Algebraic Topology

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Contents

1 Introduction

Let us introduce/recall some basic definitions which will be used throughout this course.

Definition 1.1 (Path)**.** A path in a topological space X is a continuous map $\gamma : [0,1] \subseteq \mathbb{R} \to X$. In the case that $\gamma(0) = \gamma(1)$, we call γ a loop/closed path.

Definition 1.2 (Homotopy)**.** Given two paths $\gamma_0, \gamma_1 : [0,1] \to X$ with the same end points (i.e. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are said to be homotopic with fixed endpoints if there exists a continuous map

$$H : [0,1] \times [0,1] \to X$$

such that

- $H(t,0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t,1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0,u) = \gamma_0(0) = \gamma_1(0)$ and $H(1,u) = \gamma_0(1) = \gamma_1(1)$.

Thus, graphically, two paths are homotopic if you can continuously deform a path into the other without moving the starting and ending points (see second year complex analysis for more details).

Definition 1.3 (Free Homotopy)**.** The loops γ_0, γ_1 in X is said to be freely homotopic if there exists a continuous $H : [0,1] \times [0,1] \to X$ such that

- $H(t,0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t,1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0,u) = H(1,u)$.

Definition 1.4 (Simply Connected)**.** X is said to be simply connected if any loop in X is freely homotopic to a constant loop.

Thus, informally, in a simply connected space, any loop can be contracted into a single point.

Proposition 1.1. S^2 is simply connected.

Simply connectedness is a important notion and relates to many difficult problems in geometry.

Theorem 1. S^2 and $\mathbb{R}^2$ are, up to homeomorphism, the only two simply-connected 2-dimensional manifolds.

Theorem 2 (Poincaré Conjecture)**.** The only compact, simply connected 3-dimensional manifold is the sphere S^3 (up to homeomorphism).

1.1 The Torus

An important example in algebraic topology is the torus. We will now provide a proof that the torus is not simply connected.

Definition 1.5 (Torus)**.** The 2-torus T^2 is the product topological space $S^1 \times S^1$.

We will now provide an alternative method of constructing the torus. Define the homeomorphisms

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1 + 1, x_2); T_2 : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1, x_2 + 1).$$

It is clear that T_1 and T_2 commutes and the map

$$\psi : \mathbb{Z}^2 \to \mathrm{Homeo}(\mathbb{R}^2) : (n, m) \mapsto T_1^n \circ T_2^m$$

is a group homomorphism. As this map is injective, we have in some sense embedded $\mathbb{Z}$ inside of $\mathrm{Homeo}(\mathbb{R}^2)$. Now, defining the equivalence relation on $\mathbb{R}^2$ by

$$(x_1, x_2) \sim (y_1, y_2) \iff \exists (n, m) \in \mathbb{Z}^2, \psi(n, m)(x_1, x_2) = (y_1, y_2),$$

or equivalently, there exists $(n, m) \in \mathbb{Z}^2$ such that $(x_1 + n, x_2 + m) = (y_1, y_2)$, we define the quotient topology $X := \mathbb{R}^2 / \sim$.

Lemma 1.1. Let X and Y be topological spaces and $\sim$ be an equivalence relation on X. Then, if $p : X \to X/\sim : x \mapsto [x]_\sim$ is the quotient map, any map $f : X/\sim \to Y$ is continuous if and only if $f \circ p$ is continuous.

Proof. Exercise. □

We see that the above lemma together with the universal property for quotients provides the universal property for topological spaces. Namely, if $f : X \to Y$ is a continuous map and for all $x \sim y \in X$, $f(x) = f(y)$, then the unique map $\tilde{f}$ obtained such that $\tilde{f} \circ p = f$ is continuous.

Lemma 1.2. Let X be a compact space and Y Hausdorff. Then, any continuous bijective map $f : X \to Y$ is a homeomorphism.

Proof. Exercise. □

Proposition 1.2. X is homeomorphic to the torus T^2.

Proof. Define the map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ such that $\pi(x, y) = (e^{2\pi i x}, e^{2\pi i y})$ for all $(x, y) \in \mathbb{R}^2$. We observe that $\pi(x, y) = \pi(u, v)$ if and only if $(x, y) \sim (u, v)$, and so, by the universal property for topological spaces, we obtain the unique continuous map defined by

$$\tilde{\pi} : X \to S^1 \times S^1 : [(x, y)] \mapsto \pi(x, y).$$

This map is clearly bijective and continuous by the universal property, and thus, by the above lemma, it suffices to show X is compact. But, this is clear since the $p([0, 1]^2) = \mathbb{R}^2 / \sim$ and the continuous image of a compact set is compact. □

While the map π as described above is not injective, it is locally so (exercise). Thus, given a path on the torus, we may think of lifting it to $\mathbb{R}^2$ by lifting the paths piecewise via. the local homeomorphisms induced by π.

Lemma 1.3 (Pasting Lemma)**.** Let X, Y be both open subsets of a topological space and let B be another topological space, then $f : X \cup Y \to B$ is continuous if and only if $f\mid_A$ and $f\mid_B$ are continuous.

Proof. Exercise. $\square$

Corollary 2.1. If $f_1 : X \to B, f_2 Y \to B$ are continuous and agree on $X \cap Y$, then the map

$$f : X \cup Y \to B : x \mapsto \begin{cases} f_1(x) & \text{if } x \in X \\ f_2(x) & \text{otherwise} \end{cases}$$

is continuous.

Proposition 1.3 (Lifting of Paths)**.** For any $\gamma : [0,1] \to T^2$ a path, $\tilde{x} \in \mathbb{R}^2$ such that $\pi(\tilde{x}) =: x = \gamma(0)$, there exists a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ such that $\gamma = \pi \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = \tilde{x}$.

Proof. As π restricts to local homeomorphisms, we obtain an open cover of the path. Invoking compactness, we obtain a finite subcover for which we may lift the path locally such that the paths are compatible on intersections. With this, we obtain the required path by the pasting lemma.

For uniqueness, we observe that if $\pi \circ \tilde{\gamma}_1 = \pi \circ \tilde{\gamma}_2$, then, for all $t, \pi(\tilde{\gamma}_1(t)) = \pi(\tilde{\gamma}_2(t))$ and hence, $\tilde{\gamma}_1(t) \sim \tilde{\gamma}_2(t)$. Thus, the map

$$\delta := \tilde{\gamma}_1 - \tilde{\gamma}_2$$

must take value in $\mathbb{Z}^2$, and so, is a constant as continuous maps are constant on connected components. Now, as $\tilde{\gamma}_1(0) = \tilde{x} = \tilde{\gamma}_2(0)$ and so, $\delta = 0$ and $\tilde{\gamma}_1 = \tilde{\gamma}_2$. $\square$

Proposition 1.4 (Free Homotopy Classes of Loops on T^2)**.** Given a loop $\gamma : [0,1] \to T^2$ and its lift onto $\mathbb{R}^2$ the number

$$\rho(\gamma) := \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^2$$

is well-defined and for all γ_1 freely homotopic to γ_2, $\rho(\gamma_1) = \rho(\gamma_2)$.

Proof. Since for a loop, $\gamma(0) = \gamma(1)$, $\tilde{\gamma}(0) - \tilde{\gamma}(1) \in \mathbb{Z}^2$ is well-defined since two lifts differ only by a constant.

Suppose now γ_1, γ_2 are two freely homotopic loops on the torus, i.e. there exists some continuous map $H : [0,1] \times [0,1] \to T^2$ such that

$$H(0, \cdot) = \gamma_0; H(1, \cdot) = \gamma_1$$

and for all $u \in [0,1]$, the map $t \mapsto H(u,t)$ is closed. Let $\tilde{x}_0 \in \pi^{-1}(\gamma_0(0))$ and consider the map

$$\delta : [0,1] \to T^2 : t \mapsto H(t,0),$$

(i.e. the path of base points of the free homotopy) let $\tilde{\delta}$ to be the lift of δ starting at $\tilde{x}_0$. Now, define $\tilde{\gamma}_t$ to be the lift of $u \mapsto H(t,u) =: \gamma_t$ based at $\tilde{\delta}(t)$, I claim, $\tilde{H} : [0,1]^2 \to \mathbb{R}^2 : (t,u) \mapsto \tilde{\gamma}_t(u)$ is a free homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$. Thus, as $t \mapsto \rho(\gamma_t) = \tilde{\gamma}_t(1) - \tilde{\gamma}_t(0)$ is continuous and take value in $\mathbb{Z}^2$, it must be a constant, concluding the proof. $\square$

Suppose now we denote $P := \{\text{loops on } T^2\}$ and $\sim$ the freely homotopic equivalence relation on P, we have the following proposition.

Proposition 1.5. The map $\rho : L := P/\sim \to \mathbb{Z}^2 : [\gamma] \mapsto \rho(\gamma)$ is a bijection.

Proof. Surjectivity follows by considering the loop $\gamma : t \mapsto \pi(tu, tm)$ for all $(n, m) \in \mathbb{Z}^2$. Then, $\rho(\gamma) = (n, m)$ and hence the map is surjective.

Suppose on the other hand γ_0, γ_1 are loops on the torus such that $\rho(\gamma_0) = \rho(\gamma_1)$, injectivity follows by showing the loops are freely homotopic. Let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be lifts of γ_0 and γ_1 respectively. Then, define the homotopy from γ_0 to γ_1 by

$$\tilde{H}(t, u) := (1-t)\tilde{\gamma}_0(u) + t\tilde{\gamma}_1,$$

and define $H = \pi \circ \tilde{H}$. For all t, $H(t, \cdot)$ is a closed path since

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = (1-t)(\tilde{\gamma}_0(1) - \tilde{\gamma}_0(0)) + t(\tilde{\gamma}_1(1) - \tilde{\gamma}_1(0)) = (1-t)\rho(\gamma_0) + t(\rho(\gamma_1)).$$

By assumption, we have $\rho(\gamma_0) = \rho(\gamma_1)$ and so,

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = \rho(\gamma_0) \in \mathbb{Z}^2$$

implying that the loop is closed. Hence, H is a free homotopy between γ_1 and γ_2 as required. $\square$

Theorem 3. The torus T^2 is not simply connected.

Proof. If the loop γ on the torus is freely homotopic to the constant path, $\rho(\gamma) = \rho(c) = 0$. But we have provided examples where this is not the case, and hence, not all loops are freely homotopic to a constant path. $\square$

This procedure for proving a space is not simply connected will is common. In particular, we will provide a proposition for situations where we have a space X, a simply connected space $\tilde{X}$ and a group Γ characterising the lack of simply connectedness of X.

2 Fundamental Group

2.1 Definition

Given a topological space X, we will consider the set of all loops on X quotiented by the homotopy relation. Then, by equipping this quotient with the operation of gluing paths together, we obtain a group on this quotient. This group is known as the Fundamental group.

Definition 2.1 (Concatenation of Paths)**.** Let $\gamma_1, \gamma_2 : [0,1] \to X$ be paths such that $\gamma_1(1) = \gamma_2(0)$, then, we define the path

$$\gamma_1 * \gamma_2 : [0,1] \to X : t \mapsto \begin{cases} \gamma_1(2t), & t \leq 1/2, \\ \gamma_2(2t-1), & t > 1/2. \end{cases}$$

It is not difficult to see that this is continuous and hence is a path.

Definition 2.2. We define the equivalence relation $\sim$ on the space of paths with the same end points such that for $\gamma_1, \gamma_2 : [0,1] \to X$ with the same end points, $\gamma_1 \sim \gamma_2$ if and only if γ_1 is homotopic to γ_2.

Definition 2.3 (Fundamental Group)**.** Let X be a path-connected space and let $x_0 \in X$. Then, we define the fundamental group as

$$\pi_1(X, x_0) := \{\text{loops at } x_0\} / \sim .$$

We would like to equip the above quotient with a group structure using the concatenation operation. To achieve this, we need to check that, for loops at x_0, γ_1 and γ_2, we have

$$[\gamma_1 * \gamma_2]_\sim = [\gamma_1]_\sim * [\gamma_2]_\sim$$

and the equivalence class is compatible to concatenation independently of the end representative of the equivalence class.

Proposition 2.1. Let γ_1, γ_1' be paths from x to y and let γ_2, γ_2' be paths from y to z, then if $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$ then $\gamma_1 * \gamma_2 \sim \gamma_1' * \gamma_2'$.

Proof. Concatenate the homotopies. Namely, if H_1 is a homotopy between γ_1 and γ_1' and H_2 is a homotopy between γ_2 and γ_2', then define $H(t,u) := H_1(t,u) * H_2(t,u)$. $\square$

With the above proposition, we define the group operation on $\pi_1(X, x_0)$ such that $[\gamma_1] * [\gamma_2] := [\gamma_1 * \gamma_2]$.

Lemma 2.1. Let $\gamma : [0,1] \to X$ be a path and let $\phi : [0.1] \to [0,1]$ be a continuous function such that $\phi(0) = 0$ and $\phi(1) = 1$ (such ϕ is known as a reparametrisation. Then, $\gamma \circ \phi \sim \gamma$.

Proof. We define the homotopy $H(t,u) := \gamma((1-u)t + u\phi(t))$. H is clearly continuous, $H(0,u) = \gamma(0), H(1,u) = \gamma(1)$ and $\gamma(t) = H(t,0), \gamma(\phi(t)) = H(t,1)$. $\square$

Proposition 2.2. $(\pi_1(X, x_0), *)$ form a group.

Proof. The identity element of this group is the class of the constant loop $\mathrm{id} : t \mapsto x_0$. Indeed, for any loops γ at x_0, we have

$$\mathrm{id} * \gamma : t \mapsto \begin{cases} \gamma(2t), & t \leq 1/2, \\ x_0, & t > 1/2. \end{cases}$$

Thus, we may construct the homotopy between γ and $\mathrm{id} * \gamma$ by defining

$$H(t, u) := \begin{cases} \gamma((1+u)t), & t \leq 1/(1+u), \\ x_0, & t > 1/(1+u). \end{cases}$$

This is a homotopy by the glueing lemma and thus, $[\mathrm{id} * \gamma] = [\gamma]$. Alternatively, we observe $\mathrm{id} * \gamma$ is a reparametrisation of γ with $\phi(t) = 2t$ for $t \leq 1/2$ and $\phi(t) = 1$ for all $t > 1/2$.

It is to check that, with the above definition of the identity, the inverse of the loop $[\gamma]$ is simply $[t \mapsto \gamma(1-t)]$. Thus, it remains to check associativity.

Let $\gamma_1, \gamma_2, \gamma_3$ be loops at x_0. We note that $(\gamma_1 * \gamma_2) * \gamma_3 \neq \gamma_1 * (\gamma_2 * \gamma_3)$ though the two paths remain to be homotopic. Indeed, we see that the two paths are simply reparametrisations of each other with

$$\phi : [0,1] \to [0,1] : t \mapsto \begin{cases} 2t, & t \leq 1/4, \\ t + 1/4, & 1/4 < t \leq 3/4, \\ t/2 + 3/4, & t > 3/4, \end{cases}$$

such that $(\gamma_1 * \gamma_2) * \gamma_3 \circ \phi = \gamma_1 * (\gamma_2 * \gamma_3)$. Hence, by the above lemma, the two paths are homotopic and so,

$$([\gamma_1] * [\gamma_2]) * [\gamma_2] = [\gamma_1] * ([\gamma_2] * [\gamma_2]),$$

as required. ◻

Proposition 2.3. Let $x_0, x_1 \in X$, and let $\delta : [0,1] \to X$ be a path from x_0 to x_1. Then δ induces and isomorphism between $\pi_1(X_, x_0)$ and $\pi_1(X_, x_1)$ by

$$[\gamma] \mapsto [\delta^{-1} * \gamma * \delta].$$

Proof. Clearly, for the constant path, $\delta^{-1} * \mathrm{id} * \delta$ is an reparametrisation of $\delta^{-1} * \delta$ which is homotopic to id.

Now, given γ_1, γ_2 loops at x_0,

$$\begin{aligned} [\delta^{-1} * \gamma_1 * \gamma_2 * \delta] &= [\delta^{-1} * \gamma_1] * [\gamma_2 * \delta] \\ &= [\delta^{-1} * \gamma_1] * [\delta * \delta^{-1}] * [\gamma_2 * \delta] \\ &= [\delta^{-1} * \gamma_1 * \delta] * [\delta^{-1} * \gamma_2 * \delta]. \end{aligned}$$

Finally, as δ^{-1} by symmetry induces a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$ and these two homomorphisms are inverses, the induced map in an isomorphism as required. ◻

With the above proposition in mind, we see that the fundamental group of a space is independent (up to isomorphism) of the choice of the base point. So, we have also that a space is simply connected if and only if its fundamental group is trivial.

2.2 Covering Spaces

In the section we generalize the method introduced for toruses to general spaces.

Definition 2.4 (Covering Map)**.** A map $\pi : \tilde{X} \to X$ is a covering map if there exists an open cover $(U_\alpha)_{\alpha \in A}$ of X for all $\alpha \in A$, $\pi^{-1}(U_\alpha)$ is a disjoint union of open sets of $\tilde{X}$ each of which is homeomorphic to U_α with π (i.e. the π restricts on such an open set is homeomorphic to U_α).

Given a covering map π, we call X the base space and $\tilde{X}$ the covering map.

In some sense, the covering map provides a local homeomorphism for some specific open cover which each open set of the cover is represented in the domain as disjoint copies.

Proposition 2.4. If $\pi : \tilde{X} \to X$ is a covering map for some connected X, then the cardinalities of the fibres of π is constant. Namely the map

$$s : X \to \overline{\mathbb{N}} : x \mapsto \#\pi^{-1}(\{x\})$$

is constant. We call this constant the number of sheets of π.

Proof. It is clear that for $x, y \in U_\alpha$ where $\alpha \in A$, the fibres of x and y have the same cardinalities. Thus, as $(U_\alpha)_{\alpha \in A}$ is an open cover, for all $x \in X$,

$$s^{-1}\{s(x)\} = \bigcup_{\substack{\exists x' \in U_\alpha, \\ s(x') = s(x)}} U_\alpha.$$

Hence, $s^{-1}\{s(x)\}$ is open. Then, then we have the disjoint open cover of X

$$\{s^{-1}\{s(x)\} \mid x \in X\}.$$

But, since X is connected, any disjoint open cover of X can have at most 1 element, and thus, s is a constant as required. $\square$

Proposition 2.5. Suppose $\pi : \tilde{X} \to X$ be a covering map and let Y be a topological space and $f : [0,1] \times Y \to X$ is a continuous map such that there exists some $\tilde{f}_0 : Y \to \tilde{X}$ such that $\pi \circ \tilde{f}_0 = f(0, \cdot)$. Then, there exists a unique $\tilde{f} : [0,1] \times Y \to \tilde{X}$ such that $\tilde{f}(0, \cdot) = \tilde{f}_0$ and $\pi \circ \tilde{f} = f$.

Proof. For all $y \in Y$, there exists a neighbourhood N of y such that for all $t \in [0,1]$, there exists some $I_t \subseteq [0,1]$ containing t such that $f(I_t \times N) \subseteq U_\alpha$ for some α since f is continuous on the compact set $[0,1]$.

With this in mind, we can build the lift on $[0,1] \times N$. for $\{0\} \times N$, by assumption, the lift must be $\tilde{f}_0$. By the construction of N, there exists some U_α such that $f(\{0\} \times N) \subseteq U_\alpha$. Then, by continuity, there exists a neighbourhood $I_0 \subseteq [0,1]$ such that $f(I_0 \times N) \subseteq U_\alpha$ and so, we may define

$$\tilde{f} : N \times I_0 \to \tilde{X} : (t, y) \mapsto (\pi|_{\tilde{U}_\alpha(\beta)})^{-1} \circ f$$

as π^{-1} is a homeomorphism on $\tilde{U}_\alpha(\beta)$ where $\pi^{-1}(U_\alpha) = \bigsqcup_{\beta \in B} \tilde{U}_\alpha(\beta)$.

Now, by the compactness of $[0,1]$, there exists

$$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1,$$

such that for all $k = 0, \cdots, n$, $f([t_k, t_{k+1}] \times N) \subseteq U_{\alpha_k}$ for some $\alpha_k \in A$. Now, by induction, we may extend $\tilde{f}$ from $[0, t_k]$ to $[0, t_{k+1}]$ resulting in an extension of $\tilde{f}$ to the whole set of $[0,1]$.

Suppose now both $\tilde{f}, \bar{f}$ lifts f with $\tilde{f}(0, \cdot) = \bar{f}(0, \cdot) = \tilde{f}_0$. Let

$$A := \{z \in [0,1] \times N \mid \tilde{f}(z) = \bar{f}(z)\}.$$

Then, for all $z \in A$, there exists some $V_2 \subseteq N \times [0,1]$ such that $\tilde{f}(V_z), \bar{f}(V_z) \subseteq \tilde{U}_\alpha(\beta)$. On V_z, it is clear that $\tilde{f} = \bar{f} = (\pi \mid_{\tilde{U}_\alpha(\beta)})^{-1} \circ f$. Now, since N is a union of connected components, we obtain uniqueness. ☐

2.3 Induced Maps

In the case that we have a map between to topological spaces, we can define an induced map on their fundamental group by composing the map with the loops.

Definition 2.5 (Induced Map)**.** Let X, Y be topological spaces and let x, y be elements of X and Y respectively, then if $f : X \to Y$ is a continuous map such that $f(x) = y$, then, the induced map f_* is defined to be

$$f_* : \pi_1(X, x) \to \pi_1(Y.y) : [\gamma] \mapsto [f \circ \gamma].$$

It is clear that this map is well-defined since if H is a homotopy between γ_1, γ_2 loops in X based at x, then $f \circ H$ is a homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$.

Proposition 2.6. The induced map $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ of f is a group homomorphism.

Proof. Clearly, $f \circ \mathrm{id}$ is the constant loop based at y and thus, f_* maps the identity to the identity.

Now, let γ_1, γ_2 be loops based at x, it suffices to show that $(f \circ \gamma_1) * (f \circ \gamma_2)$ is homotopic to $f \circ (\gamma_1 * \gamma_2)$. But, in fact, the two paths above are equal. Hence, homotopic and thus, f_* is a group homomorphism as required. ☐

Proposition 2.7. Let $\pi : \tilde{X} \to X$ be a covering space and suppose $\pi(\tilde{x}) = x$ for some $\tilde{x} \in \tilde{X}$ and $x \in X$. Then, the induced map of π,

$$\pi_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)$$

is injective.

Since π_* is a injective group homomorphism, in some sense, the covering space allows us to consider a fundamental group of the covering space as a subgroup of the fundamental group of the original space.

In this sense, the induced map of the covering map provides a correspondence between the covering space (quotiented by some relation) and the subgroups of the fundamental group.

Proof. We will show that $\ker \pi_*$ is trivial. Suppose γ is a loop based at $\tilde{x}$ such that $\pi \circ \gamma$ is trivial and let H be the homotopy between $\pi \circ \gamma$ and the constant loop based at x. We will lift H along π.

Let $\tilde{H} : [0,1] \times [0,1] \to \tilde{X}$ be the lift of H such that $\gamma(t) = \tilde{H}(0,t)$ and $\pi \circ \tilde{H} = H$. As π is locally homeomorphic and $\tilde{H}$ is continuous, as $\tilde{H}(0,0) = \tilde{H}(0,1) = \gamma(0) = \tilde{x}$, we have $\tilde{H}(u,0) = \tilde{H}(u,1) = \tilde{x}$. Now, by continuity, $\tilde{H}(1,t)$ is constant with $H(1,t) = x$ we have $\tilde{H}(1,t) = \tilde{x}$ and thus, $\tilde{H}$ is a homotopy between γ and the constant path as required. $\square$

An important example is the covering map $\pi : S^1 \to S^1 : z \mapsto z^n$, then the induced map $\pi_* : \mathbb{Z} \to \mathbb{Z}$ (as $\pi_1(S^1, 1)$ is isomorphic to $\mathbb{Z}$) is the map $k \mapsto nk$.

2.4 Universal Covers

Definition 2.6 (Universal Cover)**.** Let X be a topological space. A universal cover of X is a space $\tilde{X}$ that covers X (i.e. there exists a covering map $\pi : \tilde{X} \to X$) and is simply connected.

Theorem 4. If a topological space X is path connected, locally path connected and locally simply connected, then X has a unique universal cover.

We say a space has a property locally if every basis of neighbourhood has that property.

Although this is a powerful theorem, without any specific construction of the universal cover, we will not be able to make much conclusion about the space. Therefore, this theorem is not very useful for our purpose. We will provide universal covers for most spaces we work with explicitly. For this reason, we will only provide the proof for the uniqueness.

Lemma 2.2. Let Y be a topological space that is simply connected and locally path connected and suppose $\pi : \bar{X} \to X$ is a covering map, $f : Y \to X$ be continuous such that $f(y) = \pi(\bar{x}) = x$ for some $y \in Y, \bar{x} \in \bar{X}, x \in X$. Then, there exists a continuous unique map $\bar{f} : Y \to \bar{X}$ such that $\pi \circ \bar{f} = f$. i.e. the following diagram commutes

$$\begin{tikzcd}
 & \bar{X} \ni \bar{x} \arrow[d, "\pi"] \\
y \in Y \arrow[ur, dashed, "\exists ! \bar{f}"] \arrow[r, "f"'] & X \ni x
\end{tikzcd}$$

Proof. For all $y' \in Y$, let γ a path from y to y', take $\bar{f}(y')$ be the end point of the unique lift of $f \circ \gamma$ starting at $\bar{x}$. This is well-defined since if γ' is another path from y to y', then γ' is homotopic to γ' and thus, $f \circ \gamma$ and $f \circ \gamma'$ are homotopic and hence, their lifts are also homotopic with the same end points.

We will now show $\bar{f}$ is continuous. Let $U \subseteq X$ be a open neighbourhood of $f(y')$ such that $\pi^{-1}(U)$ is a disjoint copies of sets $\{U_\alpha\}_{I'}$ such that, on each copy, the restriction of π form a homeomorphism onto U. Suppse $\bar{U}$ is one such copy and let V be a sufficiently small neighbourhood of y' such that $V \subseteq f^{-1}(U)$. I claim $V \subseteq \bar{f}^{-1}(\bar{U})$. Indeed, any paths from y to y' can be concatenated to a path with end point y'' in V. By definition, $f(y'')$ is the end point of the lift of this concatenated path composed with f based at $\bar{x}$, and thus, since

$\bar{f}(y') \in \bar{U}$, the lift remains in U and so $y'' \in V$. Hence, as $\bar{U}$ form a basis of open sets of $\bar{X}$, $\bar{f}$ is continuous.

Finally, to show uniqueness, we observe that, for any $y' \in Y$ and a path γ from y to y', $f \circ \gamma = \pi \circ \bar{f} \circ \gamma$, i.e. $\bar{f} \circ \gamma$ equals the lift of $f \circ \gamma$ based at some point. Now, as we require $\bar{f}(y) = \bar{x}$, this lift must be based at $\bar{x}$ and hence uniqueness follows by the uniqueness of the lift. $\square$

Proposition 2.8. If $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ are covering maps from simply connected spaces and X is path-connected such that for some $x \in X, \tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$, $\pi_1(\tilde{x}_1) = x = \pi_2(\tilde{x}_2)$, then there exists a unique homeomorphism $\phi : \tilde{X}_1 \to \tilde{X}_2$ such that the following diagram commutes.

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "\exists!\phi"] \arrow[dr, "\pi_1"'] & & \tilde{X}_2 \arrow[dl, "\pi_2"] \\
& X &
\end{tikzcd}$$

Proof. Take ϕ to be the lift of π_1 to $\tilde{X}_2$ and one may check that the lift of π_2 to $\tilde{X}_1$ provides the inverse. $\square$

Proposition 2.9. Let $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ be a universal covering map where X is locally path-connected and let $p : (\bar{X}, \bar{x}) \to (X, x)$ be another covering map. Then, there exists a unique $f : (\tilde{X}, \tilde{x}) \to (\bar{X}, \bar{x})$ such that $\pi = p \circ f$. i.e. the following diagram commutes.

Proof. Clear by lifting π to $\tilde{X} \to \bar{X}$. $\square$