Classical Mechanics for Geometric Mechanics

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This document serves as a review for classical mechanics in order to appreciate geometric mechanics. In particular, we will recall...

Definitions

We define the notion of motion on smooth manifolds with particular focus on its interpretation for mechanics. For a more mathematical approach to these notions, see the manifolds course.

Definition 0.1 (Space)**.** Throughout this course, we will take space to be a smooth manifold Q with points $q \in Q$.

Intuitively, we may imagine smooth manifolds as general spaces on which we may do calculus on. As we shall see later, the manifold Q may also be identified with a Lie group G especially whenever we would like to consider rotation and translation.

Definition 0.2 (Time)**.** Time is a manifold T with $t \in T$. Usually $T = \mathbb{R}$ although we will also consider $T = \mathbb{R}^2$ or even more exotic cases in which T and Q are taken to be complex manifolds.

Definition 0.3 (Motion)**.** Motion is a map $\phi : T \to Q \to Q$ where we write $\phi(t) = \phi_t$ whenever there is no confusion. In the case that $T = \mathbb{R}$, the motion is a curve such that $q(t) = \phi_t(q(0))$. A motion ϕ is called a flow if $\phi_{t+s} = \phi_t \circ \phi_s$ for all $s, t \in \mathbb{R}$ and $\phi_0 = \mathrm{Id}$.

Definition 0.4 (Tangent Space)**.** The tangent space of a manifold Q at $q \in Q$, denoted by $T_q Q$, is the set of vectors $v_q = \dot{q}(t) \in T_q Q$ tangent to the curve $q(t) \in Q$ at the point q. Furthermore, we call v_q the tangent lift vector/velocity at q.

Mathematically, the tangent space is defined to be the set of curves through the point q quotiented by the tangency of two curves at q. In particular, two curves σ and τ are said to be tangent as q if $D\sigma \mid_0 = D\tau \mid_0$. This definition is independent of the chart though, if we fix a chart as q, we may transport the tangent space into $\mathbb{R}^{\dim Q}$.

Definition 0.5 (Tangent Bundle)**.** The tangent bundle is the union of all tangent spaces, i.e.

$$TQ = \bigcup_{q \in Q} T_q Q.$$

The tangent bundle is a useful concept as $\dot{q}(t)$ lives inside of TQ.

Definition 0.6 (Motion Equation)**.** The motion equation f is a vector field over Q such that $\dot{q}_t = f(q_t)$.

The Metric

The metric is a useful notion generally found in tensor calculus. We will quickly introduce them here with the goal of defining the Riemannian metric. We will work with $\mathbb{R}^n$ in this section.

Let $\{v^i\}$ be a coordinate system for $\mathbb{R}^n$ and (employing Einstein's notation) let

$$\mathbf{r} = x_i(v^i, ...)\mathbf{x}^i,$$

where $\{\mathbf{x}^i\}$ are the standard basis of $\mathbb{R}^n$ and x_i are functions of $\{v^i\}$. Then we define the set of vectors

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial v^i}, \mathbf{e}^i = \nabla v^i.$$

By calculation, we find $\mathbf{e}^i \cdot \mathbf{e}_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta function. This establishes a orthogonality relation between the vectors and it follows both $\{\mathbf{e}^i\}$ and $\{\mathbf{e}_i\}$ are linearly independent and hence are both basis of $\mathbb{R}^n$. Thus, for all vectors $\mathbf{X} \in \mathbb{R}^n$, there exists uniquely $\{X^i\}$ and $\{X_i\}$ such that $\mathbf{X} = X^i\mathbf{e}_i = X_i\mathbf{e}^i$. We call X^i the contravariant components of $\mathbf{X}$ and X_i to covariant components of $\mathbf{X}$ and it is not hard to see

$$\mathbf{e}_i \cdot \mathbf{X} = \mathbf{e}_i \cdot X_j\mathbf{e}^j = X_j\delta_{ij} = X_i.$$

Similarly, $X^i = \mathbf{e}^i \cdot \mathbf{X}$.

Definition 0.7 (Metric)**.** We define the metrics $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ and $g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j$.

Proposition 0.1. We have $g_{ij} = g_{ji}, g^{ij} = g^{ji}$ and for $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$,

$$\mathbf{X} \cdot \mathbf{Y} = g_{ij}X^iY^j = g^{ij}X_iY_j = X^iY_i = X_iY^i.$$

From this, since $\mathbf{X}$ is arbitrary, we deduce $Y^i = g^{ij}Y_j$ and $Y_i = g_{ij}Y^j$. Furthermore, as $Y_i = g^{ij}Y_j = g^{ij}g_{jk}Y^k$, we have $g^{ij}g_{jk} = \delta_{ik}$.

Variational Principles

Definition 0.8 (Kinetic Energy)**.** The kinetic energy is defined to be the function

$$KE : TM \to \mathbb{R} : \dot{q} \mapsto \frac{1}{2}g_q(\dot{q}, \dot{q}),$$

where $g_q : TM^2 \to \mathbb{R}$ is a Riemannian metric.

Definition 0.9 (Lagrangian)**.** The Lagrangian is a function $L : TM \to \mathbb{R}$ which is commonly chosen to be the kinetic energy.

Definition 0.10 (Hamilton's Principle)**.** Given a family of curves $q_\epsilon(t)$ for $(\epsilon, t) \in \mathbb{R}^2$ such that q is smooth with respect to both parameters, we say the system satisfies Hamilton's principle if $\delta S = 0$ where $S = \int_\gamma L(q, \dot{q})\mathrm{d}t$, $\gamma = q(\mathbb{R}^2)$ and $\delta = \partial/\partial\epsilon \mid_{\epsilon=0}$.

δS is known as the variational derivative of S near the identity $\epsilon = 0$. Intuitively, we interpret Hamilton's principle as the notion that some quantity along some curve does not change given a small perturbation to said curve.

Theorem 1. If Hamilton's principle is satisfied, i.e. $\delta S = 0$, then so is the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} = \frac{\partial L(q,\dot{q})}{\partial q}.$$

Definition 0.11 (Legendre Transform)**.** The Legendre transform is a function $LT_q : TQ \to T^*Q : \dot{q} \mapsto p$ that defined the momentum p as the fibre derivative of L, namely,

$$p := \frac{\partial L(q,\dot{q})}{\partial \dot{q}}.$$

The Legendre transform is invertible for $\dot{q} = f(q,p)$ provided the Hessian $\partial^2 L(q,\dot{q}) \big/ \partial \dot{q}^2$ has a non-zero determinant.

Definition 0.12 (Hamiltonian)**.** The Hamiltonian is the function

$$H_q : T^*Q \to \mathbb{R} : p \mapsto \langle p, \dot{q} \rangle - L(q,\dot{q}).$$