Functional Analysis

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1 Introduction

We have thus far looked at abstract vector spaces in linear algebra and (metric) topological spaces in topology. In this course, we will combine these concepts and study linear metric space. In particular, we will study vector spaces equipped with a topology such that certain properties are satisfies.

In this course, we will often study the space of functions and hence the name of the course. As we have seen before, given that the codomain space possesses a certain structure, it is possible to define point-wise addition and scalar multiplications on functions, and thus, possible to equip the space with a vector space structure.

Let us recall some definitions.

Definition 1.1 (Metric)**.** A metric ρ on a non-empty set X is a function with type signature $X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

- $\rho(x, y) = 0 \iff x = y$;
- $\rho(x, y) = \rho(y, x)$;
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Definition 1.2 (Translation Invariant)**.** A metric space (V, ρ) where V is equipped with the binary operation $(+) : V \times V \to V$ is translational invariant if for all $w, z, v \in V$,

$$\rho(w + v, z + v) = \rho(w, z).$$

Definition 1.3 (Norm)**.** A norm $\|\cdot\|$ on the vector space V (over the field $\mathbb{K}$ equipped with a modulus $|\cdot|$) is a function with type signature $X \to \mathbb{R}^+$ such that for all $x, y \in V, k \in \mathbb{K}$,

- $\|x\| = 0 \iff x = 0$;
- $\|k \cdot x\| = |k|\|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

We recall that a norm induces a metric by defining $\rho(x, y) = \|x - y\|$. In this case, it is possible to show that $(+)$ and $(\cdot)$ are continuous with respect to this metric and ρ is translational invariant.

Definition 1.4 (Banach Space)**.** A normed space is said to be a Banach space if it is complete, i.e. every Cauchy sequence converge.

Definition 1.5 (Separable)**.** A topological space is said to be separable if there exists a dense countable subset.

As we shall see, for $0 < 0 < \infty$, ℓ_p is separable while ℓ_∞ is not.

Definition 1.6 (Compact)**.** A topological space is said to be compact if every open cover has a finite sub-cover.

Unlike what we have seen before, as we consider infinite dimensional spaces, we will see that the Heine-Borel property will no longer hold, i.e. closed and bounded is no longer equivalent to compact.

2 Linear Spaces

Definition 2.1 (Equivalent Norms and Metrics)**.** Two norms $\|\cdot\|_k$ for $k = 1, 2$ are said to be equivalent if there exists some $M > 0$ such that for all x,

$$\frac{1}{M}\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Similarly, two metrics ρ_k are said to be equivalent if there exists some $M > 0$ such that for all x, y,

$$\frac{1}{M}\rho_1(x, y) \leq \rho_2(x, y) \leq M\rho_1(x, y).$$

It is clear that equivalent is a symmetric relation and as we have seen before, all norms on a finite dimensional space are equivalent.

Definition 2.2 (Concave and Convex Function)**.** A function $f : V \to \mathbb{R}$ is

- concave if for all $s \in [0, 1], x, y \in V$, we have
$$sf(x) + (1 - s)f(y) \leq f(sx + (1 - s)y);$$
- convex if for all $s \in [0, 1], x, y \in V$, we have
$$sf(x) + (1 - s)f(y) \geq f(sx + (1 - s)y).$$

Proposition 2.1. If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is concave and $f(0) = 0$. Then

$$f(x + y) \leq f(x) + f(y).$$

Proof. Clear by taking $s = \frac{y}{x+y}$, we have

$$(1 - s)f(x + y) = sf(0) + (1 - s)f(x + y) \leq f(s \cdot 0 + (1 - s)(x + y)) = f(x),$$

and

$$sf(x + y) = sf(x + y) + (1 - s)f(0) \leq f(s(x + y) + (1 - s) \cdot 0) = f(y).$$

Adding the two equations, we have

$$f(x + y) = (1 - s)f(x + y) + sf(x + y) \leq f(x) + f(y).$$

□

Corollary 2.1. If ρ is a metric and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave and vanishing at 0, then $\rho \circ \eta$ is also a metric.

Definition 2.3 (Linear Metric Space)**.** A vector space V over the field $\mathbb{K}$ equipped with a metric ρ on V and a metric $|\cdot - \cdot|$ on $\mathbb{K}$ is a linear metric space if $(+) : V \times V \to V$ and $(\cdot) : \mathbb{K} \times V \to V$ are continuous with respect to the induced metric.

Proposition 2.2. Any normed space is a linear metric space.

Proof. Let $(x_n, y_n) \to (x, y)$ in V^2, then we have

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0.$$

Thus, $(+)$ is continuous.

Similarly, if $(\lambda_n, x_n) \to (\lambda, x)$ in $\mathbb{K} \times V$,

$$\begin{aligned}\|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x\| \\ &\leq \|\lambda_n x_n - \lambda_n x\| + \|\lambda_n x - \lambda x\| \\ &= |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\|.\end{aligned}$$

Now, since (λ_n) is convergent, it is bounded by some $M > 0$ and thus,

$$\|\lambda_n x_n - \lambda x\| \leq |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\| \leq M\|x_n - x\| + |\lambda_n - \lambda|\|x\| \to 0,$$

implying $(\cdot)$ is continuous. $\square$

2.1 Sequence Spaces

We recall the L_p spaces from second year measure theory, and in particular, when we consider the counting measure μ, we have the nice property that

$$\int f \mathrm{d}\mu = \sum_{n=0}^{\infty} f(n),$$

and we no longer require a quotient to define the linear space as the only null-set is the empty set (thus, two function are a.e equal if and only if they are equal). In this special case, we call the resulting space ℓ_p with the p-norm

$$\|f\|_p = \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{\infty} |f(n)|^p\right)^{\frac{1}{p}}.$$

We will use the sequence notation and write $a_n := a(n)$ for $a \in \ell_p$.

As this is simply a special case of the L_p space, the inequalities proved on the L_p space remains. We will recall them here for ℓ_p spaces.

Proposition 2.3 (Hölder's Inequality)**.** Let $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q \in (1, \infty)$. Then for $a = (a_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}} \in \ell_p$, we have

$$|\langle a, b \rangle| \leq \|a\|_p \|b\|_p,$$

where $\langle a, b \rangle := \sum_{i \in \mathbb{N}} a_i b_i$.

Proposition 2.4 (Minkowski's Inequality)**.** Let $a, b \in \ell_p$ for some $1 \leq p \leq \infty$. Then $a + b \in \ell_p$ and

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p.$$

Lemma 2.1. Let $I \subseteq \mathbb{R}$ be an interval and $\phi : I \to \mathbb{R}^+$ be a function. Then ϕ is convex if and only if for all $y \in I$, there exists a $\gamma \in \mathbb{R}$, such that for all $x \in I$,

$$\gamma(x - y) \leq \phi(x) - \phi(y).$$

Proposition 2.5 (Jensen's Inequality)**.** let $\phi \geq 0$ be convex and suppose $\sum_{i \in \mathbb{N}} \eta_i = 1$, $|\langle \alpha \rangle| < \infty$ and $\langle \phi(\alpha) \rangle$ (where $\langle \beta \rangle = \sum_{i \in \mathbb{N}} \eta_i \beta_i$), then

$$\phi(\langle \alpha \rangle) \leq \langle \phi(\alpha) \rangle.$$

Proof. By the above lemma, there exists some γ such that

$$\gamma(\alpha_j - \langle \alpha \rangle) \leq \phi(\alpha_j) - \phi(\langle \alpha \rangle).$$

Thus,

$$\begin{aligned} 0 = \sum \eta_i \gamma(\alpha_j - \langle \alpha \rangle) &\leq \sum \eta_i (\phi(\alpha_j) - \phi(\langle \alpha \rangle)) \\ &= \sum \eta_i \phi(\alpha_j) - \phi(\langle \alpha \rangle) \sum \eta_i \\ &= \langle \phi(\alpha_j) \rangle - \phi(\langle \alpha \rangle) \end{aligned}$$

where the first equality follows as γ is independent of the index i. $\square$

Proposition 2.6. For $p < p'$, $\ell_p \subseteq \ell_{p'}$. On the other hand, if $\sum \eta_i = 1$, we have $\ell_p(\eta) \supseteq \ell_{p'}(\eta)$.

Definition 2.4. Let ϕ be a convex function such that $\phi(0) = 0$, $\phi(x) \to \infty$ as $x \to \infty$. If ϕ has the doubling property such that there exists some $M > 0$ such that for all $x \in \mathbb{R}$, $\phi(2|x|) \leq M\phi(|x|)$, then

$$V := \left\{ a : \mathbb{N} \to \mathbb{R} \mid \sum \eta_i \phi(|a_i|) < \infty \right\}$$

where $\sum \eta_i = 1$ is a vector space with point-wise operations.

Proposition 2.7. Given a metric space (X, ρ), there exists a complete metric space $(\tilde{X}, \tilde{\rho})$ and an isometric embedding $\iota : X \to \tilde{X}$ such that for all $x, x' \in X$, $\tilde{\rho}(\iota(x), \iota(x')) = \rho(x, x')$.

Proof. We have seen similar ideas in the completion of $\mathbb{Q}$ though completing the space with equivalence classes of mutually Cauchy sequences as elements of $\tilde{X}$. $\square$

As we have seen last year, the L_p spaces are complete, and thus are Banach spaces. Thus, we have ℓ_p spaces are also complete and are Banach spaces. In fact, the proof that ℓ_p spaces are complete is easier, in that one may show completeness through showing the point-wise limit of the sequences indeed belong to ℓ_p.

Definition 2.5. We define

- $c_0 := \{x \in \ell_\infty \mid \lim_{n \to \infty} x_n = 0\}$,
- $c := \{x \in \ell_\infty \mid \exists \lim_{n \to \infty} x_n\}$,

be subspaces of ℓ_∞.

Proposition 2.8. c_0 is complete.

Proof. Let $(x_i^n) \subseteq c_0$ be a Cauchy sequence and let $x_i = \lim_{n\to\infty} x_i^n$, then it suffices to show $\lim_{i\to\infty} x_i = 0$.

Since (x_i^n) is Cauchy, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|x^n - x^N\|_\infty < \frac{\epsilon}{2}.$$

Furthermore, as $x_i^N \to 0$ as $i \to \infty$, there exists some $I \in \mathbb{N}$ such that for all $i \geq I$, $|x_i^N| < \epsilon/2$. Thus, for all $i \geq I$, we have

$$\frac{\epsilon}{2} > \|x^n - x^N\|_\infty > |x_i^n - x_i^N| > |x_i^n| - \frac{\epsilon}{2} \implies \epsilon > |x_i^n|,$$

for all $n \geq N$. Hence, taking $n \to \infty$, we have

$$\epsilon > \lim_{n\to\infty} |x_i^n| = |x_i|,$$

implying $x_i \to 0$ as $i \to \infty$. $\square$