Group Representation Theory

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1 Introduction

Group representation theory is a field of mathematics that applies linear algebra to study properties of groups. The field itself originated through a letter from Dedekind to Frobenius in which he noted that, given $f = \det A$, where A is the Cayley table of a group of n elements, by factorising f into irreducible polynomials, $f = \prod_i f_i^{d_i}$, we have $d_i = \deg f_i$. And this led Frobenius to invent group representation theory.

Group representation theory is applicable in many different areas.

- Group theory arises in Klein's "Erlangen program" as symmetries of geometric spaces.
- Burnside in 1904 proves the following using representation theory (and so shall we later on)

  Proposition 1.1. Let G be a group such that $|G| = p^r q^s$ where p, q are prime and $r + s \geq 2$, then G is not simple.

- In number theory, representations of Galois groups arises in the number field case

  $$\overline{F}/F, \mathbb{Q} \subseteq F, [F : \mathbb{Q}] < \infty,$$

  which has implications in Wiles' proof of Fermat's last theorem.

- In chemistry the symmetry and rotation of molecules can be represented by group actions.
- In quantum mechanics, spherical symmetry gives rise to discrete energy levels, orbitals, etc.
- In differential geometry, the vector space of solutions is a representation of the symmetry group of an equation.

Recalling the definition of a group, informally, the representation of a group G is a way if writing group elements as linear transformations of a vector space such that the natural group properties are satisfied.

Some examples of group representations are the following:

- For all group G, the trivial representation of G is ρ such that $\rho(g) = \mathrm{id}$ for all $g \in G$.
- Let $\zeta \in \mathbb{C}$ be a n-th root of 1 and let $G = C_n = \{1, g, \cdots, g^{n-1}\}$. Then $\rho : g^i \mapsto (\zeta^i)$ is a representation of G.
- In the case $G = S_n$, the mapping of $\sigma \in S_n$ to its corresponding permutation matrix P_σ is a representation of G.
- Another representation of S_n is $\sigma \in S_n \mapsto (\mathrm{sign}(\sigma))$[1].
- Let $G = D_n$ the dihedral group of order $2n$. Then, a representation D_n maps elements of D_n to the corresponding 2×2 matrices which rotates/reflects $\mathbb{R}^2$ by the appropriate amount.

We shall in this module study and construct representations, and furthermore, classify up to isomorphism finite-dimensional complex representations of every finite group G.

[1] $\mathrm{sign}(\sigma) = \det P_\sigma$

2 Fundamentals of Group Representation

Definition 2.1 (Representation)**.** Let G be a group, then a representation of G is the pair (V, ρ) where V is a (finite-dimensional) vector space and $\rho : G \mapsto GL(V)$ is a group homomorphism.

Alternatively, we may consider a group representation of G is a group action $(\cdot) : G \times V \to V : (g, v) \mapsto v$ such that $(\cdot)$ is linear with respect to the second parameter. In particular, we recall a group action $(\cdot)$ satisfies $e \cdot v = v$ and $g \cdot (h \cdot v) = gh \cdot v$.

Definition 2.2 (Dimension of a Representation)**.** If (V, ρ) is a representation of G, then the dimension of (V, ρ) is $\dim(V, \rho) = \dim V$.

Similar to other objects in mathematics, we introduce a notion of morphisms between representations.

Definition 2.3 (Homomorphism of Representation)**.** Let G be a group and (V, ρ_V) and (W, ρ_W) be two representations of G. Then a homomorphism of representations is a linear map $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho_V(g) = \rho_W(g) \circ T.$$

Furthermore, we say T is an isomorphism is bijective (or equivalently, it has an inverse which is also a homomorphism).

In particular, one might imagine the homomorphism as a linear map such that the following diagram commute.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

As with any definitions which work with finite-dimensional vector spaces, there are equivalent but "worse" (as we will have to choose a basis) corresponding definitions in terms of matrices. Nonetheless, these definitions with matrices are easier computationally and we shall recall the contrast here.

Clearly, if G is a group and $(\mathbb{C}^n, \rho)$ is a representation, we have $\rho(e) = I_n$. Furthermore, we have a natural isomorphism between $GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$ and more generally $\mathrm{Mat}_{n,m}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. Similarly, given a representation (V, ρ), with $\dim V < \infty$, we may choose a basis B of V and write the representation as a matrix which we denote $\rho^B(g) = [\rho(g)]_B$. Thus, we may use first year linear algebra methods to manipulate representations.

Definition 2.4. Given two matrix representations $\rho, \rho' : G \mapsto GL_n(\mathbb{C})$, we say ρ and ρ' are equivalent/isomorphic if there exists $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho'(g) = P^{-1}\rho(g)P$.

This definition is motivated by the following.

Proposition 2.1. Given (V, ρ_V) and (W, ρ_W) representations of G, we have $\rho_V \cong \rho_W$ if and only if there exists some $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho_W^C(g) = P^{-1}\rho_V^B P(g)P$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

Proposition 2.2. Given a cyclic group $C_n = \langle g \rangle$ with representations (V, ρ_V) and (W, ρ_W) of equal dimensions, we have $\rho_V \cong \rho_W$ if and only if $\rho_V^B(g)$ is conjugate to $\rho_W^C(g)$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

In fact the proposition above holds for the infinite cyclic group $C_\infty \cong \mathbb{Z}$.

2.1 Regular Representation

Let us first recall some definition about group actions though we will omit stabilizers, the orbit-stabilizer theorem and transitive actions (though it might be helpful to recall them from last year).

Definition 2.5 (Group Action)**.** Let G be a group and X a set, then a group action $(\cdot)$ of G on X is a function $G \times X \to X$ such that for all $g, h \in G$, $x \in X$, we have

- $g \cdot (h \cdot x) = gh \cdot x$,
- $1 \cdot x = x$.

Equivalently, a group action can be represented by a group homomorphism between G to S_n if $|X| = n < \infty$. We note that there exists an bijection between $\mathrm{Perm}(X)$ (a.k.a $\mathrm{Aut}(X)$ though we will avoid this term in case X has additional structures) and S_n with depends on a choice of $X \simeq \{1, \cdots, |X|\}$.

Definition 2.6 (Kernel)**.** A kernel of a representation (or group action) is simply the kernel of the corresponding group homomorphism, i.e. if ρ is a representation (or group action),

$$\ker \rho := \{g \in G \mid \rho(g) = \mathrm{id}\}.$$

We say a representation (or group action) is faithful if $\ker \rho = \{e\}$, i.e. ρ is injective.

Definition 2.7 (Morphism of Group Actions)**.** A morphism $T : X \to Y$ of group actions on X and Y is a map such that $T(g \cdot x) = g \cdot T(x)$ for all $g \in G$, $x \in X$.

This is also called a "G-equivariant map" from X to Y and one can see the resemblance of this definition and the definition for homomorphisms between representations.

For any group G, it acts on itself in three different ways. In particular, we have the left regular action $g \cdot h = gh$, the right regular action $g \cdot h = hg^{-1}$ (where the inverse is required for associativity) and the adjoint action $g \cdot h = ghg^{-1}$. One can see that the left and right regular actions are isomorphic via $T(g) = g^{-1}$. On the other hand, they are not isomorphic to the adjoint action (consider $\rho_{\mathrm{ad}}(g)(e) = e$ for all $g \in G$).

Proposition 2.3. Given two actions (or representations) ρ, ρ' on G, $g \mapsto \rho(g)\rho'(g)$ is an action (or representation) if and only if $\rho(g)\rho'(g) = \rho'(g)\rho(g)$, that is ρ and ρ' are commuting actions.

Definition 2.8. A subset $Y \subseteq X$ is said to be stable under an action $(\cdot)$ of G on X if $g \cdot y \in Y$ for all $y \in Y, g \in G$.

In the case that $Y \subseteq X$ is stable, then we may restrict the action on Y to obtain a new action of G on Y.

Definition 2.9 (Orbit)**.** Let $x \in X$, then $G \cdot x := \{g \cdot x \mid g \in G\}$ is called an orbit of x and we denote this by $\mathrm{orb}(x)$.

It is not difficult to see that orbits are stable and in fact, as an exercise, one might show that $Y \subseteq X$ is stable if and only if it is a union of orbits.

In a group G under the adjoint action, we see that the orbits are the conjugacy classes[2]. Thus, for every conjugacy class, we obtain a action on that class from the adjoint action on the whole group.

Example 2.1. Let $G = S_4$ and let $c = \{(12)(34), (13)(24), (14)(23)\}$. Then as c is a conjugacy class, we have the adjoint action on c

$$\phi : S_4 \to \mathrm{Perm}(c) \cong S_3.$$

It is not difficult to show that ϕ is surjective and $\ker \phi = c \cup \{e\} \cong K_4 \cong C_2 \times C_2$. Thus, by the first isomorphism theorem we have

$$S_3 \cong S_4 / K_4.$$

Definition 2.10. Given a finite set X, let

$$\mathbb{C}[X] := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{C} \right\},$$

equipped with the addition $\sum_{x \in X} a_x x + \sum_{x \in X} b_x x = \sum_{x \in X} (a_x + b_x) x$ and scalar multiplication $c \cdot \sum_{x \in X} a_x x = \sum_{x \in X} (c a_x) x$. This sum here does not represent some addition operation on X but a notational trick. One might instead consider elements of $\mathbb{C}[X]$ as functions $a : X \to \mathbb{C}$ equipped with point-wise addition and scalar multiplication.

We observe that $\mathbb{C}[X] \cong \mathbb{C}^{|X|}$ depending on a choice of $X \cong \{1, \cdots, |X|\}$. Furthermore, we have $X \subseteq \mathbb{C}[X]$ and is a basis (if we interpret $\mathbb{C}[X]$ as a space of functions, the canonical basis is $\{a_x : y \mapsto \chi_{\{x\}} \mid x \in X\}$). In the case that X is infinite we can still define $\mathbb{C}[X]$ allowing only finite sums.

Proposition 2.4. If $(\cdot)$ is a group action of G on X, then, the map $(g, \sum a_x x) \mapsto \sum a_x (g \cdot x)$ is a group action of G on $\mathbb{C}[X]$.

Definition 2.11. The left regular, right regular, adjoint representations are representations

$$\tilde{\rho}_L, \tilde{\rho}_R, \tilde{\rho}_{\mathrm{ad}} : G \to GL(\mathbb{C}[G])$$

obtained from the left regular, right regular and adjoint actions

$$\rho_L, \rho_R, \rho_{\mathrm{ad}} : G \to \mathrm{Perm}(G).$$

Proposition 2.5. If X is any set with a G-actin, then for all $g \in G$, $[\rho_{\mathbb{C}[X]}(g)]_B$ is always a permutation matrix.

Proof. Exercise. □

[2] What are the orbits of the left action?

Definition 2.12. Given $(V, \rho_V), (W, \rho_W)$ representations of G, we denote

$$\mathrm{Hom}_G(V, W) := \{T : V \to W \mid G\text{-linear}\}.$$

Proposition 2.6. Let (V, ρ_V) be a representation of G and $v \in V$. Then, there exists a unique homomorphism of representations $\mathbb{C}[G] \to V$ where $\mathbb{C}[G]$ is equipped with the left regular representation such that $e_G \mapsto v$ and thus,

$$\mathrm{Hom}_G(\mathbb{C}[G], V) \cong (V, \rho_V).$$

Proof. For all $g \in G$, $c = \sum_{h \in G} a_h h$, we have

$$\begin{aligned} T(g \cdot c) = \rho_V(g)(Tc) &\iff T\left(\sum a_h gh\right) = \rho_V(g)\left(T\left(\sum a_h h\right)\right) \\ &\iff \sum a_h T(gh) = \sum a_h \rho_V(g)(Th) \\ &\iff T(gh) = \rho_V(g)(Th), \ \forall h \in G, \end{aligned}$$

where the second if and only if follows as both T and ρ_V are linear. Then choosing $h = e_G$, we have $T(g) = \rho_V(g)(v)$ and thus T is uniquely determined on G and hence is unique as G is a basis of $\mathbb{C}[G]$.

It remains to show that the map T defined by $g \mapsto \rho_V(g)(v)$ is a homomorphism of representations. This is clear since

$$\begin{aligned} T(g \cdot c) &= T\left(\sum a_h gh\right) = \sum a_h T(gh) \\ &= \sum a_h \rho_V(gh)(v) = \sum a_h \rho_V(g)(\rho_V(h)(v)) \\ &= \sum a_h \rho_V(g)(Th) = \rho_V(g)\left(T\left(\sum a_h h\right)\right), \end{aligned}$$

where the fourth equality follows by the associativity of group actions. $\square$

2.2 Subrepresentation and Quotient Representation

Definition 2.13 (Subrepresentation)**.** A subrepresentation of a representation (V, ρ_V) is a subspace $W \le V$ such that $\rho_V(g)(W) \subseteq W$ for all $g \in G$.

Clearly, both $\{0\}$ and V are subrepresentations of (V, ρ_V), and we say a representation is irreducible if these two subrepresentations are the only subrepresentations. We say a representation is reducible if it is not irreducible. In general, every 1-dimension representation is irreducible.

Proposition 2.7. Irreducibility is invariant under isomorphisms.

Proof. Exercise. $\square$

Proposition 2.8. Let G be finite and (V, ρ_V) is an irreducible representation of G. Then $\dim V < \infty$.

Proof. Let $w \in V \ \{0\}$ and let $W := \mathrm{span}(\{\rho_V(g)(w) \mid g \in G\})$ which is a finite dimensional subrepresentation as G is finite and for all $h \in G$, $\rho_V(h)(\rho_V(g)(w)) = \rho_V(hg)(w)$. Thus, if $\dim V$ is not finite, we have found a proper subrepresentation which contradicts the irreducibility of (V, ρ_V). $\square$

Definition 2.14 (Quotient Representation)**.** For $W \leq V$ a subrepresentation, the quotient representation is $(V/W, \rho_{V/W})$ given by

$$\rho_{V/W}(g)(v + W) := \rho_V(g)(v) + W.$$

This is well-defined as W is stable under ρ_V.

Proposition 2.9. For $T : (V, \rho_V) \to (W, \rho_W)$ a G-linear map, $\ker T$ and $\mathrm{Im} T$ are subrepresentations.

Proof. Let $v \in \ker T$, then $T(\rho_V(g)(v)) = \rho_W(g)(Tv) = \rho_W(g)(0) = 0$ implying $\rho_V(g)(v) \in \ker T$ and thus, $\ker T$ is a subrepresentation. On the other hand, for all $w \in \mathrm{Im} T$, there exists some $v \in V$ such that $Tv = w$. Then $\rho_W(g)(w) = \rho_W(g)(Tv) = T\rho_W(g)(v)$ implying $\rho_W(g)(w) \in \mathrm{Im} T$ showing $\mathrm{Im} T$ is also a subrepresentation. $\square$

Proposition 2.10. For $T : (V, \rho_V) \to (W, \rho_W)$ a G-linear map, we have

$$\mathrm{Im} T \cong V / \ker T.$$

Proof. Follows from the first isomorphism for vector spaces and it remains to check $V / \ker T \to \mathrm{Im} T$ is G-linear. $\square$

Proposition 2.11. If $T \in \mathrm{End}_G V$ is a G-linear projection (i.e. $T^2 = T$), then V is a direct sum of subrepresentations $\ker T \oplus \mathrm{Im} T$.

Proof. Follows from the vector space case. $\square$

2.3 Maschke's Theorem

Recalling internal and external direct sums of vector spaces, we will in this section introduce and prove a powerful result in representation theory known as the Maschke's theorem.

Definition 2.15 (Decomposable)**.** The representation (V, ρ_V) is decomposable if there exists a decomposition $V = U \oplus W$ where U, W are non-zero subrepresentations.

Definition 2.16 (Semisimple)**.** The representation (V, ρ_V) is semisimple if there exists a irreducible subrepresentations $W_1, \cdots, W_n$ such that

$$V = \bigoplus_{i=1}^{n} W_i.$$

Theorem 1 (Maschke's Theorem)**.** If G is finite, then for all $W \leq V$ subrepresentations of (V, ρ_V), there exists a complementary subrepresentation U, $V = W \oplus U$.

A direct consequence of Maschke's theorem is that every finite-dimensional representation of G is semisimple.

Maschke's theorem does not hold in the case that G is not finite. Consider $G = \mathbb{Z}$ and let

$$\rho : g \to GL_2(\mathbb{C}) : m \mapsto \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

Then the only non-zero proper subrepresentation is $\text{span}\{e_1\}$ since e_1 is the only eigenvector and as ρ is a 2-dimensional representation, the only non-zero proper subrepresentation is 1-dimensional, hence an eigenspace. Thus, ρ is indecomposable but not irreducible, and hence not semisimple.