The Formalisation and Applications of the Stone-Weierstrass Theorem

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The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem states that, given an unital subalgebra of M, M_0 that is closed under lattice operations and separates points, $\bar{M_0} = M$, and we say M_0 is dense in M [2].

This theorem is important as not only does it offer a strong result regarding function spaces, it also offers insight into interpolation as it demonstrates which set of functions can act as basis for a particular function space.

Variables and Definitions

Throughout this poster, we will let X be a compact metric space, M to be the set of all bounded and continuous functions from X to $\mathbb{R}$, M_0 a subset of M and $\bar{M_0}$ the closure of M_0 *under uniform convergence to the limit*.

We say M_0 *separates points* if and only if for all distinct $x, y \in X$, there exists some $f \in M_0$ such that $f(x) \neq f(y)$.

Given $x, y \in X$, the boundary points of M_0 is defined to be $\{(f(x), f(y)) \mid f \in M_0\}$.

Lattice Operations

We define two lattice operations $\vee, \wedge : (X \to \mathbb{R})^2 \to (X \to \mathbb{R})$ such that, for all $f, g : X \to \mathbb{R}$, where f, g are bounded continuous functions

$$f \vee g = \max\{f, g\}; \quad f \wedge g = \min\{f, g\}.$$

R-algebra and Subalgebra

An *R-algebra* is a mathematical object consisting of a commutative ring R and a semi-ring A such that there exists scalar multiplication $\cdot : R \times A \to A$ and a homomorphism from R to A, $\phi : R \to A$ such that

$$r \cdot a = \phi(r) \times a,$$

and

$$\phi(r) \times a = a \times \phi(r),$$

are satisfied for all $r \in R$ and $a \in A$ [3].

A *unital subalgebra* S of an R-algebra A is a subset of A that's closed under the induced operations carried from A and contains 1 [3].

It was shown that both M and $\mathbb{R}^2$ form a R-algebra over $\mathbb{R}$ (see: http://bit.ly/3eL7LEC).

Application: Weierstrass Theorem

The Stone-Weierstrass theorem is a generalisation of the Weierstrass approximation theorem which states that any continuous functions on a closed interval can be uniformly approximated by a polynomial. This can be deduced straightaway by the Stone-Weierstrass theorem.

Consider the Taylor polynomial $P_n(x)$ of $s(x) = \sqrt{1-x}$ for $x \in [-1, 1]$. Using analysis, we can show that $P_n \to s$ uniformly, and thus, $P_n(1 - x^2) \to s(1 - x^2) = |x|$ uniformly for $x \in [-1, 1]$. Now, as

$$f \vee g = \max\{f, g\} = \frac{1}{2}(f + g + |f - g|);$$

$$f \wedge g = \min\{f, g\} = \frac{1}{2}(f + g - |f - g|),$$

we see that if $\bar{\mathcal{P}}$ is the closure of the set of polynomials under uniform convergence to the limit, $\bar{\mathcal{P}}$ would also be closed under the lattice operations. Thus, as $\bar{\mathcal{P}}$ form an unital subalgebra of all real to real functions, (as its closed under addition and multiplication), and as $\bar{\mathcal{P}}$ separates points trivially, $\bar{\bar{\mathcal{P}}} = \bar{\mathcal{P}} = [-1, 1]^{\mathbb{R}}$ which can be trivially extended to any closed intervals as required.

However, this can also be proved by constructing the approximant directly. By using the weak law of large numbers, we can prove that the Bernstein polynomials can be used to find a sequence of polynomials that uniformly converges on any continuous function. (A full proof was typed up and is presented here: http://bit.ly/2Mz8ug8)

Application: Trig. Polynomials

Another set of functions we might be interested in are the trigonometric polynomials $\mathcal{T}$ where trigonometric polynomials are functions of the form

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx); \; a_i, b_i \in \mathbb{R}.$$

As the trigonometric polynomials $\mathcal{T}$ form an unital subalgebra of the bounded continuous functions on $[-\pi, \pi]$ (by considering the identities of multiplication between trigonometric functions), and by considering the Fourier series of $|x|$, by the same argument as presented above, $\bar{\mathcal{T}}$ is closed under lattice operations and hence, by Stone-Weierstrass, $\bar{\bar{\mathcal{T}}} = \bar{\mathcal{T}} = [-\pi, \pi]^{\mathbb{R}}$

Outline and Formalisation

The Stone-Weierstrass theorem was proved and formalised using the *Lean*, the source code of which can be found in my GitHub repository: http://github.com/JasonKYi/stone-weierstrass.

The theorem itself relies on two central lemmas:

Lemma 1. For all $f \in M$, $f \in \bar{M_0}$ if and only if for all $x, y \in X$, $\epsilon > 0$, there exists $g \in M_0$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$.

Lemma 2. Given S, a subalgebra of $\mathbb{R}^2$, S must be $\{(0,0)\}$, $\{(x, 0) \mid x \in \mathbb{R}\}$, $\{(0.y) \mid y \in \mathbb{R}\}$, $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ itself.

Lemma 1 was formalised and is represented in Lean as `in_closure₂_iff_dense_at_points` in `main.lean` by constructing constructing mappings from X to set of X while considering the compactness of X; while *Lemma 2* was also formalised and is represented in Lean as `subalgebra_of_R2` and can be found in `ralgebra.lean` which proof involved evoking the law of the excluded middle on different propositions. multiple times.

Now, let M_0, M_1 be closed subalgebras of M under lattice operations, then, by considering lemma 1, it was deduced $\bar{M_0} = \bar{M_1}$ if and only if at for all distinct x, y, $\bar{M_0}$ and $\bar{M_1}$ have the same boundary points. This was formalised in `eq_iff_boundary_points_eq` in `main.lean`

Finally. by showing that the boundary points of M_0 form an unital subalgebra of $\mathbb{R}^2$, we see that the boundary points of $\bar{M_0}$ must be $\mathbb{R}^2$ by lemma 2 and hence, as the boundary points of M is $\mathbb{R}^2$ is $\mathbb{R}^2$, it follows $\bar{M_0} = M$. As required!

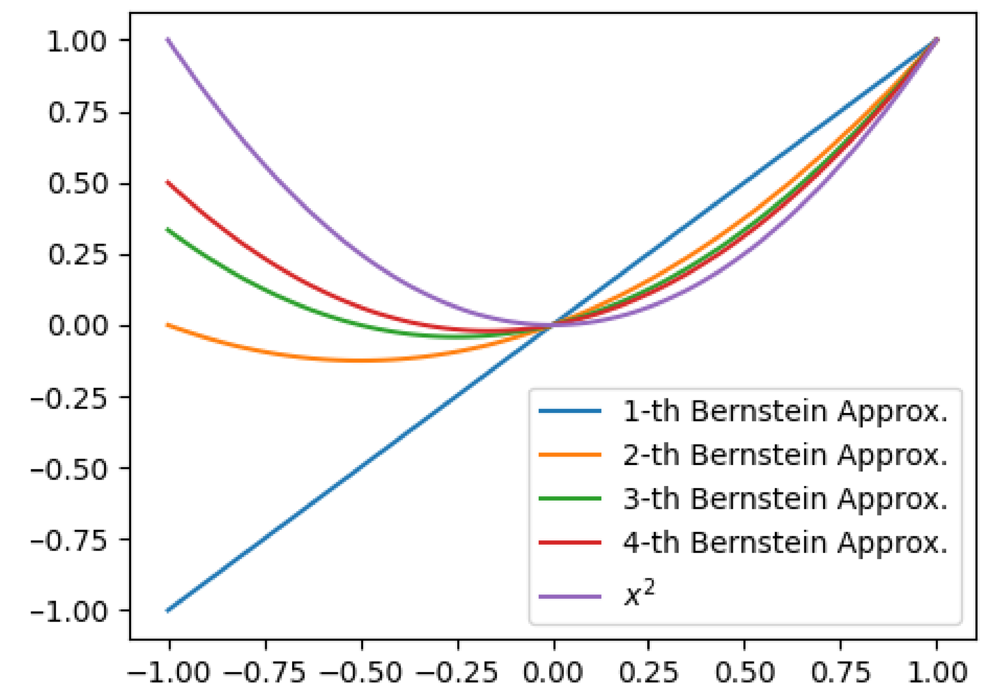
This was formalised in Lean in `main.lean` with the theorem statement being

```
theorem weierstrass_stone {M₀' :
subalgebra ℝ (X → ℝ)} (hc : closure₀
M₀'.carrier = M₀'.carrier) (hsep :
has_seperate_points M₀'.carrier) :
closure₂ M₀'.carrier = univ
```

Further Discussion & Interpolation

While the Stone-Weierstrass theorem proved existence, and the Bernstein polynomials offerd us a method to find a polynomial arbitrarily close to any continuous function with respect to the uniform norm, it is not necessarily the best approximation.

Using some simple *Python* scripts, we can easily see that for functions such as x^2, the Bernstein polynomials does not offer as much accuracy as just using x^2 as its own approximation.



Consider instead the interpolation theorem:

Theorem. Given $\{(x_0, y_0), \cdots, (x_n, y_n)\} \subseteq \mathbb{R}^2$, using Lagrange polynomials, we can construct polynomial that interpolates the set. [4]

So, if we let $f : [a, b] \to \mathbb{R}$ be a continuous function, we can approximate f using the polynomial that interpolates the set of equidistant points from a to b. By implementing this algorithm in Python, we find that for $n = 3$ for $f : [-1, 1] \to \mathbb{R} : x \mapsto x^2$, this method results in a polynomial approximation of x^2 (Very Good!).

However, unlike the Bernstein polynomial method, this method does not guarantee that the uniform norm of the difference of the function and its approximation tends to zero as demonstrated by interpolating *Runge's function* (it in fact tends to $+\infty$). To mitigate this, one can instead use Spline interpolation however this results in an approximant that is a piecewise polynomial instead of just a polynomial.

References

[1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem. Mathematics Magazine 21, no. 5 : 237-54.

[2] Gaddy, P. The Stone-Weierstrass Theorem and its Applications to L^2 Spaces.

[3] Lau, K. and Kudryashov, Y. (2018) Algebra over Commutative Semiring (under category) [Online]. Available at: http://bit.ly/3gLIPyS (Assessed: 02 June 2020)

[4] Humpherys, J. and Jarvis, T. J. (2020) Foundations of Applied Mathematics Volume 2: Algorithms, Approximation, Optimization Society for Industrial and Applied Mathematics, pp. 418