Specification and Verification

Lecture 6: Model checking TSs against LTL

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TL;DR: This lecture in short

What is model checking? Why study it?

Essentially checking whether a transition system satisfies some formal specification

Main references

- Christel Baier, Joost-Pieter Katoen: **Principles of Model Checking.** MIT Press 2018.
- Mickael Randour: Verification course @ UMONS.

Required and target competences

What tools do we need?

Discrete maths, formal language theory, computational models

What skills will we obtain?

- theory: the automata-theoretic tools usual for verification
- practice: algorithms used at every step of the model-checking pipeline

How will these skills be useful?

Model checking is the most-widely adopted automatic verification technique

LTL model checking: decision problem

Definition: LTL model checking problem

Given a TS $\mathcal{T}$ and an LTL formula φ, decide if $\mathcal{T} \models \varphi$ or not.

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$\Longrightarrow$ Model checking algorithm via **automata-based approach** (Vardi and Wolper, 1986)

Intuition.

- Represent φ as an NBA
- Use it to try to find a path π in $\mathcal{T}$ such that $\pi \not\models \varphi$
- If one is found, a prefix of it is an *error trace*; otherwise, $\mathcal{T} \models \varphi$

LTL model checking: key observation

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\mathcal{T} \models \varphi \quad & \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \subseteq \mathrm{Words}(\varphi) \\
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Line 3 uses negation for paths.

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& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}_{\neg\varphi}) = \varnothing
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Line 4 uses the existence of an NBA for any ω-regular language and the fact that **all LTL formulas describe ω-regular languages**.

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 & \text{iff} & \mathcal{T} \otimes \mathcal{A}_{\neg\varphi} \models \Diamond\Box\neg F
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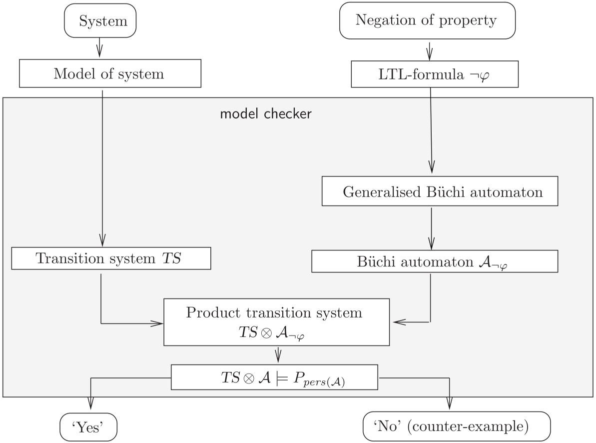
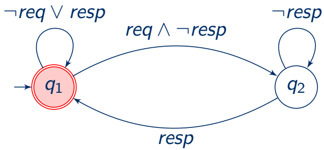
Line 5 reduces the language intersection problem to the satisfaction of a persistence property over the product TS $\mathcal{T} \otimes \mathcal{A}_{\neg\varphi}$. The idea is to **check that no trace yielded by $\mathcal{T}$ will satisfy the acceptance condition of the NBA $\mathcal{A}_{\neg\varphi}$**.

Overview of the algorithm



From LTL to GNBA: examples

- NBA for $\Box(req \rightarrow \Diamond resp)$



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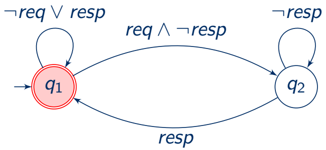
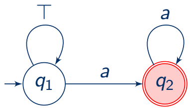
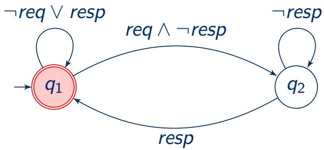


- NBA for $\Diamond\Box a$



From LTL to GNBA: examples

- NBA for $\square(req \rightarrow \lozenge resp)$



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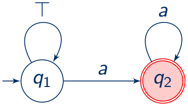
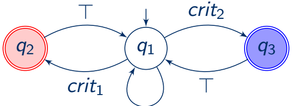


- GNBA for $\square\lozenge crit_1 \wedge \square\lozenge crit_2$



From LTL to GNBA: intuition (1/3)

Goal

For an LTL formula φ, build GNBA $\mathcal{G}_\varphi$ over alphabet 2^P such that $\mathcal{L}(\mathcal{G}_\varphi) = \mathrm{Words}(\varphi)$.

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- **What will the states of $\mathcal{G}_\varphi$ be?**
  - Let $w = a_0 a_1 a_2 \cdots \in \mathrm{Words}(\varphi)$. Idea: **"extend" the sets $a_i \subseteq P$ with subformulas ψ of φ.**

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  - Obtain $\overline{w} = B_0 B_1 B_2 \ldots$ such that

$$\psi \in B_i \quad \Longleftrightarrow \quad a_i a_{i+1} a_{i+2} \ldots \models \psi.$$

  - $\overline{w}$ will be a **run for w** in the GNBA $\mathcal{G}_\varphi$.

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From LTL to GNBA: intuition (2/3)

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- $a_0 = \{a\}$ is extended with $\neg b$, $\neg(\neg a \wedge b)$ and φ as they hold in w and no other subformula holds

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$$\underbrace{\{a, \neg b, \neg(\neg a \wedge b), \varphi\}}_{B_0}\ \underbrace{\{a, b, \neg(\neg a \wedge b), \varphi\}}_{B_1}\ \underbrace{\{\neg a, b, \neg a \wedge b, \varphi\}}_{B_2}\ \ldots$$

$=\overline{w}$

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  - $\mathcal{U}$ split according to the *expansion law* into *local condition (encoded in states)* and *next-step one (encoded in transitions)*
  - Meaning of $\mathcal{U}$ is the *least solution* of the expansion law $\Longrightarrow$ reflected in the choice of *acceptance sets for $\mathcal{G}_\varphi$*

From LTL to GNBA: closure of a formula

Definition: closure of φ

The set $\mathrm{Closure}(\varphi)$ consists of all sub-formulas ψ of φ and their negation $\neg\psi$

E.g., for $\varphi = a\,\mathcal{U}\,(\neg a \wedge b)$,

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Sets B_i are elementary subsets of $\mathrm{Closure}(\varphi)$

Intuition: a set B is *elementary* if there is a path π such that B is the set of **all** formulas $\psi \in \mathrm{Closure}(\varphi)$ with $\pi \models \psi$

From LTL to GNBA: elementary sets

Definition: elementary set

A set of sub-formulas $B \subseteq \mathrm{Closure}(\varphi)$ is *elementary* if:

1. B is **logically consistent**, i.e., for all $\varphi_1 \wedge \varphi_2, \psi \in \mathrm{Closure}(\varphi)$
   - $\varphi_1 \wedge \varphi_2 \in B \iff \varphi_1 \in B \;\wedge\; \varphi_2 \in B$
   - $\psi \in B \implies \neg\psi \notin B$
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Elementary sets: examples (1/2)

Let $\varphi = a \, \mathcal{U} \, (\neg a \wedge b)$:

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All elementary sets?

Elementary sets: examples (2/2)

Let $\varphi = a\,\mathcal{U}\,(\neg a \wedge b)$:

$$\mathrm{Closure}(\varphi) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi\}$$

All elementary sets?

$$\begin{aligned}
B_1 &= \{a, b, \neg(\neg a \wedge b), \varphi\} \\
B_2 &= \{a, b, \neg(\neg a \wedge b), \neg\varphi\} \\
B_3 &= \{a, \neg b, \neg(\neg a \wedge b), \varphi\} \\
B_4 &= \{a, \neg b, \neg(\neg a \wedge b), \neg\varphi\} \\
B_5 &= \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\} \\
B_6 &= \{\neg a, b, \neg a \wedge b, \varphi\}
\end{aligned}$$

From LTL to GNBA: $\mathcal{G}_\varphi$ (1/2)

For formula φ over P, let $\mathcal{G}_\varphi = (Q, A = 2^P, \delta, I, \mathcal{F})$ where:

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- *Intuition: for any run $B_0 B_1 B_2 \ldots$, if $\varphi_1 \,\mathcal{U}\, \varphi_2 \in B_0$, then φ_2 must eventually become true ($\rightsquigarrow$ ensured by the acceptance condition)*

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Observe that $\mathcal{F} = \varnothing$ if no until in φ.

$\Longrightarrow$ All runs are accepting in this case.

From LTL to GNBA: $\mathcal{G}_\varphi$ (2/2)

The transition relation $\delta\colon Q \times 2^P \to 2^Q$ is given by:

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- If $a = B \cap P$, then $\delta(B, a)$ is the set of all elementary sets of formulas B' satisfying
  (i) for every $\bigcirc\psi \in \mathrm{Closure}(\varphi)$, $\bigcirc\psi \in B \iff \psi \in B'$, and
  (ii) for every $\varphi_1 \,\mathcal{U}\, \varphi_2 \in \mathrm{Closure}(\varphi)$,

$$\varphi_1 \,\mathcal{U}\, \varphi_2 \in B \iff (\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \,\mathcal{U}\, \varphi_2 \in B'))$$

- *Intuition: (i) and (ii) reflect the semantics of $\bigcirc$ and $\mathcal{U}$ operators, (ii) is based on the expansion law*

From LTL to GNBA: e.g. $\varphi = \bigcirc a$

- $\text{Closure}(\varphi) = \{a, \neg a, \bigcirc a, \neg\bigcirc a\}$

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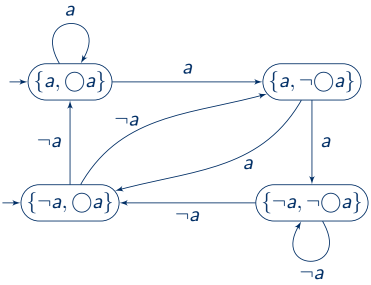

- $Q = \{\{a, \bigcirc a\}, \{a, \neg\bigcirc a\}, \{\neg a, \bigcirc a\}, \{\neg a, \neg\bigcirc a\}\}$
- $I = \{\{a, \bigcirc a\}, \{\neg a, \bigcirc a\}\}$
- $\mathcal{F} = \varnothing$

From LTL to GNBA: $\varphi = a\,\mathcal{U}\,b$ (1/3)

- $\mathrm{Closure}(\varphi) = \{a, \neg a, b, \neg b, a\,\mathcal{U}\,b, \neg(a\,\mathcal{U}\,b)\}$

$\Longrightarrow$ **Blackboard construction of the GNBA**

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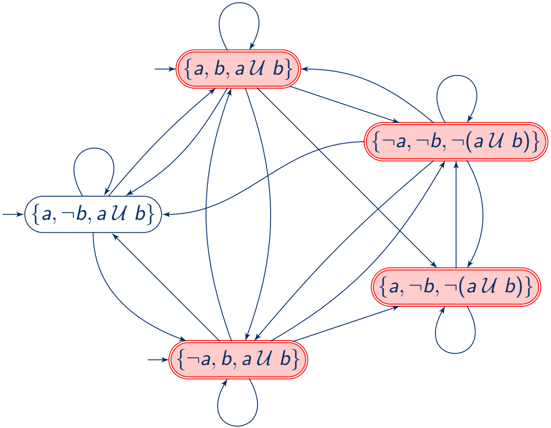


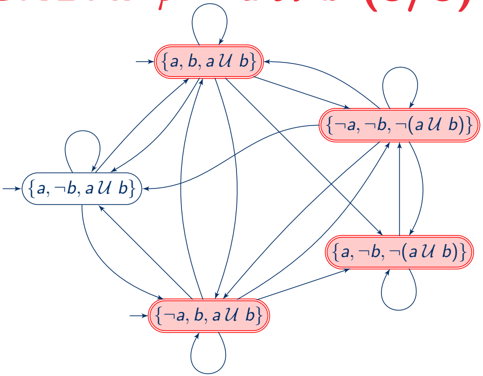
From LTL to GNBA: $\varphi = a \,\mathcal{U}\, b$ (2/3)

Some explanations

Let $B_1 = \{a, b, a \,\mathcal{U}\, b\}$, $B_2 = \{\neg a, b, a \,\mathcal{U}\, b\}$, $B_3 = \{a, \neg b, a \,\mathcal{U}\, b\}$, $B_4 = \{\neg a, \neg b, \neg(a \,\mathcal{U}\, b)\}$ and $B_5 = \{a, \neg b, \neg(a \,\mathcal{U}\, b)\}$.

- $Q = \{B_1, B_2, B_3, B_4, B_5\}$, $I = \{B_1, B_2, B_3\}$
- $\mathcal{F} = \{F_{a\mathcal{U}b}\} = \{\{B_1, B_2, B_4, B_5\}\}$.
  $\hookrightarrow$ $\mathcal{G}_\varphi$ is actually a **simple NBA**
- Labels omitted for readability (recall label is $B \cap P$)
- From B_1 (resp. B_2), we can go anywhere because $a \,\mathcal{U}\, b$ is already fulfilled by $b \in B_1$ (resp. B_2)
- From B_3, we need to go where $a \,\mathcal{U}\, b$ holds: B_1, B_2 or B_3
- From B_4, we can go anywhere because $\neg(a \,\mathcal{U}\, b)$ is already fulfilled by $\neg a, \neg b \in B_4$
- From B_5, we need to go where $\neg(a \,\mathcal{U}\, b)$ holds: B_4 or B_5

From LTL to GNBA: $\varphi = a \,\mathcal{U}\, b$ (3/3)



Sample words/runs:

- $\{a\}\,\{a\}\,\{b\}^\omega \in \mathrm{Words}(\varphi)$ has accepting run $B_3 B_3 B_2^\omega$ in $\mathcal{G}_\varphi$

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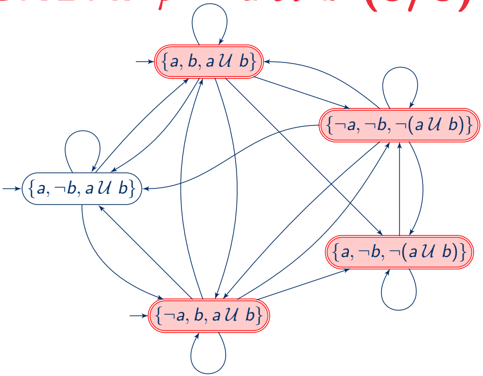

Sample words/runs:

- $\{a\}\,\{a\}\,\{b\}^\omega \in \mathrm{Words}(\varphi)$ has accepting run $B_3 B_3 B_2^\omega$ in $\mathcal{G}_\varphi$
- $\{a\}^\omega \notin \mathrm{Words}(\varphi)$ has only one run B_3^ω in $\mathcal{G}_\varphi$ and it is not accepting since $B_3 \notin F_{a\mathcal{U}b}$

From LTL to. . . NBA: construction

Idea: LTL $\rightsquigarrow$ GNBA $\rightsquigarrow$ NBA

From LTL to... NBA: construction

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Theorem: LTL to NBA

For any LTL formula φ over propositions P, there exists an NBA $\mathcal{A}_\varphi$ with $\mathrm{Words}(\varphi) = \mathcal{L}(\mathcal{A}_\varphi)$ which can be constructed in time and space $2^{\mathcal{O}(|\varphi|)}$.

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Sketch

1. Construct the GNBA $\mathcal{G}_\varphi$
   - $|\mathrm{Closure}(\varphi)| = \mathcal{O}(|\varphi|)$ and $|Q| \leq 2^{|\mathrm{Closure}(\varphi)|} = 2^{\mathcal{O}(|\varphi|)}$
   - $\#$ accepting sets of $\mathcal{G}_\varphi = \#$ until-operators in $\varphi \leq \mathcal{O}(|\varphi|)$

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   - $|\mathrm{Closure}(\varphi)| = \mathcal{O}(|\varphi|)$ and $|Q| \leq 2^{|\mathrm{Closure}(\varphi)|} = 2^{\mathcal{O}(|\varphi|)}$
   - $\#$ accepting sets of $\mathcal{G}_\varphi$ $=$ $\#$ until-operators in $\varphi \leq \mathcal{O}(|\varphi|)$
2. Construct the NBA $\mathcal{A}_\varphi$
   - $\#$ states of $\mathcal{A}_\varphi = |Q| \times \#$ accepting sets of $\mathcal{G}_\varphi$
   - $\#$ states of $\mathcal{A}_\varphi \leq 2^{\mathcal{O}(|\varphi|)} \cdot \mathcal{O}(|\varphi|) = 2^{\mathcal{O}(|\varphi|)} \cdot 2^{\log(\mathcal{O}(|\varphi|))} = 2^{\mathcal{O}(|\varphi|)}$

From LTL to...NBA: better? (1/3)

The algorithm presented here is conceptually simple but may lead to unnecessary large GNBAs (and thus NBAs)

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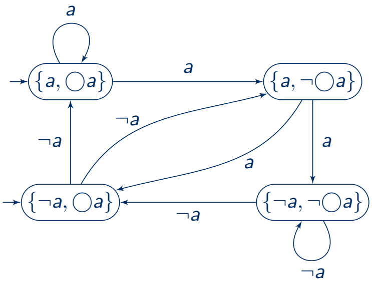
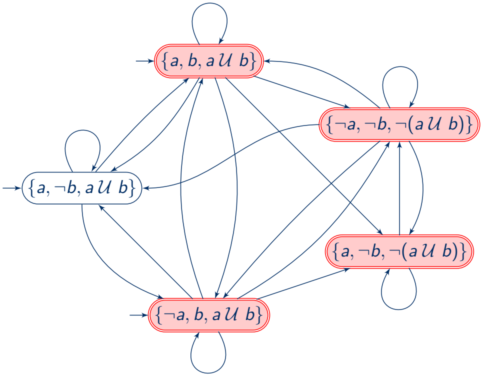
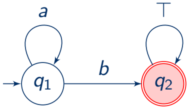


Example: the right NBA also recognizes $\bigcirc a$ but is *smaller*

From LTL to... NBA: better? (2/3)





Example: the right NBA also recognizes $a\,\mathcal{U}\,b$ but is *much smaller*

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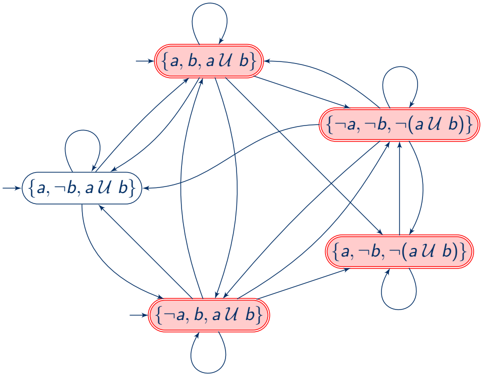

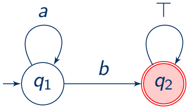

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Can we always do better?

From LTL to…NBA: better? (3/3)

In practice, there exist more efficient (but more complex) algorithms in the literature

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Still, **the exponential blowup cannot be avoided** in the worst-case!

Theorem: lower bound for NBA from LTL formula

There exists a family of LTL formulas φ_n with $|\varphi_n| = \mathcal{O}(poly(n))$ such that every NBA $\mathcal{A}_{\varphi_n}$ for φ_n has at least 2^n states.

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$\Longrightarrow$ **Proof in the next slides**

From LTL to...NBA: lower bound (1/2)

Let P be arbitrary and *non-empty*, i.e., $2^{|P|} \geq 2$. Let

$$\mathcal{L}_n = \left\{ a_1 \ldots a_n a_1 \ldots a_n \tau \;\middle|\; a_i \subseteq P \wedge \tau \in (2^P)^\omega \right\} \quad \text{for } n \geq 0.$$

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This language is expressible in LTL, i.e., $\mathcal{L}_n = \mathrm{Words}(\varphi_n)$ for

$$\varphi_n = \bigwedge_{a \in P} \; \bigwedge_{0 \leq i < n} (\bigcirc^i a \longleftrightarrow \bigcirc^{n+i} a).$$

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Claim: any NBA $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}_n$ has at least 2^n states.

From LTL to. . . NBA: lower bound (2/2)

Assume $\mathcal{A}$ is such an automaton. Words $a_1 \ldots a_n a_1 \ldots a_n \varnothing^\omega$ belong to $\mathcal{L}_n$, hence are accepted by $\mathcal{A}$.

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- Hence, **the NBA has at least 2^n states**.

LTL vs. NBAs

What have we learned?

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Corollary

Every LTL formula expresses an ω-regular property, i.e., for all LTL formula φ, $\mathrm{Words}(\varphi)$ is an ω-regular language.

Why? Because LTL can be transformed to NBA and NBAs coincide with ω-regular languages.

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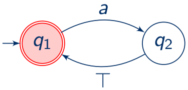
The converse is false!

Recall $\mathcal{L} = \left\{ a_0 a_1 a_2 \cdots \in (2^{\{a\}})^\omega \;\middle|\; \forall\, i \geq 0,\; a \in a_{2i} \right\}$.



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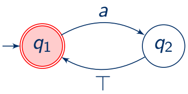

$\Longrightarrow$ **There are ω-regular properties not expressible in LTL.**

Model checking algorithm for LTL

$$
\begin{aligned}
\mathcal{T} \models \varphi \quad & \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \subseteq \mathrm{Words}(\varphi) \\
& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap ((2^{P})^{\omega} \setminus \mathrm{Words}(\varphi)) = \varnothing \\
& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap \mathrm{Words}(\neg\varphi) = \varnothing \\
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\end{aligned}
$$

It remains to consider the last line

Two remaining questions:

1. How to compute the product TS $\mathcal{T} \otimes \mathcal{A}_{\neg\varphi}$?
2. How to check persistence, i.e., $\mathcal{T} \otimes \mathcal{A}_{\neg\varphi} \models \Diamond\Box\neg F$?

Product of TS and NBA

Definition: product of TS and NBA

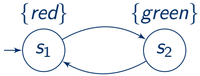
Let $\mathcal{T} = (S, A, \longrightarrow, I, P, L)$ be a TS without terminal states and $\mathcal{A} = (Q, A = 2^P, \delta, I_\mathcal{A}, F)$ a non-blocking NBA. Then, $\mathcal{T} \otimes \mathcal{A}$ is the following TS:

$$\mathcal{T} \otimes \mathcal{A} = (S', A, \longrightarrow', I', P', L') \text{ where}$$

- $S' = S \times Q$, $P' = Q$ and $L'(\langle s, q\rangle) = \{q\}$,
- $\longrightarrow'$ is the smallest relation such that if $s \xrightarrow{a} t$ and $q \xrightarrow{L(t)} p$, then $\langle s, q\rangle \xrightarrow{a}' \langle t, p\rangle$,
- $I' = \{\langle s_0, q\rangle \mid s_0 \in I \wedge \exists\, q_0 \in I_\mathcal{A},\ q_0 \xrightarrow{L(s_0)} q\}$.

Product of TS and NBA: example

Simple traffic light with two modes: *red* and *green*. LTL formula to check $\varphi = \Box\Diamond green$.



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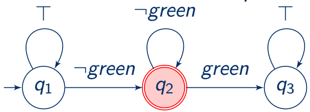

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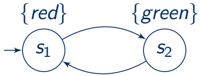

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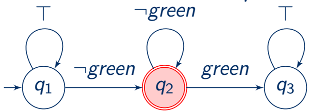

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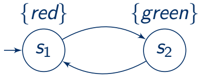

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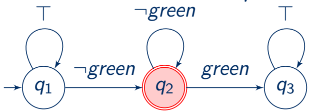

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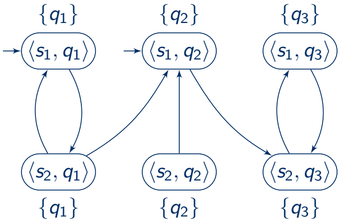

Persistence checking: illustration (1/2)

It remains to check $\mathcal{T} \otimes \mathcal{A}_{\neg\varphi} \models \Diamond\Box\neg F$ to see that $\mathcal{T} \models \varphi$.

University of Antwerp

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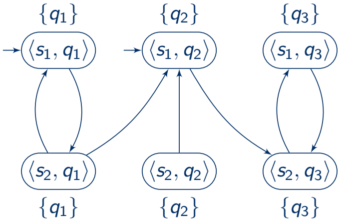

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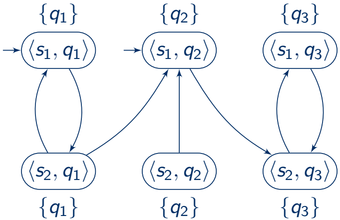

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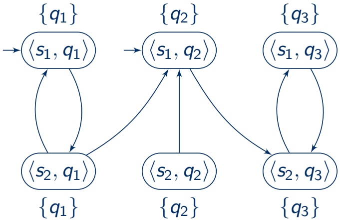

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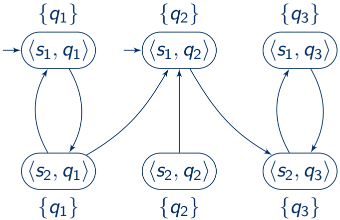

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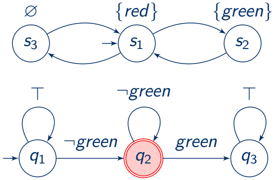
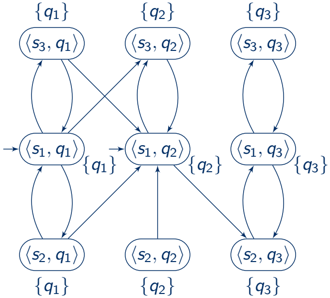
$\Longrightarrow \mathcal{T} \models \varphi$.

Persistence checking: illustration (2/2)

Slightly revised traffic light: can switch off to save energy. Same formula φ (hence same NBA $\mathcal{A}_{\neg\varphi}$).





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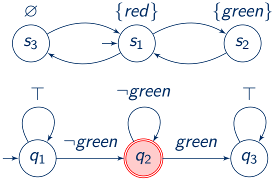

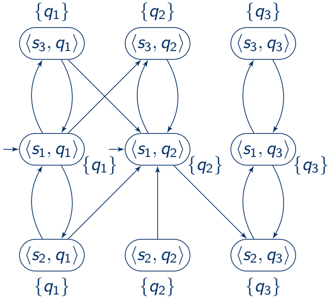

Here, $\mathcal{T} \otimes \mathcal{A}_{\neg\varphi} \not\models \Diamond\Box\neg F$ with $F = \{q_2\}$. See for example path $\langle s_1, q_1\rangle\,(\langle s_3, q_2\rangle\,\langle s_1, q_2\rangle)^{\omega}$ that visits q_2 infinitely often.

Persistence checking: cycle detection

As for checking language non-emptiness of NBA, we reduce the problem to a cycle detection problem.

Persistence checking and cycle detection

Let $\mathcal{T}$ be a TS without terminal states over P and φ a *propositional* formula over P, then

$$\mathcal{T} \not\models \Diamond\Box\varphi$$
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In particular, it holds for $\varphi = \neg F$ as needed for LTL model checking (with F the acceptance set of the NBA $\mathcal{A}_{\neg\varphi}$).

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1. Compute the reachable SCCs and check if one contains a state satisfying $\neg\varphi$.
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Still, the complexity of LTL model checking remains high!

Wrap-up of the automata-based approach

$$
\begin{aligned}
\mathcal{T} \models \varphi \quad & \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \subseteq \mathrm{Words}(\varphi) \\
& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap ((2^P)^\omega \setminus \mathrm{Words}(\varphi)) = \varnothing \\
& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap \mathrm{Words}(\neg\varphi) = \varnothing \\
& \text{iff} \quad \mathrm{Traces}(\mathcal{T}) \cap \mathcal{L}(\mathcal{A}_{\neg\varphi}) = \varnothing \\
& \text{iff} \quad \mathcal{T} \otimes \mathcal{A}_{\neg\varphi} \models \Diamond\Box\neg F
\end{aligned}
$$

Complexity of this approach

The time and space complexity is $\mathcal{O}(|\mathcal{T}|) \cdot 2^{\mathcal{O}(|\varphi|)}$.

Complexity of LTL model checking

Complexity of the model checking problem for LTL

The LTL model checking problem is PSPACE-complete.

$\Longrightarrow$ **See the book for a proof by reduction from the membership problem for polynomial-space deterministic Turing machines.**

Summary and conclusions

Automata, languages, expressions, and logic

- We have introduced an automata-based framework that allows us to check whether a system satisfies an LTL specification.
- Our main tool is a translation from LTL to NBAs.

Model checking

A TS can be model checked against an LTL formula in PSPACE using an on-the-fly algorithm.

- The sizes of the state-sets are huge in the automaton, thus huge in the product!
- How does one implement some of these algorithms?