Distribution vs. Signed Measures

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Distributions are interpreted as generalized functions with the motivating example of the δ-function. The δ-function, arising naturally from physics is a function with the property that it is 0 everywhere but at a single point and has integral 1. Namely a function $\delta_x : \mathbb{R} \to \mathbb{R}$ which satisfies

- $\delta_x(y) = 0$ for all $y \neq x$ and,
- $\int \delta_x = 1$.

It is clear such a function does not exist as the first property implies $\delta_x = 0$ almost everywhere which implies it has integral 0. Distributions are suppose to solve this problem by generalizing functions to the space of distributions.

Definition 1 (Test Function)**.** The space of test function $\mathcal{D}$ is the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with compact support.

Definition 2 (Distribution)**.** A distribution on $\mathbb{R}$ is a continuous linear function of the space of test functions $\mathcal{D}$. Namely, the space of distributions is the dual of $\mathcal{D}$ and we denote it by $\mathcal{D}'$.

Functions can be naturally embedded into the space of distributions via the map $f \mapsto T_f$ where

$$T_f : \mathcal{D} \to \mathbb{R} : \phi \mapsto \int f\phi \mathrm{d}\lambda,$$

where λ is the Lebesgue measure. For this embedding to make sense, we require f to be locally integrable. This is a rather relaxed requirement as any measurable function which is bounded on a compact set is locally integrable.

The $delta$ function can be represented naturally as a distribution. In particular, we take $\delta_x \in \mathcal{D}'$ such that $\delta'(\phi) := \phi(x)$. This represents the δ-function we had in mind by considering that *if* such a function f which satisfy the aforementioned two properties,

$$T_f(\phi) = \int f\phi \mathrm{d}\lambda = \int \mathbf{1}_{\{x\}} f\phi \mathrm{d}\lambda = \phi(x) \int f \mathrm{d}\lambda = \phi(x).$$

However, we now note a different method of representing the δ-function. Namely, the Dirac measures. The Dirac measure at x is a probability measure defined as $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise. This is similar to the distribution version as for all ϕ measurable,

$$\int \phi \mathrm{d}\delta_x = \phi(x)$$

and so, by considering the map $\mu \mapsto T_\mu \in \mathcal{D}'$ where

$$T_\mu : \mathcal{D} \mapsto \mathbb{R} : \phi \mapsto \int \phi \mathrm{d}\mu,$$

we have a natural embedding of measures into the space of distributions.

However, unlike the space of distributions, only non-negative functions $f : \mathbb{R} \to \mathbb{R}_+$ induces a measure with $f\lambda$. Thus, in order to embed general functions, we need to consider signed measures. This process is easy by mapping

$$f \mapsto A \mapsto \int_A f \mathrm{d}\lambda = \int_A f^+ \mathrm{d}\lambda - \int_A f^- \mathrm{d}\lambda,$$

i.e. $f \mapsto f^+\lambda - f^-\lambda$. With this, we have the following embeddings

$$(\mathbb{R} \to \mathbb{R}) \hookrightarrow \mathbb{R} - \text{signed measures} \hookrightarrow \mathcal{D}'.$$

One can show that every distribution which is non-negative on non-negative functions defines a measure which commutes with our embedding. However, not all distributions arises from a signed measure. Indeed, the derivative of the δ distribution ($f \mapsto -f'(0)$) cannot be represented as a signed measure.