# Group Representation Theory

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# 1 Introduction

Group representation theory is a field of mathematics that applies linear algebra to study properties of groups. The field itself originated through a letter from Dedekind to Frobenius in which he noted that, given $f = \det A$, where $A$ is the Cayley table of a group of $n$ elements, by factorising $f$ into irreducible polynomials, $f = \prod_i f_i^{d_i}$, we have $d_i = \deg f_i$. And this led Frobenius to invent group representation theory.

Group representation theory is applicable in many different areas.

- Group theory arises in Klein's "Erlangen program" as symmetries of geometric spaces.
- Burnside in 1904 proves the following using representation theory (and so shall we later on)

  **Proposition 1.1.** Let $G$ be a group such that $|G| = p^r q^s$ where $p, q$ are prime and $r + s \geq 2$, then $G$ is not simple.
- In number theory, representations of Galois groups arises in the number field case

  $$\overline{F}/F, \mathbb{Q} \subseteq F, [F : \mathbb{Q}] < \infty,$$

  which has implications in Wiles' proof of Fermat's last theorem.
- In chemistry the symmetry and rotation of molecules can be represented by group actions.
- In quantum mechanics, spherical symmetry gives rise to discrete energy levels, orbitals, etc.
- In differential geometry, the vector space of solutions is a representation of the symmetry group of an equation.

Recalling the definition of a group, informally, the representation of a group $G$ is a way if writing group elements as linear transformations of a vector space such that the natural group properties are satisfied.

Some examples of group representations are the following:

- For all group $G$, the trivial representation of $G$ is $\rho$ such that $\rho(g) = \mathrm{id}$ for all $g \in G$.
- Let $\zeta \in \mathbb{C}$ be a $n$-th root of 1 and let $G = C_n = \{1, g, \cdots, g^{n-1}\}$. Then $\rho : g^i \mapsto (\zeta^i)$ is a representation of $G$.
- In the case $G = S_n$, the mapping of $\sigma \in S_n$ to its corresponding permutation matrix $P_\sigma$ is a representation of $G$.
- Another representation of $S_n$ is $\sigma \in S_n \mapsto (\mathrm{sign}(\sigma))$[1].
- Let $G = D_n$ the dihedral group of order $2n$. Then, a representation $D_n$ maps elements of $D_n$ to the corresponding $2 \times 2$ matrices which rotates/reflects $\mathbb{R}^2$ by the appropriate amount.

We shall in this module study and construct representations, and furthermore, classify up to isomorphism finite-dimensional complex representations of every finite group $G$.

[1] $\mathrm{sign}(\sigma) = \det P_\sigma$

## 2 Fundamentals of Group Representation

**Definition 2.1** (Representation)**.** Let $G$ be a group, then a representation of $G$ is the pair $(V, \rho)$ where $V$ is a (finite-dimensional) vector space and $\rho : G \mapsto GL(V)$ is a group homomorphism.

Alternatively, we may consider a group representation of $G$ is a group action $(\cdot) : G \times V \to V : (g, v) \mapsto v$ such that $(\cdot)$ is linear with respect to the second parameter. In particular, we recall a group action $(\cdot)$ satisfies $e \cdot v = v$ and $g \cdot (h \cdot v) = gh \cdot v$.

**Definition 2.2** (Dimension of a Representation)**.** If $(V, \rho)$ is a representation of $G$, then the dimension of $(V, \rho)$ is $\dim(V, \rho) = \dim V$.

Similar to other objects in mathematics, we introduce a notion of morphisms between representations.

**Definition 2.3** (Homomorphism of Representation)**.** Let $G$ be a group and $(V, \rho_V)$ and $(W, \rho_W)$ be two representations of $G$. Then a homomorphism of representations is a linear map $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho_V(g) = \rho_W(g) \circ T.$$

Furthermore, we say $T$ is an isomorphism is bijective (or equivalently, it has an inverse which is also a homomorphism).

In particular, one might imagine the homomorphism as a linear map such that the following diagram commute.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

As with any definitions which work with finite-dimensional vector spaces, there are equivalent but "worse" (as we will have to choose a basis) corresponding definitions in terms of matrices. Nonetheless, these definitions with matrices are easier computationally and we shall recall the contrast here.

Clearly, if $G$ is a group and $(\mathbb{C}^n, \rho)$ is a representation, we have $\rho(e) = I_n$. Furthermore, we have a natural isomorphism between $GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$ and more generally $\mathrm{Mat}_{n,m}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. Similarly, given a representation $(V, \rho)$, with $\dim V < \infty$, we may choose a basis $B$ of $V$ and write the representation as a matrix which we denote $\rho^B(g) = [\rho(g)]_B$. Thus, we may use first year linear algebra methods to manipulate representations.

**Definition 2.4.** Given two matrix representations $\rho, \rho' : G \mapsto GL_n(\mathbb{C})$, we say $\rho$ and $\rho'$ are equivalent/isomorphic if there exists $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho'(g) = P^{-1}\rho(g)P$.

This definition is motivated by the following.

**Proposition 2.1.** Given $(V, \rho_V)$ and $(W, \rho_W)$ representations of $G$, we have $\rho_V \cong \rho_W$ if and only if there exists some $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho_W^C(g) = P^{-1}\rho_V^B P(g)P$ for some basis $B, C$ of $V$ and $W$ respectively.

*Proof.* Exercise. $\square$

**Proposition 2.2.** Given a cyclic group $C_n = \langle g \rangle$ with representations $(V, \rho_V)$ and $(W, \rho_W)$ of equal dimensions, we have $\rho_V \cong \rho_W$ if and only if $\rho_V^B(g)$ is conjugate to $\rho_W^C(g)$ for some basis $B, C$ of $V$ and $W$ respectively.

*Proof.* Exercise. $\square$

In fact the proposition above holds for the infinite cyclic group $C_\infty \cong \mathbb{Z}$.

## 2.1 Regular Representation

Let us first recall some definition about group actions though we will omit stabilizers, the orbit-stabilizer theorem and transitive actions (though it might be helpful to recall them from last year).

**Definition 2.5** (Group Action)**.** Let $G$ be a group and $X$ a set, then a group action $(\cdot)$ of $G$ on $X$ is a function $G \times X \to X$ such that for all $g, h \in G$, $x \in X$, we have

- $g \cdot (h \cdot x) = gh \cdot x$,
- $1 \cdot x = x$.

Equivalently, a group action can be represented by a group homomorphism between $G$ to $S_n$ if $|X| = n < \infty$. We note that there exists an bijection between $\mathrm{Perm}(X)$ (a.k.a $\mathrm{Aut}(X)$ though we will avoid this term in case $X$ has additional structures) and $S_n$ with depends on a choice of $X \simeq \{1, \cdots, |X|\}$.

**Definition 2.6** (Kernel)**.** A kernel of a representation (or group action) is simply the kernel of the corresponding group homomorphism, i.e. if $\rho$ is a representation (or group action),

$$\ker \rho := \{g \in G \mid \rho(g) = \mathrm{id}\}.$$

We say a representation (or group action) is faithful if $\ker \rho = \{e\}$, i.e. $\rho$ is injective.

**Definition 2.7** (Morphism of Group Actions)**.** A morphism $T : X \to Y$ of group actions on $X$ and $Y$ is a map such that $T(g \cdot x) = g \cdot T(x)$ for all $g \in G$, $x \in X$.

This is also called a "$G$-equivariant map" from $X$ to $Y$ and one can see the resemblance of this definition and the definition for homomorphisms between representations.

For any group $G$, it acts on itself in three different ways. In particular, we have the left regular action $g \cdot h = gh$, the right regular action $g \cdot h = hg^{-1}$ (where the inverse is required for associativity) and the adjoint action $g \cdot h = ghg^{-1}$. One can see that the left and right regular actions are isomorphic via $T(g) = g^{-1}$. On the other hand, they are not isomorphic to the adjoint action (consider $\rho_{\mathrm{ad}}(g)(e) = e$ for all $g \in G$).

**Proposition 2.3.** Given two actions (or representations) $\rho, \rho'$ on $G$, $g \mapsto \rho(g)\rho'(g)$ is an action (or representation) if and only if $\rho(g)\rho'(g) = \rho'(g)\rho(g)$, that is $\rho$ and $\rho'$ are commuting actions.

**Definition 2.8.** A subset $Y \subseteq X$ is said to be stable under an action $(\cdot)$ of $G$ on $X$ if $g \cdot y \in Y$ for all $y \in Y, g \in G$.

In the case that $Y \subseteq X$ is stable, then we may restrict the action on $Y$ to obtain a new action of $G$ on $Y$.

**Definition 2.9** (Orbit)**.** Let $x \in X$, then $G \cdot x := \{g \cdot x \mid g \in G\}$ is called an orbit of $x$ and we denote this by $\mathrm{orb}(x)$.

It is not difficult to see that orbits are stable and in fact, as an exercise, one might show that $Y \subseteq X$ is stable if and only if it is a union of orbits.

In a group $G$ under the adjoint action, we see that the orbits are the conjugacy classes[2]. Thus, for every conjugacy class, we obtain a action on that class from the adjoint action on the whole group.

**Example 2.1.** Let $G = S_4$ and let $c = \{(12)(34), (13)(24), (14)(23)\}$. Then as $c$ is a conjugacy class, we have the adjoint action on $c$

$$\phi : S_4 \to \mathrm{Perm}(c) \cong S_3.$$

It is not difficult to show that $\phi$ is surjective and $\ker \phi = c \cup \{e\} \cong K_4 \cong C_2 \times C_2$. Thus, by the first isomorphism theorem we have

$$S_3 \cong S_4/K_4.$$

**Definition 2.10.** Given a finite set $X$, let

$$\mathbb{C}[X] := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{C} \right\},$$

equipped with the addition $\sum_{x \in X} a_x x + \sum_{x \in X} b_x x = \sum_{x \in X} (a_x + b_x) x$ and scalar multiplication $c \cdot \sum_{x \in X} a_x x = \sum_{x \in X} (c a_x) x$. This sum here does not represent some addition operation on $X$ but a notational trick. One might instead consider elements of $\mathbb{C}[X]$ as functions $a : X \to \mathbb{C}$ equipped with point-wise addition and scalar multiplication.

We observe that $\mathbb{C}[X] \cong \mathbb{C}^{|X|}$ depending on a choice of $X \cong \{1, \cdots, |X|\}$. Furthermore, we have $X \subseteq \mathbb{C}[X]$ and is a basis (if we interpret $\mathbb{C}[X]$ as a space of functions, the canonical basis is $\{a_x : y \mapsto \chi_{\{x\}} \mid x \in X\}$). In the case that $X$ is infinite we can still define $\mathbb{C}[X]$ allowing only finite sums.

**Proposition 2.4.** If $(\cdot)$ is a group action of $G$ on $X$, then, the map $(g, \sum a_x x) \mapsto \sum a_x (g \cdot x)$ is a group action of $G$ on $\mathbb{C}[X]$.

**Definition 2.11.** The left regular, right regular, adjoint representations are representations

$$\tilde{\rho}_L, \tilde{\rho}_R, \tilde{\rho}_{\mathrm{ad}} : G \to GL(\mathbb{C}[G])$$

obtained from the left regular, right regular and adjoint actions

$$\rho_L, \rho_R, \rho_{\mathrm{ad}} : G \to \mathrm{Perm}(G).$$

**Proposition 2.5.** If $X$ is any set with a $G$-actin, then for all $g \in G$, $[\rho_{\mathbb{C}[X]}(g)]_B$ is always a permutation matrix.

*Proof.* Exercise. □

[2] What are the orbits of the left action?

**Definition 2.12.** Given $(V, \rho_V), (W, \rho_W)$ representations of $G$, we denote

$$\operatorname{Hom}_G(V, W) := \{T : V \to W \mid G\text{-linear}\}.$$

**Proposition 2.6.** Let $(V, \rho_V)$ be a representation of $G$ and $v \in V$. Then, there exists a unique homomorphism of representations $\mathbb{C}[G] \to V$ where $\mathbb{C}[G]$ is equipped with the left regular representation such that $e_G \mapsto v$ and thus,

$$\operatorname{Hom}_G(\mathbb{C}[G], V) \cong (V, \rho_V).$$

*Proof.* For all $g \in G$, $c = \sum_{h \in G} a_h h$, we have

$$\begin{aligned} T(g \cdot c) = \rho_V(g)(Tc) &\iff T\left(\sum a_h gh\right) = \rho_V(g)\left(T\left(\sum a_h h\right)\right) \\ &\iff \sum a_h T(gh) = \sum a_h \rho_V(g)(Th) \\ &\iff T(gh) = \rho_V(g)(Th),\ \forall h \in G, \end{aligned}$$

where the second if and only if follows as both $T$ and $\rho_V$ are linear. Then choosing $h = e_G$, we have $T(g) = \rho_V(g)(v)$ and thus $T$ is uniquely determined on $G$ and hence is unique as $G$ is a basis of $\mathbb{C}[G]$.

It remains to show that the map $T$ defined by $g \mapsto \rho_V(g)(v)$ is a homomorphism of representations. This is clear since

$$\begin{aligned} T(g \cdot c) &= T\left(\sum a_h gh\right) = \sum a_h T(gh) \\ &= \sum a_h \rho_V(gh)(v) = \sum a_h \rho_V(g)(\rho_V(h)(v)) \\ &= \sum a_h \rho_V(g)(Th) = \rho_V(g)\left(T\left(\sum a_h h\right)\right), \end{aligned}$$

where the fourth equality follows by the associativity of group actions. □

## 2.2 Subrepresentation and Quotient Representation

**Definition 2.13** (Subrepresentation)**.** A subrepresentation of a representation $(V, \rho_V)$ is a subspace $W \leq V$ such that $\rho_V(g)(W) \subseteq W$ for all $g \in G$.

Clearly, both $\{0\}$ and $V$ are subrepresentations of $(V, \rho_V)$, and we say a representation is irreducible if these two subrepresentations are the only subrepresentations. We say a representation is reducible if it is not irreducible. In general, every 1-dimension representation is irreducible.

**Proposition 2.7.** Irreducibility is invariant under isomorphisms.

*Proof.* Exercise. □

**Proposition 2.8.** Let $G$ be finite and $(V, \rho_V)$ is an irreducible representation of $G$. Then $\dim V < \infty$.

*Proof.* Let $w \in V \;\; \{0\}$ and let $W := \operatorname{span}(\{\rho_V(g)(w) \mid g \in G\})$ which is a finite dimensional subrepresentation as $G$ is finite and for all $h \in G$, $\rho_V(h)(\rho_V(g)(w)) = \rho_V(hg)(w)$. Thus, if $\dim V$ is not finite, we have found a proper subrepresentation which contradicts the irreducibility of $(V, \rho_V)$. □

**Definition 2.14** (Quotient Representation)**.** For $W \leq V$ a subrepresentation, the quotient representation is $(V/W, \rho_{V/W})$ given by

$$\rho_{V/W}(g)(v + W) := \rho_V(g)(v) + W.$$

This is well-defined as $W$ is stable under $\rho_V$.

**Proposition 2.9.** For $T : (V, \rho_V) \to (W, \rho_W)$ a $G$-linear map, $\ker T$ and $\mathrm{Im} T$ are subrepresentations.

*Proof.* Let $v \in \ker T$, then $T(\rho_V(g)(v)) = \rho_W(g)(Tv) = \rho_W(g)(0) = 0$ implying $\rho_V(g)(v) \in \ker T$ and thus, $\ker T$ is a subrepresentation. On the other hand, for all $w \in \mathrm{Im} T$, there exists some $v \in V$ such that $Tv = w$. Then $\rho_W(g)(w) = \rho_W(g)(Tv) = T\rho_W(g)(v)$ implying $\rho_W(g)(w) \in \mathrm{Im} T$ showing $\mathrm{Im} T$ is also a subrepresentation. $\square$

**Proposition 2.10.** For $T : (V, \rho_V) \to (W, \rho_W)$ a $G$-linear map, we have

$$\mathrm{Im} T \cong V / \ker T.$$

*Proof.* Follows from the first isomorphism for vector spaces and it remains to check $V / \ker T \to \mathrm{Im} T$ is $G$-linear. $\square$

**Proposition 2.11.** If $T \in \mathrm{End}_G V$ is a $G$-linear projection (i.e. $T^2 = T$), then $V$ is a direct sum of subrepresentations $\ker T \oplus \mathrm{Im} T$.

*Proof.* Follows from the vector space case. $\square$

## 2.3 Maschke's Theorem and Schur's Lemma

Recalling internal and external direct sums of vector spaces, we will in this section introduce and prove a powerful result in representation theory known as the Maschke's theorem.

**Definition 2.15** (Decomposable)**.** The representation $(V, \rho_V)$ is decomposable if there exists a decomposition $V = U \oplus W$ where $U, W$ are non-zero subrepresentations.

**Definition 2.16** (Semisimple)**.** The representation $(V, \rho_V)$ is semisimple if there exists a irreducible subrepresentations $W_1, \cdots, W_n$ such that

$$V = \bigoplus_{i=1}^{n} W_i.$$

**Theorem 1** (Maschke's Theorem)**.** If $G$ is finite, then for all $W \leq V$ subrepresentations of $(V, \rho_V)$, there exists a complementary subrepresentation $U$, $V = W \oplus U$.

A direct consequence of Maschke's theorem is that every finite-dimensional representation of $G$ is semisimple.

Maschke's theorem does not hold in the case that $G$ is not finite. Consider $G = \mathbb{Z}$ and let

$$\rho : g \to GL_2(\mathbb{C}) : m \mapsto \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

Then the only non-zero proper subrepresentation is span$\{e_1\}$ since $e_1$ is the only eigenvector and as $\rho$ is a 2-dimensional representation, the only non-zero proper subrepresentation is 1-dimensional, hence an eigenspace. Thus, $\rho$ is indecomposable but not irreducible, and hence not semisimple.

*Proof of Maschke's Theorem.* To prove the theorem, we will attempt to find some $G$-linear map $T : V \to V$ that is a projection, i.e. $T^2 = T$, such that im$T = W$ and so $V = \ker T \oplus \mathrm{Im} T = \ker T \oplus W$.

In the case of linear maps, a map satisfying the above proposition must map

$$T(u + w) = Tu + Tw = 0 + w = w,$$

where $u \in \ker T$ and $w \in W$. As, $\ker T \oplus W = V$, this property uniquely identifies $T$ on $V$. However, this map is not $G$-linear and so we will modify $T$ such that it is $G$-linear.

By recalling that a linear map is $G$-linear if and only if it is conjugate with it self by $\rho(g)$ for all $g \in G$, let us define

$$\tilde{T} := \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1},$$

such that it is in some sense the average of all conjugates over all $g$.

We will now show that $\tilde{T}$ is a $G$-linear projection and im$\tilde{T} = W$. Indeed, for all $h \in G$, we have

$$\rho_V(h) \circ \frac{1}{|G|} \left( \sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1} \right) \circ \rho_V(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg) \circ T \circ \rho_V((hg)^{-1}) = \tilde{T},$$

as $g \mapsto h \cdot g$ is bijective as it has the inverse $g \mapsto h^{-1}g$. On the other hand, it is clear that $\tilde{T}(V) \subseteq W$ as for all $v \in V$, $T(\rho_V(g)^{-1}v) \in W$, and as $W$ is a subrepresentation, we have $\rho_V(g)T(\rho_V(g)^{-1}v) \in W$. Thus, as $\tilde{T}\mid_W = \mathrm{id}\mid_W$, since

$$\tilde{T}w = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)T(\rho_V(g)^{-1}w) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)\rho_V(g)^{-1}w = \frac{1}{|G|} \sum_{g \in G} w = w,$$

we have Im$\tilde{T} = W$ and $\tilde{T}$ is a projection. $\square$

We note that we used the property of $\mathbb{C}$ precisely when we needed $|G|^{-1}$ and thus, the same proof works for all field in which $|G|$ is invertible, i.e. of characteristic not a factor of $|G|$.

This decomposition needs not be unique. Indeed, if $V, \rho_V$ is a trivial representation with dimension $> 1$. Then any vector space decomposition is a decomposition of representations. For example, if $V = \mathbb{C}^2$, then

$$\mathbb{C}^2 = \{(a, 0)\} \oplus \{(0, b)\} = \{(a, a)\} \oplus \{(b, -b)\},$$

and in fact any pair of subspaces spanned by two linearly independent basis form a decomposition.

**Proposition 2.12.** For $G = C_m = \langle g \rangle$, $(V, \rho_V)$ a representation of $G$, there exists a unique decomposition of $(V, \rho_V)$ into 1-dimensional subrepresentations if and only if $\rho_V(g)$ has distinct eigenvalues.

*Proof.* Exercise. $\square$

**Lemma 2.1** (Schur's Lemma)**.** Let $V$ and $W$ be irreducible representations of $G$, then

- every $G$-linear map $T : V \to W$ is invertible or zero;
- let $V = W$ be finite-dimensional, then every $G$-linear map $T : V \to V$ is a multiple of the identity, i.e. $\mathrm{End}_G V = \mathbb{C} \cdot \mathrm{id}$.

We note that the first property does not require $V, W$ to be finite dimensional, and in fact it is true for arbitrary fields. On the other hand the second property only works for algebraically closed fields.

*Proof.* The first property is rather trivial. Indeed, if $T \neq 0$ then $\ker T$ must be $\{0\}$ as $\ker T$ is a subrepresentation and $V$ is irreducible. Similarly, for the same reason $\mathrm{Im} T = W$ and thus, $T$ is bijective.

For the second property, we recall that the eigenvalues of $T$ are the roots of the characteristic polynomial of $T$. Now since $\mathbb{C}$ is algebraically closed, there exists some $\lambda \in \mathbb{C}$ such that $\ker(\lambda I - T) \neq \{0\}$. Now since $\ker(\lambda I - T)$ is a subrepresentation of $V$, as $V$ is irreducible, we have $\ker(\lambda I - T) = V$. Thus, for all $v \in V$,

$$\lambda v - Tv = 0 \implies Tv = \lambda v \implies T = \lambda \cdot \mathrm{id}$$

as required. $\square$

**Theorem 2.** Up to isomorphism (and reordering), the representation decomposition is unique. That is, if $T : V := V_1 \oplus \cdots \oplus V_m \cong W := W_1 \oplus \cdots \oplus W_n$, then $V_i \cong W_i$ up to ordering.

*Proof.* We have $T : V \to W$ is a $G$-isomorphism map and so $T(V_i)$ is a subrepresentation of $W$. Then, as $W = W_1 \oplus \cdots \oplus W_n$, there exists some $j$ such that $W_j \cap T(V_i) \neq \emptyset$. Thus, we have $T \mid_{V_i} : V_i \to W_j$ is a $G$-linear map between two irreducible representations. As $T \mid_{V_i} \neq 0$, by Schur's lemma, it follows $T(V_i) = W_i$. Now, since for $i \neq k$, $T(V_i) \neq T(V_k)$ as $T$ is bijective and $V_i \cap V_k = \emptyset$, by pairing the $V_i$ and $W_j$, we are able to correspond each $V_i$ with a $W_j$. Reversing this process with $T^{-1}$, we are able to pair each $W_j$ with a $V_i$ and thus, we have $V_i \cong W_i$ up to ordering as required. $\square$

**Theorem 3.** In the case that $V$ is finite dimensional, there is a unique decomposition $V = \bigoplus_{i=1}^n V_m$ (up to reordering) if and only if in some decomposition, $V_i$ are all non-isomorphic.

*Proof.* Suppose first that $V_1 \oplus \cdots \oplus V_n = V$, and $V_1, \cdots, V_n$ are all non-isomorphic. Then, for all $G$-linear maps $T : V_i \to V$, we have by a similar argument as above, if $T \neq 0$, there exists some $j$ such that $T : V_i \cong V_j$. But as $V_i, V_j$ are non-isomorphic for $i \neq j$, we have $T \in \mathrm{End}_G V_i$. Thus, by the second part of Schur's lemma, there exists some $\lambda$ such that $T = \lambda \cdot \mathrm{id}\mid_{V_i}$. Now, if $V = W_1 \oplus \cdots \oplus W_m$, by the above theorem, WLOG, we have $T_i : V_i \cong W_i \subseteq V$. But we just shown $T_i = \lambda \cdot \mathrm{id}\mid_{V_i}$ and so, $V_i = W_i$ and the decomposition is unique.

Conversely, consider that for any representation $V_i$, there exists infinitely many subrepresentations of $V_i \oplus V_i$ by taking

$$V_a := \{(v, av) \mid v \in V\}.$$

Thus, if $V_i \cong V_j$, we have

$$V_a \leq V_i \oplus_{ext} V_i \cong V_i \oplus_{ext} V_j \cong V_i \oplus_{int} V_j.$$

Denoting the isomorphism from $V_i \oplus_{ext} V_i$ to $V_i \oplus_{int} V_j$ as $T$, we have $T(V_a) \leq V_i \oplus_{int} V_j$ and by Maschke's theorem, there exists some subrepresentation $U$ such that, $U$ is complement to $T(V_a)$ and

$$T(V_a) \oplus_{int} U = V_i \oplus_{int} V_j.$$

Thus, as $T$ is an isomorphism, $a \neq b$ implies $T(V_a) \neq T(V_b)$, we have found infinitely many non-equal decompositions of $V_i \oplus_{int} V_j$ and hence, also $V$. $\square$

**Corollary 3.1.** For $V_1, \cdots, V_n$ irreducible, non-isomorphic, subrepresentations of $V$ such that $V = V_1 \oplus \cdots \oplus V_n$ all subrepresentations of $V$ are of the form $V_{i_1} \oplus \cdots \oplus V_{i_m}$. In particular $V$ has $2^n$ different subrepresentations.

### 2.3.1 Representation of Abelian Groups

**Definition 2.17** (Centre)**.** The centre of a group $G$ is the subgroup with the underlying set

$$Z(G) := \{z \in G \mid zg = gz, \forall g \in G\}.$$

**Proposition 2.13.** Let $(V, \rho_V)$ be a finite dimensional irreducible representation and let $z \in Z(G)$. Then $\rho_V(z)$ is a scalar, i.e. $\rho_V(z) = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.

*Proof.* For all $g \in G$, we have $\rho_V(z)\rho_V(g) = \rho_V(zg) = \rho_V(gz) = \rho_V(g)\rho_V(z)$. So $\rho_V(z) : V \to V$ is $G$-linear. Thus, by the second property of Schur's lemma, there exists some $\lambda$ such that $\lambda \cdot \mathrm{id}_V$. $\square$

**Corollary 3.2.** If $G$ is abelian and if $(V, \rho_V)$ is an irreducible representation of $G$, then $\dim V = 1$.

*Proof.* As for abelian groups $G$, $Z(G) = G$, we have $\rho_V(g) = \lambda_g \cdot \mathrm{id}_V$ for all $g \in G$. Then, for all subspaces $W$ of $V$, $W$ is a subrepresentation. However, as $V$ is irreducible, $W$ must either be zero or $V$, and thus, $\dim V = 1$. $\square$

**Corollary 3.3.** If $G$ is finite and abelian, then every finite dimensional representation is a direct sum of 1-dimensional subrepresentations.

The finite dimensional condition is necessary. Indeed, if $F$ be a field $C \subsetneq F$ (then $\dim F = \infty$), e.g.

$$F := \left\{ \frac{P(X)}{Q(X)} \mid P, Q \in \mathbb{C}[X], Q \neq 0 \right\},$$

then $F$ is an irreducible $\infty$-dimensional representation over $\mathbb{C}$ of $G = F^\times$ although the latter is abelian.

**Corollary 3.4.** The irreducible representations of $C_m$ are up to isomorphism $(\mathbb{C}, \rho_\zeta)$ where $\zeta$ is a $m$-th root of unity. Furthermore, the irreducible representations of $C_{m_1} \times \cdots \times C_{m_l}$ are up to isomorphism,

$$\rho_{\zeta_1, \cdots, \zeta_l}(g_1^{j_1}, \cdots, g_l^{j_l}) = (\zeta_1^{j_1}, \cdots, \zeta_l^{j_l}).$$

The above corollary is important as every finite abelian group is an internal direct product of cyclic groups, we have classified all irreducible representations of finite abelian groups. So, if $G$ is finite abelian, then it has $|G|$ number of 1-dimensional representations.

**Proposition 2.14.** Every 1-dimensional representation of a group $G$ is of the form $(V, \rho_V)$, $\rho_V(g) = \lambda_g \mathrm{id}_V$ for some $\lambda_g \in \mathbb{C}$. Furthermore, two such representations are isomorphic if and only if the $\lambda_g$ are the same.

*Proof.* The first statement is clear while the second state follows since, if $(V, \rho_V : g \mapsto \lambda_g \mathrm{id}_V), (W, \rho_W : g \mapsto \lambda'_g \mathrm{id}_W)$ are two isomorphic 1-dimensional representations of $G$ along $T : V \to W$, then given some $v \in V \quad \{0\}$, we have $T(v) \neq 0$ and

$$\lambda_g T(v) = T(\lambda_g v) = T(\rho_V(g)(v)) = \rho_W(g)(T(v)) = \lambda'_g T(v).$$

Thus, $\lambda_g = \lambda'_g$. $\square$

As all 1-dimensional vector spaces over $\mathbb{C}$ are isomorphic to $GL_1(\mathbb{C}) \cong \mathbb{C}^\times$, we can simply only consider $\mathbb{C}^\times$ as the vector space of all 1-dimensional representations.

Consider the 1-dimensional representations on $S_n$. Let $(\mathbb{C}^\times, \rho)$ be a representation of $S_n$, then by considering $\sigma(a_1, \cdots, a_n)\sigma^{-1} = (\sigma(a_1), \cdots, \sigma(a_n))$, we have

$$(ij) = ((1i)(2j))(12)((2j)(1i)),$$

and so $(ij)$ is conjugate to $(12)$ for all $i, j$. Furthermore, by considering $\rho(ij)^2 = \rho((ij)^2) = \rho(e) = \mathrm{id}$, $\rho(ij) = \pm\mathrm{id}$. Now, by recalling that $\sigma = \tau\sigma'\tau^{-1} \iff \mathrm{sign}(\sigma) = \mathrm{sign}(\sigma')$, and the fact that every permutation can be represented as a product of transpositions, either, $\rho(\sigma) = \mathrm{sign}(\sigma) \cdot \mathrm{id}$ or $\rho(\sigma) = \mathrm{id}$. Thus, there exists exactly two 1-dimensional representation up to isomorphism.

## 2.4 Abelianisation

We have so far showed that if $G$ is abelian, then every finite dimensional representation of $G$ must be 1-dimensional (and in fact, we may drop the finite dimensional condition if $G$ is finite). Now we would like to reduce the study of 1-dimensional representations of any group $G$ to that of an abelian group.

**Definition 2.18** (Commutator)**.** Given a group $G$, the commutator subgroup is the subgroup $[G, G]$ generated by $[g, h] := ghg^{-1}h^{-1}$ for all $g, h \in G$.

**Proposition 2.15.** Given a group $G$, then

- $[G, G] \trianglelefteq G$ ($[G, G]$ is a normal subgroup of $G$);
- $G/[G, G]$ is abelian;
- for a normal subgroup $N \trianglelefteq G$, $G/N$ is abelian if and only if $[G, G] \leq N$.

*Proof.*

- For all $g, h_1, h_2 \in G$, we have

$$g[h_1, h_2]g^{-1} = [gh_1g^{-1}, gh_2g^{-1}] \in [G, G].$$

- Denote $\overline{g} := g[G, G]$, then $\overline{g}\overline{h} = gh[G, G]$, now since $h^{-1}g^{-1}hg \in [G, G]$, we have

$$\overline{g}\overline{h} = gh[G, G] = gh(h^{-1}g^{-1}hg)[G, G] = hg[G, G] = \overline{h}\overline{g}.$$

- If $G/N$ is abelian, then

$$gNhN = hNgN \implies N = gNhNg^{-1}h^{-1}N = ghg^{-1}h^{-1}N,$$

  thus, $ghg^{-1}h^{-1} \in N$ and so, $[G, G] \leq N$.

$\square$

**Definition 2.19** (Abelianisation)**.** The abelianisation of a group $G$ is the abelian group $G_{ab} := G/[G, G]$.

We note that as for all normal subgroup $N$ for which $G/N$ is abelian, we have $[G, G] \trianglelefteq N \trianglelefteq G$, by the third isomorphism theorem, we have

$$\frac{G_{ab}}{N/[G, G]} \equiv \frac{G/[G, G]}{N/[G, G]} \cong \frac{G}{N}.$$

**Proposition 2.16.** Let $A$ be an abelian group, the map

$$\mathrm{Hom}(G_{ab}, A) \to \mathrm{Hom}(G, A) : \phi \mapsto \phi \circ q_{ab},$$

is a bijection, where $q_{ab} : G \mapsto G_{ab}$ is the quotient map.

In other words, every homomorphism $\psi : G \to A$ uniquely corresponds to the homomorphism $\phi : G_{ab} \to A$ such that $\psi = \phi \circ q_{ab}$.

With this proposition, we see that for all normal subgroups $N \trianglelefteq G$ such that $G/N$ is abelian, we have the following diagram.

$$\begin{tikzcd} G \arrow[r, "q_N"] \arrow[d, "q_{ab}"'] & G/N \\ G_{ab} \arrow[ur, "\exists!"'] & \end{tikzcd}$$

*Proof.* Suppose that $\phi : G \to A$ is a homomorphism, then $\phi([g,h]) = \phi(ghg^{-1}h^{-1}) = e_A.$, and thus, $[G,G] \leq \ker \phi$. Hence, using the universal property of quotient, there exists

$$\overline{\phi} : G_{ab} \to A : g[G,G] = \phi(g).$$

By construction, $\overline{\phi} \circ q_{ab} = \phi$, and hence, the map is surjective.

On the other hand, if $\phi \circ q_{ab} = \psi \circ q_{ab}$, then for all $g \in G$, $\phi(g) = (\phi \circ q_{ab})(g) = (\psi \circ q_{ab})(g) = \psi(g)$ and so $\phi = \psi$. Thus, the map is bijective as required. □

**Corollary 3.5.** If $|G_{ab}| < \infty$, then the number of 1-dimensional representations of $G$ equals $|G_{ab}|$.

*Proof.* Follows as the 1-dimensional representations of $G$ are $\mathrm{Hom}(G, GL_1(\mathbb{C}))$ which bijects $\mathrm{Hom}(G_{ab}, GL_1(\mathbb{C}))$, and as $G_{ab}$ is abelian, it has $|G_{ab}|$ number of 1-dimensional representations. □

**Corollary 3.6.** If $|G| < \infty$, then the number of 1-dimensional representations of $G$ divides $|G|$.

*Proof.* Follows as $|G_{ab}| \mid |G|$ by Lagrange's theorem. □

In some sense we have found a bijection between $G_{ab}$ and the set of 1-dimensional representations of $G$. However, while $G_{ab}$ is a group, the 1-dimensional representations of $G$ are not. In fact, there is a natural group structure on the 1-dimensional representations of $G$, and for $G_{ab}$ finite, the two groups are isomorphic (exercise).

**Proposition 2.17.** If $G$ is simple nonabelian, then $G_{ab} = \{e\}$.

*Proof.* Clear as the only normal subsgroups of a simple groups are the trivial subgroup or the entire group. □

**Definition 2.20** (Perfect)**.** A group $G$ is perfect if $[G,G] = G$ (or equivalently $G_{ab} = \{e\}$).

From the above proposition, we see that simple nonabelian implies perfect. Furthermore, $G$ is perfect implies all 1-dimensional representations of $G$ are trivial. In the case the $G$ is finite, the converse is also true.

## 2.5 Decomposition of Representations

### 2.5.1 Regular Representation

Recall that for linear maps, the space of all linear maps between two vector spaces form a linear map and we have the following isomorphisms.

**Lemma 2.2.**

$$\mathrm{Hom}(V, W_1 \oplus W_2) \cong \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2),$$

and similarly,

$$\mathrm{Hom}(V_1 \oplus V_2, W) \cong \mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W).$$

*Proof.* Clear using the natural bijections. □

Similarly to the situation for vector spaces, the space of representations also form a vector space and similar isomorphisms hold. In particular,

$$\mathrm{Hom}_G(V, W_1 \oplus W_2) \cong \mathrm{Hom}_G(V, W_1) \oplus \mathrm{Hom}_G(V, W_2),$$

and

$$\mathrm{Hom}_G(V_1 \oplus V_2, W) \cong \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W),$$

where the space $(\mathrm{Hom}_G(V, W), \rho_{\mathrm{Hom}(V,W)})$ is defined such that

$$\rho_{\mathrm{Hom}(V,W)}(g)(\phi) = \rho_W(g) \circ \rho \circ \rho_V(g)^{-1}.$$

If $G$ is finite, By Maschke's theorem, $\mathbb{C}[G]$ is finite dimensional, and so, $\mathbb{C}[G]$ is always decomposable into irreducible representations. We will now prove some properties about this decomposition.

**Proposition 2.18.** Let $V_1, \cdots, V_m$ be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m},$$

where $W^k = W \oplus \cdots \oplus W$ $k$-times. Then, $\dim V_i = r_i$.

*Proof.* Recall that $\mathrm{Hom}_G(\mathbb{C}[G], V) \cong V$ by the bijection $\phi \mapsto \phi(e)$. So, by this and the above isomorphism,

$$V_i \cong \mathrm{Hom}_G(\mathbb{C}[G], V_i) \cong \bigoplus_{j=1}^{m} \mathrm{Hom}_G(V_j, V_i)^{r_j}.$$

By Schur's lemma, for all $i \neq j$, we have $\mathrm{Hom}_G(V_i, V_j) = 0$ and so,

$$\bigoplus_{j=1}^{m} \mathrm{Hom}_G(V_j, V_i)^{r_j} \cong (\mathbb{C} \cdot \mathrm{Id}_{V_i})^{r_i} \cong \mathbb{C}^{r_i}.$$

Thus, we have $\dim V_i = r_i$. □

**Corollary 3.7.** Let $V_1, \cdots, V_m$ be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}.$$

Then, $r_i = 1$ for all $i$ if $G$ is abelian.

As we shall see, converse is also true.

**Corollary 3.8.** Let $V_1, \cdots, V_m$ be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m},$$

then

$$|G| = \sum_{i=1}^{m} (\dim V_i)^2.$$

*Proof.*

$$|G| = \dim \mathbb{C}[G] = \dim(V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}) = \dim(V_1^{\dim V_1} \oplus \cdots \oplus V_m^{\dim V_m}) = \sum_{i=1}^{m} (\dim V_i)^2.$$

□

This is the sum of squares formula though it provided that every irreducible representation of $G$ is isomorphic to exactly one of the $V_i$.

**Proposition 2.19.** Let $V_1, \cdots, V_m$ be irreducible representations that are non-isomorphic such that

$$\mathbb{C}[G] \cong V_1^{r_1} \oplus \cdots \oplus V_m^{r_m}.$$

Then for all irreducible representations $W$ of $G$, there exists uniquely $i$ such that $W \cong V_i$.

*Proof.* By assumption $V_i \ncong V_j$ for $i \neq j$ and so, it remains to show $W$ must be isomorphic at some $V_i$. Suppose otherwise, then,

$$W \cong \mathrm{Hom}_G(\mathbb{C}[G], W) \cong \bigoplus_i \mathrm{Hom}_G(V_i, W) = 0,$$

by Schur's lemma, which is a contradiction. □

**Corollary 3.9.** For non-trivial $G$, every irreducible representation has dimension strictly less than $\sqrt{|G|}$.

*Proof.* Given $(V, \rho)$ a non-trivial representation,

$$|G| = \dim \mathbb{C}[G] = \dim(\mathbb{C} \oplus V^{\dim V} \oplus \cdots) \geq \dim(\mathbb{C} \oplus V^{\dim V}) = 1 + (\dim V)^2.$$

□

### 2.5.2 Revisiting Maschke's Theorem

**Definition 2.21** ($G$-invariant Subspace)**.** Given a representation $(V, \rho_V)$ of $g$, then the $G$-invariant subspace of $V$ is

$$V^G := \{v \in V \mid \rho_V(g)(v) = v, \ \forall g \in G\}.$$

A useful note is that, $V^G$ is the intersection of all eigenspaces of $\rho_V(g)$ of eigenvalue 1 for all $g$.

**Lemma 2.3.** $V^G \subseteq V$ is the largest trivial subrepresentation of $V$.

*Proof.* $V^G$ is a trivial subrepresentation as it sends every element of $V^G$ to itself and thus, $\rho_{V^G}(g) = \mathrm{Id}_{V^G}$ for all $g \in G$. Lastly, if $W \subseteq V$ is a trivial subrepresentation, for all $v \in W$, $\rho_W(g)(v) = \mathrm{Id}_W v = v$ for all $g \in G$, and so $v \in V^G$ and hence, $W \subseteq V^G$. □

**Definition 2.22** (Averaging Map)**.** If $G$ is finite and $(V, \rho_V)$ is a representation of $G$, then the averaging map is the function

$$\mathrm{avg} : V \to V : v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v).$$

**Proposition 2.20.** The averaging map is a $G$-linear projection with the image $V^G$.

*Proof.* Clearly, $\mathrm{Im}(\mathrm{avg}) \subseteq V^G$ since for all $h \in G$, $v \in V$,

$$\rho(h)(\mathrm{avg}(v)) = \rho(h)\left(\frac{1}{|G|}\sum_{g\in G}\rho(g)(v)\right) = \frac{1}{|G|}\sum_{g\in G}\rho(hg)(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(v) = \mathrm{avg}(v).$$

Furthermore, as $\mathrm{avg}\mid_{V^G} = \mathrm{Id}_{V^G}$, we have $\mathrm{Im}(\mathrm{avg}) = V^G$ and thus, avg is a projection. Finally, by the same calculation, we have

$$\rho(h)(\mathrm{avg}(v)) = \frac{1}{|G|}\sum_{g\in G}\rho(hg)(v) = \frac{1}{|G|}\sum_{g\in G}\rho(gh)(v) = \frac{1}{|G|}\sum_{g\in G}\rho(g)(\rho(h)(v)) = \mathrm{avg}(\rho(h)(v)),$$

which shows avg is a $G$-linear map. □

With this proposition, we obtain that $V = \ker(\mathrm{avg}) \oplus V^G$ with $\ker(\mathrm{avg})$ being a subrepresentation with no $G$-invariant vectors, i.e. $\ker(\mathrm{avg})^G = \{0\}$. With these definitions in mind, let us revisit Maschke's theorem.

**Proposition 2.21.** $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$.

*Proof.* Definition check. □

**Corollary 3.10.** The map

$$\mathrm{avg} : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$$

is a $G$-linear projection.

Thus, with this corollary, we see that the averaging map provides a ($G$-linear) map between $\mathrm{End}(V)$ and $\mathrm{End}_G(W)$. Nonetheless, to obtain Maschke's theorem, it remains to show that it avg maps linear projections to $G$-linear projections (with the same image).

### 2.5.3 Adjoint Representation

Consider $V_1, \cdots, V_m$ be all irreducible representations (up to isomorphism) of $G$ with $|G|$. Then, we may define the map

$$G \to \bigoplus_{i=1}^{m} \mathrm{End}(V_i) : g \mapsto (\rho_{V_1}(g), \cdots, \rho_{V_m}(g)).$$

Linearly extending, we obtain a map $\phi : \mathbb{C}[G] \to \bigoplus \mathrm{End}(V_i)$. Then, by the sum of squares formula, since $\dim(\mathrm{End}(W)) = (\dim W)^2$, the source and target of $\phi$ have the same dimension.

**Proposition 2.22.** The map $\phi : \mathbb{C}[G] \to \bigoplus \mathrm{End}(V_i)$ is a linear isomorphism and is $G$-linear with respect to the adjoint representation.

*Proof.* It suffices to show $G$-linearity. For all $h \in G$, we have

$$\begin{aligned}\phi(\rho_{ad}(g)(h)) &= \phi(ghg^{-1}) = (\rho_{V_1}(ghg^{-1}), \cdots, \rho_{V_m}(ghg^{-1})) \\ &= (\rho_{\mathrm{End}(V_1)}(g)(\rho_{V_1}(h)), \cdots, \rho_{\mathrm{End}(V_m)}(g)(\rho_{V_m}(h))) \\ &= \rho(g)(\phi(h)).\end{aligned}$$

□

Thus, we have found found an isomorphism $\mathbb{C}[G]_{ad} \cong \bigoplus \mathrm{End}(V_i)$ and thus, taking the dimension of both sides, we recover the sum of squares formula $|G| = \sum \dim(V_i)^2$. In particular, we have replaced an equality by an isomorphism and in general, this process is called "categorification".

Continuing with $V_1, \cdots, V_m$ as the irreducible representations of $G$, $|G| < \infty$ up to isomorphism, we have the following corollaries.

**Corollary 3.11.** Since,

$$\mathbb{C}[G]_{ad} \cong \bigoplus_{i=1}^{m} \mathrm{End}(V_i).$$

By taking $G$-invariant subspaces, we have

$$\mathbb{C}[G]_{ad}^G \cong \bigoplus_{i=1}^{m} \mathrm{End}(V_i)^G = \bigoplus \mathrm{End}_G(V_i) = \bigoplus \mathbb{C} \cdot \mathrm{Id}_{V_i} \cong \mathbb{C}^m$$

by Schur's lemma.

**Proposition 2.23.** $(\mathbb{C}[G], \rho_{ad})^G$ have a basis $f_1, \cdots, f_{m'}$ given by

$$f_i = \sum_{g \in \mathcal{C}_i} g,$$

where $\mathcal{C}_1, \cdots, \mathcal{C}_{m'}$ are the conjugacy classes of $G$.

*Proof.* Follows by considering $\sum a_g g \in \mathbb{C}[G]_{ad}^G \iff a_h = a_{ghg^{-1}}$ for all $g, h \in G$. □

**Corollary 3.12.** The number of conjugacy classes of $G$ equals the number of irreducible representation up to isomorphisms.

*Proof.* Follows by taking the dimensions on both sides of the isomorphism $\mathbb{C}[G]_{ad}^G \cong \mathbb{C}^m$. □