

# Group Representation Theory

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# 1 Introduction

Group representation theory is a field of mathematics that applies linear algebra to study properties of groups. The field itself originated through a letter from Dedekind to Frobenius in which he noted that, given  $f = \det A$ , where  $A$  is the Cayley table of a group of  $n$  elements, by factorising  $f$  into irreducible polynomials,  $f = \prod_i f_i^{d_i}$ , we have  $d_i = \deg f_i$ . And this led Frobenius to invent group representation theory.

Group representation theory is applicable in many different areas.

- Group theory arises in Klein's "Erlangen program" as symmetries of geometric spaces.
- Burnside in 1904 proves the following using representation theory (and so shall we later on)

**Proposition 1.1.** Let  $G$  be a group such that  $|G| = p^r q^s$  where  $p, q$  are prime and  $r + s \geq 2$ , then  $G$  is not simple.

- In number theory, representations of Galois groups arises in the number field case

$$\overline{F}/F, \mathbb{Q} \subseteq F, [F : \mathbb{Q}] < \infty,$$

which has implications in Wiles' proof of Fermat's last theorem.

- In chemistry the symmetry and rotation of molecules can be represented by group actions.
- In quantum mechanics, spherical symmetry gives rise to discrete energy levels, orbitals, etc.
- In differential geometry, the vector space of solutions is a representation of the symmetry group of an equation.

Recalling the definition of a group, informally, the representation of a group  $G$  is a way of writing group elements as linear transformations of a vector space such that the natural group properties are satisfied.

Some examples of group representations are the following:

- For all group  $G$ , the trivial representation of  $G$  is  $\rho$  such that  $\rho(g) = \text{id}$  for all  $g \in G$ .
- Let  $\zeta \in \mathbb{C}$  be a  $n$ -th root of 1 and let  $G = C_n = \{1, g, \dots, g^{n-1}\}$ . Then  $\rho : g^i \mapsto (\zeta^i)$  is a representation of  $G$ .
- In the case  $G = S_n$ , the mapping of  $\sigma \in S_n$  to its corresponding permutation matrix  $P_\sigma$  is a representation of  $G$ .
- Another representation of  $S_n$  is  $\sigma \in S_n \mapsto (\text{sign}(\sigma))^1$ .
- Let  $G = D_n$  the dihedral group of order  $2n$ . Then, a representation  $D_n$  maps elements of  $D_n$  to the corresponding  $2 \times 2$  matrices which rotates/reflects  $\mathbb{R}^2$  by the appropriate amount.

We shall in this module study and construct representations, and furthermore, classify up to isomorphism finite-dimensional complex representations of every finite group  $G$ .

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<sup>1</sup> $\text{sign}(\sigma) = \det P_\sigma$

## 2 Fundamentals of Group Representation

**Definition 2.1** (Representation). Let  $G$  be a group, then a representation of  $G$  is the pair  $(V, \rho)$  where  $V$  is a (finite-dimensional) vector space and  $\rho : G \mapsto GL(V)$  is a group homomorphism.

Alternatively, we may consider a group representation of  $G$  is a group action  $(\cdot) : G \times V \rightarrow V : (g, v) \mapsto v$  such that  $(\cdot)$  is linear with respect to the second parameter. In particular, we recall a group action  $(\cdot)$  satisfies  $e \cdot v = v$  and  $g \cdot (h \cdot v) = gh \cdot v$ .

**Definition 2.2** (Dimension of a Representation). If  $(V, \rho)$  is a representation of  $G$ , then the dimension of  $(V, \rho)$  is  $\dim(V, \rho) = \dim V$ .

Similar to other objects in mathematics, we introduce a notion of morphisms between representations.

**Definition 2.3** (Homomorphism of Representation). Let  $G$  be a group and  $(V, \rho_V)$  and  $(W, \rho_W)$  be two representations of  $G$ . Then a homomorphism of representations is a linear map  $T : V \rightarrow W$  such that for all  $g \in G$ ,

$$T \circ \rho_V(g) = \rho_W(g) \circ T.$$

Furthermore, we say  $T$  is an isomorphism is bijective (or equivalently, it has an inverse which is also a homomorphism).

In particular, one might imagine the homomorphism as a linear map such that the following diagram commute.

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \rho_V(g) \downarrow & & \downarrow \rho_W(g) \\ V & \xrightarrow{T} & W \end{array}$$

As with any definitions which work with finite-dimensional vector spaces, there are equivalent but “worse” (as we will have to choose a basis) corresponding definitions in terms of matrices. Nonetheless, these definitions with matrices are easier computationally and we shall recall the contrast here.

Clearly, if  $G$  is a group and  $(\mathbb{C}^n, \rho)$  is a representation, we have  $\rho(e) = I_n$ . Furthermore, we have a natural isomorphism between  $GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$  and more generally  $\text{Mat}_{n,m}(\mathbb{C}) \cong \text{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ . Similarly, given a representation  $(V, \rho)$ , with  $\dim V < \infty$ , we may choose a basis  $B$  of  $V$  and write the representation as a matrix which we denote  $\rho^B(g) = [\rho(g)]_B$ . Thus, we may use first year linear algebra methods to manipulate representations.

**Definition 2.4.** Given two matrix representations  $\rho, \rho' : G \mapsto GL_n(\mathbb{C})$ , we say  $\rho$  and  $\rho'$  are equivalent/isomorphic if there exists  $P \in GL_n(\mathbb{C})$  such that for all  $g \in G$ ,  $\rho'(g) = P^{-1}\rho(g)P$ .

This definition is motivated by the following.

**Proposition 2.1.** Given  $(V, \rho_V)$  and  $(W, \rho_W)$  representations of  $G$ , we have  $\rho_V \cong \rho_W$  if and only if there exists some  $P \in GL_n(\mathbb{C})$  such that for all  $g \in G$ ,  $\rho_W^C(g) = P^{-1}\rho_V^B(g)P$  for some basis  $B, C$  of  $V$  and  $W$  respectively.

*Proof.* Exercise. □

**Proposition 2.2.** Given a cyclic group  $C_n = \langle g \rangle$  with representations  $(V, \rho_V)$  and  $(W, \rho_W)$  of equal dimensions, we have  $\rho_V \cong \rho_W$  if and only if  $\rho_V^B(g)$  is conjugate to  $\rho_W^C(g)$  for some basis  $B, C$  of  $V$  and  $W$  respectively.

*Proof.* Exercise. □

In fact the proposition above holds for the infinite cyclic group  $C_\infty \cong \mathbb{Z}$ .

## 2.1 Regular Representation

Let us first recall some definition about group actions though we will omit stabilizers, the orbit-stabilizer theorem and transitive actions (though it might be helpful to recall them from last year).

**Definition 2.5** (Group Action). Let  $G$  be a group and  $X$  a set, then a group action  $(\cdot)$  of  $G$  on  $X$  is a function  $G \times X \rightarrow X$  such that for all  $g, h \in G, x \in X$ , we have

- $g \cdot (h \cdot x) = gh \cdot x$ ,
- $1 \cdot x = x$ .

Equivalently, a group action can be represented by a group homomorphism between  $G$  to  $S_n$  if  $|X| = n < \infty$ . We note that there exists a bijection between  $\text{Perm}(X)$  (a.k.a  $\text{Aut}(X)$  though we will avoid this term in case  $X$  has additional structures) and  $S_n$  with depends on a choice of  $X \simeq \{1, \dots, |X|\}$ .

**Definition 2.6** (Kernel). A kernel of a representation (or group action) is simply the kernel of the corresponding group homomorphism, i.e. if  $\rho$  is a representation (or group action),

$$\ker \rho := \{g \in G \mid \rho(g) = \text{id}\}.$$

We say a representation (or group action) is faithful if  $\ker \rho = \{e\}$ , i.e.  $\rho$  is injective.

**Definition 2.7** (Morphism of Group Actions). A morphism  $T : X \rightarrow Y$  of group actions on  $X$  and  $Y$  is a map such that  $T(g \cdot x) = g \cdot T(x)$  for all  $g \in G, x \in X$ .

This is also called a “ $G$ -equivariant map” from  $X$  to  $Y$  and one can see the resemblance of this definition and the definition for homomorphisms between representations.

For any group  $G$ , it acts on itself in three different ways. In particular, we have the left regular action  $g \cdot h = gh$ , the right regular action  $g \cdot h = hg^{-1}$  (where the inverse is required for associativity) and the adjoint action  $g \cdot h = ghg^{-1}$ . One can see that the left and right regular actions are isomorphic via  $T(g) = g^{-1}$ . On the other hand, they are not isomorphic to the adjoint action (consider  $\rho_{\text{ad}}(g)(e) = e$  for all  $g \in G$ ).

**Proposition 2.3.** Given two actions (or representations)  $\rho, \rho'$  on  $G$ ,  $g \mapsto \rho(g)\rho'(g)$  is an action (or representation) if and only if  $\rho(g)\rho'(g) = \rho'(g)\rho(g)$ , that is  $\rho$  and  $\rho'$  are commuting actions.

**Definition 2.8.** A subset  $Y \subseteq X$  is said to be stable under an action  $(\cdot)$  of  $G$  on  $X$  if  $g \cdot y \in Y$  for all  $y \in Y, g \in G$ .

In the case that  $Y \subseteq X$  is stable, then we may restrict the action on  $Y$  to obtain a new action of  $G$  on  $Y$ .

**Definition 2.9** (Orbit). Let  $x \in X$ , then  $G \cdot x := \{g \cdot x \mid g \in G\}$  is called an orbit of  $x$  and we denote this by  $\text{orb}(x)$ .

It is not difficult to see that orbits are stable and in fact, as an exercise, one might show that  $Y \subseteq X$  is stable if and only if it is a union of orbits.

In a group  $G$  under the adjoint action, we see that the orbits are the conjugacy classes<sup>2</sup>. Thus, for every conjugacy class, we obtain an action on that class from the adjoint action on the whole group.

**Example 2.1.** Let  $G = S_4$  and let  $c = \{(12)(34), (13)(24), (14)(23)\}$ . Then as  $c$  is a conjugacy class, we have the adjoint action on  $c$

$$\phi : S_4 \rightarrow \text{Perm}(c) \cong S_3.$$

It is not difficult to show that  $\phi$  is surjective and  $\ker \phi = c \cup \{e\} \cong K_4 \cong C_2 \times C_2$ . Thus, by the first isomorphism theorem we have

$$S_3 \cong S_4/K_4.$$

**Definition 2.10.** Given a finite set  $X$ , let

$$\mathbb{C}[X] := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{C} \right\},$$

equipped with the addition  $\sum_{x \in X} a_x x + \sum_{x \in X} b_x x = \sum_{x \in X} (a_x + b_x)x$  and scalar multiplication  $c \cdot \sum_{x \in X} a_x x = \sum_{x \in X} (ca_x)x$ . This sum here does not represent some addition operation on  $X$  but a notational trick. One might instead consider elements of  $\mathbb{C}[X]$  as functions  $a : X \rightarrow \mathbb{C}$  equipped with point-wise addition and scalar multiplication.

We observe that  $\mathbb{C}[X] \cong \mathbb{C}^{|X|}$  depending on a choice of  $X \cong \{1, \dots, |X|\}$ . Furthermore, we have  $X \subseteq \mathbb{C}[X]$  and is a basis (if we interpret  $\mathbb{C}[X]$  as a space of functions, the canonical basis is  $\{a_x : y \mapsto \chi_{\{x\}} \mid x \in X\}$ ). In the case that  $X$  is infinite we can still define  $\mathbb{C}[X]$  allowing only finite sums.

**Proposition 2.4.** If  $(\cdot)$  is a group action of  $G$  on  $X$ , then, the map  $(g, \sum a_x x) \mapsto \sum a_x (g \cdot x)$  is a group action of  $G$  on  $\mathbb{C}[X]$ .

**Definition 2.11.** The left regular, right regular, adjoint representations are representations

$$\tilde{\rho}_L, \tilde{\rho}_R, \tilde{\rho}_{\text{ad}} : G \rightarrow GL(\mathbb{C}[G])$$

obtained from the left regular, right regular and adjoint actions

$$\rho_L, \rho_R, \rho_{\text{ad}} : G \rightarrow \text{Perm}(G).$$

**Proposition 2.5.** If  $X$  is any set with a  $G$ -actin, then for all  $g \in G$ ,  $[\rho_{\mathbb{C}[X]}(g)]_B$  is always a permutation matrix.

*Proof.* Exercise. □

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<sup>2</sup>What are the orbits of the left action?

**Definition 2.12.** Given  $(V, \rho_V), (W, \rho_W)$  representations of  $G$ , we denote

$$\text{Hom}_G(V, W) := \{T : V \rightarrow W \mid G\text{-linear}\}.$$

**Proposition 2.6.** Let  $(V, \rho_V)$  be a representation of  $G$  and  $v \in V$ . Then, there exists a unique homomorphism of representations  $\mathbb{C}[G] \rightarrow V$  where  $\mathbb{C}[G]$  is equipped with the left regular representation such that  $e_G \mapsto v$  and thus,

$$\text{Hom}_G(\mathbb{C}[G], V) \cong (V, \rho_V).$$

*Proof.* For all  $g \in G$ ,  $c = \sum_{h \in G} a_h h$ , we have

$$\begin{aligned} T(g \cdot c) = \rho_V(g)(Tc) &\iff T\left(\sum a_h gh\right) = \rho_V(g)\left(T\left(\sum a_h h\right)\right) \\ &\iff \sum a_h T(gh) = \sum a_h \rho_V(g)(Th) \\ &\iff T(gh) = \rho_V(g)(Th), \quad \forall h \in G, \end{aligned}$$

where the second if and only if follows as both  $T$  and  $\rho_V$  are linear. Then choosing  $h = e_G$ , we have  $T(g) = \rho_V(g)(v)$  and thus  $T$  is uniquely determined on  $G$  and hence is unique as  $G$  is a basis of  $\mathbb{C}[G]$ .

It remains to show that the map  $T$  defined by  $g \mapsto \rho_V(g)(v)$  is a homomorphism of representations. This is clear since

$$\begin{aligned} T(g \cdot c) &= T\left(\sum a_h gh\right) = \sum a_h T(gh) \\ &= \sum a_h \rho_V(gh)(v) = \sum a_h \rho_V(g)(\rho_V(h)(v)) \\ &= \sum a_h \rho_V(g)(Th) = \rho_V(g)\left(T\left(\sum a_h h\right)\right), \end{aligned}$$

where the fourth equality follows by the associativity of group actions.  $\square$

## 2.2 Subrepresentation and Quotient Representation

**Definition 2.13** (Subrepresentation). A subrepresentation of a representation  $(V, \rho_V)$  is a subspace  $W \leq V$  such that  $\rho_V(g)(W) \subseteq W$  for all  $g \in G$ .

Clearly, both  $\{0\}$  and  $V$  are subrepresentations of  $(V, \rho_V)$ , and we say a representation is irreducible if these two subrepresentations are the only subrepresentations. We say a representation is reducible if it is not irreducible. In general, every 1-dimension representation is irreducible.

**Proposition 2.7.** Irreducibility is invariant under isomorphisms.

*Proof.* Exercise.  $\square$

**Proposition 2.8.** Let  $G$  be finite and  $(V, \rho_V)$  is an irreducible representation of  $G$ . Then  $\dim V < \infty$ .

*Proof.* Let  $w \in V \setminus \{0\}$  and let  $W := \text{span}(\{\rho_V(g)(w) \mid g \in G\})$  which is a finite dimensional subrepresentation as  $G$  is finite and for all  $h \in G$ ,  $\rho_V(h)(\rho_V(g)(w)) = \rho_V(hg)(w)$ . Thus, if  $\dim V$  is not finite, we have found a proper subrepresentation which contradicts the irreducibility of  $(V, \rho_V)$ .  $\square$

**Definition 2.14** (Quotient Representation). For  $W \leq V$  a subrepresentation, the quotient representation is  $(V/W, \rho_{V/W})$  given by

$$\rho_{V/W}(g)(v + W) := \rho_V(g)(v) + W.$$

This is well-defined as  $W$  is stable under  $\rho_V$ .

**Proposition 2.9.** For  $T : (V, \rho_V) \rightarrow (W, \rho_W)$  a  $G$ -linear map,  $\ker T$  and  $\text{Im} T$  are subrepresentations.

*Proof.* Let  $v \in \ker T$ , then  $T(\rho_V(g)(v)) = \rho_W(g)(Tv) = \rho_W(g)(0) = 0$  implying  $\rho_V(g)(v) \in \ker T$  and thus,  $\ker T$  is a subrepresentation. On the other hand, for all  $w \in \text{Im} T$ , there exists some  $v \in V$  such that  $Tv = w$ . Then  $\rho_W(g)(w) = \rho_W(g)(Tv) = T\rho_V(g)(v)$  implying  $\rho_W(g)(w) \in \text{Im} T$  showing  $\text{Im} T$  is also a subrepresentation.  $\square$

**Proposition 2.10.** For  $T : (V, \rho_V) \rightarrow (W, \rho_W)$  a  $G$ -linear map, we have

$$\text{Im} T \cong V / \ker T.$$

*Proof.* Follows from the first isomorphism for vector spaces and it remains to check  $V / \ker T \rightarrow \text{Im} T$  is  $G$ -linear.  $\square$

**Proposition 2.11.** If  $T \in \text{End}_G V$  is a  $G$ -linear projection (i.e.  $T^2 = T$ ), then  $V$  is a direct sum of subrepresentations  $\ker T \oplus \text{Im} T$ .

*Proof.* Follows from the vector space case.  $\square$

### 2.3 Maschke's Theorem and Schur's Lemma

Recalling internal and external direct sums of vector spaces, we will in this section introduce and prove a powerful result in representation theory known as the Maschke's theorem.

**Definition 2.15** (Decomposable). The representation  $(V, \rho_V)$  is decomposable if there exists a decomposition  $V = U \oplus W$  where  $U, W$  are non-zero subrepresentations.

**Definition 2.16** (Semisimple). The representation  $(V, \rho_V)$  is semisimple if there exists a irreducible subrepresentations  $W_1, \dots, W_n$  such that

$$V = \bigoplus_{i=1}^n W_i.$$

**Theorem 1** (Maschke's Theorem). If  $G$  is finite, then for all  $W \leq V$  subrepresentations of  $(V, \rho_V)$ , there exists a complementary subrepresentation  $U$ ,  $V = W \oplus U$ .

A direct consequence of Maschke's theorem is that every finite-dimensional representation of  $G$  is semisimple.

Maschke's theorem does not hold in the case that  $G$  is not finite. Consider  $G = \mathbb{Z}$  and let

$$\rho : g \rightarrow GL_2(\mathbb{C}) : m \mapsto \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

Then the only non-zero proper subrepresentation is  $\text{span}\{e_1\}$  since  $e_1$  is the only eigenvector and as  $\rho$  is a 2-dimensional representation, the only non-zero proper subrepresentation is 1-dimensional, hence an eigenspace. Thus,  $\rho$  is indecomposable but not irreducible, and hence not semisimple.

*Proof of Maschke's Theorem.* To prove the theorem, we will attempt to find some  $G$ -linear map  $T : V \rightarrow V$  that is a projection, i.e.  $T^2 = T$ , such that  $\text{im}T = W$  and so  $V = \ker T \oplus \text{Im}T = \ker T \oplus W$ .

In the case of linear maps, a map satisfying the above proposition must map

$$T(u + w) = Tu + Tw = 0 + w = w,$$

where  $u \in \ker T$  and  $w \in W$ . As,  $\ker T \oplus W = V$ , this property uniquely identifies  $T$  on  $V$ . However, this map is not  $G$ -linear and so we will modify  $T$  such that it is  $G$ -linear.

By recalling that a linear map is  $G$ -linear if and only if it is conjugate with it self by  $\rho(g)$  for all  $g \in G$ , let us define

$$\tilde{T} := \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1},$$

such that it is in some sense the average of all conjugates over all  $g$ .

We will now show that  $\tilde{T}$  is a  $G$ -linear projection and  $\text{im}\tilde{T} = W$ . Indeed, for all  $h \in G$ , we have

$$\rho_V(h) \circ \frac{1}{|G|} \left( \sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1} \right) \circ \rho_V(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg) \circ T \circ \rho_V(hg)^{-1} = \tilde{T},$$

as  $g \mapsto h \cdot g$  is bijective as it has the inverse  $g \mapsto h^{-1}g$ . On the other hand, it is clear that  $\tilde{T}(V) \subseteq W$  as for all  $v \in V$ ,  $T(\rho_V(g)^{-1}v) \in W$ , and as  $W$  is a subrepresentation, we have  $\rho_V(g)T(\rho_V(g)^{-1}v) \in W$ . Thus, as  $\tilde{T}|_W = \text{id}|_W$ , since

$$\tilde{T}w = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)T(\rho_V(g)^{-1}w) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)\rho_V(g)^{-1}w = \frac{1}{|G|} \sum_{g \in G} w = w,$$

we have  $\text{Im}\tilde{T} = W$  and  $\tilde{T}$  is a projection.  $\square$

We note that we used the property of  $\mathbb{C}$  precisely when we needed  $|G|^{-1}$  and thus, the same proof works for all field in which  $|G|$  is invertible, i.e. of characteristic not a factor of  $|G|$ .

This decomposition needs not be unique. Indeed, if  $V, \rho_V$  is a trivial representation with dimension  $> 1$ . Then any vector space decomposition is a decomposition of representations. For example, if  $V = \mathbb{C}^2$ , then

$$\mathbb{C}^2 = \{(a, 0)\} \oplus \{(0, b)\} = \{(a, a)\} \oplus \{(b, -b)\},$$

and in fact any pair of subspaces spanned by two linearly independent basis form a decomposition.



**Proposition 2.12.** For  $G = C_m = \langle g \rangle$ ,  $(V, \rho_V)$  a representation of  $G$ , there exists a unique decomposition of  $(V, \rho_V)$  into 1-dimensional subrepresentations if and only if  $\rho_V(g)$  has distinct eigenvalues.

*Proof.* Exercise. □

**Lemma 2.1** (Schur's Lemma). Let  $V$  and  $W$  be irreducible representations of  $G$ , then

- every  $G$ -linear map  $T : V \rightarrow W$  is invertible or zero;
- let  $V = W$  be finite-dimensional, then every  $G$ -linear map  $T : V \rightarrow V$  is a multiple of the identity, i.e.  $\text{End}_G V = \mathbb{C} \cdot \text{id}$ .

We note that the first property does not require  $V, W$  to be finite dimensional, and in fact it is true for arbitrary fields. On the other hand the second property only works for algebraically closed fields.

*Proof.* The first property is rather trivial. Indeed, if  $T \neq 0$  then  $\ker T$  must be  $\{0\}$  as  $\ker T$  is a subrepresentation and  $V$  is irreducible. Similarly, for the same reason  $\text{Im} T = W$  and thus,  $T$  is bijective.

For the second property, we recall that the eigenvalues of  $T$  are the roots of the characteristic polynomial of  $T$ . Now since  $\mathbb{C}$  is algebraically closed, there exists some  $\lambda \in \mathbb{C}$  such that  $\ker(\lambda I - T) \neq \{0\}$ . Now since  $\ker(\lambda I - T)$  is a subrepresentation of  $V$ , as  $V$  is irreducible, we have  $\ker(\lambda I - T) = V$ . Thus, for all  $v \in V$ ,

$$\lambda v - Tv = 0 \implies Tv = \lambda v \implies T = \lambda \cdot \text{id}$$

as required. □

**Theorem 2.** Up to isomorphism (and reordering), the representation decomposition is unique. That is, if  $T : V := V_1 \oplus \cdots \oplus V_m \cong W := W_1 \oplus \cdots \oplus W_n$ , then  $V_i \cong W_i$  up to ordering.

*Proof.* We have  $T : V \rightarrow W$  is a  $G$ -isomorphism map and so  $T(V_i)$  is a subrepresentation of  $W$ . Then, as  $W = W_1 \oplus \cdots \oplus W_n$ , there exists some  $j$  such that  $W_j \cap T(V_i) \neq \emptyset$ . Thus, we have  $T|_{V_i} : V_i \rightarrow W_j$  is a  $G$ -linear map between two irreducible representations. As  $T|_{V_i} \neq 0$ , by Schur's lemma, it follows  $T(V_i) = W_j$ . Now, since for  $i \neq k$ ,  $T(V_i) \neq T(V_k)$  as  $T$  is bijective and  $V_i \cap V_k = \emptyset$ , by pairing the  $V_i$  and  $W_j$ , we are able to correspond each  $V_i$  with a  $W_j$ . Reversing this process with  $T^{-1}$ , we are able to pair each  $W_j$  with a  $V_i$  and thus, we have  $V_i \cong W_i$  up to ordering as required. □

**Theorem 3.** In the case that  $V$  is finite dimensional, there is a unique decomposition  $V = \bigoplus_{i=1}^n V_m$  (up to reordering) if and only if in some decomposition,  $V_i$  are all non-isomorphic.

*Proof.* Suppose first that  $V_1 \oplus \cdots \oplus V_n = V$ , and  $V_1, \dots, V_n$  are all non-isomorphic. Then, for all  $G$ -linear maps  $T : V_i \rightarrow V$ , we have by a similar argument as above, if  $T \neq 0$ , there exists some  $j$  such that  $T : V_i \cong V_j$ . But as  $V_i, V_j$  are non-isomorphic for  $i \neq j$ , we have  $T \in \text{End}_G V_i$ . Thus, by the second part of Schur's lemma, there exists some  $\lambda$  such

that  $T = \lambda \cdot \text{id} \mid_{V_i}$ . Now, if  $V = W_1 \oplus \dots \oplus W_m$ , by the above theorem, WLOG, we have  $T_i : V_i \cong W_i \subseteq V$ . But we just shown  $T_i = \lambda \cdot \text{id} \mid_{V_i}$  and so,  $V_i = W_i$  and the decomposition is unique.

Conversely, consider that for any representation  $V_i$ , there exists infinitely many subrepresentations of  $V_i \oplus V_i$  by taking

$$V_a := \{(v, av) \mid v \in V\}.$$

Thus, if  $V_i \cong V_j$ , we have

$$V_a \leq V_i \oplus_{\text{ext}} V_i \cong V_i \oplus_{\text{ext}} V_j \cong V_i \oplus_{\text{int}} V_j.$$

Denoting the isomorphism from  $V_i \oplus_{\text{ext}} V_i$  to  $V_i \oplus_{\text{int}} V_j$  as  $T$ , we have  $T(V_a) \leq V_i \oplus_{\text{int}} V_j$  and by Maschke's theorem, there exists some subrepresentation  $U$  such that,  $U$  is complement to  $T(V_a)$  and

$$T(V_a) \oplus_{\text{int}} U = V_i \oplus_{\text{int}} V_j.$$

Thus, as  $T$  is an isomorphism,  $a \neq b$  implies  $T(V_a) \neq T(V_b)$ , we have found infinitely many non-equal decompositions of  $V_i \oplus_{\text{int}} V_j$  and hence, also  $V$ .  $\square$

**Corollary 3.1.** For  $V_1, \dots, V_n$  irreducible, non-isomorphic, subrepresentations of  $V$  such that  $V = V_1 \oplus \dots \oplus V_n$  all subrepresentations of  $V$  are of the form  $V_{i_1} \oplus \dots \oplus V_{i_m}$ . In particular  $V$  has  $2^n$  different subrepresentations.

### 2.3.1 Representation of Abelian Groups

**Definition 2.17** (Centre). The centre of a group  $G$  is the subgroup with the underlying set

$$Z(G) := \{z \in G \mid zg = gz, \forall g \in G\}.$$

**Proposition 2.13.** Let  $(V, \rho_V)$  be a finite dimensional irreducible representation and let  $z \in Z(G)$ . Then  $\rho_V(z)$  is a scalar, i.e.  $\rho_V(z) = \lambda \cdot \text{id}_V$  for some  $\lambda \in \mathbb{C}$ .

*Proof.* For all  $g \in G$ , we have  $\rho_V(z)\rho_V(g) = \rho_V(zg) = \rho_V(gz) = \rho_V(g)\rho_V(z)$ . So  $\rho_V(z) : V \rightarrow V$  is  $G$ -linear. Thus, by the second property of Schur's lemma, there exists some  $\lambda$  such that  $\lambda \cdot \text{id}_V$ .  $\square$

**Corollary 3.2.** If  $G$  is abelian and if  $(V, \rho_V)$  is an irreducible representation of  $G$ , then  $\dim V = 1$ .

*Proof.* As for abelian groups  $G$ ,  $Z(G) = G$ , we have  $\rho_V(g) = \lambda_g \cdot \text{id}_V$  for all  $g \in G$ . Then, for all subspaces  $W$  of  $V$ ,  $W$  is a subrepresentation. However, as  $V$  is irreducible,  $W$  must either be zero or  $V$ , and thus,  $\dim V = 1$ .  $\square$

**Corollary 3.3.** If  $G$  is finite and abelian, then every finite dimensional representation is a direct sum of 1-dimensional subrepresentations.

The finite dimensional condition is necessary. Indeed, if  $F$  be a field  $C \subsetneq F$  (then  $\dim F = \infty$ ), e.g.

$$F := \left\{ \frac{P(X)}{Q(X)} \mid P, Q \in \mathbb{C}[X], Q \neq 0 \right\},$$

then  $F$  is an irreducible  $\infty$ -dimensional representation over  $\mathbb{C}$  of  $G = F^\times$  although the latter is abelian.

**Corollary 3.4.** The irreducible representations of  $C_m$  are up to isomorphism  $(\mathbb{C}, \rho_\zeta)$  where  $\zeta$  is a  $m$ -th root of unity. Furthermore, the irreducible representations of  $C_{m_1} \times \cdots \times C_{m_l}$  are up to isomorphism,

$$\rho_{\zeta_1, \dots, \zeta_l}(g_1^{j_1}, \dots, g_l^{j_l}) = (\zeta_1^{j_1}, \dots, \zeta_l^{j_l}).$$

The above corollary is important as every finite abelian group is an internal direct product of cyclic groups, we have classified all irreducible representations of finite abelian groups. So, if  $G$  is finite abelian, then it has  $|G|$  number of 1-dimensional representations.

**Proposition 2.14.** Every 1-dimensional representation of a group  $G$  is of the form  $(V, \rho_V)$ ,  $\rho_V(g) = \lambda_g \text{id}_V$  for some  $\lambda_g \in \mathbb{C}$ . Furthermore, two such representations are isomorphic if and only if the  $\lambda_g$  are the same.

*Proof.* The first statement is clear while the second state follows since, if  $(V, \rho_V : g \mapsto \lambda_g \text{id}_V)$ ,  $(W, \rho_W : g \mapsto \lambda'_g \text{id}_W)$  are two isomorphic 1-dimensional representations of  $G$  along  $T : V \rightarrow W$ , then given some  $v \in V \setminus \{0\}$ , we have  $T(v) \neq 0$  and

$$\lambda_g T(v) = T(\lambda_g v) = T(\rho_V(g)(v)) = \rho_W(g)(T(v)) = \lambda'_g T(v).$$

Thus,  $\lambda_g = \lambda'_g$ . □

As all 1-dimensional vector spaces over  $\mathbb{C}$  are isomorphic to  $GL_1(\mathbb{C}) \cong \mathbb{C}^\times$ , we can simply only consider  $\mathbb{C}^\times$  as the vector space of all 1-dimensional representations.

Consider the 1-dimensional representations on  $S_n$ . Let  $(\mathbb{C}^\times, \rho)$  be a representation of  $S_n$ , then by considering  $\sigma(a_1, \dots, a_n)\sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_n))$ , we have

$$(ij) = ((1i)(2j))(12)((2j)(1i)),$$

and so  $(ij)$  is conjugate to  $(12)$  for all  $i, j$ . Furthermore, by considering  $\rho(ij)^2 = \rho((ij)^2) = \rho(e) = \text{id}$ ,  $\rho(ij) = \pm \text{id}$ . Now, by recalling that  $\sigma = \tau\sigma'\tau^{-1} \iff \text{sign}(\sigma) = \text{sign}(\sigma')$ , and the fact that every permutation can be represented as a product of transpositions, either,  $\rho(\sigma) = \text{sign}(\sigma) \cdot \text{id}$  or  $\rho(\sigma) = \text{id}$ . Thus, there exists exactly two 1-dimensional representation up to isomorphism.

## 2.4 Abelianisation

We have so far showed that if  $G$  is abelian, then every finite dimensional representation of  $G$  must be 1-dimensional (and in fact, we may drop the finite dimensional condition if  $G$  is finite). Now we would like to reduce the study of 1-dimensional representations of any group  $G$  to that of an abelian group.

**Definition 2.18** (Commutator). Given a group  $G$ , the commutator subgroup is the subgroup  $[G, G]$  generated by  $[g, h] := ghg^{-1}h^{-1}$  for all  $g, h \in G$ .

**Proposition 2.15.** Given a group  $G$ , then

- $[G, G] \trianglelefteq G$  ( $[G, G]$  is a normal subgroup of  $G$ );
- $G/[G, G]$  is abelian;
- for a normal subgroup  $N \trianglelefteq G$ ,  $G/N$  is abelian if and only if  $[G, G] \leq N$ .

*Proof.*

- For all  $g, h_1, h_2 \in G$ , we have

$$g[h_1, h_2]g^{-1} = [gh_1g^{-1}, gh_2g^{-1}] \in [G, G].$$

- Denote  $\bar{g} := g[G, G]$ , then  $\bar{g}\bar{h} = gh[G, G]$ , now since  $h^{-1}g^{-1}hg \in [G, G]$ , we have

$$\bar{g}\bar{h} = gh[G, G] = gh(h^{-1}g^{-1}hg)[G, G] = hg[G, G] = \bar{h}\bar{g}.$$

- If  $G/N$  is abelian, then

$$gNhN = hNgN \implies N = gNhNg^{-1}h^{-1}N = ghg^{-1}h^{-1}N,$$

thus,  $ghg^{-1}h^{-1} \in N$  and so,  $[G, G] \leq N$ .

□

**Definition 2.19** (Abelianisation). The abelianisation of a group  $G$  is the abelian group  $G_{ab} := G/[G, G]$ .

We note that as for all normal subgroup  $N$  for which  $G/N$  is abelian, we have  $[G, G] \trianglelefteq N \trianglelefteq G$ , by the third isomorphism theorem, we have

$$\frac{G_{ab}}{N/[G, G]} \equiv \frac{G/[G, G]}{N/[G, G]} \cong \frac{G}{N}.$$

**Proposition 2.16.** Let  $A$  be an abelian group, the map

$$\text{Hom}(G_{ab}, A) \rightarrow \text{Hom}(G, A) : \phi \mapsto \phi \circ q_{ab},$$

is a bijection, where  $q_{ab} : G \mapsto G_{ab}$  is the quotient map.

In other words, every homomorphism  $\psi : G \rightarrow A$  uniquely corresponds to the homomorphism  $\phi : G_{ab} \rightarrow A$  such that  $\psi = \phi \circ q_{ab}$ .

With this proposition, we see that for all normal subgroups  $N \trianglelefteq G$  such that  $G/N$  is abelian, we have the following diagram.

$$\begin{array}{ccc} G & \xrightarrow{q_N} & G/N \\ q_{ab} \downarrow & \nearrow \exists! & \\ G_{ab} & & \end{array}$$

*Proof.* Suppose that  $\phi : G \rightarrow A$  is a homomorphism, then  $\phi([g, h]) = \phi(ghg^{-1}h^{-1}) = e_A$ , and thus,  $[G, G] \leq \ker \phi$ . Hence, using the universal property of quotient, there exists

$$\bar{\phi} : G_{ab} \rightarrow A : g[G, G] = \phi(g).$$

By construction,  $\bar{\phi} \circ q_{ab} = \phi$ , and hence, the map is surjective.

On the other hand, if  $\phi \circ q_{ab} = \psi \circ q_{ab}$ , then for all  $g \in G$ ,  $\phi(g) = (\phi \circ q_{ab})(g) = (\psi \circ q_{ab})(g) = \psi(g)$  and so  $\phi = \psi$ . Thus, the map is bijective as required.  $\square$

**Corollary 3.5.** If  $|G_{ab}| < \infty$ , then the number of 1-dimensional representations of  $G$  equals  $|G_{ab}|$ .

*Proof.* Follows as the 1-dimensional representations of  $G$  are  $\text{Hom}(G, GL_1(\mathbb{C}))$  which bijects  $\text{Hom}(G_{ab}, GL_1(\mathbb{C}))$ , and as  $G_{ab}$  is abelian, it has  $|G_{ab}|$  number of 1-dimensional representations.  $\square$

**Corollary 3.6.** If  $|G| < \infty$ , then the number of 1-dimensional representations of  $G$  divides  $|G|$ .

*Proof.* Follows as  $|G_{ab}| \mid |G|$  by Lagrange's theorem.  $\square$

In some sense we have found a bijection between  $G_{ab}$  and the set of 1-dimensional representations of  $G$ . However, while  $G_{ab}$  is a group, the 1-dimensional representations of  $G$  are not. In fact, there is a natural group structure on the 1-dimensional representations of  $G$ , and for  $G_{ab}$  finite, the two groups are isomorphic (exercise).

**Proposition 2.17.** If  $G$  is simple nonabelian, then  $G_{ab} = \{e\}$ .

*Proof.* Clear as the only normal subgroups of a simple groups are the trivial subgroup or the entire group.  $\square$

**Definition 2.20** (Perfect). A group  $G$  is perfect if  $[G, G] = G$  (or equivalently  $G_{ab} = \{e\}$ ).

From the above proposition, we see that simple nonabelian implies perfect. Furthermore,  $G$  is perfect implies all 1-dimensional representations of  $G$  are trivial. In the case the  $G$  is finite, the converse is also true.

## 2.5 Decomposition of Representations

### 2.5.1 Regular Representation

Recall that for linear maps, the space of all linear maps between two vector spaces form a linear map and we have the following isomorphisms.

**Lemma 2.2.**

$$\text{Hom}(V, W_1 \oplus W_2) \cong \text{Hom}(V, W_1) \oplus \text{Hom}(V, W_2),$$

and similarly,

$$\text{Hom}(V_1 \oplus V_2, W) \cong \text{Hom}(V_1, W) \oplus \text{Hom}(V_2, W).$$

*Proof.* Clear using the natural bijections.  $\square$

Similarly to the situation for vector spaces, the space of representations also form a vector space and similar isomorphisms hold. In particular,

$$\mathrm{Hom}_G(V, W_1 \oplus W_2) \cong \mathrm{Hom}_G(V, W_1) \oplus \mathrm{Hom}_G(V, W_2),$$

and

$$\mathrm{Hom}_G(V_1 \oplus V_2, W) \cong \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W),$$

where the space  $(\mathrm{Hom}_G(V, W), \rho_{\mathrm{Hom}(V, W)})$  is defined such that

$$\rho_{\mathrm{Hom}(V, W)}(g)(\phi) = \rho_W(g) \circ \rho \circ \rho_V(g)^{-1}.$$

If  $G$  is finite, By Maschke's theorem,  $\mathbb{C}[G]$  is finite dimensional, and so,  $\mathbb{C}[G]$  is always decomposable into irreducible representations. We will now prove some properties about this decomposition.

**Proposition 2.18.** Let  $V_1, \dots, V_m$  be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m},$$

where  $W^k = W \oplus \dots \oplus W$   $k$ -times. Then,  $\dim V_i = r_i$ .

*Proof.* Recall that  $\mathrm{Hom}_G(\mathbb{C}[G], V) \cong V$  by the bijection  $\phi \mapsto \phi(e)$ . So, by this and the above isomorphism,

$$V_i \cong \mathrm{Hom}_G(\mathbb{C}[G], V_i) \cong \bigoplus_{j=1}^m \mathrm{Hom}_G(V_j, V_i)^{r_j}.$$

By Schur's lemma, for all  $i \neq j$ , we have  $\mathrm{Hom}_G(V_i, V_j) = 0$  and so,

$$\bigoplus_{j=1}^m \mathrm{Hom}_G(V_j, V_i)^{r_j} \cong (\mathbb{C} \cdot \mathrm{Id}_{V_i})^{r_i} \cong \mathbb{C}^{r_i}.$$

Thus, we have  $\dim V_i = r_i$ . □

**Corollary 3.7.** Let  $V_1, \dots, V_m$  be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m}.$$

Then,  $r_i = 1$  for all  $i$  if  $G$  is abelian.

As we shall see, converse is also true.

**Corollary 3.8.** Let  $V_1, \dots, V_m$  be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m},$$

then

$$|G| = \sum_{i=1}^m (\dim V_i)^2.$$

*Proof.*

$$|G| = \dim \mathbb{C}[G] = \dim(V_1^{r_1} \oplus \dots \oplus V_m^{r_m}) = \dim(V_1^{\dim V_1} \oplus \dots \oplus V_m^{\dim V_m}) = \sum_{i=1}^m (\dim V_i)^2.$$

□

This is the sum of squares formula though it provided that every irreducible representation of  $G$  is isomorphic to exactly one of the  $V_i$ .

**Proposition 2.19.** Let  $V_1, \dots, V_m$  be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \dots \oplus V_m^{r_m}.$$

Then for all irreducible representations  $W$  of  $G$ , there exists uniquely  $i$  such that  $W \cong V_i$ .

*Proof.* By assumption  $V_i \not\cong V_j$  for  $i \neq j$  and so, it remains to show  $W$  must be isomorphic at some  $V_i$ . Suppose otherwise, then,

$$W \cong \text{Hom}_G(\mathbb{C}[G], W) \cong \bigoplus_i \text{Hom}_G(V_i, W) = 0,$$

by Schur's lemma, which is a contradiction.

□

**Corollary 3.9.** For non-trivial  $G$ , every irreducible representation has dimension strictly less than  $\sqrt{|G|}$ .

*Proof.* Given  $(V, \rho)$  a non-trivial representation,

$$|G| = \dim \mathbb{C}[G] = \dim(\mathbb{C} \oplus V^{\dim V} \oplus \dots) \geq \dim(\mathbb{C} \oplus V^{\dim V}) = 1 + (\dim V)^2.$$

□

### 2.5.2 Revisiting Maschke's Theorem

**Definition 2.21** ( $G$ -invariant Subspace). Given a representation  $(V, \rho_V)$  of  $g$ , then the  $G$ -invariant subspace of  $V$  is

$$V^G := \{v \in V \mid \rho_V(g)(v) = v, \forall g \in G\}.$$

A useful note is that,  $V^G$  is the intersection of all eigenspaces of  $\rho_V(g)$  of eigenvalue 1 for all  $g$ .

**Lemma 2.3.**  $V^G \subseteq V$  is the largest trivial subrepresentation of  $V$ .

*Proof.*  $V^G$  is a trivial subrepresentation as it sends every element of  $V^G$  to itself and thus,  $\rho_{V^G}(g) = \text{Id}_{V^G}$  for all  $g \in G$ . Lastly, if  $W \subseteq V$  is a trivial subrepresentation, for all  $v \in W$ ,  $\rho_W(g)(v) = \text{Id}_W v = v$  for all  $g \in G$ , and so  $v \in V^G$  and hence,  $W \subseteq V^G$ . □

**Definition 2.22** (Averaging Map). If  $G$  is finite and  $(V, \rho_V)$  is a representation of  $G$ , then the averaging map is the function

$$\text{avg} : V \rightarrow V : v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v).$$

**Proposition 2.20.** The averaging map is a  $G$ -linear projection with the image  $V^G$ .

*Proof.* Clearly,  $\text{Im}(\text{avg}) \subseteq V^G$  since for all  $h \in G, v \in V$ ,

$$\rho(h)(\text{avg}(v)) = \rho(h) \left( \frac{1}{|G|} \sum_{g \in G} \rho(g)(v) \right) = \frac{1}{|G|} \sum_{g \in G} \rho(hg)(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(v) = \text{avg}(v).$$

Furthermore, as  $\text{avg}|_{V^G} = \text{Id}_{V^G}$ , we have  $\text{Im}(\text{avg}) = V^G$  and thus,  $\text{avg}$  is a projection. Finally, by the same calculation, we have

$$\rho(h)(\text{avg}(v)) = \frac{1}{|G|} \sum_{g \in G} \rho(hg)(v) = \frac{1}{|G|} \sum_{g \in G} \rho(gh)(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(\rho(h)(v)) = \text{avg}(\rho(h)(v)),$$

which shows  $\text{avg}$  is a  $G$ -linear map.  $\square$

With this proposition, we obtain that  $V = \ker(\text{avg}) \oplus V^G$  with  $\ker(\text{avg})$  being a subrepresentation with no  $G$ -invariant vectors, i.e.  $\ker(\text{avg})^G = \{0\}$ . With these definitions in mind, let us revisit Maschke's theorem.

**Proposition 2.21.**  $\text{Hom}_G(V, W) = \text{Hom}(V, W)^G$ .

*Proof.* Definition check.  $\square$

**Corollary 3.10.** The map

$$\text{avg} : \text{Hom}(V, W) \rightarrow \text{Hom}(V, W)^G = \text{Hom}_G(V, W)$$

is a  $G$ -linear projection.

Thus, with this corollary, we see that the averaging map provides a ( $G$ -linear) map between  $\text{End}(V)$  and  $\text{End}_G(W)$ . Nonetheless, to obtain Maschke's theorem, it remains to show that it  $\text{avg}$  maps linear projections to  $G$ -linear projections (with the same image).

### 2.5.3 Adjoint Representation

Consider  $V_1, \dots, V_m$  be all irreducible representations (up to isomorphism) of  $G$  with  $|G|$ . Then, we may define the map

$$G \rightarrow \bigoplus_{i=1}^m \text{End}(V_i) : g \mapsto (\rho_{V_1}(g), \dots, \rho_{V_m}(g)).$$

Linearly extending, we obtain a map  $\phi : \mathbb{C}[G] \rightarrow \bigoplus \text{End}(V_i)$ . Then, by the sum of squares formula, since  $\dim(\text{End}(W)) = (\dim W)^2$ , the source and target of  $\phi$  have the same dimension.



**Proposition 2.22.** The map  $\phi : \mathbb{C}[G] \rightarrow \bigoplus \text{End}(V_i)$  is a linear isomorphism and is  $G$ -linear with respect to the adjoint representation.

*Proof.* It suffices to show  $G$ -linearity. For all  $h \in G$ , we have

$$\begin{aligned} \phi(\rho_{ad}(g)(h)) &= \phi(ghg^{-1}) = (\rho_{V_1}(ghg^{-1}), \dots, \rho_{V_m}(ghg^{-1})) \\ &= (\rho_{\text{End}(V_1)}(g)(\rho_{V_1}(h)), \dots, \rho_{\text{End}(V_m)}(g)(\rho_{V_m}(h))) \\ &= \rho(g)(\phi(h)). \end{aligned}$$

□

Thus, we have found an isomorphism  $\mathbb{C}[G]_{ad} \cong \bigoplus \text{End}(V_i)$  and thus, taking the dimension of both sides, we recover the sum of squares formula  $|G| = \sum \dim(V_i)^2$ . In particular, we have replaced an equality by an isomorphism and in general, this process is called “categorification”.

Continuing with  $V_1, \dots, V_m$  as the irreducible representations of  $G$ ,  $|G| < \infty$  up to isomorphism, we have the following corollaries.

**Corollary 3.11.** Since,

$$\mathbb{C}[G]_{ad} \cong \bigoplus_{i=1}^m \text{End}(V_i).$$

By taking  $G$ -invariant subspaces, we have

$$\mathbb{C}[G]_{ad}^G \cong \bigoplus_{i=1}^m \text{End}(V_i)^G = \bigoplus \text{End}_G(V_i) = \bigoplus \mathbb{C} \cdot \text{Id}_{V_i} \cong \mathbb{C}^m$$

by Schur’s lemma.

**Proposition 2.23.**  $(\mathbb{C}[G], \rho_{ad})^G$  have a basis  $f_1, \dots, f_{m'}$  given by

$$f_i = \sum_{g \in \mathcal{C}_i} g,$$

where  $\mathcal{C}_1, \dots, \mathcal{C}_{m'}$  are the conjugacy classes of  $G$ .

*Proof.* Follows by considering  $\sum a_g g \in \mathbb{C}[G]_{ad}^G \iff a_h = a_{ghg^{-1}}$  for all  $g, h \in G$ . □

**Corollary 3.12.** The number of conjugacy classes of  $G$  equals the number of irreducible representation up to isomorphisms.

*Proof.* Follows by taking the dimensions on both sides of the isomorphism  $\mathbb{C}[G]_{ad}^G \cong \mathbb{C}^m$ . □