Manifolds

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Contents

1 Introduction

This module introduces the notion of manifolds and provides the infrastructure for generalizing theorems from calculus to manifolds. In particular, we will talk about

- Smooth manifolds and smooth functions;
- Tangent spaces and vector fields;
- Differential forms, integrations and Stoke's theorem.

In contrast to the curves and spaces module, instead of working on Euclidean spaces, we will define these notions for general manifolds. Thus, many definitions such as the tangent space will be defined in a more intrinsic point of view, without requiring our manifold to be within a Euclidean space.

Furthermore, a goal of this module is to differentiate between different manifolds, that is determine whether or not two manifolds are diffeomorphic with one another. This is achieved through introducing invariants such as the notion of differential forms and these notions will appear in many other places especially in geometry.

Manifolds is the subject of studying geometric shapes, and in mathematics, there are in general two ways of doing this. The first of which is by embedding the object into an ambient space such as $\mathbb{R}^2$ or $\mathbb{R}^3$. An example of this is studying the unit circle through the parametrisation

$$\{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

and is the more common method of what we have done thus far. On the other hand, one may study the object independently of the ambient space. This is the approach we shall take throughout this course. In particular, we will study spaces which at a local level "looks like" a Euclidean space directly without embedding the structure into $\mathbb{R}^n$.

2 Topological and Smooth Manifolds

Let us first recall some notions from topology.

Definition 2.1. Let X, Y be topological spaces and let $f : X \to Y$ be a function, then

- f is continuous if $f^{-1}(U)$ is open in X for all U open in Y.
- f is a homeomorphism if it is continuous and has a continuous inverse.

Definition 2.2. A topological space X is

- Hausdorff if for all $x, y \in X$, $x \neq y$, there exists open sets U, V in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.
- second-countable if there exists countable $\mathcal{F} \subseteq \mathcal{T}_X$ such that any open set in X can be written as a union of elements of $\mathcal{F}$, i.e. $\mathcal{F}$ is a countable basis of X.

In general, in this module, we will assume our topology is Hausdorff and second-countable in order to avoid pathological examples in smooth and topological manifolds.

Definition 2.3 (Co-ordinate Chart)**.** Let X be a topological space. A co-ordinate chart on X is the collection of

- an open set $U \subseteq X$,
- an open set $\tilde{U} \subseteq \mathbb{R}^n$ for some $n \geq 0$,
- a homeomorphism $f : U \to \tilde{U}$.

We denote a co-ordinate chart by (U, f).

Definition 2.4. Let X be a (Hausdorff and second-countable) topological space. We say that X is a topological manifold of dimension n if for all $x \in X$, there exists a co-ordinate chart (U, f) with $\tilde{U} \subseteq \mathbb{R}^n$ such that $x \in U$.

The classical example of a topological manifold is the circle, in particular

$$S^1 := \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

is a 1 dimensional topological manifold. Consider $U_1 = S^1 \;\; \{(0, -1)\}$, and we define the stereographic projection $f_1 : U_1 \to \mathbb{R}$,

$$f : (x, y) \mapsto \frac{x}{y + 1} := \tilde{x}.$$

It is not difficult to see that f_1 is invertible with the inverse

$$f_1^{-1} : \tilde{x} \mapsto \left(\frac{2\tilde{x}}{1 + \tilde{x}^2}, \frac{1 - \tilde{x}^2}{1 + \tilde{x}^2} \right).$$

Furthermore, as f_1 and f^{-1} are continuous, we have (U_1, f_1) is a co-ordinate chart. Similarly, we define $U_2 = S^1 \;\; \{(0, 1)\}$, and we my show the existence of a homeomorphism $f_2 : U_2 \to \mathbb{R}$, providing the second co-ordinate chart (U_2, f_2). Thus, as $S^1 = U_1 \cup U_2$, we have S^1 is a 1 dimensional topological manifold.

The above example can be expanded to n-dimensional sphere

$$S^n := \{(x_0, \cdots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n.$$

Similarly as before, we can construct two co-ordinate charts covering all points on the sphere except for the poles allowing us to conclude S^n is a n-dimensional topological manifold.

Definition 2.5 (Transition Function)**.** Let X be a topological manifold and let (U_1, f_1) and (U_2, f_2) be two co-ordinate charts on X such that $U_1 \cap U_2 \neq \emptyset$. Then the transition function between these two co-ordinate charts is the function

$$\phi_{21} := f_2 \circ f_1^{-1} : f_1(U_1 \cap U_2) \to f_2(U_1 \cap U_2).$$

Let X be a topological manifold with co-ordinate charts (U_i, f_i) for $i = 1, 2, 3$ such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$. Then it is clear that $\phi_{21} := f_2 \circ f_1^{-1}$ is a homeomorphism with the inverse $\phi_{12} := f_1 \circ f_2^{-1}$. Furthermore, by considering $\phi_{31} := f_3 \circ f_1^{-1}$ we observe

$$\phi_{31} = (f_3 \circ f_2^{-1}) \circ (f_2 \circ f_1^{-1}) = \phi_{32} \circ \phi_{21}.$$

This is known as the cocycle property and explains the subscript notation.

Definition 2.6 (Atlas)**.** Let X be a topological manifold. An atlas for X is the collection of co-ordinate charst $\{(U_i, f_i)\}_{i \in I}$ such that

$$\bigcup_{i \in I} U_i = X.$$

We note that we do not require the index set I to be finite. Although, since $\{U_i\}_{i \in I}$ is an open cover, if X is compact, it is possible to obtain a finite sub-cover, and hence a finite atlas. Nonetheless, since we assumed X is second-countable, we can always choose I to be countable.

2.1 Smooth Manifolds

So far, we have only considered ourselves with the topological structure. As we would like to do calculus on our manifolds, we will now equip our manifolds with the property of smoothness. Recall the following definition for Euclidean spaces.

Definition 2.7. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (or C^∞) if all the partial derivatives of F of any order exists.

Of course, this is technically a property not a definition though it will suffice for our purposes.

Definition 2.8 (Smooth Atlas)**.** Let X be a topological manifold of dimension n. Then an atlas $\{(U_i, f_i)\}_{i \in I}$ on X is smooth if for all $i, j \in I$, the transition function

$$\phi_{ij} : f_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to f_i(U_i \cap U_j) \subseteq \mathbb{R}^n$$

is smooth.

Since ϕ_{ij} is a (bijective) map between open subsets of Euclidean spaces, it makes sense to ask whether or not ϕ_{ij} is smooth.

Definition 2.9 (Diffeomorphism)**.** Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $f : U \to V$. Then f is a diffeomorphism if f is smooth and has a smooth inverse.

As $(\phi_{ij})^{-1} = \phi_{ji}$, and both ϕ_{ij} and ϕ_{ji} are smooth, the transition functions of any smooth manifold are diffeomorphisms.

Definition 2.10 (Compatible)**.** Let X be a topological manifold and let $\mathcal{A} := \{(U_i, f_i)\}$ be a smooth atlas. Let (U, f) be any co-ordinate chart on X, then (U, f) is compatible with the atlas $\mathcal{A}$ if the transition function between (U, f) and any chart in $\mathcal{A}$ is a diffeomorphism.

Clearly, any chart in a smooth atlas is compatible with that atlas, and if (U, f) is compatible with the smooth atlas $\mathcal{A}$, then $(U, f) \cup \mathcal{A}$ is also a smooth atlas.

Definition 2.11. Let X be a topological manifold and $\mathcal{A}, \mathcal{B}$ be two atlases on X. Then $\mathcal{A}$ is compatible with $\mathcal{B}$ if every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$.

Similarly as before, if $\mathcal{A}, \mathcal{B}$ are compatible, then $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas on X.

Lemma 2.1. Let X be a topological manifold and let

$$\mathcal{A} := \{(U_i, f_i)\}_{i \in I}, \mathcal{B} := \{(U_j, f_j)\}_{j \in J},$$

be two compatible smooth atlases on X. Then for all (U, f) co-ordinate charts compatible with $\mathcal{A}$, (U, f) is compatible with $\mathcal{B}$.

Proof. It suffices to show that for all $(U_j, f_j) \in \mathcal{B}$, $U \cap U_j \neq \emptyset$, the transition map

$$\phi := f_j \circ f^{-1} : f(U \cap U_j) \to f_j(U \cap U_j)$$

and its inverse are smooth.

Let $y \in f(U \cap U_j)$, then there exist some $x \in U \cap U_j$ such that $f(x) = y$. As $\mathcal{A}$ is an atlas, it contains a co-ordinate chart $(U_i, f_i) \in \mathcal{A}$ such that $x \in U_i$. Then, defining $W := U \cap U_i \cap U_j \neq \emptyset$, we have the homomorphisms $f : W \to f(W), f_i : W \to f_i(W)$ and $f_j : W \to f_j(W)$. As remarked before, we have

$$\phi = (f^{-1} \circ f_i) \circ (f_i^{-1} \circ f_j)$$

on W. Now, by compatibility, the right hand side is smooth, and so we have ϕ is smooth on W implying it is smooth at y. Thus, as $y \in f(U \cap U_j)$ was arbitrary, ϕ is smooth (by a similar argument ϕ^{-1} is also smooth) and (U, f) is compatible with $\mathcal{B}$. $\square$

With this lemma it is easy to see that compatibility defines an equivalence relation on the set of smooth atlases and with this we can define smooth manifolds.

Definition 2.12 (Smooth Manifold)**.** A smooth manifold is a topological manifold with an equivalence class $[\mathcal{A}]$ of compatible smooth atlases on X. The equivalence class of atlases is called a smooth structure on X.

The reason for the definition considering only the equivalence class of compatible smooth atlases is because we do not want to distinguish between compatible smooth atlases. Indeed, recalling our example of a sphere, we would like to not consider the atlases which projects the sphere with respect to two other points that are not the poles as an alternative manifold.

From this point forward, we will always work with smooth manifolds and thus, omit the word "smooth" whenever it is clear from the context, i.e. a manifold is a smooth manifold and a atlas is a smooth atlas.

2.2 Submanifolds

Definition 2.13 (Affine Subspace)**.** An affine subspace $A \subseteq \mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$, i.e. there exists some $v \in V$ and $W \leq \mathbb{R}^n$ such that

$$A := v + W = \{v + w \mid w \in W\}.$$

Definition 2.14 (Submanifold)**.** Let X be an n-dimensional manifold and let $Y \subseteq X$. Then Y is an m-dimensional submanifold of X if for all $y \in Y$, there exists a

- a co-ordinate chart (U, f) of X which is compatible with the smooth structure of X such that $y \in U$ and,
- an m-dimensional affine subspace $A \subseteq \mathbb{R}^n$

$$f(U \cap Y) = f(U) \cap A.$$

Proposition 2.1. Let X be an n-dimensional manifold and Y an m-dimensional submanifold of X, then Y is an m-dimensional manifold.

Proof. As Y is a topological subspace of X, it is Hausdorff and second-countable. Thus, it remains to show that Y is equipped with a smooth structure.

By linear algebra, it is easy to see that the linear map $\tau : A = v + W \to W : a \mapsto a - v$ is continuously invertible, and thus, for all $y \in Y$ there exists a chart $(U, f' := \tau \circ f)$ of X such that $y \in U$ and $f'(U \cap Y) = f'(U) \cap W$. Let $T : W \cong \mathbb{R}^m$, then defining the atlas

$$\{(U_y, \tilde{f}_y)\}_{y \in Y} := \{(U_y, T \circ f')\}_{y \in Y},$$

for all $a, b \in Y$, its transition map

$$\phi_{ab} = (T \circ \tau \circ f_b) \circ (T \circ \tau \circ f_a)^{-1} = T \circ \tau \circ (f_b \circ f_a^{-1}) \circ \tau^{-1} \circ T^{-1},$$

is a composition of smooth functions, and thus is smooth. Hence Y is a smooth manifold. $\square$

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, then define the set $s_f := \{(x, y) \mid y = f(x)\} \subseteq \mathbb{R}^2$ and I claim that s_f is a submanifold of $\mathbb{R}^2$. Define the chart (U, g) on $\mathbb{R}^2$ where $U = \mathbb{R}^2$ and

$$g(x, y) = (x, y - f(x)).$$

It is clear that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism as it is invertible with the inverse $g^{-1}(x, y) = (x, y + f(x))$ and so, $\{(U, g)\}$ is a smooth atlas of $\mathbb{R}^2$. Now considering $g \mid_{s_f} : s_f \to g(s_f) : (x, f(x)) \mapsto (x, 0)$ we have s_f is a smooth submanifold of $\mathbb{R}^2$.

Let us recall the following proposition from year-two analysis.

Proposition 2.2 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset and let $F : U \to \mathbb{R}^n$ be smooth. Let $x \in U$ such that the Jacobian as x, $DF\mid_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists an open neighbourhood $V \subseteq U$ of x such that $F\mid_V : V \to F(V) \subseteq \mathbb{R}^n$ is a diffeomorphism.

Corollary 0.1. A smooth, bijective function $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ which has non-zero Jacobian everywhere has a smooth inverse.

The inverse function theorem is useful for showing whether a subset of a manifold is a submanifold. Consider the circle $S_1 := \{x^2 + y^2 = 1\}$ as a subset of the manifold $\mathbb{R}^2$. Then, let

$$U = \mathbb{R}^2 \quad \{(x,0) \mid x \leq 0\} \text{ and } f : U \to \mathbb{R}^2 : (r\cos\theta, r\sin\theta) \mapsto (r,\theta).$$

As $f : U \to f(U)$ is smooth, bijective and has non-zero Jacobian on U, then $f^{-1} : f(U) \to U$ is also smooth. Thus, (U, f) is a smooth chart on $U \to \tilde{U} := \mathbb{R}^+ \times (-\pi, \pi) \subseteq \mathbb{R}^2$. Then, for all $(\cos\theta, \sin\theta) \in S_1 \quad \{(-1,0)\}$, we have $f(\cos\theta, \sin\theta) = (1,\theta)$ implying

$$f(U \cap S_1) = \{(1,\theta) \mid \theta \in (-\pi,\pi)\} = f(U) \cap A,$$

where A is the affine subspace $(1,0) + \{(0,y) \mid y \in \mathbb{R}\}$. Hence S_1 is a submanifold of $\mathbb{R}^2$.

Definition 2.15 (Level Sets)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a function and let $\alpha \in \mathbb{R}^k$. Then the level set of h at α is

$$h^{-1}(\{\alpha\}) = \{x \in \mathbb{R}^n \mid h(x) = \alpha\} \subseteq \mathbb{R}^n.$$

Definition 2.16 (Regular Points and Values)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. A point $x \in \mathbb{R}^n$ is called a regular point of h if the Jacobian of h at x

$$Dh\mid_x : \mathbb{R}^n \to \mathbb{R}^k$$

is surjective.

$\alpha \in \mathbb{R}^k$ is called a regular value if every point of the α-level set $h^{-1}(\{\alpha\})$ is regular.

If $x \in \mathbb{R}^n$ is not a regular point, then it is called a critical point. Similarly, if $\alpha \in \mathbb{R}^k$ is not a regular value, then it is called a critical value.

Definition 2.17 (Standard Projection)**.** Let $k \leq n$. The standard projection is the morphism

$$\pi : \mathbb{R}^n \to \mathbb{R}^k : (x_1, \cdots, x_n) \mapsto (x_{n-k+1}, \cdots, x_n).$$

That is π forgets the first $n-k$ entries.

Level sets are a useful tool for constructing submanifolds.

Theorem 1. Let $U \subseteq \mathbb{R}^n$ be an open subset and let $h : U \to \mathbb{R}^k$ be a smooth function where $k \leq n$. Let $z \in U$ be a regular point of h. Then there exists an open neighbourhood $V \subseteq U$ of z and a diffeomorphism

$$f : V \to f(V) \subseteq \mathbb{R}^n \text{ s.t. } h \circ f^{-1} = \pi : f(V) \to \mathbb{R}^k.$$

Informally, this theorem states that a smooth function around a regular point looks like the standard projection.

Proof. Let $x_1, \cdots, x_n$ be co-ordinates on $\mathbb{R}^n$ and let us write

$$h(x) = (h_1(x), \cdots, h_k(x)).$$

As z is regular, we have $Dh\mid_z : \mathbb{R}^n \to \mathbb{R}^k$ is surjective and thus, possibly by reordering, the set

$$\left\{ \frac{\partial h(z)}{\partial x_{n-k+1}}, \cdots, \frac{\partial h(z)}{\partial x_n} \right\}$$

form a basis of $\mathbb{R}^k$ and the matrix

$$M := \begin{pmatrix} \frac{\partial h_1(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_1(z)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_k(z)}{\partial x_{n-k+1}} & \cdots & \frac{\partial h_k(z)}{\partial x_n} \end{pmatrix}$$

is invertible. Then, by defining

$$f : U \to f(U) : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^{n-k}, h_1(x), \cdots, h_k(x)),$$

we have,

$$Df\mid_z = \left(\begin{array}{c|c} I_{n-k} & 0 \\ \hline \star & M \end{array} \right)$$

which is invertible as $\det Df\mid_z = \det I_{n-k} \det M = \det M \neq 0$. Thus, by the inverse function theorem, there exists some open $V \subseteq U$ such that $f : V \to f(V)$ is a diffeomorphism. Then, by considering $\pi \circ f = h$, we have $\pi = h \circ f^{-1}$. $\square$

Corollary 1.1. If $h : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function, and α is a regular value, then the level set of h at α is a submanifold of $\mathbb{R}^n$ of dimension $n - k$.

Proof. For all $z \in h^{-1}(\{\alpha\})$, we have z is a regular point. Thus, by the above theorem, there exists an open neighbourhood V of z and a diffeomorphism $f : V \to f(V)$ such that $h \circ f^{-1} = \pi$. Then,

$$f(h^{-1}(\{\alpha\}) \cap V) = f(h^{-1}(\{\alpha\})) \cap f(V) = \pi^{-1}(\{\alpha\}) \cap f(V).$$

Hence, as $\pi^{-1}(\{\alpha\}) = \{(x_1, \cdots, x_{n-k}, \alpha_1, \cdots, \alpha_k)\} = \alpha + A_{n-k}$, we have $h^{-1}(\{\alpha\})$ is a submanifold of dimension $n - k$. $\square$

This corollary is extremely useful. Consider the sphere $S^n = \{x_0^2 + \cdots + x_n^2 = \alpha\}$, by defining $h : \mathbb{R}^n \to \mathbb{R} : (x_0, \cdots, x_n) \mapsto x_0^2 + \cdots + x_n^2$, we see that h is smooth with the the Jacobian

$$Dh\mid_x = (2x_0, \cdots, 2x_n).$$

Thus, α is a regular value of h for all $\alpha > 0$. Hence, $S^n = \{h(x) = \alpha\}$ is a submanifold of $\mathbb{R}^{n+1}$ for all $\alpha > 0$.

Theorem 2 (Sard's Theorem)**.** Let $h : \mathbb{R}^n \to \mathbb{R}^k$ be a smooth function. Then the set of regular values $Z \subseteq \mathbb{R}^k$ is dense. Furthermore, $\mathbb{R}^k \ \ Z$ has Lebesgue measure zero.