

# Algebraic Topology Revision Notes

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## Algebra

Fundamental groups is a topological invariant. Abelianization is a group invariant.

The abelianization of a presentation  $\langle S \mid R \rangle$  is simply that

$$\langle S \mid R \rangle = \langle S \mid R \cup [a, b], a, b \in S \rangle.$$

## Notes on Shapes

**Proposition.** If  $A$  is a deformation retract of  $X$ , then the inclusion map  $i : A \hookrightarrow X$  induces a group isomorphism  $i_* : \pi_1(A) \xrightarrow{\sim} \pi_1(X)$ .

**Proposition.** If  $A$  is a deformation retract of  $X$ , then for all  $n$ , the inclusion map  $i : A \hookrightarrow X$  induces a group isomorphism  $i_* : H_n(A) \xrightarrow{\sim} H_n(X)$ .

**Definition.** Given  $(X, x)$  a pointed space, there is the natural map

$$\phi : \pi_1(X, x) \rightarrow H_1(X)$$

by interpreting each loop as a 1-chain.

**Proposition.** As  $H_1(X)$  is abelian, by the universal property, there exists a unique  $\Phi$  such that

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{\phi} & H_1(X) \\ \downarrow \text{ab} & \searrow \Phi & \\ \pi_1(X, x)_{\text{ab}} & & \end{array}$$

commutes. Furthermore, if  $X$  is path-connected,  $\Phi$  is an isomorphism.

- $S^n$ : simply connected for  $n \geq 1$  so  $\pi_1(S^n) = 0$ .

$$H_k(S^n) = \begin{cases} \mathbb{Z}, & k = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

- $T^2 = S^1 \times S^1$ :  $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$  as  $\pi_1(X \times Y, (x, y)) = \pi_1(X, x) \times \pi_1(Y, y)$ .

$$H_0(T^2) = H_2(T^2) = \mathbb{Z}, H_1(T^2) = \mathbb{Z}^2, 0 \text{ otherwise.}$$

- $S^1 \times D^2$  (filled torus): deformation retracts to  $S^1$ .
- $S^1 \cup_f S^1$  where  $f$  identifies two points from each  $S^1$ , (i.e. a venn diagram): collapsing two points from one side provides a homotopy equivalence. This results in  $S^1 \vee S^1 \vee S^1$ .
- $S\{1, \dots, n\}$  (suspension of  $n$  elements): possible to deform into  $n - 1$ -circles in the shape of 8. Thus,  $\pi_1(S\{1, \dots, n\}) = F_{n-1}$ . Generalizes the above point.

Let  $X \subseteq \mathbb{R}^n \subseteq S^n = \mathbb{R}^n \cup \{\infty\}$ . Under some conditions  $S^n \setminus X^\circ \simeq X$ . This is true for  $X$  a solid sphere or solid torus (probably true for more spaces).

## General Facts

Path-connected does not imply locally path connected.

If  $f$  and  $g$  are homotopic, then their induced maps are equal at both the fundamental group level as well as the homology level.

A simply connected space only has homeomorphisms as its covering maps (**Use converse to show not simply connected**). Induced map of a covering map is injective by the homotopy lifting theorem and so, the space is simply connected, the covering space must also be simply connected, namely the covering is universal. Thus, by uniqueness, this must be a homeomorphism as the space covers itself universally.

Covering maps are open maps since they can be written as a quotient map.

**A method to find the fundamental group.** Let  $X$  be some space and suppose it has the universal cover  $\tilde{X}$ . Then, if  $\Gamma$  is a group acting nicely on  $\tilde{X}$  and  $X \simeq \tilde{X}/\Gamma$ ,  $X$  has fundamental group  $\Gamma$ .

**Theorem (Lifting Criterion).** Let  $\pi : (Y, y) \rightarrow (X, x)$  be a covering map and let  $f : (Z, z) \rightarrow (X, x)$  be continuous where  $Z$  path-connected and locally path-connected. Then, there exists a **unique**  $\tilde{f} : (Z, z) \rightarrow (Y, y)$  such that

$$\begin{array}{ccc} & & (Y, y) \\ & \nearrow \tilde{f} & \downarrow \pi \\ (Z, z) & \xrightarrow{f} & (X, x) \end{array}$$

commutes **if and only if**

$$f_* \pi_1(Z, z) \subseteq p_* \pi_1(Y, y)$$

as subgroups.

For  $X$  path-connected,  $H_0(X, A) = 0$  for all  $\emptyset \neq A \subseteq X$  by considering paths from  $A$  to any point in  $X$  as 1-chains.

**Theorem (Long Exact Sequence for Quotients).** Given  $A \subseteq X$  where  $(X, A)$  is a good pair. We have the long exact sequence

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X/A) \rightarrow H_{n-1}(A) \rightarrow \cdots$$

with the end

$$\cdots \rightarrow H_1(A) \rightarrow H_1(X) \rightarrow H_1(X/A) \rightarrow \tilde{H}_0(A) \rightarrow \tilde{H}_0(X) \rightarrow \tilde{H}_0(X/A),$$

where  $\tilde{H}_0(X) \simeq 0 \simeq \tilde{H}_0(X/A)$  if  $X$  is path connected.