Markov Processes Revision Notes

Kexing Ying

April 20, 2022

Markov Chains

- The strong Markov property implies that for a finite stopping time T,
$$\mathbb{P}(x_{n+T} \in A \mid \mathcal{F}_T) = P^n(x_T, A).$$
- Given a chain $\{x_n\}$ with initial distribution μ and recurrent state $j \in \mathcal{X}$ where $\mathbb{P}_\mu(T_j < \infty) = 1$, the intervals
$$\{T_j^n - T_j^{n-1}\}_{n \geq 1}$$
are independent and $\mathbb{P}(T_j^{k+1} - T_j^k = m) = \mathbb{P}_j(T_j = m)$.
- A state $j \in \mathcal{X}$ is transient if and only if $\sum_{n=1}^\infty P_{jj}^n < \infty$.
- For THMCLLN, we proved the lemma
$$\frac{1}{n} \sum_{l=0}^{T_i^n} f(x_l) \to \mathbb{E}_i T_i \int f \mathrm{d}\pi.$$

Feller and Strong Feller

Proposition. If $g : \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and in L^1 with $\int g \mathrm{d}\lambda = 1$, then the transition operator
$$Tf(x) := \int f(y) g(x - y) \lambda(\mathrm{d}y) = (f * g)(x)$$
is strong Feller.

Proposition. If $x \mapsto P(x, \cdot) : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ is continuous with respect to the total variation norm, then P is strong Feller.

Proposition. Given measures $\mu, \nu \in \mathcal{P}(\mathcal{X})$, $\mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) = \varnothing$ implies $\mu \perp \nu$.

Furthermore, if μ, ν are mutually singular, invariant probability measures of a strong Feller process, then $\mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) = \varnothing$.

Proposition. A strong Feller process admitting a common point of support in every invariant probability measure has at most one invariant probability measure (Caution: result requires mastery material).

Invariant Measures in the General Case

Theorem (Krylov-Bogoliubov)**.** Let (x_n) be a Feller process with transition probability P on a compact separable metric space $\mathcal{X}$, then, if there exists some $x_0 \in \mathcal{X}$ such that $\{P^n(x_0, \cdot) \mid n \in \mathbb{N}\}$ is tight, P admits an invariant probability measure.

Corollary. If $\mathcal{X}$ is compact, every Feller process admits an invariant probability measure as every family of probability measures on a compact set is tight.

Corollary. Any a.e. bounded Feller Markov process (x_n) admits an invariant probability measure.

Proof. Suppose $|x_n| \leq M$ almost everywhere. Then, for all $\epsilon > 0$, defining $K = [-M.M]$ (which is compact), we have

$$P^n(x_0, K) = \mathbb{P}(x_n \in K \mid x_0) = \mathbb{E}(\mathbf{1}_K(x_n) \mid x_0).$$

Thus, as $\mathbf{1}_K(x_n) = \mathbf{1}$ a.e. $\mathbb{E}(\mathbf{1}_K(x_n) \mid x_0)$ is a $\sigma(x_0)$-measurable function which is $\mathbf{1}$ a.e. So, taking any $\omega \in \mathbb{E}(\mathbf{1}_K(x_n) \mid x_0)^{-1}\{1\}$,

$$P^n(x_0(\omega), K) = \mathbb{E}(\mathbf{1}_K(x_n) \mid x_0)(\omega) = 1 \geq 1 - \epsilon$$

implying tightness of $\{P^n(x_0, \cdot)\}$. Hence, we may conclude by Krylov-Bogoliubov. $\square$

The Lyapunov function test builds upon the Krylov-Bogoliubov theorem. In particular, the existence of a Lyapunov function V implies $\mathcal{X}_0 := \{V < \infty\}$ has measure 1, and furthermore, as $TV(x) \leq \gamma V(x) + C$, for every x satisfying $V(x) < \infty$, $P^n(x, \mathcal{X}_0) = 1$ implying tightness as required.

Definition (Lyapunov Function)**.** A function $V : \mathcal{X} \to \overline{\mathbb{R}_+}$ is a Lyapunov function if

- $V^{-1}(\mathbb{R}_+) \neq \varnothing$ (i.e. V is not always ∞);
- for all $a \in \mathbb{R}_+$, $\{V \leq a\}$ is compact;
- there exists $0 < \gamma < 1$ and some C such that

  $$TV(x) = \int_{\mathcal{X}} V(y) P(x, \mathrm{d}y) \leq \gamma V(x) + C$$

  for all x such that $V(x) < \infty$.

Note that V continuous and $\lim_{x \to \pm\infty} V(x) = +\infty$ implies point 2.

Common Lyapunov functions include $V(x) = |x|^p$ and $V(x) = \log |x|$.

Theorem (Lyapunov Function Test)**.** A Feller process which admits a Lyapunov function admits an invariant probability measure.

Useful inequalities:

- $(x + y)^2 \leq (1 + \epsilon^{-2})x^2 + (1 + \epsilon^2)y^2$.
- for any $p \geq 1, \delta > 0$, there exists some $K > 1$,

  $$|1 + x|^p \leq K|x|^p + 1 + \delta.$$

  Note that for $x \leq 0$, $|1 + x|^p \leq 1 + |x|^p$;

- (Young's inequality) for $a, b \geq 0, p, q > 1$ such that $1/p + 1/q = 1$ (i.e. p, q are Hoelder conjugates),

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Uniqueness of the Invariant Measure

Definition (Coupling)**.** Given probability measures π_1, π_2 on $\mathcal{X}$, a measure μ on $\mathcal{X}^2$ is a coupling of π_1 and π_2 if $(p_i)_*\mu = \pi_i$ for $i = 1, 2$ where $p_i : \mathcal{X}^2 \to \mathcal{X}$ is the projection map onto the i-th component.

Proposition. If π_1, π_2 are probability measures on $\mathcal{X}$ and $\Delta := \{(x, x) \mid x \in \mathcal{X}\} \subseteq \mathcal{X}^2$. Then $\pi_1 = \pi_2$ if there exists a coupling μ of π_1, π_2 such that $\mu(\Delta) = 1$. Equivalently,

$$\int_{\mathcal{X}^2} 1 \wedge d(x, y)\mu(\mathrm{d}x, \mathrm{d}y) = 0,$$

as $\Delta = \{(x, y) \in \mathcal{X}^2 \mid 1 \wedge d(x, y) = 0\}$.

Theorem (Deterministic Contraction)**.** If there exists a constant $\gamma \in (0, 1)$ such that for all $x, y \in \mathcal{X}$,

$$\mathbb{E}d(F(x, \xi_1), F(y, \xi_1)) \leq \gamma d(x, y),$$

then the random dynamical system defined by $x_{n+1} = F(x_n, \xi_{n+1})$ where $\{\xi_n\}$ are i.i.d. random variables has at most one invariant probability measure.

The proof of this follows by constructing two synchronised coupling, for which their joint laws are couplings (and hence, are tight). Taking a convergent subsequence, we know its limit is also a coupling. Thus, it suffices to show the condition in the above proposition applies.

Minorisation also provides a method for showing uniqueness.

Definition (Minorised)**.** The transitional probabilities $P = \{P(x, \cdot)\}_{x \in \mathcal{X}}$ is said to be minorised by $\eta \in \mathcal{P}(\mathcal{X})$ if there exists some $a > 0$ such that for all $x \in \mathcal{X}$,

$$P(x, \cdot) \geq a\eta.$$

Also known as Doeblin's condition.

Theorem. Let P be a transition probability on a space $\mathcal{X}$ which is minorised by $\eta \in \mathcal{P}(\mathcal{X})$ (by some $a > 0$). Then,

- P has a unique invariant probability measure π;
- for all $\mu, \nu \in \mathcal{P}(\mathcal{X})$,

$$\|T^{n+1}\mu - T^{n+1}\nu\|_{TV} \leq (1 - a)^n \|\mu - \nu\|_{TV}.$$

Clearly, the second point implies the first by Banach fixed point theorem.

As with Banach fixed point theorem, the above result remains to hold if there exists some n_0 such that $P^{n_0}(x, \cdot) \geq a\eta$ for all x.

Invariant Sets

Definition (P-Invariant Set)**.** Given a transition probability P, a set A is P-invariant if $P(x, A) = 1$ for all $x \in A$.

One may restrict a transition probability P on a P-invariant set A by taking $P|_A := \{P(x, \cdot) \mid x \in A\}$. Then,

$$P(x, B \cap A) = P|_A(x, B)$$

for all $x \in A$, $B \in \mathcal{B}(\mathcal{X})$.

In general, we would like to find P-invariant sets which are closed as we would like the state space A of $P|_A$ to be *complete* separable. Hence, if $A \subseteq \mathcal{X}$ is compact, we may apply Krylov-Bogoliubov should $P|_A$ be Feller.

Proposition. Let $\pi^0 \in \mathcal{P}(\mathcal{X})$, $\pi := \pi^0 \cap A \in \mathcal{P}(\mathcal{X})$ where A is a P-invariant set. Then, π^0 restricted on A is invariant for $P|_A$ if and only if π is invariant on P.

Theorem. Let A be P-invariant where P is Feller. Then $P|_A$ is also Feller, and if A is compact, there exists an invariant probability measure for P by Krylov-Bogoliubov and the above proposition.

Definition. Given a P-invariant set A, we define inductively $A_0 := A$ and

$$A_{n+1} := \{x \in \mathcal{X} \mid P(x, A_n) > 0\}.$$

This sequence is monotonically increasing.

Proposition. Let A be P-invariant such that $\bigcup_{n \geq 0} A_n = \mathcal{X}$. Then, every invariant probability measure π is concentrated on A, i.e. $\pi(A) = 1$.

With this proposition, one may use the above criterions to show uniqueness of invariant probability measure on A which implies unique invariant probability measure on $\mathcal{X}$.

To show a set A has measure 1 under all invariant measures (useful to find support), it suffices to show it is P-invariant and $\bigcup A_n = \mathcal{X}$.