Manifolds

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Contents

1 Introduction

This module introduces the notion of manifolds and provides the infrastructure for generalizing theorems from calculus to manifolds. In particular, we will talk about

- Smooth manifolds and smooth functions;
- Tangent spaces and vector fields;
- Differential forms, integrations and Stoke's theorem.

In contrast to the curves and spaces module, instead of working on Euclidean spaces, we will define these notions for general manifolds. Thus, many definitions such as the tangent space will be defined in a more intrinsic point of view, without requiring our manifold to be within a Euclidean space.

Furthermore, a goal of this module is to differentiate between different manifolds, that is determine whether or not two manifolds are diffeomorphic with one another. This is achieved through introducing invariants such as the notion of differential forms and these notions will appear in many other places especially in geometry.

Manifolds is the subject of studying geometric shapes, and in mathematics, there are in general two ways of doing this. The first of which is by embedding the object into an ambient space such as $\mathbb{R}^2$ or $\mathbb{R}^3$. An example of this is studying the unit circle through the parametrisation

$$\{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

and is the more common method of what we have done thus far. On the other hand, one may study the object independently of the ambient space. This is the approach we shall take throughout this course. In particular, we will study spaces which at a local level "looks like" a Euclidean space directly without embedding the structure into $\mathbb{R}^n$.

2 Topological and Smooth Manifolds

Let us first recall some notions from topology.

Definition 2.1. Let X, Y be topological spaces and let $f : X \to Y$ be a function, then

- f is continuous if $f^{-1}(U)$ is open in X for all U open in Y.
- f is a homeomorphism if it is continuous and has a continuous inverse.

Definition 2.2. A topological space X is

- Hausdorff if for all $x, y \in X$, $x \neq y$, there exists open sets U, V in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.
- second-countable if there exists countable $\mathcal{F} \subseteq \mathcal{T}_X$ such that any open set in X can be written as a union of elements of $\mathcal{F}$, i.e. $\mathcal{F}$ is a countable basis of X.

In general, in this module, we will assume our topology is Hausdorff and second-countable in order to avoid pathological examples in smooth and topological manifolds.

Definition 2.3 (Co-ordinate Chart)**.** Let X be a topological space. A co-ordinate chart on X is the collection of

- an open set $U \subseteq X$,
- an open set $\tilde{U} \subseteq \mathbb{R}^n$ for some $n \geq 0$,
- a homeomorphism $f : U \to \tilde{U}$.

We denote a co-ordinate chart by (U, f).

Definition 2.4. Let X be a (Hausdorff and second-countable) topological space. We say that X is a topological manifold of dimension n if for all $x \in X$, there exists a co-ordinate chart (U, f) with $\tilde{U} \subseteq \mathbb{R}^n$ such that $x \in U$.

The classical example of a topological manifold is the circle, in particular

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

is a 1 dimensional topological manifold. Consider $U_1 = S^1 \setminus \{(0, -1)\}$, and we define the stereographic projection $f_1 : U_1 \to \mathbb{R}$,

$$f : (x, y) \mapsto \frac{x}{y+1} := \tilde{x}.$$

It is not difficult to see that f_1 is invertible with the inverse

$$f_1^{-1} : \tilde{x} \mapsto \left(\frac{2\tilde{x}}{1+\tilde{x}^2}, \frac{1-\tilde{x}^2}{1+\tilde{x}^2}\right).$$

Furthermore, as f_1 and f^{-1} are continuous, we have (U_1, f_1) is a co-ordinate chart. Similarly, we define $U_2 = S^1 \setminus \{(0, 1)\}$, and we my show the existence of a homeomorphism $f_2 : U_2 \to \mathbb{R}$, providing the second co-ordinate chart (U_2, f_2). Thus, as $S^1 = U_1 \cup U_2$, we have S^1 is a 1 dimensional topological manifold.

The above example can be expanded to n-dimensional sphere

$$S^n := \{(x_0, \cdots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n.$$

Similarly as before, we can construct two co-ordinate charts covering all points on the sphere except for the poles allowing us to conclude S^n is a n-dimensional topological manifold.

Definition 2.5 (Transition Function)**.** Let X be a topological manifold and let (U_1, f_1) and (U_2, f_2) be two co-ordinate charts on X such that $U_1 \cap U_2 \neq \emptyset$. Then the transition function between these two co-ordinate charts is the function

$$\phi_{21} := f_2 \circ f_1^{-1} : f_1(U_1 \cap U_2) \to f_2(U_1 \cap U_2).$$

Let X be a topological manifold with co-ordinate charts (U_i, f_i) for $i = 1, 2, 3$ such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$. Then it is clear that $\phi_{21} := f_2 \circ f_1^{-1}$ is a homeomorphism with the inverse $\phi_{12} := f_1 \circ f_2^{-1}$. Furthermore, by considering $\phi_{31} := f_3 \circ f_1^{-1}$ we observe

$$\phi_{31} = (f_3 \circ f_2^{-1}) \circ (f_2 \circ f_1^{-1}) = \phi_{32} \circ \phi_{21}.$$

This is known as the cocycle property and explains the subscript notation.

Definition 2.6 (Atlas)**.** Let X be a topological manifold. An atlas for X is the collection of co-ordinate charst $\{(U_i, f_i)\}_{i \in I}$ such that

$$\bigcup_{i \in I} U_i = X.$$

We note that we do not require the index set I to be finite. Although, since $\{U_i\}_{i \in I}$ is an open cover, if X is compact, it is possible to obtain a finite sub-cover, and hence a finite atlas. Nonetheless, since we assumed X is second-countable, we can always choose I to be countable.

2.1 Smooth Manifolds

So far, we have only considered ourselves with the topological structure. As we would like to do calculus on our manifolds, we will now equip our manifolds with the property of smoothness. Recall the following definition for Euclidean spaces.

Definition 2.7. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (or C^∞) if all the partial derivatives of F of any order exists.

Of course, this is technically a property not a definition though it will suffice for our purposes.

Definition 2.8 (Smooth Atlas)**.** Let X be a topological manifold of dimension n. Then an atlas $\{(U_i, f_i)\}_{i \in I}$ on X is smooth if for all $i, j \in I$, the transition function

$$\phi_{ij} : f_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to f_i(U_i \cap U_j) \subseteq \mathbb{R}^n$$

is smooth.

Since ϕ_{ij} is a (bijective) map between open subsets of Euclidean spaces, it makes sense to ask whether or not ϕ_{ij} is smooth.

Definition 2.9 (Diffeomorphism)**.** Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $f : U \to V$. Then f is a diffeomorphism if f is smooth and has a smooth inverse.

As $(\phi_{ij})^{-1} = \phi_{ji}$, and both ϕ_{ij} and ϕ_{ji} are smooth, the transition functions of any smooth manifold are diffeomorphisms.

Definition 2.10 (Compatible)**.** Let X be a topological manifold and let $\mathcal{A} := \{(U_i, f_i)\}$ be a smooth atlas. Let (U, f) be any co-ordinate chart on X, then (U, f) is compatible with the atlas $\mathcal{A}$ if the transition function between (U, f) and any chart in $\mathcal{A}$ is a diffeomorphism.

Clearly, any chart in a smooth atlas is compatible with that atlas, and if (U, f) is compatible with the smooth atlas $\mathcal{A}$, then $(U, f) \cup \mathcal{A}$ is also a smooth atlas.

Definition 2.11. Let X be a topological manifold and $\mathcal{A}, \mathcal{B}$ be two atlases on X. Then $\mathcal{A}$ is compatible with $\mathcal{B}$ if every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$.

Similarly as before, if $\mathcal{A}, \mathcal{B}$ are compatible, then $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas on X.

Lemma 2.1. Let X be a topological manifold and let

$$\mathcal{A} := \{(U_i, f_i)\}_{i \in I}, \mathcal{B} := \{(U_j, f_j)\}_{j \in J},$$

be two compatible smooth atlases on X. Then for all (U, f) co-ordinate charts compatible with $\mathcal{A}$, (U, f) is compatible with $\mathcal{B}$.

Proof. It suffices to show that for all $(U_j, f_j) \in \mathcal{B}$, $U \cap U_j \neq \emptyset$, the transition map

$$\phi := f_j \circ f^{-1} : f(U \cap U_j) \to f_j(U \cap U_j)$$

and its inverse are smooth.

Let $y \in f(U \cap U_j)$, then there exist some $x \in U \cap U_j$ such that $f(x) = y$. As $\mathcal{A}$ is an atlas, it contains a co-ordinate chart $(U_i, f_i) \in \mathcal{A}$ such that $x \in U_i$. Then, defining $W := U \cap U_i \cap U_j \neq \emptyset$, we have the homomorphisms $f : W \to f(W), f_i : W \to f_i(W)$ and $f_j : W \to f_j(W)$. As remarked before, we have

$$\phi = (f^{-1} \circ f_i) \circ (f_i^{-1} \circ f_j)$$

on W. Now, by compatibility, the right hand side is smooth, and so we have ϕ is smooth on W implying it is smooth at y. Thus, as $y \in f(U \cap U_j)$ was arbitrary, ϕ is smooth (by a similar argument ϕ^{-1} is also smooth) and (U, f) is compatible with $\mathcal{B}$. $\square$

With this lemma it is easy to see that compatibility defines an equivalence relation on the set of smooth atlases and with this we can define smooth manifolds.

Definition 2.12 (Smooth Manifold)**.** A smooth manifold is a topological manifold with an equivalence class $[\mathcal{A}]$ of compatible smooth atlases on X. The equivalence class of atlases is called a smooth structure on X.

The reason for the definition considering only the equivalence class of compatible smooth atlases is because we do not want to distinguish between compatible smooth atlases. Indeed, recalling our example of a sphere, we would like to not consider the atlases which projects the sphere with respect to two other points that are not the poles as an alternative manifold.

From this point forward, we will always work with smooth manifolds and thus, omit the word "smooth" whenever it is clear from the context, i.e. a manifold is a smooth manifold and a atlas is a smooth atlas.

2.2 Submanifolds

Definition 2.13 (Affine Subspace)**.** An affine subspace $A \subseteq \mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$, i.e. there exists some $v \in V$ and $W \leq \mathbb{R}^n$ such that

$$A := v + W = \{v + w \mid w \in W\}.$$

Definition 2.14 (Submanifold)**.** Let X be an n-dimensional manifold and let $Y \subseteq X$. Then Y is an m-dimensional submanifold of X if for all $y \in Y$, there exists a

- a co-ordinate chart (U, f) of X which is compatible with the smooth structure of X such that $y \in U$ and,
- an m-dimensional affine subspace $A \subseteq \mathbb{R}^n$

$$f(U \cap Y) = f(U) \cap A.$$

Proposition 2.1. Let X be an n-dimensional manifold and Y an m-dimensional submanifold of X, then Y is an m-dimensional manifold.

Proof. As Y is a topological subspace of X, it is Hausdorff and second-countable. Thus, it remains to show that Y is equipped with a smooth structure.

By linear algebra, it is easy to see that the linear map $\tau : A = v + W \to W : a \mapsto a - v$ is continuously invertible, and thus, for all $y \in Y$ there exists a chart $(U, f' := \tau \circ f)$ of X such that $y \in U$ and $f'(U \cap Y) = f'(U) \cap W$. Let $T : W \cong \mathbb{R}^m$, then defining the atlas

$$\{(U_y, \tilde{f}_y)\}_{y \in Y} := \{(U_y, T \circ f')\}_{y \in Y},$$

for all $a, b \in Y$, its transition map

$$\phi_{ab} = (T \circ \tau \circ f_b) \circ (T \circ \tau \circ f_a)^{-1} = T \circ \tau \circ (f_b \circ f_a^{-1}) \circ \tau^{-1} \circ T^{-1},$$

is a composition of smooth functions, and thus is smooth. Hence Y is a smooth manifold. □

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, then define the set $s_f := \{(x, y) \mid y = f(x)\} \subseteq \mathbb{R}^2$ and I claim that s_f is a submanifold of $\mathbb{R}^2$. Define the chart (U, g) on $\mathbb{R}^2$ where $U = \mathbb{R}^2$ and

$$g(x, y) = (x, y - f(x)).$$

It is clear that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism as it is invertible with the inverse $g^{-1}(x, y) = (x, y + f(x))$ and so, $\{(U, g)\}$ is a smooth atlas of $\mathbb{R}^2$. Now considering $g \mid_{s_f} : s_f \to g(s_f) : (x, f(x)) \mapsto (x, 0)$ we have s_f is a smooth submanifold of $\mathbb{R}^2$.

Let us recall the following proposition from year-two analysis.

Proposition 2.2 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and let $F : U \to \mathbb{R}^n$ be smooth. Let $x \in U$ such that the Jacobian as x, $DF\,|_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists an open neighbourhood $V \subseteq U$ of x such that $F\,|_V : V \to F(V) \subseteq \mathbb{R}^n$ is a diffeomorphism.

Corollary 0.1. A smooth, bijective function $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ which has non-zero Jacobian everywhere has a smooth inverse.

The inverse function theorem is useful for showing whether a subset of a manifold is a submanifold. Consider the circle $S_1 := \{x^2 + y^2 = 1\}$ as a subset of the manifold $\mathbb{R}^2$. Then, let

$$U = \mathbb{R}^2 \setminus \{(x, 0) \mid x \leq 0\} \text{ and } f : U \to \mathbb{R}^2 : (r\cos\theta, r\sin\theta) \mapsto (r, \theta).$$

As $f : U \to f(U)$ is smooth, bijective and has non-zero Jacobian on U, then $f^{-1} : f(U) \to U$ is also smooth. Thus, (U, f) is a smooth chart on $U \to \tilde{U} := \mathbb{R}^+ \times (-\pi, \pi) \subseteq \mathbb{R}^2$. Then, for all $(\cos\theta, \sin\theta) \in S_1 \setminus \{(-1, 0)\}$, we have $f(\cos\theta, \sin\theta) = (1, \theta)$ implying

$$f(U \cap S_1) = \{(1, \theta) \mid \theta \in (-\pi, \pi)\} = f(U) \cap A,$$

where A is the affine subspace $(1, 0) + \{(0, y) \mid y \in \mathbb{R}\}$. Hence S_1 is a submanifold of $\mathbb{R}^2$.

Definition 2.15 (Level Sets)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a function and let $\alpha \in \mathbb{R}^k$. Then the level set of h at α is

$$h^{-1}(\{\alpha\}) = \{x \in \mathbb{R}^n \mid h(x) = \alpha\} \subseteq \mathbb{R}^n.$$

Definition 2.16 (Regular Points and Values)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. A point $x \in \mathbb{R}^n$ is called a regular point of h if the Jacobian of h at x

$$Dh\,|_x : \mathbb{R}^n \to \mathbb{R}^k$$

is surjective.

$\alpha \in \mathbb{R}^k$ is called a regular value if every point of the α-level set $h^{-1}(\{\alpha\})$ is regular.

If $x \in \mathbb{R}^n$ is not a regular point, then it is called a critical point. Similarly, if $\alpha \in \mathbb{R}^k$ is not a regular value, then it is called a critical value.

Definition 2.17 (Standard Projection)**.** Let $k \leq n$. The standard projection is the morphism

$$\pi : \mathbb{R}^n \to \mathbb{R}^k : (x_1, \cdots, x_n) \mapsto (x_{n-k+1}, \cdots, x_n).$$

That is π forgets the first $n - k$ entries.

Level sets are a useful tool for constructing submanifolds.

Theorem 1 (Implicit Function theorem)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and let $h : U \to \mathbb{R}^k$ be a smooth function where $k \leq n$. Let $z \in U$ be a regular point of h. Then there exists an open neighbourhood $V \subseteq U$ of z and a diffeomorphism

$$f : V \to f(V) \subseteq \mathbb{R}^n \text{ s.t. } h \circ f^{-1} = \pi : f(V) \to \mathbb{R}^k.$$

Informally, this theorem states that a smooth function around a regular point looks like the standard projection.

Proof. Let $x_1, \cdots, x_n$ be co-ordinates on $\mathbb{R}^n$ and let us write

$$h(x) = (h_1(x), \cdots, h_k(x)).$$

As z is regular, we have $Dh\mid_z : \mathbb{R}^n \to \mathbb{R}^k$ is surjective and thus, possibly by reordering, the set

$$\left\{ \frac{\partial h(z)}{\partial x_{n-k+1}}, \cdots, \frac{\partial h(z)}{\partial x_n} \right\}$$

form a basis of $\mathbb{R}^k$ and the matrix

$$M := \begin{pmatrix} \frac{\partial h_1(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_k(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_k(z)}{\partial x_n} \end{pmatrix}$$

is invertible. Then, by defining

$$f : U \to f(U) : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^{n-k}, h_1(x), \cdots, h_k(x)),$$

we have,

$$Df\mid_z = \left(\begin{array}{c|c} I_{n-k} & 0 \\ \hline \star & M \end{array} \right)$$

which is invertible as $\det Df\mid_z = \det I_{n-k} \det M = \det M \neq 0$. Thus, by the inverse function theorem, there exists some open $V \subseteq U$ such that $f : V \to f(V)$ is a diffeomorphism. Then, by considering $\pi \circ f = h$, we have $\pi = h \circ f^{-1}$. □

Corollary 1.1. If $h : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function, and α is a regular value, then the level set of h at α is a submanifold of $\mathbb{R}^n$ of dimension $n - k$.

Proof. For all $z \in h^{-1}(\{\alpha\})$, we have z is a regular point. Thus, by the above theorem, there exists an open neighbourhood V of z and a diffeomorphism $f : V \to f(V)$ such that $h \circ f^{-1} = \pi$. Then,

$$f(h^{-1}(\{\alpha\}) \cap V) = f(h^{-1}(\{\alpha\})) \cap f(V) = \pi^{-1}(\{\alpha\}) \cap f(V).$$

Hence, as $\pi^{-1}(\{\alpha\}) = \{(x_1, \cdots, x_{n-k}, \alpha_1, \cdots, \alpha_k)\} = \alpha + A_{n-k}$, we have $h^{-1}(\{\alpha\})$ is a submanifold of dimension $n - k$. □

This corollary is extremely useful. Consider the sphere $S^n = \{x_0^2 + \cdots + x_n^2 = \alpha\}$, by defining $h : \mathbb{R}^n \to \mathbb{R} : (x_0, \cdots, x_n) \mapsto x_0^2 + \cdots + x_n^2$, we see that h is smooth with the the Jacobian

$$Dh\mid_x = (2x_0, \cdots, 2x_n).$$

Thus, α is a regular value of h for all $\alpha > 0$. Hence, $S^n = \{h(x) = \alpha\}$ is a submanifold of $\mathbb{R}^{n+1}$ for all $\alpha > 0$.

Theorem 2 (Sard's Theorem)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. Then the set of regular values $Z \subseteq \mathbb{R}^k$ is dense. Furthermore, $\mathbb{R}^k \setminus Z$ has Lebesgue measure zero.

2.3 Smooth Functions

We know what a smooth function between two Euclidean spaces is. We will extend this notion to functions between two manifolds.

Definition 2.18. Let X be a manifold and let $h : X \to \mathbb{R}$ be a function. Then h is said to be smooth at $x \in X$ if for any chart (U, f) containing x such that it is compatible with the smooth structure of X, the function

$$h \circ f^{-1} : f(U) \to \mathbb{R}$$

is smooth at the point $f(x)$.

We say h is smooth if it is smooth at all points in X.

It is not difficult to see that smoothness is independent of the chart we pick, i.e. h is smooth at x as long as there exists a compatible chart (U, f) containing x such that $h \circ f^{-1}$ is smooth at x.

Proposition 2.3. Let X be a manifold and let $h : X \to \mathbb{R}$ be a function. Then, if $(U_1, f_1), (U_2, f_2)$ are two compatible charts on X such that $x \in U_1 \cap U_2$, $h \circ f_1^{-1}$ is smooth at x if and only if $h \circ f_2^{-1}$ is smooth at x.

Proof. Since the two charts are compatible the transition function $\phi_{12} = f_1 \circ f_2^{-1}$ is smooth. Thus, if $h \circ f_1^{-1}$ is smooth at $f_1(x)$, so is

$$h \circ f_1^{-1} \circ \phi_{12} = h \circ f_1^{-1} \circ f_1 \circ f_2^{-1} = h \circ f_2^{-1}.$$

Similar argument for the other direction. □

Thus, to show that h is smooth at some x, it suffices to find a compatible chart (U, f) at x such that $h \circ f^{-1}$ is smooth at $f(x)$.

Definition 2.19 (Smooth)**.** Let X, Y be manifolds of dimension n and m. Then a function $H : X \to Y$ is smooth at $x \in X$ if there exists a chart (U, f) compatible with the smooth structure of X such that $x \in U$ and a chart (V, g) compatible with the smooth structure of Y such that $H(x) \in V$ and $H(U) \subseteq V$ and

$$g \circ H \circ f^{-1} : f(U) \subseteq \mathbb{R}^n \to g(V) \subseteq \mathbb{R}^m$$

is smooth at $f(x)$.

We say H is smooth if it is smooth at all points in X.

In the case that H is a continuous function, we see that the condition of $H(U) \subseteq V$ can be relaxed by considering the chart on X, $(U \cap H^{-1}(V), f)$ in which $U \cap H^{-1}(V) \subseteq V$ is open by the continuity of H.

Definition 2.20 (Diffeomorphism)**.** A function $H : X \to Y$ between manifolds is said to be a diffeomorphism if it is smooth, a bijection, and H^{-1} is smooth.

Similar to before, the definition of smoothness is independent of the choice of the charts (consider $\phi_{21}^Y \circ g_1 \circ h \circ f_1 \circ \phi_{12}^X$).

Proposition 2.4. Let $Y \subseteq X$ be a submanifold of X and let

$$\iota_Y : Y \hookrightarrow X$$

be the inclusion map from Y to X. Then ι_Y is smooth.

Proof. Let $y \in Y$, then by definition, there exists a chart (V, g) on X containing y such that $g(V \cap Y) = g(V) \cap A$ for some A an affine space. Then defining $U = V \cap Y$ and $f = g\mid_U$, we have (U, f) is a chart on Y and $g \circ \iota_Y \circ f^{-1}$ is the identity on $f(U)$. Thus ι_Y is smooth. □

Proposition 2.5. Let X, Y, Z be manifolds and let $H : X \to Y$ and $G : Y \to Z$ be smooth, then $G \circ H$ is also smooth.

Proof. Follows by considering

$$g \circ G \circ f_1^{-1} \circ \phi_{12} \circ f_2 \circ H \circ h = g \circ (G \circ H) \circ h,$$

for some appropriately chosen charts which is restricted whenever necessary. □

From the two propositions above, we see that the restriction of any smooth maps on a submanifold is smooth as $F\mid_Y = F \circ \iota_Y$. In particular, we have that any smooth maps between Euclidean spaces restricted on some submanifolds of that Euclidean space is smooth (e.g. any smooth map restricted on the n-sphere is smooth).

Definition 2.21 (Product Manifold)**.** Given X, Y manifolds of dimension n and m. Then the Cartesian product $X \times Y$ is a manifold of dimension $n + m$.

To see why this is a topological manifold, consider for all $(x, y) \in X \times Y$ we may choose a chart (U, f) on X such that $x \in U$ and a chart (V, g) in Y such that $y \in V$. Then, if we define $W := U \times V$ and $h : W \to h(W) \subseteq \mathbb{R}^{n+m} := (x, y) \mapsto (f(x), g(y))$, we have (W, h) is a chart of $X \times Y$ containing (x, y). Similarly, using the same construction, if X, Y are smooth, one may show that $X \times Y$ is also smooth.

Definition 2.22 (Lie Group)**.** A Lie group is a manifold G which has a group structure $(G, \cdot)$ such that the multiplication and the inverse are both smooth.

An important example of a Lie group is the general linear group. In particular, as the space $M_n(\mathbb{R})$ of all $n \times n$ matrices of real coefficients is a vector space of dimension n^2, it is an n^2-dimensional manifold. Now, as $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$, it follows that it is also an n^2-dimensional manifold. Now, as $GL_n(\mathbb{R})$ is a group equipped with matrix multiplication, one may show that it is a Lie group by checking that the multiplication and the inverse are smooth.

2.3.1 Rank of a Smooth Function

For linear maps between Euclidean spaces, we have a notion of a rank by considering the dimension of the image of that map. Similarly, for smooth functions between Euclidean spaces, its rank is defined by considering the rank of its derivative. We will in this section extend this notion for smooth functions between manifolds.

Definition 2.23 (Rank)**.** Let X, Y be manifolds of dimension n and k, and let $H : X \to Y$ be a smooth function. Let $x \in X$ and let (U, f) be a chart on X such that $x \in U$, and (V, g) be a chart on Y at $H(x)$ such that $H(U) \subseteq V$. Then, we may define

$$\tilde{H} := g \circ H \circ f^{-1} : f(U) \to g(V).$$

As $\tilde{H}$ is a smooth function between Euclidean spaces, it has a Jacobian at $f(x)$,

$$D\tilde{H} \mid_{f(x)} : \mathbb{R}^n \to \mathbb{R}^k.$$

Then the rank of H at x is simply the rank of $D\tilde{H} \mid_{f(x)}$.

It is clear that the rank of a smooth function at a point is independent of the choice of the chart as the transition functions are diffeomorphisms. In particular, if $\tilde{H}$ and $\tilde{H}'$ results from two different choices of charts, we have $\tilde{H}' = \psi^{-1} \circ \tilde{H} \circ \phi$ where ψ and ϕ are appropriate transition maps. Then, by the chain rule, we have

$$D\tilde{H}' \mid_{f'(x)} = D\psi^{-1} \mid_{g(H(x))} \circ D\tilde{H} \mid_{f(x)} \circ D\phi \mid_{f'(x)} .$$

Now, as ϕ, ψ are diffeomorphisms, both $D\psi^{-1} \mid_{g(H(x))}$ and $D\phi \mid_{f'(x)}$ are invertible, and hence, the rank of $D\tilde{H}' \mid_{f'(x)}$ is the same as $D\tilde{H} \mid_{f(x)}$.

Definition 2.24 (Regular)**.** Let X, Y be manifolds of dimension n and k, and let $F : X \to Y$ be a smooth function. Then $x \in X$ is said to be a regular point if the rank of F at x is k. If x is not regular, then it is called a critical point.

Furthermore, $y \in Y$ is called a regular value if every point $x \in F^{-1}(y)$ is regular. Otherwise, it is called a critical value.

Lemma 2.2. Let X, Y be manifolds of dimension n and k, and let $F : X \to Y$ be a smooth function. Let $y \in Y$ be a regular value of F. Then the level set

$$Z_y := F^{-1}(y) \subseteq X$$

is a submanifold of X of dimension $n - k$.

Proof. Let $x \in Z_y$, then there exists a chart (U, f) on X containing x and a chart (V, g) on Y containing $F(U)$. Then we have the smooth function

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \to g(V).$$

Then by construction, we have $\tilde{F}^{-1}(g(y)) = f(Z_y \cap U)$. Now, since y is a regular value of F, x is a regular point of F, and so $f(x)$ is a regular point of $\tilde{F}$. Then, as $x \in Z_y \cap U$ was chosen arbitrarily, $\tilde{F}^{-1}(g(y))$ is a regular set (by choosing the same chart for all $x \in \tilde{F}^{-1}(g(y))$). Thus, the set $\tilde{F}^{-1}(g(y))$ is a submanifold of $\mathbb{R}^n$, and by definition, there exists a chart (W, h) containing $f(x)$ such that

$$h(\tilde{F}^{-1}(g(y))) = h(W) \cap A,$$

for some affine subspace A. Hence, it follows $(f^{-1}(W), h \circ f)$ is a co-ordinate chart on X such that

$$h \circ f(f^{-1}(W) \cap Z_y) = h \circ f(f^{-1}(W)) \cap A.$$

$\square$

2.3.2 Immersion and Submersion

Definition 2.25. Let X, Y be manifolds of dimension n and k, and let $F : X \to Y$ be a smooth function. Then F is said to be a submersion if the rank of F at any $x \in X$ is k. Furthermore, F is said to be an immersion if the rank of F at any $x \in X$ is n.

Thus, F is a submersion if its Jacobian at any point x is surjective. Similarly, F is a immersion of its Jacobian at any point x is injective.

Lemma 2.3. Let $Y \subseteq X$ be a m-dimensional submanifold, then the inclusion map

$$\iota : Y \hookrightarrow X$$

is an immersion.

Proof. By the definition of a submanifold, for all $y \in Y$, there exists some chart (V, g) of X containing y such that

$$g(V \cap Y) = g(Y) \cap A,$$

for some affine subspace A. Then, restricting the chart on to Y, we have $(V \cap Y, g\mid_Y)$ is a chart on Y containing y. Thus,

$$\tilde{\iota} := g \circ \iota \circ g\mid_Y^{-1} = \mathrm{Id}_{g(V \cap Y)}.$$

Hence, as $D\tilde{\iota}\mid_{g(y)} = D\mathrm{Id}_{g(V\cap Y)}\mid_{g(y)}$ has dimension m, ι is a immersion. □

Proposition 2.6. Let X, Y be n-dimensional manifolds, and let $F : X \to Y$ be a function which is smooth, bijective and of rank n at any point $x \in X$. Then F is a diffeomorphism.

Proof. It suffices to show that F^{-1} is smooth. For all $y \in Y$, let (U, f) be a chart of $F^{-1}(y)$ on X and (V, g) be a chart of Y containing $f(U)$. Then, by definition,

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \to g(V)$$

is smooth with invertible derivative $D\tilde{F}\mid_{f(F^{-1}(y))}$. By the inverse function theorem, $\tilde{F}^{-1} = f \circ F^{-1} \circ g^{-1}$ is locally smooth at $g^{-1}(y)$ with derivative $DF^{-1}\mid_{g(y)} = D\tilde{F}\mid^{-1}_{f(F^{-1}(y))}$. Thus, as $y \in Y$ was chosen arbitrarily, F^{-1} is smooth at any $y \in Y$, and so F^{-1} is smooth. □

Proposition 2.7. If X, Y are manifolds of dimension n and k, the projection map

$$p_1 : X \times Y \to X : (x, y) \mapsto x$$

is a submersion.

Proof. Exercise. □

Proposition 2.8. Let X, Y be manifolds of dimension n and k, and let $F : X \to Y$ be a submersion (so $k \leq n$). Then F is an open map, i.e. for any open subset $W \subseteq X$, $F(W)$ is open in Y.

Proof. Let $y \in F(W)$ and we will show that $F(W)$ is locally open as y. Let $x \in F^{-1}(y) \cap W$ and suppose (U, f) be a chart at x contained in W and (V, g) be a chart at y such that $F(U) \subseteq V$. Since F is a submersion, the function

$$\tilde{F} := g \circ F \circ f^{-1} : f(U) \to g(V)$$

is smooth and has rank k at $f(x)$, and so $f(x)$ is a regular point of $\tilde{F}$. Then, by the implicit function theorem, there exists an open set $U' \subseteq f(U)$ containing $f(x)$ and a diffeomorphism $h : U' \to h(U')$ such that $\tilde{F} \circ h^{-1} = \pi : h(U') \to \mathbb{R}^k$ is the standard projection.

Now as the standard projection and diffeomorphisms are an open maps, we have $\tilde{F} = \pi \circ h$ is also an open map, and in particular, $\tilde{F}(U')$ is open. Hence, as g is a homeomorphism, $\tilde{F}(U') = g(F(f^{-1}(U')))$ is open in $g(V)$ implies $F(f^{-1}(U')) \subseteq F(U) \subseteq F(W)$ is open in Y. Thus, $F(W)$ is locally open at y, and so $F(W)$ is open and F is an open map. $\square$

3 Tangent Spaces

The tangent space of a surface is a intuitive concept and for a surface in $\mathbb{R}^n$, we may often define it to be an affine subspace of $\mathbb{R}^n$ tangent to the surface at a point x. This definition in requires an ambient space and so is not generalizable to manifolds. We will in this section define a new notion of tangent spaces which applies to manifolds.

Let $U \subseteq \mathbb{R}^n$. Let $x \in U$, then a curve σ through x is a smooth function from $(-\epsilon, \epsilon)$ to U for some $\epsilon > 0$ and $\sigma(0) = x$. Then, we define the tangent vector of σ at the point x to be the Jacobian $D\sigma \mid_0$.

The space of curves is massive though we really only care about their tangent vectors, and so we introduce an equivalence relation which we will take the quotient by. Let σ_1, σ_2 are two curves through x, then we say $\sigma_1 \sim \sigma_2$ if and only if $D\sigma_1 \mid_0 = D\sigma_2 \mid_0$. Then, we define the tangent space of U at x to be,

$$T_x U := \{\sigma : (-\epsilon, \epsilon) \to U \mid \sigma \text{ smooth curve through } x\}/\sim .$$

We have a natural bijection

$$\Delta_x : T_x U \to \mathbb{R}^n : [\sigma] \mapsto D\sigma \mid_0 .$$

In particular, for all $v \in \mathbb{R}^n$, we can define $\sigma : (-\epsilon, \epsilon) \to U : t \mapsto x + tv$ such that $D\sigma \mid_0 = v$ and so Δ_x is surjective.

This definition can be extended to manifolds easily.

Definition 3.1 (Curves)**.** Let X be a n-dimensional manifold and let $x \in X$. Then a curve through x is a smooth function

$$\sigma : (-\epsilon, \epsilon) \to X,$$

such that $\sigma(0) = x$.

Definition 3.2. Let X be a n-dimensional manifold and let $x \in X$ and σ be a curve through x. Then, given a chart (U, f) containing x, we may define $\tilde{\sigma} := f \circ \sigma : (-\epsilon, \epsilon) \to f(U)$, i.e. $\tilde{\sigma}$ is a curve through $f(x)$. Finally, if σ_1, σ_2 are curves through x, we say

$$\sigma_1 \sim \sigma_2 \iff \tilde{\sigma}_1 \sim \tilde{\sigma}_2.$$

For the above definition to be well-defined, we will need to check it is independent of the choice of the chart. Let $(U_1, f_1), (U_2, f_2)$ be two charts of X containing x. Then, if $f_1 \circ \sigma_1 \sim f_1 \circ \sigma_2$, we have

$$\begin{aligned} D(f_2 \circ \sigma_1) \mid_0 &= D(\phi_{12} \circ f_1 \circ \sigma_1)_0 = D(\phi_{12}) \mid_{f_1(x)} \circ D(f_1 \circ \sigma_1)_0 \\ &= D(\phi_{12}) \mid_{f_1(x)} \circ D(f_1 \circ \sigma_2)_0 = D(\phi_{12} \circ f_1 \circ \sigma_2)_0 = D(f_2 \circ \sigma_2) \mid_0 . \end{aligned}$$

Definition 3.3 (Tangent Space)**.** Let X be a n-dimensional manifold and let $x \in X$, then the tangent space of X at x is

$$T_x X := \{\sigma : (-\epsilon, \epsilon) \to X \mid \sigma \text{ is a curve through } x\}/\sim .$$

We call elements of $T_x X$ tangent vectors.

Let (U, f) be a chart of X containing x. Then, we have a bijection

$$T_x X \to T_{f(x)} f(U) : [\sigma] \mapsto [f \circ \sigma].$$

Now as $f(U)$ is an open subset of $\mathbb{R}^n$, there exists a bijection $\Delta_{f(x)} : T_{f(x)} f(U) \to \mathbb{R}^n$. Composing the two functions, we obtain a bijection

$$\Delta_f : T_x X \to \mathbb{R}^n : [\sigma] \mapsto D(f \circ \sigma) \mid_0 .$$

With this function, we may equip the tangent space with a vector space structure. In particular, for all $\alpha \in \mathbb{R}, v, w \in T_x X$, we define $\alpha \cdot v := \Delta_f^{-1}(\alpha \cdot \Delta_f(v))$ and $v + w := \Delta_f^{-1}(\Delta_f(v) + \Delta_f(w))$.

We note that Δ_f depends on the choice of the chart. On the other hand, the vector space structure on $T_x X$ is independent of the choice of the chart (Hint : $\Delta_{f_2} = D\phi_{21} \mid_{f_1(0)} \circ \Delta_{f_1}$).

The tangent space allow us to talk about the Jacobian of a smooth function between to manifolds.

Definition 3.4 (Jacobian of Smooth Functions)**.** Let X, Y be manifolds and let $F : X \to Y$ be a smooth function. We define the Jacobian at $x \in X$ as

$$DF \mid_x T_x X \to T_{f(x)} Y : [\sigma] \mapsto [F \circ \sigma].$$

One may check that $DF \mid_x$ is well-defined and is a linear map using similar arguments as above (hint: for linearity, show $DF \mid_x = \Delta_g^{-1} \circ D\tilde{F} \mid_x \circ \Delta_f$ where $\tilde{F} = g \circ F \circ f^{-1}$). Furthermore, as one might expect, the rank of a smooth function is simply the rank of $DF \mid_x$.

3.1 Tangent Space to Submanifolds

Let Z is a m-dimensional submanifold of X, then as we have shown before, the inclusion map $\iota : Z \hookrightarrow X$ is a smooth immersion. Then, each curve in Z is a curve in X and so, for all $z \in Z$, we may define the linear injection

$$D\iota \mid_z : T_z Z \hookrightarrow T_z X : [\sigma] \mapsto [\sigma].$$

In this sense, we can view $T_z Z$ as a subspace of $T_z X$, and we can see this explicitly in co-ordinates. Let (U, f) be a chart of X containing z such that

$$f(U \cap Z) = f(U) \cap \mathbb{R}^m.$$

Then, by considering for all curves σ in Z through z, we have

$$\Delta_f(\sigma) = f \circ \sigma : (-\epsilon, \epsilon) \to Z \to \mathbb{R}^m,$$

for sufficiently small $\epsilon > 0$; and hence, $\Delta_f(T_z Z) \subseteq \mathbb{R}^m$. Now, as Δ_f is linear isomorphism between $T_z X$ and $\mathbb{R}^n$, $\Delta_f(T_z Z)$ must be a subspace of $\mathbb{R}^n$ of m dimension, and so $\Delta_f(T_z Z) = \mathbb{R}^m$.

Lemma 3.1. If X, Y are manifolds of dimension n and m respectively, and $F : X \to Y$ is a smooth function. Then, for all regular points $x \in X$, there exists a chart (U, f) of X containing x and (V, g) of Y containing $F(x)$ such that,

$$\pi = g \circ F \circ f^{-1} : f(U) \to g(V),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the standard projection.

Proof. Choose a chart (W, h) at x, and (V, g) at $F(x)$, then by the definition of smoothness, we have $g \circ F \circ h^{-1}$ is smooth. Now, as x is a regular point, $D(g \circ F \circ h^{-1}) \mid_{h(x)}$ is a surjection, and so, by the implicit function theorem, there exists an open set $U' \subseteq h(W)$ containing $h(x)$ and a diffeomorphism $f' : U' \to f'(U')$ such that

$$g \circ F \circ h^{-1} \circ f'^{-1} : f'(U') \to g(V),$$

is the standard projection. Thus, defining $U := h^{-1}(U')$ and $f := h^{-1} \circ f'^{-1}$, we have found a chart (U, f) of X containing x such that

$$g \circ F \circ f^{-1} : f(U) \to g(V)$$

is the restriction of the standard projection. □

Lemma 3.2. Let X be an n-dimensional manifold and let Y be an m-dimensional manifold where $m \leq n$. Let $F : X \to Y$ be a smooth function and let $y \in Y$ be a regular value of F. Then, if $Z = F^{-1}(\{y\})$ and $z \in Z$, $T_z Z$ is the kernel of the linear map

$$DF \mid_z : T_z X \to T_y Y.$$

Proof. By the above lemma, there exists a chart (U, f) of X containing z and a chart (V, g) of Y containing $y = F(z)$ and (Wlog. by translating) $g(y) = 0$, such that $\tilde{F} = g \circ F \circ f^{-1} : f(U) \to g(V)$ is the restriction of the standard projection and so,

$$DF \mid_z = \Delta_g^{-1} \circ D\tilde{F} \mid_z \circ \Delta_f = \Delta_g^{-1} \circ \pi \mid_z \circ \Delta_f,$$

where π has the kernel $\mathbb{R}^{n-m}$. Now, by considering

$$f(Z \cap U) = \tilde{F}^{-1}(g(y)) = f(U) \cap \pi^{-1}(y),$$

as the kernel of $D\tilde{F}$ is simply $T_z Z$ as $T_z Z \simeq \mathbb{R}^{n-m}$ and $D\tilde{F} \simeq \pi$ using the chosen chart. □

Consider the Lie group $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$. It is not difficult to show that the determinant map

$$\det GL_n(\mathbb{R}) \to \mathbb{R} : A \mapsto \det A$$

is a smooth function. We will show that 1 is a regular value of det and so the special linear group (note that $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$)

$$SL_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$$

is a submanifold of $GL_n(\mathbb{R})$ and so is a Lie group.

Recall that the determinant of the matrix A is $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$, for any $i = 1, \cdots, n$, where a_{ij} is the (i,j)-th entry of A and A_{ij} is the (i,j)-th minor of A. Thus,

$$\frac{\partial \det}{\partial a_{ij}} \mid_A = (-1)^{i+j} \det A_{ij}$$

since the minor $\det A_{ij}$ is independent of a_{ij}. Hence, it follows that if $A \in \det^{-1}(1)$ is critical, then $\det A_{ij} = 0$ for all i, j which implies $\det A = 0$, a contradiction! # Thus, $SL_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$ of dimension $n^2 - 1$. With this, we may compute the tangent space of $SL_n(\mathbb{R})$ at some A by

$$T_A SL_n(\mathbb{R}) = \ker D(\det) \mid_A = \{x \in GL_n(\mathbb{R}) \mid x \cdot [(-1)^{i+j} \det A_{ij}]_{ij} = 0\}.$$

3.2 Vector Fields and Flows

In the case of $\mathbb{R}^n$, we have a good intuition for what a vector field is, i.e. a smooth function from $U \subseteq \mathbb{R}^n$ open which maps $x \in U$ to a vector in $T_x U \simeq \mathbb{R}^n$. In this sense the codomain of a vector field is a union of all tangent spaces. This motivates the following definition.

Definition 3.5. The tangent bundle of the open set $U \subseteq \mathbb{R}^n$ is defined to be

$$TU := \bigcup_{x \in U} T_x U \simeq U \times \mathbb{R}^n,$$

where the homeomorphism follows as $T_x U \simeq \mathbb{R}^n$ for all $x \in U$.

As TU is the product of two n-dimensional manifolds, if is a manifold of dimension $2n$.

Definition 3.6. A vector field on the open set $U \subseteq \mathbb{R}^n$ is a smooth function $\xi : U \to TU$ such that $\pi \circ \xi = \mathrm{id}_U$ where $\pi : TU \to U$ is the projection map $(x, v) \mapsto x$.

Thus, for all $x \in U$, if ξ is a vector field of U then $\xi(x) = (x, v(x))$ where $v(x) \in T_x U$.

We may extend this idea to general manifolds.

Definition 3.7 (Tangent Bundle)**.** Let X be a manifold. We define the tangent bundle TX of X as

$$TX := \bigcup_{x \in U} T_x X.$$

We note that unlike the definition in $\mathbb{R}^n$, this union is disjoint as individual tangent vectors lives in different tangent spaces. Furthermore, we may not express the tangent bundle as a product without choosing a chart.

Similar to the projection map in $\mathbb{R}^n$, there is induced map from the tangent bundle $\pi : TX \to X : v \mapsto x$ for all $v \in T_x X$. As v is uniquely identified by the tangent space it lives in, this function is well-defined.

Proposition 3.1. Let X be a n-dimensional manifold. Then TX is a $2n$-dimensional manifold and the projection $\pi : TX \to X$ is smooth.

Proof. We will first define a topology on TX by defining a topology on individual charts.

Given (U, f) a chart on X, let $TU := \pi^{-1}(U) = \bigcup_{x \in U} T_x X$. Recall that, for all $x \in U$, we have the linear isomorphism $\Delta_f : T_x X \to \mathbb{R}^n$, and thus, we may construct the bijection

$$F_U : TU = \bigcup_{x \in U} T_x X \to f(U) \times \mathbb{R}^n : v \in T_x X \mapsto (f(x), \Delta_f(v)).$$

It is easy to check that this is a bijection. We see that F_U is injective as f is an homeomorphism, and for all $(f(x), w) \in f(U) \times \mathbb{R}^n$, as $\Delta_f : T_x X \to \mathbb{R}^n$ is a linear isomorphism, there exists some $v \in T_x X$ such that $\Delta_f(v) = w$ and so, $F_U(v) = (f(x), w)$ as required. With this, F_U induces a topology on TU be defining $W \subseteq TU$ is open if $F_U(W)$ is open in $f(U) \times \mathbb{R}^n$.

Now, consider an atlas $\mathcal{A} = \{(U_i, f_i)\}_{i \in \mathcal{I}}$ on X. Through the above procedure, we may define a topology for each $TU_i = \pi^{-1}(U_i)$. Then, as $\bigcup_{i \in I} TU_i = TX$ (note that this union is not necessary disjoint as U_i might not be), we may define the topology on TX where a $W \subseteq TX$ is open if $U \cap TU_i$ is open for all $i \in \mathcal{I}$.

It is easy to see that this procedure produces a Hausdorff and second countable topology (exercise).

Now to see that TX is a topological manifold we will need to construct an atlas on TX. This is clear as for each $T_xX \in TX$, there exists some chart (U, f) on X containing x. Then, by construction

$$F_U : TU \to f(U) \times \mathbb{R}^n \subseteq \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$$

is a homeomorphism, (TU, F_U) is a valid chart on TX containing x.

Finally, to show that TX is a smooth manifold, it suffices to show that the transition functions from the above lattice are smooth. But this is clear since by construction

$$(F_{U_i} \circ F_{U_j}^{-1})(x, v) = ((f_{U_i} \circ f_{U_j}^{-1})(x), (\Delta_{f_i} \circ \Delta_{f_j}^{-1})(v)) = (\phi_{ij}(x), D\phi_{ij} \mid_x (v)).$$

Thus, the transition function is smooth as its components are smooth.

To see that $\pi : TX \to X$ is a smooth function, let $w \in TX$ so there exists some i such that $w \in TU_i$, then we will show

$$\tilde{\pi} := f_i \circ \pi \circ F_{U_i}^{-1} : F_{U_i}(TU_i) \to f_i(U_i)$$

is smooth at $F_{U_i}(w)$ with appropriately chosen charts. But since $\tilde{\pi} : f_i(U_i) \times \mathbb{R}^n \to f_i(U_i)$ is simply the projection function, it is smooth, and hence π is smooth. □

Proposition 3.2. The projection $\pi : TX \to X$ is a submersion.

Proof. Define $\tilde{\pi}$ as above. But this is simply a projection function from $\mathbb{R}^{2n} \to \mathbb{R}^n$ and thus, has rank n. □

Definition 3.8 (Vector Field)**.** Let X be a manifold. A vector field on X is a smooth function $\xi : X \to TX$ such that $\pi \circ \xi = \mathrm{id}_X$.

In particular, we see that $\xi(x) \in T_xX$ for all $x \in X$.

Lemma 3.3. Let X be a manifold and let $\xi : X \to TX$ be a function such that $\pi \circ \xi = \mathrm{id}_X$. Then ξ is a vector field if and only if for every chart (U, f) on X, we have

$$F_U \circ \xi \mid_U \circ f^{-1} : f(U) \to f(U) \times \mathbb{R}^n$$

is smooth (where F_U is the bijection between $TU = \bigcup_{x \in U} T_xX$ and $f(U) \times \mathbb{R}^n$).

Proof. By the definition of smooth functions (only need to check $\xi(U) \subseteq TU$ so that (TU, F_U) is a chart at $\xi(x)$). □

Proposition 3.3. Let X be an n-dimensional manifold and let $Z \subseteq X$ be a submanifold of X with the inclusion map $\iota : Z \hookrightarrow X$. Then the image of $Di : TZ \to TX$ is a submanifold of TX.

Proof. Denote $\pi_Z : TZ \to Z$ and $\pi_X : TX \to X$ for the respective projective maps. Let $w \in TZ$ and $z = \pi_Z(w)$. Since Z is a submanifold, there exists a chart (U, f) on X containing z such that

$$f(U \cap Z) = f(U) \cap A$$

for some affine subspace $A \subseteq \mathbb{R}^n$. Then, it follows

$$F_U(TU \cap Di(TZ)) = F_U(TU) \cap (A \times A)$$

where $A \times A$ is a affine subspace of $\mathbb{R}^{2n}$. $\square$

Lemma 3.4. Let X be a manifold and Z is a submanifold of X. If ξ is a vector field on X such that $\xi(z) \in T_z Z$ for all $z \in Z$, then $\xi\mid_Z$ is a vector field on Z.

Proof. As the inclusion map $\iota : Z \hookrightarrow X$ is smooth, so is $\xi\mid_Z = \xi \circ \iota$. By assumption, $\xi\mid_Z(z) \in TZ$ for all $z \in Z$, and thus the induced morphism $\xi\mid_Z : Z \to TZ$ is also smooth. Finally, as $\pi_Z \circ \xi\mid_Z = \mathrm{id}_X\mid_Z = \mathrm{id}_Z$, we have $\xi\mid_Z$ is a vector field on Z. $\square$

Theorem 3 (Hairy Ball Theorem)**.** Any vector field on S^2 is zero at some point.

Proof. Requires algebraic topology or differential topology. $\square$

Definition 3.9 (Flow)**.** Let X be a manifold. A one-parameter family of diffeomorphisms or flow on X is a smooth function

$$F : (-\epsilon, \epsilon) \times X \to X$$

for some $\epsilon > 0$, such that

- for all $s \in (-\epsilon, \epsilon)$, the function

  $$F_s : X \to X : x \mapsto F(s, x)$$

  is a diffeomorphism,
- $F_0 = \mathrm{id}$,
- $F_t \circ F_s = F_{s+t}$ for all $t, s \in (-\epsilon, \epsilon)$ and $s + t \in (-\epsilon, \epsilon)$.

Let X be a manifold and let $F : (-\epsilon, \epsilon) \times X \to X$ be a flow on X. Then instead of fixing $s \in (-\epsilon, \epsilon)$, fixing $x \in X$, we obtain a smooth function

$$\sigma_x : (-\epsilon, \epsilon) \to X : t \mapsto F(t, x).$$

Straight away, we observe, as $F_0 = \mathrm{id}$, we have

$$\sigma_x(0) = F_0(x) = x,$$

and so σ_x is a curve through x. By recalling that the tangent space at x is defined to be the set of equivalent classes of the curves through x, this provides a function

$$\xi^F : X \to TX : x \mapsto [\sigma_x].$$

This is a vector field on X and is known as the infinitesimal version of the flow F.

Proposition 3.4. Let X be a compact manifold and let ξ be a vector field on X. Then for any $x \in X$, there exists an open subset $U \subseteq X$ containing x and a flow

$$F : (-\epsilon, \epsilon) \times U \to X,$$

such that $\xi \mid_U$ is the infinitesimal version of F, i.e. $\xi \mid_U = \xi^F$. That is to say, any vector field is locally a flow.

Proof. The proof of this proposition requires results from PDEs. □

Let X be an n-dimensional manifold and let $F : (-\epsilon, \epsilon) \times X \to X$ be a flow on X. Let (U, f) be a chart on X such that for all $s \in (-\epsilon, \epsilon)$, we have $F_s(U) \subseteq U$. Then we may define the smooth function

$$\tilde{F} : (-\epsilon, \epsilon) \times f(U) \to f(U) : (t, y) \mapsto (f \circ F_t \circ f^{-1})(y).$$

Since $f(U) \subseteq \mathbb{R}^n$, we may write

$$\tilde{F} = (\tilde{F}_i)_{i=1}^n, \tilde{F}_i : (-\epsilon, \epsilon) \times f(U) \to \mathbb{R}.$$

Then the infinitesimal version of $f(U)$ is $\tilde{\xi}^F = \frac{\partial \tilde{F}}{\partial s} \mid_0$. Thus,

$$\xi^F = \Delta_f^{-1} \circ \tilde{\xi}^F \circ f : U \to TU,$$

is the corresponding vector field on U.

3.3 Cotangent Space

Definition 3.10 (Cotangent Space)**.** Let X be a manifold and let $x \in X$ and define

$$R_x(X) := \{h : X \to \mathbb{R} \mid \mathrm{rk}(h) \mid_x = 0\} \leq C^\infty(X),$$

where $C^\infty(X)$ is the space of smooth maps between X and $\mathbb{R}$. Then the cotangent space is the quotient

$$T_x^*(X) = C^\infty(X)/R_x(X).$$

The cotangent space can be thought of as the dual space of the tangent space, and hence the notation. As one might expect, the dimension of a cotangent space equals the dimension of the tangent space and hence, the dimension of the manifold. As with before, we will first prove this for $\mathbb{R}^n$.

Lemma 3.5. Let $U \subseteq \mathbb{R}^n$ be open and let $x \in U$. Then $\dim T_x^* X = n$.

Proof. Define the map

$$\nabla_x : C^\infty(U) \to \mathbb{R}^n : h \mapsto Dh \mid_x .$$

Clearly ∇_x is linear and $\ker \nabla_x = R_x(U)$. Then, by the first isomorphism, we have $T_x^* U \cong \mathrm{Im}(\nabla_x)$ and it suffices to show ∇_x is surjective. But this is clear as for all $v := (v_1, \cdots, v_n) \in \mathbb{R}^n$, defining $h(w) := v \cdot w$, we have $\nabla_x h = v$ and so ∇_x is surjective. □

Now to extend this idea to general manifolds, we quickly realise we will need to be able to extend a smooth function defined on a subset to the whole space. We will achieve this with the bump function.

Let

$$\phi : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} e^{-1/x}, \ x > 0 \\ 0, \ x \leq 0. \end{cases}$$

It is not difficult to show that ϕ is smooth and $\phi^(k)(0) = 0$ for all $k \geq 0$. Now, define

$$\psi(x) := \frac{\phi(x)}{\phi(x) - \phi(1-x)},$$

as $\phi(x) - \phi(1-x) > 0$ for all $x \in \mathbb{R}$, we see that ψ is also a smooth function such that $\psi(x) = 0$ for $x \leq 0$ and $\psi(x) = 1$ for $x \geq 1$. Now, for real numbers $0 < r < r'$, we define

$$\tilde{\psi} : \mathbb{R}^n \to \mathbb{R} : x \mapsto \psi\left(\frac{|x| - r'}{r - r'}\right).$$

One can show that $\tilde{\psi}$ is smooth and for $x \notin B_{r'}(0)$ $\tilde{\psi}(x) = 0$ and for $x \in B_r(1)$, $\tilde{\psi}(x) = 1$. This function is known as the bump function for $\mathbb{R}^n$.

This idea can be generalized to general manifolds. Let X be a manifold and let $x \in X$ and suppose that (U, f) is a chart on X containing x such that $f(x) = 0$. Now, as $f(U)$ is open, there exists some $r' > 0$, such that $\overline{B_{r'}(0)} \subseteq f(U)$. Taking $0 < r < r'$, we define

$$\rho : X \to \mathbb{R} : y \mapsto \begin{cases} (\tilde{\psi} \circ f)(y), \ y \in U, \\ 0, \ y \notin U. \end{cases}$$

It is clear that ρ is smooth and $\rho(y) = 0$ for all $y \notin U$ and $\rho(y) = 1$ for all $y \in f^{-1}(B_r(0))$. We call this function a bump function with respect to the chart (U, f).

Proposition 3.5. Let X be an n-dimensional manifold and let $x \in X$. Then $\dim T_x^* X = n$.

Proof. Let (U, f) be a chart on X containing x and $f(x) = 0$. Then for all $h \in C^\infty(X)$, define $\tilde{h} := h \circ f^{-1} : f(U) \to \mathbb{R}$. Then $h \in R_x(X)$ if and only if the rank of $\tilde{h}$ is zero at $f(x)$, in particular, $h \in \ker \nabla_{f,x}$ where

$$\nabla_{f,x} : C^\infty(X) \to \mathbb{R} : h \mapsto D\tilde{h} \mid_{f(x)}.$$

Again, by the first isomorphism theorem, $T_x^* X \cong \mathrm{Im}(\nabla_{f,x})$ and again, it suffices to show $\nabla_{f,x}$ is surjective. However, this is a bit tricky since we now require a smooth function on $C^\infty(X)$ rather than $C^\infty(U)$ and so, we will use the bump function.

For all $v := (v_1, \cdots, v_n)$, define $\tilde{h}(w) := v \cdot w$ so that $\nabla_0 \tilde{h} = v$. Now, define

$$h : X \to \mathbb{R} : y \mapsto \begin{cases} \rho(\tilde{h}(f(y))), \ y \in U, \\ 0, \ y \notin U, \end{cases}$$

we see that for some open $W \subseteq U$, $x \in W$, $\tilde{h} \mid_W = (h \circ f^{-1}) \mid_W$ and so

$$\nabla_{f,x}(h) = D\tilde{h} \mid_{f(v)} = v,$$

and $\nabla_{f,x}$ is surjective as required. $\square$

For all $h \in C^\infty(X)$, we denote $dh \mid_x := [h] \in T_x^* X$.

Suppose $h \in C^\infty(X)$, $x \in X$ and $(U, f), (U', f')$ are two charts on X containing x. Then $\nabla_f (dh \mid_x) = D\tilde{h} \mid_{f(x)}$ where $\tilde{h} = h \circ f^{-1}$ and $\nabla_{f'}(dh \mid_x) = D\tilde{h}' \mid_{f'(x)}$ where $\tilde{h}' = h \cdot f'^{-1}$. Thus,

$$\tilde{h}' = \tilde{h} \circ f \circ f' {-1} = \tilde{h} \circ \phi^{-1},$$

where ϕ is the transition map between f and f'. Then,

$$\nabla_{f'}(dh \mid_x) = D(\tilde{h} \circ \phi^{-1}) \mid_{f'(x)} = D\tilde{h} \mid_{f(x)} \circ D\phi^{-1} \mid_{f'(x)} = \nabla_f (dh \mid_x) \circ D\phi^{-1} \mid_{f'(x)} .$$

As mentioned previously, we may think about the cotangent space as the dual of the tangent space. This notion can be made rigorous with the following.

Let $U \subseteq \mathbb{R}^n$ be an open set at let $x \in U$. Then for $v \in T_x U \simeq \mathbb{R}^n$, we may write $v = (v_1, \cdots, v_n)$ and so, we may define the partial derivative at x in the direction of v as the linear map

$$\partial_{x,v} : C^\infty(U) \to \mathbb{R} : h \mapsto Dh \mid_x (v).$$

Then, for all $h \in R_x(U)$, $\partial_{x,v}(h) = 0$, and so $R_x(X) \subseteq \ker \partial_{x,v}$ and the map $\partial_{x,v}$ induces a injection from $T_x^* U$ to $\mathbb{R}$. As $\partial_{x,v}$ is linear $\partial_{x,v} \in (T_x^* U)^* =: \mathrm{Hom}(T_x^* U, \mathbb{R})$. Thus, we have a linear map

$$T_x U \to (T_x^* U)^* : v \mapsto \partial_{x,v},$$

and we will show this is an isomorphism.

Suppose $v \in T_x X$ such that $\partial_{x,v} = 0$. Then, taking $h(y) := v \cdot y$, we have $0 = \partial_{x,v}(h) = v \cdot v$ implying $v = 0$ and so, the map is injective. On the other hand, as $\dim (T_x^* U)^* = \dim T_x^* U = \dim T_x U$, the map is indeed an isomorphism. Thus, in some sense, the tangent space is canonically isomorphic to the dual of the cotangent space. Hence, we have a canonical isomorphism

$$(T_x U)^* \simeq ((T_x^* U)^*)^* \simeq T_x^* U.$$

Now, suppose X is an n-dimensional manifold and $x \in X$. Then, for any $v = [\sigma] \in T_x X$, we define

$$\partial_{x,v} : C^\infty(X) \to \mathbb{R} : h \mapsto \frac{\mathrm{d}(h \circ \sigma)}{\mathrm{d}t}\Big|_0.$$

Then given a chart (U, f) on X containing x, we have a curve

$$\tilde{\sigma} := f \circ \sigma : (-\epsilon, \epsilon) \to f(U) \subseteq \mathbb{R}^n.$$

In particular, we may write $\tilde{\sigma} = (\tilde{\sigma}_1, \cdots, \tilde{\sigma}_n)$. Then, $h \circ \sigma = \tilde{h} \circ \tilde{\sigma}$ where $\tilde{h} = h \circ f^{-1} \in C^\infty(f(U))$. Thus, we have

$$\frac{\mathrm{d}(h \circ \sigma)}{\mathrm{d}t}\Big|_0 = \frac{\mathrm{d}(\tilde{h} \circ \tilde{\sigma})}{\mathrm{d}t}\Big|_0 = D\tilde{h} \mid_{f(x)} \circ D\tilde{\sigma} \mid_0 = \sum_{i=1}^n \frac{\partial \tilde{h}}{\partial x_i}\Big|_{f(x)} \frac{\mathrm{d}\tilde{\sigma}_i}{\mathrm{d}t}\Big|_0 = \nabla_f (dh \mid_x)^T \cdot \Delta_f([\sigma]).$$

Now, by considering if $h \in R_x(X)$ then $\partial_{x,v}(h) = \frac{\mathrm{d}(h \circ \sigma)}{\mathrm{d}t}\Big|_0 = 0$, we have $R_x(X) \subseteq \ker \partial_{x,v}$ and so, we obtain a map

$$T_x X \to (T_x^* X)^* : v \mapsto \partial_{x,v}.$$

Proposition 3.6. Let X be an n-dimensional manifold and let $x \in X$. Then the map

$$T_x X \to (T_x^* X)^* : v \mapsto \partial_{x,v}$$

is an isomorphism of vector spaces.

Proof. Let (U, f) be a chart on X containing x and recall that

$$\Delta_f : T_x \to \mathbb{R}^n, \ \nabla_f : T_x^* X \to \mathbb{R}^n$$

and both isomorphisms of vector spaces. Let $\sigma : (-\epsilon, \epsilon) \to X$ be a curve through x and let $h \in C^\infty(X)$. Then as we have shown above, defining $v = [\sigma]$, we have

$$\partial_{x,v}(dh \mid_x) = \nabla_f (dh \mid_x)^T \cdot \Delta_f(v).$$

In particular, the map

$$(\nabla_f)^* \circ (v \mapsto \partial_{x,v}) \circ \Delta_f^{-1} : \mathbb{R}^m \to (\mathbb{R}^n)^* : w \mapsto v = [\sigma] \mapsto \partial_{x,v} \mapsto \tilde{\partial}_{x,v}$$

maps any vector w to the linear functional $u \mapsto u \cdot w$. Thus, similar to the Euclidean case, the map is an isomorphism. $\square$

With this, a similar conclusion can be made by taking the dual on both sides of the isomorphism, obtaining the canonical isomorphism

$$(T_x X)^* \simeq ((T_x^* X)^*)^* \simeq T_x^* X.$$

3.4 Derivation

So far, we have defined the cotangent space and showed the existence of an isomorphism between $T_x X$ and $(T_x^* X)^*$ via $v \mapsto \partial_{x,v}$. We will again treat this topic but this time through an algebraic point of view with derivations.

Definition 3.11 (Derivation)**.** Let X be an n-dimensional manifold and let $x \in X$. A derivation of X at x is a linear map $\mathfrak{d} : C^\infty(X) \to \mathbb{R}$ such that

$$\mathfrak{d}(h_1 h_2) = h_1(x)\mathfrak{d}(h_2) + h_2(x)\mathfrak{d}(h_1).$$

We call this property the Leibniz rule and note the similarity of the statement with the Leibniz rule (or product rule) from calculus. We denote

$$\mathrm{Der}_x(X) := \{\text{derivations of } X \text{ at } x\}.$$

We note that the definition of the Leibniz rule depends on the choice of x though later, we shall remove this dependence.

Theorem 4. Let X be a manifold and let $x \in X$. A linear map $\mathfrak{d} : C^\infty \to \mathbb{R}$ is a derivation of X at x if and only if $\mathfrak{d}(h) = 0$ for all $h \in R_x(X) = \{h \in C^\infty \mid \mathrm{rk}(h) \mid_x = 0\}$.

Proof of the reverse direction of theorem 4. The forward direction requires some further definitions so let us prove the reverse direction first.

Suppose $\mathfrak{d}$ satisfies $\mathfrak{d}(h) = 0$ for all $h \in R_x(X)$. Let $h_1, h_2 \in C^\infty(X)$ and define

$$h(y) := h_1(y)h_2(y) - h_1(x)h_2(y) - h_2(x)h_1(y).$$

We see that (as the Jacobian satisfies the Leibniz rule),

$$Dh \mid_x = h_1(x)Dh_2 \mid_x + h_2(x)Dh_1 \mid_x - h_1(x)Dh_2 \mid_x - h_2(x)Dh_1 \mid_x = 0$$

and so $h \in R_x(X)$. By assumption, $\mathfrak{d}(h) = 0$ and by linearity of $\mathfrak{d}$,

$$0 = \mathfrak{d}(h) = \mathfrak{d}(h_1 h_2) - h_1(x)\mathfrak{d}(h_2) - h_2(x)\mathfrak{d}(h_1)$$

implying

$$\mathfrak{d}(h_1 h_2) = h_1(x)\mathfrak{d}(h_2) + h_2(x)\mathfrak{d}(h_1)$$

as required. □

As with other manifold proofs, we will first consider the statement in $\mathbb{R}^n$ and then generalize.

Lemma 3.6. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $x \in U$ and $\mathfrak{d}$ be a derivation as x. Then for all $h \in R_x(X)$, $\mathfrak{d}(h) = 0$.

Proof. Consider the constant function 1. By Leibniz rule, we have

$$\mathfrak{d}(1) = \mathfrak{d}(1 \cdot 1) = \mathfrak{d}(1) + \mathfrak{d}(1) = 2\mathfrak{d}(1)$$

implying $\mathfrak{d}(1) = 0$. Now, since $\mathfrak{d}$ is linear, it follows $\mathfrak{d}(c) = 0$ for all $c \in \mathbb{R}$. Now, let $h \in R_x(U)$, then Taylor's theorem implies

$$h(y) = h(x) + \sum_{i=1}^{n} (y_i - x_i)G_i(y),$$

for some $G_i \in C^\infty(X)$. Furthermore, by the choice of h, $G_i(x) = 0$ for all $i = 1, \cdots, n$. Thus,

$$\mathfrak{d}(h) = \mathfrak{d}(h(x)) + \sum_{i=1}^{n} \mathfrak{d}((y_i - x_i)G_i(y)) = \sum_{i=1}^{n} (x_i - x_i)\mathfrak{d}(G_i(y)) + G_i(x)\mathfrak{d}(y_i - x_i) = 0$$

as required. □

Lemma 3.7. Let X be a manifold, let $x \in X$ and $\mathfrak{d}$ be a derivation at x. Furthermore, let U be an open neighbourhood of x in X, $h \in C^\infty(X)$ be a smooth function such that $h(y) = 0$ for all $y \in U$. Then $\mathfrak{d}(h) = 0$.

Proof. With the bump function, we showed the existence of a function $\rho \in C^\infty(X)$ such that there exist open sets,

$$x \in V \subseteq W \subseteq U$$

so, $\rho \mid_V = 1$ and $\rho \mid_{X \setminus W} = 0$. Define $\psi = 1 - \rho$ so that we have $\psi \cdot h = h$. Then, by the Leibniz rule, we have

$$\mathfrak{d}(h) = \mathfrak{d}(\psi \cdot h) = h(x)\mathfrak{d}(\psi) + \psi(x)\mathfrak{d}(h) = 0$$

since $h(x) = \psi(x) = 0$. □

Definition 3.12 (Germ)**.** Let X be a manifold and let $x \in X$. Then denote $W \subseteq C^\infty(X)$ the set

$$W := \{h \in C^\infty(X) \mid \exists \text{ open } U \subseteq X, x \in U \wedge h\mid_U = 0\}.$$

it is not difficult to show that W is a linear subspace of $C^\infty(X)$ and so, we define the germ to be the quotient

$$\widehat{C_x^\infty}(X) := C^\infty(X)/W.$$

Elements of this space are called germs of smooth functions of X at x and we see that two smooth functionals in $C^\infty(X)$ are the same germ if their difference is locally zero around x.

We see that the evaluation map $\mathrm{ev}_x : \widehat{C_x^\infty}(X) \to \mathbb{R} : [h] \mapsto h(x)$ is well-defined. Moreover, the above lemma implies that, for all $\mathfrak{d} \in \mathrm{Der}_x(X)$, the induced linear map

$$\hat{\mathfrak{d}} : \widehat{C_x^\infty}(X) \to \mathbb{R}$$

is well-defined (since $W \leq \ker \mathfrak{d}$) and it satisfies the Leibniz rule

$$\hat{\mathfrak{d}}([h_1 \cdot h_2]) = h_1(x)\hat{\mathfrak{d}}([h_2]) + h_2(x)\hat{\mathfrak{d}}([h_1]).$$

Lemma 3.8. Let X be a manifold and let $x \in X$ and $U \subseteq X$ be an open subset containing x. Then, we have an linear isomorphism

$$\widehat{C_x^\infty}(X) \tilde{\to} \widehat{C_x^\infty}(U) : [h] \mapsto [h\mid_U].$$

Proof. The map is well-defined as the intersection of two open sets is open. As the map is clearly linear and injective, it remains to show surjectivity. With the bump function, there exists some $\rho \in C^\infty(X)$ and open sets

$$x \in V \subseteq W \subseteq U$$

such that $\rho\mid_V = 1$ and $\rho\mid_{X \setminus W} = 0$. Then, for all $[h] \in \widehat{C_x^\infty}(U)$, define

$$\tilde{h}(x) := \begin{cases} \rho(x)h(x), & x \in U, \\ 0, & x \notin U. \end{cases}$$

It is clear that $\tilde{h}$ is smooth and, furthermore, $\tilde{h}\mid_U \in [h]$ since $h - \tilde{h}$ is zero on V. Thus, $[\tilde{h}] = [h]$ and the map is surjective as claimed. □

Lemma 3.9. Let X be a manifold and let $x \in X$ and $U \subseteq X$ be an open subset containing x. Then, we have a linear isomorphism

$$\mathrm{Der}_x(U) \tilde{\to} \mathrm{Der}_x(X) : \mathfrak{d}' \mapsto \mathfrak{d}$$

where $\mathfrak{d}(h) := \mathfrak{d}'(h\mid_U)$.

Proof. As we have seen, a derivation $\mathfrak{d}'$ on U induces a map $\hat{\mathfrak{d}}' : \widehat{C_x^\infty}(U) \to \mathbb{R}$ which satisfy the Leibniz rule. Then, by the above lemma, we have the isomorphism $\phi : \widehat{C_x^\infty}(X) \cong \widehat{C_x^\infty}(U)$. Thus, we may define

$$\hat{\mathfrak{d}} := \hat{\mathfrak{d}}' \circ \phi : \widehat{C_x^\infty}(X) \to \widehat{C_x^\infty}(U) \to \mathbb{R}.$$

Hence, taking $\mathfrak{d} := \hat{\mathfrak{d}} \circ q$ where q is quotient map suffices. These steps can be reversed and thus, this map is an isomorphism. □

Finally, we may prove the forward direction of theorem 4.

Proof of the forward direction of theorem 4. Suppose $\mathfrak{d} : C^\infty(X) \to \mathbb{R}$ is a derivation at x and we will show that $\mathfrak{d}(h) = 0$ for all $h \in R_x(X)$.

Let $h \in R_x(X)$ and suppose (U, f) is a chart on X containing x. By the above lemma, there exists some derivation $\mathfrak{d}' \in \mathrm{Der}_x(U)$ such that $\mathfrak{d}(h) = \mathfrak{d}'(h \mid_U)$. Now, as f is a homeomorphism between U and $f(U)$, by composing with f, we have the isomorphisms

$$C^\infty(U) \tilde{\to} C^\infty(f(U)) \text{ and } \mathrm{Der}_x(U) \tilde{\to} \mathrm{Der}_{f(x)}(f(U)).$$

Then, taking $\tilde{h} := h \circ f^{-1}$, we have the rank of $\tilde{h}$ at $f(x)$ is 0 and thus, by lemma 3.6, the claim follows. $\square$

With theorem 4, the canonical isomorphism between $T_x X$ and $(T_x^* X)^*$ becomes an isomorphism between $T_x X$ and $\mathrm{Der}_x(X)$, and thus, this gives another definition of the tangent space of a manifold at a point.

3.4.1 Derivation and Vector Fields

Let $U \subseteq \mathbb{R}^n$ be an open set and let $\xi : U \to TU$ be a vector field. Since $TU = U \times \mathbb{R}^n$, ξ induces a smooth map $\tilde{\xi} : U \to \mathbb{R}^n := \pi_2 \circ \xi$. Then, for all $x \in U$, we have a derivation along the direction $\tilde{\xi}(x)$ with

$$\partial_{x, \tilde{\xi}(x)} : C^\infty(U) \to \mathbb{R}.$$

Thus, for all $h \in C^\infty(X)$, we have

$$\tilde{\xi}(h) : U \to \mathbb{R} : x \mapsto \partial_{x, \tilde{\xi}(x)}(h).$$

If we write $\tilde{\xi}$ in coordinates so that $\tilde{\xi}(x) = (\tilde{\xi}_1(x), \cdots, \tilde{\xi}_n(x))$. Then,

$$\tilde{\xi}(h) = \partial_{x, \tilde{\xi}(x)}(h) = \tilde{\xi}_1 \frac{\partial h}{\partial x_1} + \cdots + \tilde{\xi}_n \frac{\partial h}{\partial x_n}.$$

For $h_1, h_2 \in C^\infty(U)$, we see that the Leibniz rule apply, i.e.

$$\tilde{\xi}(h_1, h_2) = h_1 \tilde{\xi}(h_2) + h_2 \tilde{\xi}(h_1),$$

and for this reason, we call ξ a derivation on X.

Now, let $\xi : X \to TX$ be a vector field on the n-dimensional manifold X. Similarly, for any $x \in X$, we have an element in $\mathrm{Der}_x(X)$ given by the linear map

$$\partial_{x, \xi(x)} : C^\infty(X) \to \mathbb{R}.$$

Thus, defining ξ such that for all $h \in C^\infty(X)$,

$$\xi(h) = x \mapsto \partial_{x, \xi(x)}(h).$$

We will show $\xi(h)$ is smooth. Let $x \in X$ and take (U, f) be a chart on X containing x and let $v := \Delta_f(\xi(x))$ and $h := h \circ f^{-1}$. Then, as we have already show,

$$\partial_{x, \xi(x)}(h) = \partial_{f(x), v}(\tilde{h}) = \nabla_f (dh \mid_x)^T \cdot v.$$

Then, let $\tilde{\xi}$ be the corresponding function from $f(U)$ to $\mathbb{R}^n$ corresponding to ξ, i.e. $\tilde{\xi}(y) := \Delta_f(\xi \mid_{f^{-1}(y)})$. Then, $\xi(h) \circ f^{-1}\tilde{\xi}(\tilde{h})$ is smooth on $f(U)$ since $f(U)$ is an open subset of $\mathbb{R}^n$. Thus, $\xi(h)$ is smooth implying $\xi(h) \in C^\infty(X)$.

With this, we see that ξ is a linear mapping from $C^\infty(X)$ to itself, and so, it is a derivation.

Definition 3.13 (Derivation)**.** Let X be a manifold. A derivation is a linear map

$$\mathfrak{D} : C^\infty(X) \to C^\infty(X)$$

which satisfies the Leibniz rule

$$\mathfrak{D}(h_1 h_2) = h_1 \mathfrak{D}(h_2) + h_2 \mathfrak{D}(h_1),$$

for all $h_1, h_2 \in C^\infty(X)$.

We denote the set of all derivations on X by $\mathrm{Der}(X)$.

We note that our previous definition of the derivation depended on the choice of x which is definition is independent of that choice.

We have so far seen that a vector field induces a derivation. We shall show the reverse is also true.

Definition 3.14. Let X be an n-dimensional manifold. Then, any derivation $\mathfrak{D}$ on X defines a vector field on X.

Proof. Let $x \in X$, we have the linear map

$$\mathfrak{D} \mid_x : C^\infty(X) \to \mathbb{R} : h \mapsto \mathfrak{D}(h)(x).$$

The Leibniz rule implies that $\mathfrak{D} \mid_x \in \mathrm{Der}_x(X)$. However, we know that $\mathrm{Der}_x(X) \cong T_x X$ and so, we obtain a map

$$\xi : X \to TX : x \mapsto \mathfrak{D} \mid_x .$$

It remains to show that ξ is smooth. Let (U, f) be a chart on X containing x. Then ξ induced a map

$$\tilde{\xi} : f(U) \to f(U) \times \mathbb{R}^n$$

where the second component of $\tilde{\xi} = \Delta_f \circ \xi \circ f^{-1}$. Write $\tilde{\xi} = \sum_{i=1}^n \tilde{\xi}_i \frac{\partial}{\partial x_i}$ and we will show ξ_i is smooth for all $i = 1, \cdots, n$. Recall that, using the bump function, there exists a smooth function ψ and open sets

$$f(x) \in W \subseteq V \subseteq f(U)$$

with $\psi \mid_W = 1$ and $\psi \mid_{X/V} = 0$. Then, defining

$$\chi(x) := \begin{cases} ((x_i \psi) \circ f)(x), & x \in U, \\ 0, & x \notin U. \end{cases}$$

In particular, $\tilde{\chi}_i := \chi_i \circ f^{-1} = x_i$ inside W and we have for all $y \in f^{-1}(W)$,

$$\mathfrak{D}(\chi_i) \mid_y = \sum_{j=1}^n \tilde{\xi}_j \mid_{f(y)} \frac{\partial \tilde{\chi}_i}{\partial x_j}\Big|_{f(y)} = \tilde{\xi}_i \mid_{f(y)} .$$

$\square$

3.5 Vector Bundle

Vector bundle is an important tool in both differential geometry and algebraic geometry and is somehow a generalization of the tangent bundles. We will in this small section define the vector bundle and prove some properties about it.

Definition 3.15 (Vector Bundle)**.** Let X be an n-dimensional manifold. A vector bundle over X consists of the data

- a manifold E of dimension $n+r$;
- a smooth surjection $\pi : E \to X$;
- an atlas $\{(U_i, f_i)\}_{i\in I}$ for X and an atlas $\{(V_i, g_i)\}_{i\in I}$ for E such that,
  - $V_i = \pi^{-1}(U_i)$ for all $i \in I$;
  - $g_i(V_i) = f_i(U_i) \times \mathbb{R}^r \subseteq \mathbb{R}^{n+r}$ for all $i \in I$;
  - $\mathrm{pr}_1 \circ g_i(v) = f_i(\pi(v))$ for all $i \in I$ and $v \in V_i$;
  - for each $x \in X$, the level set $E_x := \pi^{-1}(\{x\})$ is a vector space of dimension r such that for all $i \in I$, the map $g_i \mid_{E_x} : E_x \to \mathbb{R}^r$ is an isomorphism.

The number r is called the rank of the vector bundle and the vector spaces E_x are known as fibres of the vector bundle.

We see that in some sense vector bundles embed a manifold inside a larger manifold and for each point in the original manifold, we associate it with a real vector space in the larger manifold. As one might expect, the tangent bundle is a vector bundle and with a similar construction ($T^*X := \bigcup_{x\in X} T^*_x X$) we may define the cotangent bundle which is also a vector bundle. Another important vector bundle is the trivial vector bundle $E := X \times \mathbb{R}^r$ with the map $(x, v) \in X \times \mathbb{R}^r \mapsto x$.

Definition 3.16 (Section)**.** Let X be an n-dimensional manifold and let $\pi : E \to X$ be a vector bundle. A section of E is a smooth morphism $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$.

A section of the tangent bundle is a vector field.

By definition, if $(E, \pi : E \to X)$ is a vector bundle of X, for all $x \in X$, the level set $\pi^{-1}(\{x\})$ is a vector space, and thus, it makes sense to map $x \mapsto 0_{\pi^{-1}(\{x\})}$. This is clearly a section and is called the zero section.

Definition 3.17. Let X be a manifold and let $\pi_1 : E_1 \to X, \pi_2 : E_2 \to X$ be two vector bundles. An isomorphism between E_1, E_2 is a diffeomorphism $F : E_1 \to E_2$ such that $\pi_2 \circ F = \pi_1$ and the induced function $F_x : (E_1)_x \to (E_2)_x$ is an isomorphism of vector spaces for all $x \in X$.

We say a vector bundle (E, π) is trivial if it is isomorphic to the trivial vector bundle $X \times \mathbb{R}^r$ for some $r > 0$.

Proposition 3.7. Two isomorphic vector bundles have the same rank.

Proof. Clear as a linear isomorphism preserves dimension. □

Definition 3.18 (Line Bundle)**.** A line bundle is a vector bundle of rank 1.

Proposition 3.8. Let X be a manifold and let $\pi : L \to X$ be a line bundle on X which admits a section $s : X \to L$ which is nowhere zero, i.e. $s(x) \neq 0 \in L_x$ for all $x \in X$. Then L is trivial.

Proof. Consider the function $F : X \times \mathbb{R} \to L : (x, v) \mapsto v \cdot s(x)$. It is clear that F commutes with the projection map $X \times \mathbb{R} \to X$ and $L \to X$ and for $x \in X$ as $s(x) \neq 0$, it induces an isomorphism of vector spaces.

We will now show F, F^{-1} is smooth. Since L is a vector bundle, for each $x \in X$, there exists a chart (U, f) in X containing x and a chart (V, g) on Y such that $V = \pi^{-1}(U)$ and $g(V) = f(U) \times \mathbb{R}$. Then

$$\tilde{s} := g \circ s \circ f^{-1} : f(U) \to f(U) \times \mathbb{R} = g(V)$$

is smooth (as s is smooth). Thus,

$$\tilde{F} := g \circ (f^{-1}, \mathrm{id}_{\mathbb{R}}) : f(U) \times \mathbb{R} \to f(U) \times \mathbb{R} : (y, v) \mapsto (y, v \cdot \mathrm{pr}_2 \circ \tilde{s}(y)),$$

is smooth and hence, it follows $\tilde{F}^{-1}$ is smooth and both F, F^{-1} are smooth. Hence, F is an isomorphism of vector bundles between L and the trivial vector bundle. □

This proposition can be generalized into the following.

Proposition 3.9. Let X be a manifold and let $\pi : L \to X$ be a vector bundle of rank r which admits sections $s_1, \cdots, s_r : X \to L$ such that for all $x \in X$, $\{s_1(x), \cdots, s_r(x)\}$ is linearly independent in E_x (and so form a basis). Then L is trivial.

Proof. See problem sheet. □

4 Forms and Integration

4.1 1-form

Definition 4.1. Let X be a manifold. A 1-form on X is a section of T^*X, i.e. a smooth function

$$\alpha : X \to T^*X$$

such that $\pi \circ \alpha = \mathrm{id}_X$.

Let X be an n-dimensional manifold and suppose $x \in X$ and let α be a 1-form on X and (U, f) be a chart on X containing x. We recall that we have the linear isomorphism $\nabla_f : T^*_x X \to \mathbb{R}^n$. Thus, we obtain a smooth morphism

$$\tilde{\alpha} : f(U) \to \mathbb{R}^n : y \mapsto \nabla_f(\alpha \mid_{f^{-1}(y)}).$$

On the other hand, if (U', f') is also a chart on X containing x, then the corresponding 1-form on $f'(U')$ $\tilde{\alpha}'$ can be written as

$$\tilde{\alpha}'(f'(x)) = \tilde{\alpha}(f(x)) D\phi^{-1} \mid_{f'(x)}$$

by recalling $\nabla_{f'}(dh \mid_x) = \nabla_f(dh \mid_x) D\phi^{-1} \mid_{f'(x)}$ where $h \in C^\infty(X)$ and ϕ is the transition function.

We will now construct a 1-form dh on X from any smooth function $h \in C^\infty(X)$. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $h \in C^\infty(U)$. Then, for any $x \in U$, we recall $dh \mid_x \in T^*_x U = C^\infty / R_x(U)$. Since $T^*_x U \cong \mathbb{R}^n$ canonically, we can write $dh \mid_x = (\partial_{x_1} h, \cdots, \partial_{x_n} h) \mid_x \in \mathbb{R}^n$. Thus, we have a smooth 1-form on U

$$dh : U \to U \times \mathbb{R}^n : x \mapsto (x, dh \mid_x).$$

This construction provides some 1-forms from the co-ordinate functions $x_i : U \to \mathbb{R}$ resulting in the 1-forms dx_i for $i = 1, \cdots, n$. We note that $\{dx_i(x)\}$ form a basis of $T^*_x U$, and so, if α is a 1-form on U, we may write $\alpha = \alpha_1 dx_1 + \cdots + \alpha_n dx_n$ where $\alpha_1, \cdots, \alpha_n$ are smooth functions on U. Applying this to dh for $h \in C^\infty(U)$, we have

$$dh = \frac{\partial h}{\partial x_1} dx_1 + \cdots + \frac{\partial h}{\partial x_n} dx_n.$$

This is known as the exact 1-forms.

Let us now generalize this to manifolds. Let X be a manifold and let $h \in C^\infty(X)$. Then for any $x \in X$, we have $dh \mid_x \in T^*_x X$, and so, we obtain a function

$$dh : X \to T^*X : x \mapsto dh \mid_x .$$

We will show dh is smooth. Let (U, f) be a chart on X containing x. Define $\tilde{h} := h \circ f^{-1} \in C^\infty(f(U))$ and if (V, g) is the corresponding chart on T^*X, we have $d\tilde{h} = g \circ dh \circ f^{-1}$ which is a smooth 1-form on $f(U)$. Thus, dh is smooth as required and hence, dh is a 1-form on X.

Suppose now that $F : X \to Y$ is a smooth function between manifolds, then the Jacobian of F defines a map $DF \mid_x : T_x X \to T_{F(x)} Y$. Recalling that the cotangent space is canonically isomorphic to the dual of the tangent space, the dual map of DF can be interpreted as a map

$$DF^* \mid_x : T^*_{F(x)} Y \to T^*_x X.$$

Now, let $h \in C^\infty(Y)$, then $h \circ F \in C^\infty(X)$ and

$$DF^* \mid_x (dh \mid_{F(x)}) = d(h \circ F) \mid_x \in T^*_x X.$$

Hence, glueing these maps together, we obtain a 1-form on X.

Definition 4.2 (Pull-back of 1-form on Euclidean Spaces)**.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$ be open and let $F : U \to V$ be a smooth function. Then, if α be a 1-form on V,

$$F^*\alpha : U \to T^*U : x \mapsto DF^* \mid_x (\alpha \mid_{F(x)})$$

is a 1-form on U.

To show that this is indeed a 1-form on X, we will show it is smooth. Let us first consider a smooth map $F = (F_1, \cdots, F_k)$ between $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^k$ where $F_i \in C^\infty(U)$. Let α be a 1-form on V, then

$$F^*\alpha : U \to U \times \mathbb{R}^n : z \mapsto (DF \mid_z)^T \alpha \mid_{F(z)} .$$

$F^*\alpha$ is clearly smooth, and so $F^*\alpha$ is a 1-form on U.

It is not difficult to see that F^* is linear, i.e. for α_1, α_2 1-forms on V and $\lambda_1, \lambda_2 \in \mathbb{R}$, then

$$F^*(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda_1 F^*\alpha_1 + \lambda_2 F^*\alpha_2.$$

Thus, writing the form α on V in terms of the basis representation, $\alpha = \alpha_1 dy_1 + \cdots + \alpha_k dy_k$, we have $F^*\alpha = (\alpha_1 \circ F)F^*dy_1 + \cdots + (\alpha_k \circ F)F^*dy_k$. Furthermore, for each i, we see

$$F^*dy_i = \frac{\partial F_j}{\partial x_1} dx_1 + \cdots + \frac{\partial F_j}{\partial x_n} dx_n.$$

Definition 4.3 (Pull-back of 1-form)**.** Let X and Y be manifolds and $F : X \to Y$ be a smooth function. Then, if α be a 1-form on Y,

$$F^*\alpha : X \to T^*X : x \mapsto DF^* \mid_x (\alpha \mid_{F(x)})$$

is a 1-form on X. Moreover, if $h \in C^\infty(Y)$, then $F^*dh = d(h \circ F)$.

4.2 p-form

In order to generalize 1-forms to p-forms, we will construct an operator called the wedge product. This is a linear algebra construction and is also used else where than manifolds. As a result, the proofs required for all linear algebra constructions and propositions are left as an exercise.

Definition 4.4 (Multilinear Map)**.** Let V be a $\mathbb{R}$-vector space and for any integer $p \geq 0$, a multilinear map $\omega : V^p \to \mathbb{R}$ is a function which is linear with respect to each term (by convention, we denote V^0 as $\mathbb{R}$).

Definition 4.5 (Signature of Permutation)**.** Given $\sigma \in S_p$ where S_p is the p-permutation group, we denote the siganature of σ by

$$\epsilon(\sigma) := (-1)^m,$$

where m is the number of transpositions in a decomposition of σ.

We recall that in group theory we refer to this as the sign of a permuation.

Definition 4.6 (Alternating p-form)**.** A multilinear map $\omega : V^p \to \mathbb{R}$ is called an alternating p-form if for all $v_1, \cdots, v_p \in V$ and for all $\sigma \in S_p$, we have

$$\omega(v_{\sigma(1)}, \cdots, v_{\sigma(n)}) = \epsilon(\sigma)\omega(v_1, \cdots, v_n).$$

We denote $\wedge^p V^*$ the set of all alternating p-forms. It is clear that $\wedge^p V^*$ form a vector space under the natural operations and we call this the p-th exterior power of V.

We note that for $p = 1$, $\wedge V^* = V^* = \mathrm{Hom}(V, \mathbb{R})$ and hence the notation. On the other hand, for $p = 0$, $\wedge^0 V^*$ is the set of all linear maps from $\mathbb{R}$ to $\mathbb{R}$ which is isomorphic to $\mathbb{R}$.

Definition 4.7 (Exterior Product)**.** Let p, q be positive integers. Then, we denote

$$S_{p,q} := \{\sigma \in S_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+1)\}.$$

Then, given $\omega_1 \in \wedge^p V^*$ and $\omega_2 \in \wedge^q V^*$, we define the exterior product $\omega_1 \wedge \omega_2 \in \wedge^{p+q} V^*$ by definition, for each $v_1, \cdots, v_{p+q} \in V$,

$$\omega_1 \wedge \omega_2(v_1, \cdots, v_{p+q}) := \sum_{\sigma \in S_{p,q}} \epsilon(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(p)})\omega(v_{\sigma(p+1)}, \cdots, v_{\sigma(p+q)}).$$

This definition is not very useful when computing and as we shall see, there is a much more natural characterization of the exterior product which allows us to compute the exterior product more effectively.

Proposition 4.1. Let $\omega_1, \omega_2 \in \wedge^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v, w) = \omega_1(v)\omega_2(w) - \omega_1(w)\omega_2(v).$$

More generally, if $\omega_1, \cdots, \omega_p \in \wedge^1 V^*$, then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \cdots, v_p) = \det \begin{pmatrix} \omega_1(v_1) & \omega_1(v_2) & \cdots & \omega_1(v_p) \\ \omega_2(v_1) & \omega_2(v_2) & \cdots & \omega_2(v_p) \\ \vdots & \vdots & & \vdots \\ \omega_p(v_1) & \omega_p(v_2) & \cdots & \omega_p(v_p) \end{pmatrix}$$

Proposition 4.2. Let $\omega_i \in \wedge^{p_i} V^*$ for $i = 1, \cdots, 3$. Then

- (associativity) $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$;
- (distributivity) if $p_2 = p_3$, then $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$;
- (super-commutativity) $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$.

Definition 4.8 (Pull-back of Alternating Form)**.** Let $\Phi : V \to W$ be a linear map between vector spaces and suppose $\omega \in \wedge^p W^*$. Then, the pull-back of ω along Φ is the p-form $\Phi^*\omega \in \wedge^p V^*$ such that

$$(\Phi^*\omega)(v_1, \cdots, v_p) := \omega(\Phi(v_1), \cdots, \Phi(v_p)).$$

Proposition 4.3. Let $\Phi : V \to W$ and $\psi : W \to Z$ be linear maps between vector spaces. Then

- the pull-back $\Phi^* : \wedge^p W^* \to \wedge^p V^* : \omega \mapsto \Phi^*\omega$ is linear and preserves the exterior product, i.e.
$$\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*\omega_1 \wedge \Phi^*\omega_2;$$
- $(\Psi \circ \Phi)^*\omega = \Phi^*\Psi^*\omega$;
- if $V = W$ and $p = \dim V$, then $\Phi^*\omega = \det(\Phi)\omega$.

Proposition 4.4. If $p > n := \dim V$, then $\wedge^p V^* = 0$.

Proposition 4.5. Let $v_1, \cdots, v_n \in V$ be a basis and let $v_1^\vee, \cdots, v_n^\vee$ be its dual basis. Then, for all $0 < p \leq n$,
$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} \mid i_1 < \cdots < i_p\}$$
form a basis of $\wedge^p V^*$ (so $\dim \wedge^p V^*$ is n choose p).

Finally, with the above constructions, we may return to the theory of manifolds and we will define the differential forms.

Definition 4.9 (p-th Exterior Bundle)**.** Let X be an n-dimensional manifold and let $x \in X$. We denote
$$\wedge^p T_x^* X := \wedge^p (T_x X)^*.$$
Furthermore, we define
$$\wedge^p T^* X := \bigcup_{x \in X} \wedge^p T_x^* X.$$
We call this the p-th exterior bundle of X and with other bundles, there exists a projection $\pi : \wedge^p T^* X \to X : \omega \in \wedge^p T_x^* X \to x$.

We note that for $p = 1$, $\wedge^1 T_x^* X = (T_x X)^* = T_x^* X$.

Definition 4.10 (Differential p-form)**.** A differential p-form on X is a section of $\wedge^p T^* X$.

We introduce the following notations

$$\Omega^p(X) := \{\omega \mid \omega \text{ is a differential } p\text{-form on } X\},$$

and
$$\Omega^\bullet X := \bigoplus_{p=0}^{n} \Omega^p(X).$$
We note that $\Omega^0(X) = C^\infty(X)$ and if X has dimension n, then $\Omega^p(X) = 0$ for all $p > n$.

Definition 4.11. Given $\omega_1 \in \Omega^p(X)$ and $\omega_2 \in \Omega^q(X)$, we define $\omega_1 \wedge \omega_2 \in \Omega^{p+q} X$ where

$$\omega_1 \wedge \omega_2 : X \to \wedge^{p+q} T^* X : x \mapsto \omega_1(x) \wedge \omega_2(x).$$

It follows by the definition that associativity, distributivity and super-commutativity still hold.

Again, similar to alternating forms, we may define the pull-back of differential forms.

Definition 4.12 (Pull-back of Differential p-form)**.** Let X, Y be manifolds and suppose $F : X \to Y$ is smooth. Then, for all $x \in X$, we have the Jacobian of F at x is a linear map

$$DF \mid_x : T_x X \to T_{F(x)} Y.$$

Thus, we may define

$$F^* : \wedge^p T^*_{F(x)} Y \to \wedge^p T^*_x X$$

such that, for all $\omega \in \wedge^p T^*_{F(x)} Y$ and $v_1, \cdots, v_p \in T_x X$,

$$F^* \omega(x)(v_1, \cdots, v_p) := \omega(F(x))(DF \mid_x v_1, \cdots, DF \mid_x v_p).$$

Thus, as we may define the pull-back given any $x \in X$, given a differential p-form ω on Y, we have a differential p-form $F^* \omega$ on X given by

$$F^* \omega : X \mapsto \wedge^p T^* X : x \mapsto F^* \omega(x).$$

With this in mind, we have define $F^* : \Omega^p(Y) \to \Omega^p(X)$ where it is not difficult to show F^* is linear. Moreover, given $\omega_1 \in \Omega^p(Y)$ and $\omega_2 \in \Omega^q(Y)$, we have

$$F^*(\omega_1 \wedge \omega_2) = F^*(\omega_1) \wedge F^*(\omega_2).$$

Finally, we see that if $G : Y \to Z$ is also smooth and $\omega \in \Omega^p(Z)$, then

$$(G \circ F)^* \omega = F^* G^* \omega.$$

Definition 4.13. Let $U \subseteq \mathbb{R}^n$ be open and let $x \in U$. Then $\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\}$ is a basis of $T_x U$ and we define $dx_i := \left(\frac{\partial}{\partial x_1} \right)^\vee$, i.e. $\{dx_i\}$ is the dual basis of $(T_x U)^* \cong T^*_x U$. Then, a basis of $\wedge^p T^*_x U$ is given by

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\},$$

and a differential p-form on U is locally given by

$$\omega = \sum_{|I|=p} h_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where $h_I : U \to \mathbb{R}$ is smooth for all $I = (i_1, \cdots, i_p)$.

By recalling that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for all $i, j \in \{1, \cdots, n\}$, we have $dx_i \wedge dx_i = 0$.

Let $F : U \to V$ be a smooth function between open sets of $\mathbb{R}^n$ and suppose $\omega : X \to \wedge^n T^* X$ is an n-form on V. Then, locally, we may write $\omega = h dy_1 \wedge \cdots \wedge h dy_n$ where $h : U \to \mathbb{R}$ is a smooth function. With this in mind, we have the pull-back of ω along F is given locally by

$$F^* \omega(x) = h \circ F(x) \cdot \det DF \mid_x \cdot dx_1 \wedge \cdots \wedge dx_n.$$

4.3 de Rham Differential

Recall that given $U \subseteq \mathbb{R}^n$ is open, and $h \in C^\infty(U)$, the differential of h is defined locally as

$$dh := \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i,$$

i.e. $dh(\xi) = \xi(h)$ for any vector field $\xi : U \to TU$. Alternatively, we have $dh = h^* dx$ where dx is the standard 1-form on $\mathbb{R}$.

More generally, recall that if $\omega \in \Omega^p(U)$ is a differential p-form, it is locally given by

$$\omega = \sum_{|I|=p} h_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Then, the differential of ω is locally given by

$$d\omega = \sum_{|I|=p} dh_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

and so, $d\omega \in \Omega^{p+1}(U)$ is a differential $p+1$-form. This process induces a map $d : \Omega^p(U) \to \Omega^{p+1}(U)$ and is called the de Rham differential of U.

For manifolds the idea is similar. Let X be an n-dimensional manifold and $\pi : E \to X$ be a vector bundle of rank r. Then we recall that there exists an atlas $\{(U_i, f_i)\}_{i\in I}$ for X and an atlas $\{(V_i, g_i)\}_{i\in I}$ for E such that

$$V_i = \pi^{-1}(U_i) \text{ and } g_i(V_i) = f_i(U_i) \times \mathbb{R}^r,$$

for all $i \in I$. Then, if $s : X \to E$ is a section, i.e. smooth and $\pi \circ s = \mathrm{id}_X$, we may define

$$\tilde{s}_i := g_i \circ s \circ f_i^{-1} : f_i(U_i) \to f_i(U_i) \times \mathbb{R}^r$$

for all $i \in I$. Furthermore, for any $i, j \in I$ where $U_i \cap U_j \neq \emptyset$, we have the transition functions

$$\phi_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j), \text{ and } \Phi_{ij} : f_j(U_i \cap U_j) \times \mathbb{R}^r \to f_i(U_i \cap U_j) \times \mathbb{R}^r.$$

In particular, the following diagram is commutative,

$$\begin{tikzcd}
f_j(U_i \cap U_j) \arrow[r, "\tilde{s}_j"] \arrow[d, "\phi_{ij}"] & f_j(U_i \cap U_j) \times \mathbb{R}^r \arrow[d, "\Phi_{ij}"] \\
f_i(U_i \cap U_j) \arrow[r, "\tilde{s}_i"] & f_i(U_i \cap U_j) \times \mathbb{R}^r
\end{tikzcd}$$

and so $\Phi_{ij} \circ \tilde{s}_j = \tilde{s}_i \circ \phi_{ij}$. It is easy to check the reverse is also true, i.e. if there exists smooth $\tilde{s}_i : f_i(U_i) \to f_i(U_i) \times \mathbb{R}^r$ for all $i \in I$ such that $\Phi_{ij} \circ \tilde{s}_j = \tilde{s}_i \circ \phi_{ij}$ for all $i, j \in I$, there exists a unique section s such that $\tilde{s}_i = g_i \circ s \circ f_i^{-1}$ for all $i \in I$.

With this in mind, we may define the de Rham differential on a manifold. Let X be an n-dimensional manifold and let $\omega \in \Omega^p(X)$ be a p-form. Let $\{(U_i, f_i)\}_{i\in I}$ be an atlas on X. Then, for each $i \in I$, we can define

$$\omega_i := (f_i^{-1})^* \omega \in \Omega^p(f_i(U_i)).$$

As these are p-forms on subsets of $\mathbb{R}^n$, if makes sense to take their de Rham differential $d\omega_i$ to be $p+1$-forms. Finally, as it is clear that these $p+1$-forms satisfies the commutative diagram as presented above, there exists a unique section $d\omega$ such that satisfying the commutative diagram for all charts. We call $d\omega$ the de Rham differential of ω and we denote the de Rham operator by $d : \Omega^p(X) \to \Omega^{p+1}(X)$.

With this construction in mind, in particular, we note that this construction is local, to prove properties about the de Rham differential, it suffice to prove them locally. The following three propositions illustrate this.

Proposition 4.6 (Leibniz Rule)**.** Given $\omega_1 \in \Omega_1^p(X)$ and $\omega_2 \in \Omega_2^q(X)$ we have,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2.$$

Proof. Let $x \in X$, and take (U_i, f_i) be a chart on X containing x, then by construction, $d\omega(x) = d\omega_i(x) = d(f_i^{-1})^*\omega(f_i(x)) \in \Omega^p(f_i(U_i))$. Thus, WLOG. we may assume $X = U \subseteq \mathbb{R}^n$ where U is open.

Write $\omega_1 = h_1 dx_{i_1} \wedge \cdots dx_{i_p}$ and $\omega_2 = h_2 dx_{j_1} \wedge \cdots dx_{j_q}$ where $h_1, h_2 \in C^\infty(U)$ (we note that we omitted the sum since we see that if the Leibniz rule is true for one term of the sum, it is true for the sum as the Leibniz rule is "linear"). Then, by definition, we have

$$\begin{aligned}
d(\omega_1 \wedge \omega_2) &= d(h_1 h_2 dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&= d(h_1 h_2) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\
&= (h_2 dh_1 + h_1 dh_2) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \\
&= (dh_1 \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (h_2 dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&\qquad + (-1)^p (h_1 \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (dh_2 \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}) \\
&= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2,
\end{aligned}$$

where the second to last equality is due to super-commutativity where we move dh_2 p-times to the right. □

Proposition 4.7. For all $p \geq 0$, the map $d \circ d : \Omega^p(X) \to \Omega^{p+2}(X)$ is the zero map.

Proof. Again, WLOG. assume $X = U \subseteq \mathbb{R}^n$ and write $\omega = h dx_{i_1} \wedge \cdots dx_{i_p} \in \Omega^p(U)$ where $h \in C^\infty(U)$. Then,

$$d(d\omega) = \sum_{k,j=1}^n \frac{\partial^2 h}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Now, since $h \in C^\infty(U)$, we have

$$\frac{\partial^2 h}{\partial x_k \partial x_j} dx_k \wedge dx_j = -\frac{\partial^2 h}{\partial x_j \partial x_k} dx_j dx_k,$$

and so, by the symmetry of the sum,

$$d(d\omega) = -d(d\omega)$$

implying $d(d\omega) = 0$. As ω was chosen arbitrarily, this implies $d \circ d = 0$ as required. □

Proposition 4.8. If $F : X \to Y$ is a smooth function between manifolds, then for any $\omega \in \Omega^p(Y)$, $F^* d\omega = d(F^*\omega)$.

Proof. Assuming $Y = U$ we write $\omega = h dy_{i_1} \wedge \cdots \wedge dy_{i_p} \in \Omega^p(U)$ where $h \in C^\infty(U)$. Consider

$$F^*\omega = h \circ F(F^* dy_{i_1}) \wedge \cdots \wedge (F^* dy_{i_p}).$$

By recalling $F^* dg = d(g \circ F)$ for all $g \in C^\infty(Y)$, we have $d(F^* dy_{i_j}) = d^2(y_{i_j} \circ F) = 0$. Thus,

$$\begin{aligned} d(F^*\omega) &= d(h \circ F) \wedge (F^* dy_{i_1}) \wedge \cdots \wedge (F^* dy_{i_p}) \\ &= F^* dh \wedge (F^* dy_{i_1}) \wedge \cdots \wedge (F^* dy_{i_p}) = F^*(dh \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_p}) = F^*(d\omega). \end{aligned}$$

by the Leibniz rule where terms containing $d(F^* dy_{i_j})$ vanishes. □

Definition 4.14. Let $\omega \in \Omega^p(X)$ be a differential p-form. Then,

- ω is said to be closed if $d\omega = 0$;
- ω is said to be exact if there exists some $\omega' \in \Omega^{p-1}(X)$ such that $d\omega' = \omega$.

It is clear that if ω is exact, then $\omega = d\omega'$ and so, $d\omega = d^2\omega' = 0$ and hence, it is closed.

4.4 Integration

Recall that, for a smooth function $F : X \to X$ where X is an n-dimensional manifold, we may define the pull-back of $\omega \in \Omega^n(X)$ along F such that for all $x \in X$,

$$F^*\omega(x) = \det(DF \mid_x) \cdot \omega(F(x)).$$

Now, assume that $\{U_i, f_i\}$ is a smooth atlas on X and for $i, j \in I$, ϕ_{ij} is a transition function from (U_j, f_j) to (U_i, f_i). Then, given an n-form $\omega \in \Omega^n(f_i(U_i))$ such that

$$\omega = h dy_1 \wedge \cdots \wedge dy_n$$

for some $h \in C^\infty(f_i(U_i))$, we may consider its pull-back along ϕ_{ij}. In particular,

$$\phi_{ij}^*\omega(x) = (h \circ \phi_{ij})(x) \det\big(D\phi_{ij} \mid_x\big) dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(f_j(U_i \cap U_j)).$$

We note the similarity of this equation to the change of variable formula in multivariable calculus and indeed, this will be our motivation for integration on manifolds.

In the case of $\mathbb{R}^n$, we have the following proposition.

Proposition 4.9. Let $D \subseteq \mathbb{R}^n$ be compact with border of measure zero. Then, given $h : \mathbb{R}^n \to \mathbb{R}$ smooth and $\phi : D \to \phi(D) \subseteq \mathbb{R}^n$ an diffeomorphism, we have

$$\int_{\phi(D)} h dy_1 \wedge \cdots \wedge dy_n = \int_D h \circ \phi |\det d\phi| dx_1 \wedge \cdots \wedge dx_n$$

Given $\omega \in \Omega^n(D)$, where $\omega = h dx_1 \wedge \cdots \wedge dx_n$, we define

$$\int_D \omega := \int_D h dx_1 \wedge \cdots \wedge dx_n.$$

Corollary 4.1. If $\Phi : V \to U$ is a diffeomorphism such that $\det D\Phi\mid_x > 0$ for all $x \in V$ (equivalently, there exists some $x \in V$ such that $\det D\Phi\mid_x > 0$). Then,

$$\int_U \omega = \int_V \Phi^* \omega.$$

Before generalizing the above, let us first consider the notion of orientation.

Definition 4.15 (Orientation)**.** Let V be a vector space over $\mathbb{R}$ of dimension n and let $B_1 = (v_1, \cdots, v_n)$ and $B_2 = (w_1, \cdots, w_n)$ be two ordered basis of V, then B_1, B_2 are said to have the same orientation if $\det T > 0$ where $T : V \to V := v_i \mapsto w_i$ (i.e. T is the change of basis matrix from B_1 to B_2).

It is clear that $\det T \neq 0$ since T is a linear isomorphism.

Definition 4.16. Given $\omega \in \wedge^n V^*$. Then an orientation of $\wedge$ of V is the set of all ordered basis $(v_1, \cdots, v_n)$ of V such that $\omega(v_1, \cdots, v_n) > 0$.

Proposition 4.10. Given $B_1 = (v_1, \cdots, v_n)$ and $B_2 = (w_1, \cdots, w_n)$ ordered basis of V with the same orientation, then $\omega(v_1, \cdots, v_n) > 0$ if and only if $\omega(w_1 \cdots w_n) > 0$.

Definition 4.17 (Orientation Preserving)**.** Let $\Phi : V \to W$ be an isomorphism of vector spaces with orientations $\wedge_1, \wedge_2$ on V and W respectively. Then Φ is said to be orientation preserving if an ordered basis of $V \in \wedge_1$ induces an ordered basis of $W \in \wedge_2$.

In the cases that $V = \mathbb{R}^n$, we define $\wedge^+$ to be the orientation induced by the standard ordered basis $e_1, \cdots, e_n$. This is called the positive orientation of V.

We would like to generalize this notion of positive orientation to manifolds. In particular, for each $x \in X$, we would like to define an orientation $\wedge_x$ of $T_x X$ such that it is compatible with the structure of the manifold on X.

In the case $X = U \subseteq \mathbb{R}^n$, for all $x \in U$, we have a canonical isomorphism $T_x U \simeq \mathbb{R}^n$, and so, we define the positive orientation $\wedge_U^+$ on U to be the collection of orientations $\wedge_x^+$ on $T_x V$ where $T_x U \simeq \mathbb{R}^n$ induces the positive orientation $\wedge^+$ on $\mathbb{R}^n$.

Definition 4.18 (Positive Orientation)**.** Let X be an n-dimensional manifold and let (U, f) be a chart. The positive orientation $\wedge^+$ on (U, f) is the collection of orientations on $T_x X$ for all $x \in U$ such that, considering the positive orientation on $f(U)$, the isomorphism

$$Df\mid_x : T_x U \to T_{f(x)}(f(U))$$

is orientation preserving.

In other words, X is oriented if given an n-form, there exists an atlas $\{U_i, f_i\}$, such that for all positively oriented basis $[\sigma_1], \cdots, [\sigma_n]$ of $T_x X$, where $x \in U_i$, we have $[f_i \circ \sigma_j] \in T_{f(x)}(f(U))$ is positively oriented in $T_{f(x)}(f(U))$.

Definition 4.19 (Orientable)**.** A manifold X is called orientable if there exists an atlas $\{(U_i, f_i)\}$ of positively oriented charts, such that for $x \in U_i \cap U_j$, the orientation induced by (U_i, f_i) coincides with the orientation induced by (U_j, f_j).

Alternatively, we see that X is orientable if and only if all the transition functions have differential with positive determinant.

Definition 4.20 (Orientation Preserving)**.** A smooth function $F : X \to Y$ between oriented manifolds is said to be orientation preserving if for all $x \in X$, the linear map

$$DF\,|_x \colon T_x X \to T_{F(x)} Y$$

is orientation preserving with respect to the orientations induced on $T_x X$ and $T_{F(x)} Y$ respectively.

With these definitions, we may finally consider integration on manifolds.

Let X be a manifold. Then for any $p \geq 0$, we denote Ω_c^p the set of differential p-forms with compact support.

Definition 4.21. Let $\omega \in \Omega_c^n(X)$ and let $\mathrm{supp}(\omega) \subseteq U$ where (U, f) is a positively oriented chart of X. Then,

$$(f^{-1})^* \omega \in \Omega_c^n(f(U))$$

and

$$\int_X \omega := \int_{f(U)} (f^{-1})^* \omega.$$

We see that this definition is well-defined, i.e. is independent of the chart (U, f). Indeed, if (V, g) is another positively oriented chart containing $\mathrm{supp}(\omega)$, then

$$\begin{aligned} \int_{g(U\cap V)} (g^{-1})^* \omega &= \int_{f(U\cap V)} (g \circ f^{-1})^* (g^{-1})^* \omega \\ &= \int_{f(U\cap V)} (f^{-1})^* g^* (g^{-1})^* \omega = \int_{f(U\cap V)} (f^{-1})^* \omega, \end{aligned}$$

where the second equality is due to corollary 4.1.

With this definition in mind, in some sense, we will partition the support of a form into parts contained in charts and then define the integral of a form over a manifold by glueing these parts together.

Definition 4.22 (Partition of Unity)**.** Let X be a manifold and let $\mathcal{U} = \{U_i\}$ be a open cover of X. A partition of unity with respect to $\mathcal{U}$ is a collection of smooth functions $h_i : X \to [0, 1]$ such that

- $\mathrm{supp}(h_i) \subseteq U_i$ for all i;
- $\sum_i h_i(x) = 1$ for all $x \in X$;
- for all $x \in X$, there exists an open neighbourhood V of x such that $\mathrm{supp}(h_i) \cap V \neq 0$ for finitely many i.

With the third property in mind, we see that the sum in the second property is well-defined as all but finitely many $h_i(x)$ are zero.

Proposition 4.11. If X is a manifold, then any open cover of X induces a partition of unity.

Proof. Omitted. (The idea is to use bump function which are scaled appropriately). $\square$

4.4.1 Volume Form

Proposition 4.12. Let X be a manifold of dimension n. X is orientable if and only if there exists a non-vanishing n-form $\omega \in \Omega^n(X)$. In this case, ω is called a volume form.

Proof. Suppose $\omega \in \Omega^n(X)$ is non-vanishing, then, we would like to find an atlas $\{(U_i, f_i)\}$ of X, such that the positive orientation on $T_x X$ induced by ω is preserved by f_i if $x \in U_i$.

Let $\{(U_i, f_i)\}$ be any atlas on X. Then, for each i, there exists some $g_i \in C^\infty(f_i(U_i))$ such that

$$(f_i^{-1})^* \omega = g_i dx_1 \wedge \cdots \wedge dx_n.$$

Since ω is non-vanishing, it follows that g_i is either positive and negative on $f_i(U_i)$ and so, by possibly replacing f_i by the composition $t \circ f_i$ where

$$t : \mathbb{R}^n \to \mathbb{R}^n : (x_1, x_2, \cdots, x_n) \mapsto (-x_1, x_2, \cdots, x_n)$$

we may assume g_i is positive for all i (after the replacement). Thus, for i, j, if $x \in U_i \cap U_j$, we see $(U_i, f_i), (U_j, f_j)$ define the same orientation on x implying X is orientable.

On the other hand, if X is orientable then, we may choose $\{U_i, f_i\}$ an atlas of positively oriented charts. Let $h_i : X \to [0, 1]$ be a partition of unity with respect to this atlas and define

$$\omega_i = f^*(dx_1 \wedge \cdots \wedge dx_n).$$

Then, taking $\tilde{\omega}_i$ to be the n-form defined by ω_i inside U_i and zero otherwise, we define $\omega := \sum_i h_i \tilde{\omega}_i$. Then, for each $x \in X$ and any positively oriented basis $v_1, \cdots, v_p$, we have that $\omega(x)(v_1, \cdots, v_p) > 0$. Thus, ω is a volume form on X. $\square$

With this proposition in mind, an alternative definition of orientation can be introduced. In particular, this definition might be more popular in literatures.

With the above proposition, we have showed that X being orientable is equivalent to the existence of a non-vanishing n-form. While this is the case, two non-vanishing n-forms do not necessary induce the same orientation. Indeed, it is clear that if $\omega \in \Omega^n(X)$, $-\omega$ induces a different orientation on X as bases of positive orientation on $T_x X$ with respect to ω will have negative orientation on $-\omega$. Thus, it makes sense to define an equivalence relation characterizing this property.

Definition 4.23. Given $\omega, \eta \in \Omega^n(X)$ are non-vanishing, we say $\omega \sim \eta$ if and only if $\omega(x)$ and $\eta(x)$ induces the same orientation on $T_x X$ for all $x \in X$.

One may show that there exists a non-vanishing $f \in C^\infty(X)$ such that $\omega = f\eta$. Since both ω and η are non-vanishing, it follows f must be positive or negative everywhere. Thus, it is clear that ω and η induces the same orientation if and only if f is positive everywhere.

Definition 4.24. An orientation on X is an equivalence class of the equivalence relation $\sim$.

It follows that if a non-vanishing n-form exists on X, then there are two choices of orientation on X while if such a form does not exist, we may not choose an orientation.

Moving back to our original definition, let us generalize the definition of an integral to the case where $\omega \in \Omega^n(X)$ which does not necessarily have support contained within a chart.

Definition 4.25. Let X be an oriented manifold and suppose $\omega \in \Omega_c^n(X)$ and $\{(U_i, f_i)\}$ be an atlas on X with positively oriented smooth charts. Then, given $h_i : X \to [0,1]$ a partition of unity with respect to the atlas, we define the integral of ω over X to be

$$\int_X \omega := \sum_{i \in I} \int_X h_i \omega.$$

Thus, for each i, the support of $h_i\omega$ is contained in U_i and in particular,

$$\int_X \omega := \sum_i \int_X h_i \omega = \sum_i \int_{f_i(U_i)} (f_i^{-1})^* h_i \omega.$$

Proposition 4.13. The integral $\int_X \omega$ is independent on the choice of the atlas $\{U_i, f_i\}$ and the choice of the partition of unity h_i.

Proof. We have already shown that $\int_{U_i} h_i\omega$ is independent on the choice of the chart and so, it suffices to show that the definition is independent of the choice of the partition of unity.

Let $\{(U_i, f_i)\}$ and $\{(V_j, g_j)\}$ be atlases on X with positively oriented charts and suppose $h_i, \bar{h}_j : X \to [0,1]$ are partitions of unity with respect to these atlases. Then, we have

$$\sum_i h_i = \sum_j \bar{h}_j = 1,$$

and so

$$\int_X h_i \omega = \int_X \left(\sum_j \bar{h}_j \right) h_i \omega = \sum_j \int_X \bar{h}_j h_i \omega.$$

Thus,

$$\sum_i \int_X h_i \omega = \sum_{i,j} \int_X \bar{h}_j h_i \omega = \sum_j \int_X \bar{h}_j \left(\sum_i h_i \right) \omega = \sum_j \int_X \bar{h}_j \omega.$$

□

Proposition 4.14. Let X, Y be oriented manifolds of dimension n and let $\omega, \eta \in \Omega_c^n(X)$. Then,

- $\int_X (a\omega + b\eta) = a \int_X \omega + b \int_X \eta$ for all $a, b \in \mathbb{R}$;
- if $\bar{X}$ is the manifold X with the opposite orientation, then $\int_{\bar{X}} \omega = - \int_X \omega$;
- if ω is a volume form, then $\int_X \omega > 0$;
- if $F : Y \to X$ be an orientation preserving diffeomorphism, then $\int_X \omega = \int_Y F^* \omega$.

Proof. The first two properties follows directly by the definition and properties of the Riemann integral and so we will prove the third and fourth property.

Let ω be a volume form and let $\{(U_i, f_i)\}$ be an atlas of oriented charts with the partition of unity h_i. Then, for each i, we may write

$$(f_i^{-1})^* \omega = g_i dx_1 \wedge \cdots \wedge dx_n$$

locally for some $g_i \in C^\infty(f_i(U_i))$ with $g_i > 0$. Then,

$$\int_X \omega = \sum_i \int_{f_i(U_i)} (f_i^{-1})^* h_i \omega = \sum_i \int_{f_i(U_i)} h_i g_i dx_1 \wedge \cdots \wedge dx_n > 0$$

as we are summing over non-negative terms at least one of which is positive.

Let $F : Y \to X$ be an orientation preserving diffeomorphism. Then, we may assume that the support of ω is contained in a unique positively oriented chart (U, f). Then, by assumption, $(F^{-1}(U), f \circ F)$ is also a positively oriented chart which contains the support of $F^*\omega$. Then,

$$\int_U \omega = \int_{F^{-1}(U)} F^*\omega,$$

and thus, taking the sum over all charts, it follows

$$\int_X \omega = \int_Y F^*\omega.$$

□

4.5 Stokes' Theorem

Let us denote

$$\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\},$$

and it boundary

$$\partial\mathbb{R}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n = 0\}.$$

Definition 4.26 (Manifolds With Boundary)**.** A manifold with boundary of dimension n is a Hausdorff and second countable topological space X with an atlas $\{(U_i, f_i)\}$ such that the functions $f_u : U_i \to \mathbb{R}^n_+$ is a homeomorphism for all i, and furthermore, for all i, j, the induced morphism

$$f_i \circ f_j^{-i} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$

is smooth.

Definition 4.27. The boundary ∂X of a manifold with boundary X is the set

$$\partial X := \{x \in X \mid f_i(x) \in \partial\mathbb{R}^n_+\}.$$

One may check that this definition is independent of the choice of i.

We also define the interior of X to be $\mathrm{int}(X) = X \setminus \partial X$.

It is clear that if X is a manifold with boundary, then the boundary ∂X is closed in X. Thus, if X is compact, then so is ∂X. Furthermore, if X is a manifold with boundary of dimension n, then $\mathrm{int}(X)$ is a manifold of dimension n.

We note that these definitions are different from the topological definitions as manifolds are defined intrinsically.

Many definitions from ordinary manifolds transfer directly to manifolds with boundaries. In particular, notions such as the tangent space, tensor fields, orientability, differential forms, partition of unity and many more are defined exactly the same.

In the case that X is an oriented manifold, then so is ∂X is also oriented. Similarly, for each point in the interior of $\mathbb{R}^n_+$, the volume form $dx_1 \wedge \cdots \wedge dx_n$ defines a positive orientation which as a convention, induces the positive volume form on ∂X given by $(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$.

Theorem 5 (Stokes' Theorem)**.** Let X be a smooth oriented manifold with boundary of dimension n. Then for all $\omega \in \Omega_c^{n-1}(X)$,

$$\int_X d\omega = \int_{\partial X} \omega.$$

Proof. Let $\{(U_i, f_i)\}$ be an atlas on X of positively oriented charts and let h_i be a partition of unity with respect to the atlas. Then,

$$\int_X d\omega = \sum_{i \in I} \int_X d(h_i \omega) = \sum_i \int_{f_i(U_i)} (f_i^{-1})^* d(h_i \omega).$$

Recall that $(f_i^{-1})^* d(h_i \omega) = d(f_i^{-1})^* h_i \omega$, and so, denoting $\tilde{h}_i := h_i \circ f_i^{-1}$, we have $(f_i^{-1})^* h_i \omega \in \Omega^{n-1}(\mathbb{R}^n_+)$ have local coordinates on $f_i(U_i)$

$$\sum_j^n \tilde{h}_i \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where $\omega_j \in C^\infty(f_i(U_i))$ and $\widehat{dx_j}$ means we omit the component dx_j. Taking the de Rham differential, we obtain

$$\begin{aligned} d\left(\sum_j^n \tilde{h}_i \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) &= \sum_j^n \frac{\partial \tilde{h}_i \omega_j}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \left(\sum_j^n (-1)^{j-1} \frac{\partial \tilde{h}_i \omega_j}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Thus, we have

$$\int_X d\omega = \sum_i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_j^n (-1)^{j-1} \frac{\partial \tilde{h}_i \omega_j}{\partial x_j} \, dx_1 \cdots dx_n$$

where the i-th integral of the first $(n-1)$ integrals corresponds to the integration over dx_i for $i = 1, \cdots, n-1$ and the last integral corresponds to the integration over dx_n (since we are integrating over $\mathbb{R}^n_+$).

Then, by the fundamental theorem of calculus, as ω is compactly supported, all integrands at infinity vanishes and thus, the only term remaining the integral at the boundary. Thus,

$$\int_X d\omega = \sum_i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^n \tilde{h}_i(x_1, \cdots, x_{n-1}, 0) \omega_n dx_1 \wedge \cdots \wedge dx_{n-1} = \int_{\partial X} \omega$$

as required. $\square$

Stokes' theorem is a powerful tool and we will demonstrate some direct consequences.

Corollary 5.1 (Integration by Parts)**.** Let X be a manifold with boundary of dimension n and let $p \geq 0$. Then, given $\omega \in \Omega_c^p(X)$ and $\eta \in \Omega_c^{n-p-1}(X)$, we have

$$\int_{\partial X} \omega \wedge \eta = \int_X d\omega \wedge \eta + (-1)^p \int_X \omega \wedge d\eta.$$

Proof. Follows by applying the Leibniz rule on Stokes' theorem. □

Corollary 5.2 (Brouwer's Fixed Point Theorem)**.** Let $D := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the closed disc and let $f : D \to D$ be smooth. Then f admits a fixed point, i.e. there exists some $x \in D$ such that $f(x) = x$.

Proof. Suppose that for all $x \in D$, $f(x) \neq x$. Then the ray beginning at $f(x)$ passing through x is well-defined. Now, defining $g(x)$ to be the point this ray meets with the boundary of D, we have $g : D \to \partial D$ is a smooth function such that $g(x) = x$ for all $x \in \partial D$. Then, as ∂D is the $n-1$-dimensional sphere, it is orientable and hence admits a volume form $\omega \in \Omega^{n-1}(\partial D)$. Thus,

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D d(g^* \omega) = \int_D g^* d\omega$$

by Stokes' theorem. But $d\omega$ is an n-form on the $n-1$-dimensional manifold ∂D, and thus is zero, implying $\int_D g^* d\omega = 0$, contradiction! □

Corollary 5.3. Let $\omega \in \Omega_c^n(X)$ be an exact form on an oriented manifold X of dimension n without boundary, then $\int_X \omega = 0$.

Similarly, if X is an oriented manifold with boundary of dimension n, and $\omega \in \Omega_c^{n-1}(X)$ is closed, then $\int_{\partial X} \omega = 0$.

Proof. Both claims follows directly by Stokes' theorem. □

Let X be an oriented manifold of dimension n and let $Z \subseteq X$ be an oriented submanifold of dimension k. Then, if $\omega \in \Omega_c^k(X)$, we define the integral of ω on Z to be

$$\int_Z \omega := \int_Z i^* \omega$$

where $i : Z \to X$ is the inclusion map. As $i^* \omega$ is then a k-form on Z, the integral makes sense. We denote $i^* \omega$ by $\omega \mid_Z$.

Corollary 5.4. Let X be an oriented manifold of dimension n and let Z be a compact oriented submanifold without boundary of X of dimension k. Then, given $\omega \in \Omega_c^k(X)$ such that $\int_Z \omega \neq 0$,

- ω is not exact on X and $\omega \mid_Z$ is not exact on Z;
- Z is not the boundary of a compact oriented submanifold $Y \subseteq X$ of dimension $k + 1$.