The Stone-Weierstrass Theorem and its Formalisation

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The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem states that, given an unital subalgebra of M, M_0 that is closed under lattice operations and separates points, $\bar{M_0} = M$, and we say M_0 is dense in M [2].

Variables and Definitions

Throughout this poster, we will let X be a compact metric space, M to be the set of all bounded and continuous functions from X to $\mathbb{R}$, M_0 a subset of M and $\bar{M_0}$ the closure of M_0 *under uniform convergence to the limit*.

We say M_0 *separates points* if and only if for all distinct $x, y \in X$, there exists some $f \in M_0$ such that $f(x) \neq f(y)$.
Given $x, y \in X$, the boundary points of M_0 is defined to be $\{(f(x), f(y)) \mid f \in M_0\}$.

Lattice Operations

We define two lattice operations $\vee, \wedge : (X \to \mathbb{R})^2 \to (X \to \mathbb{R})$ such that, for all $f, g : X \to \mathbb{R}$, where f, g are bounded continuous functions

$$f \vee g = \max\{f, g\},$$

and

$$f \wedge g = \min\{f, g\}.$$

R-algebra and Subalgebra

An *R-algebra* is a mathematical object consisting of a commutative ring R and a semi-ring A such that there exists scalar multiplication $\cdot : R \times A \to A$ and a homomorphism from R to A, $\phi : R \to A$ such that

$$r \cdot a = \phi(r) \times a,$$

and

$$\phi(r) \times a = a \times \phi(r),$$

are satisfied for all $r \in R$ and $a \in A$ [3].

A *unital subalgebra* S of an R-algebra A is a subset of A that's closed under the induced operations carried from A and contains 1 [4].

It was shown that both M and $\mathbb{R}^2$ form a R-algebra over $\mathbb{R}$ (see: http://bit.ly/3eL7LEC).

Approximation Theorem

The Stone-Weierstrass theorem is a generalisation of the Weierstrass approximation theorem which states that any continuous functions on a closed interval can be uniformly approximated by a polynomial. This can be deduced straightaway by the Stone-Weierstrass theorem.

Consider the Taylor polynomial $P_n(x)$ of $s(x) = \sqrt{1-x}$ for $x \in [-1, 1]$. Using analysis, we can show that $P_n \to s$ uniformly, and thus, $P_n(1 - x^2) \to s(1 - x^2) = |x|$ uniformly for $x \in [-1, 1]$. Now, as

$$f \vee g = \max\{f, g\} = \frac{1}{2}(f + g + |f - g|);$$

$$f \wedge g = \min\{f, g\} = \frac{1}{2}(f + g - |f - g|),$$

we see that if $\bar{\mathcal{P}}$ is the closure of the set of polynomials under uniform convergence to the limit, $\bar{\mathcal{P}}$ would also be closed under the lattice operations. Thus, as $\bar{\mathcal{P}}$ form an unital subalgebra of all real to real functions, (as its closed under addition and multiplication), and as $\bar{\mathcal{P}}$ separates points trivially, $\bar{\bar{\mathcal{P}}} = \bar{\mathcal{P}} = \mathbb{R}^{\mathbb{R}}$ as required.

However, this can also be proved by constructing the approximant directly. By using the weak law of large numbers, we can prove that the Bernstein polynomials can be used to find a sequence of polynomials that uniformly converges on any continuous function (see: http://bit.ly/2Mz8ug8)

Trigonometric Polynomials

Another set of functions we might be interested in are the trigonometric polynomials $\mathcal{T}$ where trigonometric polynomials are functions of the form

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx); \; a_i, b_i \in \mathbb{R}.$$

As the trigonometric polynomials $\mathcal{T}$ form an unital subalgebra of the bounded continuous functions on $[-\pi, \pi]$ (by considering the identities of multiplication between trigonometric functions), and by considering the Fourier series of $|x|$, by the same argument as presented above, $\bar{\mathcal{T}}$ is closed under lattice operations and hence, by Stone-Weierstrass, $\bar{\bar{\mathcal{T}}} = \bar{\mathcal{T}} = [-\pi, \pi]^{\mathbb{R}}$

Outline and Formalisation

The Stone-Weierstrass theorem was proved and formalised using the *Lean*, the source code of which can be found in my GitHub repository: http://github.com/JasonKYi/stone-weierstrass.

The theorem itself relies on two central lemmas:
Lemma 1. For all $f \in M$, $f \in \bar{M_0}$ if and only if for all $x, y \in X$, $\epsilon > 0$, there exists $g \in M_0$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$.
Lemma 2. Given S, a subalgebra of $\mathbb{R}^2$, S must be $\{(0, 0)\}$, $\{(x, 0) \mid x \in \mathbb{R}\}$, $\{(0.y) \mid y \in \mathbb{R}\}$, $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ itself.

Lemma 1 was formalised and is represented in Lean as `in_closure₂_iff_dense_at_points` in `main.lean` the method of which we will discuss below. The forward direction of the proof is trivial so we will discuss the reverse.

Let us fix x and ϵ and define a mapping to set of X

$$S : X \to \texttt{set } X := \lambda y, \{z | f(z) - g_y(z) < \epsilon\},$$

where g_y was chosen such that $|f(x) - g_y(x)| < \epsilon$ and $|f(y) - g_y(y)| < \epsilon$.
Then for all $y \in X$, $y \in S(y)$ so $\bigcup_{y \in X} S(y) = X$. But as X is compact, $\bigcup_{y \in X} S(y)$ admits a finite subcover; so, there exists a finite index set I such that $\bigcup_{i \in I} S(y_i) = X$. Thus, by letting $p_x = \bigvee_{i \in I} g_{y_i}$, we have constructed a function $p_x \in \bar{M_0}$ such that

$$p_x(z) \geq g_{y_i}(z) > f(z) - \epsilon \text{ and } p_x(z) < f(z) + \epsilon$$

for all $z \in X$ and $i \in I$.
Now, by defining a similar mapping to set of X,

$$T : X \to \texttt{set } X := \lambda x, \{z | p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of X and thus can create the required function with $\bigwedge_{j \in J} p_{x_j}$ where J is the index set such that $\bigcup_{j \in J} T(p_{x_i}) = X$.

Lemma 2 was also formalised and is represented in Lean as `subalgebra_of_R2` and can be found in `ralgebra.lean`. The proof this lemma is rather tedious and follows directly by evoking the law of the excluded middle on different propositions multiple times.

Now, by considering *lemma 1*, it can be deduced that, for M_0, M_1, closed subalgebras of M under lattice operations and uniform convergence to the limit, $M_0 = M_1$ if and only if at for all distinct x, y, M_0 and M_1 have the same boundary points. This was formalised in `eq_iff_boundary_points_eq` by constructing the notion of `closure'` in `definitions.lean`

Lastly, as the boundary points of M_0 form an unital subalgebra of $\mathbb{R}^2$ as demonstrated shown by `subalgebra_of_boundary_points` in `main.lean`, we can utilise *lemma 2* to deduce that the boundary points must either be $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ (the first three possibilities in lemma 2 are not possible since $(1, 1)$ is in the boundary points as it is unital). Now, if M_0 separates points then there must exist $f \in M_0$, $f(x) \neq f(y)$ so that excludes $\{(z, z) \mid z \in \mathbb{R}\}$ and hence the boundary points is $\mathbb{R}^2$ and the theorem follows. This was formalised in `main.lean` with the statement being

```
theorem weierstrass_stone {M₀'
: subalgebra ℝ (X → ℝ)} (hc :
closure₀ M₀'.carrier = M₀'.carrier)
(hsep : has_seperate_points
M₀'.carrier) : closure₂ M₀'.carrier =
 univ
```

with the `M₀'.carrier` referring to the underlying subset of `M₀'` the subalgebra, `hc` the hypothesis that `M₀'.carrier` is closed under lattice operations and `closure₂ M₀'.carrier` the closure of `M₀'.carrier` under uniform convergence.

References

[1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem. Mathematics Magazine 21, no. 5 : 237-54.
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[4] Bourbaki, N. (1989) 'Subalgebras. Ideals. Quotient Algebras' in Elements of mathematics, Algebra I. Paris: Herman, Publisher of Science, pp. 429