Functional Analysis

Kexing Ying

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1 Introduction

We have thus far looked at abstract vector spaces in linear algebra and (metric) topological spaces in topology. In this course, we will combine these concepts and study linear metric space. In particular, we will study vector spaces equipped with a topology such that certain properties are satisfies.

In this course, we will often study the space of functions and hence the name of the course. As we have seen before, given that the codomain space possesses a certain structure, it is possible to define point-wise addition and scalar multiplications on functions, and thus, possible to equip the space with a vector space structure.

Let us recall some definitions.

Definition 1.1 (Metric)**.** A metric ρ on a non-empty set X is a function with type signature $X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

- $\rho(x, y) = 0 \iff x = y$;
- $\rho(x, y) = \rho(y, x)$;
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Definition 1.2 (Translation Invariant)**.** A metric space (V, ρ) where V is equipped with the binary operation $(+) : V \times V \to V$ is translational invariant if for all $w, z, v \in V$,

$$\rho(w + v, z + v) = \rho(w, z).$$

Definition 1.3 (Norm)**.** A norm $\| \cdot \|$ on the vector space V (over the field $\mathbb{K}$ equipped with a modulus $| \cdot |$) is a function with type signature $X \to \mathbb{R}^+$ such that for all $x, y \in V, k \in \mathbb{K}$,

- $\|x\| = 0 \iff x = 0$;
- $\|k \cdot x\| = |k|\|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

We recall that a norm induces a metric by defining $\rho(x, y) = \|x - y\|$. In this case, it is possible to show that $(+)$ and $(\cdot)$ are continuous with respect to this metric and ρ is translational invariant.

Definition 1.4 (Banach Space)**.** A normed space is said to be a Banach space if it is complete, i.e. every Cauchy sequence converge.

Definition 1.5 (Separable)**.** A topological space is said to be separable if there exists a dense countable subset.

As we shall see, for $0 < 0 < \infty$, ℓ_p is separable while ℓ_∞ is not.

Definition 1.6 (Compact)**.** A topological space is said to be compact if every open cover has a finite sub-cover.

Unlike what we have seen before, as we consider infinite dimensional spaces, we will see that the Heine-Borel property will no longer hold, i.e. closed and bounded is no longer equivalent to compact.

2 Linear Spaces

Definition 2.1 (Equivalent Norms and Metrics)**.** Two norms $\|\cdot\|_k$ for $k = 1, 2$ are said to be equivalent if there exists some $M > 0$ such that for all x,

$$\frac{1}{M}\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Similarly, two metrics ρ_k are said to be equivalent if there exists some $M > 0$ such that for all x, y,

$$\frac{1}{M}\rho_1(x, y) \leq \rho_2(x, y) \leq M\rho_1(x, y).$$

It is clear that equivalent is a symmetric relation and as we have seen before, all norms on a finite dimensional space are equivalent.

Definition 2.2 (Concave and Convex Function)**.** A function $f : V \to \mathbb{R}$ is

- concave if for all $s \in [0, 1], x, y \in V$, we have

$$sf(x) + (1 - s)f(y) \leq f(sx + (1 - s)y);$$

- convex if for all $s \in [0, 1], x, y \in V$, we have

$$sf(x) + (1 - s)f(y) \geq f(sx + (1 - s)y).$$

Proposition 2.1. If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is concave and $f(0) = 0$. Then

$$f(x + y) \leq f(x) + f(y).$$

Proof. Clear by taking $s = \frac{y}{x+y}$, we have

$$(1 - s)f(x + y) = sf(0) + (1 - s)f(x + y) \leq f(s \cdot 0 + (1 - s)(x + y)) = f(x),$$

and

$$sf(x + y) = sf(x + y) + (1 - s)f(0) \leq f(s(x + y) + (1 - s) \cdot 0) = f(y).$$

Adding the two equations, we have

$$f(x + y) = (1 - s)f(x + y) + sf(x + y) \leq f(x) + f(y).$$

□

Corollary 2.1. If ρ is a metric and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave and vanishing at 0, then $\rho \circ \eta$ is also a metric.

Definition 2.3 (Linear Metric Space)**.** A vector space V over the field $\mathbb{K}$ equipped with a metric ρ on V and a metric $|\cdot - \cdot|$ on $\mathbb{K}$ is a linear metric space if $(+) : V \times V \to V$ and $(\cdot) : \mathbb{K} \times V \to V$ are continuous with respect to the induced metric.

Proposition 2.2. Any normed space is a linear metric space.

Proof. Let $(x_n, y_n) \to (x, y)$ in V^2, then we have

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0.$$

Thus, $(+)$ is continuous.

Similarly, if $(\lambda_n, x_n) \to (\lambda, x)$ in $\mathbb{K} \times V$,

$$\begin{aligned} \|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x\| \\ &\leq \|\lambda_n x_n - \lambda_n x\| + \|\lambda_n x - \lambda x\| \\ &= |\lambda_n| \|x_n - x\| + |\lambda_n - \lambda| \|x\|. \end{aligned}$$

Now, since (λ_n) is convergent, it is bounded by some $M > 0$ and thus,

$$\|\lambda_n x_n - \lambda x\| \leq |\lambda_n| \|x_n - x\| + |\lambda_n - \lambda| \|x\| \leq M \|x_n - x\| + |\lambda_n - \lambda| \|x\| \to 0,$$

implying $(\cdot)$ is continuous. $\square$

2.1 Classical Spaces

We recall the L_p spaces from second year measure theory, and in particular, when we consider the counting measure μ, we have the nice property that

$$\int f \mathrm{d}\mu = \sum_{n=0}^{\infty} f(n),$$

and we no longer require a quotient to define the linear space as the only null-set is the empty set (thus, two function are a.e equal if and only if they are equal). In this special case, we call the resulting space ℓ_p with the p-norm

$$\|f\|_p = \left(\int |f|^p \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\sum_{n=0}^{\infty} |f(n)|^p \right)^{\frac{1}{p}}.$$

We will use the sequence notation and write $a_n := a(n)$ for $a \in \ell_p$.

As this is simply a special case of the L_p space, the inequalities proved on the L_p space remains. We will recall them here for ℓ_p spaces.

Proposition 2.3 (Hölder's Inequality)**.** Let $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q \in (1, \infty)$. Then for $a = (a_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}} \in \ell_p$, we have

$$|\langle a, b \rangle| \leq \|a\|_p \|b\|_q,$$

where $\langle a, b \rangle := \sum_{i \in \mathbb{N}} a_i b_i$.

Proposition 2.4 (Minkowski's Inequality)**.** Let $a, b \in \ell_p$ for some $1 \leq p \leq \infty$. Then $a + b \in \ell_p$ and

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p.$$

Lemma 2.1. Let $I \subseteq \mathbb{R}$ be an interval and $\phi : I \to \mathbb{R}^+$ be a function. Then ϕ is convex if and only if for all $y \in I$, there exists a $\gamma \in \mathbb{R}$, such that for all $x \in I$,

$$\gamma(x - y) \leq \phi(x) - \phi(y).$$

Proposition 2.5 (Jensen's Inequality)**.** let $\phi \geq 0$ be convex and suppose $\sum_{i \in \mathbb{N}} \eta_i = 1$, $|\langle \alpha \rangle| < \infty$ and $\langle \phi(\alpha) \rangle$ (where $\langle \beta \rangle = \sum_{i \in \mathbb{N}} \eta_i \beta_i$), then

$$\phi(\langle \alpha \rangle) \leq \langle \phi(\alpha) \rangle.$$

Proof. By the above lemma, there exists some γ such that

$$\gamma(\alpha_j - \langle \alpha \rangle) \leq \phi(\alpha_j) - \phi(\langle \alpha \rangle).$$

Thus,

$$\begin{aligned} 0 = \sum \eta_i \gamma(\alpha_j - \langle \alpha \rangle) &\leq \sum \eta_i (\phi(\alpha_j) - \phi(\langle \alpha \rangle)) \\ &= \sum \eta_i \phi(\alpha_j) - \phi(\langle \alpha \rangle) \sum \eta_i \\ &= \langle \phi(\alpha_j) \rangle - \phi(\langle \alpha \rangle) \end{aligned}$$

where the first equality follows as γ is independent of the index i. $\square$

Proposition 2.6. For $p < p'$, $\ell_p \subseteq \ell_{p'}$. On the other hand, if $\sum \eta_i = 1$, we have $\ell_p(\eta) \supseteq \ell_{p'}(\eta)$.

Definition 2.4. Let ϕ be a convex function such that $\phi(0) = 0$, $\phi(x) \to \infty$ as $x \to \infty$. If ϕ has the doubling property such that there exists some $M > 0$ such that for all $x \in \mathbb{R}$, $\phi(2|x|) \leq M\phi(|x|)$, then

$$V := \left\{ a : \mathbb{N} \to \mathbb{R} \mid \sum \eta_i \phi(|a_i|) < \infty \right\}$$

where $\sum \eta_i = 1$ is a vector space with point-wise operations.

Proposition 2.7. Given a metric space (X, ρ), there exists a complete metric space $(\tilde{X}, \tilde{\rho})$ and an isometric embedding $\iota : X \to \tilde{X}$ such that for all $x, x' \in X$, $\tilde{\rho}(\iota(x), \iota(x')) = \rho(x, x')$.

Proof. We have seen similar ideas in the completion of $\mathbb{Q}$ though completing the space with equivalence classes of mutually Cauchy sequences as elements of $\tilde{X}$. $\square$

As we have seen last year, the L_p spaces are complete, and thus are Banach spaces. Thus, we have ℓ_p spaces are also complete and are Banach spaces. In fact, the proof that ℓ_p spaces are complete is easier, in that one may show completeness through showing the point-wise limit of the sequences indeed belong to ℓ_p.

Definition 2.5. We define

- $c_0 := \{x \in \ell_\infty \mid \lim_{n \to \infty} x_n = 0\}$,
- $c := \{x \in \ell_\infty \mid \exists \lim_{n \to \infty} x_n\}$,

be subspaces of ℓ_∞.

Proposition 2.8. c_0 is complete.

Proof. Let $(x_i^n) \subseteq c_0$ be a Cauchy sequence and let $x_i = \lim_{n\to\infty} x_i^n$, then it suffices to show $\lim_{i\to\infty} x_i = 0$.

Since (x_i^n) is Cauchy, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|x^n - x^N\|_\infty < \frac{\epsilon}{2}.$$

Furthermore, as $x_i^N \to 0$ as $i \to \infty$, there exists some $I \in \mathbb{N}$ such that for all $i \geq I$, $|x_i^N| < \epsilon/2$. Thus, for all $i \geq I$, we have

$$\frac{\epsilon}{2} > \|x^n - x^N\|_\infty > |x_i^n - x_i^N| > |x_i^n| - \frac{\epsilon}{2} \implies \epsilon > |x_i^n|,$$

for all $n \geq N$. Hence, taking $n \to \infty$, we have

$$\epsilon > \lim_{n\to\infty} |x_i^n| = |x_i|,$$

implying $x_i \to 0$ as $i \to \infty$. $\square$

Proposition 2.9. c is complete.

Proof. Similar to above, let (x_i^n) be Cauchy and define (x^n) such that for all $n \in \mathbb{N}$, $x^n = \lim_{i\to\infty} x_i^n$ and (x_i) such that $(x_i^n) \to (x_i)$ in ℓ_∞. It is not difficult to see that (x^n) is Cauchy and thus converges to some x.

$$\begin{array}{ccc} (x_i^n) & \xrightarrow{n\to\infty} & (x_i) \\ {\scriptstyle i\to\infty}\downarrow & & \downarrow{\scriptstyle i\to\infty} \\ (x^n) & \xrightarrow{n\to\infty} & x \end{array}$$

Indeed, as (x_i^n) is Cauchy, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $p, q \geq N$, $\|x_i^p - x_i^q\|_\infty < \epsilon/3$. Furthermore, there exists some $I \in \mathbb{N}$ such that for all $j \geq I$, $|x_j^p - x^p|, |x_j^q - x^q| < \epsilon/3$. Thus,

$$|x^p - x^q| < |x_j^p - x^p| + |x_j^p - x_j^q| + |x_j^q - x^q| < \frac{\epsilon}{3} + \|x_i^p - x_i^q\|_\infty + \frac{\epsilon}{3} = \epsilon.$$

Hence, it remains to show $x_i \to x$ as $i \to \infty$. Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x^n - x| < \epsilon/3$. Furthermore, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $\|x_i^n - x_i\|_\infty < \epsilon/3$. Then, for all $p \geq \max\{N, M\}$, there exists $I \in \mathbb{N}$ such that for all $j \geq I$, $|x_j^p - x^p| < \epsilon/3$. Finally,

$$|x_j - x| \leq |x_j - x_j^p| + |x_j^p - x^p| + |x^p - x| \leq \|x_i - x_i^p\|_\infty + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,$$

implying $x_i \to x$ as $i \to \infty$ and $(x_i) \in c$. $\square$

2.1.1 Separability

In this section we will consider the separability of different classical spaces.

Proposition 2.10. The space ℓ_p is separable for all $p \geq 1$.

Proof. It is easy to see that the subspace of all finite sequences is dense in ℓ_p. Indeed, if $x \in \ell_p$ and x^n is the sequence such that $x_i^n = x_i$ for all $i \leq n$ and $x_i^n = 0$ for all $i > n$, we have

$$\|x - x^n\|_p^p = \sum_{i=0}^{\infty} |x_i - x_i^n|^p = \sum_{i=n+1}^{\infty} |x_i|^p$$

which tends to 0 as $n \to \infty$. Now, by defining $\ell_{\mathbb{Q},n}$ as the set of rational sequences with length n, we have $d := \bigcup_{n \in \mathbb{N}} \ell_{\mathbb{Q},n}$ is dense in the space of all finite sequences. Now, since d is countable as it is a countable union of countable sets, we have found a countable dense set of ℓ_p. $\square$

Lemma 2.2. A metric space is X not separable if there exists an uncountable subset S such that for some some $k > 0$, $d(x, y) > k$ for all $x, y \in S$.

Proof. As subset of a separable *metric* space is separable, S, must be separable. But as $d(x, y) > k$ for all $x, y \in S$, no sequences of S but the constant sequence can converge. Thus, for all countable subsets of S, simply picking a point of S not in that subset suffices. $\square$

Proposition 2.11. ℓ_∞ is not separable.

Proof. Clearly the set containing all sequences of only 0 and 1s is a subset of ℓ_∞. Furthermore, this sequence is uncountable as it bijects $\mathbb{R}$ be considering the binary representation of a real number. Thus, since all distinct elements in this set have distance 1 apart, the conclusion follows by the above lemma. $\square$

Proposition 2.12. $C^k([a, b])$ (the set of k-differentiable functions from the interval $[a, b]$) for all $k \in \mathbb{N}$ is separable.

Proof. Recalling the Weierstass approximation theorem, we have for all continuous function f, there exists a sequence of polynomials f_n such that $f_n \to f$ uniformly. Thus, as the set of polynomials with rational coefficients is countable, and dense in the space of all polynomials, we have found a countable dense set of $C^0([a, b])$. Now as $C^k([a, b]) \subseteq C^0([a, b])$ for all $k \in \mathbb{N}$, we have $C^k([a, b])$ is separable as required. $\square$

Definition 2.6 (Absolutely Convergent)**.** Let X be a normed space and let (x_n) be a sequence of X, then a series $\sum_{n \in \mathbb{N}} x_n$ is said to be absolutely convergent if $\sum_{n \in \mathbb{N}} \|x_n\| < \infty$.

Proposition 2.13. A normed space X is complete if and only if for all sequences (x_n) of X such that $\sum x_n$ is absolutely convergent implies x_n converges. Thus, a normed space is Banach if and only if this property is satisfied.

Proof. See problem sheet. $\square$

By recalling the proof of the completeness of L_p spaces, we note that this property was used extensively. As a consequence, we see that $L_p(\mu)$ is separable if the measure μ is separable. In particular, the spaces $L_p(\mathbb{R}^n, \lambda)$ are separable for all n. On the other hand, L_∞ is in general not separable.

2.2 Hamel and Schauder Basis

In this small section we will introduce two new notions of basis for infinite dimensional spaces. In particular, we will introduce the Hamel basis which is a natural extension of the definition of basis for the finite dimensional case and the Schauder basis which is a notion of basis that incorporates the topological properties of the space.

Definition 2.7 (Linear Independent)**.** A set $W \subseteq V$ is linear independent if for all $(\lambda_i)_{i=1}^m \subseteq \mathbb{K}$, $(w_i)_{i=1}^n \subseteq W$,

$$\sum \lambda_i w_i = 0 \implies \lambda_i = 0$$

for all $i = 1, \cdots, n$.

Definition 2.8 (Hamel Basis)**.** A set $W \subseteq V$ is a Hamel basis for a linear space V if W is linearly independent and for all $x \in V$, there exists a unique finite linear combination of vectors $(w_i)_{i=1}^n \subseteq W$, $(\lambda_i)_{i=1}^m \subseteq \mathbb{K}$ such that

$$x = \sum \lambda_i w_i.$$

Proposition 2.14. Every linear space has a Hamel basis.

We will come back to the proof of this proposition after discussing an important result known as the Hahn-Banach theorem.

Definition 2.9 (Schauder Basis)**.** A set $W \subseteq V$ is a Schauder basis for a normed space V if W is countable, linearly independent, and for all $x \in V$, there exists a unique (possibly infinite)-sequence $(\lambda_i)_{i=1}^\infty \subseteq \mathbb{K}$ and $(w_i)_{i=1}^\infty \subseteq W$ such that

$$x = \sum_{i=1}^\infty \lambda_i w_i.$$

Proposition 2.15. If a Banach space X has a Schauder basis, then it is separable.

The proof of the above proposition is left as an exercise. It is notable that the reverse of the above is not true and was in fact a Scottish book problem which an counterexample was given by Per Enflo in 1972 who was awarded a live goose.

It is clear that for $p \geq 1$, the set $W := \{e_j \mid j \in \mathbb{N}\}$ where $(e_j)_i = \delta_{ij}$ is a Schauder basis of ℓ_p.

Recall that $c \subseteq \ell_\infty$ is the set of sequences which has a limit. By considering the sequence $x_n = 1$, we see that the set of standard basis vectors no longer form a basis of c as

$$\left\| x - \sum_{i=1}^n e_i \right\|_\infty = 1,$$

for all n. Defining $W := \{e_0 := (1, 1, \cdots)\} \cup \{e_j \mid j \in \mathbb{N}\}$, for all $(a_n) \in c$, let $\lambda_0 := \lim_{n \to \infty} a_n$ and $\lambda_n = a_n - \lambda_0$. Then, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - \lambda_0| \leq \epsilon$. Thus,

$$\left\| \lambda_0 e_0 + \sum_{i=1}^n \lambda_i e_i - a \right\|_\infty = \|(0, \cdots, 0, \lambda_0 - a_n, \lambda_0 - a_{n+1}, \cdots)\|_\infty < \epsilon,$$

implying W is a Schauder basis of c.

Furthermore, one may show that $C([0,1])$ has a Schauder basis consisting of all the "spike" functions at the points $k2^{-n}$ for all $n \in \mathbb{N}$, $k = 1, \cdots 2^{-n}$.

2.3 Hilbert Spaces

Recall the definition of sesquilinear forms over some vector space $\mathbb{H}$.

Definition 2.10 (Sesquilinear Form)**.** A sesquilinear form $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ on $\mathbb{H}$ where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ is a function such that

- it is linear with respect to the second argument;
- and is conjugate symmetric.

We say $\langle \cdot, \cdot \rangle$ is nondegenerate if $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in \mathbb{H}$. Nondegenerate sesquilinear forms are called scalar products and a vector space equipped with a scalar product is called a unitary space (or a scalar product space).

We see that a scalar product induces a norm by defining $\|f\|^2 := \langle f, f \rangle$. In particular, we recall the $\ell_2, C([a,b])$ and L_2 are all scalar product spaces.

Proposition 2.16. Let $\mathbb{H}$ be a unitary space, then

- $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$ (parallelogram identity);
- $\langle f, g \rangle = \frac{1}{4}(\|f+g\|^2 - |f-g\|^2)$ if $\mathbb{K} = \mathbb{R}$ and $\langle f, g \rangle = \frac{1}{4}\sum_{k=0,\cdots 3} i^k \|f + i^k g\|^2$ if $\mathbb{K} = \mathbb{C}$ (polarisation identity).

Definition 2.11 (Hilbert Space)**.** A unitary space is a Hilbert space if it is complete with respect to its induced norm.

Definition 2.12 (Convex Set)**.** A set S is said to be convex if for all $x, y \in S$, $(1-t)x + ty \in S$ for all $t \in [0,1]$.

Proposition 2.17 (Nearest Point Property)**.** Every nonempty closed convex set $\mathcal{K}$ in a Hilbert space $\mathbb{H}$ contains a vector of the smallest norm. Moreover, if $h \in \mathbb{H}$, there exists a unique $h_0 \in \mathcal{K}$ such that

$$\|h - h_0\| = \operatorname{dist}(h, \mathcal{K}) := \inf_{k \in K} \|h - k\|.$$

Proof. By the definition of infimum, there exists a sequence $(k_n) \subseteq \mathcal{K}$ such that

$$\lim_{n \to \infty} \|k_n\| \to d := \inf_{k \in \mathbb{K}} \|k\|.$$

Consider, for all $n, m \in \mathbb{N}$, by the parallelogram identity,

$$\left\|\frac{1}{2}(k_n - k_m)\right\|^2 = \frac{1}{2}(\|k_n\|^2 + \|k_m\|^2) - \left\|\frac{1}{2}(k_n + k_m)\right\|^2.$$

Then, as $\mathbb{K}$ is convex, $\frac{1}{2}(k_n + k_m) \in \mathbb{K}$ and so $\|(k_n + k_m)/2\|^2 \geq d^2$ and

$$\left\|\frac{1}{2}(k_n - k_m)\right\|^2 \leq \frac{1}{2}(\|k_n\|^2 + \|k_m\|^2) - d^2.$$

Thus, taking $n, m \to \infty$, the right hand side tends to zero and so (k_n) is Cauchy and hence convergent. Now as $\mathcal{K}$ is closed, the limit is in $\mathcal{K}$ and hence the statement. $\square$