Group Representation Revision Notes

Kexing Ying

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Group Representation

Finding 1-dimensional subrepresentations in $(\mathbb{C}[G], \rho_{\text{reg}})$: Denote the 1-dimensional representations of G by $(\mathbb{C}, \theta)$, then, define $v_\theta := \sum_{g \in G} \overline{\theta(g)} g \in \mathbb{C}[G]$, we observe, for all $h \in G$, $hv_\theta = \theta(h) v_\theta$. Hence, v_θ is a shared eigenvector of $\rho_{\text{reg}}(g)$ with eigenvalue $\theta(g)$, implying $\langle v_\theta \rangle$ is a 1-dimensional subrepresentation of $\mathbb{C}[G]$ isomorphic to θ.

Symmetric and Dihedral Groups

The dihedral group D_n is the finite group with the presentation

$$D_n = \langle x, y \mid x^n = 1 = y^2, yxy^{-1} = 1 \rangle.$$

It has the following important properties:

- $|D_n| = 2n$;
- D_n has $(n+3)/2$ conjugacy classes if n is odd and $(n+6)/2$ if n is even (this tells us how many irreducible representations there are);
- $(D_n)_{ab} = C_2$ if n is odd and $(D_n)_{ab} = C_2 \times C_2$ if n is even;
- geometrically, elements of the dihedral group corresponds to rotations and reflections. In particular, for n even, this includes all reflections along opposite vertices and edges;
- D_n always has the two-dimensional irreducible representation $(\mathbb{C}^2, \rho_{\mathbb{C}^2})$ given by

$$\rho_{\mathbb{C}^2}(x) = \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}, \ \rho_{\mathbb{C}^2}(y) = \begin{pmatrix} \cos \frac{4\pi}{n} & \sin \frac{4\pi}{n} \\ \sin \frac{4\pi}{n} & -\cos \frac{4\pi}{n} \end{pmatrix}.$$

- The elements which commute with elements of End$(\mathbb{C}^2)$ ($(\mathbb{C}^2, \rho_{\mathbb{C}^2})$ is the representation in the above point) are the identity and rotation by $\pi/2$;
- $D_3 \simeq S_3$.

To find all irreducible representations of D_n, we can construct the following homomorphisms (note that these might not be automorphisms),

$$\phi_k : D_n \to D_n : x^a y^b \mapsto x^{ka} y^b.$$

It is clear that $\rho_{\mathbb{C}^2} \circ \phi_k$ is an irreducible representation. Furthermore, these are non-isomorphic for $1 \leq k < n/2$ by considering their characters. These and the aforementioned one-dimensional representations must be all of the irreducible ones by sum of squares.

Denoting $\mathbf{n} := \{1, \cdots, n\}$, the symmetric group S_n is the set of bijections between $\mathbf{n}$ to itself. It has the following properties:

- $|S_n| = n!$;
- for all $\sigma \in S_n$, the conjugacy class $[\sigma]$ contains all elements of S_n which have the same cycle type as σ. Thus, to find the number of conjugacy classes, one count the number of possible cycle types/partitions;
- $\text{sgn} : S_n \to \{\pm 1\} : \sigma \mapsto \text{sgn}(\sigma)$ is a group homomorphism;
- $\ker \text{sgn} = A_n$ where A_n is the alternating group;
- $C_2 \simeq \{\pm 1\} \simeq S_n/A_n \simeq (S_n)_{ab}$;
- hence S_n has only two one-dimensional representations, namely the trivial and the sign representation;
- defining $P_\sigma \in GL_n(\mathbb{C})$ the permutation matrix corresponding to σ, $(\mathbb{C}^n, \rho_{\text{perm}})$ where $\rho_{\text{perm}} : \sigma \mapsto P_\sigma$ is a representation known as the permutation representation;
- the permutation representation is reducible and, in particular,
$$(\mathbb{C}^n, \rho_{\text{perm}}) = (\mathbb{C}, \rho_{\text{triv}}) \oplus (\mathbb{C}^{n-1}, \rho_{\text{refl}}),$$
where the reflection representation $(\mathbb{C}^{n-1}, \rho_{\text{refl}})$ is a subrepresentation of the permutation representation on the sub-linear space $\{x \in \mathbb{C}^n \mid \sum_{i=1}^n x_i = 0\}$.

As A_n is a subgroup of S_n, we may compute the number of conjugacy classes of A_n from that of S_n.

Theorem. A conjugacy class of S_n splits into two disjoint conjugacy classes of A_n if and only if its cycle type consists of distinct odd integers. Otherwise, its simply remains a single conjugacy class in A_n.

We have the following surjection $q : S_4 \to S_3$ such that
$$q((12)) = (12), q((23)) = (23), q((34)) = (12)$$
which can be restricted such that $q|_{A_4} : A_4 \to A_3 \simeq C_3$ is a surjection.

Tensor and Dual

For arbitrary representations $(V_1, \rho_1), (V_2, \rho_2), (W, \rho_W)$ of G, we have the linear isomorphisms
$$\text{Hom}_G(V_1 \oplus V_2, W) \simeq \text{Hom}_G(V_1, W) \oplus \text{Hom}_G(V_2, W),$$
$$\text{Hom}_G(W, V_1 \oplus V_2) \simeq \text{Hom}_G(W, V_1) \oplus \text{Hom}_G(W, V_2).$$

There is a canonical linear injection
$$V^* \otimes W \to \text{Hom}(V, W)$$
which is an isomorphism if V is finite dimensional.

Character Theory

Characters of g with order n is the sum (some, possibly all) of n-th roots of unity.

Inner product on class functions (which contains characters) is defined by

$$\langle \chi_1, \chi_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)}.$$

The set of characters of irreducible representations form an orthonormal basis with respect to this inner product. Thus, by Maschke's, for all representations $V = \bigoplus_i V_i^{\oplus n_i}$, where V_i are irreducible representations, we can find the multiplicity n_i with

$$\langle \chi_V, \chi_{V_i} \rangle = \langle \sum_j n_j \chi_{V_j}, \chi_{V_i} \rangle = \sum_j n_j \delta_{ji} = n_i.$$

To find the last row of the character table, we have the following identity

$$\chi_{V_j}(g) = -(\dim V_j)^{-1} \sum_{i \neq j} \dim V_i \chi_{V_i}(g).$$

To find the size of the conjugacy classes given a character table, we have

$$\frac{|G|}{|\mathcal{C}_i|} = \sum_{k=1}^{m} |\chi_{V_k}(g_i)|^2$$

where $V_1, \cdots, V_m$ are all the irreducible representations and $g_i \in \mathcal{C}_i$.

If (V, ρ) is the regular representation of G, it has character $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$.

A group G is **not** simple iff there exists a nontrivial character χ such that $\chi(g) = \chi(e)$ for some $g \neq e$.

For $g \in G$ of finite order, $|\chi_V(g)| \leq \dim V$ with equality iff $\rho_V(g)$ is a scalar multiple of the identity.

Normal subgroups of G are precisely the subgroups N_J of the form

$$N_J := \{n \in G \mid \chi_{V_j}(n) = \chi_{V_j}(e), \forall j \in J\},$$

for $J \subseteq \{1, \cdots, m\}$.

Algebra Representations

If (V, ρ) is a finite dimensional representation of G, then $\rho(G)$ spans End(V) if and only if (V, ρ) is irreducible. The reverse direction requires semisimple algebras.

Similar to the group case, if $\rho_V : A \to$ End(V) is surjective for a A-module (V, ρ_V), the (V, ρ_V) is simple. If V is finite dimensional, the converse also holds.

For modules V, W, $V \simeq W$ implies $\chi_V = \chi_W$. The converse is true if A is semisimple.

For $W \leq V$ a submodule, we have $\chi_V = \chi_W + \chi_{V/W}$. This provides a counter example to the converse of the above statement if V is not semisimple.