

## The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem states that, given an unital subalgebra of  $M$ ,  $M_0$  that is closed under lattice operations and separates points,  $\bar{M}_0 = M$ , and we say  $M_0$  is dense in  $M$  [2].

## Variables and Definitions

Throughout this poster, we will let  $X$  be a compact metric space,  $M$  to be the set of all bounded and continuous functions from  $X$  to  $\mathbb{R}$ ,  $M_0$  a subset of  $M$  and  $\bar{M}_0$  the closure of  $M_0$  under uniform convergence to the limit.

We say  $M_0$  separates points if and only if for all distinct  $x, y \in X$ , there exists some  $f \in M_0$  such that  $f(x) \neq f(y)$ . Given  $x, y \in X$ , the boundary points of  $M_0$  is defined to be  $\{(f(x), f(y)) \mid f \in M_0\}$ .

## Lattice Operations

We define two lattice operations  $\vee, \wedge : (X \rightarrow \mathbb{R})^2 \rightarrow (X \rightarrow \mathbb{R})$  such that, for all  $f, g : X \rightarrow \mathbb{R}$ , where  $f, g$  are bounded continuous functions

$$f \vee g = \max\{f, g\},$$

and

$$f \wedge g = \min\{f, g\}.$$

## $R$ -algebra and Subalgebra

An  $R$ -algebra is a mathematical object consisting of a commutative ring  $R$  and a semi-ring  $A$  such that there exists scalar multiplication  $\cdot : R \times A \rightarrow A$  and a homomorphism from  $R$  to  $A$ ,  $\phi : R \rightarrow A$  such that

$$r \cdot a = \phi(r) \times a,$$

and

$$\phi(r) \times a = a \times \phi(r),$$

are satisfied for all  $r \in R$  and  $a \in A$  [3].

A unital subalgebra  $S$  of an  $R$ -algebra  $A$  is a subset of  $A$  that's closed under the induced operations carried from  $A$  and contains 1 [4].

It was shown that both  $M$  and  $\mathbb{R}^2$  form a  $R$ -algebra over  $\mathbb{R}$  (see: <http://bit.ly/3eL7LEC>).

## Approximation Theorem

The Stone-Weierstrass theorem is a generalisation of the Weierstrass approximation theorem which states that any continuous functions on a closed interval can be uniformly approximated by a polynomial. This can be deduced straightaway by the Stone-Weierstrass theorem.

Consider the Taylor polynomial  $P_n(x)$  of  $s(x) = \sqrt{1-x}$  for  $x \in [-1, 1]$ . Using analysis, we can show that  $P_n \rightarrow s$  uniformly, and thus,  $P_n(1-x^2) \rightarrow s(1-x^2) = |x|$  uniformly for  $x \in [-1, 1]$ . Now, as

$$f \vee g = \max\{f, g\} = \frac{1}{2}(f + g + |f - g|);$$

$$f \wedge g = \min\{f, g\} = \frac{1}{2}(f + g - |f - g|),$$

we see that if  $\bar{\mathcal{P}}$  is the closure of the set of polynomials under uniform convergence to the limit,  $\bar{\mathcal{P}}$  would also be closed under the lattice operations. Thus, as  $\bar{\mathcal{P}}$  form an unital subalgebra of all real to real functions, (as its closed under addition and multiplication), and as  $\bar{\mathcal{P}}$  separates points trivially,  $\bar{\bar{\mathcal{P}}} = \bar{\mathcal{P}} = [-1, 1]^{\mathbb{R}}$  which can be trivially extended to any closed intervals as required.

However, this can also be proved by constructing the approximant directly. By using the weak law of large numbers, we can prove that the Bernstein polynomials can be used to find a sequence of polynomials that uniformly converges on any continuous function (see: <http://bit.ly/2Mz8ug8>)

## Trigonometric Polynomials

Another set of functions we might be interested in are the trigonometric polynomials  $\mathcal{T}$  where trigonometric polynomials are functions of the form

$$f(x) = a_0 + \sum_{k=1}^n a_k \cos(kx) + b_k \sin(kx); \quad a_i, b_i \in \mathbb{R}.$$

As the trigonometric polynomials  $\mathcal{T}$  form an unital subalgebra of the bounded continuous functions on  $[-\pi, \pi]$  (by considering the identities of multiplication between trigonometric functions), and by considering the Fourier series of  $|x|$ , by the same argument as presented above,  $\bar{\mathcal{T}}$  is closed under lattice operations and hence, by Stone-Weierstrass,  $\bar{\bar{\mathcal{T}}} = \bar{\mathcal{T}} = [-\pi, \pi]^{\mathbb{R}}$

## Outline and Formalisation

The Stone-Weierstrass theorem was proved and formalised using the Lean, the source code of which can be found in my GitHub repository: <http://github.com/JasonKYi/stone-weierstrass>.

The theorem itself relies on two central lemmas:

**Lemma 1.** For all  $f \in M$ ,  $f \in \bar{M}_0$  if and only if for all  $x, y \in X$ ,  $\epsilon > 0$ , there exists  $g \in M_0$  such that  $|f(x) - g(x)| < \epsilon$  and  $|f(y) - g(y)| < \epsilon$ .

**Lemma 2.** Given  $S$ , a subalgebra of  $\mathbb{R}^2$ ,  $S$  must be  $\{(0, 0)\}$ ,  $\{(x, 0) \mid x \in \mathbb{R}\}$ ,  $\{(0, y) \mid y \in \mathbb{R}\}$ ,  $\{(z, z) \mid z \in \mathbb{R}\}$ , or  $\mathbb{R}^2$  itself.

Lemma 1 was formalised and is represented in Lean as `in_closure2_iff_dense_at_points` in `main.lean` the method of which we will discuss below. The forward direction of the proof is trivial so we will discuss the reverse.

Let us fix  $x$  and  $\epsilon$  and define a mapping to set of  $X$

$$S : X \rightarrow \text{set } X := \lambda y, \{z \mid |f(z) - g_y(z)| < \epsilon\},$$

where  $g_y$  was chosen such that  $|f(x) - g_y(x)| < \epsilon$  and  $|f(y) - g_y(y)| < \epsilon$ . Then for all  $y \in X$ ,  $y \in S(y)$  so  $\bigcup_{y \in X} S(y) = X$ . But as  $X$  is compact,  $\bigcup_{y \in X} S(y)$  admits a finite subcover; so, there exists a finite index set  $I$  such that  $\bigcup_{i \in I} S(y_i) = X$ . Thus, by letting  $p_x = \bigvee_{i \in I} g_{y_i}$ , we have constructed a function  $p_x \in \bar{M}_0$  such that

$$p_x(z) \geq g_{y_i}(z) > f(z) - \epsilon \text{ and } p_x(x) < f(x) + \epsilon$$

for all  $z \in X$  and  $i \in I$ .

Now, by defining a similar mapping to set of  $X$ ,

$$T : X \rightarrow \text{set } X := \lambda x, \{z \mid p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of  $X$  and thus can create the required function with  $\bigwedge_{j \in J} p_{x_j}$  where  $J$  is the index set such that  $\bigcup_{j \in J} T(p_{x_j}) = X$ .

## References

- [1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem. Mathematics Magazine 21, no. 5 : 237-54.
- [2] Gaddy, P. The Stone-Weierstrass Theorem and its Applications to  $L^2$  Spaces.
- [3] Lau, K. and Kudryashov, Y. (2018) Algebra over Commutative Semiring (under category) [Online]. Available at: <http://bit.ly/3gLiPyS> (Assessed: 02 June 2020)
- [4] Bourbaki, N. (1989) 'Subalgebras. Ideals. Quotient Algebras' in Elements of mathematics, Algebra I. Paris: Herman, Publisher of Science, pp. 429

Lemma 2 was also formalised and is represented in Lean as `subalgebra_of_R2` and can be found in `ralgebra.lean`. The proof this lemma is rather tedious and follows directly by evoking the law of the excluded middle on different propositions multiple times.

Now, by considering lemma 1, it can be deduced that, for  $M_0, M_1$ , closed subalgebras of  $M$  under lattice operations and uniform convergence to the limit,  $M_0 = M_1$  if and only if at for all distinct  $x, y$ ,  $M_0$  and  $M_1$  have the same boundary points. This was formalised in `eq_iff_boundary_points_eq` by constructing the notion of `closure'` in `definitions.lean`

Lastly, as the boundary points of  $M_0$  form an unital subalgebra of  $\mathbb{R}^2$  as demonstrated shown by `subalgebra_of_boundary_points` in `main.lean`, we can utilise lemma 2 to deduce that the boundary points must either be  $\{(z, z) \mid z \in \mathbb{R}\}$ , or  $\mathbb{R}^2$  (the first three possibilities in lemma 2 are not possible since  $(1, 1)$  is in the boundary points as it is unital). Now, if  $M_0$  separates points then there must exist  $f \in M_0$ ,  $f(x) \neq f(y)$  so that excludes  $\{(z, z) \mid z \in \mathbb{R}\}$  and hence the boundary points is  $\mathbb{R}^2$  and the theorem follows. This was formalised in `main.lean` with the statement being

```
theorem weierstrass_stone {M0' : subalgebra ℝ (X → ℝ)} (hc : closure0 M0'.carrier = M0'.carrier) (hsep : has_separate_points M0'.carrier) : closure2 M0'.carrier = univ
```

with the `M0'.carrier` referring to the underlying subset of  $M'_0$  the subalgebra, `hc` the hypothesis that `M0'.carrier` is closed under lattice operations and `closure2 M0'.carrier` the closure of `M0'.carrier` under uniform convergence.