Fourier Analysis and the Theory of Distributions

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Contents

1 Orthonormal System

We will in this section recall some results about orthonormal systems in Euclidean spaces[1] and generalize them to complex spaces.

1.1 Euclidean Space

Definition 1.1. A system of nonzero vectors $\{X_\alpha\} \subseteq R$ where R is an Euclidean space is called orthogonal if $\langle X_\alpha, X_\beta \rangle = 0$ for all $\alpha \neq \beta$.

In addition, if for all α, $\langle X_\alpha, X_\alpha \rangle = 1$, we say the system is orthonormal.

Clearly, given an orthogonal system $\{X_\alpha\}$, we may normalize the vector such that $\{X_\alpha / \|X_\alpha\|\}$ is an orthonormal system. Furthermore, recall that a system of orthogonal vectors is linearly independent.

Definition 1.2. A complete (i.e. the smallest closed subspace containing the system is R) orthogonal system $\{X_\alpha\} \subseteq R$ is said to be an orthogonal basis of R.

Some important spaces we shall study in this course include $\mathbb{R}^2$ (equipped with the Euclidean norm), l_2, $\mathcal{C}([-\pi, \pi])$ (the space of continuous functions on $[-\pi, \pi]$ equipped with the L_2 norm).

Proposition 1.1. Let R be a separable Euclidean space. Then any orthogonal system in R is countable.

Proof. By normalizing, we may assume the system $\{X_\alpha\}$ is orthonormal. Then, for $\alpha \neq \beta$,

$$\|X_\alpha - X_\beta\|^2 = \|X_\alpha\|^2 - 2\langle X_\alpha, X_\beta \rangle + \|X_\beta\|^2 = \|X_\alpha\|^2 + \|X_\beta\|^2 = 2.$$

Then, $B_{1/2}(X_\alpha) \cap B_{1/2}(X_\beta) = \emptyset$ for all $\alpha \neq \beta$. Thus, if the system is not countable, we have found a uncountable number of disjoint open balls, contradicting the separability of R. $\square$

Proposition 1.2. Let $f_1, f_2, \cdots$ be a linearly independent system in a Euclidean space R. Then, there exists an orthonormal system $\phi_1, \phi_2, \cdots$ such that

$$\phi_n = a_{n_1} f_1 + \cdots + a_{n_n} f_n$$

and

$$f_n = b_{n_1} \phi_1 + \cdots + b_{n_n} \phi_n$$

for some $a_{n_k}, b_{n_k} \in \mathbb{R}$ and $a_{n_n}, b_{n_n} \neq 0$. Furthermore, the system $\phi_1, \phi_2, \cdots$ is uniquely determined up to a multiplication by ± 1.

Proof. Use Gram-Schmidt. $\square$

Corollary 0.1. A separable Euclidean space R possesses an orthonormal basis.

Proof. Simply obtain the orthonormal system corresponding to the countable dense system of R. The resulting system is complete since the two systems have the same linear closure. $\square$

[1] In this course, we shall call real inner product spaces Euclidean spaces.

Definition 1.3 (Fourier Coefficients)**.** Let $\phi_1, \phi_2, \cdots$ be an orthonormal system in R and let $f \in R$. Consider the sequence $c_k = \langle f, \phi_k \rangle$ for all $k = 1, 2, \cdots$. Then c_k are called the coordinates or Fourier coefficients of f with respect to the system $\{\phi_k\}$ and $\sum_{k=1}^{\infty} c_k \phi_k$ is called the Fourier series of f.

Note that this series in the definition is a formal series as we do not yet know whether or not the series converges.

In the finite case, it is not difficult to see that the sequence α_k for $k = 1, \cdots, n$ which minimizes $\|f - S_n^{(\alpha)}\|$ where $S_n^{(\alpha)} := \sum_{k=1}^{n} \alpha_k \phi_k$ is the Fourier coefficients. Indeed, we have

$$\begin{aligned}\|f - S_n^{(\alpha)}\|^2 &= \langle f, f \rangle - 2\langle f, S_n^{(\alpha)} \rangle + \langle S_n^{(\alpha)}, S_n^{(\alpha)} \rangle \\ &= \|f\|^2 - 2\sum \alpha_k c_k + \sum \alpha_k^2 \\ &= \|f\|^2 - \sum c_k^2 + \sum (\alpha_k - c_k)^2.\end{aligned}$$

Hence, $\|f - S_n^{(\alpha)}\|$ is minimized when $\alpha_k = c_k$ for all $k = 1, \cdots, n$. With this in mind, choosing α to be the Fourier coefficients, we have

$$\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2.$$

Geometrically, $f - S_n^{(\alpha)}$ is orthogonal to the subspace generated by $\phi_1, \cdots, \phi_n$ if and only if $\alpha = c$.

Furthermore, by noting $0 \leq \|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, we have

$$\sum_{k=1}^{n} c_k^2 \leq \|f\|^2 < \infty,$$

and hence, taking $n \to \infty$, we have $\sum_{k=1}^{\infty} c_k^2$ exists and is bounded above by $\|f\|^2$. This inequality is known as the Bessel inequality.

Definition 1.4 (Closed Orthonormal System)**.** The orthonormal system $\{\phi_k\}$ is closed if for any $f \in R$, we have

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

This property is called the Parseval equality.

Again, by observing $\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, the system is closed if and only if for any f, the partial sums of the Fourier series converge to f, i.e. $f = \sum_{k=1}^{\infty} c_k \phi_k$.

Proposition 1.3. In a separable Euclidean space R, an orthonormal system is complete if and only if it is closed.

Proof. Suppose first that $\{\phi_k\}$ is closed. Then, for all $f \in R$, $f = \sum_{k=1}^{\infty} c_k \phi_k$. Thus, the finite linear combinations of $\{\phi_k\}$ is dense in R and thus, $\{\phi_k\}$ is complete.

On the other hand, suppose that $\{\phi_k\}$ is complete (it is countable as R is separable), for any $f \in R$, there exists some α^k such that $\|f - S_\infty^{(\alpha^k)}\| \to 0$. As we have seen, for any partial sum $S_n^{(\alpha^k)}$, we have $\|f - S_n^{(c)}\| \leq \|f - S_n^{(\alpha^k)}\|$ and so,

$$\|f - S_\infty^{(c)}\| \leq \|f - S_\infty^{(\alpha^k)}\| \to 0$$

implying $\|f - S_\infty^{(c)}\| = 0$ and the system is closed. □

Proposition 1.4. Given $f, g \in R$ and a closed orthonormal system $\{\phi_k\}$,

$$\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$$

where $(c_k), (d_k)$ are the Fourier coefficients of f and g with respect to $\{\phi_k\}$ respectively.

Proof. We have, by Parseval's identity, $\|f\|^2 = \sum c_k^2$, $\|g\|^2 = \sum d_k^2$ and $\|f + g\|^2 = \sum (c_k + d_k)^2 = \sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2$, we have

$$\sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2 = \|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2.$$

Thus, cancelling using $\|f\|^2 = \sum c_k^2$ and $\|g\|^2 = \sum d_k^2$, we have $\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$ as required. □

In the case the system is only orthogonal but not necessary orthonormal, we may normalize the Fourier coefficients, i.e. given an orthogonal system $\{\phi_k\}$, we have $\{\phi / \|\phi_k\|\}$ is an orthonormal system, and so, we define

$$c_k = \left\langle f, \frac{\phi_k}{\|\phi_k\|} \right\rangle = \frac{1}{\|\phi_k\|} \langle f, \phi_k \rangle.$$

Similarly, the Fourier series of f is becomes

$$\sum_{k=1}^{\infty} c_k \frac{\phi_k}{\|\phi_k\|} = \sum \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2} \phi_k.$$

Substituting this definition of the Fourier coefficients into the Bessel inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{\langle f, \phi_k \rangle^2}{\|\phi_k\|^2} \leq \|f\|^2,$$

for any orthogonal system $\{\phi_k\}$.

Theorem 1 (Riesz)**.** Let $\{\phi_k\}$ be a orthonormal system in a complete Euclidean space R (i.e. a real Hilbert space) and let $c \in \ell_2$ (i.e. $\sum_{k=1}^{\infty} c_k^2 < \infty$). Then, there exists some $f \in R$ such that $c_k = \langle f, \phi_k \rangle$ and Parseval's identity holds, i.e.

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

Proof. Let $f_n := \sum_{k=1}^n c_k\phi_k$. Then, by definition, we have $c_k = \langle f_n, \phi_k\rangle$ for all $k = 1, \cdots, n$. Then, for all $p \geq 1$, we have

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\phi_{n+1} + \cdots + c_{n+p}\phi_{n+p}\|^2 = \sum_{k=n+1}^{n+p} c_k^2.$$

Now, as $\sum c_k^2 < \infty$, we have $\{f_n\}$ is Cauchy, and thus, as R is complete, there exists some $f \in R$ such that $f_n \to f$. Thus, by noting,

$$\langle f, \phi_k\rangle = \langle f_n \phi_k\rangle + \langle f - f_n, \phi_k\rangle = c_k + \langle f - f_n, \phi_k\rangle,$$

where $\langle f - f_n, \phi_k\rangle \to 0$ as $n \to \infty$ since $|\langle f - f_n, \phi_k\rangle| \leq \|f - f_n\|\|\phi_k\|$ by the Cauchy-Schwarz inequality, we have $c_k = \langle f, \phi_k\rangle$.

Finally, Parseval's identity, follows as $\|\cdot\|^2$ is continuous in a normed space. □

Let us recall the following result from functional analysis.

Proposition 1.5. Any separable Hilbert space is isomorphic to ℓ_2 (thus, any two separable Hilbert spaces are isomorphic).

Proof. Let H be a separable Hilbert space and choose $\{\phi_k\}$ a complete orthonormal system (which exists as H is separable). Then, for any $f \in H$, we map f to the sequence corresponding to its Fourier coefficients, i.e.

$$\psi : f \mapsto (c_1, c_2, \cdots)$$

which is well-defined by Bessel's inequality. On the other hand, by Riesz's theorem, for any $x \in \ell_2$, $\sum x_k^2 < \infty$ and so, there exists a unique $f \in H$, such that $\psi(f) = x$. Thus, as ψ is clearly linear (as the inner products are linear with respect to the left component), we have the isomorphism between H and ℓ_2. □

1.2 Complex Inner Product Space

We will in the course take the complex inner product to be anti-linear in the second component. As promised earlier, most definitions can be generalized from the real case to the complex directly.

Definition 1.5 (Fourier Coefficients)**.** Let R be a complex inner product space. Then, for an orthonormal system (ϕ_n) and $f \in R$, we define its Fourier coefficients to be $c_k := \langle f, \phi_k\rangle$ for all $k = 1, \cdots, n$. Similarly, we define the Fourier series of f to be the formal series $\sum_{k=1}^\infty c_k\phi_k$.

Going through the same argument as the real case, we obtain the complex version of Bessel's inequality.

Proposition 1.6 (Bessel's Inequality)**.** Given an orthonormal system (ϕ_n) and $f \in R$, we have

$$\sum_{k=1}^\infty |c_k|^2 \leq \|f\|^2.$$

Going through all proved theorems for real spaces, we find they also hold for complex spaces (with trivial modifications).

2 Trigonometric Series

We will consider the space $L_2[-\pi,\pi]$ (the space of square-integrable functions from $[-\pi,\pi]$ quotiented by the a.e.-equal equivalence relation equipped with the inner product $\langle f, g\rangle := \int_{[-\pi,\pi]} fg \mathrm{d}\lambda$), and the trigonometric system

$$\{\mathbf{1}, \cos(nx), \sin(nx) \mid n = 1, 2, \cdots\}.$$

It is not difficult to see that this system is orthogonal, but in fact, it is also complete. Indeed, completeness follows by the Weierstrass approximation theorem for trigonometric polynomials (we will discuss this later / recall the Stone-Weierstrass theorem and observe that the trigonometric system seperates points).

Nonetheless, this system is not orthonormal, and thus, we normalise the system such that the system becomes

$$\left\{\frac{1}{\sqrt{2\pi}}\mathbf{1}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \mid n = 1, 2, \cdots\right\}.$$

Hence, the Fourier series of an element $f \in L_2[-\pi,\pi]$ becomes the famous formula

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$a_k := \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\cos(kx)\lambda(\mathrm{d}x), b_k := \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\sin(kx)\lambda(\mathrm{d}x).$$

By recalling the above theory, the n-th partial sum of this series provides the best (in L_2 metric) approximation of f among all trigonometric polynomials of degree n. Hence, as the trigonometric system is complete, Parseval's identity holds, and so,

$$\|f - S_n\|_2 \to 0, \text{ as } n \to \infty.$$

By observing that $e^{ix} = \cos x + i \sin x$, we may rewrite the Fourier series can be written in the complex form. In particular, $L_2[-\pi,\pi]$ has the orthogonal system $\{e^{inx} \mid n \in \mathbb{Z}\}$ and the Fourier series of $f \in L_2[-\pi,\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where } c_n = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x)e^{-inx}\lambda(\mathrm{d}x).$$

- Since a function on $[-\pi,\pi]$ can be extended to $\mathbb{R}$ by periodicity, we can, instead of functions on $[-\pi,\pi]$ consider periodic functions with period 2π on $\mathbb{R}$.
- Since $\cos nx, \sin nx$ are bounded functions, the integrals defining the trigonometric Fourier coefficients exists for any function in $L_1[-\pi,\pi]$, i.e. if $f \in L_1[-\pi,\pi]$, then

$$\int f\cos(nx), \int f\sin(nx) < \int |f| < \infty.$$

- $L_2[-\pi,\pi] \subseteq L_1[-\pi,\pi]$ by Hölder's inequality and thus, with the above remark in mind, the definition of Fourier series is also well-defined for any integrable functions (though convergence is much opaque in this case).

While the Fourier series of f converges to f in L_2 though it is not clear that the Fourier series converges point-wise to f (it might be interesting to recall that convergence in L_p implies convergence in measure and the existence of a subsequence which converges almost everywhere).

Consider the partial sum of the Fourier series of $f \in L_2[-\pi, \pi]$,

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\ &= \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}(\cos kx \cos kt + \sin kx \sin kt)\right)\lambda(\mathrm{d}t) \\ &= \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right)\lambda(\mathrm{d}t). \end{aligned}$$

By noting the identity

$$\frac{1}{2} + \sum_{k=1}^{n}\cos ku = \frac{\sin\frac{2n+1}{2}u}{2\sin\frac{u}{2}},$$

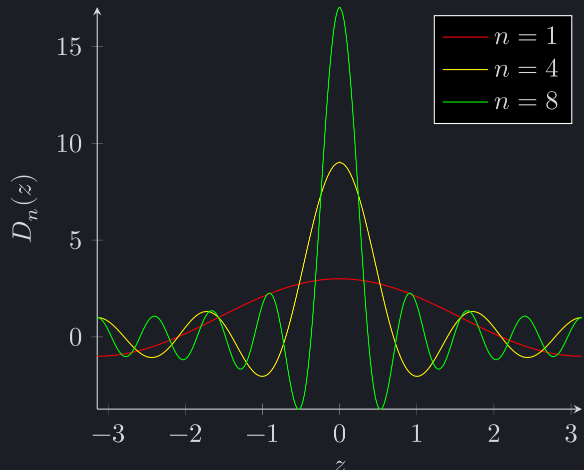
we obtain

$$S_n(x) = \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\frac{\sin\frac{2n+1}{2}(t-x)}{2\sin\frac{t-x}{2}}\lambda(\mathrm{d}t).$$

Finally, by noting the periodicity of f, by change of variable $z = t - x$, we obtain

$$S_n(x) = \int_{[-\pi,\pi]} f(x+z)D_n(z)\lambda(\mathrm{d}z), \text{ where } D_n(z) := \frac{1}{2\pi}\frac{\sin\frac{2n+1}{2}z}{\sin\frac{z}{2}}$$

and D_n is known as the Dirichlet kernel. We remark that the Dirichlet kernel $D_n(z)$ tends to $\frac{2n+1}{2\pi}$ as $z \to 0$ and rapidly osculates for large n though this does not impact our calculation as we are dealing with a point which has measure 0.



By observing the graph of the Dirichlet kernel, in some heuristic sense, we note that $D_n(z) \to \delta(z)$ for some function where δ is 0 at all points but $z = 0$ while $\int_{[-\epsilon,\epsilon]} \delta \mathrm{d}\lambda = 1$ for all $\epsilon > 0$.

Such an function cannot exist, however it motivates the second part of the course - the theory of distributions.

2.1 Conditions for Point-wise Convergence

We observe that $\|D_n\|_1 = 1$ and so we may write

$$S_n(x) - f(x) = \int_{[-\pi,\pi]} (f(x+z) - f(x))D_n(z)\lambda(\mathrm{d}z).$$

Clearly, $S_n(x) \to f(x)$ as $n \to \infty$ if and only if the right hand side of the above tends to 0.

Lemma 2.1 (Riemann-Lebesgue)**.** If $\phi \in L_1[a,b]$ for some $a < b$, then both

$$\int_{[a,b]} \phi(x)\sin(\gamma x)\lambda(\mathrm{d}x), \int_{[a,b]} \phi(x)\cos(\gamma x)\lambda(\mathrm{d}x)$$

tends to 0 as $\gamma \to \infty$.

Proof. We will prove the statement for the sin case. We observe that if ϕ is continuously differentiable, by integration by parts, we have

$$\int_{[a,b]} \phi(x)\sin(\gamma x)\lambda(\mathrm{d}x) = \left[-\phi(x)\frac{\cos\gamma x}{\gamma}\right]_a^b + \int_{[a,b]} \phi'(x)\frac{\cos\gamma x}{\gamma}\lambda(\mathrm{d}x),$$

which tends to 0 as $\gamma \to \infty$ (we note that ϕ' is continuous on a compact set, and hence bounded above). Now, in the general case, we observe that continuously differentiable functions are everywhere dense in $L_1[a,b]$, and so, for every $\epsilon > 0$, there exists some continuously differentiable $\phi_\epsilon(x)$, such that

$$\int_{[a,b]} |\phi - \phi_\epsilon|\mathrm{d}\lambda = \|\phi - \phi_\epsilon\|_1 < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}
\left|\int_{[a,b]} \phi(x)\sin(\gamma x)\lambda(\mathrm{d}x)\right| &\le \left|\int_{[a,b]} (\phi(x) - \phi_\epsilon(x))\sin(\gamma x)\lambda(\mathrm{d}x)\right| + \left|\int_{[a,b]} \phi_\epsilon(x)\sin(\gamma x)\lambda(\mathrm{d}x)\right| \\
&\le \|\phi - \phi_\epsilon\|_1 + \left|\int_{[a,b]} \phi_\epsilon(x)\sin(\gamma x)\lambda(\mathrm{d}x)\right| \\
&< \frac{\epsilon}{2} + \left|\int_{[a,b]} \phi_\epsilon(x)\sin(\gamma x)\lambda(\mathrm{d}x)\right|.
\end{aligned}$$

Since ϕ_ϵ is continuously differentiable, the last term tends to 0 as $\gamma \to \infty$ implying it is less than $\epsilon/2$ for sufficiently large γ. Thus, we have established the required limit. □

Corollary 1.1. If $f \in L_1[-\pi,\pi]$, then, its Fourier coefficients $a_k, b_k \to 0$ as $k \to \infty$.

With the above in mind, we will now provide a sufficient condition for convergence of Fourier series at a point x.

Theorem 2. If $f \in L_1[-\pi, \pi]$ and for any $x \in [-\pi, \pi]$, $\delta > 0$,

$$\int_{[-\delta,\delta]} \left| \frac{f(x+t) - f(t)}{t} \right| \lambda(\mathrm{d}t) < \infty$$

exists (this is called Dini's condition), then $S_n(x) \to f(x)$ as $n \to \infty$.

Proof. We observe

$$\begin{aligned} S_n(x) - f(x) &= \int_{[-\pi,\pi]} (f(x+z) - f(x)) D_n(z) \lambda(\mathrm{d}z) \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} \frac{f(x+z) - f(x)}{z} \frac{z}{\sin \frac{z}{2}} \sin\left(\frac{2n+1}{2} z \right) \lambda(\mathrm{d}z). \end{aligned}$$

Then, if Dini's condition is satisfied, as $\sin \frac{z}{2}$ is bounded on $[-\pi, \pi]$, we have

$$\frac{f(x+z) - f(x)}{z} \frac{z}{\sin \frac{z}{2}} \in L_1[-\pi, \pi]$$

and hence, by the Riemann-Lebesgue lemma, $S_n(x) - f(x) \to 0$ as $n \to \infty$. $\square$

We observe an trivial sufficient condition for which Dini's condition holds. In particular, the Dini condition holds if f is continuous at x and its derivative at x exists (or the limit of the derivative exists from the right and the left).

Suppose on the other hand that f has a discontinuity of the first kind at x (i.e. both the limit from the left and the right exists at x). Then, the argument in the proof remains to hold if we replace the Dini condition by

$$\int_{[-\delta,0]} \left| \frac{f(x+t) - f(t)}{t} \right| \lambda(\mathrm{d}t) < \infty \text{ and } \int_{[0,\delta]} \left| \frac{f(x+t) - f(t)}{t} \right| \lambda(\mathrm{d}t) < \infty.$$

Then, denoting $f(x+0), f(x-0)$ the limit of f at x from the right and the left respectively, we have

$$\begin{aligned} &S_n(x) - \frac{f(x+0) + f(x-0)}{2} = \\ &\int_{[-\pi,0]} (f(x+z) - f(x-0)) D_n(z) \lambda(\mathrm{d}z) + \int_{[0,\pi]} (f(x+z) - f(x+0)) D_n(z) \lambda(\mathrm{d}z) \end{aligned}$$

Hence, the $S_n(x)$ converges to the average of the limit f at x from the left and from the right.

Let us summarise the above in the following statement.

Proposition 2.1. If f is a bounded function of period 2π with only discontinuities of the first kind, and also possesses at each point left and right derivatives, Then, its Fourier series converges everywhere and its sum equals $f(x)$ at points of continuity and equals

$$\frac{f(x+0) + f(x-0)}{2}$$

at points of discontinuity.

We remark there exists continuous functions whose Fourier series diverge at some points. More curiously, there exists L_1 functions which diverge at all points.

2.2 Continuous Functions

Suppose now f is a continuous function with period 2π on $\mathbb{R}$, then it is uniquely determined by its Fourier series (exercise). Nonetheless, as we have seen, the Fourier series of f does not necessarily equal to f at every point. However, we can still reconstruct f from its Fourier series. Consider the partial sums

$$S_k(x) = \frac{a_0}{2} + \sum_{j=1}^{k} a_j \cos(jx) + b_j \sin(jx).$$

Then, we define the Feje's sums

$$\sigma_n(x) := \frac{1}{n}(S_0(x) + S_1(x) + \cdots S_{n-1}(x)),$$

we have the following theorem.

Definition 2.1 (Fejer's Kernel)**.** Fejer's kernel is defined to be the function

$$\Phi_n(z) = \frac{1}{2\pi n}\left(\frac{\sin(nz/2)}{\sin(z/2)}\right)^2$$

for all $n \geq 0$.

Lemma 2.2. Denoting Φ_n as Fejer's kernel, we have

- $\Phi_n \geq 0$;
- $\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda = 1$;
- for all $\delta \in (0,\pi]$, $\int_{[-\pi,-\delta]} \Phi_n \mathrm{d}\lambda = \int_{[\delta,\pi]} \Phi_n \mathrm{d}\lambda \to 0$ as $n \to \infty$.

Proof. Exercise. □

Theorem 3 (Fejer's)**.** If f is a continuous function with period 2π, then the sequence (σ_n) of its Fejer's's sums converges to f uniformly on $\mathbb{R}$.

Proof. Recall that

$$S_k(x) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x+z) D_k(z) \lambda(\mathrm{d}z),$$

and so,

$$\sigma_n(x) = \frac{1}{2\pi n}\int_{[-\pi,\pi]} f(x+z) \sum_{k=0}^{n-1} D_k(z) \lambda(\mathrm{d}z).$$

Then, using the identity that

$$\sum_{k=0}^{n-1} \sin(2k+1)u = \frac{\sin^2(nu)}{\sin u},$$

we obtain

$$\sigma_n(x) = \int_{[-\pi,\pi]} f(x+z)\Phi_n(z)\lambda(\mathrm{d}z)$$

where Φ_n is Fejer's kernel. Now, by nothing that f is continuous and periodic, f is bounded and uniformly continuous on $\mathbb{R}$. Thus, there exists some $M > 0$, such that for all $\sup_x |f(x)| \leq M$ and for all $\epsilon > 0$, there exists some $\delta \in (0, \pi]$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}\left(\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda\right)^{-1}$ for all $|x - y| < 2\delta$. Now,

$$\begin{aligned}
f(x) - \sigma_n(x) &= \int_{[-\pi,\pi]} (f(x) - f(x+z))\Phi_n(z)\lambda(\mathrm{d}z) \\
&= \left(\int_{[-\pi,-\delta]} + \int_{[-\delta,\delta]} + \int_{[\delta,\pi]}\right)(f(x) - f(x+z))\Phi_n(z)\lambda(\mathrm{d}z).
\end{aligned}$$

Then, by the above lemma, there exists some N such that for all $n \geq N$, $\int_{[-\pi,-\delta]} \Phi_n \mathrm{d}\lambda < \frac{\epsilon}{8M}$ and so, for all $n \geq N$, we have $\int_{[-\pi,-\delta]}(f(x)-f(x+z))\Phi_n(z)\lambda(\mathrm{d}z) \leq \int_{[-\pi,-\delta]} 2M\Phi_n(z)\lambda(\mathrm{d}z) < 2M\frac{\epsilon}{8M} = \epsilon/4$

$$\begin{aligned}
f(x) - \sigma_n(x) &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \int_{[-\delta,\delta]} (f(x) - f(x+z))\Phi_n \mathrm{d}\lambda \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}\left(\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda\right)^{-1} \int_{[-\delta,\delta]} \Phi_n \mathrm{d}\lambda \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $|f(x) - \sigma_n(x)| = f(x) - \sigma_n(x) < \epsilon$ for all $x \in \mathbb{R}$ implying uniform convergence as required. $\square$

While we had used the Weierstass approximation to obtain the above, Fejer's theorem also implies Weierstass approximation theorem straight away. Nonetheless, while the Stone-Weierstass theorem is more general, Fejer's theorem provides an explicit construction of such an series.

Corollary 3.1 (Weierstass Approximation Theorem)**.** Any continuous periodic function is a limit of a uniformly convergent sequence of trigonometirc polynomials.

Corollary 3.2. The trigonometric system $\{1, \sin nx, \cos nx \mid n = 1, 2, \cdots\}$ is complete in $L_2[-\pi, \pi]$ as uniform convergence implies convergence in L_2.

We remark that convergence in $L_p[-\pi, \pi]$ implies the existence of a subsequence which converges almost everywhere and hence, by Egorov's theorem, there exists a subsequence which converges uniformly on an arbitrarily large space.

By recalling that uniform convergence is definitionally equal to the supremum norm, Fejer's theorem tells us that for all $f \in \mathcal{C}_\infty[-\pi, \pi]$, the sequence of its Fejer's sums converges f in $\mathcal{C}_\infty[-\pi, \pi]$. Furthermore, if $f \in L_1[-\pi, \pi]$, then $\sigma_n \to f$ in L_1.

Corollary 3.3. For all $f \in L_1[-\pi, \pi]$, it is (a.e.) uniquely determined by its Fourier coefficients.

Proof. Suppose $f, g \in L_1[-\pi, \pi]$ have the same Fourier coefficients. Then, f and g have the same Fejer's sums. But, as L_1 is a metric space, the Fejer's sums has a unique limit and hence, $f = g$ in L_1. $\square$

2.3 Fourier Transform

Thus far, we have considered periodic functions in L_1 is represented uniquely by its Fourier series. We would now like to generalize this theory to non-periodic functions.

As an intuition (we will formally justify this later), consider the case first that f is periodic on $[-l, l]$ for some $l > 0$. From the exercise sheet, we were able to extend the Fourier series and conclude that, if $f \in L_1$ and satisfies Dini's condition at each point of $[-l, l]$, then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{l}x\right) + b_k \sin\left(\frac{k\pi}{l}x\right),$$

where

$$a_k = \frac{1}{l}\int_{[-l,l]} f(t)\cos\left(\frac{k\pi}{l}t\right)\lambda(\mathrm{d}t) \text{ and } b_k = \frac{1}{l}\int_{[-l,l]} f(t)\sin\left(\frac{k\pi}{l}t\right)\lambda(\mathrm{d}t).$$

Thus,

$$f(x) = \frac{1}{2l}\int_{[-l,l]} f\mathrm{d}\lambda + \frac{1}{l}\sum_{k=1}^{\infty} f(t)\cos\left(\frac{k\pi}{l}(t-x)\right)\lambda(\mathrm{d}t).$$

Hence, setting $y_k = \pi k/l$, $\Delta y = \pi/l$ and taking $l \to \infty$, we expect

$$f(x) = \frac{1}{\pi}\int_{(0,\infty)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t).$$

This equation is known as the Fourier integral. To see the similarity of this with the Fourier series, we may write the Fourier integral as

$$f(x) = \int_{(0,\infty)} (a_y\cos(yx) + b_y\sin(y_x))\lambda(\mathrm{d}y)$$

where

$$a_y = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\cos(yt)\lambda(\mathrm{d}t) \text{ and } b_y = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\sin(yt)\lambda(\mathrm{d}t).$$

Theorem 4. Let $f \in L_1$ and satisfy Dini's condition at every point. Then for all $x \in \mathbb{R}$, we have

$$f(x) = \frac{1}{\pi}\int_{(0,\infty)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t).$$

Proof. Let us denote

$$J_x(A) := \frac{1}{\pi}\int_{(0,A)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t),$$

and we will show $J_x(A) \to f(x)$ as $A \to \infty$. By assumption, $f \in L_1$ and so, J_x converges absolutely, and hence, by Fubini's theorem,

$$J_x(A) = \frac{1}{\pi}\int_{\mathbb{R}} \lambda(\mathrm{d}t)f(t)\int_{(0,A)} \cos(y(t-x))\lambda(\mathrm{d}y) = \frac{1}{\pi}\int_{\mathbb{R}} \lambda(\mathrm{d}t)f(t)\frac{\sin A(t-x)}{t-x}.$$

By change of variables $z = t - x$, we obtain

$$J_x(A) = \frac{1}{\pi} \int_{\mathbb{R}} f(x+z) \frac{\sin Az}{z} \lambda(\mathrm{d}z).$$

Now, by observing that $\frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin Az}{z} \lambda(\mathrm{d}z) = 1$, we have

$$\begin{aligned} J_x(A) - f(x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \\ &= \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \\ &+ \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \\ &- \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \end{aligned}$$

for all $N > 0$. By observing that the last two integrand are bounded, both integrals tends to 0 as $N \to \infty$. On the other hand, the first integral tends to 0 as $A \to \infty$ for fixed N by the Riemann-Lebesgue lemma. With this in mind, let $\epsilon > 0$, then, there exists some $N > 0$ such that

$$\left| \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \right|, \left| \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3}.$$

Then, by the Riemann-Lebesgue lemma (and Dini's condition), there exists some B, such that for all $A \geq B$,

$$\left| \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3}.$$

Hence, for all $A \geq B$, we have

$$\begin{aligned} |J_x(A) - f(x)| \leq &\left| \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \right| \\ &+ \left| \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \right| \\ &- \left| \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

implying $J_x(A) \to f(x)$ as $A \to \infty$ as required. □

Similar to the finite case, we may write the Fourier integral using the complex exponential function. In particular, as cos is even, we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda(\mathrm{d}y) \int_{\mathbb{R}} f(t) \cos(y(t-x)) \lambda(\mathrm{d}t).$$

On the other hand, as sin is odd,

$$\frac{1}{2\pi}\int_{\mathbb{R}}\lambda(\mathrm{d}y)\int_{\mathbb{R}}f(t)\sin(y(t-x))\lambda(\mathrm{d}t)=0.$$

Summing the two integrals, we obtain that, if $f\in L_1$ and satisfies the Dini condition, then

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\lambda(\mathrm{d}y)\int_{\mathbb{R}}f(t)e^{-iy(t-x)}\lambda(\mathrm{d}t).$$

Definition 2.2 (Fourier Transform)**.** Let $f\in L_1$. We define the Fourier transform of f to be

$$g(y)=\mathcal{F}[f](y):=\int_{\mathbb{R}}f(t)e^{-iyt}\lambda(\mathrm{d}t).$$

Proposition 2.2 (Inversion of the Fourier Transform)**.** In the case that f satisfies Dini's condition at some point x, then

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[f](y)e^{iyx}\lambda(\mathrm{d}y).$$

Proof. See above. □

From the above process, we observe that the Fourier transform is very similar to the Fourier series in which the Fourier transform is analogous to the Fourier coefficients while the inversion formula is analogous to the Fourier series both of which holds under Dini's condition.

Theorem 5. Let $f\in L_1$ and suppose $\mathcal{F}[f]=0$, then $f=0$ almost everywhere.

Thus, similar to the case of the Fourier series, the Fourier transform (a.e.) uniquely determines a function.

Proof. We note that we have not required Dini's condition and thus, we may not use the inversion formula.

By change of variable, we note that $\int f(x+t)e^{-iyx}\lambda(\mathrm{d}x)=0$ for all t,y, and let us define

$$\phi_\mu(x):=\int_{(0,\mu)}f(x+t)\lambda(\mathrm{d}t)$$

for some $\mu>0$. As ϕ_μ is integrable, by Fubini's theorem, it follows that $\mathcal{F}[\phi_\mu]=0$. Furthermore, as ϕ_μ is an integral function, it is absolutely continuous on any finite interval, and hence, has a derivative a.e. and so, it satisfies Dini's condition a.e. Hence, by the Fourier inversion formula, we have

$$\phi_u(x)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[\phi_\mu](y)e^{iyx}\lambda(\mathrm{d}y)=0,$$

almost everywhere. But, as ϕ_μ is continuous (as its absolutely continuous), and so $\phi_\mu=0$ everywhere. Thus, it follows that $f=0$ a.e. □

Lemma 2.3. Let $f, f_n \in L_1$ for $n = 1, 2, \cdots$ such that $f_n \to f$ in L_1. Then, $g_n(y) := \mathcal{F}[f_n](y) \to \mathcal{F}[f](y)$ uniformly on $\mathbb{R}$.

Proof. Follows as

$$|g_n - \mathcal{F}[f]| \leq \int_{\mathbb{R}} |f_n - f| \mathrm{d}\lambda.$$

□

Lemma 2.4. Let $f \in L_1$, then $\mathcal{F}[f]$ is a bounded continuous function and $\mathcal{F}[f](y) \to 0$ as $|y| \to \infty$.

Proof. Again, boundedness follows from the bound $|\mathcal{F}[f]| \leq \|f\|_1 < \infty$.

For the other two properties, we will first prove for indicator functions. Consider

$$\mathcal{F}[\mathbf{1}_{[a,b]}](y) = \int_{[a,b]} e^{-iyx} \lambda(\mathrm{d}x) = \frac{e^{-iyb} - e^{-iya}}{-iy}$$

which is continuous and tends to 0 as $|y| \to \infty$.

By the linearity of the Lebesgue integral, and hence, the Fourier transform is linear, we have that the Fourier transform of any simple function is also continuous and tends to 0 as $|y| \to \infty$. Then, as the simple function is dense in L_1, for any $f \in L_1$, there exists a sequence (f_n) of simple functions such that $f_n \to f$ in L_1. This, by the above lemma, we obtain $\mathcal{F}[f_n] \to \mathcal{F}[f]$ uniformly. Hence, $\mathcal{F}[f]$ is continuous and tends to 0 as $|y| \to \infty$. □