Functional Analysis Revision Notes

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If d is a metric and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is concave (and increasing?) and $\eta(0) = 0$, then $\eta \circ d$ is also a metric.

Beware the properties of **inner product spaces**. Notably:

- (Cauchy-Schwarz) $|\langle x, y \rangle| \leq \|x\|\|y\|$.
- (Parallelogram law) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
- (Polarization identity) $\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ over $\mathbb{R}$ and $\langle x, y \rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$ over $\mathbb{C}$.
- (Ptolemy's inequality) $\|x - y\|\|z\| + \|y - z\|\|x\| \geq \|x - z\|\|y\|$.

$C[0,1]^*$ is not separable as $\{\delta_x \mid x \in [0,1]\}$ where $\delta_x : \phi \mapsto \phi(x)$ is a uncountable set in $C[0,1]^*$, and for any $\delta_x \neq \delta_y$, there exists $\phi \in C[0,1]^*$ such that $\phi(x) = 1, \phi(y) = 0$ implying $\|\delta_x - \delta_y\| \geq 1$.

If X is reflexive, X is separable iff. X^* is (reverse direction does not require reflexivity. c.f. Prop 3.11 in notes).

A Banach space with a Schauder basis is separable. The converse is not necessarily true.

For $p \leq q$, $\ell^p \subseteq \ell^q$.

For finite measure space Ω, $p \leq q$, $L^q(\Omega) \subseteq L^p(\Omega)$.

Banach Fixed Point Theorem

For operators in the form $T : u(x) \mapsto \nu(x) + \beta \int \kappa(x,y)\alpha(u(y))\lambda(\mathrm{d}y)$, to show strict contraction, we use the FTC to observe

$$\alpha(u(y)) - \alpha(v(y)) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau u(x) + (1-\tau)v(x))\mathrm{d}\tau,$$

should $|\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}|$ be bounded.

Baire Category and Banach-Steinhaus

Any proper *closed* linear subspace of a normed space is nowhere dense (in fact true in any TVS).

There cannot be a countable Hamel basis as a result.

The space of polynomials is not complete wrt. any norm by considering $A_n := \langle 1, x, \cdots, x^n \rangle$ and Baire's category theorem.

Closed graph theorem useful to show equivalence of norms by considering the identity

$$\text{id} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2).$$

Compact Operators

The following properties are exercieses from sheet 8:

- The dual and adjoint of a compact operator is compact.
- Bounded, self adjoint operator T on a Hilbert space is compact if T^n is compact for some n $(*)$.
- Any bounded sequence (x_l) in a reflexive space has a weakly convergent subsequence.
- **Corollary**. Any bounded sequence (x_l) in a Hilbert space has a weakly convergent subsequence.
- Finite rank operators are compact.
- T is compact in a Hilbert space if and only if $x_n \rightharpoonup x$ implies $Tx_n \to Tx$. The backwards direction is from lectures while the forward is from the problem sheet.

$(*)$ One method of proof is follows. Clearly T compact implies T^n is compact for any n and so, T^{2^N} is compact for sufficiently large N. Then, it suffices to show T^{2^N} is compact implies $T^{2^{N-1}}$ is compact.

Weak Convergence

A sequence (x_n) converges weakly to x in a Hilbert space if and only if it is bounded and $\langle x_n, e_i \rangle \to \langle x, e_i \rangle$ for all i where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for the space.