# Manifolds

### Kexing Ying

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## Contents

# 1 Introduction

This module introduces the notion of manifolds and provides the infrastructure for generalizing theorems from calculus to manifolds. In particular, we will talk about

- Smooth manifolds and smooth functions;
- Tangent spaces and vector fields;
- Differential forms, integrations and Stoke's theorem.

In contrast to the curves and spaces module, instead of working on Euclidean spaces, we will define these notions for general manifolds. Thus, many definitions such as the tangent space will be defined in a more intrinsic point of view, without requiring our manifold to be within a Euclidean space.

Furthermore, a goal of this module is to differentiate between different manifolds, that is determine whether or not two manifolds are diffeomorphic with one another. This is achieved through introducing invariants such as the notion of differential forms and these notions will appear in many other places especially in geometry.

Manifolds is the subject of studying geometric shapes, and in mathematics, there are in general two ways of doing this. The first of which is by embedding the object into an ambient space such as $\mathbb{R}^2$ or $\mathbb{R}^3$. An example of this is studying the unit circle through the parametrisation

$$\{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

and is the more common method of what we have done thus far. On the other hand, one may study the object independently of the ambient space. This is the approach we shall take throughout this course. In particular, we will study spaces which at a local level "looks like" a Euclidean space directly without embedding the structure into $\mathbb{R}^n$.

# 2  Topological and Smooth Manifolds

Let us first recall some notions from topology.

**Definition 2.1.** Let $X, Y$ be topological spaces and let $f : X \to Y$ be a function, then

- $f$ is continuous if $f^{-1}(U)$ is open in $X$ for all $U$ open in $Y$.

- $f$ is a homeomorphism if it is continuous and has a continuous inverse.

**Definition 2.2.** A topological space $X$ is

- Hausdorff if for all $x, y \in X$, $x \neq y$, there exists open sets $U, V$ in $X$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

- second-countable if there exists countable $\mathcal{F} \subseteq \mathcal{T}_X$ such that any open set in $X$ can be written as a union of elements of $\mathcal{F}$, i.e. $\mathcal{F}$ is a countable basis of $X$.

In general, in this module, we will assume our topology is Hausdorff and second-countable in order to avoid pathological examples in smooth and topological manifolds.

**Definition 2.3** (Co-ordinate Chart). Let $X$ be a topological space. A co-ordinate chart on $X$ is the collection of

- an open set $U \subseteq X$,

- an open set $\tilde{U} \subseteq \mathbb{R}^n$ for some $n \geq 0$,

- a homeomorphism $f : U \to \tilde{U}$.

We denote a co-ordinate chart by $(U, f)$.

**Definition 2.4.** Let $X$ be a (Hausdorff and second-countable) topological space. We say that $X$ is a topological manifold of dimension $n$ if for all $x \in X$, there exists a co-ordinate chart $(U, f)$ with $\tilde{U} \subseteq \mathbb{R}^n$ such that $x \in U$.

The classical example of a topological manifold is the circle, in particular

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

is a 1 dimensional topological manifold. Consider $U_1 = S^1 \setminus \{(0, -1)\}$, and we define the stereographic projection $f_1 : U_1 \to \mathbb{R}$,

$$f : (x, y) \mapsto \frac{x}{y + 1} := \tilde{x}.$$

It is not difficult to see that $f_1$ is invertible with the inverse

$$f_1^{-1} : \tilde{x} \mapsto \left( \frac{2\tilde{x}}{1 + \tilde{x}^2}, \frac{1 - \tilde{x}^2}{1 + \tilde{x}^2} \right).$$

Furthermore, as $f_1$ and $f^{-1}$ are continuous, we have $(U_1, f_1)$ is a co-ordinate chart. Similarly, we define $U_2 = S^1 \setminus \{(0, 1)\}$, and we my show the existence of a homeomorphism $f_2 : U_2 \to \mathbb{R}$, providing the second co-ordinate chart $(U_2, f_2)$. Thus, as $S^1 = U_1 \cup U_2$, we have $S^1$ is a 1 dimensional topological manifold.

The above example can be expanded to $n$-dimensional sphere

$$S^n := \{(x_0, \cdots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n.$$

Similarly as before, we can construct two co-ordinate charts covering all points on the sphere except for the poles allowing us to conclude $S^n$ is a $n$-dimensional topological manifold.

**Definition 2.5** (Transition Function). Let $X$ be a topological manifold and let $(U_1, f_1)$ and $(U_2, f_2)$ be two co-ordinate charts on $X$ such that $U_1 \cap U_2 \neq \emptyset$. Then the transition function between these two co-ordinate charts is the function

$$\phi_{21} := f_2 \circ f_1^{-1} : f_1(U_1 \cap U_2) \to f_2(U_1 \cap U_2).$$

Let $X$ be a topological manifold with co-ordinate charts $(U_i, f_i)$ for $i = 1, 2, 3$ such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$. Then it is clear that $\phi_{21} := f_2 \circ f_1^{-1}$ is a homeomorphism with the inverse $\phi_{12} := f_1 \circ f_2^{-1}$. Furthermore, by considering $\phi_{31} := f_3 \circ f_1^{-1}$ we observe

$$\phi_{31} = (f_3 \circ f_2^{-1}) \circ (f_2 \circ f_1^{-1}) = \phi_{32} \circ \phi_{21}.$$

This is known as the cocycle property and explains the subscript notation.

**Definition 2.6** (Atlas). Let $X$ be a topological manifold. An atlas for $X$ is the collection of co-ordinate charst $\{(U_i, f_i)\}_{i \in I}$ such that

$$\bigcup_{i \in I} U_i = X.$$

We note that we do not require the index set $I$ to be finite. Although, since $\{U_i\}_{i \in I}$ is an open cover, if $X$ is compact, it is possible to obtain a finite sub-cover, and hence a finite atlas. Nonetheless, since we assumed $X$ is second-countable, we can always choose $I$ to be countable.

## 2.1 Smooth Manifolds

So far, we have only considered ourselves with the topological structure. As we would like to do calculus on our manifolds, we will now equip our manifolds with the property of smoothness. Recall the following definition for Euclidean spaces.

**Definition 2.7.** A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (or $C^\infty$) if all the partial derivatives of $F$ of any order exists.

Of course, this is technically a property not a definition though it will suffice for our purposes.

**Definition 2.8** (Smooth Atlas). Let $X$ be a topological manifold of dimension $n$. Then an atlas $\{(U_i, f_i)\}_{i \in I}$ on $X$ is smooth if for all $i, j \in I$, the transition function

$$\phi_{ij} : f_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to f_i(U_i \cap U_j) \subseteq \mathbb{R}^n$$

is smooth.

Since $\phi_{ij}$ is a (bijective) map between open subsets of Euclidean spaces, it makes sense to ask whether or not $\phi_{ij}$ is smooth.

**Definition 2.9** (Diffeomorphism)**.** Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $f : U \to V$. Then $f$ is a diffeomorphism if $f$ is smooth and has a smooth inverse.

As $(\phi_{ij})^{-1} = \phi_{ji}$, and both $\phi_{ij}$ and $\phi_{ji}$ are smooth, the transition functions of any smooth manifold are diffeomorphisms.

**Definition 2.10** (Compatible)**.** Let $X$ be a topological manifold and let $\mathcal{A} := \{(U_i, f_i)\}$ be a smooth atlas. Let $(U, f)$ be any co-ordinate chart on $X$, then $(U, f)$ is compatible with the atlas $\mathcal{A}$ if the transition function between $(U, f)$ and any chart in $\mathcal{A}$ is a diffeomorphism.

Clearly, any chart in a smooth atlas is compatible with that atlas, and if $(U, f)$ is compatible with the smooth atlas $\mathcal{A}$, then $(U, f) \cup \mathcal{A}$ is also a smooth atlas.

**Definition 2.11.** Let $X$ be a topological manifold and $\mathcal{A}, \mathcal{B}$ be two atlases on $X$. Then $\mathcal{A}$ is compatible with $\mathcal{B}$ if every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$.

Similarly as before, if $\mathcal{A}, \mathcal{B}$ are compatible, then $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas on $X$.

**Lemma 2.1.** Let $X$ be a topological manifold and let

$$\mathcal{A} := \{(U_i, f_i)\}_{i \in I}, \mathcal{B} := \{(U_j, f_j)\}_{j \in J},$$

be two compatible smooth atlases on $X$. Then for all $(U, f)$ co-ordinate charts compatible with $\mathcal{A}$, $(U, f)$ is compatible with $\mathcal{B}$.

*Proof.* It suffices to show that for all $(U_j, f_j) \in \mathcal{B}$, $U \cap U_j \neq \emptyset$, the transition map

$$\phi := f_j \circ f^{-1} : f(U \cap U_j) \to f_j(U \cap U_j)$$

and its inverse are smooth.

Let $y \in f(U \cap U_j)$, then there exist some $x \in U \cap U_j$ such that $f(x) = y$. As $\mathcal{A}$ is an atlas, it contains a co-ordinate chart $(U_i, f_i) \in \mathcal{A}$ such that $x \in U_i$. Then, defining $W := U \cap U_i \cap U_j \neq \emptyset$, we have the homomorphisms $f : W \to f(W), f_i : W \to f_i(W)$ and $f_j : W \to f_j(W)$. As remarked before, we have

$$\phi = (f^{-1} \circ f_i) \circ (f_i^{-1} \circ f_j)$$

on $W$. Now, by compatibility, the right hand side is smooth, and so we have $\phi$ is smooth on $W$ implying it is smooth at $y$. Thus, as $y \in f(U \cap U_j)$ was arbitrary, $\phi$ is smooth (by a similar argument $\phi^{-1}$ is also smooth) and $(U, f)$ is compatible with $\mathcal{B}$. $\square$

With this lemma it is easy to see that compatibility defines an equivalence relation on the set of smooth atlases and with this we can define smooth manifolds.

**Definition 2.12** (Smooth Manifold)**.** A smooth manifold is a topological manifold with an equivalence class $[\mathcal{A}]$ of compatible smooth atlases on $X$. The equivalence class of atlases is called a smooth structure on $X$.

The reason for the definition considering only the equivalence class of compatible smooth atlases is because we do not want to distinguish between compatible smooth atlases. Indeed, recalling our example of a sphere, we would like to not consider the atlases which projects the sphere with respect to two other points that are not the poles as an alternative manifold.

From this point forward, we will always work with smooth manifolds and thus, omit the word "smooth" whenever it is clear from the context, i.e. a manifold is a smooth manifold and a atlas is a smooth atlas.

## 2.2   Submanifolds

**Definition 2.13** (Affine Subspace). An affine subspace $A \subseteq \mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$, i.e. there exists some $v \in V$ and $W \leq \mathbb{R}^n$ such that

$$A := v + W = \{v + w \mid w \in W\}.$$

**Definition 2.14** (Submanifold). Let $X$ be an $n$-dimensional manifold and let $Y \subseteq X$. Then $Y$ is an $m$-dimensional submanifold of $X$ if for all $y \in Y$, there exists a

- a co-ordinate chart $(U, f)$ of $X$ which is compatible with the smooth structure of $X$ such that $y \in U$ and,

- an $m$-dimensional affine subspace $A \subseteq \mathbb{R}^n$

$$f(U \cap Y) = f(U) \cap A.$$

**Proposition 2.1.** Let $X$ be an $n$-dimensional manifold and $Y$ an $m$-dimensional submanifold of $X$, then $Y$ is an $m$-dimensional manifold.

*Proof.* As $Y$ is a topological subspace of $X$, it is Hausdorff and second-countable. Thus, it remains to show that $Y$ is equipped with a smooth structure.

By linear algebra, it is easy to see that the linear map $\tau : A = v + W \to W : a \mapsto a - v$ is continuously invertible, and thus, for all $y \in Y$ there exists a chart $(U, f' := \tau \circ f)$ of $X$ such that $y \in U$ and $f'(U \cap Y) = f'(U) \cap W$. Let $T : W \cong \mathbb{R}^m$, then defining the atlas

$$\{(U_y, \tilde{f}_y)\}_{y \in Y} := \{(U_y, T \circ f')\}_{y \in Y},$$

for all $a, b \in Y$, its transition map

$$\phi_{ab} = (T \circ \tau \circ f_b) \circ (T \circ \tau \circ f_a)^{-1} = T \circ \tau \circ (f_b \circ f_a^{-1}) \circ \tau^{-1} \circ T^{-1},$$

is a composition of smooth functions, and thus is smooth. Hence $Y$ is a smooth manifold. □

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, then define the set $s_f := \{(x, y) \mid y = f(x)\} \subseteq \mathbb{R}^2$ and I claim that $s_f$ is a submanifold of $\mathbb{R}^2$. Define the chart $(U, g)$ on $\mathbb{R}^2$ where $U = \mathbb{R}^2$ and

$$g(x, y) = (x, y - f(x)).$$

It is clear that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism as it is invertible with the inverse $g^{-1}(x, y) = (x, y + f(x))$ and so, $\{(U, g)\}$ is a smooth atlas of $\mathbb{R}^2$. Now considering $g \mid_{s_f} : s_f \to g(s_f) : (x, f(x)) \mapsto (x, 0)$ we have $s_f$ is a smooth submanifold of $\mathbb{R}^2$.

Let us recall the following proposition from year-two analysis.

**Proposition 2.2** (Inverse Function Theorem). Let $U \subseteq \mathbb{R}^n$ be an open subset and let $F : U \to \mathbb{R}^n$ be smooth. Let $x \in U$ such that the Jacobian as $x$, $DF \mid_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists an open neighbourhood $V \subseteq U$ of $x$ such that $F \mid_V : V \to F(V) \subseteq \mathbb{R}^n$ is a diffeomorphism.

**Corollary 0.1.** A smooth, bijective function $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ which has non-zero Jacobian everywhere has a smooth inverse.

The inverse function theorem is useful for showing whether a subset of a manifold is a submanifold. Consider the circle $S_1 := \{x^2 + y^2 = 1\}$ as a subset of the manifold $\mathbb{R}^2$. Then, let

$$U = \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\} \text{ and } f : U \to \mathbb{R}^2 : (r\cos\theta, r\sin\theta) \mapsto (r, \theta).$$

As $f : U \to f(U)$ is smooth, bijective and has non-zero Jacobian on $U$, then $f^{-1} : f(U) \to U$ is also smooth. Thus, $(U, f)$ is a smooth chart on $U \to \tilde{U} := \mathbb{R}^+ \times (-\pi, \pi) \subseteq \mathbb{R}^2$. Then, for all $(\cos\theta, \sin\theta) \in S_1 \setminus \{(-1,0)\}$, we have $f(\cos\theta, \sin\theta) = (1, \theta)$ implying

$$f(U \cap S_1) = \{(1, \theta) \mid \theta \in (-\pi, \pi)\} = f(U) \cap A,$$

where $A$ is the affine subspace $(1,0) + \{(0,y) \mid y \in \mathbb{R}\}$. Hence $S_1$ is a submanifold of $\mathbb{R}^2$.

**Definition 2.15** (Level Sets). Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a function and let $\alpha \in \mathbb{R}^k$. Then the level set of $h$ at $\alpha$ is

$$h^{-1}(\{\alpha\}) = \{x \in \mathbb{R}^n \mid h(x) = \alpha\} \subseteq \mathbb{R}^n.$$

**Definition 2.16** (Regular Points and Values). Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. A point $x \in \mathbb{R}^n$ is called a regular point of $h$ if the Jacobian of $h$ at $x$

$$Dh \mid_x : \mathbb{R}^n \to \mathbb{R}^k$$

is surjective.

$\alpha \in \mathbb{R}^k$ is called a regular value if every point of the $\alpha$-level set $h^{-1}(\{\alpha\})$ is regular.

If $x \in \mathbb{R}^n$ is not a regular point, then it is called a critical point. Similarly, if $\alpha \in \mathbb{R}^k$ is not a regular value, then it is called a critical value.

**Definition 2.17** (Standard Projection). Let $k \leq n$. The standard projection is the morphism

$$\pi : \mathbb{R}^n \to \mathbb{R}^k : (x_1, \cdots, x_n) \mapsto (x_{n-k+1}, \cdots, x_n).$$

That is $\pi$ forgets the first $n - k$ entries.

Level sets are a useful tool for constructing submanifolds.

**Theorem 1** (Implicit Function theorem). Let $U \subseteq \mathbb{R}^n$ be an open subset and let $h : U \to \mathbb{R}^k$ be a smooth function where $k \leq n$. Let $z \in U$ be a regular point of $h$. Then there exists an open neighbourhood $V \subseteq U$ of $z$ and a diffeomorphism

$$f : V \to f(V) \subseteq \mathbb{R}^n \text{ s.t. } h \circ f^{-1} = \pi : f(V) \to \mathbb{R}^k.$$

Informally, this theorem states that a smooth function around a regular point looks like the standard projection.

*Proof.* Let $x_1, \cdots, x_n$ be co-ordinates on $\mathbb{R}^n$ and let us write

$$h(x) = (h_1(x), \cdots, h_k(x)).$$

As $z$ is regular, we have $Dh \mid_z: \mathbb{R}^n \to \mathbb{R}^k$ is surjective and thus, possibly by reordering, the set

$$\left\{ \frac{\partial h(z)}{\partial x_{n-k+1}}, \cdots, \frac{\partial h(z)}{\partial x_n} \right\}$$

form a basis of $\mathbb{R}^k$ and the matrix

$$M := \begin{pmatrix} \frac{\partial h_1(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_k(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_k(z)}{\partial x_n} \end{pmatrix}$$

is invertible. Then, by defining

$$f : U \to f(U) : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^{n-k}, h_1(x), \cdots, h_k(x)),$$

we have,

$$Df \mid_z = \left( \begin{array}{c|c} I_{n-k} & 0 \\ \hline \star & M \end{array} \right)$$

which is invertible as $\det Df \mid_z = \det I_{n-k} \det M = \det M \neq 0$. Thus, by the inverse function theorem, there exists some open $V \subseteq U$ such that $f : V \to f(V)$ is a diffeomorphism. Then, by considering $\pi \circ f = h$, we have $\pi = h \circ f^{-1}$. $\qquad\square$

**Corollary 1.1.** If $h : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function, and $\alpha$ is a regular value, then the level set of $h$ at $\alpha$ is a submanifold of $\mathbb{R}^n$ of dimension $n - k$.

*Proof.* For all $z \in h^{-1}(\{\alpha\})$, we have $z$ is a regular point. Thus, by the above theorem, there exists an open neighbourhood $V$ of $z$ and a diffeomorphism $f : V \to f(V)$ such that $h \circ f^{-1} = \pi$. Then,

$$f(h^{-1}(\{\alpha\}) \cap V) = f(h^{-1}(\{\alpha\})) \cap f(V) = \pi^{-1}(\{\alpha\}) \cap f(V).$$

Hence, as $\pi^{-1}(\{\alpha\}) = \{(x_1, \cdots, x_{n-k}, \alpha_1, \cdots, \alpha_k)\} = \alpha + A_{n-k}$, we have $h^{-1}(\{\alpha\})$ is a submanifold of dimension $n - k$. $\qquad\square$

This corollary is extremely useful. Consider the sphere $S^n = \{x_0^2 + \cdots + x_n^2 = \alpha\}$, by defining $h : \mathbb{R}^n \to \mathbb{R} : (x_0, \cdots, x_n) \mapsto x_0^2 + \cdots + x_n^2$, we see that $h$ is smooth with the the Jacobian

$$Dh \mid_x = (2x_0, \cdots, 2x_n).$$

Thus, $\alpha$ is a regular value of $h$ for all $\alpha > 0$. Hence, $S^n = \{h(x) = \alpha\}$ is a submanifold of $\mathbb{R}^{n+1}$ for all $\alpha > 0$.

**Theorem 2** (Sard's Theorem). Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. Then the set of regular values $Z \subseteq \mathbb{R}^k$ is dense. Furthermore, $\mathbb{R}^k \setminus Z$ has Lebesgue measure zero.

## 2.3 Smooth Functions

We know what a smooth function between two Euclidean spaces is. We will extend this notion to functions between two manifolds.

**Definition 2.18.** Let $X$ be a manifold and let $h : X \to \mathbb{R}$ be a function. Then $h$ is said to be smooth at $x \in X$ if for any chart $(U, f)$ containing $x$ such that it is compatible with the smooth structure of $X$, the function

$$h \circ f^{-1} : f(U) \to \mathbb{R}$$

is smooth at the point $f(x)$.

We say $h$ is smooth if it is smooth at all points in $X$.

It is not difficult to see that smoothness is independent of the chart we pick, i.e. $h$ is smooth at $x$ as long as there exists a compatible chart $(U, f)$ containing $x$ such that $h \circ f^{-1}$ is smooth at $x$.

**Proposition 2.3.** Let $X$ be a manifold and let $h : X \to \mathbb{R}$ be a function. Then, if $(U_1, f_1), (U_2, f_2)$ are two compatible charts on $X$ such that $x \in U_1 \cap U_2$, $h \circ f_1^{-1}$ is smooth at $x$ if and only if $h \circ f_2^{-1}$ is smooth at $x$.

*Proof.* Since the two charts are compatible the transition function $\phi_{12} = f_1 \circ f_2^{-1}$ is smooth. Thus, if $h \circ f_1^{-1}$ is smooth at $f_1(x)$, so is

$$h \circ f_1^{-1} \circ \phi_{12} = h \circ f_1^{-1} \circ f_1 \circ f_2^{-1} = h \circ f_2^{-1}.$$

Similar argument for the other direction. $\qquad\square$

Thus, to show that $h$ is smooth at some $x$, it suffices to find a compatible chart $(U, f)$ at $x$ such that $h \circ f^{-1}$ is smooth at $f(x)$.

**Definition 2.19** (Smooth). Let $X, Y$ be manifolds of dimension $n$ and $m$. Then a function $H : X \to Y$ is smooth at $x \in X$ if there exists a chart $(U, f)$ compatible with the smooth structure of $X$ such that $x \in U$ and a chart $(V, g)$ compatible with the smooth structure of $Y$ such that $H(x) \in V$ and $H(U) \subseteq V$ and

$$g \circ H \circ f^{-1} : f(U) \subseteq \mathbb{R}^n \to g(V) \subseteq \mathbb{R}^m$$

is smooth at $f(x)$.

We say $H$ is smooth if it is smooth at all points in $X$.

In the case that $H$ is a continuous function, we see that the condition of $H(U) \subseteq V$ can be relaxed by considering the chart on $X$, $(U \cap H^{-1}(V), f)$ in which $U \cap H^{-1}(V) \subseteq V$ is open by the continuity of $H$.

**Definition 2.20** (Diffeomorphism). A function $H : X \to Y$ between manifolds is said to be a diffeomorphism if it is smooth, a bijection, and $H^{-1}$ is smooth.

Similar to before, the definition of smoothness is independent of the choice of the charts (consider $\phi_{21}^Y \circ g_1 \circ h \circ f_1 \circ \phi_{12}^X$).

**Proposition 2.4.** Let $Y \subseteq X$ be a submanifold of $X$ and let

$$\iota_Y : Y \hookrightarrow X$$

be the inclusion map from $Y$ to $X$. Then $\iota_Y$ is smooth.

*Proof.* Let $y \in Y$, then by definition, there exists a chart $(V, g)$ on $X$ containing $y$ such that $g(V \cap Y) = g(V) \cap A$ for some $A$ an affine space. Then defining $U = V \cap Y$ and $f = g \mid_U$, we have $(U, f)$ is a chart on $Y$ and $g \circ \iota_Y \circ f^{-1}$ is the identity on $f(U)$. Thus $\iota_Y$ is smooth. $\square$

**Proposition 2.5.** Let $X, Y, Z$ be manifolds and let $H : X \to Y$ and $G : Y \to Z$ be smooth, then $G \circ H$ is also smooth.

*Proof.* Follows by considering

$$g \circ G \circ f_1^{-1} \circ \phi_{12} \circ f_2 \circ H \circ h = g \circ (G \circ H) \circ h,$$

for some appropriately chosen charts which is restricted whenever necessary. $\square$

From the two propositions above, we see that the restriction of any smooth maps on a submanifold is smooth as $F \mid_Y = F \circ \iota_Y$. In particular, we have that any smooth maps between Euclidean spaces restricted on some submanifolds of that Euclidean space is smooth (e.g. any smooth map restricted on the $n$-sphere is smooth).

**Definition 2.21** (Product Manifold). Given $X, Y$ manifolds of dimension $n$ and $m$. Then the Cartesian product $X \times Y$ is a manifold of dimension $n + m$.

To see why this is a topological manifold, consider for all $(x, y) \in X \times Y$ we may choose a chart $(U, f)$ on $X$ such that $x \in U$ and a chart $(V, g)$ in $Y$ such that $y \in V$. Then, if we define $W := U \times V$ and $h : W \to h(W) \subseteq \mathbb{R}^{n+m} := (x, y) \mapsto (f(x), g(y))$, we have $(W, h)$ is a chart of $X \times Y$ containing $(x, y)$. Similarly, using the same construction, if $X, Y$ are smooth, one may show that $X \times Y$ is also smooth.

**Definition 2.22** (Lie Group). A Lie group is a manifold $G$ which has a group structure $(G, \cdot)$ such that the multiplication and the inverse are both smooth.

An important example of a Lie group is the general linear group. In particular, as the space $M_n(\mathbb{R})$ of all $n \times n$ matrices of real coefficients is a vector space of dimension $n^2$, it is an $n^2$-dimensional manifold. Now, as $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$, it follows that it is also an $n^2$-dimensional manifold. Now, as $GL_n(\mathbb{R})$ is a group equipped with matrix multiplication, one may show that it is a Lie group by checking that the multiplication and the inverse are smooth.

### 2.3.1   Rank of a Smooth Function

For linear maps between Euclidean spaces, we have a notion of a rank by considering the dimension of the image of that map. Similarly, for smooth functions between Euclidean spaces, its rank is defined by considering the rank of its derivative. We will in this section extend this notion for smooth functions between manifolds.

**Definition 2.23** (Rank). Let $X, Y$ be manifolds of dimension $n$ and $k$, and let $H : X \to Y$ be a smooth function. Let $x \in X$ and let $(U, f)$ be a chart on $X$ such that $x \in U$, and $(V, g)$ be a chart on $Y$ at $H(x)$ such that $H(U) \subseteq V$. Then, we may define

$$\tilde{H} := g \circ H \circ f^{-1} : f(U) \to g(V).$$

As $\tilde{H}$ is a smooth function between Euclidean spaces, it has a Jacobian at $f(x)$,

$$D\tilde{H} \mid_{f(x)} \colon \mathbb{R}^n \to \mathbb{R}^k.$$

Then the rank of $H$ at $x$ is simply the rank of $D\tilde{H} \mid_{f(x)}$.

It is clear that the rank of a smooth function at a point is independent of the choice of the chart as the transition functions are diffeomorphisms. In particular, if $\tilde{H}$ and $\tilde{H}'$ results from two different choices of charts, we have $\tilde{H}' = \psi^{-1} \circ \tilde{H} \circ \phi$ where $\psi$ and $\phi$ are appropriate transition maps. Then, by the chain rule, we have

$$D\tilde{H}' \mid_{f'(x)} = D\psi^{-1} \mid_{g(H(x))} \circ D\tilde{H} \mid_{f(x)} \circ D\phi \mid_{f'(x)}.$$

Now, as $\phi, \psi$ are diffeomorphisms, both $D\psi^{-1} \mid_{g(H(x))}$ and $D\phi \mid_{f'(x)}$ are invertible, and hence, the rank of $D\tilde{H}' \mid_{f'(x)}$ is the same as $D\tilde{H} \mid_{f(x)}$.

**Definition 2.24** (Regular). Let $X, Y$ be manifolds of dimension $n$ and $k$, and let $F : X \to Y$ be a smooth function. Then $x \in X$ is said to be a regular point if the rank of $F$ at $x$ is $k$. If $x$ is not regular, then it is called a critical point.

Furthermore, $y \in Y$ is called a regular value if every point $x \in F^{-1}(y)$ is regular. Otherwise, it is called a critical value.

**Lemma 2.2.** Let $X, Y$ be manifolds of dimension $n$ and $k$, and let $F : X \to Y$ be a smooth function. Let $y \in Y$ be a regular value of $F$. Then the level set

$$Z_y := F^{-1}(y) \subseteq X$$

is a submanifold of $X$ of dimension $n - k$.

*Proof.* Let $x \in Z_y$, then there exists a chart $(U, f)$ on $X$ containing $x$ and a chart $(V, g)$ on $Y$ containing $F(U)$. Then we have the smooth function

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \to g(V).$$

Then by construction, we have $\tilde{F}^{-1}(g(y)) = f(Z_y \cap U)$. Now, since $y$ is a regular value of $F$, $x$ is a regular point of $F$, and so $f(x)$ is a regular point of $\tilde{F}$. Then, as $x \in Z_y \cap U$ was chosen arbitrarily, $\tilde{F}^{-1}(g(y))$ is a regular set (by choosing the same chart for all $x \in \tilde{F}^{-1}(g(y))$). Thus, the set $\tilde{F}^{-1}(g(y))$ is a submanifold of $\mathbb{R}^n$, and by definition, there exists a chart $(W, h)$ containing $f(x)$ such that

$$h(\tilde{F}^{-1}(g(y))) = h(W) \cap A,$$

for some affine subspace $A$. Hence, it follows $(f^{-1}(W), h \circ f)$ is a co-ordinate chart on $X$ such that

$$h \circ f(f^{-1}(W) \cap Z_y) = h \circ f(f^{-1}(W)) \cap A.$$

$\square$

### 2.3.2 Immersion and Submersion

**Definition 2.25.** Let $X, Y$ be manifolds of dimension $n$ and $k$, and let $F : X \to Y$ be a smooth function. Then $F$ is said to be a submersion if the rank of $F$ at any $x \in X$ is $k$. Furthermore, $F$ is said to be an immersion if the rank of $F$ at any $x \in X$ is $n$.

Thus, $F$ is a submersion if its Jacobian at any point $x$ is surjective. Similarly, $F$ is a immersion of its Jacobian at any point $x$ is injective.

**Lemma 2.3.** Let $Y \subseteq X$ be a $m$-dimensional submanifold, then the inclusion map

$$\iota : Y \hookrightarrow X$$

is an immersion.

*Proof.* By the definition of a submanifold, for all $y \in Y$, there exists some chart $(V, g)$ of $X$ containing $y$ such that

$$g(V \cap Y) = g(Y) \cap A,$$

for some affine subspace $A$. Then, restricting the chart on to $Y$, we have $(V \cap Y, g \mid_Y)$ is a chart on $Y$ containing $y$. Thus,

$$\tilde{\iota} := g \circ \iota \circ g \mid_Y^{-1} = \mathrm{Id}_{g(V \cap Y)}.$$

Hence, as $D\tilde{\iota} \mid_{g(y)} = D\mathrm{Id}_{g(V \cap Y)} \mid_{g(y)}$ has dimension $m$, $\iota$ is a immersion. $\square$

**Proposition 2.6.** Let $X, Y$ be $n$-dimensional manifolds, and let $F : X \to Y$ be a function which is smooth, bijective and of rank $n$ at any point $x \in X$. Then $F$ is a diffeomorphism.

*Proof.* It suffices to show that $F^{-1}$ is smooth. For all $y \in Y$, let $(U, f)$ be a chart of $F^{-1}(y)$ on $X$ and $(V, g)$ be a chart of $Y$ containing $f(U)$. Then, by definition,

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \to g(V)$$

is smooth with invertible derivative $D\tilde{F} \mid_{f(F^{-1}(y))}$. By the inverse function theorem, $\tilde{F}^{-1} = f \circ F^{-1} \circ g^{-1}$ is locally smooth at $g^{-1}(y)$ with derivative $DF^{-1} \mid_{g(y)} = D\tilde{F} \mid_{f(F^{-1}(y))}^{-1}$. Thus, as $y \in Y$ was chosen arbitrarily, $F^{-1}$ is smooth at any $y \in Y$, and so $F^{-1}$ is smooth. $\square$

**Proposition 2.7.** If $X, Y$ are manifolds of dimension $n$ and $k$, the projection map

$$p_1 : X \times Y \to X : (x, y) \mapsto x$$

is a submersion.

*Proof.* Exercise. $\square$

**Proposition 2.8.** Let $X, Y$ be manifolds of dimension $n$ and $k$, and let $F : X \to Y$ be a submersion (so $k \leq n$). Then $F$ is an open map, i.e. for any open subset $W \subseteq X$, $F(W)$ is open in $Y$.

*Proof.* Let $y \in F(W)$ and we will show that $F(W)$ is locally open as $y$. Let $x \in F^{-1}(y) \cap W$ and suppose $(U, f)$ be a chart at $x$ contained in $W$ and $(V, g)$ be a chart at $y$ such that $F(U) \subseteq V$. Since $F$ is a submersion, the function

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \rightarrow g(V)$$

is smooth and has rank $k$ at $f(x)$, and so $f(x)$ is a regular point of $\tilde{F}$. Then, by the implicit function theorem, there exists an open set $U' \subseteq f(U)$ containing $f(x)$ and a diffeomorphism $h : U' \rightarrow h(U')$ such that $\tilde{F} \circ h^{-1} = \pi : h(U') \rightarrow \mathbb{R}^k$ is the standard projection.

Now as the standard projection and diffeomorphisms are an open maps, we have $\tilde{F} = \pi \circ h$ is also an open map, and in particular, $\tilde{F}(U')$ is open. Hence, as $g$ is a homeomorphism, $\tilde{F}(U') = g(F(f^{-1}(U')))$ is open in $g(V)$ implies $F(f^{-1}(U')) \subseteq F(U) \subseteq F(W)$ is open in $Y$. Thus, $F(W)$ is locally open at $y$, and so $F(W)$ is open and $F$ is an open map. $\square$

# 3    Tangent Spaces

The tangent space of a surface is a intuitive concept and for a surface in $\mathbb{R}^n$, we may often define it to be an affine subspace of $\mathbb{R}^n$ tangent to the surface at a point $x$. This definition in requires an ambient space and so is not generalizable to manifolds. We will in this section define a new notion of tangent spaces which applies to manifolds.

Let $U \subseteq \mathbb{R}^n$. Let $x \in U$, then a curve $\sigma$ through $x$ is a smooth function from $(-\epsilon, \epsilon)$ to $U$ for some $\epsilon > 0$ and $\sigma(0) = x$. Then, we define the tangent vector of $\sigma$ at the point $x$ to be the Jacobian $D\sigma \mid_0$.

The space of curves is massive though we really only care about their tangent vectors, and so we introduce an equivalence relation which we will take the quotient by. Let $\sigma_1, \sigma_2$ are two curves through $x$, then we say $\sigma_1 \sim \sigma_2$ if and only if $D\sigma_1 \mid_0 = D\sigma_2 \mid_0$. Then, we define the tangent space of $U$ at $x$ to be,

$$T_x U := \{\sigma : (-\epsilon, \epsilon) \to U \mid \sigma \text{ smooth curve through } x\}/ \sim .$$

We have a natural bijection

$$\Delta_x : T_x U \to \mathbb{R}^n : [\sigma] \mapsto D\sigma \mid_0 .$$

In particular, for all $v \in \mathbb{R}^n$, we can define $\sigma : (-\epsilon, \epsilon) \to U : t \mapsto x + tv$ such that $D\sigma \mid_0 = v$ and so $\Delta_x$ is surjective.

This definition can be extended to manifolds easily.

**Definition 3.1** (Curves). Let $X$ be a $n$-dimensional manifold and let $x \in X$. Then a curve through $x$ is a smooth function

$$\sigma : (-\epsilon, \epsilon) \to X,$$

such that $\sigma(0) = x$.

**Definition 3.2.** Let $X$ be a $n$-dimensional manifold and let $x \in X$ and $\sigma$ be a curve through $x$. Then, given a chart $(U, f)$ containing $x$, we may define $\tilde{\sigma} := f \circ \sigma : (-\epsilon, \epsilon) \to f(U)$, i.e. $\tilde{\sigma}$ is a curve through $f(x)$. Finally, if $\sigma_1, \sigma_2$ are curves through $x$, we say

$$\sigma_1 \sim \sigma_2 \iff \tilde{\sigma}_1 \sim \tilde{\sigma}_2.$$

For the above definition to be well-defined, we will need to check it is independent of the choice of the chart. Let $(U_1, f_1), (U_2, f_2)$ be two charts of $X$ containing $x$. Then, if $f_1 \circ \sigma_1 \sim f_1 \circ \sigma_2$, we have

$$D(f_2 \circ \sigma_1) \mid_0 = D(\phi_{12} \circ f_1 \circ \sigma_1)_0 = D(\phi_{12}) \mid_{f_1(x)} \circ D(f_1 \circ \sigma_1)_0$$
$$= D(\phi_{12}) \mid_{f_1(x)} \circ D(f_1 \circ \sigma_2)_0 = D(\phi_{12} \circ f_1 \circ \sigma_2)_0 = D(f_2 \circ \sigma_2) \mid_0 .$$

**Definition 3.3** (Tangent Space). Let $X$ be a $n$-dimensional manifold and let $x \in X$, then the tangent space of $X$ at $x$ is

$$T_x X := \{\sigma : (-\epsilon, \epsilon) \to X \mid \sigma \text{ is a curve through } x\}/ \sim .$$

We call elements of $T_x X$ tangent vectors.

Let $(U, f)$ be a chart of $X$ containing $x$. Then, we have a bijection

$$T_x X \to T_{f(x)} f(U) : [\sigma] \mapsto [f \circ \sigma].$$

Now as $f(U)$ is an open subset of $\mathbb{R}^n$, there exists a bijection $\Delta_{f(x)} : T_{f(x)} f(U) \to \mathbb{R}^n$. Composing the two functions, we obtain a bijection

$$\Delta_f : T_x X \to \mathbb{R}^n : [\sigma] \mapsto D(f \circ \sigma)\mid_0 .$$

With this function, we may equip the tangent space with a vector space structure. In particular, for all $\alpha \in \mathbb{R}, v, w \in T_x X$, we define $\alpha \cdot v := \Delta_f^{-1}(\alpha \cdot \Delta_f(v))$ and $v + w := \Delta_f^{-1}(\Delta_f(v) + \Delta_f(w))$.

We note that $\Delta_f$ depends on the choice of the chart. On the other hand, the vector space structure on $T_x X$ is independent of the choice of the chart (Hint : $\Delta_{f_2} = D\phi_{21}\mid_{f_1(0)} \circ \Delta_{f_1}$).

The tangent space allow us to talk about the Jacobian of a smooth function between to manifolds.

**Definition 3.4** (Jacobian of Smooth Functions)**.** Let $X, Y$ be manifolds and let $F : X \to Y$ be a smooth function. We define the Jacobian at $x \in X$ as

$$DF\mid_x T_x X \to T_{f(x)} Y : [\sigma] \mapsto [F \circ \sigma].$$

One may check that $DF\mid_x$ is well-defined and is a linear map using similar arguments as above (hint: for linearity, show $DF\mid_x = \Delta_g^{-1} \circ D\tilde{F}\mid_x \circ \Delta_f$ where $\tilde{F} = g \circ F \circ f^{-1}$). Furthermore, as one might expect, the rank of a smooth function is simply the rank of $DF\mid_x$.

## 3.1   Tangent Space to Submanifolds

Let $Z$ is a $m$-dimensional submanifold of $X$, then as we have shown before, the inclusion map $\iota : Z \hookrightarrow X$ is a smooth immersion. Then, each curve in $Z$ is a curve in $X$ and so, for all $z \in Z$, we may define the linear injection

$$D\iota\mid_z : T_z Z \hookrightarrow T_z X : [\sigma] \mapsto [\sigma].$$

In this sense, we can view $T_z Z$ as a subspace of $T_z X$, and we can see this explicitly in co-ordinates. Let $(U, f)$ be a chart of $X$ containing $z$ such that

$$f(U \cap Z) = f(U) \cap \mathbb{R}^m.$$

Then, by considering for all curves $\sigma$ in $Z$ through $z$, we have

$$\Delta_f(\sigma) = f \circ \sigma : (-\epsilon, \epsilon) \to Z \to \mathbb{R}^m,$$

for sufficiently small $\epsilon > 0$; and hence, $\Delta_f(T_z Z) \subseteq \mathbb{R}^m$. Now, as $\Delta_f$ is linear isomorphism between $T_z X$ and $\mathbb{R}^n$, $\Delta_f(T_z Z)$ must be a subspace of $\mathbb{R}^n$ of $m$ dimension, and so $\Delta_f(T_z Z) = \mathbb{R}^m$.

**Lemma 3.1.** If $X, Y$ are manifolds of dimension $n$ and $m$ respectively, and $F : X \to Y$ is a smooth function. Then, for all regular points $x \in X$, there exists a chart $(U, f)$ of $X$ containing $x$ and $(V, g)$ of $Y$ containing $F(x)$ such that,

$$\pi = g \circ F \circ f^{-1} : f(U) \to g(V),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the standard projection.

*Proof.* Choose a chart $(W, h)$ at $x$, and $(V, g)$ at $F(x)$, then by the definition of smoothness, we have $g \circ F \circ h^{-1}$ is smooth. Now, as $x$ is a regular point, $D(g \circ F \circ h^{-1}) \mid_{h(x)}$ is a surjection, and so, by the implicit function theorem, there exists an open set $U' \subseteq h(W)$ containing $h(x)$ and a diffeomorphism $f' : U' \to f'(U')$ such that

$$g \circ F \circ h^{-1} \circ f'^{-1} : f'(U') \to g(V),$$

is the standard projection. Thus, defining $U := h^{-1}(U')$ and $f := h^{-1} \circ f'^{-1}$, we have found a chart $(U, f)$ of $X$ containing $x$ such that

$$g \circ F \circ f^{-1} : f(U) \to g(V)$$

is the restriction of the standard projection. $\qquad\square$

**Lemma 3.2.** Let $X$ be an $n$-dimensional manifold and let $Y$ be an $m$-dimensional manifold where $m \leq n$. Let $F : X \to Y$ be a smooth function and let $y \in Y$ be a regular value of $F$. Then, if $Z = F^{-1}(\{y\})$ and $z \in Z$, $T_z Z$ is the kernel of the linear map

$$DF \mid_z : T_z X \to T_y Y.$$

*Proof.* By the above lemma, there exists a chart $(U, f)$ of $X$ containing $z$ and a chart $(V, g)$ of $Y$ containing $y = F(z)$ and (Wlog. by translating) $g(y) = 0$, such that $\tilde{F} = g \circ F \circ f^{-1} : f(U) \to g(V)$ is the restriction of the standard projection and so,

$$DF \mid_z = \Delta_g^{-1} \circ D\tilde{F} \mid_z \circ \Delta_f = \Delta_g^{-1} \circ \pi \mid_z \circ \Delta_f,$$

where $\pi$ has the kernel $\mathbb{R}^{n-m}$. Now, by considering

$$f(Z \cap U) = \tilde{F}^{-1}(g(y)) = f(U) \cap \pi^{-1}(y),$$

as the kernel of $D\tilde{F}$ is simply $T_z Z$ as $T_z Z \simeq \mathbb{R}^{n-m}$ and $D\tilde{F} \simeq \pi$ using the chosen chart. $\quad\square$

Consider the Lie group $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$. It is not difficult to show that the determinant map

$$\det GL_n(\mathbb{R}) \to \mathbb{R} : A \mapsto \det A$$

is a smooth function. We will show that 1 is a regular value of det and so the special linear group (note that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$)

$$SL_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$$

is a submanifold of $GL_n(\mathbb{R})$ and so is a Lie group.

Recall that the determinant of the matrix $A$ is $\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$, for any $i = 1, \cdots, n$, where $a_{ij}$ is the $(i,j)$-th entry of $A$ and $A_{ij}$ is the $(i,j)$-th minor of $A$. Thus,

$$\frac{\partial \det}{\partial a_{ij}} \mid_A = (-1)^{i+j} \det A_{ij}$$

since the minor $\det A_{ij}$ is independent of $a_{ij}$. Hence, it follows that if $A \in \det^{-1}(1)$ is critical, then $\det A_{ij} = 0$ for all $i, j$ which implies $\det A = 0$, a contradiction! # Thus, $SL_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$ of dimension $n^2 - 1$. With this, we may compute the tangent space of $SL_n(\mathbb{R})$ at some $A$ by

$$T_A SL_n(\mathbb{R}) = \ker D(\det) \mid_A = \{x \in GL_n(\mathbb{R}) \mid x \cdot [(-1)^{i+j} \det A_{ij}]_{ij} = 0\}.$$

## 3.2 Vector Fields and Flows

In the case of $\mathbb{R}^n$, we have a good intuition for what a vector field is, i.e. a smooth function from $U \subseteq \mathbb{R}^n$ open which maps $x \in U$ to a vector in $T_x U \simeq \mathbb{R}^n$. In this sense the codomain of a vector field is a union of all tangent spaces. This motivates the following definition.

**Definition 3.5.** The tangent bundle of the open set $U \subseteq \mathbb{R}^n$ is defined to be

$$TU := \bigcup_{x \in U} T_x U \simeq U \times \mathbb{R}^n,$$

where the homeomorphism follows as $T_x U \simeq \mathbb{R}^n$ for all $x \in U$.

As $TU$ is the product of two $n$-dimensional manifolds, if is a manifold of dimension $2n$.

**Definition 3.6.** A vector field on the open set $U \subseteq \mathbb{R}^n$ is a smooth function $\xi : U \to TU$ such that $\pi \circ \xi = \mathrm{id}_U$ where $\pi : TU \to U$ is the projection map $(x, v) \mapsto x$.

Thus, for all $x \in U$, if $\xi$ is a vector field of $U$ then $\xi(x) = (x, v(x))$ where $v(x) \in T_x U$.

We may extend this idea to general manifolds.

**Definition 3.7** (Tangent Bundle)**.** Let $X$ be a manifold. We define the tangent bundle $TX$ of $X$ as

$$TX := \bigcup_{x \in U} T_x X.$$

We note that unlike the definition in $\mathbb{R}^n$, this union is disjoint as individual tangent vectors lives in different tangent spaces. Furthermore, we may not express the tangent bundle as a product without choosing a chart.

Similar to the projection map in $\mathbb{R}^n$, there is induced map from the tangent bundle $\pi : TX \to X : v \mapsto x$ for all $v \in T_x X$. As $v$ is uniquely identified by the tangent space it lives in, this function is well-defined.

**Proposition 3.1.** Let $X$ be a $n$-dimensional manifold. Then $TX$ is a $2n$-dimensional manifold and the projection $\pi : TX \to X$ is smooth.

*Proof.* We will first define a topology on $TX$ by defining a topology on individual charts.

Given $(U, f)$ a chart on $X$, let $TU := \pi^{-1}(U) = \bigcup_{x \in U} T_x X$. Recall that, for all $x \in U$, we have the linear isomorphism $\Delta_f : T_x X \to \mathbb{R}^n$, and thus, we may construct the bijection

$$F_U : TU = \bigcup_{x \in U} T_x X \to f(U) \times \mathbb{R}^n : v \in T_x X \mapsto (f(x), \Delta_f(v)).$$

It is easy to check that this is a bijection. We see that $F_U$ is injective as $f$ is an homeomorphism, and for all $(f(x), w) \in f(U) \times \mathbb{R}^n$, as $\Delta_f : T_x X \to \mathbb{R}^n$ is a linear isomorphism, there exists some $v \in T_x X$ such that $\Delta_f(v) = w$ and so, $F_U(v) = (f(x), w)$ as required. With this, $F_U$ induces a topology on $TU$ be defining $W \subseteq TU$ is open if $F_U(W)$ is open in $f(U) \times \mathbb{R}^n$.

Now, consider an atlas $\mathcal{A} = \{(U_i, f_i)\}_{i \in \mathcal{J}}$ on $X$. Through the above procedure, we may define a topology for each $TU_i = \pi^{-1}(U_i)$. Then, as $\bigcup_{i \in I} TU_i = TX$ (note that this union is not necessary disjoint as $U_i$ might not be), we may define the topology on $TX$ where a $W \subseteq TX$ is open if $U \cap TU_i$ is open for all $i \in \mathcal{J}$.

It is easy to see that this procedure produces a Hausdorff and second countable topology (exercise).

Now to see that $TX$ is a topological manifold we will need to construct an atlas on $TX$. This is clear as for each $T_x X \in TX$, there exists some chart $(U, f)$ on $X$ containing $x$. Then, by construction

$$F_U : TU \to f(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$$

is a homeomorphism, $(TU, F_U)$ is a valid chart on $TX$ containing $x$.

Finally, to show that $TX$ is a smooth manifold, it suffices to show that the transition functions from the above lattice are smooth. But this is clear since by construction

$$(F_{U_i} \circ F_{U_j}^{-1})(x, v) = ((f_{U_i} \circ f_{U_j}^{-1})(x), (\Delta_{f_i} \circ \Delta_{f_j}^{-1})(v)) = (\phi_{ij}(x), D\phi_{ij} \mid_x (v)).$$

Thus, the transition function is smooth as its components are smooth.

To see that $\pi : TX \to X$ is a smooth function, let $w \in TX$ so there exists some $i$ such that $w \in TU_i$, then we will show

$$\tilde{\pi} := f_i \circ \pi \circ F_{U_i}^{-1} : F_{U_i}(TU_i) \to f_i(U_i)$$

is smooth at $F_{U_i}(w)$ with appropriately chosen charts. But since $\tilde{\pi} : f_i(U_i) \times \mathbb{R}^n \to f_i(U_i)$ is simply the projection function, it is smooth, and hence $\pi$ is smooth. $\square$

**Proposition 3.2.** The projection $\pi : TX \to X$ is a submersion.

*Proof.* Define $\tilde{\pi}$ as above. But this is simply a projection function from $\mathbb{R}^{2n} \to \mathbb{R}^n$ and thus, has rank $n$. $\square$

**Definition 3.8** (Vector Field). Let $X$ be a manifold. A vector field on $X$ is a smooth function $\xi : X \to TX$ such that $\pi \circ \xi = \mathrm{id}_X$.

In particular, we see that $\xi(x) \in T_x X$ for all $x \in X$.

**Lemma 3.3.** Let $X$ be a manifold and let $\xi : X \to TX$ be a function such that $\pi \circ \xi = \mathrm{id}_X$. Then $\xi$ is a vector field if and only if for every chart $(U, f)$ on $X$, we have

$$F_U \circ \xi \mid_U \circ f^{-1} : f(U) \to f(U) \times \mathbb{R}^n$$

is smooth (where $F_U$ is the bijection between $TU = \bigcup_{x \in U} T_x X$ and $f(U) \times \mathbb{R}^n$).

*Proof.* By the definition of smooth functions (only need to check $\xi(U) \subseteq TU$ so that $(TU, F_U)$ is a chart at $\xi(x)$). $\square$

**Proposition 3.3.** Let $X$ be an $n$-dimensional manifold and let $Z \subseteq X$ be a submanifold of $X$ with the inclusion map $\iota : Z \hookrightarrow X$. Then the image of $D\iota : TZ \to TX$ is a submanifold of $TX$.

*Proof.* Denote $\pi_Z : TZ \to Z$ and $\pi_X : TX \to X$ for the respective projective maps. Let $w \in TZ$ and $z = \pi_Z(w)$. Since $Z$ is a submanifold, there exists a chart $(U, f)$ on $X$ containing $z$ such that

$$f(U \cap Z) = f(U) \cap A$$

for some affine subspace $A \subseteq \mathbb{R}^n$. Then, it follows

$$F_U(TU \cap Di(TZ)) = F_U(TU) \cap (A \times A)$$

where $A \times A$ is a affine subspace of $\mathbb{R}^{2n}$. $\qquad\square$

**Lemma 3.4.** Let $X$ be a manifold and $Z$ is a submanifold of $X$. If $\xi$ is a vector field on $X$ such that $\xi(z) \in T_z Z$ for all $z \in Z$, then $\xi \mid_Z$ is a vector field on $Z$.

*Proof.* As the inclusion map $\iota : Z \hookrightarrow X$ is smooth, so is $\xi \mid_Z = \xi \circ \iota$. By assumption, $\xi \mid_Z (z) \in TZ$ for all $z \in Z$, and thus the induced morphism $\xi \mid_Z : Z \to TZ$ is also smooth. Finally, as $\pi_Z \circ \xi \mid_Z = \mathrm{id}_X \mid_Z = \mathrm{id}_Z$, we have $\xi \mid_Z$ is a vector field on $Z$. $\qquad\square$

**Theorem 3** (Hairy Ball Theorem). Any vector field on $S^2$ is zero at some point.

*Proof.* Requires algebraic topology or differential topology. $\qquad\square$

**Definition 3.9** (Flow). Let $X$ be a manifold. A one-parameter family of diffeomorphisms or flow on $X$ is a smooth function

$$F : (-\epsilon, \epsilon) \times X \to X$$

for some $\epsilon > 0$, such that

- for all $s \in (-\epsilon, \epsilon)$, the function

$$F_s : X \to X : x \mapsto F(s, x)$$

  is a diffeomorphism,

- $F_0 = \mathrm{id}$,

- $F_t \circ F_s = F_{s+t}$ for all $t, s \in (-\epsilon, \epsilon)$ and $s + t \in (-\epsilon, \epsilon)$.

Let $X$ be a manifold and let $F : (-\epsilon, \epsilon) \times X \to X$ be a flow on $X$. Then instead of fixing $s \in (-\epsilon, \epsilon)$, fixing $x \in X$, we obtain a smooth function

$$\sigma_x : (-\epsilon, \epsilon) \to X : t \mapsto F(t, x).$$

Straight away, we observe, as $F_0 = \mathrm{id}$, we have

$$\sigma_x(0) = F_0(x) = x,$$

and so $\sigma_x$ is a curve through $x$. By recalling that the tangent space at $x$ is defined to be the set of equivalent classes of the curves through $x$, this provides a function

$$\xi^F : X \to TX : x \mapsto [\sigma_x].$$

This is a vector field on $X$ and is known as the infinitesimal version of the flow $F$.

**Proposition 3.4.** Let $X$ be a compact manifold and let $\xi$ be a vector field on $X$. Then for any $x \in X$, there exists an open subset $U \subseteq X$ containing $x$ and a flow

$$F : (-\epsilon, \epsilon) \times U \to X,$$

such that $\xi \mid_U$ is the infinitesimal version of $F$, i.e. $\xi \mid_U = \xi^F$. That is to say, any vector field is locally a flow.

*Proof.* The proof of this proposition requires results from PDEs. $\qquad\square$

Let $X$ be an $n$-dimensional manifold and let $F : (-\epsilon, \epsilon) \times X \to X$ be a flow on $X$. Let $(U, f)$ be a chart on $X$ such that for all $s \in (-\epsilon, \epsilon)$, we have $F_s(U) \subseteq U$. Then we may define the smooth function

$$\tilde{F} : (-\epsilon, \epsilon) \times f(U) \to f(U) : (t, y) \mapsto (f \circ F_t \circ f^{-1})(y).$$

Since $f(U) \subseteq \mathbb{R}^n$, we may write

$$\tilde{F} = (\tilde{F}_i)_{i=1}^n, \tilde{F}_i : (-\epsilon, \epsilon) \times f(U) \to \mathbb{R}.$$

Then the infinitesimal version of $f(U)$ is $\tilde{\xi}^F = \frac{\partial \tilde{F}}{\partial s} \mid_0$. Thus,

$$\xi^F = \Delta_f^{-1} \circ \tilde{\xi}^F \circ f : U \to TU,$$

is the corresponding vector field on $U$.

## 3.3 Cotangent Space

**Definition 3.10** (Cotangent Space)**.** Let $X$ be a manifold and let $x \in X$ and define

$$R_x(X) := \{h : X \to \mathbb{R} \mid \mathrm{rk}(h) \mid_x = 0\} \le C^\infty(X),$$

where $C^\infty(X)$ is the space of smooth maps between $X$ and $\mathbb{R}$. Then the cotangent space is the quotient
$$T_x^*(X) = C^\infty(X)/R_x(X).$$

The cotangent space can be thought of as the dual space of the tangent space, and hence the notation. As one might expect, the dimension of a cotangent space equals the dimension of the tangent space and hence, the dimension of the manifold. As with before, we will first prove this for $\mathbb{R}^n$.

**Lemma 3.5.** Let $U \subseteq \mathbb{R}^n$ be open and let $x \in U$. Then $\dim T_x^* X = n$.

*Proof.* Define the map
$$\nabla_x : C^\infty(U) \to \mathbb{R}^n : h \mapsto Dh \mid_x .$$

Clearly $\nabla_x$ is linear and $\ker \nabla_x = R_x(U)$. Then, by the first isomorphism, we have $T_x^* U \cong \mathrm{Im}(\nabla_x)$ and it suffices to show $\nabla_x$ is surjective. But this is clear as for all $v := (v_1, \cdots, v_n) \in \mathbb{R}^n$, defining $h(w) := v \cdot w$, we have $\nabla_x h = v$ and so $\nabla_x$ is surjective. $\qquad\square$

Now to extend this idea to general manifolds, we quickly realise we will need to be able to extend a smooth function defined on a subset to the whole space. We will achieve this with the bump function.

Let

$$\phi : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} e^{-1/x}, \ x > 0 \\ 0, \ x \leq 0. \end{cases}$$

It is not difficult to show that $\phi$ is smooth and $\phi^{\langle} k)(0) = 0$ for all $k \geq 0$. Now, define

$$\psi(x) := \frac{\phi(x)}{\phi(x) - \phi(1-x)},$$

as $\phi(x) - \phi(1-x) > 0$ for all $x \in \mathbb{R}$, we see that $\psi$ is also a smooth function such that $\psi(x) = 0$ for $x \leq 0$ and $\psi(x) = 1$ for $x \geq 1$. Now, for real numbers $0 < r < r'$, we define

$$\tilde{\psi} : \mathbb{R}^n \to \mathbb{R} : x \mapsto \psi\left(\frac{|x| - r'}{r - r'}\right).$$

One can show that $\tilde{\psi}$ is smooth and for $x \notin B_{r'}(0)$ $\tilde{\psi}(x) = 0$ and for $x \in B_r(1)$, $\tilde{\psi}(x) = 1$. This function is known as the bump function for $\mathbb{R}^n$.

This idea can be generalized to general manifolds. Let $X$ be a manifold and let $x \in X$ and suppose that $(U, f)$ is a chart on $X$ containing $x$ such that $f(x) = 0$. Now, as $f(U)$ is open, there exists some $r' > 0$, such that $\overline{B_{r'}(0)} \subseteq f(U)$. Taking $0 < r < r'$, we define

$$\rho : X \to \mathbb{R} : y \mapsto \begin{cases} (\tilde{\psi} \circ f)(y), \ y \in U, \\ 0, \ y \notin U. \end{cases}$$

It is clear that $\rho$ is smooth and $\rho(y) = 0$ for all $y \notin U$ and $\rho(y) = 1$ for all $y \in f^{-1}(B_r(0))$. We call this function a bump function with respect to the chart $(U, f)$.

**Proposition 3.5.** Let $X$ be an $n$-dimensional manifold and let $x \in X$. Then $\dim T_x^* X = n$.

*Proof.* Let $(U, f)$ be a chart on $X$ containing $x$ and $f(x) = 0$. Then for all $h \in C^\infty(X)$, define $\tilde{h} := h \circ f^{-1} : f(U) \to \mathbb{R}$. Then $h \in R_x(X)$ if and only if the rank of $\tilde{h}$ is zero at $f(x)$, in particular, $h \in \ker \nabla_{f,x}$ where

$$\nabla_{f,x} : C^\infty(X) \to \mathbb{R} : h \mapsto D\tilde{h} \mid_{f(x)}.$$

Again, by the first isomorphism theorem, $T_x^* X \cong \text{Im}(\nabla_{f,x})$ and again, it suffices to show $\nabla_{f,x}$ is surjective. However, this is a bit tricky since we now require a smooth function on $C^\infty(X)$ rather than $C^\infty(U)$ and so, we will use the bump function.

For all $v := (v_1, \cdots, v_n)$, define $\tilde{h}(w) := v \cdot w$ so that $\nabla_0 \tilde{h} = v$. Now, define

$$h : X \to \mathbb{R} : y \mapsto \begin{cases} \rho(\tilde{h}(f(y))), \ y \in U, \\ 0, \ y \notin U, \end{cases}$$

we see that for some open $W \subseteq U$, $x \in W$, $\tilde{h} \mid_W = (h \circ f^{-1}) \mid_W$ and so

$$\nabla_{f,x}(h) = D\tilde{h} \mid_{f(v)} = v,$$

and $\nabla_{f,x}$ is surjective as required. $\qquad \square$

For all $h \in C^\infty(X)$, we denote $dh \mid_x := [h] \in T_x^* X$.

Suppose $h \in C^\infty(X)$, $x \in X$ and $(U, f), (U', f')$ are two charts on $X$ containing $x$. Then $\nabla_f(dh \mid_x) = D\tilde{h} \mid_{f(x)}$ where $\tilde{h} = h \circ f^{-1}$ and $\nabla_{f'}(dh \mid_x) = D\tilde{h}' \mid_{f'(x)}$ where $\tilde{h}' = h \cdot f'^{-1}$. Thus,

$$\tilde{h}' = \tilde{h} \circ f \circ f' - 1 = \tilde{h} \circ \phi^{-1},$$

where $\phi$ is the transition map between $f$ and $f'$. Then,

$$\nabla_{f'}(dh \mid_x) = D(\tilde{h} \circ \phi^{-1}) \mid_{f'(x)} = D\tilde{h} \mid_{f(x)} \circ D\phi^{-1} \mid_{f'(x)} = \nabla_f(dh \mid_x) \circ D\phi^{-1} \mid_{f'(x)}.$$

As mentioned previously, we may think about the cotangent space as the dual of the tangent space. This notion can be made rigorous with the following.

Let $U \subseteq \mathbb{R}^n$ be an open set at let $x \in U$. Then for $v \in T_x U \simeq \mathbb{R}^n$, we may write $v = (v_1, \cdots, v_n)$ and so, we may define the partial derivative at $x$ in the direction of $v$ as the linear map

$$\partial_{x,v} : C^\infty(U) \to \mathbb{R} : h \mapsto Dh \mid_x (v).$$

Then, for all $h \in R_x(U)$, $\partial_{x,v}(h) = 0$, and so $R_x(X) \subseteq \ker \partial_{x,v}$ and the map $\partial_{x,v}$ induces a injection from $T_x^* U$ to $\mathbb{R}$. As $\partial_{x,v}$ is linear $\partial_{x,v} \in (T_x^* U)^* =: \operatorname{Hom}(T_x^* U, \mathbb{R})$. Thus, we have a linear map

$$T_x U \to (T_x^* U)^* : v \mapsto \partial_{x,v},$$

and we will show this is an isomorphism.

Suppose $v \in T_x X$ such that $\partial_{x,v} = 0$. Then, taking $h(y) := v \cdot y$, we have $0 = \partial_{x,v}(h) = v \cdot v$ implying $v = 0$ and so, the map is injective. On the other hand, as $\dim(T_x^* U)^* = \dim T_x^* U = \dim T_x U$, the map is indeed an isomorphism. Thus, in some sense, the tangent space is canonically isomorphic to the dual of the cotangent space. Hence, we have a canonical isomorphism

$$(T_x U)^* \simeq ((T_x^* U)^*)^* \simeq T_x^* U.$$

Now, suppose $X$ is an $n$-dimensional manifold and $x \in X$. Then, for any $v = [\sigma] \in T_x X$, we define

$$\partial_{x,v} : C^\infty(X) \to \mathbb{R} : h \mapsto \frac{d(h \circ \sigma)}{dt}\bigg|_0.$$

Then given a chart $(U, f)$ on $X$ containing $x$, we have a curve

$$\tilde{\sigma} := f \circ \sigma : (-\epsilon, \epsilon) \to f(U) \subseteq \mathbb{R}^n.$$

In particular, we may write $\tilde{\sigma} = (\tilde{\sigma}_1, \cdots, \tilde{\sigma}_n)$. Then, $h \circ \sigma = \tilde{h} \circ \tilde{\sigma}$ where $\tilde{h} = h \circ f^{-1} \in C^\infty(f(U))$. Thus, we have

$$\frac{d(h \circ \sigma)}{dt}\bigg|_0 = \frac{d(\tilde{h} \circ \tilde{\sigma})}{dt}\bigg|_0 = D\tilde{h} \mid_{f(x)} \circ D\tilde{\sigma} \mid_0 = \sum_{i=1}^n \frac{\partial \tilde{h}}{\partial x_i}\bigg|_{f(x)} \frac{d\tilde{\sigma}_i}{dt}\bigg|_0 = \nabla_f(dh \mid_x)^T \cdot \Delta_f([\sigma]).$$

Now, by considering if $h \in R_x(X)$ then $\partial_{x,v}(h) = \frac{d(h \circ \sigma)}{dt}\big|_0 = 0$, we have $R_x(X) \subseteq \ker \partial_{x,v}$ and so, we obtain a map

$$T_x X \to (T_x^* X)^* : v \mapsto \partial_{x,v}.$$

**Proposition 3.6.** Let $X$ be an $n$-dimensional manifold and let $x \in X$. Then the map

$$T_x X \to (T_x^* X)^* : v \mapsto \partial_{x,v}$$

is an isomorphism of vector spaces.

*Proof.* Let $(U, f)$ be a chart on $X$ containing $x$ and recall that

$$\Delta_f : T_x \to \mathbb{R}^n, \ \nabla_f : T_x^* X \to \mathbb{R}^n$$

and both isomorphisms of vector spaces. Let $\sigma : (-\epsilon, \epsilon) \to X$ be a curve through $x$ and let $h \in C^\infty(X)$. Then as we have shown above, defining $v = [\sigma]$, we have

$$\partial_{x,v}(dh \mid_x) = \nabla_f(dh \mid_x)^T \cdot \Delta_f(v).$$

In particular, the map

$$(\nabla_f)^* \circ (v \mapsto \partial_{x,v}) \circ \Delta_f^{-1} : \mathbb{R}^m \to (\mathbb{R}^n)^* : w \mapsto v = [\sigma] \mapsto \partial_{x,v} \mapsto \tilde{\partial}_{x,v}$$

maps any vector $w$ to the linear functional $u \mapsto u \cdot w$. Thus, similar to the Euclidean case, the map is an isomorphism. $\square$

With this, a similar conclusion can be made by taking the dual on both sides of the isomorphism, obtaining the canonical isomorphism

$$(T_x X)^* \simeq ((T_x^* X)^*)^* \simeq T_x^* X.$$

## 3.4 Derivation

So far, we have defined the cotangent space and showed the existence of an isomorphism between $T_x X$ and $(T_x^* X)^*$ via $v \mapsto \partial_{x,v}$. We will again treat this topic but this time through an algebraic point of view with derivations.

**Definition 3.11** (Derivation)**.** Let $X$ be an $n$-dimensional manifold and let $x \in X$. A derivation of $X$ at $x$ is a linear map $\mathfrak{d} : C^\infty(X) \to \mathbb{R}$ such that

$$\mathfrak{d}(h_1 h_2) = h_1(x)\mathfrak{d}(h_2) + h_2(x)\mathfrak{d}(h_1).$$

We call this property the Leibniz rule and note the similarity of the statement with the Leibniz rule (or product rule) from calculus. We denote

$$\mathrm{Der}_x(X) := \{\text{derivations of } X \text{ at } x\}.$$

We note that the definition of the Leibniz rule depends on the choice of $x$ though later, we shall remove this dependence.

**Theorem 4.** Let $X$ be a manifold and let $x \in X$. A linear map $\mathfrak{d} : C^\infty \to \mathbb{R}$ is a derivation of $X$ at $x$ if and only if $\mathfrak{d}(h) = 0$ for all $h \in R_x(X) = \{h \in C^\infty \mid \mathrm{rk}(h) \mid_x = 0\}$.

*Proof of the reverse direction of theorem 4.* The forward direction requires some further definitions so let us prove the reverse direction first.

Suppose $\mathfrak{d}$ satisfies $\mathfrak{d}(h) = 0$ for all $h \in R_x(X)$. Let $h_1, h_2 \in C^\infty(X)$ and define

$$h(y) := h_1(y)h_2(y) - h_1(x)h_2(y) - h_2(x)h_1(y).$$

We see that (as the Jacobian satisfies the Leibniz rule),

$$Dh\mid_x = h_1(x)Dh_2\mid_x + h_2(x)Dh_1\mid_x - h_1(x)Dh_2\mid_x - h_2(x)Dh_1\mid_x = 0$$

and so $h \in R_x(X)$. By assumption, $\mathfrak{d}(h) = 0$ and by linearity of $\mathfrak{d}$,

$$0 = \mathfrak{d}(h) = \mathfrak{d}(h_1 h_2) - h_1(x)\mathfrak{d}(h_2) - h_2(x)\mathfrak{d}(h_1)$$

implying

$$\mathfrak{d}(h_1 h_2) = h_1(x)\mathfrak{d}(h_2) + h_2(x)\mathfrak{d}(h_1)$$

as required. $\qquad\square$

As with other manifold proofs, we will first consider the statement in $\mathbb{R}^n$ and then generalize.

**Lemma 3.6.** Let $U \subseteq \mathbb{R}^n$ be an open subset and let $x \in U$ and $\mathfrak{d}$ be a derivation as $x$. Then for all $h \in R_x(X)$, $\mathfrak{d}(h) = 0$.

*Proof.* Consider the constant function 1. By Leibniz rule, we have

$$\mathfrak{d}(1) = \mathfrak{d}(1 \cdot 1) = \mathfrak{d}(1) + \mathfrak{d}(1) = 2\mathfrak{d}(1)$$

implying $\mathfrak{d}(1) = 0$. Now, since $\mathfrak{d}$ is linear, it follows $\mathfrak{d}(c) = 0$ for all $c \in \mathbb{R}$. Now, let $h \in R_x(U)$, then Taylor's theorem implies

$$h(y) = h(x) + \sum_{i=1}^{n}(y_i - x_i)G_i(y),$$

for some $G_i \in C^\infty(X)$. Furthermore, by the choice of $h$, $G_i(x) = 0$ for all $i = 1, \cdots, n$. Thus,

$$\mathfrak{d}(h) = \mathfrak{d}(h(x)) + \sum_{i=1}^{n}\mathfrak{d}((y_i - x_i)G_i(y)) = \sum_{i=1}^{n}(x_i - x_i)\mathfrak{d}(G_i(y)) + G_i(x)\mathfrak{d}(y_i - x_i) = 0$$

as required. $\qquad\square$

**Lemma 3.7.** Let $X$ be a manifold, let $x \in X$ and $\mathfrak{d}$ be a derivation at $x$. Furthermore, let $U$ be an open neighbourhood of $x$ in $X$, $h \in C^\infty(X)$ be a smooth function such that $h(y) = 0$ for all $y \in U$. Then $\mathfrak{d}(h) = 0$.

*Proof.* With the bump function, we showed the existence of a function $\rho \in C^\infty(X)$ such that there exist open sets,

$$x \in V \subseteq W \subseteq U$$

so, $\rho\mid_V = 1$ and $\rho\mid_{X\setminus W} = 0$. Define $\psi = 1 - \rho$ so that we have $\psi \cdot h = h$. Then, by the Leibniz rule, we have

$$\mathfrak{d}(h) = \mathfrak{d}(\psi \cdot h) = h(x)\mathfrak{d}(\psi) + \psi(x)\mathfrak{d}(h) = 0$$

since $h(x) = \psi(x) = 0$. $\qquad\square$

**Definition 3.12** (Germ). Let $X$ be a manifold and let $x \in X$. Then denote $W \subseteq C^\infty(X)$ the set

$$W := \{h \in C^\infty(X) \mid \exists \text{ open } U \subseteq X, x \in U \wedge h \mid_U = 0\}.$$

it is not difficult to show that $W$ is a linear subspace of $C^\infty(X)$ and so, we define the germ to be the quotient

$$\widehat{C_x^\infty}(X) := C^\infty(X)/W.$$

Elements of this space are called germs of smooth functions of $X$ at $x$ and we see that two smooth functionals in $C^\infty(X)$ are the same germ if their difference is locally zero around $x$.

We see that the evaluation map $\mathrm{ev}_x : \widehat{C_x^\infty}(X) \to \mathbb{R} : [h] \mapsto h(x)$ is well-defined. Moreover, the above lemma implies that, for all $\mathfrak{d} \in \mathrm{Der}_x(X)$, the induced linear map

$$\hat{\mathfrak{d}} : \widehat{C_x^\infty}(X) \to \mathbb{R}$$

is well-defined (since $W \leq \ker \mathfrak{d}$) and it satisfies the Leibniz rule

$$\hat{\mathfrak{d}}([h_1 \cdot h_2]) = h_1(x)\hat{\mathfrak{d}}([h_2]) + h_2(x)\hat{\mathfrak{d}}([h_1]).$$

**Lemma 3.8.** Let $X$ be a manifold and let $x \in X$ and $U \subseteq X$ be an open subset containing $x$. Then, we have an linear isomorphism

$$\widehat{C_x^\infty}(X) \xrightarrow{\sim} \widehat{C_x^\infty}(U) : [h] \mapsto [h \mid_U].$$

*Proof.* The map is well-defined as the intersection of two open sets is open. As the map is clearly linear and injective, it remains to show surjectivity. With the bump function, there exists some $\rho \in C^\infty(X)$ and open sets

$$x \in V \subseteq W \subseteq U$$

such that $\rho \mid_V = 1$ and $\rho \mid_{X \setminus W} = 0$. Then, for all $[h] \in \widehat{C_x^\infty}(U)$, define

$$\tilde{h}(x) := \begin{cases} \rho(x)h(x), & x \in U, \\ 0, & x \notin U. \end{cases}$$

It is clear that $\tilde{h}$ is smooth and, furthermore, $\tilde{h} \mid_U \in [h]$ since $h - \tilde{h}$ is zero on $V$. Thus, $[\tilde{h}] = [h]$ and the map is surjective as claimed. $\qquad \square$

**Lemma 3.9.** Let $X$ be a manifold and let $x \in X$ and $U \subseteq X$ be an open subset containing $x$. Then, we have a linear isomorphism

$$\mathrm{Der}_x(U) \xrightarrow{\sim} \mathrm{Der}_x(X) : \mathfrak{d}' \mapsto \mathfrak{d}$$

where $\mathfrak{d}(h) := \mathfrak{d}'(h \mid_U)$.

*Proof.* As we have seen, a derivation $\mathfrak{d}'$ on $U$ induces a map $\hat{\mathfrak{d}}' : \widehat{C_x^\infty}(U) \to \mathbb{R}$ which satisfy the Leibniz rule. Then, by the above lemma, we have the isomorphism $\phi : \widehat{C_x^\infty}(X) \cong \widehat{C_x^\infty}(U)$. Thus, we may define

$$\hat{\mathfrak{d}} := \hat{\mathfrak{d}}' \circ \phi : \widehat{C_x^\infty}(X) \to \widehat{C_x^\infty}(U) \to \mathbb{R}.$$

Hence, taking $\mathfrak{d} := \hat{\mathfrak{d}} \circ q$ where $q$ is quotient map suffices. These steps can be reversed and thus, this map is an isomorphism. $\qquad \square$

Finally, we may prove the forward direction of theorem 4.

*Proof of the forward direction of theorem 4.* Suppose $\mathfrak{d} : C^\infty(X) \to \mathbb{R}$ is a derivation at $x$ and we will show that $\mathfrak{d}(h) = 0$ for all $h \in R_x(X)$.

Let $h \in R_x(X)$ and suppose $(U, f)$ is a chart on $X$ containing $x$. By the above lemma, there exists some derivation $\mathfrak{d}' \in \mathrm{Der}_x(U)$ such that $\mathfrak{d}(h) = \mathfrak{d}'(h \mid_U)$. Now, as $f$ is a homeomorphism between $U$ and $f(U)$, by composing with $f$, we have the isomorphisms

$$C^\infty(U) \tilde{\to} C^\infty(f(U)) \text{ and } \mathrm{Der}_x(U) \tilde{\to} \mathrm{Der}_{f(x)}(f(U)).$$

Then, taking $\tilde{h} := h \circ f^{-1}$, we have the rank of $\tilde{h}$ at $f(x)$ is 0 and thus, by lemma 3.6, the claim follows. $\qquad\square$

With theorem 4, the canonical isomorphism between $T_x X$ and $(T_x^* X)^*$ becomes an isomorphism between $T_x X$ and $\mathrm{Der}_x(X)$, and thus, this gives another definition of the tangent space of a manifold at a point.

### 3.4.1 Derivation and Vector Fields

Let $U \subseteq \mathbb{R}^n$ be an open set and let $\xi : U \to TU$ be a vector field. Since $TU = U \times \mathbb{R}^n$, $\xi$ induces a smooth map $\tilde{\xi} : U \to \mathbb{R}^n := \pi_2 \circ \xi$. Then, for all $x \in U$, we have a derivation along the direction $\tilde{\xi}(x)$ with

$$\partial_{x, \tilde{\xi}(x)} : C^\infty(U) \to \mathbb{R}.$$

Thus, for all $h \in C^\infty(X)$, we have

$$\tilde{\xi}(h) : U \to \mathbb{R} : x \mapsto \partial_{x, \tilde{\xi}(x)}(h).$$

If we write $\tilde{\xi}$ in coordinates so that $\tilde{\xi}(x) = (\tilde{\xi}_1(x), \cdots, \tilde{\xi}_n(x))$. Then,

$$\tilde{\xi}(h) = \partial_{x, \tilde{\xi}(x)}(h) = \tilde{\xi}_1 \frac{\partial h}{\partial x_1} + \cdots + \tilde{\xi}_n \frac{\partial h}{\partial x_n}.$$

For $h_1, h_2 \in C^\infty(U)$, we see that the Leibniz rule apply, i.e.

$$\tilde{\xi}(h_1, h_2) = h_1 \tilde{\xi}(h_2) + h_2 \tilde{\xi}(h_1),$$

and for this reason, we call $\xi$ a derivation on $X$.

Now, let $\xi : X \to TX$ be a vector field on the $n$-dimensional manifold $X$. Similarly, for any $x \in X$, we have an element in $\mathrm{Der}_x(X)$ given by the linear map

$$\partial_{x, \xi(x)} : C^\infty(X) \to \mathbb{R}.$$

Thus, defining $\xi$ such that for all $h \in C^\infty(X)$,

$$\xi(h) = x \mapsto \partial_{x, \xi(x)}(h).$$

We will show $\xi(h)$ is smooth. Let $x \in X$ and take $(U, f)$ be a chart on $X$ containing $x$ and let $v := \Delta_f(\xi(x))$ and $h := h \circ f^{-1}$. Then, as we have already show,

$$\partial_{x, \xi(x)}(h) = \partial_{f(x), v}(\tilde{h}) = \nabla_f (dh \mid_x)^T \cdot v.$$

Then, let $\tilde{\xi}$ be the corresponding function from $f(U)$ to $\mathbb{R}^n$ corresponding to $\xi$, i.e. $\tilde{\xi}(y) := \Delta_f(\xi \mid_{f^{-1}(y)})$. Then, $\xi(h) \circ f^{-1}\tilde{\xi}(\tilde{h})$ is smooth on $f(U)$ since $f(U)$ is an open subset of $\mathbb{R}^n$. Thus, $\xi(h)$ is smooth implying $\xi(h) \in C^\infty(X)$.

With this, we see that $\xi$ is a linear mapping from $C^\infty(X)$ to itself, and so, it is a derivation.

**Definition 3.13** (Derivation). Let $X$ be a manifold. A derivation is a linear map

$$\mathfrak{D} : C^\infty(X) \to C^\infty(X)$$

which satisfies the Leibniz rule

$$\mathfrak{D}(h_1 h_2) = h_1 \mathfrak{D}(h_2) + h_2 \mathfrak{D}(h_1),$$

for all $h_1, h_2 \in C^\infty(X)$.

We denote the set of all derivations on $X$ by $\mathrm{Der}(X)$.

We note that our previous definition of the derivation depended on the choice of $x$ which is definition is independent of that choice.

We have so far seen that a vector field induces a derivation. We shall show the reverse is also true.

**Definition 3.14.** Let $X$ be an $n$-dimensional manifold. Then, any derivation $\mathfrak{D}$ on $X$ defines a vector field on $X$.

*Proof.* Let $x \in X$, we have the linear map

$$\mathfrak{D} \mid_x : C^\infty(X) \to \mathbb{R} : h \mapsto \mathfrak{D}(h)(x).$$

The Leibniz rule implies that $\mathfrak{D} \mid_x \in \mathrm{Der}_x(X)$. However, we know that $\mathrm{Der}_x(X) \cong T_x X$ and so, we obtain a map

$$\xi : X \to TX : x \mapsto \mathfrak{D} \mid_x .$$

It remains to show that $\xi$ is smooth. Let $(U, f)$ be a chart on $X$ containing $x$. Then $\xi$ induced a map

$$\tilde{\xi} : f(U) \to f(U) \times \mathbb{R}^n$$

where the second component of $\tilde{\xi} = \Delta_f \circ \xi \circ f^{-1}$. Write $\tilde{\xi} = \sum_{i=1}^n \tilde{\xi}_i \frac{\partial}{\partial x_i}$ and we will show $\xi_i$ is smooth for all $i = 1, \cdots, n$. Recall that, using the bump function, there exists a smooth function $\psi$ and open sets

$$f(x) \in W \subseteq V \subseteq f(U)$$

with $\psi \mid_W = 1$ and $\psi \mid_{X/V} = 0$. Then, defining

$$\chi(x) := \begin{cases} ((x_i \psi) \circ f)(x), & x \in U, \\ 0, & x \notin U. \end{cases}$$

In particular, $\tilde{\chi}_i := \chi_i \circ f^{-1} = x_i$ inside $W$ and we have for all $y \in f^{-1}(W)$,

$$\mathfrak{D}(\chi_i) \mid_y = \sum_{j=1}^n \tilde{\xi}_j \mid_{f(y)} \frac{\partial \tilde{\chi}_i}{\partial x_j}\bigg|_{f(y)} = \tilde{\xi}_i \mid_{f(y)} .$$

$\square$

## 3.5   Vector Bundle

Vector bundle is an important tool in both differential geometry and algebraic geometry and is somehow a generalization of the tangent bundles. We will in this small section define the vector bundle and prove some properties about it.

**Definition 3.15** (Vector Bundle). Let $X$ be an $n$-dimensional manifold. A vector bundle over $X$ consists of the data

- a manifold $E$ of dimension $n + r$;

- a smooth surjection $\pi : E \to X$;

- an atlas $\{(U_i, f_i)\}_{i \in I}$ for $X$ and an atlas $\{(V_i, g_i)\}_{i \in I}$ for $E$ such that,

    - $V_i = \pi^{-1}(U_i)$ for all $i \in I$;

    - $g_i(V_i) = f_i(U_i) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ for all $i \in I$;

    - $\mathrm{pr}_1 \circ g_i(v) = f_i(\pi(v))$ for all $i \in I$ and $v \in V_i$;

    - for each $x \in X$, the level set $E_x := \pi^{-1}(\{x\})$ is a vector space of dimension $r$ such that for all $i \in I$, the map $g_i \mid_{E_x}: E_x \to \mathbb{R}^r$ is an isomorphism.

The number $r$ is called the rank of the vector bundle and the vector spaces $E_x$ are known as fibres of the vector bundle.

We see that in some sense vector bundles embed a manifold inside a larger manifold and for each point in the original manifold, we associate it with a real vector space in the larger manifold. As one might expect, the tangent bundle is a vector bundle and with a similar construction $(T^*X := \bigcup_{x \in X} T_x^* X)$ we may define the cotangent bundle which is also a vector bundle. Another important vector bundle is the trivial vector bundle $E := X \times \mathbb{R}^r$ with the map $(x, v) \in X \times \mathbb{R}^r \mapsto x$.

**Definition 3.16** (Section). Let $X$ be an $n$-dimensional manifold and let $\pi : E \to X$ be a vector bundle. A section of $E$ is a smooth morphism $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$.

A section of the tangent bundle is a vector field.

By definition, if $(E, \pi : E \to X)$ is a vector bundle of $X$, for all $x \in X$, the level set $\pi^{-1}(\{x\})$ is a vector space, and thus, it makes sense to map $x \mapsto 0_{\pi^{-1}(\{x\})}$. This is clearly a section and is called the zero section.

**Definition 3.17.** Let $X$ be a manifold and let $\pi_1 : E_1 \to X, \pi_2 : E_2 \to X$ be two vector bundles. An isomorphism between $E_1, E_2$ is a diffeomorphism $F : E_1 \to E_2$ such that $\pi_2 \circ F = \pi_1$ and the induced function $F_x : (E_1)_x \to (E_2)_x$ is an isomorphism of vector spaces for all $x \in X$.

We say a vector bundle $(E, \pi)$ is trivial if it is isomorphic to the trivial vector bundle $X \times \mathbb{R}^r$ for some $r > 0$.

**Proposition 3.7.** Two isomorphic vector bundles have the same rank.

*Proof.* Clear as a linear isomorphism preserves dimension. $\square$

**Definition 3.18** (Line Bundle). A line bundle is a vector bundle of rank 1.

**Proposition 3.8.** Let $X$ be a manifold and let $\pi : L \to X$ be a line bundle on $X$ which admits a section $s : X \to L$ which is nowhere zero, i.e. $s(x) \neq 0 \in L_x$ for all $x \in X$. Then $L$ is trivial.

*Proof.* Consider the function $F : X \times \mathbb{R} \to L : (x, v) \mapsto v \cdot s(x)$. It is clear that $F$ commutes with the projection map $X \times \mathbb{R} \to X$ and $L \to X$ and for $x \in X$ as $s(x) \neq 0$, it induces an isomorphism of vector spaces.

We will now show $F, F^{-1}$ is smooth. Since $L$ is a vector bundle, for each $x \in X$, there exists a chart $(U, f)$ in $X$ containing $x$ and a chart $(V, g)$ on $Y$ such that $V = \pi^{-1}(U)$ and $g(V) = f(U) \times \mathbb{R}$. Then

$$\tilde{s} := g \circ s \circ f^{-1} : f(U) \to f(U) \times \mathbb{R} = g(V)$$

is smooth (as $s$ is smooth). Thus,

$$\tilde{F} := g \circ \circ (f^{-1}, \mathrm{id}_{\mathbb{R}}) : f(U) \times \mathbb{R} \to f(U) \times \mathbb{R} : (y, v) \mapsto (y, v \cdot \mathrm{pr}_2 \circ \tilde{s}(y)),$$

is smooth and hence, it follows $\tilde{F}^{-1}$ is smooth and both $F, F^{-1}$ are smooth. Hence, $F$ is an isomorphism of vector bundles between $L$ and the trivial vector bundle. $\qquad \square$

This proposition can be generalized into the following.

**Proposition 3.9.** Let $X$ be a manifold and let $\pi : L \to X$ be a vector bundle of rank $r$ which admits sections $s_1, \cdots, s_r : X \to L$ such that for all $x \in X$, $\{s_1(x), \cdots, s_r(x)\}$ is linearly independent in $E_x$ (and so form a basis). Then $L$ is trivial.

*Proof.* See problem sheet. $\qquad \square$