Martingales

Kexing Ying

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1 Discrete Time Martingales

Definition 1.1 (Filtration)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space, then a filtration on this measure space is a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of increasing sub-σ-algebras of $\mathcal{A}$. If $(X, \mathcal{A}_0, \mu)$ is σ-finite, then we call $(X, \mathcal{A}, (\mathcal{A}_n), \mu)$ a σ-finite filtered measure space.

Definition 1.2 (Natural Filtration)**.** Given a sequence of functions $(f_n : X \to Y)$ where Y is a measurable space, the natural filtration is the measure space $(X, \sigma(f_n \mid n \in \mathbb{N}), \mu)$ equipped with the filtration

$$(\mathcal{A}_n) = (\sigma(f_1, \cdots, f_n)).$$

It is clear that the natural filtration is the least filtration such that f_n is $\mathcal{A}_n$-measurable. As we shall see in the continuous case, one may define filtrations more generally with a pre-order index.

Definition 1.3 (Discrete Time Martingale)**.** Let $(X, \mathcal{A}, (\mathcal{A}_n), \mu)$ be a σ-finite filtered measure space. Then a sequence of $\mathcal{A}$-measurable functions $(f_n)_{n\in\mathbb{N}}$ is a martingale if $f_n \in \mathcal{L}^1(\mathcal{A}_n)$ and

$$\int_A f_{n+1} \mathrm{d}\mu = \int_A f_n \mathrm{d}\mu$$

for all $n \in \mathbb{N}$ and $A \in \mathcal{A}_n$.

Using probabilistic notation, the above property for martingales is simply $\mathbb{E}[f_{n+1} \mid \mathcal{A}_n] = f_n$.

Definition 1.4 (Sub/Super-martingale)**.** The sequence of $\mathcal{A}$-measurable functions $(f_n)_{n\in\mathbb{N}}$ is a submartingale if $f_n \in \mathcal{L}^1(\mathcal{A}_n)$ and

$$\int_A f_{n+1} \mathrm{d}\mu \geq \int_A f_n \mathrm{d}\mu$$

for all $n \in \mathbb{N}$ and $A \in \mathcal{A}_n$. Similarly, (f_n) is a supermartingale if $f_n \in \mathcal{L}^1(\mathcal{A}_n)$ and

$$\int_A f_{n+1} \mathrm{d}\mu \leq \int_A f_n \mathrm{d}\mu$$

for all $n \in \mathbb{N}$ and $A \in \mathcal{A}_n$.

It is clear that a martingale is both a submartingale and a supermartingale and the negative of a submartingale is a supermartingale (and vice-versa).

Proposition 1.1. The sequence of $\mathcal{L}^1$ functions $(f_n)_{n\in\mathbb{N}}$ is a submartingale if and only if

$$\int \phi f_{n+1} \mathrm{d}\mu \geq \int \phi f_n \mathrm{d}\mu$$

for all $\phi \in \mathcal{L}^\infty(\mathcal{A}_n) \cap \geq 0$. Similar statements hold for martingales and supermartingales by the same proof.

Proof. The backwards implication is clear by choosing $\phi = \mathbb{1}_A$ so let us consider the forward direction. By approximation by simple functions, there exists a sequence of simple functions

$\left(s_N = \sum_{i=0}^N \alpha_i \mathbb{1}_{A_i}\right)_{N\in\mathbb{N}}$ such that $s_N \uparrow \phi$. Then, as $\phi \in \mathcal{L}^\infty$, there exists some $M \in \mathbb{R}$ such that $M \geq \phi \geq s_N$. Thus, by dominated convergence theorem,

$$\begin{aligned}\int \phi f_{n+1} = \int \lim_{N\to\infty} s_N f_{n+1} = \lim_{N\to\infty} \int s_N f_{n+1} = \lim_{N\to\infty} \sum_{i=0}^N \int_{A_i} f_{n+1} \\ \geq \lim_{N\to\infty} \sum_{i=0}^N \int_{A_i} f_n = \lim_{N\to\infty} \int s_N f_n = \int \lim_{N\to\infty} s_N f_n = \int \phi f_n.\end{aligned}$$

□

Proposition 1.2. If (f_n) and (g_n) are martingales and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n$ is also a martingale. Similar statements hold for submartingales and supermartingales (with consideration to the sign of α and β).

Let us recall some definitions from probability theory.

Definition 1.5 (Independence)**.** Let $(X, \mathcal{A}, \mu)$ be a probability space. A sequence of functions $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1$ is said to be independent if

$$\mu\left(\bigcap_{n=1}^N f_n^{-1}(B_n)\right) = \prod_{n=1}^N \mu(f_n^{-1}(B_n)),$$

for all $(B_n)_{n=1}^N \subseteq \mathcal{B}(\mathbb{R})$.

Lemma 1.1. Given a sequence of independent functions $(f_n)_{n=1}^{N+1} \subseteq \mathcal{L}^1$, for all $A \in \sigma(f_1, \cdots, f_N)$,

$$\int_A f_{N+1} \mathrm{d}\mu = \mu(A) \int f_{N+1} \mathrm{d}\mu,$$

and for all $\phi \in \mathcal{L}^1(\sigma(f_1, \cdots, f_N))$,

$$\int \phi f_{N+1} \mathrm{d}\mu = \int \phi \mathrm{d}\mu \cdot \int f_{N+1} \mathrm{d}\mu.$$

Furthermore,

$$\int \prod_{n=1}^k f_n \mathrm{d}\mu = \prod_{n=1}^k \int f_n \mathrm{d}\mu,$$

for all $k = 1, \cdots, N$.

Proposition 1.3. Given a sequence of independent functions $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1$, $(s_n) := \left(\sum_{i=1}^n f_i\right)$ is a submartingale with respect to the natural filtration if and only if $\int f_n \geq 0$ for all n.

Proof. The statement follows by the following chain of equalities,

$$\int_A s_{n+1} \mathrm{d}\mu = \int_A s_n + f_{n+1} \mathrm{d}\mu = \int_A s_n \mathrm{d}\mu + \int_A f_{n+1} \mathrm{d}\mu = \int_A s_n \mathrm{d}\mu + \mu(A) \int f_{n+1} \mathrm{d}\mu.$$

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