Probability Theory

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1 Review of Measure Theory

Modern probability theory is based on measure theory and we will in this section recall some notions from measure theory.

Definition 1.1 (Algebra)**.** Given a set Ω, a set of subsets $\mathcal{A}$ of Ω is an algebra if $\Omega \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite union and complements.

It follows straight away that an algebra is also closed under finite intersections.

Definition 1.2 (Finitely Additive Measure)**.** A function $\mu : \mathcal{A} \to [0, \infty]$ where $\mathcal{A}$ is an algebra, is a finitely additive measure if for any disjoint sets $A, B \in \mathcal{A}$,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Definition 1.3 (σ-Algebra)**.** A σ-algebra $\mathcal{F}$ is an algebra that is closed under countable unions.

Similarly, it follows that $\mathcal{F}$ is closed under countable intersections.

Definition 1.4 (Measure)**.** A function $\mu : \mathcal{F} \to [0, \infty]$ where $\mathcal{F}$ is a σ-algebra, is a σ-additive measure (or simply measure) if given a sequence of pairwise disjoint sets $A_1, A_2, \ldots$ of $\mathcal{F}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_n).$$

We call a measure a probability measure if $\mu(\Omega) = 1$.

Definition 1.5 (σ-Finite Measure)**.** A measure μ is said to be σ-finite if there exists a sequence of pairwise disjoint sets $A_1, A_2, \ldots$ of $\mathcal{F}$, such that $\bigcup_{i=1}^{\infty} A_i = \Omega$ and for all i, $\mu(A_i) < \infty$.

Definition 1.6 (Probability Space)**.** A probability space is the triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a set Ω, a σ-algebra $\mathcal{F}$ on Ω and $\mathbb{P}$ a probability measure on $\mathcal{F}$.

We call elements of $\mathcal{F}$ (i.e. a $\mathcal{F}$-measurable set) an event.

Proposition 1.1 (Continuity of Measures)**.** Let $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$, then

- (continuity from below) if (A_n) is increasing, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n).$$

- (continuity from above) if (A_n) is decreasing, then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n).$$

We recall the the finiteness of the measure is vital for continuity from below while continuity from above is also valid for general measures.

Proof. Exercise. $\square$

Proposition 1.2. A finitely additive probability measure on the σ-algebra $\mathcal{F}$ is a probability measure if and only if it is continuous at 0.

Proof. The forward direction follows from above so we will prove the reverse. Suppose μ is finitely additive and for any decreasing $(A_n) \subseteq \mathcal{F}$ with $\bigcap A_n = \emptyset$, we have $\lim_{n\to\infty} \mu(A_n) = 0$. Then, μ is continuous from below, and so for any sequence of disjoint sets (B_n), we have $(C_n) := (\bigcup_{i=1}^n B_i)$ is a sequence of increasing sets and thus,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} C_i\right) = \lim_{n\to\infty} \mathbb{P}(C_n) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(B_i)$$

implying μ is σ-additive and so, μ is a measure. $\square$

Proposition 1.3. Given a collection $\{\mathcal{F}_i\}_{i\in I}$ σ-algebras of Ω, $\bigcap_{i\in I} \mathcal{F}_i$ is also a σ-algebra on Ω.

Definition 1.7 (σ-Algebra Generated By Sets)**.** Given a collection of subsets S of Ω, the σ-algebra generated by S is

$$\sigma(S) := \bigcap \{\mathcal{F} \text{ a } \sigma\text{-algebra} \mid S \subseteq \mathcal{F}\}.$$

Definition 1.8 (Borel σ-Algebra)**.** Given a topological space $(X, \mathcal{T})$, the Borel σ-algebra on X is $\mathcal{B}(X) := \sigma(\mathcal{T})$.

Definition 1.9 (Product σ-Algebra)**.** Given measurable spaces $(\Omega_1, \mathcal{F}_1), (\Omega_2, \mathcal{F}_2)$, the product σ-algebra on $\Omega_1 \times \Omega_2$ is

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma(\mathcal{F}_1 \times \mathcal{F}_2) = \sigma(\{A_1 \times A_2 \mid A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}).$$

Definition 1.10 (Cylindrical σ-Algebra)**.** A set $C \subseteq \mathbb{R}^\infty$ is said to be cylindrical if is of the form

$$C = \{x \in \mathbb{R}^\infty \mid (x_1, \cdots, x_n) \in C_n\}$$

where $C_n \in \mathcal{B}(\mathbb{R}^n)$. The set of cylindrical sets $\mathcal{B}(\mathbb{R}^\infty)$ form a σ-algebra on $\mathbb{R}^\infty$ and is called the cylindrical σ-algebra.

Recall that a nondecreasing function g on $\mathbb{R}$ is continuous up to possibly countably many discontinuities of the first kind. Furthermore, the derivative g' exists λ-a.e. (where λ is the Lebesgue measure on $\mathbb{R}$.

Proposition 1.4. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$ be a probability space. Defining $F(x) := \mathbb{P}(-\infty, x]$, we have

- F is nondecreasing;
- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$;
- F is continuous on the right.

Proof. Clear by the monotonicity, continuity of measures (from above). $\square$

Definition 1.11 (Distribution Function)**.** Any function $F : \mathbb{R} \to [0,1]$ satisfying the above three properties is said to be a distribution function on $\mathbb{R}$.

It is clear that any probability measure induces a distribution. On the other hand the converse is also true.

Proposition 1.5. Given a distribution function F, there exists a unique probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(x) = \mathbb{P}(-\infty, x]$ for all $x \in \mathbb{R}$.

Proof. Use Caratheodory extension theorem on the algebra $\{(-\infty, x] \mid x \in \mathbb{R}\}$ mapping $(-\infty, x] \mapsto F(x)$. The uniqueness of the probability measure follows by the uniqueness of the Caratheodory extension. $\square$