Manifolds

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This module introduces the notion of manifolds and provides the infrastructure for generalizing theorems from calculus to manifolds. In particular, we will talk about

- Smooth manifolds and smooth functions;
- Tangent spaces and vector fields;
- Differential forms, integrations and Stoke's theorem.

In contrast to the curves and spaces module, instead of working on Euclidean spaces, we will define these notions for general manifolds. Thus, many definitions such as the tangent space will be defined in a more intrinsic point of view, without requiring our manifold to be within a Euclidean space.

Furthermore, a goal of this module is to differentiate between different manifolds, that is determine whether or not two manifolds are diffeomorphic with one another. This is achieved through introducing invariants such as the notion of differential forms and these notions will appear in many other places especially in geometry.

Manifolds is the subject of studying geometric shapes, and in mathematics, there are in general two ways of doing this. The first of which is by embedding the object into an ambient space such as $\mathbb{R}^2$ or $\mathbb{R}^3$. An example of this is studying the unit circle through the parametrisation

$$\{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

and is the more common method of what we have done thus far. On the other hand, one may study the object independently of the ambient space. This is the approach we shall take throughout this course. In particular, we will study spaces which at a local level "looks like" a Euclidean space directly without embedding the structure into $\mathbb{R}^n$.