Group Representation Theory

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1 Introduction

Group representation theory is a field of mathematics that applies linear algebra to study properties of groups. The field itself originated through a letter from Dedekind to Frobenius in which he noted that, given $f = \det A$, where A is the Cayley table of a group of n elements, by factorising f into irreducible polynomials, $f = \prod_i f_i^{d_i}$, we have $d_i = \deg f_i$. And this led Frobenius to invent group representation theory.

Group representation theory is applicable in many different areas.

- Group theory arises in Klein's "Erlangen program" as symmetries of geometric spaces.
- Burnside in 1904 proves the following using representation theory (and so shall we later on)

  Proposition 1.1. *Let G be a group such that $|G| = p^r q^s$ where p, q are prime and $r + s \geq 2$, then G is not simple.*

- In number theory, representations of Galois groups arises in the number field case

  $$\overline{F}/F, \mathbb{Q} \subseteq F, [F : \mathbb{Q}] < \infty,$$

  which has implications in Wiles' proof of Fermat's last theorem.

- In chemistry the symmetry and rotation of molecules can be represented by group actions.
- In quantum mechanics, spherical symmetry gives rise to discrete energy levels, orbitals, etc.
- In differential geometry, the vector space of solutions is a representation of the symmetry group of an equation.

Recalling the definition of a group, informally, the representation of a group G is a way if writing group elements as linear transformations of a vector space such that the natural group properties are satisfied.

Some examples of group representations are the following:

- For all group G, the trivial representation of G is ρ such that $\rho(g) = \mathrm{id}$ for all $g \in G$.
- Let $\zeta \in \mathbb{C}$ be a n-th root of 1 and let $G = C_n = \{1, g, \cdots, g^{n-1}\}$. Then $\rho : g^i \mapsto (\zeta^i)$ is a representation of G.
- In the case $G = S_n$, the mapping of $\sigma \in S_n$ to its corresponding permutation matrix P_σ is a representation of G.
- Another representation of S_n is $\sigma \in S_n \mapsto (\mathrm{sign}(\sigma))$[1].
- Let $G = D_n$ the dihedral group of order $2n$. Then, a representation D_n maps elements of D_n to the corresponding 2×2 matrices which rotates/reflects $\mathbb{R}^2$ by the appropriate amount.

We shall in this module study and construct representations, and furthermore, classify up to isomorphism finite-dimensional complex representations of every finite group G.

[1] $\mathrm{sign}(\sigma) = \det P_\sigma$

2 Fundamentals of Group Representation

Definition 2.1 (Representation)**.** Let G be a group, then a representation of G is the pair (V, ρ) where V is a (finite-dimensional) vector space and $\rho : G \mapsto GL(V)$ is a group homomorphism.

Alternatively, we may consider a group representation of G is a group action $(\cdot) : G \times V \to V : (g, v) \mapsto v$ such that $(\cdot)$ is linear with respect to the second parameter. In particular, we recall a group action $(\cdot)$ satisfies $e \cdot v = v$ and $g \cdot (h \cdot v) = gh \cdot v$.

Definition 2.2 (Dimension of a Representation)**.** If (V, ρ) is a representation of G, then the dimension of (V, ρ) is $\dim(V, \rho) = \dim V$.

Similar to other objects in mathematics, we introduce a notion of morphisms between representations.

Definition 2.3 (Homomorphism of Representation)**.** Let G be a group and (V, ρ_V) and (W, ρ_W) be two representations of G. Then a homomorphism of representations is a linear map $T : V \to W$ such that for all $g \in G$,

$$T \circ \rho_V(g) = \rho_W(g) \circ T.$$

Furthermore, we say T is an isomorphism is bijective (or equivalently, it has an inverse which is also a homomorphism).

In particular, one might imagine the homomorphism as a linear map such that the following diagram commute.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

As with any definitions which work with finite-dimensional vector spaces, there are equivalent but "worse" (as we will have to choose a basis) corresponding definitions in terms of matrices. Nonetheless, these definitions with matrices are easier computationally and we shall recall the contrast here.

Clearly, if G is a group and $(\mathbb{C}^n, \rho)$ is a representation, we have $\rho(e) = I_n$. Furthermore, we have a natural isomorphism between $GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$ and more generally $\mathrm{Mat}_{n,m}(\mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$. Similarly, given a representation (V, ρ), with $\dim V < \infty$, we may choose a basis B of V and write the representation as a matrix which we denote $\rho^B(g) = [\rho(g)]_B$. Thus, we may use first year linear algebra methods to manipulate representations.

Definition 2.4. Given two matrix representations $\rho, \rho' : G \mapsto GL_n(\mathbb{C})$, we say ρ and ρ' are equivalent/isomorphic if there exists $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho'(g) = P^{-1}\rho(g)P$.

This definition is motivated by the following.

Proposition 2.1. Given (V, ρ_V) and (W, ρ_W) representations of G, we have $\rho_V \cong \rho_W$ if and only if there exists some $P \in GL_n(\mathbb{C})$ such that for all $g \in G$, $\rho_W^C(g) = P^{-1}\rho_V^B P(g)P$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

Proposition 2.2. Given a cyclic group $C_n = \langle g \rangle$ with representations (V, ρ_V) and (W, ρ_W) of equal dimensions, we have $\rho_V \cong \rho_W$ if and only if $\rho_V^B(g)$ is conjugate to $\rho_W^C(g)$ for some basis B, C of V and W respectively.

Proof. Exercise. $\square$

In fact the proposition above holds for the infinite cyclic group $C_\infty \cong \mathbb{Z}$.

2.1 Regular Representation

Let us first recall some definition about group actions though we will omit stabilizers, the orbit-stabilizer theorem and transitive actions (though it might be helpful to recall them from last year).

Definition 2.5 (Group Action)**.** Let G be a group and X a set, then a group action $(\cdot)$ of G on X is a function $G \times X \to X$ such that for all $g, h \in G$, $x \in X$, we have

- $g \cdot (h \cdot x) = gh \cdot x$,
- $1 \cdot x = x$.

Equivalently, a group action can be represented by a group homomorphism between G to S_n if $|X| = n < \infty$. We note that there exists an bijection between $\mathrm{Perm}(X)$ (a.k.a $\mathrm{Aut}(X)$ though we will avoid this term in case X has additional structures) and S_n with depends on a choice of $X \simeq \{1, \cdots, |X|\}$.

Definition 2.6 (Kernel)**.** A kernel of a representation (or group action) is simply the kernel of the corresponding group homomorphism, i.e. if ρ is a representation (or group action),

$$\ker \rho := \{g \in G \mid \rho(g) = \mathrm{id}\}.$$

We say a representation (or group action) is faithful if $\ker \rho = \{e\}$, i.e. ρ is injective.

Definition 2.7 (Morphism of Group Actions)**.** A morphism $T : X \to Y$ of group actions on X and Y is a map such that $T(g \cdot x) = g \cdot T(x)$ for all $g \in G$, $x \in X$.

This is also called a "G-equivariant map" from X to Y and one can see the resemblance of this definition and the definition for homomorphisms between representations.

For any group G, it acts on itself in three different ways. In particular, we have the left regular action $g \cdot h = gh$, the right regular action $g \cdot h = hg^{-1}$ (where the inverse is required for associativity) and the adjoint action $g \cdot h = ghg^{-1}$. One can see that the left and right regular actions are isomorphic via $T(g) = g^{-1}$. On the other hand, they are not isomorphic to the adjoint action (consider $\rho_{\mathrm{ad}}(g)(e) = e$ for all $g \in G$).

Proposition 2.3. Given two actions (or representations) ρ, ρ' on G, $g \mapsto \rho(g)\rho'(g)$ is an action (or representation) if and only if $\rho(g)\rho'(g) = \rho'(g)\rho(g)$, that is ρ and ρ' are commuting actions.

Definition 2.8. A subset $Y \subseteq X$ is said to be stable under an action $(\cdot)$ of G on X if $g \cdot y \in Y$ for all $y \in Y, g \in G$.

In the case that $Y \subseteq X$ is stable, then we may restrict the action on Y to obtain a new action of G on Y.

Definition 2.9 (Orbit)**.** Let $x \in X$, then $G \cdot x := \{g \cdot x \mid g \in G\}$ is called an orbit of x and we denote this by $\operatorname{orb}(x)$.

It is not difficult to see that orbits are stable and in fact, as an exercise, one might show that $Y \subseteq X$ is stable if and only if it is a union of orbits.

In a group G under the adjoint action, we see that the orbits are the conjugacy classes[2]. Thus, for every conjugacy class, we obtain a action on that class from the adjoint action on the whole group.

Example 2.1. Let $G = S_4$ and let $c = \{(12)(34), (13)(24), (14)(23)\}$. Then as c is a conjugacy class, we have the adjoint action on c

$$\phi : S_4 \to \operatorname{Perm}(c) \cong S_3.$$

It is not difficult to show that ϕ is surjective and $\ker\phi = c \cup \{e\} \cong K_4 \cong C_2 \times C_2$. Thus, by the first isomorphism theorem we have

$$S_3 \cong S_4/K_4.$$

Definition 2.10. Given a finite set X, let

$$\mathbb{C}[X] := \left\{ \sum_{x \in X} a_x x \mid a_x \in \mathbb{C} \right\},$$

equipped with the addition $\sum_{x\in X} a_x x + \sum_{x \in X} b_x x = \sum_{x\in X}(a_x + b_x)x$ and scalar multiplication $c \cdot \sum_{x \in X} a_x x = \sum_{x\in X}(ca_x)x$. This sum here does not represent some addition operation on X but a notational trick. One might instead consider elements of $\mathbb{C}[X]$ as functions $a : X \to \mathbb{C}$ equipped with point-wise addition and scalar multiplication.

We observe that $\mathbb{C}[X] \cong \mathbb{C}^{|X|}$ depending on a choice of $X \cong \{1, \cdots, |X|\}$. Furthermore, we have $X \subseteq \mathbb{C}[X]$ and is a basis (if we interpret $\mathbb{C}[X]$ as a space of functions, the canonical basis is $\{a_x : y \mapsto \chi_{\{x\}} \mid x \in X\}$). In the case that X is infinite we can still define $\mathbb{C}[X]$ allowing only finite sums.

Proposition 2.4. If $(\cdot)$ is a group action of G on X, then, the map $(g, \sum a_x x) \mapsto \sum a_x (g \cdot x)$ is a group action of G on $\mathbb{C}[X]$.

Definition 2.11. The left regular, right regular, adjoint representations are representations

$$\tilde{\rho}_L, \tilde{\rho}_R, \tilde{\rho}_{\mathrm{ad}} : G \to GL(\mathbb{C}[G])$$

obtained from the left regular, right regular and adjoint actions

$$\rho_L, \rho_R, \rho_{\mathrm{ad}} : G \to \operatorname{Perm}(G).$$

Proposition 2.5. If X is any set with a G-actin, then for all $g \in G$, $[\rho_{\mathbb{C}[X]}(g)]_B$ is always a permutation matrix.

Proof. Exercise. $\square$

[2] What are the orbits of the left action?

Definition 2.12. Given $(V, \rho_V), (W, \rho_W)$ representations of G, we denote

$$\mathrm{Hom}_G(V, W) := \{T : V \to W \mid G\text{-linear}\}.$$

Proposition 2.6. Let (V, ρ_V) be a representation of G and $v \in V$. Then, there exists a unique homomorphism of representations $\mathbb{C}[G] \to V$ where $\mathbb{C}[G]$ is equipped with the left regular representation such that $e_G \mapsto v$ and thus,

$$\mathrm{Hom}_G(\mathbb{C}[G], V) \cong (V, \rho_V).$$

Proof. For all $g \in G$, $c = \sum_{h \in G} a_h h$, we have

$$\begin{aligned} T(g \cdot c) = \rho_V(g)(Tc) &\iff T\left(\sum a_h gh\right) = \rho_V(g)\left(T\left(\sum a_h h\right)\right) \\ &\iff \sum a_h T(gh) = \sum a_h \rho_V(g)(Th) \\ &\iff T(gh) = \rho_V(g)(Th), \ \forall h \in G, \end{aligned}$$

where the second if and only if follows as both T and ρ_V are linear. Then choosing $h = e_G$, we have $T(g) = \rho_V(g)(v)$ and thus T is uniquely determined on G and hence is unique as G is a basis of $\mathbb{C}[G]$.

It remains to show that the map T defined by $g \mapsto \rho_V(g)(v)$ is a homomorphism of representations. This is clear since

$$\begin{aligned} T(g \cdot c) &= T\left(\sum a_h gh\right) = \sum a_h T(gh) \\ &= \sum a_h \rho_V(gh)(v) = \sum a_h \rho_V(g)(\rho_V(h)(v)) \\ &= \sum a_h \rho_V(g)(Th) = \rho_V(g)\left(T\left(\sum a_h h\right)\right), \end{aligned}$$

where the fourth equality follows by the associativity of group actions. □

2.2 Subrepresentation and Quotient Representation

Definition 2.13 (Subrepresentation)**.** A subrepresentation of a representation (V, ρ_V) is a subspace $W \leq V$ such that $\rho_V(g)(W) \subseteq W$ for all $g \in G$.

Clearly, both $\{0\}$ and V are subrepresentations of (V, ρ_V), and we say a representation is irreducible if these two subrepresentations are the only subrepresentations. We say a representation is reducible if it is not irreducible. In general, every 1-dimension representation is irreducible.

Proposition 2.7. Irreducibility is invariant under isomorphisms.

Proof. Exercise. □

Proposition 2.8. Let G be finite and (V, ρ_V) is an irreducible representation of G. Then $\dim V < \infty$.

Proof. Let $w \in V \ \{0\}$ and let $W := \mathrm{span}(\{\rho_V(g)(w) \mid g \in G\})$ which is a finite dimensional subrepresentation as G is finite and for all $h \in G$, $\rho_V(h)(\rho_V(g)(w)) = \rho_V(hg)(w)$. Thus, if $\dim V$ is not finite, we have found a proper subrepresentation which contradicts the irreducibility of (V, ρ_V). □

Definition 2.14 (Quotient Representation)**.** For $W \leq V$ a subrepresentation, the quotient representation is $(V/W, \rho_{V/W})$ given by

$$\rho_{V/W}(g)(v+W) := \rho_V(g)(v) + W.$$

This is well-defined as W is stable under ρ_V.

Proposition 2.9. For $T : (V, \rho_V) \to (W, \rho_W)$ a G-linear map, $\ker T$ and $\mathrm{Im} T$ are subrepresentations.

Proof. Let $v \in \ker T$, then $T(\rho_V(g)(v)) = \rho_W(g)(Tv) = \rho_W(g)(0) = 0$ implying $\rho_V(g)(v) \in \ker T$ and thus, $\ker T$ is a subrepresentation. On the other hand, for all $w \in \mathrm{Im} T$, there exists some $v \in V$ such that $Tv = w$. Then $\rho_W(g)(w) = \rho_W(g)(Tv) = T\rho_W(g)(v)$ implying $\rho_W(g)(w) \in \mathrm{Im} T$ showing $\mathrm{Im} T$ is also a subrepresentation. $\square$

Proposition 2.10. For $T : (V, \rho_V) \to (W, \rho_W)$ a G-linear map, we have

$$\mathrm{Im} T \cong V / \ker T.$$

Proof. Follows from the first isomorphism for vector spaces and it remains to check $V / \ker T \to \mathrm{Im} T$ is G-linear. $\square$

Proposition 2.11. If $T \in \mathrm{End}_G V$ is a G-linear projection (i.e. $T^2 = T$), then V is a direct sum of subrepresentations $\ker T \oplus \mathrm{Im} T$.

Proof. Follows from the vector space case. $\square$

2.3 Maschke's Theorem and Schur's Lemma

Recalling internal and external direct sums of vector spaces, we will in this section introduce and prove a powerful result in representation theory known as the Maschke's theorem.

Definition 2.15 (Decomposable)**.** The representation (V, ρ_V) is decomposable if there exists a decomposition $V = U \oplus W$ where U, W are non-zero subrepresentations.

Definition 2.16 (Semisimple)**.** The representation (V, ρ_V) is semisimple if there exists a irreducible subrepresentations $W_1, \cdots, W_n$ such that

$$V = \bigoplus_{i=1}^n W_i.$$

Theorem 1 (Maschke's Theorem)**.** If G is finite, then for all $W \leq V$ subrepresentations of (V, ρ_V), there exists a complementary subrepresentation U, $V = W \oplus U$.

A direct consequence of Maschke's theorem is that every finite-dimensional representation of G is semisimple.

Maschke's theorem does not hold in the case that G is not finite. Consider $G = \mathbb{Z}$ and let

$$\rho : g \to GL_2(\mathbb{C}) : m \mapsto \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}.$$

Then the only non-zero proper subrepresentation is span$\{e_1\}$ since e_1 is the only eigenvector and as ρ is a 2-dimensional representation, the only non-zero proper subrepresentation is 1-dimensional, hence an eigenspace. Thus, ρ is indecomposable but not irreducible, and hence not semisimple.

Proof of Maschke's Theorem. To prove the theorem, we will attempt to find some G-linear map $T : V \to V$ that is a projection, i.e. $T^2 = T$, such that $\mathrm{im}T = W$ and so $V = \ker T \oplus \mathrm{Im}T = \ker T \oplus W$.

In the case of linear maps, a map satisfying the above proposition must map

$$T(u + w) = Tu + Tw = 0 + w = w,$$

where $u \in \ker T$ and $w \in W$. As, $\ker T \oplus W = V$, this property uniquely identifies T on V. However, this map is not G-linear and so we will modify T such that it is G-linear.

By recalling that a linear map is G-linear if and only if it is conjugate with it self by $\rho(g)$ for all $g \in G$, let us define

$$\tilde{T} := \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1},$$

such that it is in some sense the average of all conjugates over all g.

We will now show that $\tilde{T}$ is a G-linear projection and $\mathrm{im}\tilde{T} = W$. Indeed, for all $h \in G$, we have

$$\rho_V(h) \circ \frac{1}{|G|} \left(\sum_{g \in G} \rho_V(g) \circ T \circ \rho_V(g)^{-1} \right) \circ \rho_V(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho_V(hg) \circ T \circ \rho_V((hg)^{-1}) = \tilde{T},$$

as $g \mapsto h \cdot g$ is bijective as it has the inverse $g \mapsto h^{-1}g$. On the other hand, it is clear that $\tilde{T}(V) \subseteq W$ as for all $v \in V$, $T(\rho_V(g)^{-1}v) \in W$, and as W is a subrepresentation, we have $\rho_V(g)T(\rho_V(g)^{-1}v) \in W$. Thus, as $\tilde{T} \mid_W = \mathrm{id} \mid_W$, since

$$\tilde{T}w = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)T(\rho_V(g)^{-1}w) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)\rho_V(g)^{-1}w = \frac{1}{|G|} \sum_{g \in G} w = w,$$

we have $\mathrm{Im}\tilde{T} = W$ and $\tilde{T}$ is a projection. $\square$

We note that we used the property of $\mathbb{C}$ precisely when we needed $|G|^{-1}$ and thus, the same proof works for all field in which $|G|$ is invertible, i.e. of characteristic not a factor of $|G|$.

This decomposition needs not be unique. Indeed, if V, ρ_V is a trivial representation with dimension > 1. Then any vector space decomposition is a decomposition of representations. For example, if $V = \mathbb{C}^2$, then

$$\mathbb{C}^2 = \{(a, 0)\} \oplus \{(0, b)\} = \{(a, a)\} \oplus \{(b, -b)\},$$

and in fact any pair of subspaces spanned by two linearly independent basis form a decomposition.

Proposition 2.12. For $G = C_m = \langle g \rangle$, (V, ρ_V) a representation of G, there exists a unique decomposition of (V, ρ_V) into 1-dimensional subrepresentations if and only if $\rho_V(g)$ has distinct eigenvalues.

Proof. Exercise. $\square$

Lemma 2.1 (Schur's Lemma)**.** Let V and W be irreducible representations of G, then

- every G-linear map $T : V \to W$ is invertible or zero;
- let $V = W$ be finite-dimensional, then every G-linear map $T : V \to V$ is a multiple of the identity, i.e. $\mathrm{End}_G V = \mathbb{C} \cdot \mathrm{id}$.

We note that the first property does not require V, W to be finite dimensional, and in fact it is true for arbitrary fields. On the other hand the second property only works for algebraically closed fields.

Proof. The first property is rather trivial. Indeed, if $T \neq 0$ then $\ker T$ must be $\{0\}$ as $\ker T$ is a subrepresentation and V is irreducible. Similarly, for the same reason $\mathrm{Im} T = W$ and thus, T is bijective.

For the second property, we recall that the eigenvalues of T are the roots of the characteristic polynomial of T. Now since $\mathbb{C}$ is algebraically closed, there exists some $\lambda \in \mathbb{C}$ such that $\ker(\lambda I - T) \neq \{0\}$. Now since $\ker(\lambda I - T)$ is a subrepresentation of V, as V is irreducible, we have $\ker(\lambda I - T) = V$. Thus, for all $v \in V$,

$$\lambda v - Tv = 0 \implies Tv = \lambda v \implies T = \lambda \cdot \mathrm{id}$$

as required. $\square$

Theorem 2. Up to isomorphism (and reordering), the representation decomposition is unique. That is, if $T : V := V_1 \oplus \cdots \oplus V_m \cong W := W_1 \oplus \cdots \oplus W_n$, then $V_i \cong W_i$ up to ordering.

Proof. We have $T : V \to W$ is a G-isomorphism map and so $T(V_i)$ is a subrepresentation of W. Then, as $W = W_1 \oplus \cdots \oplus W_n$, there exists some j such that $W_j \cap T(V_i) \neq \emptyset$. Thus, we have $T \mid_{V_i} : V_i \to W_j$ is a G-linear map between two irreducible representations. As $T \mid_{V_i} \neq 0$, by Schur's lemma, it follows $T(V_i) = W_i$. Now, since for $i \neq k$, $T(V_i) \neq T(V_k)$ as T is bijective and $V_i \cap V_k = \emptyset$, by pairing the V_i and W_j, we are able to correspond each V_i with a W_j. Reversing this process with T^{-1}, we are able to pair each W_j with a V_i and thus, we have $V_i \cong W_i$ up to ordering as required. $\square$

Theorem 3. In the case that V is finite dimensional, there is a unique decomposition $V = \bigoplus_{i=1}^n V_m$ (up to reordering) if and only if in some decomposition, V_i are all non-isomorphic.

Proof. Suppose first that $V_1 \oplus \cdots \oplus V_n = V$, and $V_1, \cdots, V_n$ are all non-isomorphic. Then, for all G-linear maps $T : V_i \to V$, we have by a similar argument as above, if $T \neq 0$, there exists some j such that $T : V_i \cong V_j$. But as V_i, V_j are non-isomorphic for $i \neq j$, we have $T \in \mathrm{End}_G V_i$. Thus, by the second part of Schur's lemma, there exists some λ such that $T = \lambda \cdot \mathrm{id}\,|_{V_i}$. Now, if $V = W_1 \oplus \cdots \oplus W_m$, by the above theorem, WLOG, we have $T_i : V_i \cong W_i \subseteq V$. But we just shown $T_i = \lambda \cdot \mathrm{id}\,|_{V_i}$ and so, $V_i = W_i$ and the decomposition is unique.

Conversely, consider that for any representation V_i, there exists infinitely many subrepresentations of $V_i \oplus V_i$ by taking

$$V_a := \{(v, av) \mid v \in V\}.$$

Thus, if $V_i \cong V_j$, we have

$$V_a \leq V_i \oplus_{ext} V_i \cong V_i \oplus_{ext} V_j \cong V_i \oplus_{int} V_j.$$

Denoting the isomorphism from $V_i \oplus_{ext} V_i$ to $V_i \oplus_{int} V_j$ as T, we have $T(V_a) \leq V_i \oplus_{int} V_j$ and by Maschke's theorem, there exists some subrepresentation U such that, U is complement to $T(V_a)$ and

$$T(V_a) \oplus_{int} U = V_i \oplus_{int} V_j.$$

Thus, as T is an isomorphism, $a \neq b$ implies $T(V_a) \neq T(V_b)$, we have found infinitely many non-equal decompositions of $V_i \oplus_{int} V_j$ and hence, also V. $\square$

Corollary 3.1. For $V_1, \cdots, V_n$ irreducible, non-isomorphic, subrepresentations of V such that $V = V_1 \oplus \cdots \oplus V_n$ all subrepresentations of V are of the form $V_{i_1} \oplus \cdots \oplus V_{i_m}$. In particular V has 2^n different subrepresentations.

2.3.1 Representation of Abelian Groups

Definition 2.17 (Centre)**.** The centre of a group G is the subgroup with the underlying set

$$Z(G) := \{z \in G \mid zg = gz, \forall g \in G\}.$$

Proposition 2.13. Let (V, ρ_V) be a finite dimensional irreducible representation and let $z \in Z(G)$. Then $\rho_V(z)$ is a scalar, i.e. $\rho_V(z) = \lambda \cdot \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.

Proof. For all $g \in G$, we have $\rho_V(z)\rho_V(g) = \rho_V(zg) = \rho_V(gz) = \rho_V(g)\rho_V(z)$. So $\rho_V(z) : V \to V$ is G-linear. Thus, by the second property of Schur's lemma, there exists some λ such that $\lambda \cdot \mathrm{id}_V$. $\square$

Corollary 3.2. If G is abelian and if (V, ρ_V) is an irreducible representation of G, then $\dim V = 1$.

Proof. As for abelian groups G, $Z(G) = G$, we have $\rho_V(g) = \lambda_g \cdot \mathrm{id}_V$ for all $g \in G$. Then, for all subspaces W of V, W is a subrepresentation. However, as V is irreducible, W must either be zero or V, and thus, $\dim V = 1$. $\square$

Corollary 3.3. If G is finite and abelian, then every finite dimensional representation is a direct sum of 1-dimensional subrepresentations.

The finite dimensional condition is necessary. Indeed, if F be a field $C \subsetneq F$ (then $\dim F = \infty$), e.g.

$$F := \left\{ \frac{P(X)}{Q(X)} \mid P, Q \in \mathbb{C}[X], Q \neq 0 \right\},$$

then F is an irreducible ∞-dimensional representation over $\mathbb{C}$ of $G = F^\times$ although the latter is abelian.

Corollary 3.4. The irreducible representations of C_m are up to isomorphism $(\mathbb{C}, \rho_\zeta)$ where ζ is a m-th root of unity. Furthermore, the irreducible representations of $C_{m_1} \times \cdots \times C_{m_l}$ are up to isomorphism,

$$\rho_{\zeta_1, \cdots, \zeta_l}(g_1^{j_1}, \cdots, g_l^{j_l}) = (\zeta_1^{j_1}, \cdots, \zeta_l^{j_l}).$$

The above corollary is important as every finite abelian group is an internal direct product of cyclic groups, we have classified all irreducible representations of finite abelian groups.

Proposition 2.14. Every 1-dimensional representation of a group G is of the form (V, ρ_V), $\rho_V(g) = \lambda_g \mathrm{id}_V$ for some $\lambda_g \in \mathbb{C}$. Furthermore, two such representations are isomorphic if and only if the λ_g are the same.

Proof. The first statement is clear while the second state follows since, if $(V, \rho_V : g \mapsto \lambda_g \mathrm{id}_V), (W, \rho_W : g \mapsto \lambda'_g \mathrm{id}_W)$ are two isomorphic 1-dimensional representations of G along $T : V \to W$, then given some $v \in V \quad \{0\}$, we have $T(v) \neq 0$ and

$$\lambda_g T(v) = T(\lambda_g v) = T(\rho_V(g)(v)) = \rho_W(g)(T(v)) = \lambda'_g T(v).$$

Thus, $\lambda_g = \lambda'_g$. $\square$

As all 1-dimensional vector spaces over $\mathbb{C}$ are isomorphic to $GL_1(\mathbb{C}) \cong \mathbb{C}^\times$, we can simply only consider $\mathbb{C}^\times$ as the vector space of all 1-dimensional representations.

Consider the 1-dimensional representations on S_n. Let $(\mathbb{C}^\times, \rho)$ be a representation of S_n, then by considering $\sigma(a_1, \cdots, a_n)\sigma^{-1} = (\sigma(a_1), \cdots, \sigma(a_n))$, we have

$$(ij) = ((1i)(2j))(12)((2j)(1i)),$$

and so (ij) is conjugate to (12) for all i, j. Furthermore, by considering $\rho(ij)^2 = \rho((ij)^2) = \rho(e) = \mathrm{id}$, $\rho(ij) = \pm\mathrm{id}$. Now, by recalling that $\sigma = \tau\sigma'\tau^{-1} \iff \mathrm{sign}(\sigma) = \mathrm{sign}(\sigma')$, and the fact that every permutation can be represented as a product of transpositions, either, $\rho(\sigma) = \mathrm{sign}(\sigma) \cdot \mathrm{id}$ or $\rho(\sigma) = \mathrm{id}$. Thus, there exists exactly two 1-dimensional representation up to isomorphism.