Fourier Analysis and the Theory of Distributions

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1 Orthonormal Systems

We will in this section recall some results about orthonormal systems in Euclidean spaces[1] and generalize them to complex spaces.

Definition 1.1. A system of nonzero vectors $\{X_\alpha\} \subseteq R$ where R is an Euclidean space is called orthogonal if $\langle X_\alpha, X_\beta \rangle = 0$ for all $\alpha \neq \beta$.

In addition, if for all α, $\langle X_\alpha, X_\alpha \rangle = 1$, we say the system is orthonormal.

Clearly, given an orthogonal system $\{X_\alpha\}$, we may normalize the vector such that $\{X_\alpha / \|X_\alpha\|\}$ is an orthonormal system. Furthermore, recall that a system of orthogonal vectors is linearly independent.

Definition 1.2. A complete (i.e. the smallest closed subspace containing the system is R) orthogonal system $\{X_\alpha\} \subseteq R$ is said to be an orthogonal basis of R.

Some important spaces we shall study in this course include $\mathbb{R}^2$ (equipped with the Euclidean norm), l_2, $\mathcal{C}([-\pi, \pi])$ (the space of continuous functions on $[-\pi, \pi]$ equipped with the L_2 norm).

Proposition 1.1. Let R be a separable Euclidean space. Then any orthogonal system in R is countable.

Proof. By normalizing, we may assume the system $\{X_\alpha\}$ is orthonormal. Then, for $\alpha \neq \beta$,

$$\|X_\alpha - X_\beta\|^2 = \|X_\alpha\|^2 - 2\langle X_\alpha, X_\beta \rangle + \|X_\beta\|^2 = \|X_\alpha\|^2 + \|X_\beta\|^2 = 2.$$

Then, $B_{1/2}(X_\alpha) \cap B_{1/2}(X_\beta) = \emptyset$ for all $\alpha \neq \beta$. Thus, if the system is not countable, we have found a uncountable number of disjoint open balls, contradicting the separability of R. $\square$

Proposition 1.2. Let $f_1, f_2, \cdots$ be a linearly independent system in a Euclidean space R. Then, there exists an orthonormal system $\phi_1, \phi_2, \cdots$ such that

$$\phi_n = a_{n_1} f_1 + \cdots + a_{n_n} f_n$$

and

$$f_n = b_{n_1} \phi_1 + \cdots + b_{n_n} \phi_n$$

for some $a_{n_k}, b_{n_k} \in \mathbb{R}$ and $a_{n_n}, b_{n_n} \neq 0$. Furthermore, the system $\phi_1, \phi_2, \cdots$ is uniquely determined up to a multiplication by ± 1.

Proof. Use Gram-Schmidt. $\square$

Corollary 0.1. A separable Euclidean space R possesses an orthonormal basis.

Proof. Simply obtain the orthonormal system corresponding to the countable dense system of R. The resulting system is complete since the two systems have the same linear closure. $\square$

[1] In this course, we shall call real inner product spaces Euclidean spaces.

Definition 1.3 (Fourier Coefficients)**.** Let $\phi_1, \phi_2, \cdots$ be an orthonormal system in R and let $f \in R$. Consider the sequence $c_k = \langle f, \phi_k \rangle$ for all $k = 1, 2, \cdots$. Then c_k are called the coordinates or Fourier coefficients of f with respect to the system $\{\phi_k\}$ and $\sum_{k=1}^{\infty} c_k \phi_k$ is called the Fourier series of f.

Note that this series in the definition is a formal series as we do not yet know whether or not the series converges.

In the finite case, it is not difficult to see that the sequence α_k for $k = 1, \cdots, n$ which minimizes $\|f - S_n^{(\alpha)}\|$ where $S_n^{(\alpha)} := \sum_{k=1}^{n} \alpha_k \phi_k$ is the Fourier coefficients. Indeed, we have

$$\begin{aligned}\|f - S_n^{(\alpha)}\|^2 &= \langle f, f \rangle - 2\langle f, S_n^{(\alpha)} \rangle + \langle S_n^{(\alpha)}, S_n^{(\alpha)} \rangle \\ &= \|f\|^2 - 2\sum \alpha_k c_k + \sum \alpha_k^2 \\ &= \|f\|^2 - \sum c_k^2 + \sum (\alpha_k - c_k)^2.\end{aligned}$$

Hence, $\|f - S_n^{(\alpha)}\|$ is minimized when $\alpha_k = c_k$ for all $k = 1, \cdots, n$. With this in mind, choosing α to be the Fourier coefficients, we have

$$\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2.$$

Geometrically, $f - S_n^{(\alpha)}$ is orthogonal to the subspace generated by $\phi_1, \cdots, \phi_n$ if and only if $\alpha = c$.

Furthermore, by noting $0 \le \|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, we have

$$\sum_{k=1}^{n} c_k^2 \le \|f\|^2 < \infty,$$

and hence, taking $n \to \infty$, we have $\sum_{k=1}^{\infty} c_k^2$ exists and is bounded above by $\|f\|^2$. This inequality is known as the Bessel inequality.

Definition 1.4 (Closed Orthonormal System)**.** The orthonormal system $\{\phi_k\}$ is closed if for any $f \in R$, we have

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

This property is called the Parseval equality.

Again, by observing $\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, the system is closed if and only if for any f, the partial sums of the Fourier series converge to f, i.e. $f = \sum_{k=1}^{\infty} c_k \phi_k$.

Proposition 1.3. In a separable Euclidean space R, an orthonormal system is complete if and only if it is closed.

Proof. Suppose first that $\{\phi_k\}$ is closed. Then, for all $f \in R$, $f = \sum_{k=1}^{\infty} c_k \phi_k$. Thus, the finite linear combinations of $\{\phi_k\}$ is dense in R and thus, $\{\phi_k\}$ is complete.

On the other hand, suppose that $\{\phi_k\}$ is complete (it is countable as R is separable), for any $f \in R$, there exists some α^k such that $\|f - S_\infty^{(\alpha^k)}\| \to 0$. As we have seen, for any partial sum $S_n^{(\alpha^k)}$, we have $\|f - S_n^{(c)}\| \leq \|f - S_n^{(\alpha^k)}\|$ and so,

$$\|f - S_\infty^{(c)}\| \leq \|f - S_\infty^{(\alpha^k)}\| \to 0$$

implying $\|f - S_\infty^{(c)}\| = 0$ and the system is closed. $\square$

Proposition 1.4. Given $f, g \in R$ and a closed orthonormal system $\{\phi_k\}$,

$$\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$$

where $(c_k), (d_k)$ are the Fourier coefficients of f and g with respect to $\{\phi_k\}$ respectively.

Proof. We have, by Parseval's identity, $\|f\|^2 = \sum c_k^2$, $\|g\|^2 = \sum d_k^2$ and $\|f + g\|^2 = \sum (c_k + d_k)^2 = \sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2$, we have

$$\sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2 = \|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2.$$

Thus, cancelling using $\|f\|^2 = \sum c_k^2$ and $\|g\|^2 = \sum d_k^2$, we have $\langle f, g \rangle = \sum_{k=1}^\infty c_k d_k$ as required. $\square$

In the case the system is only orthogonal but not necessary orthonormal, we may normalize the Fourier coefficients, i.e. given an orthogonal system $\{\phi_k\}$, we have $\{\phi / \|\phi_k\|\}$ is an orthonormal system, and so, we define

$$c_k = \left\langle f, \frac{\phi_k}{\|\phi_k\|} \right\rangle = \frac{1}{\|\phi_k\|} \langle f, \phi_k \rangle.$$

Similarly, the Fourier series of f is becomes

$$\sum_{k=1}^{\infty} c_k \frac{\phi_k}{\|\phi_k\|} = \sum \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2} \phi_k.$$

Substituting this definition of the Fourier coefficients into the Bessel inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{\langle f, \phi_k \rangle^2}{\|\phi_k\|^2} \leq \|f\|^2,$$

for any orthogonal system $\{\phi_k\}$.

Theorem 1 (Riesz)**.** Let $\{\phi_k\}$ be a orthonormal system in a complete Euclidean space R (i.e. a real Hilbert space) and let $c \in \ell_2$ (i.e. $\sum_{k=1}^\infty c_k^2 < \infty$). Then, there exists some $f \in R$ such that $c_k = \langle f, \phi_k \rangle$ and Parseval's identity holds, i.e.

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

Proof. Let $f_n := \sum_{k=1}^{n} c_k\phi_k$. Then, by definition, we have $c_k = \langle f_n, \phi_k\rangle$ for all $k = 1, \cdots, n$. Then, for all $p \geq 1$, we have

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\phi_{n+1} + \cdots + c_{n+p}\phi_{n+p}\|^2 = \sum_{k=n+1}^{n+p} c_k^2.$$

Now, as $\sum c_k^2 < \infty$, we have $\{f_n\}$ is Cauchy, and thus, as R is complete, there exists some $f \in R$ such that $f_n \to f$. Thus, by noting,

$$\langle f, \phi_k\rangle = \langle f_n \phi_k\rangle + \langle f - f_n, \phi_k\rangle = c_k + \langle f - f_n, \phi_k\rangle,$$

where $\langle f - f_n, \phi_k\rangle \to 0$ as $n \to \infty$ since $|\langle f - f_n, \phi_k\rangle| \leq \|f - f_n\|\|\phi_k\|$ by the Cauchy-Schwarz inequality, we have $c_k = \langle f, \phi_k\rangle$.

Finally, Parseval's identity, follows as $\|\cdot\|^2$ is continuous in a normed space. □

Let us recall the following result from functional analysis.

Proposition 1.5. Any separable Hilbert space is isomorphic to ℓ_2 (thus, any two separable Hilbert spaces are isomorphic).

Proof. Let H be a separable Hilbert space and choose $\{\phi_k\}$ a complete orthonormal system (which exists as H is separable). Then, for any $f \in H$, we map f to the sequence corresponding to its Fourier coefficients, i.e.

$$\psi : f \mapsto (c_1, c_2, \cdots)$$

which is well-defined by Bessel's inequality. On the other hand, by Riesz's theorem, for any $x \in \ell_2$, $\sum x_k^2 < \infty$ and so, there exists a unique $f \in H$, such that $\psi(f) = x$. Thus, as ψ is clearly linear (as the inner products are linear with respect to the left component), we have the isomorphism between H and ℓ_2. □