Algebraic Topology

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Contents

1 Introduction

Let us introduce/recall some basic definitions which will be used throughout this course.

Definition 1.1 (Path)**.** A path in a topological space X is a continuous map $\gamma : [0,1] \subseteq \mathbb{R} \to X$. In the case that $\gamma(0) = \gamma(1)$, we call γ a loop/closed path.

Definition 1.2 (Homotopy)**.** Given two paths $\gamma_0, \gamma_1 : [0,1] \to X$ with the same end points (i.e. $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are said to be homotopic with fixed endpoints if there exists a continuous map

$$H : [0,1] \times [0,1] \to X$$

such that

- $H(t,0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t,1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0,u) = \gamma_0(0) = \gamma_1(0)$ and $H(1,u) = \gamma_0(1) = \gamma_1(1)$.

Thus, graphically, two paths are homotopic if you can continuously deform a path into the other without moving the starting and ending points (see second year complex analysis for more details).

Definition 1.3 (Free Homotopy)**.** The loops γ_0, γ_1 in X is said to be freely homotopic if there exists a continuous $H : [0,1] \times [0,1] \to X$ such that

- $H(t,0) = \gamma_0(t)$ for all $t \in [0,1]$;
- $H(t,1) = \gamma_1(t)$ for all $t \in [0,1]$;
- for all $u \in [0,1]$, $H(0,u) = H(1,u)$.

Definition 1.4 (Simply Connected)**.** X is said to be simply connected if any loop in X is freely homotopic to a constant loop.

Thus, informally, in a simply connected space, any loop can be contracted into a single point.

Proposition 1.1. S^2 is simply connected.

Simply connectedness is a important notion and relates to many difficult problems in geometry.

Theorem 1. S^2 and $\mathbb{R}^2$ are, up to homeomorphism, the only two simply-connected 2-dimensional manifolds.

Theorem 2 (Poincaré Conjecture)**.** The only compact, simply connected 3-dimensional manifold is the sphere S^3 (up to homeomorphism).

1.1 The Torus

An important example in algebraic topology is the torus. We will now provide a proof that the torus is not simply connected.

Definition 1.5 (Torus)**.** The 2-torus T^2 is the product topological space $S^1 \times S^1$.

We will now provide an alternative method of constructing the torus. Define the homeomorphisms

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1 + 1, x_2); T_2 : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1, x_2 + 1).$$

It is clear that T_1 and T_2 commutes and the map

$$\psi : \mathbb{Z}^2 \to \mathrm{Homeo}(\mathbb{R}^2) : (n, m) \mapsto T_1^n \circ T_2^m$$

is a group homomorphism. As this map is injective, we have in some sense embedded $\mathbb{Z}$ inside of $\mathrm{Homeo}(\mathbb{R}^2)$. Now, defining the equivalence relation on $\mathbb{R}^2$ by

$$(x_1, x_2) \sim (y_1, y_2) \iff \exists (n, m) \in \mathbb{Z}^2, \psi(n, m)(x_1, x_2) = (y_1, y_2),$$

or equivalently, there exists $(n, m) \in \mathbb{Z}^2$ such that $(x_1 + n, x_2 + m) = (y_1, y_2)$, we define the quotient topology $X := \mathbb{R}^2 / \sim$.

Lemma 1.1. Let X and Y be topological spaces and $\sim$ be an equivalence relation on X. Then, if $p : X \to X/\sim : x \mapsto [x]_\sim$ is the quotient map, any map $f : X/\sim \to Y$ is continuous if and only if $f \circ p$ is continuous.

Proof. Exercise. □

We see that the above lemma together with the universal property for quotients provides the universal property for topological spaces. Namely, if $f : X \to Y$ is a continuous map and for all $x \sim y \in X$, $f(x) = f(y)$, then the unique map $\tilde{f}$ obtained such that $\tilde{f} \circ p = f$ is continuous.

Lemma 1.2. Let X be a compact space and Y Hausdorff. Then, any continuous bijective map $f : X \to Y$ is a homeomorphism.

Proof. Exercise. □

Proposition 1.2. X is homeomorphic to the torus T^2.

Proof. Define the map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ such that $\pi(x, y) = (e^{2\pi i x}, e^{2\pi i y})$ for all $(x, y) \in \mathbb{R}^2$. We observe that $\pi(x, y) = \pi(u, v)$ if and only if $(x, y) \sim (u, v)$, and so, by the universal property for topological spaces, we obtain the unique continuous map defined by

$$\tilde{\pi} : X \to S^1 \times S^1 : [(x, y)] \mapsto \pi(x, y).$$

This map is clearly bijective and continuous by the universal property, and thus, by the above lemma, it suffices to show X is compact. But, this is clear since the $p([0, 1]^2) = \mathbb{R}^2/\sim$ and the continuous image of a compact set is compact. □

While the map π as described above is not injective, it is locally so (exercise). Thus, given a path on the torus, we may think of lifting it to $\mathbb{R}^2$ by lifting the paths piecewise via. the local homeomorphisms induced by π.

Lemma 1.3 (Pasting Lemma)**.** Let X, Y be both open subsets of a topological space and let B be another topological space, then $f : X \cup Y \to B$ is continuous if and only if $f\,|_A$ and $f\,|_B$ are continuous.

Proof. Exercise. $\square$

Corollary 2.1. If $f_1 : X \to B, f_2 Y \to B$ are continuous and agree on $X \cap Y$, then the map

$$f : X \cup Y \to B : x \mapsto \begin{cases} f_1(x) & \text{if } x \in X \\ f_2(x) & \text{otherwise} \end{cases}$$

is continuous.

Proposition 1.3 (Lifting of Paths)**.** For any $\gamma : [0,1] \to T^2$ a path, $\tilde{x} \in \mathbb{R}^2$ such that $\pi(\tilde{x}) =: x = \gamma(0)$, there exists a unique path $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ such that $\gamma = \pi \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = \tilde{x}$.

Proof. As π restricts to local homeomorphisms, we obtain an open cover of the path. Invoking compactness, we obtain a finite subcover for which we may lift the path locally such that the paths are compatible on intersections. With this, we obtain the required path by the pasting lemma.

For uniqueness, we observe that if $\pi \circ \tilde{\gamma}_1 = \pi \circ \tilde{\gamma}_2$, then, for all $t, \pi(\tilde{\gamma}_1(t)) = \pi(\tilde{\gamma}_2(t))$ and hence, $\tilde{\gamma}_1(t) \sim \tilde{\gamma}_2(t)$. Thus, the map

$$\delta := \tilde{\gamma}_1 - \tilde{\gamma}_2$$

must take value in $\mathbb{Z}^2$, and so, is a constant as continuous maps are constant on connected components. Now, as $\tilde{\gamma}_1(0) = \tilde{x} = \tilde{\gamma}_2(0)$ and so, $\delta = 0$ and $\tilde{\gamma}_1 = \tilde{\gamma}_2$. $\square$

Proposition 1.4 (Free Homotopy Classes of Loops on T^2)**.** Given a loop $\gamma : [0,1] \to T^2$ and its lift onto $\mathbb{R}^2$ the number

$$\rho(\gamma) := \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^2$$

is well-defined and for all γ_1 freely homotopic to γ_2, $\rho(\gamma_1) = \rho(\gamma_2)$.

Proof. Since for a loop, $\gamma(0) = \gamma(1)$, $\tilde{\gamma}(0) - \tilde{\gamma}(1) \in \mathbb{Z}^2$ is well-defined since two lifts differ only by a constant.

Suppose now γ_1, γ_2 are two freely homotopic loops on the torus, i.e. there exists some continuous map $H : [0,1] \times [0,1] \to T^2$ such that

$$H(0, \cdot) = \gamma_0; H(1, \cdot) = \gamma_1$$

and for all $u \in [0,1]$, the map $t \mapsto H(u,t)$ is closed. Let $\tilde{x}_0 \in \pi^{-1}(\gamma_0(0))$ and consider the map

$$\delta : [0,1] \to T^2 : t \mapsto H(t, 0),$$

(i.e. the path of base points of the free homotopy) let $\tilde{\delta}$ to be the lift of δ starting at $\tilde{x}_0$. Now, define $\tilde{\gamma}_t$ to be the lift of $u \mapsto H(t,u) =: \gamma_t$ based at $\tilde{\delta}(t)$, I claim, $\tilde{H} : [0,1]^2 \to \mathbb{R}^2 : (t,u) \mapsto \tilde{\gamma}_t(u)$ is a free homotopy from $\tilde{\gamma}_0$ to $\tilde{\gamma}_1$. Thus, as $t \mapsto \rho(\gamma_t) = \tilde{\gamma}_t(1) - \tilde{\gamma}_t(0)$ is continuous and take value in $\mathbb{Z}^2$, it must be a constant, concluding the proof. $\square$

Suppose now we denote $P := \{\text{loops on } T^2\}$ and $\sim$ the freely homotopic equivalence relation on P, we have the following proposition.

Proposition 1.5. The map $\rho : L := P/\sim \to \mathbb{Z}^2 : [\gamma] \mapsto \rho(\gamma)$ is a bijection.

Proof. Surjectivity follows by considering the loop $\gamma : t \mapsto \pi(tu, tm)$ for all $(n, m) \in \mathbb{Z}^2$. Then, $\rho(\gamma) = (n, m)$ and hence the map is surjective.

Suppose on the other hand γ_0, γ_1 are loops on the torus such that $\rho(\gamma_0) = \rho(\gamma_1)$, injectivity follows by showing the loops are freely homotopic. Let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be lifts of γ_0 and γ_1 respectively. Then, define the homotopy from γ_0 to γ_1 by

$$\tilde{H}(t, u) := (1 - t)\tilde{\gamma}_0(u) + t\tilde{\gamma}_1,$$

and define $H = \pi \circ \tilde{H}$. For all t, $H(t, \cdot)$ is a closed path since

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = (1 - t)(\tilde{\gamma}_0(1) - \tilde{\gamma}_0(0)) + t(\tilde{\gamma}_1(1) - \tilde{\gamma}_1(0)) = (1 - t)\rho(\gamma_0) + t(\rho(\gamma_1)).$$

By assumption, we have $\rho(\gamma_0) = \rho(\gamma_1)$ and so,

$$\tilde{H}(t, 1) - \tilde{H}(t, 0) = \rho(\gamma_0) \in \mathbb{Z}^2$$

implying that the loop is closed. Hence, H is a free homotopy between γ_1 and γ_2 as required. □

Theorem 3. The torus T^2 is not simply connected.

Proof. If the loop γ on the torus is freely homotopic to the constant path, $\rho(\gamma) = \rho(c) = 0$. But we have provided examples where this is not the case, and hence, not all loops are freely homotopic to a constant path. □

This procedure for proving a space is not simply connected will is common. In particular, we will provide a proposition for situations where we have a space X, a simply connected space $\tilde{X}$ and a group Γ characterising the lack of simply connectedness of X.