

Fourier Analysis and the Theory of Distributions

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1 Orthonormal Systems

We will in this section recall some results about orthonormal systems in Euclidean spaces¹ and generalize them to complex spaces.

Definition 1.1. A system of nonzero vectors $\{X_\alpha\} \subseteq R$ where R is an Euclidean space is called orthogonal if $\langle X_\alpha, X_\beta \rangle = 0$ for all $\alpha \neq \beta$.

In addition, if for all α , $\langle X_\alpha, X_\alpha \rangle = 1$, we say the system is orthonormal.

Clearly, given an orthogonal system $\{X_\alpha\}$, we may normalize the vector such that $\{X_\alpha/\|X_\alpha\|\}$ is an orthonormal system. Furthermore, recall that a system of orthogonal vectors is linearly independent.

Definition 1.2. A complete (i.e. the smallest closed subspace containing the system is R) orthogonal system $\{X_\alpha\} \subseteq R$ is said to be an orthogonal basis of R .

Some important spaces we shall study in this course include \mathbb{R}^2 (equipped with the Euclidean norm), l_2 , $\mathcal{C}([-\pi, \pi])$ (the space of continuous functions on $[-\pi, \pi]$ equipped with the L_2 norm).

Proposition 1.1. Let R be a separable Euclidean space. Then any orthogonal system in R is countable.

Proof. By normalizing, we may assume the system $\{X_\alpha\}$ is orthonormal. Then, for $\alpha \neq \beta$,

$$\|X_\alpha - X_\beta\|^2 = \|X_\alpha\|^2 - 2\langle X_\alpha, X_\beta \rangle + \|X_\beta\|^2 = \|X_\alpha\|^2 + \|X_\beta\|^2 = 2.$$

Then, $B_{1/2}(X_\alpha) \cap B_{1/2}(X_\beta) = \emptyset$ for all $\alpha \neq \beta$. Thus, if the system is not countable, we have found a uncountable number of disjoint open balls, contradicting the separability of R . \square

Proposition 1.2. Let f_1, f_2, \dots be a linearly independent system in a Euclidean space R . Then, there exists an orthonormal system ϕ_1, ϕ_2, \dots such that

$$\phi_n = a_{n_1}f_1 + \dots + a_{n_n}f_n$$

and

$$f_n = b_{n_1}\phi_1 + \dots + b_{n_n}\phi_n$$

for some $a_{n_k}, b_{n_k} \in \mathbb{R}$ and $a_{n_n}, b_{n_n} \neq 0$. Furthermore, the system ϕ_1, ϕ_2, \dots is uniquely determined up to a multiplication by ± 1 .

Proof. Use Gram-Schmidt. \square

Corollary 0.1. A separable Euclidean space R possesses an orthonormal basis.

Proof. Simply obtain the orthonormal system corresponding to the countable dense system of R . The resulting system is complete since the two systems have the same linear closure. \square

¹In this course, we shall call real inner product spaces Euclidean spaces.

Definition 1.3 (Fourier Coefficients). Let ϕ_1, ϕ_2, \dots be an orthonormal system in R and let $f \in R$. Consider the sequence $c_k = \langle f, \phi_k \rangle$ for all $k = 1, 2, \dots$. Then c_k are called the coordinates or Fourier coefficients of f with respect to the system $\{\phi_k\}$ and $\sum_{k=1}^{\infty} c_k \phi_k$ is called the Fourier series of f .

Note that this series in the definition is a formal series as we do not yet know whether or not the series converges.

In the finite case, it is not difficult to see that the sequence α_k for $k = 1, \dots, n$ which minimizes $\|f - S_n^{(\alpha)}\|$ where $S_n^{(\alpha)} := \sum_{k=1}^n \alpha_k \phi_k$ is the Fourier coefficients. Indeed, we have

$$\begin{aligned} \|f - S_n^{(\alpha)}\|^2 &= \langle f, f \rangle - 2\langle f, S_n^{(\alpha)} \rangle + \langle S_n^{(\alpha)}, S_n^{(\alpha)} \rangle \\ &= \|f\|^2 - 2 \sum \alpha_k c_k + \sum \alpha_k^2 \\ &= \|f\|^2 - \sum c_k^2 + \sum (\alpha_k - c_k)^2. \end{aligned}$$

Hence, $\|f - S_n^{(\alpha)}\|$ is minimized when $\alpha_k = c_k$ for all $k = 1, \dots, n$. With this in mind, choosing α to be the Fourier coefficients, we have

$$\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^n c_k^2.$$

Geometrically, $f - S_n^{(\alpha)}$ is orthogonal to the subspace generated by ϕ_1, \dots, ϕ_n if and only if $\alpha = c$.

Furthermore, by noting $0 \leq \|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^n c_k^2$, we have

$$\sum_{k=1}^n c_k^2 \leq \|f\|^2 < \infty,$$

and hence, taking $n \rightarrow \infty$, we have $\sum_{k=1}^{\infty} c_k^2$ exists and is bounded above by $\|f\|^2$. This inequality is known as the Bessel inequality.

Definition 1.4 (Closed Orthonormal System). The orthonormal system $\{\phi_k\}$ is closed if for any $f \in R$, we have

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

This property is called the Parseval equality.

Again, by observing $\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^n c_k^2$, the system is closed if and only if for any f , the partial sums of the Fourier series converge to f , i.e. $f = \sum_{k=1}^{\infty} c_k \phi_k$.

Proposition 1.3. In a separable Euclidean space R , an orthonormal system is complete if and only if it is closed.

Proof. Suppose first that $\{\phi_k\}$ is closed. Then, for all $f \in R$, $f = \sum_{k=1}^{\infty} c_k \phi_k$. Thus, the finite linear combinations of $\{\phi_k\}$ is dense in R and thus, $\{\phi_k\}$ is complete.

On the other hand, suppose that $\{\phi_k\}$ is complete (it is countable as R is separable), for any $f \in R$, there exists some α^k such that $\|f - S_\infty^{(\alpha^k)}\| \rightarrow 0$. As we have seen, for any partial sum $S_n^{(\alpha^k)}$, we have $\|f - S_n^{(c)}\| \leq \|f - S_n^{(\alpha^k)}\|$ and so,

$$\|f - S_\infty^{(c)}\| \leq \|f - S_\infty^{(\alpha^k)}\| \rightarrow 0$$

implying $\|f - S_\infty^{(c)}\| = 0$ and the system is closed. \square

Proposition 1.4. Given $f, g \in R$ and a closed orthonormal system $\{\phi_k\}$,

$$\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$$

where $(c_k), (d_k)$ are the Fourier coefficients of f and g with respect to $\{\phi_k\}$ respectively.

Proof. We have, by Parseval's identity, $\|f\|^2 = \sum c_k^2$, $\|g\|^2 = \sum d_k^2$ and $\|f + g\|^2 = \sum (c_k + d_k)^2 = \sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2$, we have

$$\sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2 = \|f + g\|^2 = \|f\|^2 + 2 \langle f, g \rangle + \|g\|^2.$$

Thus, cancelling using $\|f\|^2 = \sum c_k^2$ and $\|g\|^2 = \sum d_k^2$, we have $\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$ as required. \square

In the case the system is only orthogonal but not necessary orthonormal, we may normalize the Fourier coefficients, i.e. given an orthogonal system $\{\phi_k\}$, we have $\{\phi/\|\phi_k\|\}$ is an orthonormal system, and so, we define

$$c_k = \left\langle f, \frac{\phi_k}{\|\phi_k\|} \right\rangle = \frac{1}{\|\phi_k\|} \langle f, \phi_k \rangle.$$

Similarly, the Fourier series of f is becomes

$$\sum_{k=1}^{\infty} c_k \frac{\phi_k}{\|\phi_k\|} = \sum \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2} \phi_k.$$

Substituting this definition of the Fourier coefficients into the Bessel inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{\langle f, \phi_k \rangle^2}{\|\phi_k\|^2} \leq \|f\|^2,$$

for any orthogonal system $\{\phi_k\}$.

Theorem 1 (Riesz). Let $\{\phi_k\}$ be a orthonormal system in a complete Euclidean space R (i.e. a real Hilbert space) and let $c \in \ell_2$ (i.e. $\sum_{k=1}^{\infty} c_k^2 < \infty$). Then, there exists some $f \in R$ such that $c_k = \langle f, \phi_k \rangle$ and Parseval's identity holds, i.e.

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

Proof. Let $f_n := \sum_{k=1}^n c_k \phi_k$. Then, by definition, we have $c_k = \langle f_n, \phi_k \rangle$ for all $k = 1, \dots, n$. Then, for all $p \geq 1$, we have

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\phi_{n+1} + \dots + c_{n+p}\phi_{n+p}\|^2 = \sum_{k=n+1}^{n+p} c_k^2.$$

Now, as $\sum c_k^2 < \infty$, we have $\{f_n\}$ is Cauchy, and thus, as R is complete, there exists some $f \in R$ such that $f_n \rightarrow f$. Thus, by noting,

$$\langle f, \phi_k \rangle = \langle f_n \phi_k \rangle + \langle f - f_n, \phi_k \rangle = c_k + \langle f - f_n, \phi_k \rangle,$$

where $\langle f - f_n, \phi_k \rangle \rightarrow 0$ as $n \rightarrow \infty$ since $|\langle f - f_n, \phi_k \rangle| \leq \|f - f_n\| \|\phi_k\|$ by the Cauchy-Schwarz inequality, we have $c_k = \langle f, \phi_k \rangle$.

Finally, Parseval's identity, follows as $\|\cdot\|^2$ is continuous in a normed space. \square

Let us recall the following result from functional analysis.

Proposition 1.5. Any separable Hilbert space is isomorphic to ℓ_2 (thus, any two separable Hilbert spaces are isomorphic).

Proof. Let H be a separable Hilbert space and choose $\{\phi_k\}$ a complete orthonormal system (which exists as H is separable). Then, for any $f \in H$, we map f to the sequence corresponding to its Fourier coefficients, i.e.

$$\psi : f \mapsto (c_1, c_2, \dots)$$

which is well-defined by Bessel's inequality. On the other hand, by Riesz's theorem, for any $x \in \ell_2$, $\sum x_k^2 < \infty$ and so, there exists a unique $f \in H$, such that $\psi(f) = x$. Thus, as ψ is clearly linear (as the inner products are linear with respect to the left component), we have the isomorphism between H and ℓ_2 . \square