Fourier Analysis Revision Notes

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Inner Product Spaces

We denote R a (real or complex) inner product space (Euclidean space).

Definition (Complete System). A system $\{X_\alpha\}_{\alpha \in A}$ is said to be complete if its linear closure is R, namely $\langle X_\alpha \mid \alpha \in A \rangle = R$.

Definition (Orthogonal Basis). A system is an orthogonal basis if it is orthogonal and complete.

Proposition. If R is separable, then any orthogonal system of R is countable.

Proposition. Any separable real inner product space possesses a orthonormal basis.

Definition (Fourier Coefficients). Given an orthonormal system $\{\phi_n\}_{n=1}^\infty$ of R. The Fourier coefficients of any $f \in R$ is defined to be

$$c_k := \langle f, \phi_k \rangle$$

for all k. The formal sum $\sum_{k=1}^\infty c_k \phi_k$ is called the Fourier series of f.

Definition (Closed System). An othonormal system $\{\phi_n\}$ is closed if

$$\sum_{k=1}^\infty c_k^2 = \|f\|^2$$

for all $f \in R$. We call this property Parseval's identity.

Proposition (Bessel's Inequality). Given the orthonormal system $\{\phi_n\}$ of R, we have

$$\sum_{k=1}^\infty |c_k|^2 \leq \|f\|^2$$

for all $f \in R$.

Theorem. In a separable inner product space R, an orthonormal system is complete if and only if it is closed.

Proposition. Given f, g and a closed system $\{\phi_n\}$ of R, $\langle f, g \rangle = \sum_{k=1}^\infty c_k^f c_k^g$ where c_k^f, c_k^g are the Fourier coefficients of f, g respectively.

Contour Integration

Proposition (Jordan's Lemma)**.** If f is holomorphic except for finitely many singularities, and $f(z) \to 0$ as $|z| \to \infty$, then

$$\int_{\gamma_R} f(z)e^{i\lambda z}\mathrm{d}z \to 0$$

for all $\lambda > 0$ where γ_R is the upper half circle of radius R centred at 0 oriented counter-clockwise.

In the case Jordan's lemma fails due to $\lambda < 0$, try integrating on the lower half circle.

Proposition. $e^{iz} = 1 + O(|z|)$. Useful for integrating on small contours.

Fourier Series

Smoother functions have quicker decaying of Fourier coefficients.

Proposition. For all $f \in L^2[-\pi, \pi]$, $\|f - S_n\|_2 \to 0$ where S_n is the n-th partial sum of the Fourier series of f.

Theorem (Dini's Condition for Pointwise Convergence)**.** If $f \in L^1[-\pi, \pi]$ and for any $x \in [-\pi, \pi]$, there exists some $\delta > 0$ such that

$$\int_{[-\delta,\delta]} \left|\frac{f(x+t) - f(x)}{t}\right| \lambda(\mathrm{d}t) < \infty$$

exists, then $S_n(x) \to f(x)$ as $n \to \infty$ for all x.

Dini's condition is in some sense as strong as possible. Indeed, if $\frac{f(x+t)-f(x)}{t}$ is not locally integrable at some x, we can find a continuous function g with $|g| \leq f$ with non-convergence Fourier series at x.

If f is continuous at x and has a derivative at x (or the limit exists from either the left or the right), then Dini's condition is satisfied at x.

A continuous function with period 2π is uniquely determined by its Fourier series. Furthermore, we can reconstruct a continuous function from its Fourier series by using the Fejer sums. Indeed, denoting σ_n the n-th Fejer sum, $\sigma_n \to f$ uniformly.

Proposition (Poisson Summation Formula)**.** Given $f \in L^1$,

$$2\pi t \sum_{n=-\infty}^{\infty} f(2\pi tn) = \sum_{n=-\infty}^{\infty} \mathcal{F}[f](n/t).$$

Fourier Transform

Proposition. Let $f \in L^1$. Then $\mathcal{F}[f] = 0$ implies $f = 0$ almost everywhere.

Proposition. For $f, f_n \in L^1$ such that $f_n \to f$ in L^1, then $\mathcal{F}[f_n] \to \mathcal{F}[f]$ uniformly.

Proposition. For $f \in L^1$, $\mathcal{F}[f](y) \to 0$ as $|y| \to \infty$.

Corollary. For $f \in L^1$, $\mathcal{F}[f]$ is uniformly continuous.

Proposition. For $f \in L^1$ differentiable with $f' \in L^1$ and f absolutely continuous on any finite interval, $\mathcal{F}[f'](y) = iy\mathcal{F}[f](y)$.

Proposition. For $f \in L^1$ such that $xf(x) \in L^1$, we have $\mathcal{F}[f]$ is differentiable and $D_y\mathcal{F}[f](y) = \mathcal{F}[-ixf(x)]$.

We also have the following properties: for $f, g \in L^1$, $c, c_1, c_2 \in \mathbb{R}$,

- Linearity: $\mathcal{F}[c_1 f + c_2 g] = c_1\mathcal{F}[f] + c_2\mathcal{F}[g]$.
- Translation: $\mathcal{F}[x \mapsto f(x-a)](y) = e^{-iay}\mathcal{F}[f](y)$.
- Rephasing: $\mathcal{F}[x \mapsto e^{-icx} f(x)](y) = \mathcal{F}[f](y+c)$.
- Scaling: $\mathcal{F}[x \mapsto f(cx)](y) = \frac{1}{|c|}\mathcal{F}[f](y/c)$.
- Convolution: $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$.

Distribution

$|x|' = \text{sgn}(x), \text{sgn}(x)' = 2\delta, \text{sgn}(x-a)' = 2\delta(x-a)$.

Suppose $f \in S'$. Then,

$$\langle \mathcal{F}[f'], \phi \rangle = \langle f', \mathcal{F}[\phi] \rangle = -\langle f, \mathcal{F}[\phi]' \rangle = \langle f, \mathcal{F}[it\phi(t)] \rangle = \langle \mathcal{F}[f], it\phi(t) \rangle = \langle ix\mathcal{F}[f], \phi \rangle$$

implying $\mathcal{F}[f'] = ix\mathcal{F}[f]$. Hence, $\mathcal{F}[f] = -ix^{-1}\mathcal{F}[f']$.

Similarly,

$$\begin{aligned}\langle \mathcal{F}[f]', \phi \rangle &= -\langle \mathcal{F}[f], \phi' \rangle = -\langle f, \mathcal{F}[\phi'] \rangle \\ &= -\langle f, it\mathcal{F}[\phi](t) \rangle = -\langle ixf(x), \mathcal{F}[\phi] \rangle = -i\langle \mathcal{F}[xf(x)], \phi \rangle\end{aligned}$$

so $\mathcal{F}[xf(x)] = i\mathcal{F}[f]'$.

By a similar process, the Fourier transform of a tempered distribution satisfy all the normal properties as mentioned in the previous section.