close to f at x and y.

Lemma 2. Given S, a subalgebra of $\mathbb{R}^2$, S must be $\{(0,0)\}$, $\{(x, 0) \mid x \in \mathbb{R}\}$, $\{(0.y) \mid y \in \mathbb{R}\}$, $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ itself.

By considering *lemma 1*, it can be deduced that, for M_0, M_1, closed subalgebras of M under lattice operations and uniform convergence to the limit, $M_0 = M_1$ if and only if at for all distinct x, y, M_0 and M_1 have the same boundary points (the boundary points of M_i at x y is defined to be $\{(f(x), f(y)) \mid f \in M_i\}$).

Now by considering the fact that the boundary points of M_0 form a subalgebra of $\mathbb{R}^2$, we can utilise *lemma 2* to deduce that the boundary points must either be $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ (the first three possibilities in lemma 2 are not possible since $(1, 1)$ is in the boundary points). Now, if M_0 separates points then there must exist $f \in M_0$, $f(x) \neq f(y)$ so that excludes $\{(z, z) \mid z \in \mathbb{R}\}$ and hence the boundary points is $\mathbb{R}^2$ and the theorem follows.

Formalisation

The procedure in which the formalisation was achieved is similar to the outline, the source code of which can be found in my GitHub repository: `http://github.com/JasonKYi/stone-weierstrass`

Lemma 1 was formalised and is represented in Lean as `in_closure₂_iff_dense_at_points` in `main.lean` the method of which we will discuss below. The forward direction of the proof is trivial so we will consider the reverse.

Let us fix x and ϵ and define a mapping to set of X

$$S : X \to \texttt{set } X := \lambda y, \{z \mid f(z) - g_y(z) < \epsilon\},$$

where g_y was chosen such that $|f(x) - g_y(x)| < \epsilon$ and $|f(y) - g_y(y)| < \epsilon$.
Then for all $y \in X$, $y \in S(y)$ so $\bigcup_{y \in X} S(y) = X$. But as X is compact, $\bigcup_{y \in X} S(y)$ admits a finite subcover (cite); so, there exists a finite index set I such that $\bigcup_{i \in I} S(y_i) = X$. Thus, by letting $p_x = \bigvee_{i \in I} g_{y_i}$, we have constructed a function $p_x \in \bar{M}_0$ such that

$$p_x(z) \geq g_{y_i}(z) > f(z) - \epsilon$$

for all $z \in X$ and $i \in I$.
Now, by defining a similar mapping to set of X,

$$T : X \to \texttt{set } X := \lambda x, \{z \mid p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of X and thus can create the required function with $\bigwedge_{j \in J} p_{x_j}$ where J is the index set such that $\bigcup_{j \in J} T(p_{x_i}) = X$.

Lemma 2 was also formalised and is represented in Lean as `subalgebra_of_R2` and can be found in `ralgebra.lean`. The proof this lemma is rather tedious and follows directly by evoking the law of the excluded middle on different propositions multiple times.

Thus, with both lemma 1 and lemma 2 in our arsenal, it was shown that given two subalgebra of M, M_0 and M_1 which are closed under lattice operations, $M_0 = M_1$ if and only if they have the same boundary points with `eq_iff_boundary_points_eq` by constructing the notion of `closure′` in `definitions.lean` for sets of points in $\mathbb{R}^2$.

With that, by defining the notion of `has_seperate_points` in `definitions.lean` the Stone-Weierstrass theorem was proved with the method described in the outline with theorem statement in Lean being

```
theorem weierstrass_stone {M₀′
: subalgebra ℝ (X → ℝ)} (hc :
closure₀ M₀′.carrier = M₀′.carrier)
(hsep : has_seperate_points
M₀′.carrier) : closure₂ M₀′.carrier =
 univ
```

with the $\mathtt{M}_0'$`.carrier` notation referring to the underlying subset of $\mathtt{M}_0'$ the subalgebra and `hc` the hypothesis that $\mathtt{M}_0'$`.carrier` is closed under lattice operations.

Approximation Theorem

The Stone-Weierstrass theorem is a generalisation of the Weierstrass approximation theorem which states that any continuous functions on a closed interval can be uniformly approximated by a polynomial. While this can be proved using the weak law of large numbers (a version of which was typed up and is presented here: `http://bit.ly/3gLDk39`) it can also be deduced straightaway by the Stone-Weierstrass theorem.

Consider the Taylor polynomial $P_n(x)$ of $s(x) = \sqrt{1 - x}$ for $x \in [-1, 1]$. Using analysis, we can show that $P_n \to s$ uniformly, and thus, $P_n(1 - x^2) \to s(1 - x^2) = |x|$ uniformly for $x \in [-1, 1]$. Now, as

$$f \vee g = \max\{f, g\} = \frac{1}{2}(f + g + |f - g|); \quad f \wedge g = \min\{f, g\} = \frac{1}{2}(f + g - |f - g|),$$

we see that if $\bar{\mathcal{P}}$ is the closure of the set of polynomials under uniform convergence to the limit, $\bar{\mathcal{P}}$ would also be closed under the lattice operations. Thus, as $\bar{\mathcal{P}}$ form a subalgebra of all real to real functions, (as its closed under addition and multiplication), and as $\bar{\mathcal{P}}$ separates points trivially, $\bar{\bar{\mathcal{P}}} = \bar{\mathcal{P}} = \mathbb{R}^{\mathbb{R}}$ as required.

References

[1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem. Mathematics Magazine 21, no. 5 : 237-54.

[2] Gaddy, P. The Stone-Weierstrass Theorem and its Applications to L^2 Spaces.

[3] Lau, K. and Kudryashov, Y. (2018) Algebra over Commutative Semiring (under category) [Online]. Available at: `http://bit.ly/3gLIPyS` (Assessed: 02 June 2020)

[4] Bourbaki, N. (1989) 'SUBALGEBRAS. IDEALS. QUOTIENT ALGEBRAS' in Elements of mathematics, Algebra I. Paris: HERMANN, PUBLISHERS IN ARTS AND SCIENCE, pp. 429