Fourier Analysis and the Theory of Distributions

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Contents

1 Orthonormal System

We will in this section recall some results about orthonormal systems in Euclidean spaces[1] and generalize them to complex spaces.

1.1 Euclidean Space

Definition 1.1. A system of nonzero vectors $\{X_\alpha\} \subseteq R$ where R is an Euclidean space is called orthogonal if $\langle X_\alpha, X_\beta \rangle = 0$ for all $\alpha \neq \beta$.

In addition, if for all α, $\langle X_\alpha, X_\alpha \rangle = 1$, we say the system is orthonormal.

Clearly, given an orthogonal system $\{X_\alpha\}$, we may normalize the vector such that $\{X_\alpha / \|X_\alpha\|\}$ is an orthonormal system. Furthermore, recall that a system of orthogonal vectors is linearly independent.

Definition 1.2. A complete (i.e. the smallest closed subspace containing the system is R) orthogonal system $\{X_\alpha\} \subseteq R$ is said to be an orthogonal basis of R.

Some important spaces we shall study in this course include $\mathbb{R}^2$ (equipped with the Euclidean norm), l_2, $\mathcal{C}([-\pi, \pi])$ (the space of continuous functions on $[-\pi, \pi]$ equipped with the L_2 norm).

Proposition 1.1. Let R be a separable Euclidean space. Then any orthogonal system in R is countable.

Proof. By normalizing, we may assume the system $\{X_\alpha\}$ is orthonormal. Then, for $\alpha \neq \beta$,

$$\|X_\alpha - X_\beta\|^2 = \|X_\alpha\|^2 - 2\langle X_\alpha, X_\beta \rangle + \|X_\beta\|^2 = \|X_\alpha\|^2 + \|X_\beta\|^2 = 2.$$

Then, $B_{1/2}(X_\alpha) \cap B_{1/2}(X_\beta) = \emptyset$ for all $\alpha \neq \beta$. Thus, if the system is not countable, we have found a uncountable number of disjoint open balls, contradicting the separability of R. □

Proposition 1.2. Let $f_1, f_2, \cdots$ be a linearly independent system in a Euclidean space R. Then, there exists an orthonormal system $\phi_1, \phi_2, \cdots$ such that

$$\phi_n = a_{n_1} f_1 + \cdots + a_{n_n} f_n$$

and

$$f_n = b_{n_1} \phi_1 + \cdots + b_{n_n} \phi_n$$

for some $a_{n_k}, b_{n_k} \in \mathbb{R}$ and $a_{n_n}, b_{n_n} \neq 0$. Furthermore, the system $\phi_1, \phi_2, \cdots$ is uniquely determined up to a multiplication by ± 1.

Proof. Use Gram-Schmidt. □

Corollary 0.1. A separable Euclidean space R possesses an orthonormal basis.

Proof. Simply obtain the orthonormal system corresponding to the countable dense system of R. The resulting system is complete since the two systems have the same linear closure. □

[1] In this course, we shall call real inner product spaces Euclidean spaces.

Definition 1.3 (Fourier Coefficients)**.** Let $\phi_1, \phi_2, \cdots$ be an orthonormal system in R and let $f \in R$. Consider the sequence $c_k = \langle f, \phi_k \rangle$ for all $k = 1, 2, \cdots$. Then c_k are called the coordinates or Fourier coefficients of f with respect to the system $\{\phi_k\}$ and $\sum_{k=1}^{\infty} c_k \phi_k$ is called the Fourier series of f.

Note that this series in the definition is a formal series as we do not yet know whether or not the series converges.

In the finite case, it is not difficult to see that the sequence α_k for $k = 1, \cdots, n$ which minimizes $\|f - S_n^{(\alpha)}\|$ where $S_n^{(\alpha)} := \sum_{k=1}^{n} \alpha_k \phi_k$ is the Fourier coefficients. Indeed, we have

$$\begin{aligned} \|f - S_n^{(\alpha)}\|^2 &= \langle f, f \rangle - 2\langle f, S_n^{(\alpha)} \rangle + \langle S_n^{(\alpha)}, S_n^{(\alpha)} \rangle \\ &= \|f\|^2 - 2 \sum \alpha_k c_k + \sum \alpha_k^2 \\ &= \|f\|^2 - \sum c_k^2 + \sum (\alpha_k - c_k)^2. \end{aligned}$$

Hence, $\|f - S_n^{(\alpha)}\|$ is minimized when $\alpha_k = c_k$ for all $k = 1, \cdots, n$. With this in mind, choosing α to be the Fourier coefficients, we have

$$\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2.$$

Geometrically, $f - S_n^{(\alpha)}$ is orthogonal to the subspace generated by $\phi_1, \cdots, \phi_n$ if and only if $\alpha = c$.

Furthermore, by noting $0 \leq \|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, we have

$$\sum_{k=1}^{n} c_k^2 \leq \|f\|^2 < \infty,$$

and hence, taking $n \to \infty$, we have $\sum_{k=1}^{\infty} c_k^2$ exists and is bounded above by $\|f\|^2$. This inequality is known as the Bessel inequality.

Definition 1.4 (Closed Orthonormal System)**.** The orthonormal system $\{\phi_k\}$ is closed if for any $f \in R$, we have

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

This property is called the Parseval equality.

Again, by observing $\|f - S_n^{(c)}\| = \|f\|^2 - \sum_{k=1}^{n} c_k^2$, the system is closed if and only if for any f, the partial sums of the Fourier series converge to f, i.e. $f = \sum_{k=1}^{\infty} c_k \phi_k$.

Proposition 1.3. In a separable Euclidean space R, an orthonormal system is complete if and only if it is closed.

Proof. Suppose first that $\{\phi_k\}$ is closed. Then, for all $f \in R$, $f = \sum_{k=1}^{\infty} c_k \phi_k$. Thus, the finite linear combinations of $\{\phi_k\}$ is dense in R and thus, $\{\phi_k\}$ is complete.

On the other hand, suppose that $\{\phi_k\}$ is complete (it is countable as R is separable), for any $f \in R$, there exists some α^k such that $\|f - S_\infty^{(\alpha^k)}\| \to 0$. As we have seen, for any partial sum $S_n^{(\alpha^k)}$, we have $\|f - S_n^{(c)}\| \leq \|f - S_n^{(\alpha^k)}\|$ and so,

$$\|f - S_\infty^{(c)}\| \leq \|f - S_\infty^{(\alpha^k)}\| \to 0$$

implying $\|f - S_\infty^{(c)}\| = 0$ and the system is closed. □

Proposition 1.4. Given $f, g \in R$ and a closed orthonormal system $\{\phi_k\}$,

$$\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$$

where $(c_k), (d_k)$ are the Fourier coefficients of f and g with respect to $\{\phi_k\}$ respectively.

Proof. We have, by Parseval's identity, $\|f\|^2 = \sum c_k^2$, $\|g\|^2 = \sum d_k^2$ and $\|f + g\|^2 = \sum (c_k + d_k)^2 = \sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2$, we have

$$\sum c_k^2 + 2 \sum c_k d_k + \sum d_k^2 = \|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2.$$

Thus, cancelling using $\|f\|^2 = \sum c_k^2$ and $\|g\|^2 = \sum d_k^2$, we have $\langle f, g \rangle = \sum_{k=1}^{\infty} c_k d_k$ as required. □

In the case the system is only orthogonal but not necessary orthonormal, we may normalize the Fourier coefficients, i.e. given an orthogonal system $\{\phi_k\}$, we have $\{\phi / \|\phi_k\|\}$ is an orthonormal system, and so, we define

$$c_k = \left\langle f, \frac{\phi_k}{\|\phi_k\|} \right\rangle = \frac{1}{\|\phi_k\|} \langle f, \phi_k \rangle.$$

Similarly, the Fourier series of f is becomes

$$\sum_{k=1}^{\infty} c_k \frac{\phi_k}{\|\phi_k\|} = \sum \frac{\langle f, \phi_k \rangle}{\|\phi_k\|^2} \phi_k.$$

Substituting this definition of the Fourier coefficients into the Bessel inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{\langle f, \phi_k \rangle^2}{\|\phi_k\|^2} \leq \|f\|^2,$$

for any orthogonal system $\{\phi_k\}$.

Theorem 1 (Riesz)**.** Let $\{\phi_k\}$ be a orthonormal system in a complete Euclidean space R (i.e. a real Hilbert space) and let $c \in \ell_2$ (i.e. $\sum_{k=1}^{\infty} c_k^2 < \infty$). Then, there exists some $f \in R$ such that $c_k = \langle f, \phi_k \rangle$ and Parseval's identity holds, i.e.

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2.$$

Proof. Let $f_n := \sum_{k=1}^n c_k \phi_k$. Then, by definition, we have $c_k = \langle f_n, \phi_k \rangle$ for all $k = 1, \cdots, n$. Then, for all $p \geq 1$, we have

$$\|f_{n+p} - f_n\|^2 = \|c_{n+1}\phi_{n+1} + \cdots + c_{n+p}\phi_{n+p}\|^2 = \sum_{k=n+1}^{n+p} c_k^2.$$

Now, as $\sum c_k^2 < \infty$, we have $\{f_n\}$ is Cauchy, and thus, as R is complete, there exists some $f \in R$ such that $f_n \to f$. Thus, by noting,

$$\langle f, \phi_k \rangle = \langle f_n \phi_k \rangle + \langle f - f_n, \phi_k \rangle = c_k + \langle f - f_n, \phi_k \rangle,$$

where $\langle f - f_n, \phi_k \rangle \to 0$ as $n \to \infty$ since $|\langle f - f_n, \phi_k \rangle| \leq \|f - f_n\| \|\phi_k\|$ by the Cauchy-Schwarz inequality, we have $c_k = \langle f, \phi_k \rangle$.

Finally, Parseval's identity, follows as $\| \cdot \|^2$ is continuous in a normed space. □

Let us recall the following result from functional analysis.

Proposition 1.5. Any separable Hilbert space is isomorphic to ℓ_2 (thus, any two separable Hilbert spaces are isomorphic).

Proof. Let H be a separable Hilbert space and choose $\{\phi_k\}$ a complete orthonormal system (which exists as H is separable). Then, for any $f \in H$, we map f to the sequence corresponding to its Fourier coefficients, i.e.

$$\psi : f \mapsto (c_1, c_2, \cdots)$$

which is well-defined by Bessel's inequality. On the other hand, by Riesz's theorem, for any $x \in \ell_2$, $\sum x_k^2 < \infty$ and so, there exists a unique $f \in H$, such that $\psi(f) = x$. Thus, as ψ is clearly linear (as the inner products are linear with respect to the left component), we have the isomorphism between H and ℓ_2. □

1.2 Complex Inner Product Space

We will in the course take the complex inner product to be anti-linear in the second component. As promised earlier, most definitions can be generalized from the real case to the complex directly.

Definition 1.5 (Fourier Coefficients)**.** Let R be a complex inner product space. Then, for an orthonormal system (ϕ_n) and $f \in R$, we define its Fourier coefficients to be $c_k := \langle f, \phi_k \rangle$ for all $k = 1, \cdots, n$. Similarly, we define the Fourier series of f to be the formal series $\sum_{k=1}^\infty c_k \phi_k$.

Going through the same argument as the real case, we obtain the complex version of Bessel's inequality.

Proposition 1.6 (Bessel's Inequality)**.** Given an orthonormal system (ϕ_n) and $f \in R$, we have

$$\sum_{k=1}^\infty |c_k|^2 \leq \|f\|^2.$$

Going through all proved theorems for real spaces, we find they also hold for complex spaces (with trivial modifications).

2 Fourier Series

2.1 Trigonometric Series

We will consider the space $L_2[-\pi,\pi]$ (the space of square-integrable functions from $[-\pi,\pi]$ quotiented by the a.e.-equal equivalence relation equipped with the inner product $\langle f, g\rangle := \int_{[-\pi,\pi]} fg \mathrm{d}\lambda$), and the trigonometric system

$$\{\mathbf{1}, \cos(nx), \sin(nx) \mid n = 1, 2, \cdots\}.$$

It is not difficult to see that this system is orthogonal, but in fact, it is also complete. Indeed, completeness follows by the Weierstrass approximation theorem for trigonometric polynomials (we will discuss this later / recall the Stone-Weierstrass theorem and observe that the trigonometric system seperates points).

Nonetheless, this system is not orthonormal, and thus, we normalise the system such that the system becomes

$$\left\{\frac{1}{\sqrt{2\pi}}\mathbf{1}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx) \mid n = 1, 2, \cdots\right\}.$$

Hence, the Fourier series of an element $f \in L_2[-\pi,\pi]$ becomes the famous formula

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$a_k := \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\cos(kx)\lambda(\mathrm{d}x), b_k := \frac{1}{\pi}\int_{[-\pi,\pi]} f(x)\sin(kx)\lambda(\mathrm{d}x).$$

By recalling the above theory, the n-th partial sum of this series provides the best (in L_2 metric) approximation of f among all trigonometric polynomials of degree n. Hence, as the trigonometric system is complete, Parseval's identity holds, and so,

$$\|f - S_n\|_2 \to 0, \text{ as } n \to \infty.$$

By observing that $e^{ix} = \cos x + i \sin x$, we may rewrite the Fourier series can be written in the complex form. In particular, $L_2[-\pi,\pi]$ has the orthogonal system $\{e^{inx} \mid n \in \mathbb{Z}\}$ and the Fourier series of $f \in L_2[-\pi,\pi]$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ where } c_n = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x)e^{-inx}\lambda(\mathrm{d}x).$$

- Since a function on $[-\pi,\pi]$ can be extended to $\mathbb{R}$ by periodicity, we can, instead of functions on $[-\pi,\pi]$ consider periodic functions with period 2π on $\mathbb{R}$.
- Since $\cos nx, \sin nx$ are bounded functions, the integrals defining the trigonometric Fourier coefficients exists for any function in $L_1[-\pi,\pi]$, i.e. if $f \in L_1[-\pi,\pi]$, then

$$\int f\cos(nx), \int f\sin(nx) < \int |f| < \infty.$$

- $L_2[-\pi,\pi] \subseteq L_1[-\pi,\pi]$ by Hölder's inequality and thus, with the above remark in mind, the definition of Fourier series is also well-defined for any integrable functions (though convergence is much opaque in this case).

While the Fourier series of f converges to f in L_2 though it is not clear that the Fourier series converges point-wise to f (it might be interesting to recall that convergence in L_p implies convergence in measure and the existence of a subsequence which converges almost everywhere).

Consider the partial sum of the Fourier series of $f \in L_2[-\pi,\pi]$,

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \\ &= \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}(\cos kx \cos kt + \sin kx \sin kt)\right)\lambda(\mathrm{d}t) \\ &= \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right)\lambda(\mathrm{d}t). \end{aligned}$$

By noting the identity

$$\frac{1}{2} + \sum_{k=1}^{n} \cos ku = \frac{\sin \frac{2n+1}{2}u}{2\sin\frac{u}{2}},$$

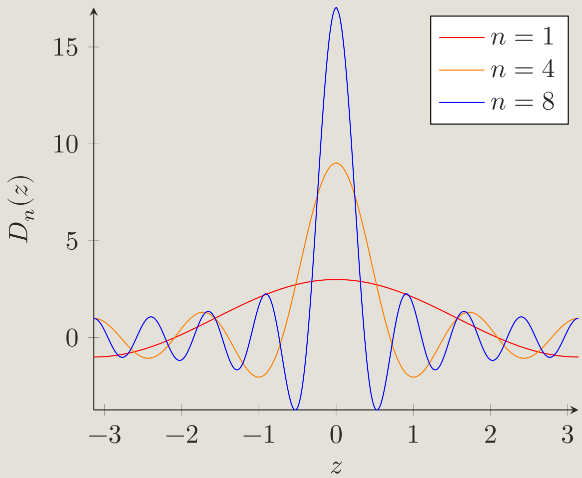
we obtain

$$S_n(x) = \frac{1}{\pi}\int_{[-\pi,\pi]} f(t)\frac{\sin\frac{2n+1}{2}(t-x)}{2\sin\frac{t-x}{2}}\lambda(\mathrm{d}t).$$

Finally, by noting the periodicity of f, by change of variable $z = t - x$, we obtain

$$S_n(x) = \int_{[-\pi,\pi]} f(x+z)D_n(z)\lambda(\mathrm{d}z), \text{ where } D_n(z) := \frac{1}{2\pi}\frac{\sin\frac{2n+1}{2}z}{\sin\frac{z}{2}}$$

and D_n is known as the Dirichlet kernel. We remark that the Dirichlet kernel $D_n(z)$ tends to $\frac{2n+1}{2\pi}$ as $z \to 0$ and rapidly osculates for large n though this does not impact our calculation as we are dealing with a point which has measure 0.



By observing the graph of the Dirichlet kernel, in some heuristic sense, we note that $D_n(z) \to \delta(z)$ for some function where δ is 0 at all points but $z = 0$ while $\int_{[-\epsilon,\epsilon]} \delta \mathrm{d}\lambda = 1$ for all $\epsilon > 0$. Such an function cannot exist, however it motivates the second part of the course - the theory of distributions.

2.2 Conditions for Point-wise Convergence

We observe that $\int D_n = 1$ and so we may write

$$S_n(x) - f(x) = \int_{[-\pi,\pi]} (f(x+z) - f(x)) D_n(z) \lambda(\mathrm{d}z).$$

Clearly, $S_n(x) \to f(x)$ as $n \to \infty$ if and only if the right hand side of the above tends to 0.

Lemma 2.1 (Riemann-Lebesgue)**.** If $\phi \in L_1[a,b]$ for some $a < b$, then both

$$\int_{[a,b]} \phi(x) \sin(\gamma x) \lambda(\mathrm{d}x), \int_{[a,b]} \phi(x) \cos(\gamma x) \lambda(\mathrm{d}x)$$

tends to 0 as $\gamma \to \infty$.

Proof. We will prove the statement for the sin case. We observe that if ϕ is continuously differentiable, by integration by parts, we have

$$\int_{[a,b]} \phi(x) \sin(\gamma x) \lambda(\mathrm{d}x) = \left[-\phi(x) \frac{\cos \gamma x}{\gamma} \right]_a^b + \int_{[a,b]} \phi'(x) \frac{\cos \gamma x}{\gamma} \lambda(\mathrm{d}x),$$

which tends to 0 as $\gamma \to \infty$ (we note that ϕ' is continuous on a compact set, and hence bounded above). Now, in the general case, we observe that continuously differentiable functions are everywhere dense in $L_1[a,b]$, and so, for every $\epsilon > 0$, there exists some continuously differentiable $\phi_\epsilon(x)$, such that

$$\int_{[a,b]} |\phi - \phi_\epsilon| \mathrm{d}\lambda = \|\phi - \phi_\epsilon\|_1 < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}
\left| \int_{[a,b]} \phi(x) \sin(\gamma x) \lambda(\mathrm{d}x) \right| &\leq \left| \int_{[a,b]} (\phi(x) - \phi_\epsilon(x)) \sin(\gamma x) \lambda(\mathrm{d}x) \right| + \left| \int_{[a,b]} \phi_\epsilon(x) \sin(\gamma x) \lambda(\mathrm{d}x) \right| \\
&\leq \|\phi - \phi_\epsilon\|_1 + \left| \int_{[a,b]} \phi_\epsilon(x) \sin(\gamma x) \lambda(\mathrm{d}x) \right| \\
&< \frac{\epsilon}{2} + \left| \int_{[a,b]} \phi_\epsilon(x) \sin(\gamma x) \lambda(\mathrm{d}x) \right|.
\end{aligned}$$

Since ϕ_ϵ is continuously differentiable, the last term tends to 0 as $\gamma \to \infty$ implying it is less than $\epsilon/2$ for sufficiently large γ. Thus, we have established the required limit. $\square$

Corollary 1.1. If $f \in L_1[-\pi,\pi]$, then, its Fourier coefficients $a_k, b_k \to 0$ as $k \to \infty$.

With the above in mind, we will now provide a sufficient condition for convergence of Fourier series at a point x.

Theorem 2. If $f \in L_1[-\pi, \pi]$ and for any $x \in [-\pi, \pi]$, there exists some $\delta > 0$,

$$\int_{[-\delta,\delta]} \left| \frac{f(x+t) - f(x)}{t} \right| \lambda(\mathrm{d}t) < \infty$$

exists (this is called Dini's condition), then $S_n(x) \to f(x)$ as $n \to \infty$.

Proof. We observe

$$\begin{aligned} S_n(x) - f(x) &= \int_{[-\pi,\pi]} (f(x+z) - f(x)) D_n(z) \lambda(\mathrm{d}z) \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} \frac{f(x+z) - f(x)}{z} \frac{z}{\sin \frac{z}{2}} \sin\left(\frac{2n+1}{2} z\right) \lambda(\mathrm{d}z). \end{aligned}$$

Then, if Dini's condition is satisfied, as $z/\sin\frac{z}{2}$ is integrable on $[-\pi, \pi]$, we have

$$\frac{f(x+z) - f(x)}{z} \frac{z}{\sin \frac{z}{2}} \in L_1[-\pi, \pi]$$

and hence, by the Riemann-Lebesgue lemma, $S_n(x) - f(x) \to 0$ as $n \to \infty$. □

We observe an trivial sufficient condition for which Dini's condition holds. In particular, the Dini condition holds if f is continuous at x and its derivative at x exists (or the limit of the derivative exists from the right and the left).

Suppose on the other hand that f has a discontinuity of the first kind at x (i.e. both the limit from the left and the right exists at x). Then, the argument in the proof remains to hold if we replace the Dini condition by

$$\int_{[-\delta,0]} \left| \frac{f(x+t) - f(t)}{t} \right| \lambda(\mathrm{d}t) < \infty \text{ and } \int_{[0,\delta]} \left| \frac{f(x+t) - f(t)}{t} \right| \lambda(\mathrm{d}t) < \infty.$$

Then, denoting $f(x+), f(x-)$ the limit of f at x from the right and the left respectively, we have

$$S_n(x) - \frac{f(x^+) + f(x^-)}{2} = \\ \int_{[-\pi,0]} (f(x+z) - f(x^-)) D_n(z) \lambda(\mathrm{d}z) + \int_{[0,\pi]} (f(x+z) - f(x^+)) D_n(z) \lambda(\mathrm{d}z)$$

Hence, the $S_n(x)$ converges to the average of the limit f at x from the left and from the right.

Let us summarise the above in the following statement.

Proposition 2.1. If f is a bounded function of period 2π with only discontinuities of the first kind, and also possesses at each point left and right derivatives, Then, its Fourier series converges everywhere and its sum equals $f(x)$ at points of continuity and equals

$$\frac{f(x^+) + f(x^-)}{2}$$

at points of discontinuity.

We remark there exists continuous functions whose Fourier series diverge at some points. More curiously, there exists L_1 functions which diverge at all points (see *Functional analysis* for the construction of a function which diverges at any given point).

2.3 Continuous Functions

Suppose now f is a continuous function with period 2π on $\mathbb{R}$, then it is uniquely determined by its Fourier series (exercise). Nonetheless, as we have seen, the Fourier series of f does not necessarily equal to f at every point. However, we can still reconstruct f from its Fourier series. Consider the partial sums

$$S_k(x) = \frac{a_0}{2} + \sum_{j=1}^{k} a_j \cos(jx) + b_j \sin(jx).$$

Then, we define the Feje's sums

$$\sigma_n(x) := \frac{1}{n}(S_0(x) + S_1(x) + \cdots S_{n-1}(x)),$$

we have the following theorem.

Definition 2.1 (Fejer's Kernel)**.** Fejer's kernel is defined to be the function

$$\Phi_n(z) = \frac{1}{2\pi n}\left(\frac{\sin(nz/2)}{\sin(z/2)}\right)^2$$

for all $n \geq 0$.

Lemma 2.2. Denoting Φ_n as Fejer's kernel, we have

- $\Phi_n \geq 0$;
- $\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda = 1$;
- for all $\delta \in (0, \pi]$, $\int_{[-\pi,-\delta]} \Phi_n \mathrm{d}\lambda = \int_{[\delta,\pi]} \Phi_n \mathrm{d}\lambda \to 0$ as $n \to \infty$.

Proof. Hint: note the identity $\sum_{k=0}^{n-1} \sin((2k+1)u) = \frac{\sin^2(nu)}{\sin u}$. $\square$

Theorem 3 (Fejer)**.** If f is a continuous function with period 2π, then the sequence (σ_n) of its Fejer's sums converges to f uniformly on $\mathbb{R}$.

Proof. Recall that

$$S_k(x) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f(x+z) D_k(z) \lambda(\mathrm{d}z),$$

and so,

$$\sigma_n(x) = \frac{1}{2\pi n}\int_{[-\pi,\pi]} f(x+z) \sum_{k=0}^{n-1} D_k(z) \lambda(\mathrm{d}z).$$

Then, using the identity that

$$\sum_{k=0}^{n-1} \sin(2k+1)u = \frac{\sin^2(nu)}{\sin u},$$

we obtain

$$\sigma_n(x) = \int_{[-\pi,\pi]} f(x+z)\Phi_n(z)\lambda(\mathrm{d}z)$$

where Φ_n is Fejer's kernel. Now, by nothing that f is continuous and periodic, f is bounded and uniformly continuous on $\mathbb{R}$. Thus, there exists some $M > 0$, such that for all $\sup_x |f(x)| \leq M$ and for all $\epsilon > 0$, there exists some $\delta \in (0, \pi]$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}\left(\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda\right)^{-1}$ for all $|x-y| < 2\delta$. Now,

$$\begin{aligned} f(x) - \sigma_n(x) &= \int_{[-\pi,\pi]} (f(x) - f(x+z))\Phi_n(z)\lambda(\mathrm{d}z) \\ &= \left(\int_{[-\pi,-\delta]} + \int_{[-\delta,\delta]} + \int_{[\delta,\pi]}\right)(f(x) - f(x+z))\Phi_n(z)\lambda(\mathrm{d}z). \end{aligned}$$

Then, by the above lemma, there exists some N such that for all $n \geq N$, $\int_{[-\pi,-\delta]} \Phi_n \mathrm{d}\lambda < \frac{\epsilon}{8M}$ and so, for all $n \geq N$, we have $\int_{[-\pi,-\delta]}(f(x) - f(x+z))\Phi_n(z)\lambda(\mathrm{d}z) \leq \int_{[-\pi,-\delta]} 2M\Phi_n(z)\lambda(\mathrm{d}z) < 2M\frac{\epsilon}{8M} = \epsilon/4$

$$\begin{aligned} f(x) - \sigma_n(x) &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \int_{[-\delta,\delta]} (f(x) - f(x+z))\Phi_n \mathrm{d}\lambda \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\left(\int_{[-\pi,\pi]} \Phi_n \mathrm{d}\lambda\right)^{-1} \int_{[-\delta,\delta]} \Phi_n \mathrm{d}\lambda \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $|f(x) - \sigma_n(x)| = f(x) - \sigma_n(x) < \epsilon$ for all $x \in \mathbb{R}$ implying uniform convergence as required. $\square$

While we had used the Weierstass approximation to obtain the above, Fejer's theorem also implies Weierstass approximation theorem straight away. Nonetheless, while the Stone-Weierstass theorem is more general, Fejer's theorem provides an explicit construction of such an series.

Corollary 3.1 (Weierstass Approximation Theorem)**.** Any continuous periodic function is a limit of a uniformly convergent sequence of trigonometric polynomials.

Corollary 3.2. The trigonometric system $\{1, \sin nx, \cos nx \mid n = 1, 2, \cdots\}$ is complete in $L_2[-\pi, \pi]$ as uniform convergence implies convergence in L_2.

We remark that convergence in $L_p[-\pi, \pi]$ implies the existence of a subsequence which converges almost everywhere and hence, by Egorov's theorem, there exists a subsequence which converges uniformly on an arbitrarily large space.

By recalling that uniform convergence is definitionally equal to the supremum norm, Fejer's theorem tells us that for all $f \in \mathcal{C}_\infty[-\pi, \pi]$, the sequence of its Fejer's sums converges f in $\mathcal{C}_\infty[-\pi, \pi]$. Furthermore, if $f \in L_1[-\pi, \pi]$, then $\sigma_n \to f$ in L_1.

Corollary 3.3. For all $f \in L_1[-\pi, \pi]$, it is (a.e.) uniquely determined by its Fourier coefficients.

Proof. Suppose $f, g \in L_1[-\pi, \pi]$ have the same Fourier coefficients. Then, f and g have the same Fejer's sums. But, as L_1 is a metric space, the Fejer's sums has a unique limit and hence, $f = g$ in L_1. $\square$

3 Fourier Transform

Thus far, we have considered periodic functions in L_1 is represented uniquely by its Fourier series. We would now like to generalize this theory to non-periodic functions.

As an intuition (we will formally justify this later), consider the case first that f is periodic on $[-l, l]$ for some $l > 0$. From the exercise sheet, we were able to extend the Fourier series and conclude that, if $f \in L_1$ and satisfies Dini's condition at each point of $[-l, l]$, then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{l}x\right) + b_k \sin\left(\frac{k\pi}{l}x\right),$$

where

$$a_k = \frac{1}{l}\int_{[-l,l]} f(t)\cos\left(\frac{k\pi}{l}t\right)\lambda(\mathrm{d}t) \text{ and } b_k = \frac{1}{l}\int_{[-l,l]} f(t)\sin\left(\frac{k\pi}{l}t\right)\lambda(\mathrm{d}t).$$

Thus,

$$f(x) = \frac{1}{2l}\int_{[-l,l]} f\mathrm{d}\lambda + \frac{1}{l}\sum_{k=1}^{\infty} f(t)\cos\left(\frac{k\pi}{l}(t-x)\right)\lambda(\mathrm{d}t).$$

Hence, setting $y_k = \pi k/l$, $\Delta y = \pi/l$ and taking $l \to \infty$, we expect

$$f(x) = \frac{1}{\pi}\int_{(0,\infty)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t).$$

This equation is known as the Fourier integral. To see the similarity of this with the Fourier series, we may write the Fourier integral as

$$f(x) = \int_{(0,\infty)} (a_y \cos(yx) + b_y \sin(y_x))\lambda(\mathrm{d}y)$$

where

$$a_y = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\cos(yt)\lambda(\mathrm{d}t) \text{ and } b_y = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\sin(yt)\lambda(\mathrm{d}t).$$

Theorem 4. Let $f \in L_1$ and satisfy Dini's condition at every point. Then for all $x \in \mathbb{R}$, we have

$$f(x) = \frac{1}{\pi}\int_{(0,\infty)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t).$$

Proof. Let us denote

$$J_x(A) := \frac{1}{\pi}\int_{(0,A)} \lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\cos(y(t-x))\lambda(\mathrm{d}t),$$

and we will show $J_x(A) \to f(x)$ as $A \to \infty$. By assumption, $f \in L_1$ and so, J_x converges absolutely, and hence, by Fubini's theorem,

$$J_x(A) = \frac{1}{\pi}\int_{\mathbb{R}} \lambda(\mathrm{d}t)f(t)\int_{(0,A)} \cos(y(t-x))\lambda(\mathrm{d}y) = \frac{1}{\pi}\int_{\mathbb{R}} \lambda(\mathrm{d}t)f(t)\frac{\sin A(t-x)}{t-x}.$$

By change of variables $z = t - x$, we obtain

$$J_x(A) = \frac{1}{\pi} \int_{\mathbb{R}} f(x+z) \frac{\sin Az}{z} \lambda(\mathrm{d}z).$$

Now, by observing that $\frac{1}{\pi} \int_{\mathbb{R}} \frac{\sin Az}{z} \lambda(\mathrm{d}z) = 1$, we have

$$\begin{aligned} J_x(A) - f(x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \\ &= \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \\ &+ \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \\ &- \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \end{aligned}$$

for all $N > 0$. By observing that the last two integrand are bounded, both integrals tends to 0 as $N \to \infty$. On the other hand, the first integral tends to 0 as $A \to \infty$ for fixed N by the Riemann-Lebesgue lemma. With this in mind, let $\epsilon > 0$, then, there exists some $N > 0$ such that

$$\left| \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \right|, \left| \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3}.$$

Then, by the Riemann-Lebesgue lemma (and Dini's condition), there exists some B, such that for all $A \geq B$,

$$\left| \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3}.$$

Hence, for all $A \geq B$, we have

$$\begin{aligned} |J_x(A) - f(x)| \leq& \left| \frac{1}{\pi} \int_{[-N,N]} \frac{f(x+z) - f(x)}{z} \sin Az \lambda(\mathrm{d}z) \right| \\ &+ \left| \frac{1}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{f(x+z)}{z} \sin Az \lambda(\mathrm{d}z) \right| \\ &- \left| \frac{f(x)}{\pi} \int_{\mathbb{R} \setminus [-N,N]} \frac{\sin Az}{z} \lambda(\mathrm{d}z) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

implying $J_x(A) \to f(x)$ as $A \to \infty$ as required. $\square$

Similar to the finite case, we may write the Fourier integral using the complex exponential function. In particular, as cos is even, we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda(\mathrm{d}y) \int_{\mathbb{R}} f(t) \cos(y(t-x)) \lambda(\mathrm{d}t).$$

On the other hand, as sin is odd,

$$\frac{1}{2\pi}\int_{\mathbb{R}}\lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)\sin(y(t-x))\lambda(\mathrm{d}t) = 0.$$

Summing the two integrals, we obtain that, if $f \in L_1$ and satisfies the Dini condition, then

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\lambda(\mathrm{d}y)\int_{\mathbb{R}} f(t)e^{-iy(t-x)}\lambda(\mathrm{d}t).$$

Definition 3.1 (Fourier Transform)**.** Let $f \in L_1$. We define the Fourier transform of f to be

$$g(y) = \mathcal{F}[f](y) := \int_{\mathbb{R}} f(t)e^{-iyt}\lambda(\mathrm{d}t).$$

Proposition 3.1 (Inversion of the Fourier Transform)**.** In the case that f satisfies Dini's condition at some point x, then

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[f](y)e^{iyx}\lambda(\mathrm{d}y).$$

Proof. See above. □

3.1 Properties of the Fourier Transform

From the above process, we observe that the Fourier transform is very similar to the Fourier series in which the Fourier transform is analogous to the Fourier coefficients while the inversion formula is analogous to the Fourier series both of which holds under Dini's condition.

Theorem 5. Let $f \in L_1$ and suppose $\mathcal{F}[f] = 0$, then $f = 0$ almost everywhere.

Thus, similar to the case of the Fourier series, the Fourier transform (a.e.) uniquely determines a function.

Proof. We note that we have not required Dini's condition and thus, we may not use the inversion formula.

By change of variable, we note that $\int f(x+t)e^{-iyx}\lambda(\mathrm{d}x) = 0$ for all t, y, and let us define

$$\phi_\mu(x) := \int_{(0,\mu)} f(x+t)\lambda(\mathrm{d}t)$$

for some $\mu > 0$. As ϕ_μ is integrable, by Fubini's theorem, it follows that $\mathcal{F}[\phi_\mu] = 0$. Furthermore, as ϕ_μ is an integral function, it is absolutely continuous on any finite interval, and hence, has a derivative a.e. and so, it satisfies Dini's condition a.e. Hence, by the Fourier inversion formula, we have

$$\phi_u(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[\phi_\mu](y)e^{iyx}\lambda(\mathrm{d}y) = 0,$$

almost everywhere. But, as ϕ_μ is continuous (as its absolutely continuous), and so $\phi_\mu = 0$ everywhere. Thus, it follows that $f = 0$ a.e. □

Lemma 3.1. Let $f, f_n \in L_1$ for $n = 1, 2, \cdots$ such that $f_n \to f$ in L_1. Then, $g_n(y) := \mathcal{F}[f_n](y) \to \mathcal{F}[f](y)$ uniformly on $\mathbb{R}$.

Proof. Follows as

$$|g_n - \mathcal{F}[f]| \leq \int_{\mathbb{R}} |f_n - f| \mathrm{d}\lambda.$$

□

Lemma 3.2. Let $f \in L_1$, then $\mathcal{F}[f]$ is a bounded continuous function and $\mathcal{F}[f](y) \to 0$ as $|y| \to \infty$.

Proof. Again, boundedness follows from the bound $|\mathcal{F}[f]| \leq \|f\|_1 < \infty$.

For the other two properties, we will first prove for indicator functions. Consider

$$\mathcal{F}[\mathbf{1}_{[a,b]}](y) = \int_{[a,b]} e^{-iyx} \lambda(\mathrm{d}x) = \frac{e^{-iyb} - e^{-iya}}{-iy}$$

which is continuous and tends to 0 as $|y| \to \infty$.

By the linearity of the Lebesgue integral, and hence, the Fourier transform is linear, we have that the Fourier transform of any simple function is also continuous and tends to 0 as $|y| \to \infty$. Then, as the simple function is dense in L_1, for any $f \in L_1$, there exists a sequence (f_n) of simple functions such that $f_n \to f$ in L_1. This, by the above lemma, we obtain $\mathcal{F}[f_n] \to \mathcal{F}[f]$ uniformly. Hence, $\mathcal{F}[f]$ is continuous and tends to 0 as $|y| \to \infty$. □

Lemma 3.3. Let $f \in L_1$ be differentiable with L_1 derivative f' and suppose f is absolutely continuous on any finite integral. Then,

$$\mathcal{F}[f'] = iy \cdot \mathcal{F}[f].$$

Proof. Since f is a.c.

$$f(x) = f(0) + \int_{[0,x]} f' \mathrm{d}\lambda.$$

Thus, as $f' \in L_1$, $\lim_{x \to \pm\infty} f(x)$ exists. Now since $f \in L_1$, $\lim_{x \to \pm\infty} f(x) = 0$ and so, by integration by parts,

$$\mathcal{F}[f'] = \int f'(x) e^{-iyx} \lambda(\mathrm{d}x) = iy \int f(x) e^{-iyx} \lambda(\mathrm{d}x) = iy\mathcal{F}[f]$$

as required. □

Corollary 5.1. In the case that f is n-times differentiable and $f^(k) \in L_1$ for all $k = 0, \cdots, n$, then applying the above lemma n-times, we have

$$\mathcal{F}[f^{(n)}] = (iy)^n \cdot \mathcal{F}[f],$$

and $|\mathcal{F}[f]| = |\mathcal{F}[f^{(n)}]|/|y|^n \leq C/|y|^n \to 0$ as $|y| \to \infty$ as $\mathcal{F}[f^{(n)}]$ is bounded.

From the above corollary, we see that, similar to the Fourier series, $\mathcal{F}[f]$ decays faster at infinity the smoother it is.

Corollary 5.2. If $f, f', f'' \in L_1$, then $\mathcal{F}[f] \in L_1$.

Proof. We have $|\mathcal{F}[f]| \leq C/|y|^2$ for some constant C and so,

$$\int_{\mathbb{R}} |\mathcal{F}[f]| \mathrm{d}\lambda \leq \int_{\mathbb{R}} \frac{C}{|y|^2} \lambda(\mathrm{d}y) < \infty.$$

$\square$

The opposite is also true, namely, the faster f decays at infinity, the smoother $\mathcal{F}[f]$ is.

Lemma 3.4. Let $f(x), xf(x) \in L_1$, then, $\mathcal{F}[f]$ is differentiable and

$$D_y \mathcal{F}[f](y) = \mathcal{F}[-ixf(x)]$$

On the other hand, if $f(x), xf(x), \cdots, x^p f(x) \in L_1$, then $\mathcal{F}[f]$ is p-times differentiable and

$$D_y^p \mathcal{F}[f](y) = \mathcal{F}[(-ix)^p f(x)].$$

Proof. It suffices to show the first statement as the second follows by induction. But this is clear by differentiating with respect to y under the integral (where we may differentiate under the integral as $|xf(x)e^{-iyx}| \leq |xf(x)|$ where $xf(x) \in L_1$ and so, we may differentiate by the dominated convergence theorem). $\square$

Corollary 5.3. If $x^p f(x) \in L_1$ for all p. Then, $\mathcal{F}[f]$ is infinitely differentiable.

Corollary 5.4. If $e^{\delta|x|} f(x) \in L_1$ for some $\delta > 0$, then $\int f(x) e^{ix\zeta} \lambda(\mathrm{d}x)$ uniformly converges for all $\zeta = y + iz$, $|z| < \delta$. Therefore,

$$g(\zeta) := \int f(x) e^{ix\zeta} \lambda(\mathrm{d}x)$$

is an analytic function on the neighbourhood $\mathbb{R} \times (-\delta, \delta)$.

3.2 Schwartz Functions

Let us now consider the space of functions which are invariant under the Fourier transform. This will lead us to define the Schwartz functions.

Since the smoothness and decay at infinity swap after an application of the Fourier transform (F.T.), we can easily indicate a class of function which are invariant under F.T.

Definition 3.2 (Schwartz Function)**.** We define the space S_∞ as the space of functions on $\mathbb{R}$ which are infinitely differentiable and such that for all $p, q = 0, 1, \cdots$, there exists some constant $C_{p,q}^f$ such that

$$|x^p f^{(q)}(x)| < C_{p,q}^f$$

for all $x \in \mathbb{R}$. If $f \in S_\infty$, then we call f a Schwartz function.

Proposition 3.2. If $f \in S_\infty$, then $g := \mathcal{F}[f] \in S_\infty$.

Proof. Since $|x^p f^{(q)}(x)| < x^{-2} C^f_{p+2,q}$ and so, is integrable for all p, q. Hence, $g = \mathcal{F}[f]$ is bounded and infinitely differentiable. Moreover, since $f^{(q)} \in L_1$, $g(y)$ tends to 0 at infinity faster than $1/|y|^q$ for all q, and so, it remains to show the same for $g^{(p)}$. But this follows by the identity

$$\mathcal{F}[((-ix)^p f)^{(q)}] = (iy)^q \mathcal{F}[(-ix)^p f] = (iy)^q g^{(p)}(y).$$

and that $g^{(p)}$ is bounded for all p. Hence, it follows $\mathcal{F}[f] \in S_\infty$. $\square$

Since Schwartz functions are infinitely differentiable, Dini's condition hold and thus, the Fourier inverse transform holds and the map $f \mapsto \mathcal{F}[f]$ is a bijection.

3.3 Convolution and Heat Equation

Definition 3.3 (Convolution)**.** Let $f_1, f_2 \in L_1$, the convolution of f_1 with f_2 is defined to be

$$f(x) = f_1 * f_2(x) := \int_{\mathbb{R}} f_1(y) f_2(x-y) \lambda(\mathrm{d}y).$$

Proposition 3.3. If $f_1, f_2 \in L_1$, then $f_1 * f_2 \in L_1$.

Proof. By Fubini's theorem, we have

$$\begin{aligned}\int |f_1 * f_2(x)| \lambda(\mathrm{d}x) &\leq \int \int |f_1(y)||f_2(x-y)| \lambda(\mathrm{d}y)\lambda(\mathrm{d}x) \\ &= \int |f_1(y)| \int |f_2(x-y)| \lambda(\mathrm{d}x)\lambda(\mathrm{d}y) < \infty.\end{aligned}$$

$\square$

Theorem 6. If $f_1, f_2 \in L_1$, then

$$\mathcal{F}[f_1 * f_2] = \mathcal{F}[f_1]\mathcal{F}[f_2].$$

Proof. Using Fubini's theorem and the change of variable $t = x - y$, we have

$$\begin{aligned}\mathcal{F}[f_1 * f_2] &= \int f_1 * f_2(x) e^{-izx} \lambda(\mathrm{d}x) = \int e^{-izx} \int f_1(y) f_2(x-y) \lambda(\mathrm{d}y)\lambda(\mathrm{d}x) \\ &= \int f_1(y) \int f_2(x-y) e^{-izx} \lambda(\mathrm{d}x)\lambda(\mathrm{d}y) = \int f_1(y) e^{-izy} \int f_2(t) e^{-izt} \lambda(\mathrm{d}t)\lambda(\mathrm{d}y) \\ &= \mathcal{F}[f_2] \int f_1 e^{-izy} \lambda(\mathrm{d}y) = \mathcal{F}[f_1]\mathcal{F}[f_2].\end{aligned}$$

$\square$

We note that since the derivative under F.T. becomes multiplication, the Fourier transform is useful for solving differential equations.

Consider a linear differential equation with constant coefficients for $f(x)$

$$f^{(n)} + a_1 f^{(n-1)} + \cdots + a_{n-1} f' + a_n f = \phi(x)$$

under the Fourier transform is mapped to (denoting $\mathcal{F}[f] = g$),

$$(iy)^n g(y) + a_1 (iy)^{n-1} g(y) + \cdots + a_{n-1} iyg(y) + a_n g(y) = \mathcal{F}[\phi](y)$$

if $f, \phi \in L_1$. Thus, solving for g, one may apply the inverse Fourier transform to obtain f.

However, we already know how to solve linear differential equations with matrix exponentials, and so, Fourier transforms are more useful for solving partial differential equations. To illustrate this, let us consider the heat equation

$$D_t u = D_x^2 u$$

where $u = u(x,t)$ is a function for $x \in \mathbb{R}, t \geq 0$. The heat equation models the evolution of the temperature over time in 1-D. We are to be provided the initial temperature distribution $u_0(x) = u(x, 0)$ and we are asked to find $u(x,t)$ for all $t > 0$.

Theorem 7. Suppose that $u_0, u_0', u_0'' \in L_1$, if there exist a solution u of the heat equation such that

- $u(x,t), D_x u(x,t), D_x^2 u(x,t) \in L_1$ for all $t \geq 0$;
- for all $T \geq 0$, there exists some $f_T \in L_1$ such that for all $0 \leq t \leq T$,

$$|D_t u(x,t)| \leq f_T(x),$$

then this solution is unique (I'm claiming this, is this true?) and can be found explicitly.

Proof. Applying F.T. to the right hand side of the heat equation, we obtain

$$\mathcal{F}[D_t u](y) = -y^2 \mathcal{F}_t[u](y)$$

where we denote $\mathcal{F}_t[u](y)$ the Fourier transform of $u(t, \cdot)$ for all $t \geq 0$. Then, by the second condition, the the dominated convergence theorem,

$$\mathcal{F}[D_t u](y) = \int \frac{\partial u}{\partial t} e^{-iyx} \lambda(\mathrm{d}x) = \frac{\partial}{\partial t} \int u e^{-iyx} \lambda(\mathrm{d}x) = D_t \mathcal{F}_t[u](y).$$

Hence, we have

$$D_t \mathcal{F}_t[u](y) = -y^2 \mathcal{F}_t[u](y).$$

Therefore, as $\mathcal{F}_t[u](y)$ is Lipschitz (as it is continuously differentiable by condition 1), this IVP has a unique solution. Furthermore, we may find this solution explicitly by integrating both sides resulting in

$$\mathcal{F}_t[u](y) = e^{-y^2 t} \mathcal{F}_0[u] = e^{-y^2 t} \mathcal{F}[u_0].$$

Thus, we may find u by taking the inverse Fourier transform. In particular, we have

$$e^{-y^2 t} = \mathcal{F}\left[\frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}\right]$$

and hence,

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[e^{-y^2 t} \mathcal{F}[u_0]] &= \mathcal{F}^{-1}\left[\mathcal{F}\left[\frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}\right] \mathcal{F}[u_0]\right] \\ &= \mathcal{F}^{-1}\left[\mathcal{F}\left[\frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} * u_0(x)\right]\right] = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} * u_0(x). \end{aligned}$$

By unfolding the definition,

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} * u_0(x) = \frac{1}{2\sqrt{\pi t}} \int e^{-y^2/4t} u_0(x-y) \lambda(\mathrm{d}y)$$

which is known as a Poisson integral. □

3.4 Fourier Transform in L_2

For convenience and generality, we will consider the complex L_2 in this section. Recall that for $f \in L_1[-\pi,\pi]$, we defined the complex Fourier coefficients

$$c_n := \frac{1}{2\pi} \int_{[-\pi,\pi]} f(x) e^{-inx} \lambda(\mathrm{d}x).$$

By Hölder's inequality, we have $L_q \subseteq L_p$ for all $p \leq q$ in a finite measure space, and thus we have $L_2[-\pi,\pi] \subseteq L_1[-\pi,\pi]$. Thus, c_n is defined for any $f \in L_2[-\pi,\pi]$. Moreover, the map $f \in L_2[-\pi,\pi]$ to its Fourier coefficients is a linear operator from $L_2[-\pi,\pi]$ to ℓ_2 satisfying Parseval's identity.

The same is not true for general measure spaces and the Fourier transform is not necessarily defined for L_2 functions. Therefore, we will need to extend the definition of Fourier transforms.

Theorem 8 (Plancherel's Theorem)**.** For any $f \in L_2$, the integral

$$g_N(y) := \int_{[-N,N]} f(x) e^{-iyx} \lambda(\mathrm{d}x)$$

is L_2 for any N. Furthermore, g_N converges in L_2 to some $g \in L_2$ as $N \to \infty$. We call this g the Fourier transform of f and Parseval's identity holds, namely

$$\int_{\mathbb{R}} |g(y)|^2 \lambda(\mathrm{d}y) = 2\pi \int_{\mathbb{R}} |f(x)|^2 \lambda(\mathrm{d}x).$$

Thus, if $f \in L_1 \cap L_2$, then g coincides with $\mathcal{F}[f]$.

Proof. We will first prove Plancherel's theorem for Schwartz functions and then use the fact that such functions are dense in L_2 (as simple functions can be approximated by Schwartz functions) to complete the proof.

Let $f_1, f_2 \in S_\infty$ and denote g_1, g_2 their Fourier transforms (which also belong in S_∞). Then, by the inverse Fourier transform and the Fubini theorem,

$$\begin{aligned}
\int_{\mathbb{R}} f_1 \overline{f_2} \mathrm{d}\lambda &= \int \frac{1}{2\pi} \int g_1(y) e^{iyx} \lambda(\mathrm{d}y) \overline{g_2(y)} \lambda(\mathrm{d}x) \\
&= \frac{1}{2\pi} \int g_1(y) \overline{\int f_2(x) e^{-iyx} \lambda(\mathrm{d}x)} \lambda(\mathrm{d}y) \\
&= \frac{1}{2\pi} \int g_1(y) \overline{g_2(y)} \lambda(\mathrm{d}y).
\end{aligned}$$

Hence, the Parseval's identity holds by setting $f_1 = f_2$.

Now, let $f \in L_2$ such that $f(x) = 0$ for all $x \notin [-a, a]$ for some $a > 0$. Then, $f \in L_2[-a, a]$ and as $[-a, a]$ has finite measure, $f \in L_1[-a, a]$ and so, as f is identically zero outside $[-a, a]$, $f \in L_1$. Hence, the Fourier transform is defined.

Now since S_∞ is dense, we may take a sequence (f_n) of Schwarz functions which are 0 outside $[-a, a]$ and $f_n \to f$ in L_2. Hence, by the same argument $f_n \to f$ in L_1. Therefore, $g_n := \mathcal{F}[f_n]$ converges to g uniformly. Moreover, (g_n) is Cauchy and hence convergent to g in L_2. Indeed, as $g_n - g_m \in S_\infty$, and by Parseval's identity for S_∞, we have

$$\int |g_n - g_m|^2 \mathrm{d}\lambda = 2\pi \int |f_n - f_m|^2 \mathrm{d}\lambda,$$

from which the Cauchy property follows. Parseval's identity follows similarly by taking the limit of $\|f_n\|_2^2 = \frac{1}{2\pi}\|g_n\|_2^2$ as $n \to \infty$ on both sides.

Finally, for any $f \in L_2$, define $f_N := \mathbf{1}_{[-N,N]} f$. Clearly, by dominated convergence, $\|f - f_N\|_2 \to 0$ as $N \to \infty$. Since $f_N \in L_1$, its Fourier transform exists and is given by

$$g_N(y) = \int f_N(x) e^{-iyx} \lambda(\mathrm{d}y) = \int_{[-N,N]} f(x) e^{-iyx} \lambda(\mathrm{d}x).$$

Then, as Parseval holds for functions vanishing outside compact interval, we have $\|f_N - f_M\|_2^2 = \frac{1}{2\pi}\|g_N - g_M\|_2^2$ and hence, (g_n) is Cauchy and thus, converges to some $g \in L_2$. Similar to above, we can take the limit $\|f_n\|_2^2 = \frac{1}{2\pi}\|g_n\|_2^2$ as $n \to \infty$ on both sides to obtain Parseval's identity. □

Corollary 8.1. Parseval's identity implies that for any $f_1, f_2 \in L_2$,

$$\int f_1 \overline{f_2} \mathrm{d}\lambda = \frac{1}{2\pi} \int g_1 \overline{g_2} \mathrm{d}\lambda.$$

Proof. Consider the Plancherel theorem for $f_1 - f_2$ and $f_1 - i f_2$. □

3.5 Laplace Transform

In the case that a function is neither L_2, we may not in general use the Fourier transform directly. Instead, we would like to consider an analogue of the Fourier transform known as the Laplace transform. In particular, the idea is that some nonintegrable functions f becomes integrable by multiplying it with $e^{-\gamma x}$ for some $\gamma \in \mathbb{R}$. Namely, we take

$$g(s) := \int_{\mathbb{R}} f(x) e^{-isx} \lambda(\mathrm{d}x) = \int_{\mathbb{R}} f(x) e^{-iyx} e^{-zx} \lambda(\mathrm{d}x)$$

where $s = y - iz$.

Suppose f satisfy the Dini condition and $f(x) = 0$ for all $x < 0$ and $|f(x)| < ce^{\gamma_0 x}$ for all $x \geq 0$ for some $c, \gamma_0 > 0$, then, we say $f \in L$. We note that for $s = y + iz$, we have

$$g(s) = \int_{\mathbb{R}} f(x) e^{-isx} = \int_{[0,\infty)} f(x) e^{zx} e^{-iyx} \lambda(\mathrm{d}x)$$

converges uniformly for $z < -\gamma_0$ and so, g is defined on the half plane $\mathbb{R} + i(-\infty, -\gamma_0)$ and is analytic on said half plane.

For fixed $z < -\gamma_0$, we note that g is simply the Fourier transform of the function $f(x)e^{zx}$ and so, as f satisfy the Dini condition, we may apply the inversion formula, namely

$$f(x)e^{zx} = \frac{1}{2\pi}\int_{\mathbb{R}} g(s)e^{iyx}\lambda(\mathrm{d}y),$$

so (noting that the integrals are principle),

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(s)e^{-zx+iyx}\lambda(\mathrm{d}y) = \frac{1}{2\pi}\int_{iz-\infty}^{iz+\infty} g(s)e^{isx}\lambda(\mathrm{d}s).$$

Now, changing the variable with $p = is$ and denote $\Phi(p) = g(s)$ and $w = -z$, we obtain,

$$\Phi(p) = \int_{[0,\infty)} f(x)e^{-px}\lambda(\mathrm{d}x)$$

and (where the integral is principle),

$$f(x) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} \Phi(p)e^{px}\lambda(\mathrm{d}x) \equiv \frac{1}{2\pi i}\lim_{T\to\infty}\int_{w-iT}^{w+iT} \Phi(p)e^{px}\lambda(\mathrm{d}p),$$

where $w > \gamma_0$. This motivates the definition of the Laplace transform.

Definition 3.4 (Laplace Transform)**.** Given $f \in L$, we define the Laplace transform of f to be

$$\Phi(p) = \int_{[0,\infty)} f(x)e^{-px}\lambda(\mathrm{d}x).$$

From the above argument, we have that Φ is analytic for $\mathrm{Re}(p) > \gamma_0$.

The Laplace transform has applications to solutions of differential equations. Consider the IVP

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x), a_1, \cdots, a_n \in \mathbb{C},$$

with initial condition $y(0) = y_0, y'(0) = y_1, \cdots, y^{(n-1)}(0) = y_{n-1}$. Furthermore, assume $b \in L$ and we will look for a solution y such that $y, y', \cdots, y^{(n)} \in L$.

Let $\mathcal{Y}(p), B(p)$ be the Laplace transform of y and b respectively. Then, by integration by parts, we observe

$$\int_{[0,\infty)} y'(x)e^{-px}\lambda(\mathrm{d}x) = y(x)e^{-px}|_0^\infty + p\int_{[0,\infty)} y(x)e^{-px}\lambda(\mathrm{d}x) = p\mathcal{Y}(p) - y_0.$$

By induction, we have

$$\int_{[0,\infty)} y^{(n)}(x)e^{-px}\lambda(\mathrm{d}x) = p^n\mathcal{Y}(p) - y_{n-1} - py_{n-2} - \cdots - p^{n-1}y_0.$$

Thus, after applying the Laplace transform to the differential equation, we obtain

$$Q(p) + R(p)\mathcal{Y}(p) = B(p)$$

where $R(p) = p^n + a_1 p^{n-1} + \cdots + a_n$ and $Q(p)$ is a polynomial of degree $n-1$ with coefficients in our initial conditions. Hence, we conclude,

$$\mathcal{Y}(p) = \frac{B(p) - Q(p)}{R(p)}$$

implying

$$y(x) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{B(p) - Q(p)}{R(p)} e^{ipx} \lambda(\mathrm{d}p),$$

by applying the inverse Laplace transform.

3.6 Fourier-Stieltjes Transform

Let us now return to the Fourier transforms. Recall the Stieltjes integral and we observe that, for $f \in L_1$, we may write its Fourier transform as a Lebesgue-Stieltjes integral,

$$\mathcal{F}[f](y) = \int_{\mathbb{R}} e^{-iyx} f(x) \lambda(\mathrm{d}x) = \int_{\mathbb{R}} e^{-iyx} \mathrm{d}F(x)$$

where $F(x) = \int_{(-\infty,x]} f(t)\lambda(\mathrm{d}t)$. Note that F is absolutely continuous with bounded variation on $\mathbb{R}$. On the other hand, the last integral remains to be well-defined for any function of bounded variation.

Let us first recall some definitions required of the Lebesgue-Stieltjes integral.

Definition 3.5 (Bounded Variation on an Interval)**.** A function F is said to have bounded variation on $[a, b]$ if the variation of F,

$$V_a^b F := \sup_{P \in \mathcal{P}} \sum_{i=1}^{n_P} |f(x_i) - f(x_{i-1})| < \infty,$$

where $\mathcal{P}$ is the set of all finite divisions of $[a, b]$. Namely, $P \in \mathcal{P}$ denotes the points

$$a_0 = x_0 \leq x_1 \leq \cdots \leq x_{n_P} = b.$$

We observe that, if F is non-decreasing, then the sum of any partition is telescoping, and thus, $V_a^b F = F(b) - F(a)$ for any interval $[a, b]$.

Definition 3.6 (Bounded Variation on $\mathbb{R}$)**.** A function F is said to have bounded variation on $\mathbb{R}$ if the limit

$$V_{-\infty}^{\infty} F := \lim_{\substack{a \to -\infty; \\ b \to \infty}} V_a^b F$$

exists and is finite.

Proposition 3.4. If F is a function of bounded variation on $\mathbb{R}$, then, there exists non-decreasing functions F_1, F_2 such that $F = F_1 - F_2$. Moreover, F is differentiable almost everywhere and can be written as

$$F = \phi + \psi + y$$

where ϕ is absolutely continuous (and $F' = \phi'$ a.e.), ψ is singular continuous (and $\psi' = 0$ a.e.), and y is a jump function (with $y' = 0$ a.e.).

With this in mind, we recall that the Lebesgue-Stieltjes measure can be defined for all functions of bounded variation and the Lebesgue-Stieltjes integral is simply the Lebesgue integral taken with respect to that measure.

Definition 3.7 (Lebesgue-Stieltjes Measure of an Interval)**.** Let F be a function of bounded variation on $[a,b]$ and suppose $F = F_1 - F_2$ where F_1, F_2 are non-decreasing. Then, the Lebesgue-Stieltjes measure of F on the interval $[a,b]$ is defined to be the unique measure obtained from the Caratheodory extension of the outer measure

$$w((s,t]) := F_1(t) - F_1(s) + F_2(t) - F_2(s).$$

Definition 3.8 (Fourier-Stieltjes Transform)**.** The integral

$$g(y) := \int_{\mathbb{R}} e^{-iyx} \mathrm{d}F(x)$$

where F is a function of bounded variation on $\mathbb{R}$, is called the Fourier-Stieltjes transform of F.

Lemma 3.5. The Fourier-Stieltjes transform (F.S.T.) of F, g is bounded and continuous on $\mathbb{R}$.

Proof. We have $|g(y)| \leq \int |e^{-iyx}| \mathrm{d}F(x) = \lambda_F(\mathbb{R})$. Now, since $\lambda_{V_{-\infty}^x F} \geq \lambda_F$ where $V_{-\infty}^x F$ is non-decreasing, by the continuity of measures from below,

$$\lambda_F(\mathbb{R}) = \lim_{x\to\infty} \lambda_F((-\infty, x]) \leq \lim_{x\to\infty} \lambda_{V_{-\infty}^x F}((-\infty, x]) = \lim_{x\to\infty} V_{-\infty}^x F = V_{-\infty}^\infty F.$$

Hence, $|g|$ is bounded by $V_{-\infty}^\infty F$.

Secondly, we observe

$$|g(y_1) - g(y_2)| \leq \int_{[-N,N]} |e^{-iy_1x} - e^{-iy_2x}| \mathrm{d}V_{-\infty}^x F + \int_{|x|\geq N} |e^{-iy_1x} - e^{-iy_2x}| \mathrm{d}V_{-\infty}^x F.$$

Thus, taking N sufficiently large, the second term is arbitrarily small (independent of y_1, y_2), and thus, as the first term tends to 0 at $|y_1 - y_2| \to 0$, we conclude g is continuous. $\square$

Unlike the Fourier transform, the Fourier-Stieltjes transform of a function does not necessarily vanish at $\pm\infty$. Indeed, defining $F = \mathbf{1}_{[0,\infty)}$, the Lebesgue-Stieltjes measure of F is $\lambda_F = \delta_0$ (Dirac measure at 0), and so, the F.S.T. of F is

$$g(y) = \int_{\mathbb{R}} e^{-iyx} \mathrm{d}F(x) = \int_{\mathbb{R}} e^{-iyx} \mathrm{d}\delta_0 = 1.$$

Recall for L_1 functions, we defined the convolution of two functions f_1, f_2 to be $f_1 * f_2(x) = \int_{\mathbb{R}} f_1(x-y) f_2(x) \lambda(\mathrm{d}y)$. We will define a similar definition for functions of bounded variations.

Let $F(x) = \int_{(-\infty,x]} f(t)\lambda(\mathrm{d}t)$ and $F_i(x) = \int_{(-\infty,x]} f_i(t)\lambda(\mathrm{d}t)$ for $i = 1, 2$, where $f = f_1 * f_2$, by the absolute integrability of f, f_1, f_2 and Fubini's theorem, we have

$$\begin{aligned} F(x) &= \int_{(-\infty,x]} f(t)\lambda(\mathrm{d}t) = \int_{(-\infty,x]} \int_{\mathbb{R}} f_1(t-y) f_2(y) \lambda(\mathrm{d}y)\lambda(\mathrm{d}t) \\ &= \int_{\mathbb{R}} \left(\int_{(-\infty,x]} f_1(t-y)\lambda(\mathrm{d}t) \right) f_2(y)\lambda(\mathrm{d}y) = \int_{\mathbb{R}} F_1(x-y) \mathrm{d}F_2(y). \end{aligned}$$

This motivates for the definition of the convolution of two functions of bounded variation.

Definition 3.9 (Convolution of Functions of Bounded Variation)**.** Given F_1, F_2 are functions of bounded variation, we define the convolution of F_1 with F_2 to be

$$F_1 * F_2(x) = \int_{\mathbb{R}} F_1(x-y)\mathrm{d}F_2(y).$$

Proposition 3.5. If F_1, F_2 are functions of bounded variation on $\mathbb{R}$, then so is $F_1 * F_2$ a function of bounded variation on $\mathbb{R}$.

Proof. Consider

$$|F(x_1) - F(x_2)| \leq \int_{\mathbb{R}} |F_1(x_1-y) - F_1(x_2-y)|\mathrm{d}V_{-\infty}^{y} F_2.$$

So,

$$\begin{aligned}
\sum_i |F(x_i) - F(x_{i-1})| &\leq \sum_i \int_{\mathbb{R}} |F_1(x_i-y) - F_1(x_{i-1}-y)|\mathrm{d}V_{-\infty}^{y} F_2 \\
&= \int_{\mathbb{R}} \left(\sum_i |F_1(x_i-y) - F_1(x_{i-1}-y)| \right) \mathrm{d}V_{-\infty}^{y} F_2 \\
&\leq \int_{\mathbb{R}} V_{-\infty}^{\infty} F_1 \mathrm{d}V_{-\infty}^{y} F_2 = V_{-\infty}^{\infty} F_1 V_{-\infty}^{\infty} F_2.
\end{aligned}$$

Hence, $V_{-\infty}^{\infty} F \leq V_{-\infty}^{\infty} F_1 V_{-\infty}^{\infty} F_2 < \infty$ implying F has bounded variation as required. $\square$

Theorem 9. Let F be the convolution of the function F_1, F_2 of bounded variations on $\mathbb{R}$ and let g, g_1, g_2 be their Fourier-Stieltjes transforms respectively. Then, $g = g_1 \cdot g_2$.

Proof. Let $F = F_1 * F_2$ and consider the division

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b$$

of the interval $[a, b]$. Then, for any y, as e^{-iyx} is continuous, the Lebesgue-Stieltjes integral of e^{-iyx} coincides with the Riemann-Stieltjes integral. In particular,

$$\begin{aligned}
\int_{[a,b]} e^{-iyx}\mathrm{d}F(x) &= \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} e^{iyx_k}(F(x_k) - F(x_{k-1})) \\
&= \lim_{\max \Delta x_k \to 0} \int_{\mathbb{R}} \sum_{k=1}^{n} e^{iy(x_k - t)}(F(x_k - t) - F(x_{k-1} - t))e^{-iyt}\mathrm{d}F_2(t) \\
&= \int_{\mathbb{R}} \left(\int_{a-y}^{b-y} e^{-iyx}\mathrm{d}F_1(x) \right) e^{-iyt}\mathrm{d}F_2(t) \\
&= \left(\int_{a-y}^{b-y} e^{-iyx}\mathrm{d}F_1(x) \right) \left(\int_{\mathbb{R}} e^{-iyt}\mathrm{d}F_2(t) \right)
\end{aligned}$$

where the third equality holds by dominated convergence. Hence, taking the limit $a \to \infty, b \to -\infty$, we obtain

$$g(y) = \int_{\mathbb{R}} e^{-iyx} \mathrm{d}F(x) = \left(\int_{\mathbb{R}} e^{-iyx} \mathrm{d}F_1(x) \right) \left(\int_{\mathbb{R}} e^{-iyt} \mathrm{d}F_2(t) \right) = g_1(y) g_2(y)$$

as required. □

This fact is applicable in probability theory and especially useful when dealing with characteristic functions. In particular, if ξ, η are two independent random variables and F_1, F_2 are their distributions, i.e. $F_1(x) = \xi_* \mathbb{P}((-\infty, x])$ and $F_2(x) = \eta_* \mathbb{P}((-\infty, x])$. Then, $\xi + \eta$ has distribution $F_1 * F_2$ (see probability theory).

The characteristic function of a random variable X is defined to be the F.S.T. of its distribution. Namely, if X has distribution F, then its characteristic function is defined to be

$$g(y) := \int_{\mathbb{R}} e^{-iyx} \mathrm{d}F(x).$$

Therefore, a corollary of the above theorem is that the characteristic function of $\xi + \eta$ is simply the product of the characteristic functions of the two.

From this, one may obtain the distribution of $\xi + \eta$ by applying the inverse Fourier-Stieltjes transform which formula is on the exercise sheet.

4 Linear Functional on Normed Space

This section is mostly a review of definitions and results from functional analysis. See functional analysis notes for more details as some proofs are omitted.

4.1 Linear Functional

Definition 4.1 (Functional)**.** Let L be a linear space (not necessarily topological) over $\mathbb{C}$ (or $\mathbb{R}$), a linear functional is simply a function $f : L \to \mathbb{C}$ (or $\mathbb{R}$).

The functional is said to be linear if for all $x, y \in L$, $\alpha \in \mathbb{C}$ (or $\mathbb{R}$),

$$f(x+y) = f(x) + f(y); f(\alpha x) = \alpha f(x).$$

If a linear space has a compatible topology, namely addition and scalar multiplication are continuous, then we have a topological linear space and we may talk about notions of continuity of the linear functionals. We will in this course restrict to the case where the norm is generated by a norm on the linear space.

Definition 4.2. A functional f on a normed linear space E is continuous if for all $x_0 \in E$, $\epsilon > 0$, there exists an open neighbourhood U of x_0 such that for all $x \in U$,

$$|f(x) - f(x_0)| < \epsilon.$$

Proposition 4.1. A linear functional on a finite dimensional normed linear space is continuous.

Proposition 4.2. A linear functional is continuous if and only if it is continuous at 0.

Proposition 4.3. A linear functional is continuous if and only if there is a neighbourhood of 0 for which it is bounded.

Corollary 9.1. A linear function is continuous if and only if it is bounded on the unit ball.

With the above propositions in mind, we see that if f is continuous, $\sup_{\|x\| \leq 1} |f(x)| < \infty$ and so, we may define a norm on the space of linear functionals by taking

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)|.$$

Equivalently, by considering for all $\|x\| \neq 0$, $\|x/\|x\|\| = 1$ and $f(x/\|x\|) = f(x)/\|x\|$, we have

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}.$$

With this, we have the inequality $|f(x)| \leq \|f\|\|x\|$ for all $x \in E$.

Proposition 4.4. If E is an inner product space, then for all $a \in E$, $a \neq 0$, defining the functional $f : E \to \mathbb{C}$ by $f(x) := \langle x, a \rangle$, f is a continuous linear functional with norm $\|a\|$.

Definition 4.3. Let L be a linear space and let p be a functional on L. Then, p is said to be convex (or a sub-linear functional) if $p(x+y) \leq p(x) + p(y)$, and $p(\alpha x) = |\alpha| p(x)$ for all $x, y \in L, \alpha \in \mathbb{C}$.

Theorem 10 (Hahn-Banach)**.** Let L be a linear space and let p be a sub-linear functional on L. Then, if f_0 is a linear functional on $M \leq L$ where M is a subspace of L such that $|f_0(x)| \leq p(x)$ for all $x \in M$. Then, there exists a linear functional f on L such that $f|_M = f_0$ and $|f(x)| \leq p(x)$ for all $x \in L$.

It is clear that the norm is a sub-linear functional on a normed linear space and so, we may apply Hahn-Banach to this special case.

Corollary 10.1. Let E be a normed linear space and let f_0 be a continuous linear functional on the subspace $E_0 \subseteq E$, then there exists a continuous linear functional f on E such that

- $f(x) = f_0(x)$ for all $x \in E_0$;
- $\|f\| = \|f_0\|$.

Proof. Let $c := \|f_0\|$. Then, defining $p(x) := c\|x\|$, by Hahn-Banach we have the extension of f_0 such that $f|_{E_0} = f_0$ and $|f(x)| \leq c\|x\|$ and so,

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq c.$$

Yet, $\|f\| \geq \|f_0\| = c$, we have $\|f\| = \|f_0\|$ as required. □

4.2 Dual Space

Definition 4.4. Given a linear space L, we define its adjoint (or dual) L^* as the space of all linear functionals on L equipped with point-wise addition and scalar multiplication.

It is easy to check that, if E is a normed linear space, then E^* is also a normed linear space. The topology induced from this norm is called the strong topology on E^*.

Theorem 11. The dual space $(E^*, \|\cdot\|)$ is complete.

Proof. Let $(f_n) \subseteq E^*$ be Cauchy. Then, f_n converges point-wise as $\mathbb{C}$ (or $\mathbb{R}$) is complete. Thus, defining $f(x) := \lim_{n\to\infty} f_n(x)$, it remains to show $f \in E^*$.

By the algebra of limits, f is linear, and so, it suffices to show it is bounded. Indeed, as (f_n) is Cauchy, there exists some N such that for all $n, m \geq N$, $\|f_n - f_m\| < 1$. Thus, for all $\|x\| = 1$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < 1.$$

Now, by taking $m \to \infty$, we have $|f_n(x) - f(x)| \leq 1$ implying $\|f\| \leq \|f_n\| + 1 < \infty$. Hence, $f \in E^*$ and E^* is complete as required. □

We note that the dual is complete irrespective of the completeness of E.

Proposition 4.5. If E is not complete, denoting $\overline{E}$ the completion of E, E^* and $\overline{E}^*$ are isometric (isomorphic in the linear space sense and preserves the norm).

Proof. Since $E \subseteq \overline{E}$ is everywhere dense in $\overline{E}$, and $f \in E^*$ can be extended uniquely to a continuous functional $\overline{f}$ in $\overline{E}^*$ and $\|f\| = \|\overline{f}\|$. Thus, by considering $f \mapsto \overline{f}$ and its inverse $\overline{f} \mapsto \overline{f}|_E$, we have our isometry. □

Proposition 4.6. Denote $c_0 \subseteq \ell_\infty$ the space of sequences which converges to 0. Then, c_0^* is isometric to ℓ_1.

Proof. Let $f \in \ell_1$. Then defining

$$\hat{f} : c_0 \to \mathbb{R} : x \mapsto \sum_{n=1}^{\infty} f_n x_n.$$

This is well-defined since as $x \to 0$ for all $x \in c_0$,

$$\|\hat{f}(x)\| \leq \|x\| \|f\|_{\ell_1} < \infty,$$

which further implies $f \to \hat{f}$ is continuous. Clearly, $f \mapsto \hat{f}$ is linear and so, it remains to show this map preserves norm and is surjective.

Consider the sequence $(x^N) \subseteq c_0$ defined by $x^N := \sum_{n=1}^N \frac{f_n}{|f_n|} e_n$ where $f_n/|f_n|$ is defined to be 0 if $f_n = 0$. It is clear that $\|x^N\| \leq 1$ and

$$\hat{f}(x^N) = \sum_{n=1}^{N} \hat{f}(e_n) \frac{f_n}{|f_n|} = \sum_{n=1}^{N} |f_n|.$$

Hence, taking $N \to \infty$, we have $\hat{f}(x^N) \to \|f\|_{\ell_1}$ implying $\|\hat{f}\| \geq \|f\|_{\ell_1}$. Thus, as we have already shown $\|\hat{f}\| \leq \|f\|_{\ell_1}$, we obtain $\|\hat{f}\| = \|f\|_{\ell_1}$.

Finally, suppose $g \in c_0^*$. We note that $x = \sum_{n=1}^\infty x_n e_n$. So, by the continuity of g, $g(x) = \sum_{n=1}^\infty x_n \hat{g}(e_n)$. Thus, defining $f \in \ell_1$ such that $f = (g(e_n))_{n=1}^\infty$, we have $\hat{f}(x) = \sum_{n=1}^\infty g(e_n) x_n = \sum_{n=1}^\infty g(x_n e_n) = g(x)$. Thus, it remains to show that $f \in \ell_1$. Indeed, we observe

$$\sum_{n=1}^{N} |g(e_n)| = \sum_{n=1}^{N} \frac{g(e_n) g(e_n)}{|g(e_n)|} = g\left(\sum_{n=1}^{N} \frac{g(e_n)}{|g(e_n)|} e_n\right) = g(x^N) \leq \|g\|,$$

implying $f \in \ell_1$ as required. □

Proposition 4.7. ℓ_1^* is isometric to ℓ_∞.

This is an important counter-example to the statement that $(E^*)^*$ is canonically isomorphic to E. Indeed, $(c_0^*)^* \cong (\ell_1)^* \cong \ell_\infty$ yet ℓ_∞ is clearly much larger than c_0. Nonetheless, the double-dual isomorphism constructed for finite-dimensional spaces remains to provide an injection (recall $x \in E \mapsto (f \in E^* \mapsto f(x))$).

Proposition 4.8. For $1 \leq p < \infty$, ℓ_p^* is isometric to ℓ_q where $1/p + 1/q = 1$.

Proposition 4.9. If H is a real Hilbert space, then H is isometric to its dual.

Proof. For all $f \in H^*$, by Riesz representation, there exists a unique $x_f \in H$ such that $f = \langle \cdot, x_f \rangle$. Thus, defining $\phi : f \mapsto x_f$ which is clearly linear, we observe ϕ is an linear isomorphism as it has inverse $x \mapsto \langle \cdot, x \rangle$.

Now, this is an isometry since $\|f\| \geq |f(x_f/\|x_f\|)| = \|x_f\|$ and by Cauchy-Schwarz, $|f(x/\|x\|)| = |\langle x, x_f \rangle / \|x\|| \leq \|x\| \|x_f\| / \|x\| = \|x_f\|$ implying $\|f\| = \|x_f\|$. □

Thus, if E is any inner product space (so not necessarily complete), and H is its completion, we have $E^* \cong H^* \cong H$.

Since the map $x \in E \mapsto \phi_x := (f \in E^* \mapsto f(x))$ is injective, it is useful to introduce the following notation. In particular, for all $x \in E, f \in E^*$, we denote

$$(f, x) := f(x) = \phi_x(f).$$

So, $(\cdot, x) \in (E^*)^*$ and $(f, \cdot) \in E^*$.

Lemma 4.1. For a normed linear space E, $\pi : x \mapsto \phi_x$ preserves the norm.

Proof. Clearly, for all $\|f\| = 1$, $|\phi_x(f)| = |f(x)| \leq \|x\|\|f\| = \|x\|$ implying $\|\phi_x\| \leq \|x\|$. On the other hand, by taking f to be the Hahn-Banach extension of f_0 where $f_0(x) = \|x\|$ and $\|f_0\| = 1$, we have $|\phi_x(f)| = |f(x)| = |f_0(x)| = \|x\|$ implying $\|\phi_x\| = \|x\|$ as required. □

Definition 4.5 (Reflexive)**.** A normed linear space E is said to be reflexive if $\pi(E) = (E^*)^*$. Thus, E is reflexive if it is canonically isometric to its double dual.

As a consequence of 4.8, we see that ℓ_p is reflexive for all $1 < p < \infty$.

4.3 Linear Topological Space

Definition 4.6. The set E is a linear topological space, also known as a topological vector space (TVS), if

- E is a real (or complex) vector space,
- E is a topological space,
- the topology on E is compatible with the vector space operations, namely, addition and scalar multiplication are continuous.

Proposition 4.10. The topology on a TVS E is uniquely determined by its open neighbourhood of 0.

Proof. Since addition is continuous, $U \subseteq E$ is open if and only if $U + x$ is open for all $x \in E$. Thus, all open neighbourhoods in E can be shifted to an open neighbourhood of 0 allowing us to conclude the claim. □

In addition to the claim that the translation operator $T_\alpha : x \mapsto \alpha + x$ is continuous, we have the following facts about TVS.

Proposition 4.11. Let $U, V \subseteq E$ be open, $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), $F \subseteq E$ closed, $x \in E \setminus F$, then

- $U + V$ is open,
- αU is open,
- αF is closed,
- x and F have non-intersecting neighbourhoods.

Proof. Exercise. □

We note that the fourth property implies Hausdorff.

If $C^\infty[a,b]$ is the set of infinitely differentiable functions on (a,b), we may equip it with the topology formed by the open neighbourhoods $U_{m,\epsilon}$ for $m \in \mathbb{N}, \epsilon > 0$ where

$$U_{m,\epsilon} := \{\phi \in C^\infty[a,b] \mid \|\phi^{(k)}\|_\infty < \epsilon, k = 0, \cdots, m\}.$$

One may show this is a topological vector space.

Definition 4.7 (Bounded)**.** Let E be a TVS. Then $M \subseteq E$ is bounded if for any open neighbourhood U of 0, there exists some $n > 0$ such that $M \subseteq \lambda U$ for all $|\lambda| \geq n$.

It is an easy exercise to show that this definition agrees with the notion of boundedness in normed spaces.

Most notions regarding linear functional remain to hold for linear functionals from a topological vector space.

Proposition 4.12. A linear functional from a TVS is continuous if and only if it is bounded on a neighbourhood of 0.

Similarly, we call the space of linear functionals as the dual. However, we do not have a norm on the dual and thus, the properties discussed previously no longer holds. Let us now construct a topology on the dual which is compatible with the induced topology in the normed case.

Recall that a system B of open neighbourhoods of 0 is *defining* (or a local base) is any open set around 0 contains a element of B. If a system B is defining, then any open set in the induced topology τ is a union of translations of elements of B, i.e.

$$\{A + x \mid A \in B, x \in E\}$$

form a basis of τ.

In the normed case a defining system is $\{U_\epsilon \mid \epsilon > 0\}$ where

$$U_\epsilon := \{f \in E^* \mid \|f\| < \epsilon\} = \{f \in E^* \mid |f(x)| < \epsilon, \|x\| \leq 1\}.$$

Thus, considering this system on arbitrary TVS, $\{U_{\epsilon,A} \mid \epsilon > 0, A \text{ bounded}\}$ where

$$U_{\epsilon,A} := \{f \in E^* \mid |f(x)| < \epsilon, x \in A\}$$

we obtain a topology on E^* and this is known as the strong topology on E^*.

Similar to before, considering the double dual $(E^*)^*$, we again can define the natural map $\pi : E \to (E^*)^* : x \mapsto (f \mapsto f(x))$.

Definition 4.8 (Reflexive)**.** A topological vector space E is reflexive if $\pi(E) = (E^*)^*$ and π is continuous.

We note that we require continuity as we can no longer conclude that π is an isometry.

4.4 Weak Topology and Weak Convergence

We will now introduce another topology on E.

Let $\epsilon > 0$, $f_1, \cdots, f_n \in E^*$, then the set

$$U := \{x \in E \mid |f_j(x)| < \epsilon, j = 1, \cdots, n\}$$

is open (since $U = \bigcap_{j=1}^n f_j^{-1}(B_\epsilon(0))$) and contains 0. Furthermore, as such sets are closed under intersection, this system is defining and so, generates a topology on E. Under this new topology, each open set is open with respect to the original topology and so, is called the weak topology on E.

One may show that this is the weakest topology on E such that the elements of E^* remain to be continuous.

Definition 4.9 (Weak Convergence)**.** Convergence within the weak topology is called weak convergence while convergence within the original topology is known as strong convergence.

Proposition 4.13. Clearly, as all open sets of the weak topology on E are open with respect to the original topology, strong convergence implies weak convergence.

A useful characterisation of weak convergence is the following.

Proposition 4.14. A sequence (x_n) in E converges weakly to $x_0 \in E$ if for all $f \in E^*$, $f(x_n) \to f(x_0)$ as $n \to \infty$.

Proof. By shifting, we may assume $x_0 = 0$ and suppose $f(x_n) \to f(x_0) = 0$ for all $f \in E^*$. Then, for any weak neighbourhood of 0

$$U = \{x \mid |f_j(x)| < \epsilon, j = 1, \cdots, k\}$$

there exists some N such that $x_n \in U$ for all $n \geq N$ by the continuity of f as k is finite.

Conversely, if $x_n \to 0$ in the weak topology, for any U, there exists some N such that $x_n \in U$ for all $n \geq N$. Then, for all $f \in E^*$, $f(x_n) \to 0$ as required. $\square$

Theorem 12. If (x_n) weakly converges to x_0 in the normed linear space E, then, there exists some $C > 0$ such that $\|x_n\| \leq C$ for all n. Namely, a weakly convergent sequence is bounded.

Proof. Defining

$$A_{k,n} = \{f \in E^* \mid |(f, x_n)| \leq K\}$$

for all $k, n = 1, 2, \cdots$. Now, since $(\cdot, x_n) \in (E^*)^*$ is continuous, $A_{k,n} = (\cdot, x_n)^{-1}[-K, K]$ is closed. Thus, the intersection

$$A_k := \bigcap_{n=1}^{\infty} A_{k,n}$$

is also closed. Now, since (x_n) is weakly convergent, $(f, x_n) = f(x_n)$ is bounded for all $f \in E^*$ and so, $E^* = \bigcup_{k=1}^\infty A_k$. Now, since E^* is complete, by the Baire category theorem it cannot be a countable union of nowhere dense sets. Thus, for some k_0, A_{k_0} is dense in some ball $B_\epsilon(f_0)$ and since, A_{k_0} is closed, $B_\epsilon(f_0) \subseteq A_{k_0}$. Thus, $f(x_n)$ is bounded for all

$f \in B_\epsilon(f_0)$ and hence, $f(x_n)$ is bounded on the unit ball. By noting that this is equivalent to $(\cdot, x_n)$ being bounded on the unit ball, and $\pi : E \to (E^*)^*$ preserves norm and hits $(\cdot, x_n)$, we have (x_n) is bounded as required. $\square$

We remark that we didn't really use the fact that (x_n) is weakly convergent. In particular, the proof remains to work that (f, x_n) is bounded for all $f \in E^*$.

Theorem 13 (Criterion for Weak Convergence)**.** A sequence (x_n) tends weakly to x in the normed linear space E if and only if (x_n) is bounded in norm and $f(x_n) \to f(x)$ for any $f \in \Delta$ where $\Delta \subseteq E^*$ is a complete system in E^* with respect to the strong topology on E^*.

Proof. The forward direction is trivial. Let $\phi \in E^*$, then, if $\phi \in \langle \Delta \rangle$, $\phi(x_n) \to \phi(x)$ as required. Now, if $\phi \in E^*$ is arbitrary, there exists a sequence $(\phi_k) \subseteq \langle \Delta \rangle$ such that $\phi_k \to \phi$ in norm. Thus, for all $\epsilon > 0$,

$$\begin{aligned} |\phi(x_n) - \phi(x)| &\leq |\phi(x_n) - \phi_N(x_n)| + |\phi_N(x_n) - \phi_N(x)| + |\phi_N(x) - \phi(x)| \\ &\leq \|\phi - \phi_N\|(\|x_n\| + \|x\|) + |\phi_N(x_n) - \phi_N(x)|. \end{aligned}$$

Now, as (x_n) is bounded in norm and $\phi_N(x_n) \to \phi_N(x)$ as $n \to \infty$, the right hand side is eventually smaller than ϵ implying $\phi(x_n) \to \phi(x)$ as so $x_n \to x$ weakly as required. $\square$

Corollary 13.1. In a finite-dimensional normed vector space E, (x_n) converges weakly if and only if it converges strongly.

Proof. By taking the dual basis of E^*, we have (x_n) converges weakly if and only if it converges point-wise by taking the coordinates according to the basis. Thus, we conclude by observing that this implies strong convergence. $\square$

Corollary 13.2. A sequence in ℓ_p is weakly convergent if and only if it is bounded and converges point-wise.

Proof. This follows from above as the system of projection functionals is complete. $\square$

This corollary provides an easy example of a weak convergent sequence which does not converge strongly. Indeed, by considering $e_i \in \ell_p$ where e_i has 1 in the i-component and 0 everywhere else, e_i converges to 0 weakly yet $\|e_i - e_j\| \geq 1$ for all $i \neq j$ and thus is not Cauchy, hence is not convergent.

Corollary 13.3. Let (x_n) be a sequence in $(C[a,b], \|\cdot\|_\infty)$. Then, if $x_n \to x$ weakly, (x_n) is bounded in uniformly and converges point-wise. One can show that the convergence is also true.

Proof. This is clear by defining $\delta_{t_0} \in C^*[a,b]$ given by $\delta_{t_0} := x(t_0)$ (and we call it a δ-function), so that $\delta_{t_0}(x_n) \to \delta_{t_0}(x)$ implies $x_n(t_0) \to x(t_0)$. $\square$

4.5 Weak Convergence in the Dual Space

Recall that we had defined a (strong) topology on the dual space E^* by a system of neighbourhoods $\{U_{\epsilon,A} \mid \epsilon > 0, A \text{ bounded}\}$, where

$$U_{\epsilon,A} := \{f \mid |f(x)| < \epsilon, x \in A\}.$$

Now, taking the subsystem where A is finite, the topology generated by this subsystem is known as the weak-$*$ topology on E^*. It is clear that the weak-$*$ topology is coarser than the strong topology and so convergence in strong topology implies convergence in the weak-$*$ topology. We call convergence in the weak-$*$ topology weak-$*$ convergence.

Proposition 4.15. A sequence $(f_n) \subseteq E^*$ converge $*$-weakly to $f \in E^*$ if for all $x \in E$, $f_n(x) \to f(x)$ as $n \to \infty$.

By noting the symmetry of weak-$*$ convergence with the weak convergence, we have the following two similar theorems.

Theorem 14. If $(f_n) \subseteq E^*$ converge $*$-weakly to $f \in E^*$ where E is a Banach space, then there exists some c, $\|f_n\| \le c$ for all n.

Theorem 15. $f_n \to f$ weak-$*$ in E^* where E is a Banach space if and only if

- (f_n) is bounded in norm and
- $(f_n, x) \to (f, x)$ for any $x \in \Delta \subseteq E$ where Δ is a complete system in E with respect to the strong topology in E.

Again, considering the space $C[a,b]$ with the uniform norm where $0 \in [a,b]$, and considering the δ-function $\delta(x) = x(0)$, taking $\phi_n \in C[a,b]$ such that $\phi_n(t) = 0$ for all $|t| > 1/n$, $\phi_n \ge 0$, and

$$\int_{[a,b]} \phi_n \mathrm{d}\lambda = 1,$$

we make the following observations.

- ϕ_n does not converge strongly in $C[a,b]$.
- Defining

  $$\Phi_n : x \mapsto \int_{[a,b]} \phi_n x \mathrm{d}\lambda = \int_{[-1/n,1/n]} \phi_n x \mathrm{d}\lambda,$$

  $\Phi_n(x) \to x(0) = \delta(x)$ as $n \to \infty$. Indeed, as x is continuous, for all $\epsilon > 0$, there exists some $\delta > 0$ such that $x(B_\delta(0)) \subseteq B_\epsilon(0)$. Thus, for sufficiently large n,

  $$\begin{aligned} |\Phi_n(x) - x(0)| = \left| \int_{[-1/n,1/n]} \phi_n x \mathrm{d}\lambda - x(0) \right| &\le \int_{[-1/n,1/n]} \phi_n |x - x(0)| \mathrm{d}\lambda \\ &< \epsilon \int_{[-1/n,1/n]} \phi_n \mathrm{d}\lambda = \epsilon. \end{aligned}$$

  Hence, $\Phi_n \to \delta$ in the weak-$*$ topology. Thus, in some sense δ is represented by a sequence of functions (ϕ_n). This motivates the theory of distributions.

We note that there is another topology on E^*, namely the weak topology on E^* by considering $(E^*)^*$. This weak topology is in general not the same as the weak-$*$ topology. Indeed, the weak-$*$ topology is coarser than the weak topology on E^* and they coincide in the case that E is reflexive.

Furthermore, in the case that E is a separable normed space, any closed ball $\overline{B}_c(f_0) := \{f \mid \|f - f_0\| \le c\} \subseteq E^*$ is compact in the weak-$*$ topology. It is useful to contrast this fact by recalling that the closed ball of any infinite dimensional normed space is not compact with respect to the strong topology.

4.6 Countably Normed Space

Definition 4.10 (Compatible)**.** Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a linear space E are said to be compatible if for any sequence $(x_n) \subseteq E$ which is Cauchy in both norms and converges to some x in E with respect to $\|\cdot\|_1$, then $x_n \to x$ in $\|\cdot\|_2$ as well.

Definition 4.11 (Countably Normed)**.** A linear space E is said to be countably normed if a countable system of pair-wise compatible norms $(\|\cdot\|_i)_{i=0}^{\infty}$ is given on E.

A countably normed space E induces a topology obtained by taking

$$U_{r,\epsilon} := \{x \in E \mid \|x\|_0 < \epsilon, \cdots, \|x\|_r < \epsilon\}$$

as the system of neighbourhood of 0 where $\epsilon > 0, r = 0, 1, \cdots$.

It is easy to check that a countably normed space is a topological vector space. Also, by the definition of these neighbourhoods, we see that a sequence (x_n) converges in this topology to some x if and only if $x_n \to x$ with respect to all $\|\cdot\|_n$.

The countably normed space is itself not a normed space. However, it is metrizable.

Lemma 4.2. A countably normed space E is metrizable with the metric

$$\rho(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x + y\|_n}.$$

Proof. Exercise. □

Some common examples of countably normed spaces comes to mind.

- The space $C^\infty[a, b]$ of infinitely differentiable functions on $[a, b]$ is countably normed with the norms
$$\|f\|_m := \sup_{0 \le k \le m; t} |f^{(k)}(t)|.$$
- The space S_∞ of Schwartz functions is countably normed with the norms
$$\|f\|_m := \sup_{\substack{0 \le k, q \le m \\ t < m}} |t^k f^{(q)}(t)|.$$

It is an exercise to check that the above are indeed compatible norms.

The norms on E can always be considered satisfying the relation $\|x\|_k \leq \|x\|_e$ for all $x \in E$, $k < e$. Indeed, if otherwise, we may define

$$\|x\|'_k := \sup\{\|x\|_0, \cdots, \|x\|_k\}$$

which is a norm inducing the same topology. Hence, for this reason, we will from this point forward, assume that the norms of a countably normed space are monotone increasing.

Proposition 4.16. A linear functional f on a countably normed space E is continuous if and only if there exists some k such that f is continuous with respect to $\|\cdot\|_k$.

Proof. Let f be a continuous linear functional on a countably normed space E (so it is bounded on some neighbourhood U of 0). By the definition of the topology on E, there is some ϵ, k such that

$$B_{k,\epsilon} = \{x \mid \|x\|_0 < \epsilon, \cdots, \|x\|_k < \epsilon\} = \{x \mid \|x\|_k < \epsilon\} \subseteq U.$$

Then, f is bounded on $B_{k,\epsilon}$ and therefore, continuous with respect of $\|\cdot\|_k$.

The converse is clear. □

Corollary 15.1. If E is a countably normed space,

$$E^* = \bigcup_{n=0}^{\infty} E_n^*$$

where E_n^* is the dual of $(E, \|\cdot\|_n)$. Moreover, the monotonicity of the norms implies $E_n^* \subseteq E_m^*$ for all $n \leq m$. With this in mind, if $f \in E^*$, there exists a smallest n such that $f \in E_n^*$. We call this n the order of f.

5 Distributions

Finally, with the set up of the last section we can now discuss the theory of distributions. However, before delving in, let us first provide some motivation.

- Physics: often in physics, we need to describe a notion of infinitely dense at a certain point. This leads to the definition of the δ-function. This was used rigorously in physics as they simply considered it as a function with some special property however, this is to be made rigorous with the theory of distributions.
- Differential Equations: we would like to generalize the notion of derivative which applies to a larger class of functions (in particular, any integrable functions). This is achieved with linear functionals.

If $f : \mathbb{R} \to \mathbb{R}$ is locally integrable (i.e. integrable on any compact set), we define the linear functional

$$T_f : C_c(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \langle f, \phi \rangle = \int_{\mathbb{R}} f\phi \mathrm{d}\lambda,$$

which is well-defined as ϕ has compact support. It is clear that $f \mapsto T_f$ is essentially injective (as $T_f - T_g = 0$ implies $f - g = 0$ almost everywhere), and thus, locally integrable functions are essentially embedded into the space of linear functionals.

However, we note that this map is not surjective. Indeed, the δ-functional, $\delta(\phi) = \phi(0)$ is not in the image of $f \mapsto T_f$ and so, the space of linear functionals is an effective extension of functions. This idea can be generalised resulting in the theory of distributions.

5.1 Definitions

We introduce the notation $C_c^\infty(A)$ as the space of infinitely differentiable functions (i.e. smooth) on $A \subseteq \mathbb{R}$ with compact support.

An important class of functions in $C_c^\infty(\mathbb{R})$ are the bump functions.

Definition 5.1 (Bump Functions)**.** For all $a < b$ we define the bump function

$$\phi_{a,b}(x) := \begin{cases} e^{-\frac{1}{(b-x)(x-a)}}, & x \in (a,b), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\phi_{a,b} \in C_c^\infty(\mathbb{R})$ with support (a, b).

Definition 5.2 (Test Functions)**.** The space of test functions on the set $A \subseteq \mathbb{R}^n$ is simply $\mathcal{D}(A) = C_c^\infty(A)$ and we denote $\mathcal{D} = \mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$.

We will later generalize this definition to a larger class of domains.

We equip the space of test functions with a topology. Define $\mathcal{D}_m \subseteq \mathcal{D}$ the space of functions with support in $[-m, m]$. Then, $\mathcal{D}_m$ is a increasing sequence of subspaces of $\mathcal{D}$ with $\mathcal{D} = \bigcup_{n=0}^\infty \mathcal{D}_n$. Now, noting that $\mathcal{D}_m$ is countably normed with the norms

$$\|\phi\|_n^{(m)} := \sup_{\substack{0 \le k \le n \\ |t| \le m}} |\phi^{(k)}(t)|.$$

Then, the topology of $\mathcal{D}$ is induced by the system of neighbourhoods of 0, where U is a neighbourhood of 0 if and only if $U \cap \mathcal{D}_m$ is a neighbourhood of 0 in $\mathcal{D}_m$ (recall $\mathcal{D}_m$ is equipped with the topology induced by the norms).

One can show $\mathcal{D}$ equipped with this topology forms a topological vector space (the proof is rather involved and so, is omitted).

Lemma 5.1. A sequence $(\phi_n)_{n=1}^\infty \subseteq \mathcal{D}$ converges to $\phi \in \mathcal{D}$ if and only if

- there is some $[a, b]$ such that for all n, ϕ_n is 0 outside of $[a, b]$ and
- for all $k = 0, 1, \cdots$, the sequence $(\phi_n^{(k)})$ converges to $\phi^{(k)}$ uniformly.

Proof. Exercise. □

Definition 5.3 (Distribution)**.** A distribution (also known as generalized function) on $\mathbb{R}$ is a continuous linear functional on the space of test functions $\mathcal{D}$, i.e. the space of distributions is $\mathcal{D}^*$ and is denoted by $\mathcal{D}'$.

Lemma 5.2. A linear functional f on $\mathcal{D}$ is continuous (i.e. $f \in \mathcal{D}'$) if and only if $f(\phi_n) \to f(\phi)$ for all $\phi_n \to \phi$ in $\mathcal{D}$.

Proof. This is simply sequential continuity. □

Let us consider some examples of distributions.

- Any locally integrable function $f : \mathbb{R} \to \mathbb{R}$ corresponds to a distribution by mapping f to
$$T_f : \mathcal{D} \to \mathbb{R} : \phi \mapsto \int_{\mathbb{R}} f\phi \mathrm{d}\lambda.$$
This is clearly linear, and is continuous since for all $\phi_n \to 0$ in $\mathcal{D}$, by the above lemma, there exists some $[a, b]$ such that $\phi_n = 0$ outside of $[a, b]$ and $\phi_n \to 0$ uniformly. Thus, as f is locally integrable, $I := \int_{[a,b]} |f| \mathrm{d}\lambda < \infty$ and for all $\epsilon > 0$, there exists some N such that $|\phi_n| < \epsilon/2I$ for all $n \geq N$. Hence, for all $n \geq N$,
$$|T_f(\phi_n)| = \left| \int f\phi_n \mathrm{d}\lambda \right| \leq \frac{\epsilon}{2I} \int_{[a,b]} |f| \mathrm{d}\lambda < \epsilon$$
implying $T_f(\phi_n) \to 0$ and so, T_f is continuous.

  With the above in mind, $T_f \in \mathcal{D}'$ and we call these distributions regular (otherwise we call them singular).
- The δ-distribution: $\delta(\phi) = \phi(0)$.
- Shifted δ defined as $\delta(x - a)(\phi) = \phi(a)$.
- The function $1/x$ is not integrable at 0. However, $\int \frac{1}{x} \phi(x) \lambda(\mathrm{d}x)$ exists as a principal value integral. Indeed, suppose $\phi \in \mathcal{D}$ is supported on $[-R, R]$, we have
$$\int \frac{1}{x} \phi(x) \lambda(\mathrm{d}x) = \int_{[-R,R]} \frac{1}{x} \phi(x) \lambda(\mathrm{d}x) = \int_{[-R,R]} \frac{\phi(x) - \phi(0)}{x} \lambda(\mathrm{d}x) + \phi(0) \int_{[-R,R]} \frac{\lambda(\mathrm{d}x)}{x}$$

which exists by integration by parts and noting that $\int_{[-R,R]} \frac{\lambda(\mathrm{d}x)}{x}$ has principal value 0, i.e.

$$\left|\int \frac{1}{x}\phi(x)\lambda(\mathrm{d}x)\right| = \left|\int_{[-R,R]} \frac{\phi(x)-\phi(0)}{x}\lambda(\mathrm{d}x)\right| = \left|\int_{[-R,R]} \phi'(x)\log|x|\lambda(\mathrm{d}x)\right|$$
$$\leq 2R|\log(R)-1| \sup_{|x|\leq R} |\phi'(x)| \to 0$$

as $\phi \to 0$. Hence, $T_{1/x}$ defines a distribution.

Lemma 5.3. A linear functional f on $\mathcal{D}$ is continuous if and only if f is continuous as a linear functional on $\mathcal{D}_m$ for all m, i.e. for all m, there exists some C, n such that

$$|f(\phi)| \leq C\|\phi\|_n^{(m)} = C \sup_{0\leq k\leq n, |x|\leq m} |\phi^{(k)}(x)|$$

for all $\phi \in \mathcal{D}_m$.

Proof. If $f \in \mathcal{D}'$ and there exists some m such that for all $n.c$, there exists some $\phi_{n,c} \in \mathcal{D}_m$ such that

$$|f(\phi_{n,c})| > c\|\phi_{n,c}\|_n^{(m)}.$$

Defining

$$\psi_{n,c} := \frac{\phi_{n,c}}{|f(\phi_{n,c})|},$$

so $1 > c\|\psi_{n,c}\|_n^{(m)}$ we have $\|\psi_{n,n}\|_n^{(m)} < 1/n$ implying $\psi_{n,n} \to 0$ in $\mathcal{D}$ as $n \to \infty$. But, as $f(\psi_{n,n}) = 1$ we have a contradiction.

The converse is clear. □

Proposition 5.1. Strong and weak-* convergence of sequences in $\mathcal{D}'$ coincide.

Definition 5.4. A sequence of distributions f_n is said to converge to a distribution f if for all $\phi \in \mathcal{D}$, $f_n(\phi) \to f(\phi)$.

5.2 Derivative of Distributions

Suppose that $T_f \in \mathcal{D}'$ is a regular distribution corresponding to the continuously differentiable function f. Then, we may define the derivative of T_f to be $T_{f'}$ where f' is the derivative of f. Namely,

$$T_f'(\phi) = T_{f'}(\phi) = \int f'\phi \mathrm{d}\lambda = -\int f\phi' \mathrm{d}\lambda = -T_f(\phi')$$

where the second to last equality is due to integration by parts (as ϕ is differentiable with compact support). This, suggests the following definition.

Definition 5.5 (Derivative)**.** A derivative of a distribution $f \in \mathcal{D}'$ is the distribution f' defined by $f'(\phi) := -f(\phi')$. Similarly, we define $f^{(k)}(\phi) := (-1)^k f(\phi^{(k)})$.

Proposition 5.2. It is clear that if $f_n \to f$ in $\mathcal{D}'$, then so do $f_n' \to f'$.

Consider the case where we take the distributional derivative of the regular distribution of a non-differentiable function. Consider the Heaviside function

$$H(x) := \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then, $T_H(\phi) = \int_{\mathbb{R}_+} \phi \mathrm{d}\lambda$ and $T_H' = -\int_{\mathbb{R}_+} \phi' \mathrm{d}\lambda = \phi(0)$ by the fundamental theorem of calculus. Hence, $T_H' = \delta$. This observation can be generalized easily.

Proposition 5.3. If $f : \mathbb{R} \to \mathbb{R}$ has jumps at $x_1, x_2, \cdots$ and is continuously differentiable elsewhere, its distributional derivative is

$$\mathbf{1}_{\mathbb{R}\setminus\{x_n\}_n} f'(x) + \sum_{i=1}^{\infty} (f(x_{i_+}) - f(x_{i_-}))\delta(x - x_i).$$

Consider now the function f with 2π-period on $\mathbb{R}$ defined by

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} \frac{\pi - x}{2}, & 0 < x \leq \pi, \\ -\frac{\pi + x}{2}, & -\pi \leq x < 0, \\ 0, & x = 0, \end{cases}$$

for $x \in [-\pi, \pi]$. Taking the distributional derivative oh the right hand side (where a jump occurs at $2\pi k$ for all k), we have

$$f' = -\frac{1}{2} + \pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k)$$

(note that only finitely many terms are non-vanishing when applied to $\phi \in \mathcal{D}$.) On the other hand, the distributional derivative of the left hand side is

$$f' = \sum_{n=1}^{\infty} \cos nx,$$

since the partial sums converges as distributions. Thus, comparing the two, we obtain

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k).$$

5.3 Application to Differential Equations

We will in this section discuss the application of distributions to differential equations.

Let us first consider the case $y' = f$.

Lemma 5.4. Let $\mathcal{D}^{(1)}$ be the set of $\phi \in \mathcal{D}$ such that ϕ is a derivative of some $\psi \in \mathcal{D}$. Then, $\phi \in \mathcal{D}^{(1)}$ if and only if $\int \phi \mathrm{d}\lambda = 0$.

Proof. If $\phi = \psi'$, then $\int \phi \mathrm{d}\lambda = \psi \mid_{-\infty}^{\infty} = 0$ by compact support. On the other hand, if $\int \phi \mathrm{d}\lambda = 0$, then we may define

$$\psi(x) := \int_{(-\infty, x]} \phi \mathrm{d}\lambda.$$

It is clear that ψ is infinitely differentiable, and sine $\int \phi \mathrm{d}\lambda = 0$, ψ has compact support, and hence, $\psi \in \mathcal{D}$. Furthermore, by the fundamental theorem of calculus, $\psi' = \phi$ proving the lemma. □

Proposition 5.4. Let y a distribution such that $y' = 0$. Then, y is a constant.

Proof. By definition, $y'(\phi) = -y(\phi') = 0$ for all $\phi \in \mathcal{D}$. As $\mathcal{D}^{(1)}$ is a linear subspace of $\mathcal{D}$, by the above lemma, $\ker y = \mathcal{D}^{(1)}$. Since the cokernel of a linear functional has dimension 1, every element $\phi \in \mathcal{D}$ can be written as $c\phi_0 + \phi_1$ where $\phi_0 \in \mathcal{D} \setminus \mathcal{D}^{(1)}$ is fixed satisfying $\int \phi_0 \mathrm{d}\lambda = 1$ and $\phi_0 \in \mathcal{D}^{(1)}$.

Integrating both sides of $\phi = c\phi_0 + \phi_1$, we obtain $c = \int \phi \mathrm{d}\lambda$ and thus, $y(\phi_0)$ uniquely defines y as

$$y(\phi) = cy(\phi_0) = y(\phi_0) \int \phi \mathrm{d}\lambda = \int y(\phi_0)\phi \mathrm{d}\lambda = T_c(\phi)$$

where T_c is the distribution corresponding to the constant function c. Thus, $y = T_c$ which is the constant distribution. □

Corollary 15.2. If $f' = g'$, then $f - g$ is a constant.

Theorem 16. Let f be a distribution, there exists some distribution $y \in \mathcal{D}'$ such that $y' = f$.

Proof. By definition, $y'(\phi) = -y(\phi') = f(\phi)$ for all $\phi \in \mathcal{D}$. Then, again by recalling that for all $\phi \in \mathcal{D}$, we may write $\phi = c\phi_0 + \phi_1$, where $\phi_1 \in \mathcal{D}^{(1)}$, we define

$$y(\phi) = y(c\phi_0 + \phi_1) := -f\left(x \mapsto \int_{(-\infty, x]} \phi_1 \mathrm{d}\lambda\right),$$

which satisfy $y' = f$ as required since for all $\phi \in \mathcal{D}$,

$$y'(\phi) = -y(\phi') = f\left(x \mapsto \int_{(-\infty, x]} \phi' \mathrm{d}\lambda\right) = f(\phi).$$

□

Any solution added by a constant is another solution and so, the equation $y' = f$ has only solutions unique up to a constant.

Theorem 17. Consider the system of homogeneous differential equations

$$y_j' = \sum_{k=1}^{n} a_{jk}(x) y_k, j = 1, \cdots, n$$

where a_{jk} are infinitely differentiable functions. Then, all solutions to such a system (in distribution) are regular.

On the other hand, if the system is not homogeneous,

$$y_j' = \sum_{k=1}^{n} a_{jk}(x) y_k + f_j, j = 1, \cdots, n$$

where a_{jk} are infinitely differentiable functions and f_j are distributions. Then, solutions of the system $y_j \in \mathcal{D}'$ exists and is determined uniquely up to an arbitrarily solution of the homogeneous system. In the case that f_j are regular, then the solution of the system is also regular.

Proof. Omitted. $\square$

5.4 Functions of Several Variables and On the Unit Circle

Consider the set of functions ϕ on $\mathbb{R}^n$ such that ϕ has partial derivatives of all order with respect to all n of its variables. Furthermore, suppose ϕ has bounded support such that ϕ vanishes outside some ball of finite radius.

One may introduce a topology on this linear space such that the convergence satisfies

$$\phi_n \to \phi \iff \exists B \text{ bounded}, \phi_k(B) = \{0\} \wedge \forall x \in B, \frac{\partial^r \phi_k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \to \frac{\partial^r \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \text{ uniformly}$$

where $r = \sum_{i=1}^n \alpha_i$ for any $\alpha_1, \cdots, \alpha_n \in 0, 1, \cdots$. We denote this space $\mathcal{D}(\mathbb{R}^n)$ and we by generalize our definition of distributions to this space.

Definition 5.6. A continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$ is called a distribution of n-variables. Similarly, we denote the space of such distributions by $\mathcal{D}'(\mathbb{R}^n)$.

As before, locally integrable functions f on $\mathbb{R}^n$ induces distributions and almost all results for single variables remains to hold for multivariate functions, e.g. the derivative of a multivariate distribution is defined as

$$\frac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = (-1)^r \left\langle f, \frac{\partial^r \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right\rangle.$$

Another class of important functions is the set of infinitely differentiable functions on $\mathbb{T}$ where

$$\mathbb{T} := \{e^{i\phi} \mid 0 \le \phi < 2\pi\}.$$

One can introduce a topology on the linear space of these functions such that $\phi_n \to \phi$ if and only if $\phi_n^{(k)} \to \phi^{(k)}$ uniformly on $\mathbb{T}$ for all $k = 0, 1, \cdots$. We note that the compact support requirement is redundant as $\mathbb{T}$ is compact. We denote this space by $D(\mathbb{T})$.

Definition 5.7. A continuous linear functional on $\mathcal{D}(\mathbb{T})$ is a distribution on the unit circle.

Functions and distributions on $\mathbb{T}$ are related to periodic functions and distribution (see definition below) on $\mathbb{R}$.

Definition 5.8. Let $f \in \mathcal{D}'$ is a periodic distribution with period a if $f(\phi(x-a)) = f(\phi)$ for all $\phi \in \mathcal{D}$.

5.5 Tempered Distribution

Recall the space of Schwarz functions S_∞. Namely, infinitely differentiable functions ϕ such that for all $p, q = 0, 1, \cdots$, there exists some $C_{p,q} > 0$, such that

$$|x^p \phi^{(q)}(x)| \le C_{p,q}$$

for all x. Note that $\mathcal{D} \subseteq S_\infty$.

S_∞ is a countably normed space with the following norms (exercise) such that for all $\phi \in S_\infty$,

$$\|\phi\|_n := \sum_{p+q=n} \sup_{\substack{x, 0\le j\le p \\ 0\le k\le q}} |(1+|x|^j)\phi^{(k)}(x)|.$$

Lemma 5.5. A sequence ϕ_n of Schwarz functions converges to $\phi \in S_\infty$ if and only if for all $q = 0, 1, \cdots$, the sequence $\phi_n^{(q)}$ converges uniformly on any bounded interval and for all $p, q = 0, 1, \cdots$, there exists a uniform bound $C_{p,q}$ such that

$$|x^p \phi_n^{(q)}(x)| < C_{p,q}$$

for all n, x.

Proof. Exercise. □

Definition 5.9. A linear continuous functional on S_∞ is a tempered distribution. We denote the space of tempered distributions as S'.

The regular functionals on S_∞ are defined similarly to that of $\mathcal{D}$. Namely, for all functions f, we define the regular tempered distribution of f (if it exists) as

$$T_f : \phi \in S_\infty \mapsto \int f\phi \mathrm{d}\lambda.$$

We note that in this case $S' \subsetneq \mathcal{D}'$ as e^{x^2} is regular in $\mathcal{D}'$ while not regular in S_∞ (as $e^{-x} \in S_\infty$ while $\int e^{x^2} e^{-x} \lambda(\mathrm{d}x) = \infty$).

Similar to $\mathcal{D}$,

- a linear functional f on S_∞ is continuous if and only if $f(\phi_n) \to f(\phi)$ for all $\phi_n \to \phi$ in S_∞.
- a sequence $(f_n) \subseteq S'$ is said to converge to $f \in S'$ if $f_n(\phi) \to f(\phi)$ for all $\phi \in S_\infty$.

5.6 Fourier Transform of Distributions

Recall that Fourier transform, as a linear operator, forms an automorphism on S_∞. This allows us to define the Fourier transform of distributions on S_∞.

Definition 5.10 (Fourier Transform of a Tempered Distribution)**.** The Fourier transform of a tempered distribution $f \in S'$ is the distribution $\mathcal{F}[f] = g \in S'$ given by

$$g(\phi) := f(\mathcal{F}[\phi])$$

for all $\phi \in S_\infty$. Namely, $\mathcal{F}[f] = f \circ \mathcal{F}$ where $\mathcal{F} : S_\infty \to S_\infty$ is considered as a linear map on the space of Schwarz functions.

We remark that $L_1(\mathbb{R}) \subseteq S'$ where we consider L_1 functions in S' as their regular distribution. For these functions, the regular distribution of their Fourier transform equals the Fourier transform of their regular distribution.

Proposition 5.5. Let $f \in L_1(\mathbb{R})$. Then, $T_f \in S'$ and

$$\mathcal{F}[T_f] = T_{\mathcal{F}[f]}.$$

Proof. First consider when $f \in S_\infty$ which result follows by Fubini. Then, by the density of S_∞ in L_1, we obtain the general result. $\square$

We leave the following computations as easy exercises.

Proposition 5.6. Let $f_1(x) = c$ be a constant function, $f_2(x) = e^{iax}$. Then, $\mathcal{F}[f_1] = 2\pi c\delta(x)$ and $\mathcal{F}[f_2] = 2\pi\delta(x-a)$.

Proposition 5.7. $\mathcal{F}[\delta(x-a)] = e^{-iax}$ for all $a \in \mathbb{R}$.

Proposition 5.8. Let $f \in S'$ such that $f(\phi) = \int \frac{\phi(x)}{x}\lambda(\mathrm{d}x)$ where the integral is taken as the principle value. Then, $\mathcal{F}[f] = -\pi i\mathrm{sign}$.

Proof. Denote $g := \mathcal{F}[f]$ and we consider g'. Indeed, for all $\phi \in S_\infty$,

$$g'(\phi) = g(-\phi') = f(-\mathcal{F}[\phi']) = f(-iy\mathcal{F}[\phi]) = -\int \mathcal{F}[\phi]\mathrm{d}\lambda = -2\pi i\phi(0).$$

Thus, $g' = -2\pi i\delta$. As we have seen from the exercises, $\mathrm{sign}' = 2\delta$, and so, as the solutions to differential equations are unique up to a constant, $g = -\pi i\mathrm{sign} + C$ for some constant C. Finally, by considering even test functions, $C = 0$, and hence $\mathcal{F}[f] = -\pi i\mathrm{sign}$ as required. $\square$

We would now like to generalise the Fourier transform to other test spaces, namely $\mathcal{D}$. Unlike S_∞, the Fourier transform is not invariant under $\mathcal{D}$, and so, let us first describe the image of the Fourier transform in more generalised settings.

Straight away we note that $\mathcal{F}[\mathcal{D}] \subseteq \mathcal{F}[S_\infty] = S_\infty$. Let $\phi \in \mathcal{D}$, with support $[-a,a]$. Then,

$$\psi(y) := \mathcal{F}[\phi](y) = \int_{[-a,a]} e^{-iyx}\phi(x)\mathrm{d}\lambda(x).$$

Since the integral is over a finite interval and the integrand $e^{-iyx}\phi(x)$ is analytic in y and continuous in x, ψ extends to an entire function on $\mathbb{C}$. Moreover, integrating by parts, we obtain

$$|y|^q|\psi(y)| = \left|\int_{[-a,a]} e^{-iyx}\phi^{(q)}(x)\mathrm{d}\lambda(x)\right| \le C_q e^{a|\mathrm{im}(y)|}.$$

Thus, the Fourier transform maps $\mathcal{D}$ into the linear space $\mathcal{Z}$ consisting of entire functions ψ which satisfy

$$|y|^q|\psi(y)| \le C_q(\psi)e^{a(\psi)|\mathrm{im}(y)|},$$

for all q and some $C_q(\psi)$ and $a(\psi)$.

Conversely, any function $\psi \in \mathcal{Z}$, is a Fourier transform of some $\phi \in \mathcal{D}$. Indeed, for all $\psi \in \mathcal{Z}$, we define ϕ to be its inverse Fourier transform, namely,

$$\phi(x) := \frac{1}{2\pi}\int \psi(y)e^{iyx}\mathrm{d}\lambda(y),$$

which exists by the condition of $\mathcal{Z}$. I claim that $\phi \in \mathcal{D}$. Indeed, formal differentiable under the sign of the integral results in absolutely and uniformly convergent integral. Therefore, ϕ is differentiable, and

$$\phi^{(q)}(x) = \frac{1}{2\pi} \int (iy)^q \psi(y) e^{iyx} \mathrm{d}\lambda(x).$$

Furthermore, ϕ is compactly supported via a construction of a contour integral (see official notes for proof).

Corollary 17.1. The Fourier transform forms a linear bijection between $\mathcal{D}$ and $\mathcal{Z}$.

With the above considerations, we can define the Fourier transform on general distributions.

Definition 5.11 (Fourier Transform of a Distribution)**.** The Fourier transform of a distribution $f \in \mathcal{D}'$ is the distribution $\mathcal{F}[f] \in \mathcal{Z}^* =: \mathcal{Z}'$ given by

$$\mathcal{F}[f](\phi) = f(\mathcal{F}[\phi])$$

for all $\phi \in \mathcal{Z}$.