Algebraic Topology

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1 Introduction

Let us introduce/recall some basic definitions which will be used throughout this course.

Definition 1.1 (Path)**.** A path in a topological space X is a continuous map $\gamma : [0,1] \subseteq \mathbb{R} \to X$. In the case that $\gamma(0) = \gamma(1)$, we call γ a loop/closed path.

Definition 1.2 (Homotopy)**.** Given two paths $\gamma_0, \gamma_1 : [0,1] \to X$ with the same end points (i.e. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are said to be homotopic with fixed endpoints if there exists a continuous map

$$H : [0,1] \times [0,1] \to X$$

such that

- $H(t, 0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t, 1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0, u) = \gamma_0(0) = \gamma_1(0)$ and $H(1, u) = \gamma_0(1) = \gamma_1(1)$.

Thus, graphically, two paths are homotopic if you can continuously deform a path into the other without moving the starting and ending points (see second year complex analysis for more details).

Definition 1.3 (Free Homotopy)**.** The loops γ_0, γ_1 in X is said to be freely homotopic if there exists a continuous $H : [0,1] \times [0,1] \to X$ such that

- $H(t, 0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t, 1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0, u) = H(1, u)$.

Definition 1.4 (Simply Connected)**.** X is said to be simply connected if any loop in X is freely homotopic to a constant loop.

Thus, informally, in a simply connected space, any loop can be contracted into a single point.

Proposition 1.1. S^2 is simply connected.

Simply connectedness is a important notion and relates to many difficult problems in geometry.

Theorem 1. S^2 and $\mathbb{R}^2$ are, up to homeomorphism, the only two simply-connected 2-dimensional manifolds.

Theorem 2 (Poincaré Conjecture)**.** The only compact, simply connected 3-dimensional manifold is the sphere S^3 (up to homeomorphism).