Poster Draft

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The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem states that, given an unitial subalgebra of M, M_0 that is closed under lattice operations and separates points, $\bar{M_0} = M$, and we say M_0 is dense in M.

Outline and Formalisation

The Stone-Weierstrass theorem was proved and formalised using the interactive theorem prover *Lean* the source code of which can be found in my GitHub repository: http:\github.com/JasonKYi/stone-weierstrass.

The theorem itself relies on two central lemmas:

Lemma 1. For all $f \in M$, $f \in \bar{M_0}$ if and only if for all $x, y \in X$, $\epsilon > 0$, there exists $g \in M_0$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$, i.e. there M_0 has a function arbitrarily close to f at x and y.

Lemma 2. Given S, a subalgebra of $\mathbb{R}^2$, S must be $\{(0,0)\}$, $\{(x,0) \mid x \in \mathbb{R}\}$, $\{(0.y) \mid y \in \mathbb{R}\}$, $\{(z,z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ itself.

Lemma 1 was formalised and is represented in Lean as `in_closure₂_iff_dense_at_points` in `main.lean` the method of which we will discuss below. The forward direction of the proof is trivial so we will consider the reverse.

Let us fix x and ϵ and define a mapping to set of X

$$S : X \to \texttt{set } X := \lambda y, \{z | f(z) - g_y(z) < \epsilon\},$$

where g_y was chosen such that $|f(x) - g_y(x)| < \epsilon$ and $|f(y) - g_y(y)| < \epsilon$.

Then for all $y \in X$, $y \in S(y)$ so $\bigcup_{y \in X} S(y) = X$. But as X is compact, $\bigcup_{y \in X} S(y)$ admits a finite subcover; so, there exists a finite index set I such that $\bigcup_{i \in I} S(y_i) = X$. Thus, by letting $p_x = \bigvee_{i \in I} g_{y_i}$, we have constructed a function $p_x \in \bar{M_0}$ such that

$$p_x(z) \geq g_{y_i}(z) > f(z) - \epsilon$$

for all $z \in X$ and $i \in I$.

Now, by defining a similar mapping to set of X,

$$T : X \to \texttt{set } X := \lambda x, \{z | p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of X and thus can create the required function with $\bigwedge_{j\in J} p_{x_j}$ where J is the index set such that $\bigcup_{j\in J} T(p_{x_i}) = X$.

Lemma 2 was also formalised and is represented in Lean as `subalgebra_of_R2` and can be found in `ralgebra.lean`. The proof this lemma is rather tedious and follows directly by evoking the law of the excluded middle on different propositions multiple times.

Now, by considering *lemma 1*, it can be deduced that, for M_0, M_1, closed subalgebras of M under lattice operations and uniform convergence to the limit, $M_0 = M_1$ if and only if at for all distinct x, y, M_0 and M_1 have the same boundary points where the boundary points of M_i at x, y is defined to be $\{(f(x), f(y)) \mid f \in M_i\}$. This was formalised in `eq_iff_boundary_points_eq` by constructing the notion of `closure'` in `definitions.lean`

Lastly, as the boundary points of M_0 form a subalgebra of $\mathbb{R}^2$, we can utilise *lemma 2* to deduce that the boundary points must either be $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ (the first three possibilities in lemma 2 are not possible since $(1, 1)$ is in the boundary points). Now, if M_0 separates points then there must exist $f \in M_0$, $f(x) \neq f(y)$ so that excludes $\{(z, z) \mid z \in \mathbb{R}\}$ and hence the boundary points is $\mathbb{R}^2$ and the theorem follows. This was formalised in `main.lean` with the statement being

```
theorem weierstrass_stone \{M₀' : subalgebra ℝ (X → ℝ)\}
(hc   : closure₀ M₀'.carrier = M₀'.carrier)
(hsep : has_seperate_points M₀'.carrier) :
closure₂ M₀'.carrier = univ
```

Trigonometric Polynomials

Similarly to normal polynomials, we can deduce that trigonometric polynomials are dense in bounded continuous functions on $[0, 2\pi]$ where trigonometric polynomials are functions of the form

$$f(x) = a_0 + \sum_{i=1}^{n} a_n \cos(nx) + b_n \sin(nx)$$

By considering the identities of multiplication between trigonometric functions, we can easily see that the trigonometric polynomials form a unitial subalgebra that seperates points, and therefore dense by Stone-Weierstrass.