Group Representation Theory

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1 Introduction

Group representation theory is a field of mathematics that applies linear algebra to study properties of groups. The field itself originated through a letter from Dedekind to Frobenius in which he noted that, given $f = \det A$, where A is the Cayley table of a group of n elements, by factorising f into irreducible polynomials, $f = \prod_i f_i^{d_i}$, we have $d_i = \deg f_i$. And this led Frobenius to invent group representation theory.

Group representation theory is applicable in many different areas.

- Group theory arises in Klein's "Erlangen program" as symmetries of geometric spaces.
- Burnside in 1904 proves the following using representation theory (and so shall we later on)

  Proposition 1.1. Let G be a group such that $|G| = p^r q^s$ where p, q are prime and $r + s \geq 2$, then G is not simple.

- In number theory, representations of Galois groups arises in the number field case

  $$\overline{F}/F, \mathbb{Q} \subseteq F, [F : \mathbb{Q}] < \infty,$$

  which has implications in Wiles' proof of Fermat's last theorem.

- In chemistry the symmetry and rotation of molecules can be represented by group actions.
- In quantum mechanics, spherical symmetry gives rise to discrete energy levels, orbitals, etc.
- In differential geometry, the vector space of solutions is a representation of the symmetry group of an equation.

Recalling the definition of a group, informally, the representation of a group G is a way if writing group elements as linear transformations of a vector space such that the natural group properties are satisfied.

Some examples of group representations are the following:

- For all group G, the trivial representation of G is ρ such that $\rho(g) = \mathrm{id}$ for all $g \in G$.
- Let $\zeta \in \mathbb{C}$ be a n-th root of 1 and let $G = C_n = \{1, g, \cdots, g^{n-1}\}$. Then $\rho : g^i \mapsto (\zeta^i)$ is a representation of G.
- In the case $G = S_n$, the mapping of $\sigma \in S_n$ to its corresponding permutation matrix P_σ is a representation of G.
- Another representation of S_n is $\sigma \in S_n \mapsto (\mathrm{sign}(\sigma))$[1].
- Let $G = D_n$ the dihedral group of order $2n$. Then, a representation D_n maps elements of D_n to the corresponding 2×2 matrices which rotates/reflects $\mathbb{R}^2$ by the appropriate amount.

We shall in this module study and construct representations, and furthermore, classify up to isomorphism finite-dimensional complex representations of every finite group G.

[1] $\mathrm{sign}(\sigma) = \det P_\sigma$

2 Fundamentals of Group Representation

Definition 2.1 (Representation)**.** Let G be a group, then a representation of G is the pair (V, ρ) where V is a (finite-dimensional) vector space and $\rho : G \mapsto GL(V)$ is a group homomorphism.

Alternatively, we may consider a group representation of G is a group action $(\cdot) : G \times V \to V : (g, v) \mapsto v$ such that $(\cdot)$ is linear with respect to the second parameter. In particular, we recall a group action $(\cdot)$ satisfies $e \cdot v = v$ and $g \cdot (h \cdot v) = gh \cdot v$.

Definition 2.2 (Dimension of a Representation)**.** If (V, ρ) is a representation of G, then the dimension of (V, ρ) is $\dim(V, \rho) = \dim V$.

Similar to other objects in mathematics, we introduce a notion of morphisms between representations.

Definition 2.3 (Homomorphism of Representation)**.** Let G be a group and (V, ρ_V) and (W, ρ_W) be two representations of G. Then a homomorphism of representations is a linear map $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho_V(g) = \rho_W(g) \circ T.$$

Furthermore, we say T is an isomorphism is bijective (or equivalently, it has an inverse which is also a homomorphism).

In particular, one might imagine the homomorphism as a linear map such that the following diagram commute.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

As with any definitions which work with finite-dimensional vector spaces, there are equivalent but "worse" (as we will have to choose a basis) corresponding definitions in terms of matrices. Nonetheless, these definitions with matrices are easier computationally and we shall recall the contrast here.

Clearly, if G is a group and $(\mathbb{C}^n, \rho)$ is a representation, we have $\rho(e) = I_n$. Furthermore, we have a natural isomorphism between $GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$ and more generally $\mathrm{Mat}_{n,m}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. Similarly, given a representation (V, ρ), with $\dim V < \infty$, we may choose a basis B of V and write the representation as a matrix which we denote $\rho^B(g) = [\rho(g)]_B$. Thus, we may use first year linear algebra methods to manipulate representations.

Definition 2.4. Given two matrix representations $\rho, \rho' : G \mapsto GL_n(\mathbb{C})$, we say ρ and ρ' are equivalent/isomorphic if there exists $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho'(g) = P^{-1}\rho(g)P$.

This definition is motivated by the following.

Proposition 2.1. Given (V, ρ_V) and (W, ρ_W) representations of G, we have $\rho_V \cong \rho_W$ if and only if there exists some $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho_W^C(g) = P^{-1}\rho_V^B P(g)P$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

Proposition 2.2. Given a cyclic group $C_n = \langle g \rangle$ with representations (V, ρ_V) and (W, ρ_W) of equal dimensions, we have $\rho_V \cong \rho_W$ if and only if $\rho_V^B(g)$ is conjugate to $\rho_W^C(g)$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

In fact the proposition above holds for the infinite cyclic group $C_\infty \cong \mathbb{Z}$.