# Functional Analysis

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## Contents

# 1 Basic Concepts

## 1.1 Basic Definitions

Unless stated otherwise, we assume all topologies are Hausdorff and we denote $\mathbb{K}$ for the field $\mathbb{R}$ or $\mathbb{C}$.

**Definition 1.1** (Banach Space)**.** A Banach space is a normed linear space $(X, \|\cdot\|)$ that is complete with respect to the induced metric.

**Definition 1.2** (Linear Hull)**.** Let $(X, \|\cdot\|)$ be a normed space and $M \subseteq X$. Then, the linear hull of $M$, denoted by $\mathrm{span}(M)$, is the intersection of all linear subspaces containing $M$.

**Definition 1.3** (Closed and Open Segment)**.** If $E$ is a vector space and $x, y \in E$, then

$$[x, y] := \{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\},$$

is called the closed segment defined by $x$ and $y$. Similarly,

$$(x, y) := \{\lambda x + (1 - \lambda)y \mid \lambda \in (0, 1)\},$$

is called the open segment defined by $x$ and $y$.

**Definition 1.4** (Convex)**.** Give a vector space $E$, a set $C \subseteq E$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

**Definition 1.5** (Convex Hull)**.** If $M \subseteq X$, the convex hull of $M$, denoted $\mathrm{conv}(M)$ is smallest convex subset of $X$ containing $M$, i.e. the intersection of all convex subsets of $X$ containing $M$.

**Definition 1.6** (Convex Function)**.** Given a convex subset $U$ of the vector space $V$, the function $f : U \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in U$ and $\lambda \in [0, 1]$. Furthermore, we say $f$ is strictly convex if

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y),$$

for all $x, y \in U$, $x \neq y$, $\lambda \in (0, 1)$.

By the triangle inequality, we observe that the norm of a normed space is a convex function on that space.

**Definition 1.7** (Symmetric and Balanced)**.** A set $M \subseteq X$ is called symmetric if $(-1)M \subseteq M$ and balanced if $\alpha M \subseteq M$ for all $\alpha \in \mathbb{K}, |\alpha| \leq 1$.

**Proposition 1.1.** Let $Y \leq X$ where $(X, \|\cdot\|)$ is a Banach space. Then $Y$ is a Banach space if and only if $Y$ is closed in $X$.

*Proof.* Clearly, if $Y$ is closed, it contains all the limits of all Cauchy sequences, and thus, is Banach. On the other hand, as all convergent sequences are Cauchy, $Y$ contains its limit points, and thus is closed. $\qquad\square$

**Definition 1.8** (Bounded)**.** A subset $M$ of a normed space $(X, \|\cdot\|)$ is called bounded if there exists some $r > 0$ such that $M \subseteq rB_X$ (where $B_X$ is the unit ball in $X$). $M$ is called totally bounded if for every $\epsilon > 0$, $M$ can be can be covered by a finite number of $\epsilon$ balls. A sequence $(x_n) \subseteq X$ is bounded if the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded.

**Proposition 1.2.** Every totally bounded set is bounded.

*Proof.* Take $\epsilon = 1$ and suppose $M$ is covered by $n$ balls of radius 1, $(B_1(x_i))_{i=1}^n$. Then, we have for all $x \in M$, there exists some $i \in \{1, \cdots, n\}$ such that $x \in B_1(x_i)$, and thus,

$$\|x\| = \|x - x_i + x_i\| \leq \|x - x_i\| + \|x_i\| < 1 + \|x_i\| \leq 1 + \max_{i=1,\cdots,n} \|x_i\|.$$

Hence, choosing $r = 1 + \max_{i=1,\cdots,n} \|x_i\|$ suffices. □

Alternatively, total boundedness can be described using $\epsilon$-nets.

**Definition 1.9** ($\epsilon$-net and $\epsilon$-separated)**.** Let $M$ be a subset of the normed space $(X, \|\cdot\|)$. Then, $A \subseteq M$ is an $\epsilon$-net in $M$ if

$$M \subseteq \bigcup_{x \in A} B_\epsilon(x).$$

That is to say, all elements of $M$ are at most $\epsilon$ away from $A$.

On the other hand $A \subseteq M$ is said to be $\epsilon$-separated in $M$ if $\|x - y\| \geq \epsilon$ for all $x, y \in A$, $x \neq y$.

Clearly, we see that a set $M \subseteq X$ is totally bounded if and only if for every epsilon there exists a finite $\epsilon$-net. Furthermore, we see that a maximal $\epsilon$-separated set is an $\epsilon$-net.

**Proposition 1.3.** Let $(X, \|\cdot\|)$ be a $n$-dimensional real normed space and suppose $(x_j)_{j=1}^N$ is an $\epsilon$-net of $B_X$. Then, $N \geq \epsilon^{-n}$.

*Proof.* Let $T$ be the natural linear isomorphism between $X$ and $\mathbb{R}^n$ and denote $\lambda$ the standard Lebesgue measure on $\mathbb{R}^n$. Then,

$$\lambda(T(B_X)) \leq \lambda \left( T \left( \bigcup_{i=1}^N B_\epsilon(x_i) \right) \right) = \lambda \left( \bigcup_{i=1}^N T(B_\epsilon(x_i)) \right) \leq \sum_{i=1}^N \lambda(T(B_\epsilon(x_i))).$$

Now, since $T(B_\epsilon(x_i)) = T(B_\epsilon(0)) + T(x_i)$, we have

$$\sum_{i=1}^N \lambda(T(B_\epsilon(x_i))) = \sum \lambda(T(B_\epsilon(0) + T(x_i))) = N\lambda(\epsilon T(B_X)) = N\epsilon^{-n}\lambda(T(B_X)),$$

and the result follows. □

**Proposition 1.4.** On the other hand, there is an $\epsilon$-net $(x_j)_{j=1}^N$ with $N \leq (1 + 2/\epsilon)^n$.

*Proof.* Since the maximal $\epsilon$-separated set is an $\epsilon$-net we will show the existence of a maximal $\epsilon$-separated set with $N \leq (1 + 2/\epsilon)^n$ elements. Let $(x_j)_{j=1}^N$ be a maximal $\epsilon$-separated set

(in which we add points sequentially until it is impossible to do so), then, it is clear that $B_{\epsilon/2}(x_i) \cap B_{\epsilon/2}(x_j) = \emptyset$ for all $i \neq j$. Thus, as $B_{\epsilon/2}(x_i) \subseteq (1 + \epsilon/2)B_X$,

$$N(2/\epsilon)^n \lambda(T(B_X)) = \lambda \left( T \left( \bigcup_{i=1}^{N} B_{\epsilon/2}(x_i) \right) \right) \lambda((1 + \epsilon/2)T(B_X)) = (1 + \epsilon/2)^n \lambda(T(B_X)).$$

With that, by rearranging, the result follows. $\qquad \square$

## 1.2 Classical Spaces

We recall some results from measure theory including Hölder and Minkowski's inequalities.

**Proposition 1.5.** $(C[0,1], \|\cdot\|_\infty)$ is a Banach space (where $C[0,1]$ is the vector space of continuous functions $f : [0,1] \to \mathbb{K}$ and $\|\cdot\|_\infty$ is the supremum norm).

*Proof.* See first year analysis. $\qquad \square$

By the same argument, the space $C(K)$ where $K$ is compact, endowed with the supremum norm is also a Banach space.

It is easy to see $C[0,1]$ is infinite dimensional (consider the standard basis for the polynomials) and more generally, $C(K)$ for compact $K$ is infinite dimensional as soon as $K$ is infinite (this follows by applying the Tietze-Urysohn theorem on distinct Kronecker delta functions).

**Definition 1.10** (Sequence Space). $\ell_\infty^n$ is the $n$-dimensional vector space of all $n$-tuples of $\mathbb{K}$ endowed with the supremum norm $\|\cdot\|_\infty$.

We note that $\ell_\infty^n$ is a special case of $C(K)$ endowed with the discrete topology.

By considering Hölder and Minkowski's inequalities for the sequence spaces, it is clear that we may define the classical $\ell_p^n$ spaces.

**Definition 1.11** (Sequence Space). $\ell_p^n$ for some $p \in [1, \infty)$ is the $n$-dimensional vector space of all $n$-tuples of $\mathbb{K}$ endowed with the norm $\|\cdot\|_p$ such that

$$\|x\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Furthermore, we denote $\ell_p = \ell_p(\mathbb{N})$ for the vector space of all scalar valued sequences $x = (x_i)_{i=1}^{\infty}$ satisfying $\sum |x_i| < \infty$ endowed with the norm

$$\|x\|_p := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

In the case $p = \infty$, we define

$$\|x\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\}.$$

We denote $c_{00} = c_{00}(\mathbb{N})$ for the subspace of $\ell_\infty$ consisting of the sequences with finite support. We also $c = c(\mathbb{N})$ for the subspace of convergent sequences. Lastly, we denote $c_0 = c_0(\mathbb{N})$ for the subspace of sequences convergent to 0.

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It is clear that $c_{00} \subseteq c_0 \subseteq c$.

**Proposition 1.6.**

- For $p \in [1, \infty]$, $\ell_p$ is a Banach space;

- $c$ and $c_0$ are closed subspaces of $\ell_\infty$ (and hence, also Banach);

- $c_{00}$ is not complete.

*Proof.* The first claim is similar to that for the function spaces while the other two are simple analysis proofs. □

We introduce the following generalisation for the above notion. We denote $\Gamma$ an arbitrary non-empty set and $\ell_p(\Gamma)$ the set of functions $f : \Gamma \to \mathbb{K}$ such that $\sum_{\gamma \in \Gamma} |f(\gamma)|^p < \infty$ equipped with the norm

$$\|f\|_p := \sum_{\gamma \in \Gamma} |f(\gamma)|^p,$$

where we define

$$\sum_{\gamma \in \Gamma} |f(\gamma)|^p := \sup \left\{ \sum_{\gamma \in F} |f(\gamma)|^p \mid F \text{ a finite subset of } \Gamma \right\}.$$

Furthermore, we define $l_\infty(\Gamma)$ the set of all bounded functions $f : \Gamma \to \mathbb{K}$ endowed with the supremum norm $\|\cdot\|_\infty$. Similarly, we define the subspaces $c_0(\Gamma)$ for the set of functions such that $\{\gamma \in \Gamma \mid |f(\gamma)| \geq \epsilon\}$ is finite for all $\epsilon > 0$, and $c_{00}(\Gamma)$ for the set of functions with finite support. Note that $c_0(\Gamma) = \overline{c_{00}(\Gamma)}$ (clearly $c_0(\Gamma) \subseteq c_{00}(\Gamma)$ and we may construct a sequence in $c_0$ converging to each element of $c_{00}$).

From the same argument as all the others, one can show $\ell_p(\Gamma)$ and $c_0(\Gamma)$ are Banach spaces.

**Definition 1.12** (Function Space). Given $p \in [1, \infty)$, we define $L^p := \mathcal{L}^p / \sim$ where $\sim$ is the equivalence relation

$$f \sim g \iff f = g \text{ a.e.}$$

and $\mathcal{L}^p$ the set of measurable functions $f$ such that

$$\int |f|^p \mathrm{d}\mu < \infty.$$

We recall that $L^p$ spaces are Banach spaces.

**Proposition 1.7.** A normed space $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series in $X$ is convergent.

*Proof.* Exercise. □

**Definition 1.13** (Essential Supremum). Given $f : [0, 1] \to \mathbb{R}$ measurable sets, we define the essential supremum of $f$ to be

$$\operatorname{ess\,sup}(f) := \inf_{\substack{N \subseteq [0,1], \\ \lambda(N)=1}} (\sup f(N)).$$

Equivalently,
$$\operatorname{ess\,sup}(f) = \inf\{\alpha \mid \lambda(f^{-1}(\alpha, \infty)) = 0\}.$$

We recall that the essential supremum forms a norm on the function space resulting in the Banach space $L^\infty$, i.e. $\|f\|_\infty = \operatorname{ess\,sup} |f|$.

## 1.3 Operators

**Proposition 1.8.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T : X \to Y$ be a linear map. Then the following are equivalent,

1. $T$ is continuous on $X$;

2. $T$ is continuous at $0_X$;

3. there exists some $C > 0$ such that $\|T(x)\|_Y \le C\|x\|_X$ for all $x \in X$;

4. $T$ is Lipschitz;

5. $T(B_X)$ is a bounded set in $Y$.

*Proof.* Clearly $1 \implies 2$, $4 \implies 1$ and $3, 4, 5$ are equivalent, it suffices to show $2 \implies 1$ and $1 \implies 5$.

Suppose $T$ is continuous at $0_X$, then, we will show $T$ is continuous at $x$ for all $x \in X$. Fix $\epsilon > 0$, as $T$ is continuous at $0_X$, there exists some $\delta > 0$ such that for all $y \in X, \|y\|_X < \delta$, $\|T(y)\|_Y < \epsilon$. Then, for all $z \in X, \|z - x\|_X < \delta$, we have $\|T(z) - T(x)\|_Y = \|T(z - x)\|_Y < \epsilon$, and so $T$ is continuous at $x$.

Suppose $T$ is continuous on $X$, then, there exists some $\delta > 0$ such that for all $\|x\|_X < \delta$, $T(x) \in B_Y$. Thus, for all $x \in B_X$, as $\|\delta x\|_X = \delta\|x\| < \delta$, $\delta T(x) = T(\delta x) \in B_X$, and hence, $T(B_X) \subseteq \delta^{-1}B_Y$ implying $T(B_X)$ is bounded. $\qquad\square$

**Definition 1.14** (Operator)**.** Let $X, Y$ be normed spaces. An operator from $X$ to $Y$ is simply a linear map $T : X \to Y$ and is called bounded if $T(B_X)$ is bounded.

We endow the space of all bounded operators from $X$ to $Y$, $\mathcal{B}(X, Y)$ with the operator norm $\|\cdot\|$ where
$$\|T\| := \sup\{\|T(x)\|_Y \mid x \in B_X\}.$$

It is easy to see that $\mathcal{B}(X, Y)$ is a normed space.

**Proposition 1.9.** Let $X, Y$ be normed spaces. Then, if $Y$ is Banach then so is $\mathcal{B}(X, Y)$.

*Proof.* Follows similarly to the proof that $C(X, Y)$ is Banach. $\qquad\square$

**Definition 1.15** (Dual)**.** Let $(X, \|\cdot\|_X)$ be a normed space and we denote $X^*$ for the dual of $X$ consisting the linear functionals of $X$, i.e. $X^* := \mathcal{B}(X, \mathbb{K})$.

**Remark.** As $X^*$ forms an additive group, it is easy to see that there exists a natural group action of $X^*$ on $X$ where given $f \in X^*$, $x \in X$, we define $f \cdot x = f(x)$.

We recall from second year linear algebra that for finite dimensional spaces $X$, $\dim X = \dim X^*$. On the other hand, it is clear that if $X$ is infinite dimensional, so is $X^*$. Indeed, if $X^*$ is finite dimensional, then $\dim X^{**} = \dim X^* < \infty$. But, by considering the canonical isomorphism from $X^{**}$ to $X$, it follows $\dim X = \dim X^{**} < \infty$.

**Proposition 1.10.** $X^*$ is a Banach space for every normed space $X$.

*Proof.* This is clear since $\mathbb{K}$ is complete. $\qquad\square$

**Definition 1.16** (Isomorphism). An operator $T \in \mathcal{B}(X,Y)$ is an linear isomorphism if it is bijective and $T^{-1} \in \mathcal{B}(Y,X)$. We denote the set of all linear isomorphisms between $X$ and $Y$ by $GL(X,Y)$.

It is clear that a linear map $T \in \mathcal{L}(X,Y)$ is a linear isomorphism if and only if there exist some $C_1, C_2$ such that for all $x \in X$,

$$C_1\|x\| \le \|T(x)\| \le C_2\|x\|.$$

Indeed, we see that $C_2$ bounds $T$ while $C_1$ bounds $T^{-1}$.

**Definition 1.17** (Equivalent Norms). Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on the vector space $X$. Then we say $\|\cdot\|_1, \|\cdot\|_2$ are equivalent if the identity map $I_X : X \to X : x \mapsto x$ from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$ is a linear isomorphism.

By the above property we see that the two norms are equivalent if and only if there exists some $C_1, C_2$ such that for all $x \in X$,

$$C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1.$$

Normed spaces $X, Y$, are called linearly isomorphic if there exists a linear isomorphism between $X$ and $Y$. It is clear that since linear isomorphisms are both vector space isomorphisms and topological homeomorphisms, a linear isomorphism preserves both vector space and topological properties. In particular, if $X, Y$ are linearly isomorphic and $X$ is Banach, so is $Y$.

**Definition 1.18** (Isometry). An operator $T \in \mathcal{B}(X,Y)$ is an linear isometry if for all $x \in X$, $\|x\| = \|T(x)\|$.

**Definition 1.19** (Banach-Mazur Distance). Let $X, Y$ be linearly isomorphic normed spaces. The Banach-Mazur distance between $X$ and $Y$ is defined to be

$$d(X,Y) := \inf\{\|T\|\|T^{-1}\| \mid T \in GL(X,Y)\}.$$

By taking the logarithm of the above, it is easy to see that $\log \circ d$ forms a metric on the space of linearly isomorphic normed spaces. In particular, it turns out, the isometry classes of $n$-dimensional normed spaces becomes a compact space with respect to this metric. This fact is called the *Banach-Mazur Compactum*.

**Definition 1.20** (Compact Operator). An operator $\mathcal{B}(X,Y)$ is called a compact operator if $\overline{T(B_X)}$ is compact. We denote the space of all compact operators from $X$ to $Y$ by $\mathcal{K}(X,Y)$.

**Definition 1.21** (Finite Rank Operator)**.** An operator $\mathcal{B}(X,Y)$ is called a finite rank operator if $\dim T(X) < \infty$. We denote the space of all finite rank operators from $X$ to $Y$ by $\mathcal{F}(X,Y)$.

**Definition 1.22** (Linear Projection)**.** Let $V$ be a vector space. Then a linear projection $P$ on $V$ is simply a idempotent $V$-endomorphism, i.e. $P : V \to V$ is a linear map such that $P \circ P = P$.

It is not difficult to see that a linear projection provides a natural algebraic decomposition of $V$. In particular, we observe $V = \ker P \oplus P(V)$ (consider $x = P(x) + (x - P(x))$). Furthermore, given a decomposition $V = X \oplus Y$, we may define a projection on $V$ by defining $P$ such that for each $x + y \in V, P(x + y) = x$.

**Definition 1.23** (Topological Direct Sum)**.** Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed spaces, we may endow the outer direct sum $X \oplus Y$ with the norm

$$\|(x,y)\| := \|x\|_X + \|y\|_Y,$$

with the resulting normed space called the topological direct sum of $X$ and $Y$.

## 1.4 Quotient Spaces

We recall the quotient vector space from year 2 linear algebra and apply its definitions directly to normed spaces.

As normed spaces are additive groups, we may quotient them by its normal subgroups.

**Definition 1.24** (Quotient Subspace)**.** Given a normed space $X$ and $Y \leq X$ a closed subspace, $X/Y$ is the space $X/\sim$ where $\sim$ is the equivalence relation

$$x_1 \sim x_2 \iff x_1 + Y = x_2 + Y,$$

endowed with the norm $\|[x]\| := \inf\{\|x\| \mid x \in [x]\}$.

**Definition 1.25** (Quotient Map)**.** The quotient map is the canonical mapping $q : X \to X/Y : x \mapsto [x]$.

It is clear that the quotient map is linear and continuous.

**Proposition 1.11.** Let $Y$ be a closed subspace of a Banach space $X$. Then $X/Y$ is also Banach.

*Proof.* It suffices to show every absolutely convergent series in $X/Y$ converges. Suppose $\sum \|[x_n]\| < \infty$. Then, since for all $n$, there exists some $x_n \in [x_n]$ such that $\|x_n\| \leq \|[x_n]\| + 2^{-n}$, we have $\sum \|x_n\| \leq \sum (\|[x_n]\| + 2^{-n}) = \sum \|[x_n]\| + 1 < \infty$. Thus, as $X$ is Banach, $\sum x_n$ converges. With that, by sequential continuity,

$$\sum [x_n] = \sum q(x_n) = q\left(\sum x_n\right)$$

also converges. $\qquad\square$

**Theorem 1.** Let $X$ be a finite dimensional vector space. Any two norms on $X$ are equivalent. In particular, all finite dimensional normed spaces are Banach spaces and every normed space of dimension $n$ is isomorphic to $\ell_2^n$.

*Proof.* Let $B = \{e_1, \cdots, e_n\}$ be a basis of $X$ and for each $x = \sum_{i=1}^n \lambda_i e_i \in X$, we define $\|x\|_1 := \sum_{i=1}^n |\lambda_i|$. By checking the axioms, it is clear that $\|\cdot\|_1$ is a norm on $X$. Let $S_1 := \{x \in X \mid \|x\|_1 = 1\}$ and I claim it suffices to show $S_1$ is compact. Indeed, if $S_1$ is compact, as $\|\cdot\|$ is continuous on $S_1$, $\|\cdot\|$ achieves its its maximum and minimum on $S_1$. Hence, there exists some $c_1, c_2 > 0$ such that

$$c_1 \leq \left\| \frac{x}{\|x\|_1} \right\| \leq c_2.$$

Now, to show $S_1$ is compact, take $x^m = \sum_{i=1}^n \lambda_i^m e_i$ a sequence in $S_1$ and we will show it has a convergent subsequence. By definition $\sum_i |\lambda_i^m| = 1$ and thus, $\lambda_i^m$ is bounded for all $m$. Thus, by Bolzanno-Weierstrass, $\lambda_i^m$ has a convergent subsequence (with respect to $m$), i.e. $\lambda_i^{m_j} \to \lambda_i$ as $j \to \infty$ for some $\lambda_i \in \mathbb{K}$. With that, defining $x := \sum_i \lambda_i e_i$, we see

$$\|x^{m_j} - x\| = \left\| \sum_i (\lambda_i^{m_j} - \lambda_i) e_i \right\| \to 0 \text{ as } j \to \infty.$$

To show that $\ell_2^n$ is linear isomorphic to $X$ simply take $T$ to be a vector space isomorphism between $X$ and $\ell_2^n$, then we may define $\|\cdot\|_2$ where $\|x\|_2 := \|T(x)\|_{\ell_2^n}$. With that $T$ forms an linear isometry between $(X, \|\cdot\|_2)$ and $\ell_2^n$ implying $\|\cdot\|_{\ell_2^n} \simeq \|\cdot\|_2 \simeq \|\cdot\|_X$. $\qquad \square$

By the above proposition, we see that given $X, Y$ finite dimensional normed spaces, $\|(x, y)\|_p := (\|x\|_X^p + \|y\|_Y^p)^{1/p}$ is a equivalent renorming of $X \oplus Y$. We denote this renormed space by $(X \oplus Y)_p$.

**Lemma 1.1** (Riesz's Lemma)**.** Let $X$ be a normed space. If $Y$ is a proper closed subspace of $X$, then for every $\epsilon > 0$, there exists some $x \in X$, $\|x\| = 1$, such that $\text{dist}(x, Y) \geq 1 - \epsilon$.

*Proof.* Since $Y$ is proper, $X/Y$ is non-trivial and thus, there exists some $[x] \in X/Y$, $\|[x]\| = 1$. Thus, as $\|\cdot\|$ is continuous, by the intermediate value theorem, there exists some $[z] \in X/Y$, $1 > \|[z]\| > 1 - \epsilon$. By definition,

$$1 > \inf\{\|z\| \mid z \in [z]\} > 1 - \epsilon,$$

and thus, there exists some $z \in [z]$, $\|z\| < 1$. Now, defining $x := z/\|z\|$, we have $\|x\| = 1$ and

$$\text{dist}(x, Y) = \inf_{y \in Y} \|x - y\| = \frac{1}{\|z\|} \inf_{y \in Y} \|z - y\| = \frac{\|[z]\|}{\|z\|} \geq \frac{1 - \epsilon}{1} = 1 - \epsilon.$$

$\qquad \square$

From Riesz's lemma, we obtain that, for proper subspaces $Y$, the quotient map $q : X \to X/Y$ satisfies $\|q\| = 1$ (since we may construct a sequence $x_n \in B_X$ such that $\|q(x_n)\| \geq 1 - \frac{1}{n}$).

**Proposition 1.12.** A normed space $X$ is finite dimensional if and only if $B_X$ is compact.

*Proof.* If $X$ is finite dimensional, then $B_X$ is compact by the same proof as theorem 1. On the other hand, if $X$ is infinitely dimensional, we construct the following sequence inductively. Choose $x_1 \in B_X$ and for $n \in \mathbb{N}$, choose $x_{n+1} \in B_X$ such that

$$\text{dist}(x_{n+1}, \text{span}\{x_1, \cdots, x_n\}) > \frac{1}{2},$$

which existence is guaranteed by Riesz's lemma. It is clear that $(x_n)$ does not have a convergent subsequence and thus, $B_X$ is not compact. $\square$

**Proposition 1.13.** Every operator $T$ from a finite-dimensional normed space $X$ into a normed space $Y$ is continuous.

*Proof.* This follows by simply bounding $T(B_X)$ which is bounded naturally. $\square$

From this, we see that every finite dimensional normed space is linearly isomorphic if they have the same dimension. Indeed, as they are already algebraically isomorphic, we obtain the isomorphisms (and its inverse) are continuous by above. Alternatively, the same result follows as they are linearly isomorphic to $\ell_2^n$

**Proposition 1.14.** Let $X, Y$ be normed spaces. Then $\mathcal{F}(X, Y)$ is a linear subspace of $\mathcal{K}(X, Y)$. Furthermore, $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$, and thus, if $Y$ is a Banach space, so is $\mathcal{K}(X, Y)$.

*Proof.* For the first claim, it suffices to show $\mathcal{F}(X, Y) \subseteq \mathcal{K}(X, Y)$. Let $T \in \mathcal{F}(X, Y)$, then, $\dim T(X) < \infty$ and $T(X)$ is homeomorphic to $\mathbb{R}^n$ where $n = \dim T(X)$. Thus, as the Heine-Borel property is topologically invariant, it follows $\overline{T(B_X)}$ is compact as it is closed and bounded. Hence, $T \in \mathcal{K}(X, Y)$ and $\mathcal{F}(X, Y) \leq \mathcal{K}(X, Y)$.

To show $\mathcal{K}(X, Y)$ is closed let $(T_n) \subseteq \mathcal{K}(X, Y)$ such that $T_n \to T \in \mathcal{B}(X, Y)$, then it suffices to show $T \in \mathcal{K}(X, Y)$, in particular, we will show $T(B_X)$ is totally bounded. Fix $\epsilon > 0$, then as $T_n \to T$ (where the convergence is uniform), there exists some $N$ such that for all $x \in B_X$, $n \geq N$, $\|T_n(x) - T(x)\| < \epsilon/2$. Now, since $T_n(B_X)$ is totally bounded, there exists a finite $\epsilon/2$-net $F$ for $T_n(B_X)$. Clearly, $F$ is a finite $\epsilon$-net for $T(B_X)$. Indeed, for all $x \in B_X$, there exists some $y \in F$ such that $\|T_n(x) - y\| < \epsilon/2$ and so,

$$\|T(x) - y\| \leq \|T(x) - T_n(x)\| + \|T_n(x) - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

## 1.5 Hilbert Spaces

We recall that Hilbert spaces are inner product spaces which is complete with respect to the induced norm. We will in this section let $H$ equipped with the inner produce $\langle \cdot, \cdot \rangle$ be a Hilbert space.

**Lemma 1.2.** For all $y \in H$, $\langle \cdot, y \rangle : H \to \mathbb{F}$ is continuous.

*Proof.* This follows straight away from the Cauchy-Schwarz inequality. $\square$

**Proposition 1.15.** If $F$ be a subspace of $H$, then $F^\perp$ is closed.

*Proof.* Let $(x_n) \subseteq F^\perp$ such that $x_n \to x \in H$, then for all $f \in F$, $0 = \langle x_n, f \rangle \to \langle x, f \rangle$, and so $\langle x, f \rangle = 0$ for all $f \in F$, i.e. $x \in F^\perp$. $\square$

**Theorem 2.** If $F$ be a closed subspace of the Hilbert space $H$, then $F \oplus F^\perp = H$. In particular, the map $T : F \oplus F^\perp \to H : (x, y) \mapsto x + y$ is a topological, and algebraic isomorphism.

*Proof.* Clearly, $F \cap F^\perp = \{0\}$ so it suffices to show $H \subseteq F + F^\perp$.

Let $h \in H$ and define

$$x_1 := \arg \min_{f \in F} \|h - f\|,$$

which exists in $F$ as it is closed. Furthermore, define $x_2 := h - x_1$ (noting that $\|x_2\|^2 = \text{dist}(x, F)^2$), it remains to show that $x_2 \in F^\perp$. Suppose otherwise, then there exists some $f \in F$ such that $\langle x_2, f \rangle > 0$. Now, for all $\epsilon > 0$, we have,

$$\|h - (x_1 + \epsilon f)\|^2 = \|x_2 - \epsilon f\|^2 = \langle x_2 - \epsilon f, x_2 - \epsilon f \rangle$$
$$= \|x_2\|^2 + \epsilon^2 \|f\| - 2\epsilon \langle x_2, f \rangle$$
$$= \text{dist}(x, F)^2 - \epsilon(2\langle x_2, f \rangle - \epsilon \|f\|).$$

Thus, as for sufficiently small $\epsilon$, we have $2\langle x_2, f \rangle - \epsilon \|f\| > 0$, we found an element of $F$ achieving better distance, a contradiction! Hence, $x_2 \in F^\perp$ and $F \oplus F^\perp = H$. $\square$

A consequence of this is that $F = (F^\perp)^\perp$ for closed subspace $F$. Indeed, it is clear that $F \subseteq (F^\perp)^\perp$ and for $h \in (F^\perp)^\perp$, there exists unique $f \in F, g \in F^\perp$ such that $h = f + g$ and so, $f = h - g$. Now, since $F^\perp$ is closed, we also have $F^\perp \oplus (F^\perp)^\perp = H$ and so, the representation $f = h + (-g)$ is unique. But, since $f \in F \subseteq (F^\perp)^\perp$, we have $f = f + 0$ and thus, $g = 0$ and finally, $h = f \in F$.

**Definition 1.26** (Orthonormal)**.** Let $H$ be a Hilbert space and $S \subseteq H$. Then $S$ is called orthonormal if for all $s_i, s_j \in S$, $\langle s_i, s_j \rangle = \delta_{ij}$.

We call an orthonormal set an orthonormal basis if it is maximal.

**Theorem 3.** Every Hilbert space has an orthonormal basis.

*Proof.* Clearly, the union of a chain of orthonormal sets is orthonormal, thus, by Zorn's lemma, there exists a maximal orthonormal set. $\square$

**Proposition 1.16.** Let $H$ be a Hilbert space and $H_0$ a closed subspace of $H$. Then, every orthonormal basis of $H_0$ can be extended to an orthonormal basis of $H_0$.

*Proof.* This follows straight away by Zorn's lemma again since every chain of the set of orthonormal sets has an upper bound. $\square$

Consequently, if $M$ is an orthonormal basis of $H$, then $\overline{\text{span}} M = H$. Indeed, if otherwise, we may simply take a vector $v$ of norm 1 in $\overline{\text{span}} M^\perp$, constructing a larger orthonormal set $M \cup v$, contradicting the maximality of $M$.

**Theorem 4.** Every separable infinite-dimensional Hilbert space $H$ has a countable orthonormal basis $(e_i)_{i=1}^\infty$. Furthermore, if $(e_i)_{i=1}^\infty$ is a countable basis of $H$, then for all $x \in H$,

$$x = \sum_{i=1}^\infty \langle x, e_i \rangle e_i,$$

where we call the coefficients $\langle x, e_i \rangle$ the Fourier coefficients and the sum itself as the Fourier expansion of $x$.

*Proof.* Let $(e_\mu)_{\mu \in M}$ be a orthonormal basis of $H$ and suppose $M$ is uncountable. Then, for all $e_\mu \neq e_\lambda$, by orthogonality, we have

$$\|e_\mu - e_\lambda\| = \sqrt{2}.$$

Thus, we have $\{B_{1/2}(e_\mu) \mid \mu \in M\}$ is a set of pairwise disjoint balls, each containing one and only one basis vector. Now, as $H$ is separable, there exists some $(h_i)_{i=1}^\infty$ which is dense in $H$. Hence, each ball contain at least one $h_i$. But then, we have a injective function from a countable set to an uncountable set, which is a contradiction. Thus, the basis must be countable. $\qquad\square$

**Proposition 1.17.** Let $(e_i)_{i=1}^\infty$ be an orthonormal set in a Hilbert space $H$ and $x \in H$. Then

- (The Bessel inequality) $\sum \langle x, e_i \rangle^2 \leq \|x\|^2$;

- (The Parseval equality) if $(e_i)_{i=1}^\infty$ is an orthonormal basis of $H$, then $\sum \langle x, e_i \rangle^2 = \|x\|^2$;

- if the Parseval equality holds for every $x \in H$, then $(e_i)_{i=1}^\infty$ is an orthonormal basis of $H$;

- if $\overline{\operatorname{span}}(e_i)_{i=1}^\infty = H$, then $(e_i)_{i=1}^\infty$ is an orthonormal basis of $H$.

**Theorem 5** (Riesz-Fischer theorem)**.** Every separable infinite-dimensional Hilbert space $H$ is linearly isometric to $\ell_2$.

# 2 Hahn-Banach Theorem and Open Mapping Theorems

**Definition 2.1.** A real-valued function $p : X \to \mathbb{R}$ is said to be subadditive if for all $x, y \in X$, $p(x + y) \leq p(x) + p(y)$; positively homogenous if for all $x \in X, a \geq 0$, we have $p(ax) = ap(x)$. Finally, if $p$ is subadditive, and for all scalars $a$, $p(ax) = |a|p(x)$, we call $p$ a seminorm.

It is clear that any positively homogenous subadditive function is convex.

**Theorem 6** (Hahn-Banach). Let $Y$ be a subspace of the real-linear space $X$, and $p$ a positively homogenous subadditive functional on $X$. Then, if $f$ is a linear functional on $Y$, such that $f(x) \leq p(x)$ for all $x \in Y$, there exists a linear functional $F$ on $X$, such that for all $x \in Y$, $F(x) = f(x)$ and $F(x) \leq p(x)$ for all $x \in F$.