

Algebraic Topology

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February 26, 2022

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1 Introduction

Let us introduce/recall some basic definitions which will be used throughout this course.

Definition 1.1 (Path). A path in a topological space X is a continuous map $\gamma : [0, 1] \subseteq \mathbb{R} \rightarrow X$. In the case that $\gamma(0) = \gamma(1)$, we call γ a loop/closed path.

Definition 1.2 (Homotopy). Given two paths $\gamma_0, \gamma_1 : [0, 1] \rightarrow X$ with the same end points (i.e. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) are said to be homotopic with fixed endpoints if there exists a continuous map

$$H : [0, 1] \times [0, 1] \rightarrow X$$

such that

- $H(t, 0) = \gamma_0(t)$ for all $t \in [0, 1]$;
- $H(t, 1) = \gamma_1(t)$ for all $t \in [0, 1]$;
- for all $u \in [0, 1]$, $H(0, u) = \gamma_0(0) = \gamma_1(0)$ and $H(1, u) = \gamma_0(1) = \gamma_1(1)$.

Thus, graphically, two paths are homotopic if you can continuously deform a path into the other without moving the starting and ending points (see second year complex analysis for more details).

Definition 1.3 (Free Homotopy). The loops γ_0, γ_1 in X is said to be freely homotopic if there exists a continuous $H : [0, 1] \times [0, 1] \rightarrow X$ such that

- $H(t, 0) = \gamma_0(t)$ for all $t \in [0, 1]$;
- $H(t, 1) = \gamma_1(t)$ for all $t \in [0, 1]$;
- for all $u \in [0, 1]$, $H(0, u) = H(1, u)$.

Definition 1.4 (Simply Connected). X is said to be simply connected if any loop in X is freely homotopic to a constant loop.

Thus, informally, in a simply connected space, any loop can be contracted into a single point.

Proposition 1.1. S^2 is simply connected.

Simply connectedness is an important notion and relates to many difficult problems in geometry.

Theorem 1. S^2 and \mathbb{R}^2 are, up to homeomorphism, the only two simply-connected 2-dimensional manifolds.

Theorem 2 (Poincaré Conjecture). The only compact, simply connected 3-dimensional manifold is the sphere S^3 (up to homeomorphism).

1.1 The Torus

An important example in algebraic topology is the torus. We will now provide a proof that the torus is not simply connected.

Definition 1.5 (Torus). The 2-torus T^2 is the product topological space $S^1 \times S^1$.

We will now provide an alternative method of constructing the torus. Define the homeomorphisms

$$T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1 + 1, x_2); T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1, x_2 + 1).$$

It is clear that T_1 and T_2 commutes and the map

$$\psi : \mathbb{Z}^2 \rightarrow \text{Homeo}(\mathbb{R}^2) : (n, m) \mapsto T_1^n \circ T_2^m$$

is a group homomorphism. As this map is injective, we have in some sense embedded \mathbb{Z} inside of $\text{Homeo}(\mathbb{R}^2)$. Now, defining the equivalence relation on \mathbb{R}^2 by

$$(x_1, x_2) \sim (y_1, y_2) \iff \exists (n, m) \in \mathbb{Z}^2, \psi(n, m)(x_1, x_2) = (y_1, y_2),$$

or equivalently, there exists $(n, m) \in \mathbb{Z}^2$ such that $(x_1 + n, x_2 + m) = (y_1, y_2)$, we define the quotient topology $X := \mathbb{R}^2 / \sim$.

Lemma 1.1. Let X and Y be topological spaces and \sim be an equivalence relation on X . Then, if $p : X \rightarrow X / \sim : x \mapsto [x]_\sim$ is the quotient map, any map $f : X / \sim \rightarrow Y$ is continuous if and only if $f \circ p$ is continuous.

Proof. Exercise. □

We see that the above lemma together with the universal property for quotients provides the universal property for topological spaces. Namely, if $f : X \rightarrow Y$ is a continuous map and for all $x \sim y \in X$, $f(x) = f(y)$, then the unique map \tilde{f} obtained such that $\tilde{f} \circ p = f$ is continuous.

Lemma 1.2. Let X be a compact space and Y Hausdorff. Then, any continuous bijective map $f : X \rightarrow Y$ is a homeomorphism.

Proof. Exercise. □

Proposition 1.2. X is homeomorphic to the torus T^2 .

Proof. Define the map $\pi : \mathbb{R}^2 \rightarrow S^1 \times S^1$ such that $\pi(x, y) = (e^{2\pi i x}, e^{2\pi i y})$ for all $(x, y) \in \mathbb{R}^2$. We observe that $\pi(x, y) = \pi(u, v)$ if and only if $(x, y) \sim (u, v)$, and so, by the universal property for topological spaces, we obtain the unique continuous map defined by

$$\tilde{\pi} : X \rightarrow S^1 \times S^1 : [(x, y)] \mapsto \pi(x, y).$$

This map is clearly bijective and continuous by the universal property, and thus, by the above lemma, it suffices to show X is compact. But, this is clear since the $p([0, 1]^2) = \mathbb{R}^2 / \sim$ and the continuous image of a compact set is compact. □

While the map π as described above is not injective, it is locally so (exercise). Thus, given a path on the torus, we may think of lifting it to \mathbb{R}^2 by lifting the paths piecewise via. the local homeomorphisms induced by π .

Lemma 1.3 (Pasting Lemma). Let X, Y be both open subsets of a topological space and let B be another topological space, then $f : X \cup Y \rightarrow B$ is continuous if and only if $f|_A$ and $f|_B$ are continuous.

Proof. Exercise. □

Corollary 2.1. If $f_1 : X \rightarrow B, f_2 : Y \rightarrow B$ are continuous and agree on $X \cap Y$, then the map

$$f : X \cup Y \rightarrow B : x \mapsto \begin{cases} f_1(x) & \text{if } x \in X \\ f_2(x) & \text{otherwise} \end{cases}$$

is continuous.

Proposition 1.3 (Lifting of Paths). For any $\gamma : [0, 1] \rightarrow T^2$ a path, $\tilde{x} \in \mathbb{R}^2$ such that $\pi(\tilde{x}) =: x = \gamma(0)$, there exists a unique path $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{R}^2$ such that $\gamma = \pi \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = \tilde{x}$.

Proof. As π restricts to local homeomorphisms, we obtain an open cover of the path. Invoking compactness, we obtain a finite subcover for which we may lift the path locally such that the paths are compatible on intersections. With this, we obtain the required path by the pasting lemma.

For uniqueness, we observe that if $\pi \circ \tilde{\gamma}_1 = \pi \circ \tilde{\gamma}_2$, then, for all $t, \pi(\tilde{\gamma}_1(t)) = \pi(\tilde{\gamma}_2(t))$ and hence, $\tilde{\gamma}_1(t) \sim \tilde{\gamma}_2(t)$. Thus, the map

$$\delta := \tilde{\gamma}_1 - \tilde{\gamma}_2$$

must take value in \mathbb{Z}^2 , and so, is a constant as continuous maps are constant on connected components. Now, as $\tilde{\gamma}_1(0) = \tilde{x} = \tilde{\gamma}_2(0)$ and so, $\delta = 0$ and $\tilde{\gamma}_1 = \tilde{\gamma}_2$. □

Proposition 1.4 (Free Homotopy Classes of Loops on T^2). Given a loop $\gamma : [0, 1] \rightarrow T^2$ and its lift onto \mathbb{R}^2 the number

$$\rho(\gamma) := \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^2$$

is well-defined and for all γ_1 freely homotopic to γ_2 , $\rho(\gamma_1) = \rho(\gamma_2)$.

Proof. Since for a loop, $\gamma(0) = \gamma(1)$, $\tilde{\gamma}(0) - \tilde{\gamma}(1) \in \mathbb{Z}^2$ is well-defined since two lifts differ only by a constant.

Suppose now γ_1, γ_2 are two freely homotopic loops on the torus, i.e. there exists some continuous map $H : [0, 1] \times [0, 1] \rightarrow T^2$ such that

$$H(0, \cdot) = \gamma_0; H(1, \cdot) = \gamma_1$$

and for all $u \in [0, 1]$, the map $t \mapsto H(u, t)$ is closed. Let $\tilde{x}_0 \in \pi^{-1}(\gamma_0(0))$ and consider the map

$$\delta : [0, 1] \rightarrow T^2 : t \mapsto H(t, 0),$$

(i.e. the path of base points of the free homotopy) let $\tilde{\delta}$ to be the lift of δ starting at \tilde{x}_0 . Now, define $\tilde{\gamma}_t$ to be the lift of $u \mapsto H(t, u) =: \gamma_t$ based at $\tilde{\delta}(t)$, I claim, $\tilde{H} : [0, 1]^2 \rightarrow \mathbb{R}^2 : (t, u) \mapsto \tilde{\gamma}_t(u)$ is a free homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$. Thus, as $t \mapsto \rho(\gamma_t) = \tilde{\gamma}_t(1) - \tilde{\gamma}_t(0)$ is continuous and take value in \mathbb{Z}^2 , it must be a constant, concluding the proof. □

Suppose now we denote $P := \{\text{loops on } T^2\}$ and \sim the freely homotopic equivalence relation on P , we have the following proposition.

Proposition 1.5. The map $\rho : L := P / \sim \rightarrow \mathbb{Z}^2 : [\gamma] \mapsto \rho(\gamma)$ is a bijection.

Proof. Surjectivity follows by considering the loop $\gamma : t \mapsto \pi(tu, tm)$ for all $(n, m) \in \mathbb{Z}^2$. Then, $\rho(\gamma) = (n, m)$ and hence the map is surjective.

Suppose on the other hand γ_0, γ_1 are loops on the torus such that $\rho(\gamma_0) = \rho(\gamma_1)$, injectivity follows by showing the loops are freely homotopic. Let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be lifts of γ_0 and γ_1 respectively. Then, define the homotopy from γ_0 to γ_1 by

$$\tilde{H}(t, u) := (1 - t)\tilde{\gamma}_0(u) + t\tilde{\gamma}_1,$$

and define $H = \pi \circ \tilde{H}$. For all t , $H(t, \cdot)$ is a closed path since

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = (1 - t)(\tilde{\gamma}_0(1) - \tilde{\gamma}_0(0)) + t(\tilde{\gamma}_1(1) - \tilde{\gamma}_1(0)) = (1 - t)\rho(\gamma_0) + t\rho(\gamma_1).$$

By assumption, we have $\rho(\gamma_0) = \rho(\gamma_1)$ and so,

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = \rho(\gamma_0) \in \mathbb{Z}^2$$

implying that the loop is closed. Hence, H is a free homotopy between γ_0 and γ_1 as required. \square

Theorem 3. The torus T^2 is not simply connected.

Proof. If the loop γ on the torus is freely homotopic to the constant path, $\rho(\gamma) = \rho(e) = 0$. But we have provided examples where this is not the case, and hence, not all loops are freely homotopic to a constant path. \square

This procedure for proving a space is not simply connected will be common. In particular, we will provide a proposition for situations where we have a space X , a simply connected space \tilde{X} and a group Γ characterising the lack of simply connectedness of X .

2 Fundamental Group

2.1 Definition

Given a topological space X , we will consider the set of all loops on X quotiented by the homotopy relation. Then, by equipping this quotient with the operation of gluing paths together, we obtain a group on this quotient. This group is known as the Fundamental group.

Definition 2.1 (Concatenation of Paths). Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$ be paths such that $\gamma_1(1) = \gamma_2(0)$, then, we define the path

$$\gamma_1 * \gamma_2 : [0, 1] \rightarrow X : t \mapsto \begin{cases} \gamma_1(2t), & t \leq 1/2, \\ \gamma_2(2t - 1), & t > 1/2. \end{cases}$$

It is not difficult to see that this is continuous and hence is a path.

Definition 2.2. We define the equivalence relation \sim on the space of paths with the same end points such that for $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$ with the same end points, $\gamma_1 \sim \gamma_2$ if and only if γ_1 is homotopic to γ_2 .

Definition 2.3 (Fundamental Group). Let X be a path-connected space and let $x_0 \in X$. Then, we define the fundamental group as

$$\pi_1(X, x_0) := \{\text{loops at } x_0\} / \sim.$$

We would like to equip the above quotient with a group structure using the concatenation operation. To achieve this, we need to check that, for loops at x_0 , γ_1 and γ_2 , we have

$$[\gamma_1 * \gamma_2]_{\sim} = [\gamma_1]_{\sim} * [\gamma_2]_{\sim}$$

and the equivalence class is compatible to concatenation independently of the end representative of the equivalence class.

Proposition 2.1. Let γ_1, γ'_1 be paths from x to y and let γ_2, γ'_2 be paths from y to z , then if $\gamma_1 \sim \gamma'_1$ and $\gamma_2 \sim \gamma'_2$ then $\gamma_1 * \gamma_2 \sim \gamma'_1 * \gamma'_2$.

Proof. Concatenate the homotopies. Namely, if H_1 is a homotopy between γ_1 and γ'_1 and H_2 is a homotopy between γ_2 and γ'_2 , then define $H(t, u) := H_1(t, u) * H_2(t, u)$. \square

With the above proposition, we define the group operation on $\pi_1(X, x_0)$ such that $[\gamma_1] * [\gamma_2] := [\gamma_1 * \gamma_2]$.

Lemma 2.1. Let $\gamma : [0, 1] \rightarrow X$ be a path and let $\phi : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $\phi(0) = 0$ and $\phi(1) = 1$ (such ϕ is known as a reparametrisation. Then, $\gamma \circ \phi \sim \gamma$.

Proof. We define the homotopy $H(t, u) := \gamma((1 - u)t + u\phi(t))$. H is clearly continuous, $H(0, u) = \gamma(0)$, $H(1, u) = \gamma(1)$ and $\gamma(t) = H(t, 0)$, $\gamma(\phi(t)) = H(t, 1)$. \square

Proposition 2.2. $(\pi_1(X, x_0), *)$ form a group.

Proof. The identity element of this group is the class of the constant loop $\text{id} : t \mapsto x_0$. Indeed, for any loops γ at x_0 , we have

$$\text{id} * \gamma : t \mapsto \begin{cases} \gamma(2t), & t \leq 1/2, \\ x_0, & t > 1/2. \end{cases}$$

Thus, we may construct the homotopy between γ and $\text{id} * \gamma$ by defining

$$H(t, u) := \begin{cases} \gamma((1+u)t), & t \leq 1/(1+u), \\ x_0, & t > 1/(1+u). \end{cases}$$

This is a homotopy by the glueing lemma and thus, $[\text{id} * \gamma] = [\gamma]$. Alternatively, we observe $\text{id} * \gamma$ is a reparametrisation of γ with $\phi(t) = 2t$ for $t \leq 1/2$ and $\phi(t) = 1$ for all $t > 1/2$.

It is to check that, with the above definition of the identity, the inverse of the loop $[\gamma]$ is simply $[t \mapsto \gamma(1-t)]$. Thus, it remains to check associativity.

Let $\gamma_1, \gamma_2, \gamma_3$ be loops at x_0 . We note that $(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3)$ though the two paths remain to be homotopic. Indeed, we see that the two paths are simply reparametrisations of each other with

$$\phi : [0, 1] \rightarrow [0, 1] : t \mapsto \begin{cases} 2t, & t \leq 1/4, \\ t + 1/4, & 1/4 < t \leq 3/4, \\ t/2 + 3/4, & t > 3/4, \end{cases}$$

such that $(\gamma_1 * \gamma_2) * \gamma_3 \circ \phi = \gamma_1 * (\gamma_2 * \gamma_3)$. Hence, by the above lemma, the two paths are homotopic and so,

$$([\gamma_1] * [\gamma_2]) * [\gamma_3] = [\gamma_1] * ([\gamma_2] * [\gamma_3]),$$

as required. \square

Proposition 2.3. Let $x_0, x_1 \in X$, and let $\delta : [0, 1] \rightarrow X$ be a path from x_0 to x_1 . Then δ induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ by

$$[\gamma] \mapsto [\delta^{-1} * \gamma * \delta].$$

Proof. Clearly, for the constant path, $\delta^{-1} * \text{id} * \delta$ is a reparametrisation of $\delta^{-1} * \delta$ which is homotopic to id .

Now, given γ_1, γ_2 loops at x_0 ,

$$\begin{aligned} [\delta^{-1} * \gamma_1 * \gamma_2 * \delta] &= [\delta^{-1} * \gamma_1] * [\gamma_2 * \delta] \\ &= [\delta^{-1} * \gamma_1] * [\delta * \delta^{-1}] * [\gamma_2 * \delta] \\ &= [\delta^{-1} * \gamma_1 * \delta] * [\delta^{-1} * \gamma_2 * \delta]. \end{aligned}$$

Finally, as δ^{-1} by symmetry induces a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$ and these two homomorphisms are inverses, the induced map is an isomorphism as required. \square

With the above proposition in mind, we see that the fundamental group of a space is independent (up to isomorphism) of the choice of the base point. So, we have also that a space is simply connected if and only if its fundamental group is trivial.

2.2 Covering Spaces

In the section we generalize the method introduced for toruses to general spaces.

Definition 2.4 (Covering Map). A map $\pi : \tilde{X} \rightarrow X$ is a covering map if there exists an open cover $(U_\alpha)_{\alpha \in A}$ of X for all $\alpha \in A$, $\pi^{-1}(U_\alpha)$ is a disjoint union of open sets of \tilde{X} each of which is homeomorphic to U_α with π (i.e. the π restricts on such an open set is homeomorphic to U_α).

Given a covering map π , we call X the base space and \tilde{X} the covering map.

In some sense, the covering map provides a local homeomorphism for some specific open cover which each open set of the cover is represented in the domain as disjoint copies.

Proposition 2.4. If $\pi : \tilde{X} \rightarrow X$ is a covering map for some connected X , then the cardinalities of the fibres of π is constant. Namely the map

$$s : X \rightarrow \overline{\mathbb{N}} : x \mapsto \#\pi^{-1}(\{x\})$$

is constant. We call this constant the number of sheets of π .

Proof. It is clear that for $x, y \in U_\alpha$ where $\alpha \in A$, the fibres of x and y have the same cardinalities. Thus, as $(U_\alpha)_{\alpha \in A}$ is an open cover, for all $x \in X$,

$$s^{-1}\{s(x)\} = \bigcup_{\substack{\exists x' \in U_\alpha, \\ s(x')=s(x)}} U_\alpha.$$

Hence, $s^{-1}\{s(x)\}$ is open. Then, then we have the disjoint open cover of X

$$\{s^{-1}\{s(x)\} \mid x \in X\}.$$

But, since X is connected, any disjoint open cover of X can have at most 1 element, and thus, s is a constant as required. \square

Proposition 2.5. Suppose $\pi : \tilde{X} \rightarrow X$ be a covering map and let Y be a topological space and $f : [0, 1] \times Y \rightarrow X$ is a continuous map such that there exists some $\tilde{f}_0 : Y \rightarrow \tilde{X}$ such that $\pi \circ \tilde{f}_0 = f(0, \cdot)$. Then, there exists a unique $\tilde{f} : [0, 1] \times Y \rightarrow \tilde{X}$ such that $\tilde{f}(0, \cdot) = \tilde{f}_0$ and $\pi \circ \tilde{f} = f$.

Proof. For all $y \in Y$, there exists a neighbourhood N of y such that for all $t \in [0, 1]$, there exists some $I_t \subseteq [0, 1]$ containing t such that $f(I_t \times N) \subseteq U_\alpha$ for some α since f is continuous on the compact set $[0, 1]$.

With this in mind, we can build the lift on $[0, 1] \times N$. for $\{0\} \times N$, by assumption, the lift must be \tilde{f}_0 . By the construction of N , there exists some U_α such that $f(\{0\} \times N) \subseteq U_\alpha$. Then, by continuity, there exists a neighbourhood $I_0 \subseteq [0, 1]$ such that $f(I_0 \times N) \subseteq U_\alpha$ and so, we may define

$$\tilde{f} : N \times I_0 \rightarrow \tilde{X} : (t, y) \mapsto (\pi|_{\tilde{U}_\alpha(\beta)})^{-1} \circ f$$

as π^{-1} is a homeomorphism on $\tilde{U}_\alpha(\beta)$ where $\pi^{-1}(U_\alpha) = \bigsqcup_{\beta \in B} \tilde{U}_\alpha(\beta)$.

Now, by the compactness of $[0, 1]$, there exists

$$0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1,$$

such that for all $k = 0, \dots, n$, $f([t_k, t_{k+1}] \times N) \subseteq U_{\alpha_k}$ for some $\alpha_k \in A$. Now, by induction, we may extend \tilde{f} from $[0, t_k]$ to $[0, t_{k+1}]$ resulting in an extension of \tilde{f} to the whole set of $[0, 1]$.

Suppose now both \tilde{f}, \bar{f} lifts f with $\tilde{f}(0, \cdot) = \bar{f}(0, \cdot) = \tilde{f}_0$. Let

$$A := \{z \in [0, 1] \times N \mid \tilde{f}(z) = \bar{f}(z)\}.$$

Then, for all $z \in A$, there exists some $V_z \subseteq N \times [0, 1]$ such that $\tilde{f}(V_z), \bar{f}(V_z) \subseteq \tilde{U}_\alpha(\beta)$. On V_z , it is clear that $\tilde{f} = \bar{f} = (\pi|_{\tilde{U}_\alpha(\beta)})^{-1} \circ f$. Now, since N is a union of connected components, we obtain uniqueness. \square

2.3 Induced Maps

In the case that we have a map between topological spaces, we can define an induced map on their fundamental group by composing the map with the loops.

Definition 2.5 (Induced Map). Let X, Y be topological spaces and let x, y be elements of X and Y respectively, then if $f : X \rightarrow Y$ is a continuous map such that $f(x) = y$, then, the induced map f_* is defined to be

$$f_* : \pi_1(X, x) \rightarrow \pi_1(Y, y) : [\gamma] \mapsto [f \circ \gamma].$$

It is clear that this map is well-defined since if H is a homotopy between γ_1, γ_2 loops in X based at x , then $f \circ H$ is a homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$.

Proposition 2.6. The induced map $f_* : \pi_1(X, x) \rightarrow \pi_1(Y, y)$ of f is a group homomorphism.

Proof. Clearly, $f \circ \text{id}$ is the constant loop based at y and thus, f_* maps the identity to the identity.

Now, let γ_1, γ_2 be loops based at x , it suffices to show that $(f \circ \gamma_1) * (f \circ \gamma_2)$ is homotopic to $f \circ (\gamma_1 * \gamma_2)$. But, in fact, the two paths above are equal. Hence, homotopic and thus, f_* is a group homomorphism as required. \square

Proposition 2.7. Let $\pi : \tilde{X} \rightarrow X$ be a covering space and suppose $\pi(\tilde{x}) = x$ for some $\tilde{x} \in \tilde{X}$ and $x \in X$. Then, the induced map of π ,

$$\pi_* : \pi_1(\tilde{X}, \tilde{x}) \rightarrow \pi_1(X, x)$$

is injective.

Since π_* is an injective group homomorphism, in some sense, the covering space allows us to consider a fundamental group of the covering space as a subgroup of the fundamental group of the original space.

In this sense, the induced map of the covering map provides a correspondence between the covering space (quotiented by some relation) and the subgroups of the fundamental group.

Proof. We will show that $\ker \pi_*$ is trivial. Suppose γ is a loop based at \tilde{x} such that $\pi \circ \gamma$ is trivial and let H be the homotopy between $\pi \circ \gamma$ and the constant loop based at x . We will lift H along π .

Let $\tilde{H} : [0, 1] \times [0, 1] \rightarrow \tilde{X}$ be the lift of H such that $\gamma(t) = \tilde{H}(0, t)$ and $\pi \circ \tilde{H} = H$. As π is locally homeomorphic and \tilde{H} is continuous, as $\tilde{H}(0, 0) = \tilde{H}(0, 1) = \gamma(0) = \tilde{x}$, we have $\tilde{H}(u, 0) = \tilde{H}(u, 1) = \tilde{x}$. Now, by continuity, $\tilde{H}(1, t)$ is constant with $H(1, t) = x$ we have $\tilde{H}(1, t) = \tilde{x}$ and thus, \tilde{H} is a homotopy between γ and the constant path as required. \square

An important example is the covering map $\pi : S^1 \rightarrow S^1 : z \mapsto z^n$, then the induced map $\pi_* : \mathbb{Z} \rightarrow \mathbb{Z}$ (as $\pi_1(S^1, 1)$ is isomorphic to \mathbb{Z}) is the map $k \mapsto nk$.

2.4 Universal Covers

Definition 2.6 (Universal Cover). Let X be a topological space. A universal cover of X is a space \tilde{X} that covers X (i.e. there exists a covering map $\pi : \tilde{X} \rightarrow X$) and is simply connected.

Theorem 4. If a topological space X is path connected, locally path connected and locally simply connected, then X has a unique universal cover.

We say a space has a property locally if every basis of neighbourhood has that property.

Although this is a powerful theorem, without any specific construction of the universal cover, we will not be able to make much conclusion about the space. Therefore, this theorem is not very useful for our purpose. We will provide universal covers for most spaces we work with explicitly. For this reason, we will only provide the proof for the uniqueness.

Lemma 2.2. Let Y be a topological space that is simply connected and locally path connected and suppose $\pi : \bar{X} \rightarrow X$ is a covering map, $f : Y \rightarrow X$ be continuous such that $f(y) = \pi(\bar{x}) = x$ for some $y \in Y, \bar{x} \in \bar{X}, x \in X$. Then, there exists a continuous unique map $\bar{f} : Y \rightarrow \bar{X}$ such that $\pi \circ \bar{f} = f$. i.e. the following diagram commutes

$$\begin{array}{ccc} & & \bar{X} \ni \bar{x} \\ & \nearrow \exists! \bar{f} & \downarrow \pi \\ y \in Y & \xrightarrow{f} & X \ni x \end{array}$$

Proof. For all $y' \in Y$, let γ a path from y to y' , take $\bar{f}(y')$ be the end point of the unique lift of $f \circ \gamma$ starting at \bar{x} . This is well-defined since if γ' is another path from y to y' , then γ' is homotopic to γ and thus, $f \circ \gamma$ and $f \circ \gamma'$ are homotopic and hence, their lifts are also homotopic with the same end points.

We will now show \bar{f} is continuous. Let $U \subseteq X$ be an open neighbourhood of $f(y')$ such that $\pi^{-1}(U)$ is a disjoint union of sets $\{U_\alpha\}_{I'}$ such that, on each copy, the restriction of π form a homeomorphism onto U . Suppose \bar{U} is one such copy and let V be a sufficiently small neighbourhood of y' such that $V \subseteq f^{-1}(U)$. I claim $V \subseteq \bar{f}^{-1}(\bar{U})$. Indeed, any paths from y to y' can be concatenated to a path with end point y'' in V . By definition, $f(y'')$ is the end point of the lift of this concatenated path composed with f based at \bar{x} , and thus, since

$\bar{f}(y') \in \bar{U}$, the lift remains in U and so $y'' \in V$. Hence, as \bar{U} form a basis of open sets of \bar{X} , \bar{f} is continuous.

Finally, to show uniqueness, we observe that, for any $y' \in Y$ and a path γ from y to y' , $\bar{f} \circ \gamma = \pi \circ \bar{f} \circ \gamma$, i.e. $\bar{f} \circ \gamma$ equals the lift of $f \circ \gamma$ based at some point. Now, as we require $\bar{f}(y) = \bar{x}$, this lift must be based at \bar{x} and hence uniqueness follows by the uniqueness of the lift. \square

Proposition 2.8. If $\pi_1 : \tilde{X}_1 \rightarrow X$ and $\pi_2 : \tilde{X}_2 \rightarrow X$ are covering maps from simply connected spaces and X is path-connected such that for some $x \in X, \tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$, $\pi_1(\tilde{x}_1) = x = \pi_2(\tilde{x}_2)$, then there exists a unique homeomorphism $\phi : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\exists! \phi} & \tilde{X}_2 \\ & \searrow \pi_1 \quad \swarrow \pi_2 & \\ & X & \end{array}$$

Proof. Take ϕ to be the lift of π_1 to \tilde{X}_2 and one may check that the lift of π_2 to \tilde{X}_1 provides the inverse. \square

Proposition 2.9. Let $\pi : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be a universal covering map where X is locally path-connected and let $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be another covering map. Then, there exists a unique $f : (\tilde{X}, \tilde{x}) \rightarrow (\tilde{X}, \tilde{x})$ such that $\pi = p \circ f$. i.e. the following diagram commutes.

Proof. Clear by lifting π to $\tilde{X} \rightarrow \tilde{X}$. \square

2.5 Classification of Covering Maps

Let us recall some definitions from group theory.

Definition 2.7 (Group Action). An action of a group Γ on a topological space X is a group homomorphism $\rho : \Gamma \rightarrow \text{Homeo}(X)$.

Equivalently, an action is a map

$$\rho : \Gamma \times X \rightarrow X : (\gamma, x) \mapsto \gamma \cdot x$$

is continuous whenever Γ is endowed with the discrete topology and for all $\gamma_1, \gamma_2 \in \Gamma$,

- $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x$;
- $e \cdot x = x$.

Definition 2.8. A group action from Γ to X is said to be “nice” if for all $x \in X$, there exists U_x a neighbourhood of x such that for all $\gamma \neq e$, $\gamma \cdot U_x \cap U_x = \emptyset$.

Proposition 2.10. If Γ acts nicely on X , then for all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, there exists some U_x a neighbourhood of x such that $\gamma_1 \cdot U_x \cap \gamma_2 \cdot U_x = \emptyset$.

Proof. Since Γ acts nicely on X , there exists some U_x a neighbourhood of x such that $\gamma_2^{-1} \gamma_1 \cdot U_x \cap U_x = \emptyset$. Thus, $\emptyset = \gamma_2 \cdot (\gamma_2^{-1} \gamma_1 \cdot U_x) \cap \gamma_2 \cdot U_x = \gamma_1 \cdot U_x \cap \gamma_2 \cdot U_x$ as required. \square

Let us introduce the equivalence relation where $x \sim y$ if and only if there exists some $\gamma \in \Gamma$ such that $\gamma \cdot x = y$. We denote the quotient topology by this equivalence relation by X/Γ .

Theorem 5. Assume that Γ acts nicely on X . Then, the projection map

$$\pi : X \rightarrow X/\Gamma$$

is a covering map.

Proof. Let $x \in X$ and take U_x to be a neighbourhood such that for all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, $\gamma_1 \cdot U_x \cap \gamma_2 \cdot U_x = \emptyset$. By the definition of the equivalence relation, $\pi(\gamma_1 \cdot U_x) = \pi(\gamma_2 \cdot U_x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and so, taking $V := \pi(U_x)$, $\pi^{-1}(V) = \bigsqcup_{\gamma \in \Gamma} \gamma \cdot U_x$ and it remains to show $\pi|_{\gamma \cdot U_x}$ is a homeomorphism for all γ . This is however clear since it is continuous and has the continuous inverse (since the action is continuous)

$$u \in U_x \mapsto \gamma \cdot u \in \gamma \cdot U_x.$$

Hence, taking the open cover to be $\{U_x\}_{x \in X}$, we have π is a covering map. \square

Proposition 2.11. Let X be simply connected, path connected and locally simply and path connected and suppose Γ acts nicely on X . Then,

- X is the universal cover of X/Γ ,
- for all $[x] \in X/\Gamma$, $\Gamma \simeq \pi_1(X/\Gamma, [x])$.

Proof. We know that $\pi : X \rightarrow X/\Gamma$ is a covering map from the above theorem. Now, since X is simply connected, by uniqueness, π is the universal covering of X/Γ .

Let us define the following map

$$\Phi : \Gamma \rightarrow \pi_1(X/\Gamma, [x]) : \gamma \mapsto [x \rightarrow \gamma \cdot x] (:= [\text{path from } x \text{ to } \gamma \cdot x]).$$

This map is well-defined since X is simply connected. This map is a group homomorphism since it clearly maps e to $[\text{id}]$ and for $\gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned} \Phi(\gamma_1 \gamma_2) &= [x \rightarrow \gamma_1 \gamma_2 \cdot x] = [x \rightarrow \gamma_1 * \gamma_1 \cdot x \rightarrow \gamma_1 \gamma_2 \cdot x] \\ &= [x \rightarrow \gamma_1 * \gamma_1(x \rightarrow \gamma_2 \cdot x)] = [x \rightarrow \gamma_1] * [x \rightarrow \gamma_2] = \phi(\gamma_1) \Phi(\gamma_2). \end{aligned}$$

Φ is injective, since for all $\gamma \in \ker \Phi$, $\pi \circ \delta$ is homotopic to the constant path for all $\delta : x \rightarrow \gamma \cdot x$. Then, taking the lift of the homotopy, we have that δ has the same fixed points implying $\gamma \cdot x = x$ and so, $\gamma = e$ (as the action is nice).

Finally, Φ is surjective since for all $\gamma \in \pi_1(X/\Gamma, [x])$, we may lift γ as a path to X based at x , $\tilde{\gamma} : x \rightarrow \gamma' \cdot x$ for some $\gamma' \in \Gamma$. Thus, $\Phi(\gamma') = [\gamma]$ implying surjectivity. \square

Let X be a connected, path-connected and locally simply and path-connected topological space. Then, if $\pi : \tilde{X} \rightarrow X$ is a covering map, we would like to construct an action of $\pi_1(X, x)$ for some $x \in X$ $\pi(\tilde{x}) = x$.

Let $\gamma \in \pi_1(X, x)$, then for any \tilde{x} , let $\tilde{\gamma}$ be the unique lift of γ based at \tilde{x} . Then, we may define the action of $\pi_1(X, x)$ on \tilde{X} by taking $\gamma \cdot \tilde{x} = \tilde{\gamma}(1)$. It is an easy exercise to check

that this is indeed a group action. This action is in fact nice. Indeed, for all $x \in X$, by the definition of covering map, let U be a neighbourhood of x satisfying the covering property. Then, given $\gamma \neq \text{id}$, $U_{\tilde{x}} \cap U_{\gamma\tilde{x}} = \emptyset$, and thus, $U_x \cap \gamma \cdot U_x = \pi(U_{\tilde{x}}) \cap \pi(U_{\gamma\tilde{x}}) = \emptyset$.

We will in the remainder of this section assume X is path-connected, locally path- and simply-connected.

Let $\pi : (\bar{X}, \bar{x}) \rightarrow (X, x)$ be a covering map. Then, as we have seen, this induces an injective group homomorphism $\pi_* : \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, x)$. As alluded to before, we may interpret the fundamental group in the covering space as a subgroup of the fundamental group in the space by taking

$$\Gamma = \pi_*(\pi_1(\bar{X}, \bar{x})) \leq \pi_1(X, x),$$

and $\Gamma \simeq \pi_1(\bar{X}, \bar{x})$. We ask whether or not the reverse is true. Namely, for all subgroups Γ of $\pi_1(X, x)$, does there exist a covering space $p : \bar{X} \rightarrow X$ such that

$$\Gamma = p_*(\pi_1(\bar{X}, \bar{x})).$$

Furthermore, if $p_1 : (\bar{X}, \bar{x}) \rightarrow (X, x)$, $p_2 : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ are two covering maps which induces the same subgroup, then the two covering spaces are homeomorphic which maps \bar{x} to \tilde{x} .

Theorem 6. If Γ is a subgroup of $\pi_1(X, x)$, then there exists a covering map $\pi : \bar{X} \rightarrow X$ such that $\Gamma = \pi_*(\pi_1(\bar{X}, \bar{x}))$.

Proof. By the assumption on X , there exists a universal cover $\pi : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ together with a nice action of $\pi_1(X, x)$ on \tilde{X} such that $X \simeq \tilde{X}/\pi_1(X, x)$. Now, by restricting the action onto Γ , we obtain an action of Γ on \tilde{X} which is nice. In particular, we have the following chain of quotients.

$$(\tilde{X}, \tilde{x}) \xrightarrow{q_\Gamma} (\tilde{X}/\Gamma, [\tilde{x}]_\Gamma) \xrightarrow{q_{\pi_1(X, x)}} (X, x)$$

π

Now, since Γ acts nicely on \tilde{X} , q_Γ is a covering map, and so, $q_{\pi_1(X, x)}$ is also a covering map. Furthermore, we have $\pi_1(\tilde{X}/\Gamma, [x]_\Gamma) \simeq \Gamma$, and thus, it suffices to show $(q_{\pi_1(X, x)})_*(\pi_1(\tilde{X}/\Gamma, [x]_\Gamma)) = \Gamma$. This follows by lifting $\gamma \in \Gamma$ to \tilde{X} to show surjectivity. \square

We will now show the uniqueness of this covering map.

Theorem 7. Suppose now $p : (\bar{X}, \bar{x}) \rightarrow (X, x)$ is a covering map such that

$$\Gamma = p_*(\pi_1(\bar{X}, \bar{x})),$$

then there exists a homeomorphism $\Phi : (\tilde{X}/\Gamma, [x]_\Gamma) \rightarrow (\bar{X}, \bar{x})$ such that the following diagram commutes.

$$\begin{array}{ccc} (\tilde{X}/\Gamma, [x]_\Gamma) & \xrightarrow{\Phi} & (\bar{X}, \bar{x}) \\ \pi \downarrow & & \downarrow p \\ (X, x) & \xrightarrow{\text{id}_X} & (X, x) \end{array}$$

where we denote $\pi : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ the universal cover.

Proof. Redo this. □

2.6 Van Kampen Theorem

Review free groups.

Given any finitely generated group $G = \langle g_1, \dots, g_n \rangle$, we can define the homomorphism

$$\Phi : \mathbb{F}_n \rightarrow G : a_i \mapsto g_i$$

where a_1, \dots, a_n are characters of \mathbb{F}_n . Thus, with this in mind, $G \simeq \mathbb{F}_n / \ker \Phi$. We note that $\ker \Phi$ encodes “relations” on elements of G . With this isomorphism, we can construct groups from \mathbb{F}_n . Let $r_i \in I$ be reduced words and denote $\langle r_i \mid i \in I \rangle$ the normal subgroup generated by $r_i \in I$. Then, we denote

$$\langle a_1, \dots, a_n \mid r_i(a_1, \dots, a_n) = 1, i \in I \rangle := \frac{\mathbb{F}_n}{\langle r_i \mid i \in I \rangle}.$$

We call this a presentation of the group. This is the construction we are referring to whenever we say to compute a group.

Definition 2.9 (Free Product). Given finitely generated groups G_1, G_2 with presentations $\langle a_1, \dots, a_n \mid r_i(a_1, \dots, a_n) = 1, i \in I \rangle$ and $\langle b_1, \dots, b_m \mid s_j(b_1, \dots, b_m) = 1, j \in J \rangle$, we define the free product between G_1 and G_2 to be

$$G_1 * G_2 := \langle a_1, \dots, a_n, b_1, \dots, b_m \mid r_i(a_1, \dots, a_n) = 1, i \in I, s_j(b_1, \dots, b_m) = 1, j \in J \rangle.$$

Definition 2.10 (Amalgamated Products). Let H be a group and let $h_1 : H \rightarrow G_1$, $h_2 : H \rightarrow G_2$ be group homomorphisms. Then, the amalgamated product of G_1 and G_2 along H is

$$G_1 *_H G_2 := \frac{G_1 * G_2}{\langle h_1(h)h_2(h)^{-1} \mid h \in H \rangle}.$$

Theorem 8 (Van Kampen Theorem). If (X, x_0) is a path-connected pointed space and $A, B \subseteq X$ are open subsets of X , $A \cup B = X$ and $x_0 \in A \cap B$ of which $A \cap B$ is path connected. Then,

$$\pi_1(X, x_0) \simeq \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0).$$

Proof. Let $j_A : A \hookrightarrow X$ and $j_B : B \hookrightarrow X$ be the inclusion maps which induces corresponding group homomorphisms. Then, by the universal property of the free product, we obtain the group homomorphism

$$\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \rightarrow \pi_1(X, x_0).$$

Thus, it suffices to show Φ is surjective with $\ker \Phi = \langle (j_A)_*(h)(j_B)_*(h)^{-1} \mid h \in \pi_1(A \cap B, x_0) \rangle$.

We will omit the injective part and only show surjectivity. Let γ be a loop on X based at x_0 . Then, since γ is continuous and $A \cup B = X$, for all t there exists an open interval $I_t \subseteq [0, 1]$ containing t such that $\gamma(I_t) \subseteq A$ or $\gamma(I_t) \subseteq B$ (compactness argument). Then, there exists some

$$t_0 = 0 < t_1 < \dots < t_n < t_{n+1} = 1$$

3 Homology

3.1 Δ -Complex

Definition 3.1 (Simplex). An n -dimensional simplex is the set

$$\Delta_n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum x_i = 1\}.$$

We see that $\Delta_0 = \{1\} \subseteq \mathbb{R}$ and so, Δ_0 is homeomorphic to a point. Similarly, $\Delta_1 = \{(x_0, x_1) \in \mathbb{R}^2 \mid x_0, x_1 \geq 0, x_0 + x_1 = 1\}$ is the line corresponding to $x_0 + x_1 = 1$ in the first quadrant. So, $\Delta_1 \simeq [0, 1]$. As an exercise, one may show that

$$\Delta_k \simeq \overline{B_0(1)} \subseteq \mathbb{R}^k.$$

While the simplex is homeomorphic to the closed ball, it processes some additional geometric properties we care about.

We observe that $\partial\Delta_k$ is a union of $k+1$ -copies of Δ_{k-1} . Indeed, defining $(e_i)_{i=0}^k$ to be the standard basis of \mathbb{R}^{k+1} , we observe

$$\Delta_k = \text{conv}(e_i \mid i = 0, \dots, k) \subseteq \mathbb{R}^{k+1},$$

where $\text{conv}(S) = \{\sum_{i=0}^k t_i s_i \mid s_i \in S, \sum_{i=0}^k t_i = 1\}$ is the convex hull of S . Then, defining

$$\partial_i \Delta_k := \{\sum_{j \neq i} t_j e_j \mid \sum t_j = 1\},$$

it is clear $\Delta_{k-1} \simeq \partial_i \Delta_k$. Thus, by observing $\partial\Delta_k = \bigcup_{i=0}^k \partial_i \Delta_{k-1}$, we have $\partial\Delta_k$ is $k+1$ copies of Δ_{k-1} . We note that this union is not disjoint and in fact, $\partial_i \Delta_k \cap \partial_k \Delta_k$ is embedded canonically in Δ_{k-2} .

Definition 3.2 (Δ -complex). A n -dimensional Δ -complex structure on a set X is the data of finitely many continuous maps

$$\phi_i^k : \Delta_k \rightarrow X$$

for $k \leq n$ such that

- for all i, k , the restriction $\phi_i^k|_{\Delta_k^\circ} : \Delta_k^\circ \rightarrow X$ is injective;
- for all i, j and $k \leq n$, the restriction $\phi_i^k|_{\partial_j \Delta_k} : \partial_j \Delta_k \rightarrow X$ is equal to $\phi_l^{k-1} : \Delta_{k-1} \rightarrow X$ for some l ;
- $\bigcup_{k=0}^n \bigcup_i \phi_i^k(\Delta_k) = X$.

As an example let us again consider the torus. We observe that, by cutting the square along its diagonal, we can embed two copies of Δ_2 inside the torus. This results in 3 copies of Δ_1 and 1 copy of Δ_0 after the quotient. This provides a Δ -complex structure on the torus by taking the maps to be the natural embeddings.

We note that the definition of Δ -complexes did not require X to have a topology. Nonetheless, we required ϕ_i^k to be continuous. Thus, to be this make sense, simple take the topology on X to be the one induced by $(\phi_i^k)_{i,k}$.

We note that there is no unique Δ -complex structure on a given space though as we shall see, there exists an invariance of the space independent of the Δ -complex.

3.2 Simplicial Homology

The idea with simplicial homology is that for any topological space X , we associate it to an abelian group $H_k(X)$ for some $k \in \mathbb{N}$ where $H_k(X)$ encodes the information of the k -dimensional shapes embedded in X .

Let us first introduce some algebraic notions we will require.

Definition 3.3 (Chain Complex). A chain complex is a sequence of abelian groups $(C_n)_{n \in \mathbb{N}}$ and a sequence of group homomorphisms $\partial_n : C_n \rightarrow C_{n-1}$,

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

such that $\ker(\partial_n) \supseteq \text{Im}(\partial_{n+1})$ for all n . We bundle the two together by writing $(C_n, \partial_n)_{n \in \mathbb{N}}$.

Definition 3.4 (Homology Group). The homology group associated to the chain complex $\mathcal{C} := (C_n, \partial_n)_{n \in \mathbb{N}}$ are

$$H_n(\mathcal{C}) := \ker(\partial_n) / \text{Im}(\partial_{n+1}).$$

Suppose now X is equipped with a Δ -complex. Then, we may define

$$C_k(X) := \bigoplus_i \mathbb{Z} \cdot [\phi_i^k],$$

that is the free **abelian** group generated by symbols corresponding to ϕ_i^k for each i . Then, we define $\partial_k : C_k(X) \rightarrow C_{k-1}(X)$ such that

$$\partial_k([\phi_i^k]) = \sum_{j=0}^k (-1)^{j+1} [\partial_j \phi_i^k]$$

where we denote $\partial_j \phi_i^k := \phi_i^k|_{\partial_j \Delta_k}$.

Lemma 3.1. For all $k \geq 1$, $\partial_k \circ \partial_{k+1} = 0$ (and so, $\text{Im}(\partial_{k+1}) \subseteq \ker(\partial_k)$).

Proof. It suffices to show that $0 = \partial_{k-1} \circ \partial_k([\phi_i^k])$. By denoting $\partial_{jl} \Delta_k$ the simplex generated by the ordered tuple $(e_0, \dots, e_{j-1}, e_{j+1}, \dots, e_{l-1}, e_{l+1}, \dots, e_k)$, we observe $\partial_l(\partial_j \phi_i^k) = \phi_i^k|_{\partial_{jl} \Delta_k}$ for all $l \leq k$ and $\partial_l(\partial_j \phi_i^k) = \phi_i^k|_{\partial_{j(l+1)} \Delta_k}$ for $l > k$. Thus,

$$\begin{aligned} \partial_{k-1}(\partial_k[\phi_i^k]) &= \partial_{k-1} \left(\sum_{j=0}^k (-1)^{j+1} [\partial_j \phi_i^k] \right) = \sum_{j=0}^k (-1)^{j+1} \partial_{k-1}([\partial_j \phi_i^k]) \\ &= \sum_{j=0}^k (-1)^{j+1} \sum_{l=0}^{k-1} (-1)^{l+1} [\partial_l(\partial_j \phi_i^k)] \\ &= \sum_{j=0}^k \sum_{l=0}^j (-1)^{j+l} [\partial_{jl} \phi_i^k] + \sum_{j=0}^k \sum_{l=j+1}^{k-1} (-1)^{j+l} [\partial_{jl} \phi_i^k] \\ &= \sum_{j=0}^k \sum_{l=0}^j (-1)^{j+l} [\phi_i^k|_{\partial_{jl} \Delta_k}] + \sum_{j=0}^k \sum_{l=j+1}^{k-1} (-1)^{j+l} [\phi_i^k|_{\partial_{j(l+1)} \Delta_k}] = 0, \end{aligned}$$

where the last equality follows by change of indices. \square

With this lemma in mind, we have $(C_k(X), \partial_k)_{k \in \mathbb{N}}$ is a chain complex and we may define a homology group on this chain.

Definition 3.5 (Homology Group of a Δ -complex on X). The simplicial homology group of X is the homology group of the chain $(C_k(X), \partial_k)_{k \in \mathbb{N}}$ where and we denote it by $H_k^\Delta(X)$.

So far, we note that the definition of the simplicial homology depends on the Δ -complex structure. However, as we shall see, the definition is in fact invariant under different Δ -complexes.

3.3 Singular Homology

We will now provide another construction of the homology group known as singular homology. This turns out to be the same homology group as the simplicial homology yet easier to manipulate formally. Yet, they are much more difficult to compute, hence our current construction.

Definition 3.6. Let X be a topological space. For all $n \in \mathbb{N}$, we define

$$C_n(X) := \bigoplus_{\substack{\Phi: \Delta_n \rightarrow X \\ \text{continuous}}} \mathbb{Z} \cdot [\Phi].$$

Recall that n -dimensional Δ -simplex is defined to be $\text{conv}(e_0, \dots, e_n) \subseteq \mathbb{R}^{n+1}$ where e_i has 1 in the i -th position and 0 everywhere else. Thus, if $A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is a linear map induced by a permutation on the e_i s, we would like to identify $[\Phi]$ with $[\Phi \circ A]$ if A preserves orientation (namely the permutation correspondence to A as positive sign) and $[\Phi]$ with $-[\Phi \circ A]$ if the permutation has negative sign.

Definition 3.7. Given the definition above, we define the group homomorphism $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ by extending

$$\partial_n([\Phi]) := \sum_{i=0}^n (-1)^i [\Phi|_{\partial_i \Delta_n}]$$

linearly.

Proposition 3.1. For all $n \geq 1$, $\partial_{n-1} \circ \partial_n = 0$.

Proof. Same proof as simplicial homology. □

Corollary 3.1. $(C_n(X), \partial_n)$ is a chain complex.

Definition 3.8 (Singular Homology). The singular homology groups of the topological space X are the homology groups of the chain complex $(C_n(X), \partial_n)$, i.e.

$$H_n(X) := \ker(\partial_n) / \text{Im}(\partial_{n+1}).$$

Definition 3.9 (Induced Map). If $f : X \rightarrow Y$ is a continuous map and $\Phi : \Delta_n \rightarrow X \in C_n(X)$. Then, $f \circ \Phi : \Delta_n \rightarrow Y \in C_n(Y)$. Thus, f induces a group homomorphism $f_* : H_n(X) \rightarrow H_n(Y)$.

With the above definitions, we may introduce a notion known as relative homology.

Let $A \subseteq X$ and we can consider for all k , $C_k(A) \subseteq C_k(X)$. We observe $\partial_k(C_k(A)) \subseteq C_{k-1}(A)$ implying ∂_k induces a group homomorphism

$$\partial_k : C_k(X)/C_k(A) \rightarrow C_{k-1}(X)/C_{k-1}(A).$$

With this in mind, we denote $C_k(X, A) := C_k(X)/C_k(A)$.

Proposition 3.2. $(C_k(X, A), \partial_k)_{k \in \mathbb{N}}$ is a chain complex.

Proof. Follows as $\partial_{n-1} \circ \partial_n = 0$ as maps from $C_n(X) \rightarrow C_{n-2}(X)$. \square

Definition 3.10 (Relative Homology). The homology groups of X relative to A are the homology groups of the chain complex $(C_k(X, A), \partial_k)_{k \in \mathbb{N}}$. We denote them by $H_n(X, A)$.

Now, by denoting $i : A \hookrightarrow X$ as the inclusion map, $\pi : C_n(X) \rightarrow C_n(X)/C_n(A)$ the quotient map, we have induced maps $i_* : H_n(A) \rightarrow H_n(X)$ and $\pi_* : H_n(X) \rightarrow H_n(X, A)$. These two maps are related with something called a long exact sequence.

Definition 3.11 (Exact Sequence). Given groups A, B, C and group homomorphisms $f : A \rightarrow B, g : B \rightarrow C$, the chain it forms is an exact sequence if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\text{Im}(f) = \ker(g).$$

Definition 3.12 (Long Exact Sequence). Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of groups and let $f_i : A_{i+1} \rightarrow A_i$ be the group homomorphisms. Then, the chain

$$\dots \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_i} A_i \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_{i-2}} \dots$$

is exact if for all i , $\text{Im}(f_i) = \ker(f_i)$.

Now, for all $u \in H_n(X, A) = \ker(\partial_n)/\text{Im}(\partial_{n+1})$, u is represented by some $c \in C_n(X)$ such that $\partial_n c \in C_{n-1}(A)$. With this in mind, we may define the group homomorphism

$$\partial : H_n(X, A) \rightarrow H_{n-1}(A) : [c] \mapsto [\partial_n c].$$

Theorem 9. The chain

$$\dots \xrightarrow{\pi_*} H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial} \dots$$

is a long exact sequence.

Proof. We will first show $\text{Im}(\partial) = \ker(i_*) \subseteq H_n(A)$. Indeed, for all $[\partial_{n+1}c] \in \text{Im}(\partial)$, $[\partial_{n+1}c] \in \ker(\partial_n)$ (as $\partial_n \circ \partial_{n+1} = 0$). Thus, as $\partial_{n+1}c \in \text{Im}(\partial_{n+1})$, we have $[\partial_{n+1}c] = 0$ implying $\text{Im}(\partial) \subseteq \ker(i_*)$. Conversely, if $a \in \ker(i_*)$, $a \in \text{Im}(\partial_{n+1})$, implying there exists some $c \in C_{n+1}(X)$ such that $a = \partial_{n+1}c = \partial c$. Thus, $\text{Im}(\partial) = \ker(i_*)$ as required.

The other two equalities are straight forward checks also. \square