The Formalisation and Applications of the Stone-Weierstrass Theorem

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The Stone-Weierstrass Theorem

Let X be a compact metric space and M the set of all continuous function from X to $\mathbb{R}$, then the *Stone-Weierstrass theorem* states that,

> Given an unital subalgebra of M, M_0 (i.e. a subset of M that contains 1 and is closed under addition, multiplication and scalar multiplication) that is closed under the lattice operations and separates points, $\bar{M}_0 = M$; [2]

where we define the lattice operations $\vee, \wedge : (X \to \mathbb{R})^2 \to (X \to \mathbb{R})$ such that, for all $f, g : X \to \mathbb{R}$, where f, g are bounded continuous functions

$$f \vee g = \max\{f, g\}; \quad f \wedge g = \min\{f, g\};$$

$\bar{M}_0$ the closure of M_0 *under uniform convergence to the limit* and we say M_0 *separates points* if and only if for all distinct $x, y \in X$, there exists some $f \in M_0$ such that $f(x) \neq f(y)$.

Application: Weierstrass Theorem

The Weierstrass approximation theorem states

> Let f be a real continuous function on a closed interval. Then for all $\epsilon > 0$, there exists a polynomial P_n such that
>
> $$\sup_{x \in [0,1]} |f(x) - P_n(x)| < \epsilon.$$

This can be deduced straightaway by the Stone-Weierstrass theorem!

From the section $|x|$ *is in the Closure!* we have shown that there exists a sequence of polynomials uniformly converging to $|x|$ on any compact interval. Now, as

$$f \vee g = \frac{1}{2}(f + g + |f - g|);$$

$$f \wedge g = \frac{1}{2}(f + g - |f - g|),$$

we see that if $\bar{\mathcal{P}}$ is the closure of the set of polynomials under uniform convergence to the limit, $\bar{\mathcal{P}}$ would also be closed under the lattice operations. Thus, as $\bar{\mathcal{P}}$ form an unital subalgebra of all real to real functions, (as it contains 1,closed under addition, multiplication and scalar multiplication), and as $\bar{\mathcal{P}}$ separates points trivially, $\bar{\bar{\mathcal{P}}} = \bar{\mathcal{P}} = [\alpha, \beta]^{\mathbb{R}}$ for all $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ As required!

Lemma: $|x|$ is in the Closure!

Let $X_1, \cdots X_n$ be a sequence of i.i.d. Bernoulli random variables with parameter x and let $S_n = \sum_{i=1}^n X_i$ so $S_n \sim Bin(n, x)$. Then by the *law of the unconscious statistician*, we see,

$$\mathbb{E}\left(\left|\alpha \frac{S_n}{n} - \beta\right|\right) = \sum_{k=0}^{n} \left|\alpha \frac{k}{n} - \beta\right| \Pr(S_n = nk)$$

$$= \sum_{k=0}^{n} \left|\alpha \frac{k}{n} - \beta\right| \binom{n}{i} x^i (1-x)^{n-i} =: P_n(x)$$

for all $\alpha, \beta \in \mathbb{R}, \alpha < \beta$. Let us now define

$$A_n(\delta) = \left\{\omega : \left|\frac{S_n(\omega)}{n} - x\right| \geq \delta\right\},$$

then, for all $\epsilon > 0$, as $f(x) : [0,1] \to \mathbb{R} := |\alpha x - \beta|$ is continuous on the compact interval $[0,1]$, it is uniformly continuous on the same interval; so there is some $\delta > 0$ such that for all $|x - y| < \delta$, $\frac{\epsilon}{2} > |f(x) - f(y)|$. Now, by LLN, $\Pr(A_n(\delta)) \to 0$, so, choose N such that $|\Pr(A_N(\delta))| < 0$, then

$$\frac{\epsilon}{2} > \mathbb{E}\left[\mathbf{1}_{(A_N)^c} \left|f(x) - f\left(\frac{S_N}{N}\right)\right| \mid A_N(\delta)^c\right]$$

$$\geq \left|f(x) - \mathbb{E}\left[f\left(\frac{S_N}{N}\right)\right]\right| \Pr((A_N(\delta))^c).$$

Now as $\mathbb{E}\left[f\left(\frac{S_N}{N}\right)\right] = P_N(x)$, we have $\epsilon/2 > |f(x) - P_N(x)|(1 - \Pr(A_N(\delta))) > |f(x) - P_N(x)|/2$, so $P_n(x) \to f(x) = |\alpha x - \beta|$ uniformly for $x \in [0,1]$ and $P_n(\frac{x+\beta}{\alpha}) \to |x|$ uniformly for all $x \in [\alpha, \beta]$.

Application: Trig. Polynomials

Another set of functions we might be interested in are the trigonometric polynomials $\mathcal{T}$ where trigonometric polynomials are functions of the form

$$f(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx); \; a_i, b_i \in \mathbb{R}.$$

As the trigonometric polynomials $\mathcal{T}$ form an unital subalgebra of the bounded continuous functions on $[-\pi, \pi]$ (by considering the identities of multiplication between trigonometric functions), and by considering the Fourier series of $|x|$, by the same argument as presented above, $\bar{\mathcal{T}}$ is closed under lattice operations and hence, by Stone-Weierstrass, $\bar{\bar{\mathcal{T}}} = \bar{\mathcal{T}} = [-\pi, \pi]^{\mathbb{R}}$

Proof Outline of Stone-Weierstrass

Lemma 1. For all $f \in M$, f is in $\bar{M}_0$ if and only if for all $x, y \in X$, $\epsilon > 0$, there exists $g \in M_0$ such that $|f(x) - g(x)| < \epsilon$ and $|f(y) - g(y)| < \epsilon$.

Proof. The forward direction is trivial so we will consider the reverse. Let us fix x and ϵ and define a mapping $S : X \to \texttt{set}\ X$,

$$S(y) := \{z \in X \mid f(z) - g_y(z) < \epsilon\},$$

where g_y was chosen such that $|f(x) - g_y(x)| < \epsilon$ and $|f(y) - g_y(y)| < \epsilon$.
Then for all $y \in X$, $y \in S(y)$ so $\bigcup_{y \in X} S(y) = X$. But as X is compact, $\bigcup_{y \in X} S(y)$ admits a finite subcover; so, there exists a finite index set I such that $\bigcup_{i \in I} S(y_i) = X$. Thus, by letting $p_x = \bigvee_{i \in I} g_{y_i}$, we have constructed a function $p_x \in \bar{M}_0$ such that

$$p_x(z) \geq g_{y_i}(z) > f(z) - \epsilon \text{ and } p_x(x) < f(x) + \epsilon$$

for all $z \in X$ and $i \in I$.
Now, by defining a similar mapping to set of X,

$$T := \{z \in X \mid p_x(z) < f(z) + \epsilon\},$$

we again create a finite subcover of X and thus can create the required function with $\bigwedge_{j \in J} p_{x_j}$ where J is the index set such that $\bigcup_{j \in J} T(x_j) = X$.

Lemma 2. Given S, a subalgebra of $\mathbb{R}^2$, S must be $\{(0,0)\}$, $\{(x, 0) \mid x \in \mathbb{R}\}$, $\{(0.y) \mid y \in \mathbb{R}\}$, $\{(z, z) \mid z \in \mathbb{R}\}$, or $\mathbb{R}^2$ itself.
Lemma 2 was proved by evoking the law of the excluded middle on different propositions.

Given $x, y \in X$, define the boundary points of M_0 to be $\{(f(x), f(y)) \mid f \in M_0\}$. Let M_0, M_1 be closed subalgebras of M under lattice operations, by lemma 1, it is deduced $\bar{M}_0 = \bar{M}_1$ iff. for distinct x, y, $\bar{M}_0$ and $\bar{M}_1$ have the same boundary points.

Now, as boundary points of M_0 form an unital subalgebra of $\mathbb{R}^2$, the boundary points of $\bar{M}_0$ must be $\mathbb{R}^2$ by lemma 2 (the first three options excluded as $1 \in M_0$ and the fourth excluded as M_0 separate points) hence, as the boundary points of M is $\mathbb{R}^2$ is $\mathbb{R}^2$, it follows $\bar{M}_0 = M$ as required!

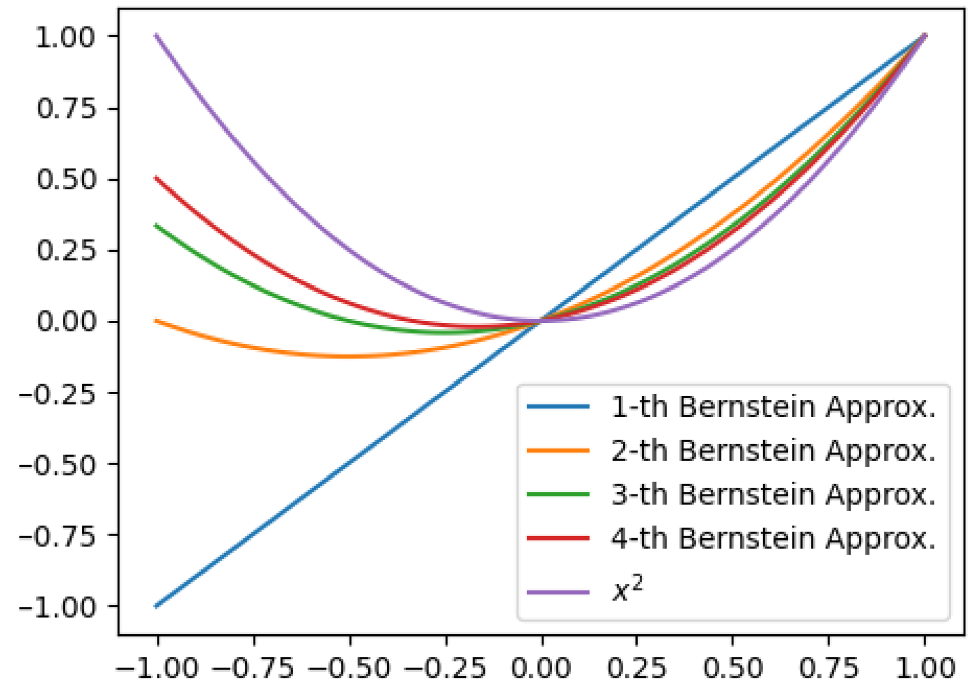
The Stone-Weierstrass theorem was proved and formalised using the *Lean*, the source code of which can be found in my GitHub repository: `http://github.com/JasonKYi/stone-weierstrass/tree/master/srcs`.

Further Discussion & Interpolation

Let us consider the approximation of x^2 using Bernstein polynomials,by calculating explicitly, we find the n-th Bernstein approximations of x^2 results in the sequence

$$B_n(x) = \frac{1}{n}x + \frac{n-1}{n}x^2,$$

resulting $\sup_{x \in [-1,1]} |x^2 - B_n(x)| = 2/n$; much worse than just using x^2 as its own approximation.



Consider instead the interpolation theorem:
Theorem. Given $\{(x_0, y_0), \cdots, (x_n, y_n)\} \subseteq \mathbb{R}^2$, using Lagrange polynomials, we can construct polynomial that interpolates the set. [3]

So, if we let $f : [a, b] \to \mathbb{R}$ be a continuous function, we can approximate f using the polynomial that interpolates the set of equidistant points from a to b. By implementing this algorithm in Python, we find that for $n = 3$ for $f : [-1, 1] \to \mathbb{R} : x \mapsto x^2$, this method results in a polynomial approximation of x^2 (a very good error of zero!).
However, unlike the Bernstein polynomial method, this method does not guarantee that the uniform norm of the difference of the function and its approximation tends to zero as demonstrated by interpolating *Runge's function* (it in fact tends to $+\infty$). To mitigate this, one can instead use Spline interpolation however this results in an approximant that is a piecewise polynomial instead of just a polynomial.

References

[1] Stone, M.H. (1948) The Generalized Weierstrass Approximation Theorem. Mathematics Magazine 21, no. 5 : 237-54.

[2] Gaddy, P. The Stone-Weierstrass Theorem and its Applications to L^2 Spaces.

[3] Humpherys, J. (2020) Foundations of Applied Mathematics Volume 2: Algorithms, Approximation, Optimization Society for Industrial and Applied Mathematics