Functional Analysis

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1 Introduction

We have thus far looked at abstract vector spaces in linear algebra and (metric) topological spaces in topology. In this course, we will combine these concepts and study linear metric space. In particular, we will study vector spaces equipped with a topology such that certain properties are satisfies.

In this course, we will often study the space of functions and hence the name of the course. As we have seen before, given that the codomain space possesses a certain structure, it is possible to define point-wise addition and scalar multiplications on functions, and thus, possible to equip the space with a vector space structure.

Let us recall some definitions.

Definition 1.1 (Metric)**.** A metric ρ on a non-empty set X is a function with type signature $X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

- $\rho(x, y) = 0 \iff x = y$;
- $\rho(x, y) = \rho(y, x)$;
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Definition 1.2 (Translation Invariant)**.** A metric space (V, ρ) where V is equipped with the binary operation $(+) : V \times V \to V$ is translational invariant if for all $w, z, v \in V$,

$$\rho(w + v, z + v) = \rho(w, z).$$

Definition 1.3 (Norm)**.** A norm $\|\cdot\|$ on the vector space V (over the field $\mathbb{K}$ equipped with a modulus $|\cdot|$) is a function with type signature $X \to \mathbb{R}^+$ such that for all $x, y \in V, k \in \mathbb{K}$,

- $\|x\| = 0 \iff x = 0$;
- $\|k \cdot x\| = |k|\|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

We recall that a norm induces a metric by defining $\rho(x, y) = \|x - y\|$. In this case, it is possible to show that $(+)$ and $(\cdot)$ are continuous with respect to this metric and ρ is translational invariant.

Definition 1.4 (Banach Space)**.** A normed space is said to be a Banach space if it is complete, i.e. every Cauchy sequence converge.

Definition 1.5 (Separable)**.** A topological space is said to be separable if there exists a dense countable subset.

As we shall see, for $0 < 0 < \infty$, ℓ_p is separable while ℓ_∞ is not.

Definition 1.6 (Compact)**.** A topological space is said to be compact if every open cover has a finite sub-cover.

Unlike what we have seen before, as we consider infinite dimensional spaces, we will see that the Heine-Borel property will no longer hold, i.e. closed and bounded is no longer equivalent to compact.

2 Linear Spaces

Definition 2.1 (Equivalent Norms and Metrics)**.** Two norms $\|\cdot\|_k$ for $k = 1, 2$ are said to be equivalent if there exists some $M > 0$ such that for all x,

$$\frac{1}{M}\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

Similarly, two metrics ρ_k are said to be equivalent if there exists some $M > 0$ such that for all x, y,

$$\frac{1}{M}\rho_1(x, y) \leq \rho_2(x, y) \leq M\rho_1(x, y).$$

It is clear that equivalent is a symmetric relation and as we have seen before, all norms on a finite dimensional space are equivalent.

Definition 2.2 (Concave and Convex Function)**.** A function $f : V \to \mathbb{R}$ is

- concave if for all $s \in [0, 1], x, y \in V$, we have

$$sf(x) + (1 - s)f(y) \leq f(sx + (1 - s)y);$$

- convex if for all $s \in [0, 1], x, y \in V$, we have

$$sf(x) + (1 - s)f(y) \geq f(sx + (1 - s)y).$$

Proposition 2.1. If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is concave and $f(0) = 0$. Then

$$f(x + y) \leq f(x) + f(y).$$

Proof. Clear by taking $s = \frac{y}{x+y}$, we have

$$(1 - s)f(x + y) = sf(0) + (1 - s)f(x + y) \leq f(s \cdot 0 + (1 - s)(x + y)) = f(x),$$

and

$$sf(x + y) = sf(x + y) + (1 - s)f(0) \leq f(s(x + y) + (1 - s) \cdot 0) = f(y).$$

Adding the two equations, we have

$$f(x + y) = (1 - s)f(x + y) + sf(x + y) \leq f(x) + f(y).$$

□

Corollary 2.1. If ρ is a metric and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave and vanishing at 0, then $\rho \circ \eta$ is also a metric.

Definition 2.3 (Linear Metric Space)**.** A vector space V over the field $\mathbb{K}$ equipped with a metric ρ on V and a metric $|\cdot - \cdot|$ on $\mathbb{K}$ is a linear metric space if $(+) : V \times V \to V$ and $(\cdot) : \mathbb{K} \times V \to V$ are continuous with respect to the induced metric.

Proposition 2.2. Any normed space is a linear metric space.

Proof. Let $(x_n, y_n) \to (x, y)$ in V^2, then we have

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0.$$

Thus, $(+)$ is continuous.

Similarly, if $(\lambda_n, x_n) \to (\lambda, x)$ in $\mathbb{K} \times V$,

$$\begin{aligned}\|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x\| \\ &\leq \|\lambda_n x_n - \lambda_n x\| + \|\lambda_n x - \lambda x\| \\ &= |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\|.\end{aligned}$$

Now, since (λ_n) is convergent, it is bounded by some $M > 0$ and thus,

$$\|\lambda_n x_n - \lambda x\| \leq |\lambda_n|\|x_n - x\| + |\lambda_n - \lambda|\|x\| \leq M\|x_n - x\| + |\lambda_n - \lambda|\|x\| \to 0,$$

implying $(\cdot)$ is continuous. □

2.1 Classical Spaces

We recall the L_p spaces from second year measure theory, and in particular, when we consider the counting measure μ, we have the nice property that

$$\int f \mathrm{d}\mu = \sum_{n=0}^{\infty} f(n),$$

and we no longer require a quotient to define the linear space as the only null-set is the empty set (thus, two function are a.e equal if and only if they are equal). In this special case, we call the resulting space ℓ_p with the p-norm

$$\|f\|_p = \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(\sum_{n=0}^{\infty} |f(n)|^p\right)^{\frac{1}{p}}.$$

We will use the sequence notation and write $a_n := a(n)$ for $a \in \ell_p$.

As this is simply a special case of the L_p space, the inequalities proved on the L_p space remains. We will recall them here for ℓ_p spaces.

Proposition 2.3 (Hölder's Inequality)**.** Let $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q \in (1, \infty)$. Then for $a = (a_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}} \in \ell_p$, we have

$$|\langle a, b\rangle| \leq \|a\|_p \|b\|_q,$$

where $\langle a, b\rangle := \sum_{i \in \mathbb{N}} a_i b_i$.

Proposition 2.4 (Minkowski's Inequality)**.** Let $a, b \in \ell_p$ for some $1 \leq p \leq \infty$. Then $a + b \in \ell_p$ and

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p.$$

Lemma 2.1. Let $I \subseteq \mathbb{R}$ be an interval and $\phi : I \to \mathbb{R}^+$ be a function. Then ϕ is convex if and only if for all $y \in I$, there exists a $\gamma \in \mathbb{R}$, such that for all $x \in I$,

$$\gamma(x - y) \leq \phi(x) - \phi(y).$$

Proposition 2.5 (Jensen's Inequality)**.** let $\phi \geq 0$ be convex and suppose $\sum_{i\in\mathbb{N}} \eta_i = 1$, $|\langle\alpha\rangle| < \infty$ and $\langle\phi(\alpha)\rangle$ (where $\langle\beta\rangle = \sum_{i\in\mathbb{N}} \eta_i\beta_i$), then

$$\phi(\langle\alpha\rangle) \leq \langle\phi(\alpha)\rangle.$$

Proof. By the above lemma, there exists some γ such that

$$\gamma(\alpha_j - \langle\alpha\rangle) \leq \phi(\alpha_j) - \phi(\langle\alpha\rangle).$$

Thus,

$$\begin{aligned} 0 = \sum \eta_i\gamma(\alpha_j - \langle\alpha\rangle) &\leq \sum \eta_i(\phi(\alpha_j) - \phi(\langle\alpha\rangle)) \\ &= \sum \eta_i\phi(\alpha_j) - \phi(\langle\alpha\rangle)\sum \eta_i \\ &= \langle\phi(\alpha_j)\rangle - \phi(\langle\alpha\rangle) \end{aligned}$$

where the first equality follows as γ is independent of the index i. □

Proposition 2.6. For $p < p'$, $\ell_p \subseteq \ell_{p'}$. On the other hand, if $\sum \eta_i = 1$, we have $\ell_p(\eta) \supseteq \ell_{p'}(\eta)$.

Definition 2.4. Let ϕ be a convex function such that $\phi(0) = 0$, $\phi(x) \to \infty$ as $x \to \infty$. If ϕ has the doubling property such that there exists some $M > 0$ such that for all $x \in \mathbb{R}$, $\phi(2|x|) \leq M\phi(|x|)$, then

$$V := \left\{a : \mathbb{N} \to \mathbb{R} \mid \sum \eta_i\phi(|a_i|) < \infty\right\}$$

where $\sum \eta_i = 1$ is a vector space with point-wise operations.

Proposition 2.7. Given a metric space (X, ρ), there exists a complete metric space $(\tilde{X}, \tilde{\rho})$ and an isometric embedding $\iota : X \to \tilde{X}$ such that for all $x, x' \in X$, $\tilde{\rho}(\iota(x), \iota(x')) = \rho(x, x')$.

Proof. We have seen similar ideas in the completion of $\mathbb{Q}$ though completing the space with equivalence classes of mutually Cauchy sequences as elements of $\tilde{X}$. □

As we have seen last year, the L_p spaces are complete, and thus are Banach spaces. Thus, we have ℓ_p spaces are also complete and are Banach spaces. In fact, the proof that ℓ_p spaces are complete is easier, in that one may show completeness through showing the point-wise limit of the sequences indeed belong to ℓ_p.

Definition 2.5. We define

- $c_0 := \{x \in \ell_\infty \mid \lim_{n\to\infty} x_n = 0\}$,
- $c := \{x \in \ell_\infty \mid \exists \lim_{n\to\infty} x_n\}$,

be subspaces of ℓ_∞.

Proposition 2.8. c_0 is complete.

Proof. Let $(x_i^n) \subseteq c_0$ be a Cauchy sequence and let $x_i = \lim_{n\to\infty} x_i^n$, then it suffices to show $\lim_{i\to\infty} x_i = 0$.

Since (x_i^n) is Cauchy, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|x^n - x^N\|_\infty < \frac{\epsilon}{2}.$$

Furthermore, as $x_i^N \to 0$ as $i \to \infty$, there exists some $I \in \mathbb{N}$ such that for all $i \geq I$, $|x_i^N| < \epsilon/2$. Thus, for all $i \geq I$, we have

$$\frac{\epsilon}{2} > \|x^n - x^N\|_\infty > |x_i^n - x_i^N| > |x_i^n| - \frac{\epsilon}{2} \implies \epsilon > |x_i^n|,$$

for all $n \geq N$. Hence, taking $n \to \infty$, we have

$$\epsilon > \lim_{n\to\infty} |x_i^n| = |x_i|,$$

implying $x_i \to 0$ as $i \to \infty$. □

Proposition 2.9. c is complete.

Proof. Similar to above, let (x_i^n) be Cauchy and define (x^n) such that for all $n \in \mathbb{N}$, $x^n = \lim_{i\to\infty} x_i^n$ and (x_i) such that $(x_i^n) \to (x_i)$ in ℓ_∞. It is not difficult to see that (x^n) is Cauchy and thus converges to some x.

$$\begin{tikzcd}
(x_i^n) \arrow[r, "n\to\infty"] \arrow[d, "i\to\infty"'] & (x_i) \arrow[d, "i\to\infty"] \\
(x^n) \arrow[r, "n\to\infty"] & x
\end{tikzcd}$$

Indeed, as (x_i^n) is Cauchy, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $p, q \geq N$, $\|x_i^p - x_i^q\|_\infty < \epsilon/3$. Furthermore, there exists some $I \in \mathbb{N}$ such that for all $j \geq I$, $|x_j^p - x^p|, |x_j^q - x^q| < \epsilon/3$. Thus,

$$|x^p - x^q| < |x_j^p - x^p| + |x_j^p - x_j^q| + |x_j^q - x^q| < \frac{\epsilon}{3} + \|x_i^p - x_i^q\|_\infty + \frac{\epsilon}{3} = \epsilon.$$

Hence, it remains to show $x_i \to x$ as $i \to \infty$. Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x^n - x| < \epsilon/3$. Furthermore, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, $\|x_i^n - x_i\|_\infty < \epsilon/3$. Then, for all $p \geq \max\{N, M\}$, there exists $I \in \mathbb{N}$ such that for all $j \geq I$, $|x_j^p - x^p| < \epsilon/3$. Finally,

$$|x_j - x| \leq |x_j - x_j^p| + |x_j^p - x^p| + |x^p - x| \leq \|x_i - x_i^p\|_\infty + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,$$

implying $x_i \to x$ as $i \to \infty$ and $(x_i) \in c$. □

2.1.1 Separability

In this section we will consider the separability of different classical spaces.

Proposition 2.10. The space ℓ_p is separable for all $p \geq 1$.

Proof. It is easy to see that the subspace of all finite sequences is dense in ℓ_p. Indeed, if $x \in \ell_p$ and x^n is the sequence such that $x_i^n = x_i$ for all $i \leq n$ and $x_i^n = 0$ for all $i > n$, we have

$$\|x - x^n\|_p^p = \sum_{i=0}^{\infty} |x_i - x_i^n|^p = \sum_{i=n+1}^{\infty} |x_i|^p$$

which tends to 0 as $n \to \infty$. Now, by defining $\ell_{\mathbb{Q},n}$ as the set of rational sequences with length n, we have $d := \bigcup_{n\in\mathbb{N}} \ell_{\mathbb{Q},n}$ is dense in the space of all finite sequences. Now, since d is countable as it is a countable union of countable sets, we have found a countable dense set of ℓ_p. □

Lemma 2.2. A metric space is X not separable if there exists an uncountable subset S such that for some some $k > 0$, $d(x, y) > k$ for all $x, y \in S$.

Proof. As subset of a separable *metric* space is separable, S, must be separable. But as $d(x, y) > k$ for all $x, y \in S$, no sequences of S but the constant sequence can converge. Thus, for all countable subsets of S, simply picking a point of S not in that subset suffices. □

Proposition 2.11. ℓ_∞ is not separable.

Proof. Clearly the set containing all sequences of only 0 and 1s is a subset of ℓ_∞. Furthermore, this sequence is uncountable as it bijects $\mathbb{R}$ be considering the binary representation of a real number. Thus, since all distinct elements in this set have distance 1 apart, the conclusion follows by the above lemma. □

Proposition 2.12. $C^k([a, b])$ (the set of k-differentiable functions from the interval $[a, b]$) for all $k \in \mathbb{N}$ is separable.

Proof. Recalling the Weierstass approximation theorem, we have for all continuous function f, there exists a sequence of polynomials f_n such that $f_n \to f$ uniformly. Thus, as the set of polynomials with rational coefficients is countable, and dense in the space of all polynomials, we have found a countable dense set of $C^0([a, b])$. Now as $C^k([a, b]) \subseteq C^0([a, b])$ for all $k \in \mathbb{N}$, we have $C^k([a, b])$ is separable as required. □

Definition 2.6 (Absolutely Convergent)**.** Let X be a normed space and let (x_n) be a sequence of X, then a series $\sum_{n\in\mathbb{N}} x_n$ is said to be absolutely convergent if $\sum_{n\in\mathbb{N}} \|x_n\| < \infty$.

Proposition 2.13. A normed space X is complete if and only if for all sequences (x_n) of X such that $\sum x_n$ is absolutely convergent implies x_n converges. Thus, a normed space is Banach if and only if this property is satisfied.

Proof. See problem sheet. □

By recalling the proof of the completeness of L_p spaces, we note that this property was used extensively. As a consequence, we see that $L_p(\mu)$ is separable if the measure μ is separable. In particular, the spaces $L_p(\mathbb{R}^n, \lambda)$ are separable for all n. On the other hand, L_∞ is in general not separable.

2.2 Hamel and Schauder Basis

In this small section we will introduce two new notions of basis for infinite dimensional spaces. In particular, we will introduce the Hamel basis which is a natural extension of the definition of basis for the finite dimensional case and the Schauder basis which is a notion of basis that incorporates the topological properties of the space.

Definition 2.7 (Linear Independent)**.** A set $W \subseteq V$ is linear independent if for all $(\lambda_i)_{i=1}^m \subseteq \mathbb{K}$, $(w_i)_{i=1}^n \subseteq W$,

$$\sum \lambda_i w_i = 0 \implies \lambda_i = 0$$

for all $i = 1, \cdots, n$.

Definition 2.8 (Hamel Basis)**.** A set $W \subseteq V$ is a Hamel basis for a linear space V if W is linearly independent and for all $x \in V$, there exists a unique finite linear combination of vectors $(w_i)_{i=1}^n \subseteq W$, $(\lambda_i)_{i=1}^m \subseteq \mathbb{K}$ such that

$$x = \sum \lambda_i w_i.$$

Proposition 2.14. Every linear space has a Hamel basis.

We will come back to the proof of this proposition after discussing an important result known as the Hahn-Banach theorem.

Definition 2.9 (Schauder Basis)**.** A set $W \subseteq V$ is a Schauder basis for a normed space V if W is countable, linearly independent, and for all $x \in V$, there exists a unique (possibly infinite)-sequence $(\lambda_i)_{i=1}^\infty \subseteq \mathbb{K}$ and $(w_i)_{i=1}^\infty \subseteq W$ such that

$$x = \sum_{i=1}^\infty \lambda_i w_i.$$

Proposition 2.15. If a Banach space X has a Schauder basis, then it is separable.

The proof of the above proposition is left as an exercise. It is notable that the reverse of the above is not true and was in fact a Scottish book problem which an counterexample was given by Per Enflo in 1972 who was awarded a live goose.

It is clear that for $p \geq 1$, the set $W := \{e_j \mid j \in \mathbb{N}\}$ where $(e_j)_i = \delta_{ij}$ is a Schauder basis of ℓ_p.

Recall that $c \subseteq \ell_\infty$ is the set of sequences which has a limit. By considering the sequence $x_n = 1$, we see that the set of standard basis vectors no longer form a basis of c as

$$\left\| x - \sum_{i=1}^n e_i \right\|_\infty = 1,$$

for all n. Defining $W := \{e_0 := (1, 1, \cdots)\} \cup \{e_j \mid j \in \mathbb{N}\}$, for all $(a_n) \in c$, let $\lambda_0 := \lim_{n\to\infty} a_n$ and $\lambda_n = a_n - \lambda_0$. Then, for all $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - \lambda_0| \leq \epsilon$. Thus,

$$\left\| \lambda_0 e_0 + \sum_{i=1}^n \lambda_i e_i - a \right\|_\infty = \|(0, \cdots, 0, \lambda_0 - a_n, \lambda_0 - a_{n+1}, \cdots)\|_\infty < \epsilon,$$

implying W is a Schauder basis of c.

Furthermore, one may show that $C([0,1])$ has a Schauder basis consisting of all the "spike" functions at the points $k2^{-n}$ for all $n \in \mathbb{N}$, $k = 1, \cdots 2^{-n}$.

2.3 Hilbert Spaces

Recall the definition of sesquilinear forms over some vector space $\mathbb{H}$.

Definition 2.10 (Sesquilinear Form)**.** A sesquilinear form $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ on $\mathbb{H}$ where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ is a function such that

- it is linear with respect to the second argument;
- and is conjugate symmetric.

We say $\langle \cdot, \cdot \rangle$ is nondegenerate if $x = 0$ if and only if $\langle x, y \rangle = 0$ for all $y \in \mathbb{H}$. Nondegenerate sesquilinear forms are called scalar products and a vector space equipped with a scalar product is called a unitary space (or a scalar product space).

We see that a scalar product induces a norm by defining $\|f\|^2 := \langle f, f \rangle$. In particular, we recall the $\ell_2, C([a,b])$ and L_2 are all scalar product spaces.

Proposition 2.16. Let $\mathbb{H}$ be a unitary space, then

- $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$ (parallelogram identity);
- $\langle f, g \rangle = \frac{1}{4}(\|f+g\|^2 - |f-g\|^2)$ if $\mathbb{K} = \mathbb{R}$ and $\langle f, g \rangle = \frac{1}{4}\sum_{k=0,\cdots 3} i^k \|f + i^k g\|^2$ if $\mathbb{K} = \mathbb{C}$ (polarisation identity).

Definition 2.11 (Hilbert Space)**.** A unitary space is a Hilbert space if it is complete with respect to its induced norm.

Definition 2.12 (Convex Set)**.** A set S is said to be convex if for all $x, y \in S$, $(1-t)x + ty \in S$ for all $t \in [0,1]$.

Proposition 2.17 (Nearest Point Property)**.** Every nonempty closed convex set $\mathcal{K}$ in a Hilbert space $\mathbb{H}$ contains a vector of the smallest norm. Moreover, if $h \in \mathbb{H}$, there exists a unique $h_0 \in \mathcal{K}$ such that

$$\|h - h_0\| = \operatorname{dist}(h, \mathcal{K}) := \inf_{k \in K} \|h - k\|.$$

Proof. By the definition of infimum, there exists a sequence $(k_n) \subseteq \mathcal{K}$ such that

$$\lim_{n \to \infty} \|k_n\| \to d := \inf_{k \in \mathbb{K}} \|k\|.$$

Consider, for all $n, m \in \mathbb{N}$, by the parallelogram identity,

$$\left\| \frac{1}{2}(k_n - k_m) \right\|^2 = \frac{1}{2}(\|k_n\|^2 + \|k_m\|^2) - \left\| \frac{1}{2}(k_n + k_m) \right\|^2.$$

Then, as $\mathbb{K}$ is convex, $\frac{1}{2}(k_n + k_m) \in \mathbb{K}$ and so $\|(k_n + k_m)/2\|^2 \geq d^2$ and

$$\left\| \frac{1}{2}(k_n - k_m) \right\|^2 \leq \frac{1}{2}(\|k_n\|^2 + \|k_m\|^2) - d^2.$$

Thus, taking $n, m \to \infty$, the right hand side tends to zero and so (k_n) is Cauchy and hence convergent. Now as $\mathcal{K}$ is closed, the limit is in $\mathcal{K}$ and hence the statement. $\square$

Definition 2.13 (Orthogonal Systems)**.** Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be an unitary space. A set of vectors $\{e_j \in \mathbb{H} \mid j \in J\}$ for some index set J is called an orthogonal system if for all distinct $i, j \in J$,

$$\langle e_i, e_j \rangle = 0.$$

If furthermore, $\langle e_i, e_i \rangle = 1$ for all $i \in J$, then we say the system is orthonormal.

Proposition 2.18. Every orthogonal system is linearly independent.

Proof. Exercise. $\square$

Using Zorn's lemma, one can show that every unitary space contains an orthogonal basis.

Definition 2.14 (Fourier Coefficients)**.** Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be an unitary space and let $\{e_j \in \mathbb{H} \mid j \in J\}$ be an orthogonal system. Then for each $f \in \mathbb{H}$, the Fourier coefficients with respect to the orthogonal system are

$$c_j := \frac{\langle e_j, f \rangle}{\|e_j\|^2}.$$

Proposition 2.19. If $f = \sum_{k \in \mathbb{N}} \alpha_k e_k$, then $a_j = c_j$.

Proof. Let $n < m$ and $S_m := \sum_{k=1}^{m} \alpha_k e_k$. Then, $\langle S_m, e_n \rangle = \overline{\alpha_n} \|e_n\|^2$. Thus, we have

$$|\overline{\alpha_n} \|e_n\|^2 - \langle f, e_n \rangle| = |\langle S_m, e_n \rangle - \langle f, e_n \rangle| = |\langle S_m - f, e_n \rangle| \leq \|S_m - f\| \|e_n\| \to 0$$

as $m \to \infty$. $\square$

Proposition 2.20. Suppose $(e_j)_{j \in \mathbb{N}}$ is orthogonal and $g(a_1, \cdots, a_n) := \left\| f - \sum_{j=1}^{n} a_j e_j \right\|^2$. Then g attains its minimum at $a_j = c_j$. Furthermore, we have

$$\sum_{j=1}^{\infty} |c_j|^2 \|e_k\|^2 \leq \|f\|^2.$$

This inequality is known as Bessel's inequality.

Proof. Exercise. $\square$

Definition 2.15. An orthogonal system $\{e_j \mid j \in J\}$ is called complete if for all $f \in \mathbb{H}$, if $\langle f, e_j \rangle = 0$ for all j, then $f = 0$.

Proposition 2.21. The following are equivalent

1. $(e_j)_{j \in \mathbb{N}}$ is complete;
2. $\left\| f - \sum_{j=1}^{n} c_j e_j \right\| \to 0$ as $k \to \infty$ where c_j are the Fourier coefficients;
3. $\|f\|^2 = \sum_{j=1}^{\infty} |c_j|^2 \|e_k\|^2$ for all $f \in \mathbb{H}$.

Proof. Exercise. $\square$

2.4 Finite Dimensional Spaces

Theorem 1. Let $(X, \|\cdot\|)$ be a finite dimensional normed space and let $\{e_i\}$ be a basis for X. Then, there exists some $M, m \in \mathbb{R}^+$ such that for all $x = \sum_{i=1}^n a_i e_i$,

$$m \sum_{i=1}^n |a_i| \leq \|x\| \leq M \sum_{i=1}^n |a_i|.$$

Proof. WLOG. assume $\sum_{i=1}^n |a_i| = 1$ since if otherwise, we may just scale x by $1/\sum |a_i|$. Then the function $f : (a_i) \mapsto \|\sum_i a_i e_i\|$ for all sequences (a_i). Now, by checking that f is continuous, and by considering that the set $\{(a_i) \mid \sum \|a_i = 1\|\}$ is a closed subspace, f must attain a minimum m implying the left hand side inequality. On the other hand, consider the inequality

$$\left\| \sum_{i=1} a_i e_i \right\| \leq \sum |a_i| \|e_j\| \leq \max_k \|e_k\| \sum |a_i| = \max_k \|e_k\|.$$

Hence, it suffices to choose $M = \max \|e_k\|$. □

A direct corollary is that all norms on finite dimensional spaces are equivalent.

Definition 2.16 (Equivalent Norms)**.** Two norms $\|\cdot\|_1, \|\cdot\|_2$ on X are said to be equivalent if there exists some $C \in \mathbb{R}^+$ such that

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1,$$

for all $x \in X$.

Proposition 2.22. Equivalence of norms is an equivalence relation.

Proof. Easy check. □

Corollary 2.2. Every norm on a finite dimensional space is equivalent.

Proof. If $\|\cdot\|_1, \|\cdot\|_2$ are two norms on X, then let $\{e_i\}$ be a basis of X. Thus, by simply defining the norm

$$\|x\| = \left\| \sum_{i=1}^n a_i e_i \right\| := \sum_i |a_i|,$$

we have $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ which is equivalent to $\|\cdot\|_2$. Hence, $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ by transitivity. □

Proposition 2.23. Every finite dimensional space over a complete field is complete.

Proof. Follows by considering the individual coefficients of a Cauchy sequence with respect to some basis is also Cauchy. □

Proposition 2.24. Every compact set of a normed space is closed and bounded.

Proof. As a normed space is a metric space, it suffices to consider sequential compactness. In particular, if a set is not closed, then it contains a sequence converging to a point not within the set. Then every sub-sequence of that sequence also converges to that point outside of the set, and so is not compact. On the other hand, if the set is not bounded, we can construct a sequence with norms tending to ∞. It is then clear that the sequence does not contain any convergent subsequence. □

We note that the converse of the above proposition is not true for infinite dimensional spaces. Indeed, we see that the canonical basis of ℓ_2 is closed and bounded but not complete.

Proposition 2.25. If $(X, \|\cdot\|)$ is a finite dimensional normed space, then every closed and bounded set is compact (this property is known as the Heine-Borel property).

Proof. Follows by choosing a subsequence for each coefficient by using Bolzano-Weierstrass. □

In fact this property is sufficient to determine whether or not a normed space is finite dimensional.

Lemma 2.3 (Riesz's lemma)**.** Let Y be a closed proper subspace of a subspace Z of X where $(X, \|\cdot\|)$ is a normed space. Then for any $\theta \in (0, 1)$, there exists some $z \in Z$ such that

$$\|z\| = 1 \text{ and } \|y - z\| \geq \theta$$

for all $y \in Y$.

Proof. Let $v \in Z \setminus Y$, and define $a := \inf_{y \in Y} \|y - v\| > 0$. Then, as Y is closed, $\|y_0 - v\|$ attains a for some $y_0 \in Y$. Thus,

$$0 < a = \|v - y_0\| \leq \frac{a}{\theta}.$$

Then, defining

$$z := \frac{v - y_0}{\|v - y_0\|} =: c(v - y_0),$$

we have for all $y \in Y$,

$$\|z - y\| = \|c(v - y_0) - y\| = c\left\|v - \left(y_0 + \frac{1}{c}y\right)\right\|.$$

Since $y_0 + \frac{1}{c}y =: y_1 \in Y$, we have $\|v - y_1\| \geq a$, and so

$$\|z - y\| = c\|v - y_1\| \geq ca = \frac{a}{\|v - y_0\|} \geq \frac{a}{a/\theta} = \theta.$$

□

Proposition 2.26. A normed space is finite dimensional if and only if $\{x \mid \|x\| = 1\}$ is compact.

Proof. Clearly, if X is finite dimensional, then as $\{x \mid \|x\| = 1\}$ is closed and bounded, it is compact.

Let $(X, \|\cdot\|)$ be an infinite dimensional normed space and suppose $\{e_i\}_{i=1}^{\infty}$ is a countable infinite linearly independent subset of X. Define $Z_n = \operatorname{span}\{e_i \mid i = 1, \cdots, n\}$, and for each n, by Riesz's lemma, define $z_n \in Z_{n+1}$ such that $\|z_n\| = 1$ and $\|z_n - z_i\| \geq 1/2$ for all $i = 1, \cdots, n$. Thus, we have defined a sequence in $\{x \mid \|x\| = 1\}$ which does not contain any convergent subsequence, and hence, $\{x \mid \|x\| = 1\}$ is not compact. $\square$

3 Linear Operators

3.1 Bounded Linear Operators

Definition 3.1 (Bounded Linear Operator)**.** A linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is a bounded linear operator if there exists some $c \in \mathbb{R}^+$ such that,

$$\|Tx\|_Y \leq C\|x\|X$$

for all $x \in X$. In this case, one may define the operator norm by,

$$\|T\| := \sup_{x \in X; x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Proposition 3.1. Every linear map from a finite dimensional normed spaces is bounded.

Proof. Simply bound T by the maximum of the norm of the basis. □

Proposition 3.2. Let X, Y be Banach spaces and let $T : X \to Y$ be a continuous linear map, then for all compact $\mathcal{K} \subseteq X$, $T(\mathcal{K}) \subseteq Y$ is also compact.

Proof. Follows by recalling that the continuous image of a compact space is compact (by pull-back of the open-cover). □

Proposition 3.3. If $T : X \to Y$ is a linear map between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, then the following are equivalent,

1. T is continuous;
2. T is Lipschitz continuous;
3. T is bounded;
4. T is continuous at some point $x_0 \in X$.

Proof. Clearly (3) $\implies$ (2) $\implies$ (1) $\implies$ (4), so let us first show (4) $\implies$ (1)). If T is continuous at x_0, then for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in B_\delta(x_0)$, $\epsilon > \|Tx - Tx_0\|$. Then, for all $\tilde{x} \in X$, we have

$$\epsilon > \|Tx - Tx_0\| = \|Tx + T(\tilde{x} - x_0) - Tx_0 - T(\tilde{x} - x_0)\| = \|T(\tilde{x} + (x - x_0)) - T\tilde{x}\|.$$

Thus, for all $x \in B_\delta(\tilde{x})$, we have $x - \tilde{x} + x_0 \in B_\delta(x_0)$ and hence,

$$\epsilon > \|T(\tilde{x} + (x - \tilde{x} + x_0 - x_0)) - T\tilde{x}\| = \|Tx - T\tilde{x}\|,$$

which implies T is continuous.

Now, it suffices to show that (1) $\implies$ (3). Since T is continuous at 0, there exists some $\delta > 0$ such that for all $x \in B_\delta(0)$, $\|Tx\| < 1$. Then for all $x \in X$, we have

$$\left\| \frac{\delta x}{2\|x\|} \right\| = \frac{\delta}{2} < \delta,$$

and so,

$$1 > \left\| T \left\| \frac{\delta x}{2\|x\|} \right\| \right\| = \frac{\delta}{2\|x\|} \|Tx\|,$$

implying $Tx < 2\|x\|/\delta$ for all x. $\square$

Definition 3.2. We denote the space of linear bounded operators between two normed spaces X, Y equipped with the operator norm by $\mathcal{L}(X, Y)$.

Proposition 3.4. Let $(X, \|\cdot\|_X)$ be a normed space and $(Y, \|\cdot\|_Y)$ be a Banach space. Then $\mathcal{L}(X, Y)$ is also a Banach space.

Proof. Exercise. $\square$

We will now recall the Banach contraction mapping theorem.

Definition 3.3 (Contraction)**.** A map $T : X \to X$ on a metric space (X, ρ) is a contraction if there exists some α, $0 < \alpha \leq 1$ such that for all $x, y \in X$,

$$\rho(Tx, Ty) \leq \alpha \rho(x, y).$$

T is a strict contraction if $a < 1$.

Definition 3.4 (Fixed Point)**.** A point $x \in X$ is a fixed point of the map $T : X \to X$ if $Tx = x$.

Theorem 2 (Banach Contraction Mapping)**.** If X is a complete metric space and $T : X \to X$ is a strict contraction, then T has a unique fixed point.

Proof. Exercise/see second year analysis. $\square$