# CS174A Lecture 3

# Announcements & Reminders

- ***Project assignment #1 due by Sunday midnight (10/6/19)***
- ***Project #2 will be discussed during tomorrow's TA session***
- ***Start forming your project teams (team size: 3-4)***
  - Remember: project expectations scale with team size
  - Project proposals & teams due by 10/29/19, final proposals due by 11/5/19
- ***I regularly update syllabus, make sure you're using the latest one***
- ***Google form in Piazza:*** https://forms.gle/Yne8PYVbfGp2vj6j9
- ***My post-lecture office location***
- ***PTE numbers***

# Last Lecture Recap

- ***A Basic Graphics System***
  - Input devices: keyboard, mouse, tablet, touchscreens
  - CPU/GPU
  - Frame Buffer: resolution, single vs. double buffering, color depth, interlaced vs. non-interlaced, refresh rate
  - Output devices: CRT (random-scan & raster), flat-panel (LED, LCD, Plasma), printers, plotters, head-mounted devices, stereo displays
- ***Linear Algebra***
  - Vectors: magnitude, unit vector, normalizing, addition, multiplication, properties
  - Linear combination of vectors: affine, convex, linear independence

# Next Up

- *Finish up vectors: basis vectors, dot product, cross product*
- *Matrices: square, zero, identity, symmetric, matrix operations*
- *Coordinate systems*
- *Homogeneous representations of points and vectors*
- *Representing shapes: lines, circles*
- *Transformations: translation, scaling, rotation, shear*

# Generators and Base Vectors

***How many vectors are needed to generate a vector space?***

- Any set of vectors that generate a vector space is called a generator set
- Given a vector space $\mathbf{R}^n$ we can prove that we need minimum $n$ vectors to generate all vectors $\mathbf{v}$ in $\mathbf{R}^n$
- A generator set with minimum size is called a basis for the given vector space

# Standard Unit Vectors

$$\mathbf{v} = (x_1, \ldots, x_n), \quad x_i \in \Re$$

$$\begin{aligned} (x_1, x_2, ..., x_n) = \ & x_1(1, 0, 0, \ldots, 0, 0) \\ & + x_2(0, 1, 0, \ldots, 0, 0) \\ & \cdots \\ & + x_n(0, 0, 0, \ldots, 0, 1) \end{aligned}$$

# Standard Unit Vectors

***For any vector space $R^n$:***

$$i_1 = (1, 0, 0, \ldots, 0, 0)$$
$$i_2 = (0, 1, 0, \ldots, 0, 0)$$
$$\ldots$$
$$i_n = (0, 0, 0, \ldots, 0, 1)$$

***The elements of a vector v in $R^n$ are the scalar coefficients of the linear combination of the basis vectors***

# Standard Unit Vectors in 2D & 3D

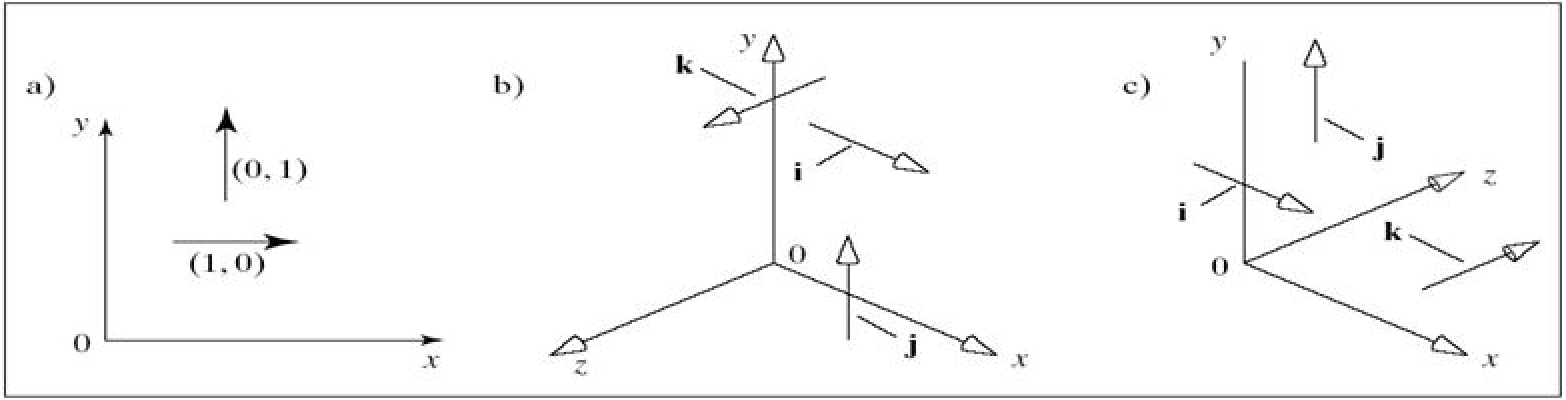
**i** = (1,0)

**j** = (0,1)

**i** = (1,0,0)

**j** = (0,1,0)

**k** = (0,0,1)




Right handed

Left handed

# Representation of Vectors Through Basis Vectors

***Given a vector space $R^n$,***
***a set of basis vectors B {$\mathrm{b}_i$ in $R^n$, i=1,…n} and***
***a vector v in $R^n$***
***we can always find scalar coefficients such that:***

$$\mathbf{v} = a_1\mathbf{b}_1+\ldots+a_n\mathbf{b}_n$$

So, vector **v** expressed with respect to *B* is:

$$\mathbf{v}_B = (a_1,\ldots,a_n)$$

# Dot Products in Graphics

- The problem dot products solve in graphics:
  - Dot with a vector of coefficients. Now you have a linear function that maps a point onto a scalar

    $3x + 4y + 5z = ?$

  - Predictable effect as you adjust a coordinate

# Dot Products and Matrices

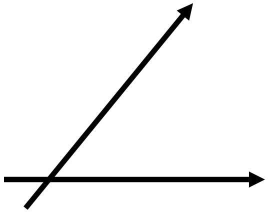
- What if we want a function that produces not a scalar, but a new point?
  - *This would become a tool for moving points somewhere new!*
- How do we generate three scalar outputs instead of one?

# Dot Products in Graphics

- Another problem dot products solve:  Comparing Vectors
  - Trig measurements!

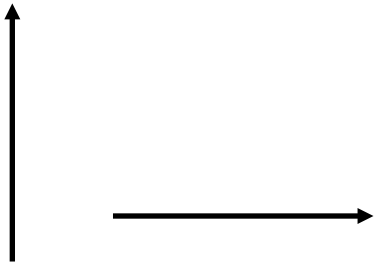
# Dot Product and Perpendicularity

***From Property 5:***



$\mathbf{a} \cdot \mathbf{b} > 0$

$\mathbf{a} \cdot \mathbf{b} = 0$



$\mathbf{a} \cdot \mathbf{b} < 0$

# Perpendicular Vectors

***Definition***

Vectors **a** and **b** are perpendicular iff $\mathbf{a}\cdot\mathbf{b}=0$

***Also called "normal" or "orthogonal" vectors***

***It is easy to see that the standard unit vectors form an orthogonal basis:***

$\mathbf{I}\cdot\mathbf{j}=0, \quad \mathbf{j}\cdot\mathbf{k}=0, \quad \mathbf{I}\cdot\mathbf{k}=0$

# Dot (Scalar) Product

***Definition:***

$$\mathbf{w}, \mathbf{v} \in \Re^n$$

$$\mathbf{w} \cdot \mathbf{v} = \sum_{i=1}^{n} w_i v_i$$

***Properties***

1. Symmetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$
5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$

# Cross (Vector) Product

***Defined only for 3D vectors and with respect to the standard unit vectors***

***Definition***

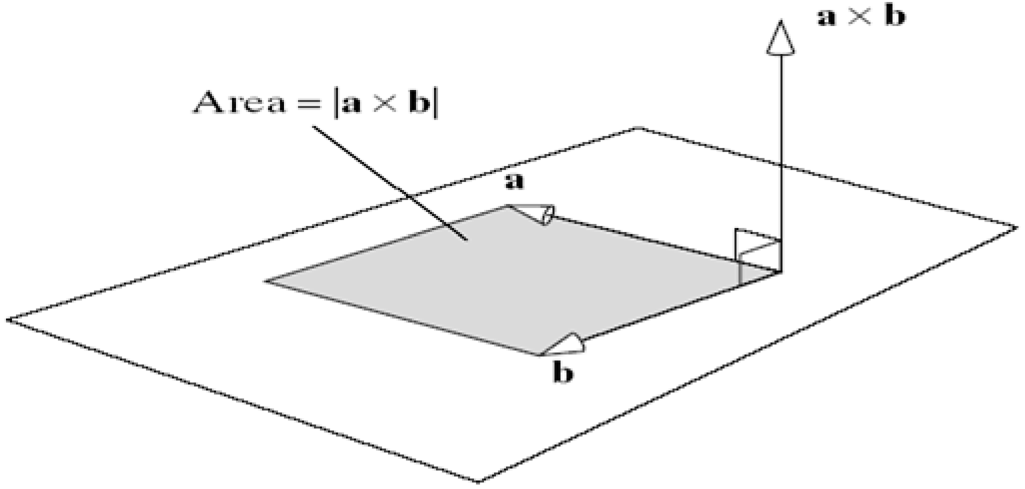
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

# Properties of the Cross Product

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$
2. Antisymmetry: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
3. Linearity: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. Homogeneity: $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b})$
5. The cross product is normal to both vectors: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$
6. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$

# Geometric Interpretation of the Cross Product



# Matrices

***Rectangular arrangement of scalar elements***

**Matrix:**
Bold upper-case

$$\mathbf{A}_{3\times 3} = \begin{pmatrix} -1 & 2.0 & 0.5 \\ 0.2 & -4.0 & 2.1 \\ 3 & 0.4 & 8.2 \end{pmatrix}$$

$$\mathbf{A} = (\mathbf{A}_{ij})$$

# Special Square ($n$ x $n$) Matrices

***Zero matrix:*** $\mathbf{A}_{ij} = 0$ *for all i,j*

***Identity matrix:*** $\mathbf{I}_n = \begin{cases} \mathbf{I}_{ii} = 1 & \text{for all } i \\ \mathbf{I}_{ij} = 0 & \text{for } i \neq j \end{cases}$

***Symmetric matrix:*** $(\mathbf{A}_{ij}) = (\mathbf{A}_{ji})$

# Operations with Matrices

**Addition:**

$$\mathbf{A}_{m\times n} + \mathbf{B}_{m\times n} = (a_{ij} + b_{ij})$$

**Properties:**

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $f(\mathbf{A} + \mathbf{B}) = f\mathbf{A} + f\mathbf{B}$
4. Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji})$

# Multiplication

**Definition:**

$$\mathbf{C}_{m\times r} = \mathbf{A}_{m\times n}\mathbf{B}_{n\times r}$$

$$(\mathbf{C}_{ij}) = (\sum_{k=1}^{n} a_{ik}b_{kj})$$

**Properties:**

1. $\mathbf{AB} \neq \mathbf{BA}$
2. $\mathbf{A(BC)} = \mathbf{(AB)C}$
3. $f(\mathbf{AB}) = (f\mathbf{A})\mathbf{B}$
4. $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$,
   $\mathbf{(B+C)A} = \mathbf{BA} + \mathbf{CA}$
5. $\mathbf{(AB)}^T = \mathbf{B}^T\mathbf{A}^T$

# Inverse of a Square Matrix

***Definition***

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

***Important property***

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$$

# Dot Product as a Matrix Multiplication

***Representing vectors as column matrices:***

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \mathbf{a}^T \mathbf{b} \\
&= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3
\end{aligned}
$$