# Fourier Analysis Revision Notes

Kexing Ying

April 4, 2022

## Inner Product Spaces

We denote $R$ a (real or complex) inner product space (Euclidean space).

**Definition** (Complete System)**.** A system $\{X_\alpha\}_{\alpha \in A}$ is said to be complete if its linear closure is $R$, namely $\langle X_\alpha \mid \alpha \in A \rangle = R$.

**Definition** (Orthogonal Basis)**.** A system is an orthogonal basis if it is orthogonal and complete.

**Proposition.** If $R$ is separable, then any orthogonal system of $R$ is countable.

**Proposition.** Any separable real inner product space possesses a orthonormal basis.

**Definition** (Fourier Coefficients)**.** Given an orthonormal system $\{\phi_n\}_{n=1}^\infty$ of $R$. The Fourier coefficients of any $f \in R$ is defined to be

$$c_k := \langle f, \phi_k \rangle$$

for all $k$. The formal sum $\sum_{k=1}^\infty c_k \phi_k$ is called the Fourier series of $f$.

**Definition** (Closed System)**.** An othonormal system $\{\phi_n\}$ is closed if

$$\sum_{k=1}^\infty c_k^2 = \|f\|^2$$

for all $f \in R$. We call this property Parseval's identity.

**Proposition** (Bessel's Inequality)**.** Given the orthonormal system $\{\phi_n\}$ of $R$, we have

$$\sum_{k=1}^\infty |c_k|^2 \le \|f\|^2$$

for all $f \in R$.

**Theorem.** In a separable inner product space $R$, an orthonormal system is complete if and only if it is closed.

**Proposition.** Given $f, g$ and a closed system $\{\phi_n\}$ of $R$, $\langle f, g \rangle = \sum_{k=1}^\infty c_k^f c_k^g$ where $c_k^f, c_k^g$ are the Fourier coefficients of $f, g$ respectively.

## Contour Integration

**Proposition** (Jordan's Lemma)**.** If $f$ is holomorphic except for finitely many singularities, and $f(z) \to 0$ as $|z| \to \infty$, then

$$\int_{\gamma_R} f(z) e^{i\lambda z} \mathrm{d}z \to 0$$

for all $\lambda > 0$ where $\gamma_R$ is the upper half circle of radius $R$ centred at 0 oriented counter-clockwise.

In the case Jordan's lemma fails due to $\lambda < 0$, try integrating on the lower half circle.

**Proposition.** $e^{iz} = 1 + O(|z|)$. Useful for integrating on small contours.

## Fourier Series

Smoother functions have quicker decaying of Fourier coefficients.

**Proposition.** For all $f \in L^2[-\pi, \pi]$, $\|f - S_n\|_2 \to 0$ where $S_n$ is the $n$-th partial sum of the Fourier series of $f$.

**Theorem** (Dini's Condition for Pointwise Convergence)**.** If $f \in L^1[-\pi, \pi]$ and for any $x \in [-\pi, \pi]$, there exists some $\delta > 0$ such that

$$\int_{[-\delta,\delta]} \left| \frac{f(x+t) - f(x)}{t} \right| \lambda(\mathrm{d}t) < \infty$$

exists, then $S_n(x) \to f(x)$ as $n \to \infty$ for all $x$.

Dini's condition is in some sense as strong as possible. Indeed, if $\frac{f(x+t)-f(x)}{t}$ is not locally integrable at some $x$, we can find a continuous function $g$ with $|g| \leq f$ with non-convergence Fourier series at $x$.

If $f$ is continuous at $x$ and has a derivative at $x$ (or the limit exists from either the left or the right), then Dini's condition is satisfied at $x$.

A continuous function with period $2\pi$ is uniquely determined by its Fourier series. Furthermore, we can reconstruct a continuous function from its Fourier series by using the Fejer sums. Indeed, denoting $\sigma_n$ the $n$-th Fejer sum, $\sigma_n \to f$ uniformly.

**Proposition** (Poisson Summation Formula)**.** Given $f \in L^1$,

$$2\pi t \sum_{n=-\infty}^{\infty} f(2\pi t n) = \sum_{n=-\infty}^{\infty} \mathcal{F}[f](n/t).$$

## Fourier Transform

**Proposition.** Let $f \in L^1$. Then $\mathcal{F}[f] = 0$ implies $f = 0$ almost everywhere.

**Proposition.** For $f, f_n \in L^1$ such that $f_n \to f$ in $L^1$, then $\mathcal{F}[f_n] \to \mathcal{F}[f]$ uniformly.

**Proposition.** For $f \in L^1$, $\mathcal{F}[f](y) \to 0$ as $|y| \to \infty$.

**Corollary.** For $f \in L^1$, $\mathcal{F}[f]$ is uniformly continuous.

**Proposition.** For $f \in L^1$ differentiable with $f' \in L^1$ and $f$ absolutely continuous on any finite interval, $\mathcal{F}[f'](y) = iy\mathcal{F}[f](y)$.

**Proposition.** For $f \in L^1$ such that $xf(x) \in L^1$, we have $\mathcal{F}[f]$ is differentiable and $D_y\mathcal{F}[f](y) = \mathcal{F}[-ixf(x)]$.

We also have the following properties: for $f, g \in L^1$, $c, c_1, c_2 \in \mathbb{R}$,

- Linearity: $\mathcal{F}[c_1 f + c_2 g] = c_1\mathcal{F}[f] + c_2\mathcal{F}[g]$.
- Translation: $\mathcal{F}[x \mapsto f(x-a)](y) = e^{-iay}\mathcal{F}[f](y)$.
- Rephasing: $\mathcal{F}[x \mapsto e^{-icx}f(x)](y) = \mathcal{F}[f](y+c)$.
- Scaling: $\mathcal{F}[x \mapsto f(cx)](y) = \frac{1}{|c|}\mathcal{F}[f](y/c)$.
- Convolution: $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$.