

# Martingales

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## 1 Discrete Time Martingales

**Definition 1** (Filtration). Let  $(X, \mathcal{A}, \mu)$  be a measure space, then a filtration on this measure space is a sequence  $(\mathcal{A}_n)_{n \in \mathbb{N}}$  of increasing sub- $\sigma$ -algebras of  $\mathcal{A}$ . If  $(X, \mathcal{A}_0, \mu)$  is  $\sigma$ -finite, then we call  $(X, \mathcal{A}, (\mathcal{A}_n), \mu)$  a  $\sigma$ -finite filtered measure space.

**Definition 2** (Natural Filtration). Given a sequence of functions  $(f_n : X \rightarrow Y)$  where  $Y$  is a measurable space, the natural filtration is the measure space  $(X, \sigma(f_n \mid n \in \mathbb{N}), \mu)$  equipped with the filtration

$$(\mathcal{A}_n) = (\sigma(f_1, \dots, f_n)).$$

It is clear that the natural filtration is the least filtration such that  $f_n$  is  $\mathcal{A}_n$ -measurable (and we call this property  $(f_n)$  is adapted to the filtration). As we shall see in the continuous case, one may define filtrations more generally with a pre-order index.

**Definition 3** (Discrete Time Martingale). Let  $(X, \mathcal{A}, (\mathcal{A}_n), \mu)$  be a  $\sigma$ -finite filtered measure space. Then a sequence of  $\mathcal{A}$ -measurable functions  $(f_n)_{n \in \mathbb{N}}$  is a martingale if  $f_n \in \mathcal{L}^1(\mathcal{A}_n)$  and

$$\int_A f_{n+1} d\mu = \int_A f_n d\mu$$

for all  $n \in \mathbb{N}$  and  $A \in \mathcal{A}_n$ .

Using probabilistic notation, the above property for martingales is simply  $\mathbb{E}[f_{n+1} \mid \mathcal{A}_n] = f_n$ .

**Definition 4** (Sub/Super-martingale). The sequence of  $\mathcal{A}$ -measurable functions  $(f_n)_{n \in \mathbb{N}}$  is a submartingale if  $f_n \in \mathcal{L}^1(\mathcal{A}_n)$  and

$$\int_A f_{n+1} d\mu \geq \int_A f_n d\mu$$

for all  $n \in \mathbb{N}$  and  $A \in \mathcal{A}_n$ . Similarly,  $(f_n)$  is a supermartingale if  $f_n \in \mathcal{L}^1(\mathcal{A}_n)$  and

$$\int_A f_{n+1} d\mu \leq \int_A f_n d\mu$$

for all  $n \in \mathbb{N}$  and  $A \in \mathcal{A}_n$ .

It is clear that a martingale is both a submartingale and a supermartingale and the negative of a submartingale is a supermartingale (and vice-versa).

**Proposition 1.1.** The sequence of  $\mathcal{L}^1$  functions  $(f_n)_{n \in \mathbb{N}}$  is a submartingale if and only if

$$\int \phi f_{n+1} d\mu \geq \int \phi f_n d\mu$$

for all  $\phi \in \mathcal{L}^\infty(\mathcal{A}_n) \cap \geq 0$ . Similar statements hold for martingales and supermartingales by the same proof.

*Proof.* The backwards implication is clear by choosing  $\phi = \mathbb{1}_A$  so let us consider the forward direction. By approximation by simple functions, there exists a sequence of simple functions  $(s_N = \sum_{i=0}^N \alpha_i \mathbb{1}_{A_i})_{N \in \mathbb{N}}$  such that  $s_N \uparrow \phi$ . Then, as  $\phi \in \mathcal{L}^\infty$ , there exists some  $M \in \mathbb{R}$  such that  $M \geq \phi \geq s_N$ . Thus, by dominated convergence theorem,

$$\begin{aligned} \int \phi f_{n+1} &= \int \lim_{N \rightarrow \infty} s_N f_{n+1} = \lim_{N \rightarrow \infty} \int s_N f_{n+1} = \lim_{N \rightarrow \infty} \sum_{i=0}^N \int_{A_i} f_{n+1} \\ &\geq \lim_{N \rightarrow \infty} \sum_{i=0}^N \int_{A_i} f_n = \lim_{N \rightarrow \infty} \int s_N f_n = \int \lim_{N \rightarrow \infty} s_N f_n = \int \phi f_n. \end{aligned}$$

□

**Proposition 1.2.** If  $(f_n)$  and  $(g_n)$  are martingales and  $\alpha, \beta \in \mathbb{R}$ , then  $\alpha f_n + \beta g_n$  is also a martingale. Similar statements hold for submartingales and supermartingales (with consideration to the sign of  $\alpha$  and  $\beta$ ).

Let us recall some definitions from probability theory.

**Definition 5** (Independence). Let  $(X, \mathcal{A}, \mu)$  be a probability space. A sequence of functions  $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}^1$  is said to be independent if

$$\mu \left( \bigcap_{n=1}^N f_n^{-1}(B_n) \right) = \prod_{n=1}^N \mu(f_n^{-1}(B_n)),$$

for all  $(B_n)_{n=1}^N \subseteq \mathcal{B}(\mathbb{R})$ .

**Lemma 1.1.** Given a sequence of independent functions  $(f_n)_{n=1}^{N+1} \subseteq \mathcal{L}^1$ , for all  $A \in \sigma(f_1, \dots, f_N)$ ,

$$\int_A f_{N+1} d\mu = \mu(A) \int f_{N+1} d\mu,$$

and for all  $\phi \in \mathcal{L}^1(\sigma(f_1, \dots, f_N))$ ,

$$\int \phi f_{N+1} d\mu = \int \phi d\mu \cdot \int f_{N+1} d\mu.$$

Furthermore,

$$\int \prod_{n=1}^k f_n d\mu = \prod_{n=1}^k \int f_n d\mu,$$

for all  $k = 1, \dots, N$ .

**Proposition 1.3.** Given a sequence of independent functions  $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}^1$ ,  $(s_n) := (\sum_{i=1}^n f_i)$  is a submartingale with respect to the natural filtration if and only if  $\int f_n \geq 0$  for all  $n$ .

*Proof.* The statement follows by the following chain of equalities,

$$\int_A s_{n+1} d\mu = \int_A s_n + f_{n+1} d\mu = \int_A s_n d\mu + \int_A f_{n+1} d\mu = \int_A s_n d\mu + \mu(A) \int f_{n+1} d\mu.$$

□