## Variables and Definitions

Throughout this poster, we will let *X* be a compact | *The Stone-Weierstrass theorem* states that, given a metric space, M to be the set of all bounded and  $\parallel$  subalgebra of M,  $M_0$  that is closed under lattice opcontinuous functions from X to  $\mathbb{R}$ ,  $M_0$  a subset of  $\blacksquare$  erations and separates points,  $\overline{M}_0 = M$ , and we say M,  $\overline{M}_0$  the closure of  $M_0$  under lattice operations and uniform convergence to the limit and unless otherwise specified, we let all functions from X to  $\begin{bmatrix} \textbf{Outline of the proof} \end{bmatrix}$  $\mathbb{R}$  be bounded and continuous.

We say  $M_0$  separates points if and only if for all distinct  $x, y \in X$ , there exists some  $f \in M_0$  such that  $f(x) \neq f(y)$ .

# Lattice Operations

While we do not form a complete lattice on the set of  $\{(0,0)\}$ ,  $\{(x,0) \mid x \in \mathbb{R}\}$ ,  $\{(0,y) \mid y \in \mathbb{R}\}$ ,  $\{(z,z) \mid x \in \mathbb{R}\}$ bounded continuous functions, we define two lat-  $z \in \mathbb{R}$ , or  $\mathbb{R}^2$  itself. tice operations  $\vee, \wedge: (X \to \mathbb{R})^2 \to (X \to \mathbb{R})$  such that, for all  $f,g:X\to\mathbb{R}$ , where f,g are bounded continuous functions

$$f \vee g = \max\{f, g\},\$$

$$f \wedge g = \min\{f, g\}.$$

# R-algebra and Subalgebra

While in many literature algebras over a field are used to examine the Stone-Weierstrass theorem [2], it is in fact not necessary to examine algebras over field. As we will demonstrate, algebras over a commutative ring (or R-algebras) will suffice.

An *R-algebra* is a mathematical object consisting of a commutative ring R and a semi-ring A such that there exists scalar multiplication  $\cdot : R \times A \rightarrow A$  and Approximation Theorem

$$r \cdot a = \phi(r) \times a,$$

and

$$\phi(r) \times a = a \times \phi(r)$$

are satisfied for all  $r \in R$  and  $a \in A$  [3].

A subalgebra S of an R-algebra A is a subset of A that's closed under the induced operations carried from *A* [4].

It was shown that both M and  $\mathbb{R}^2$  form a R-algebra over  $\mathbb{R}$  (see http://bit.ly/3eL7LEC).

### The Stone-Weierstrass Theorem

 $M_0$  is dense in M [2].

The theorem relies on two crucial lemmas:

**Lemma 1.** For all  $f \in M$ ,  $f \in \overline{M}_0$  if and only if for all  $x, y \in X$ ,  $\epsilon > 0$ , there exists  $g \in M_0$  such that  $|f(x)-g(x)|<\epsilon$  and  $|f(y)-g(y)|<\epsilon$ , i.e. there  $M_0$ has a function arbitrarily close to f at x and y.

**Lemma 2.** Given S, a subalgebra of  $\mathbb{R}^2$ , S must be

By considering lemma 1, it can be deduced that, for  $M_0$ ,  $M_1$ , closed subalgebras of M under lattice operations and uniform convergence to the limit,  $M_0 = M_1$  if and only if at for all distinct x, y,  $M_0$ and  $M_1$  have the same boundary points (the boundary points of  $M_i$  at x y is defined to be  $\{(f(x), f(y)) \mid$  $f \in M_i$ ).

Now by considering the fact that the boundary points of  $M_0$  form a subalgebra of  $\mathbb{R}^2$ , we can utilise lemma 2 to deduce that the boundary points must either be  $\{(z,z)\mid z\in\mathbb{R}\}$ , or  $\mathbb{R}^2$  (the first three possibilities in lemma 2 are not possible since (1,1) is in the boundary points). Now, if  $M_0$  separates points then there must exist  $f \in M_0$ ,  $f(x) \neq f(y)$  so that excludes  $\{(z,z) \mid z \in \mathbb{R}\}$  and hence the boundary points is  $\mathbb{R}^2$  and the theorem follows.

### Formalisation

The procedure in which the formalisation was achieved is similar to the outline, the source code of which can be found in my GitHub repository:

http://github.com/JasonKYi/stoneweierstrass

Lemma 1 was formalised and is represented in Lean as in\_closure2\_iff\_dense\_at\_points in main.lean the method of which we will discuss below. The forward direction of the proof is trivial so we will consider the reverse.

Let us fix x and  $\epsilon$  and define a mapping to set of X

$$S: X \to \operatorname{set} X := \lambda y, \{z | f(z) - g_y(z) < \epsilon\},$$

where  $g_y$  was chosen such that  $|f(x) - g_y(x)| < \epsilon$ and  $|f(y) - g_y(y)| < \epsilon$ .

Then for all  $y \in X$ ,  $y \in S(y)$  so  $\bigcup_{y \in X} S(y) = X$ . But as X is compact,  $\bigcup_{y \in X} S(y)$  admits a finite subcover (cite); so, there exists a finite index set Isuch that  $\bigcup_{i \in I} S(y_i) = X$ . Thus, by letting  $p_x = X$  $\bigvee_{i\in I}g_{y_i}$ , we have constructed a function  $p_x\in \bar{M}_0$ such that

$$p_x(z) \ge g_{y_i}(z) > f(z) - \epsilon$$

for all  $z \in X$  and  $i \in I$ .

Now, by defining a similar mapping to set of X,

$$T: X \to \operatorname{set} X := \lambda x, \{z | p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of X and thus can create the required function with  $\bigwedge_{i \in J} p_{x_i}$  where Jis the index set such that  $\bigcup_{i \in I} T(p_{x_i}) = X$ .

Lemma 2 was also formalised and is represented in Lean as subalgebra\_of\_R2 and can be found in ralgebra.lean. The proof this lemma is rather tedious and follows directly by evoking the law of the excluded middle on different propositions multiple times.

Thus, with both lemma 1 and lemma 2 in our arsenal, it was shown that given two subalgebra of M,  $M_0$  and  $M_1$  which are closed under lattice operations,  $M_0 = M_1$  if and only if they have the same boundary points with eq\_iff\_boundary\_points\_eq by constructing the notion of closure' in definitions.lean for sets of points in  $\mathbb{R}^2$ .

by defining the notion of With has\_seperate\_points in definitions.lean the Stone-Weierstrass theorem was proved with the method described in the outline with theorem statement in Lean being

theorem weierstrass\_stone  $\{M_0'\}$ : subalgebra  $\mathbb{R}$  (X  $\to$   $\mathbb{R}$ )} (hc:  $closure_0 M_0'.carrier = M_0'.carrier)$ (hsep : has\_seperate\_points  $M_0'$ .carrier) : closure<sub>2</sub>  $M_0'$ .carrier = univ

with the  $M'_0$  carrier notation referring to the underlying subset of M<sub>0</sub> the subalgebra and hc the hypothesis that  $M'_0$  carrier is closed under lattice operations.

a homomorphism from R to A,  $\phi: R \to A$  such that

The Stone-Weierstrass theorem is a generalisation of the Weierstrass approximation theorem which states

[1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem Mathematics Magazine 21, no. 5: 237-54. that any continuous functions on a closed interval can be uniformly approximated by a polynomial. While this can be proved using the weak law of large numbers (a version of which was typed up and is presented here: http://bit.ly/3gLDk39) it can also be deduced straightaway by the Stone-Weierstrass theorem.

Consider the Taylor polynomial  $P_n(x)$  of  $s(x) = \sqrt{1-x}$  for  $x \in [-1,1]$ . Using analysis, we can show that  $P_n \to s$  uniformly, and thus,  $P_n(1-x^2) \to s(1-x^2) = |x|$  uniformly for  $x \in [-1,1]$ . Now, as

$$f \lor g = \max\{f, g\} = \frac{1}{2}(f + g + |f - g|); \quad f \land g = \min\{f, g\} = \frac{1}{2}(f + g - |f - g|),$$

we see that if  $\bar{\mathcal{P}}$  is the closure of the set of polynomials under uniform convergence to the limit,  $\bar{\mathcal{P}}$  would also be closed under the lattice operations. Thus, as  $\bar{\mathcal{P}}$  form a subalgebra of all real to real functions, (as its closed under addition and multiplication), and as  $\bar{\mathcal{P}}$  separates points trivially,  $\bar{\mathcal{P}} = \bar{\mathcal{P}} = \mathbb{R}^{\mathbb{R}}$  as required.

## References

- tion Theorem. Mathematics Magazine 21, no. 5: 237-54.
- [2] Gaddy, P. The Stone-Weierstrass Theorem and its Applications to  $L^2$  Spaces.
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