University of Antwerp

# Specification and Verification

## Lecture 3: Linear temporal logic

Guillermo A. Pérez

October 7, 2024

# TL;DR: This lecture in short

## What is LTL? Why study it?

A logic to speak about linear-temporal properties

## Main references

- Christel Baier, Joost-Pieter Katoen: **Principles of Model Checking.** MIT Press 2018.
- Mickael Randour: Verification course @ UMONS.

# Required and target competences

## What tools do we need?

Discrete mathematics, Automata theory

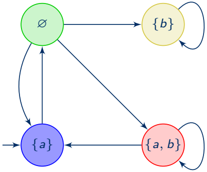
## What skills will we obtain?

- theory: a common language to speak about specifications
- practice: LTL and its variants are used for formal specifications in industry and in applied verification

## How will these skills be useful?

Having a common language will allow us to move forward in this course and will help you formalize what you want from systems you use and program.

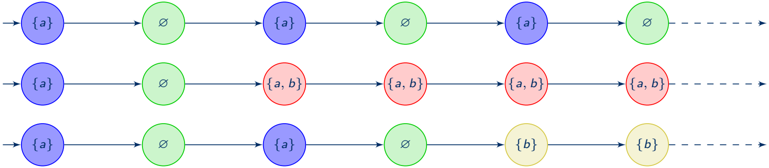
# Linear-time semantics: a reminder



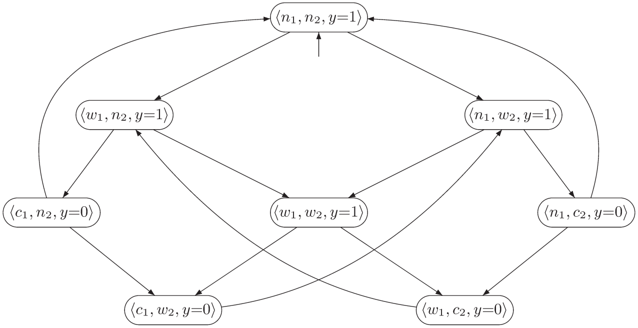

*TS $\mathcal{T}$ with atomic propositions $P = \{a, b\}$ (state and action names are omitted).*

From now on, we assume **no terminal state**

- **Linear-time semantics** deals with *traces* of executions.
  - The language of infinite words described by $\mathcal{T}$



# LT properties: safety




*TS for semaphore-based mutex*

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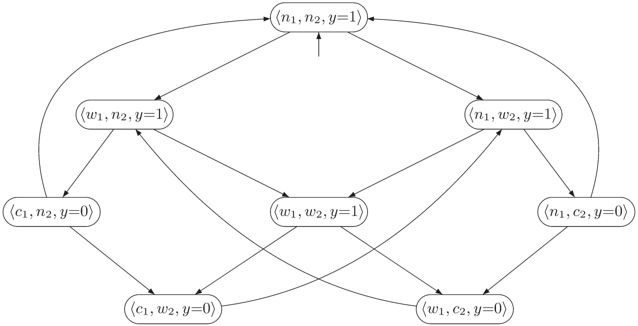


Ensure that $\langle c_1, c_2, y = \ldots\rangle \notin \mathrm{Reach}(\mathcal{T}(PG_1 \mid\mid\mid PG_2))$ or equivalently that $\neg\exists\, \pi \in \mathrm{Paths}(\mathcal{T}),\ \langle c_1, c_2, y = \ldots\rangle \in \pi$

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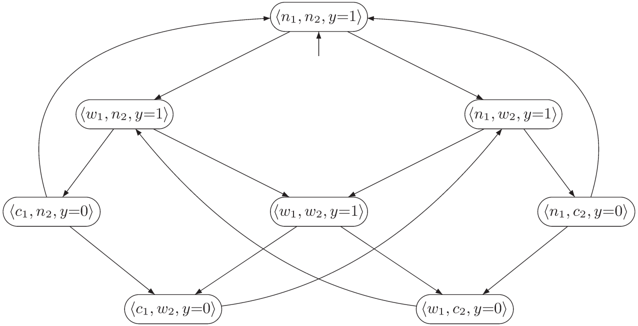


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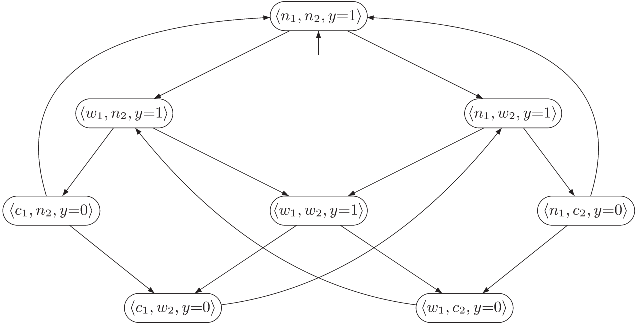
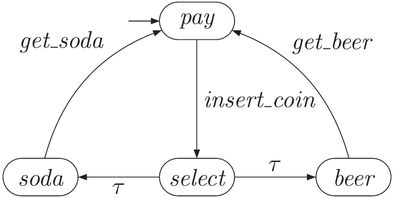



For model checking, we like to use *labels* and *traces*

- $P = \{crit_1, crit_2\}$, natural labelling
- Ensure that $\neg\exists\, w \in \mathrm{Traces}(\mathcal{T}),\ \{crit_1, crit_2\} \in w$

# LT properties: liveness




*Beverage vending machine*

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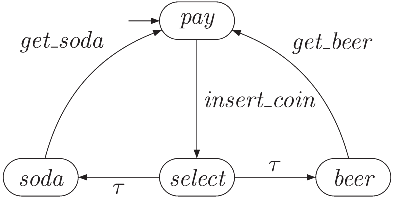


Ensure that the machine delivers a *drink* infinitely often.

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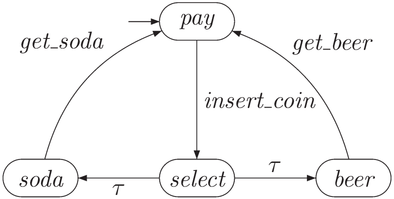


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$\hookrightarrow$ **Satisfied:** recall we consider *infinite* executions

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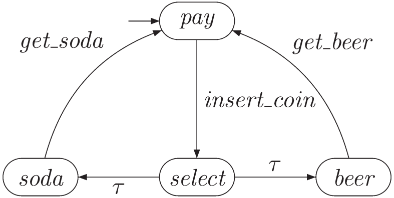

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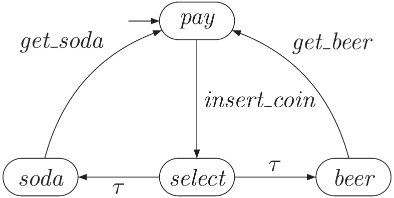


What if we ask that the machine delivers a *beer* infinitely often?

- $P = \{paid, soda, beer\}$, natural labelling
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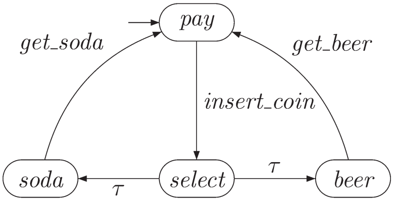


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- $\forall\, w \in \mathrm{Traces}(\mathcal{T})$, for all positions $i$ along $w$, label *beer* must appear in the future

$\hookrightarrow$ **Not satisfied:** $w = (\varnothing\,\{paid\}\,\{paid, soda\})^\omega$

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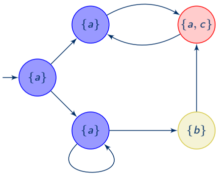
**Finite vs. infinite time**

Safety is violated by *finite* executions (i.e., the prefix up to seeing a bad state) whereas liveness is violated by *infinite* ones:

- witnessing that the good behavior never occurs.

# LT properties: persistence

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- E.g., from some point on, $a$ holds but $b$ does not

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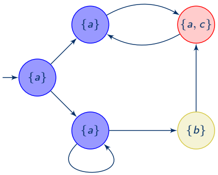


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$$\{a\}\ \left[\{a\}^\omega \,\middle|\, (\{a\}\,\{a,c\})^\omega \,\middle|\, \{a\}^+\,\{b\}\,(\{a,c\}\,\{a\})^\omega\right]$$

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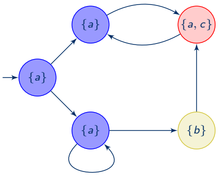


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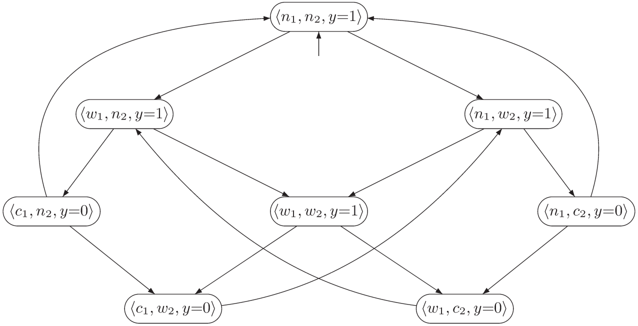
$\Longrightarrow$ **Ultimately periodic traces where $b$ is false and $a$ is true, at all steps after some point**

# LT properties: fairness (1/4)

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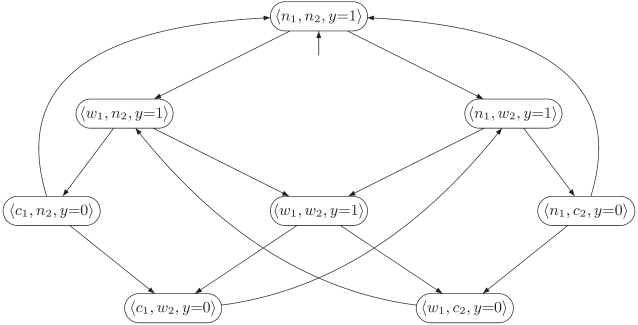


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Ensure that both processes get *fair access* to the critical section

**What is fairness?**

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$\Longrightarrow$ **All forms can be formalized in LTL**

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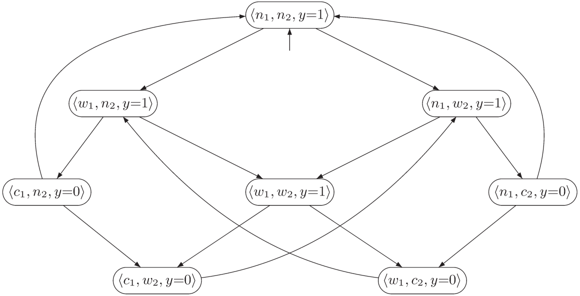


The semaphore-based mutex is **not fair** in any sense

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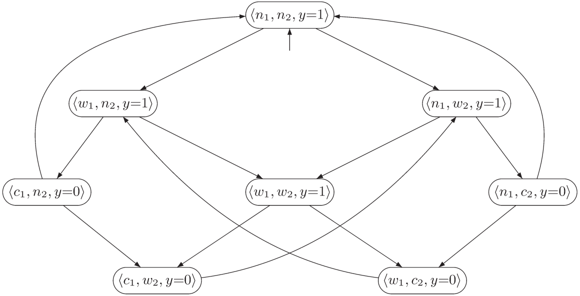
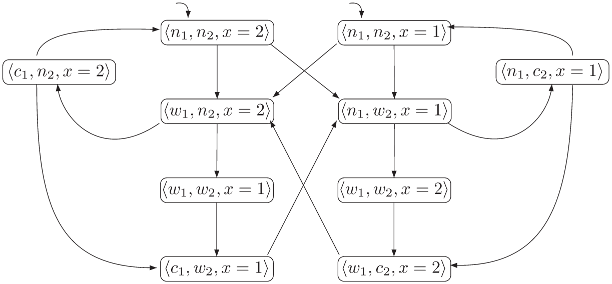



The semaphore-based mutex is **not fair** in any sense We have seen that *starvation* is possible. E.g., execution

$$\langle n_1, n_2, y = 1\rangle$$
$$\longrightarrow (\langle w_1, n_2, y = 1\rangle \longrightarrow \langle w_1, w_2, y = 1\rangle \longrightarrow \langle w_1, c_2, y = 0\rangle)^\omega$$

# LT properties: fairness (4/4)




- Peterson's mutex is **strongly fair**

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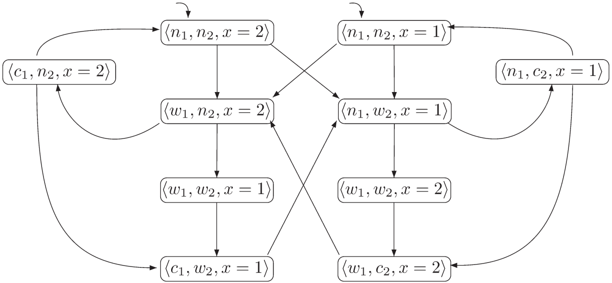


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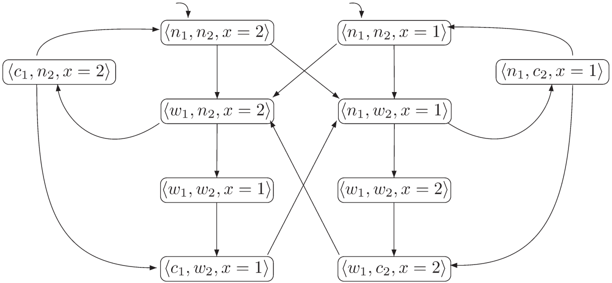


- Peterson's mutex is **strongly fair**
- We saw that it has *bounded waiting*
- A process requesting access waits at most one turn

$\hookrightarrow$ **Infinitely frequent requests $\Longrightarrow$ infinitely frequent access**

$\Longrightarrow$ **Strong fairness**

# Linear temporal logic

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$\Longrightarrow$ **Linear Temporal Logic (LTL):**

**propositional logic $+$ temporal operators**

# LTL in a nutshell

- **Atomic propositions** $a \in P$; **Boolean combinations of formulas**: $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$; **Temporal operators**:

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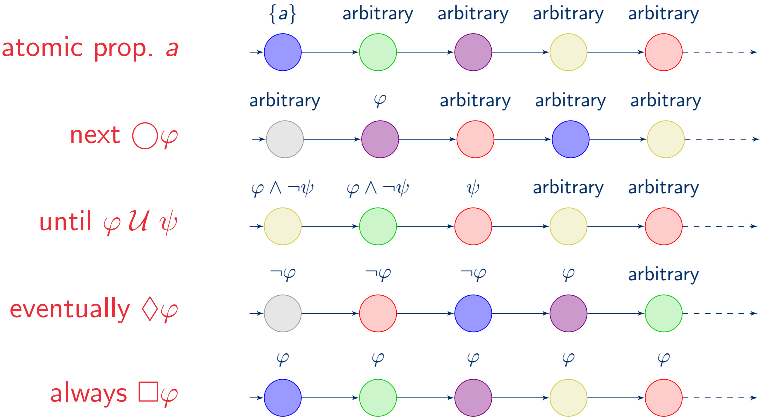

# LTL syntax

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Given the set of atomic propositions $P$, LTL formulas are formed according to the following grammar:

$$\varphi ::= \top \mid a \mid \varphi \wedge \psi \mid \neg\varphi \mid \bigcirc\varphi \mid \varphi\, \mathcal{U}\, \psi$$

where $a \in P$.

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where $a \in P$.

**$\varphi\,\mathcal{U}\,\psi$ requires that $\psi$ holds at some point!**

**(i.e., $\varphi$ forever does not suffice)**

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$$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$$ *implication*

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$$
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- Weak until $\leadsto$ until that does not require $\psi$ to be reached

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$$\varphi\,\mathcal{W}\,\psi \equiv (\varphi\,\mathcal{U}\,\psi) \vee \Box\varphi$$ **weak until**

$$\varphi\,\mathcal{R}\,\psi \equiv \neg(\neg\varphi\,\mathcal{U}\,\neg\psi)$$ **release**

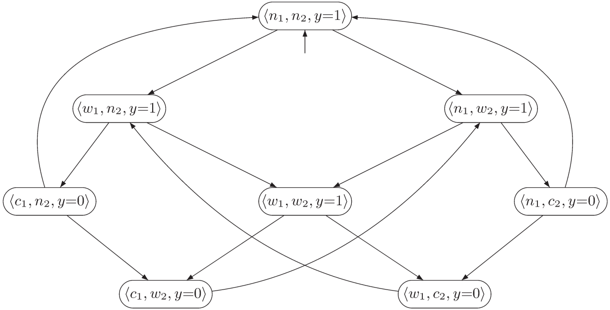
- Weak until $\rightsquigarrow$ until that does not require $\psi$ to be reached
- Release $\rightsquigarrow$ $\psi$ must hold up to the point where $\varphi$ releases it, or forever if $\varphi$ never holds

# LTL syntax: precedence order

## Precedence order

- Unary operators before binary ones,
- $\neg$ and $\bigcirc$ equally strong,
- $\mathcal{U}$ before $\wedge$, $\vee$ and $\rightarrow$

# LT properties in LTL: safety

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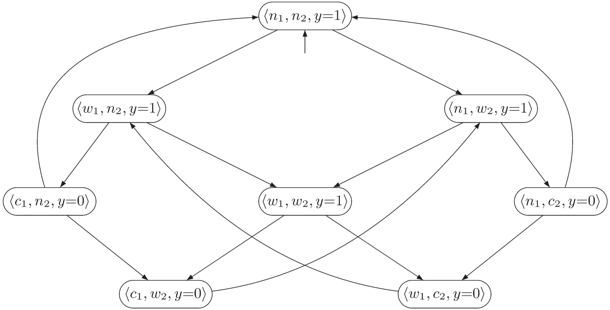


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$\hookrightarrow$ $\neg\Diamond\,(crit_1 \wedge crit_2)$ or equivalently $\Box\,(\neg crit_1 \vee \neg crit_2)$

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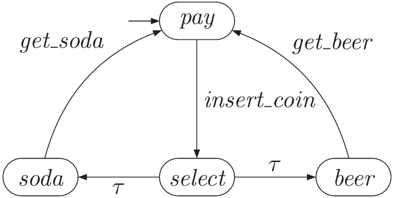


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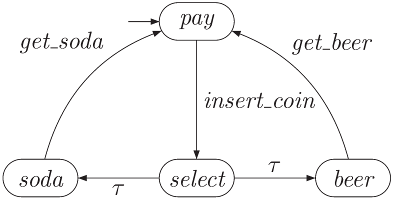


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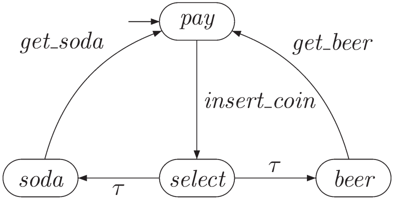


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$\Longrightarrow$ **"infinitely often"**

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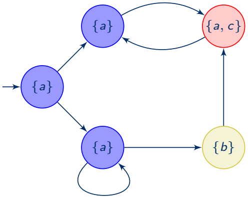


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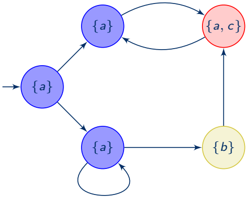


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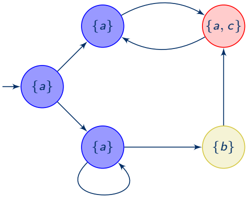


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$\Longrightarrow$ **"eventually always"**

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# LT properties in LTL: fairness

Assume $k$ processes and $P = \{wait_1, \ldots, wait_k, crit_1, \ldots, crit_k\}$

- **Unconditional fairness.** E.g., "every process gets access infinitely often"

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# LTL semantics: over words (1/2)

Given propositions $P$ and LTL formula $\varphi$, the associated LT property is the language of words:

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| $w \models \varphi \,\mathcal{U}\, \psi$ | iff | $\exists j \geq 0,\ w[j..] \models \psi$ and $\forall 0 \leq i < j,\ w[i..] \models \varphi$ |

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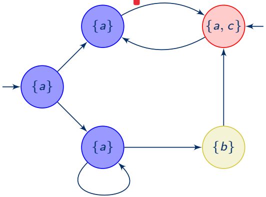
It follows that $\mathcal{T} \models \varphi$ iff $\forall\, s_0 \in I$, $s_0 \models \varphi$.

# Example




Notice the added initial state

$\mathcal{T} \not\models \Box a$

$\mathcal{T} \not\models \Diamond b$

$\mathcal{T} \models a \,\mathcal{W}\, b$

$\mathcal{T} \models \Box(b \rightarrow \Box\Diamond c)$

$\mathcal{T} \models \Diamond\Box a$

$\mathcal{T} \not\models a \,\mathcal{U}\, b$

$\mathcal{T} \not\models b \,\mathcal{R}\, a$

$\mathcal{T} \models b \rightarrow \Box c$

$\mathcal{T} \models \bigcirc(a \wedge \neg c)$

$\mathcal{T} \models \Box(c \rightarrow \bigcirc a)$

$\mathcal{T} \models \Box\neg c \rightarrow \neg\Diamond b$

$\mathcal{T} \not\models \bigcirc\bigcirc(b \vee c) \vee \Box a$

# Semantics of negation: paths

## Negation for paths

For $\pi \in \mathrm{Paths}(\mathcal{T})$ and an LTL formula $\varphi$ over $P$,

$$\pi \not\models \varphi \iff \pi \models \neg\varphi$$

because $\mathrm{Words}(\neg\varphi) = (2^P)^\omega \setminus \mathrm{Words}(\varphi)$.

# Semantics of negation: transition systems

## Negation for TSs

For TS $\mathcal{T} = (S, A, \longrightarrow, I, P, L)$ and an LTL formula $\varphi$ over $P$:

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But it may be the case that $\mathcal{T} \not\models \varphi$ and $\mathcal{T} \not\models \neg\varphi$ if

$$\mathrm{Traces}(\mathcal{T}) \cap \mathrm{Words}(\neg\varphi) \neq \varnothing \text{ and } \mathrm{Traces}(\mathcal{T}) \cap \mathrm{Words}(\varphi) \neq \varnothing.$$

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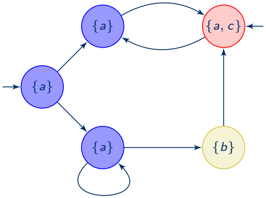


We saw that $\mathcal{T} \not\models \Diamond b$

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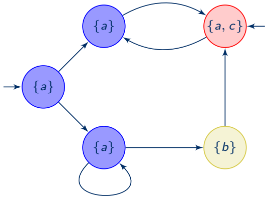


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Do we have $\mathcal{T} \models \neg\Diamond b \equiv \Box\neg b$?

$\Longrightarrow$ **No.** Because trace $w = \{a\}^2\{b\}(\{a, c\}\{a\})^\omega$ satisfies $\Diamond b$

# Equivalence of LTL formulas: definition

### Equivalence of LTL formulas

LTL formulas $\varphi$ and $\psi$ are *equivalent*, denoted $\varphi \equiv \psi$, if

$$\mathrm{Words}(\varphi) = \mathrm{Words}(\psi).$$

# Back to fairness constraints with LTL

Let $\varphi, \psi$ be LTL formulas representing that "something is enabled" ($\varphi$) and that "something is granted" ($\psi$). Recall the three types of fairness.

- *Unconditional* fairness constraint

$$\textit{ufair} = \Box\Diamond\psi$$

- *Strong* fairness constraint

$$\textit{sfair} = \Box\Diamond\varphi \rightarrow \Box\Diamond\psi$$

- *Weak* fairness constraint

$$\textit{wfair} = \Diamond\Box\varphi \rightarrow \Box\Diamond\psi$$

# Fairness assumptions

Let *fair* denote a conjunction of such assumptions. It is sometimes useful to check that all **fair executions** of a TS satisfy a formula (in contrast to all of them).

**Fair satisfaction**

Let $\varphi$ be an LTL formula and *fair* an LTL fairness assumption. We have that $\mathcal{T} \models_{fair} \varphi$ iff

$$\forall\, w \in \mathrm{Traces}(\mathcal{T}) \text{ such that } w \models \mathit{fair},\ w \models \varphi.$$

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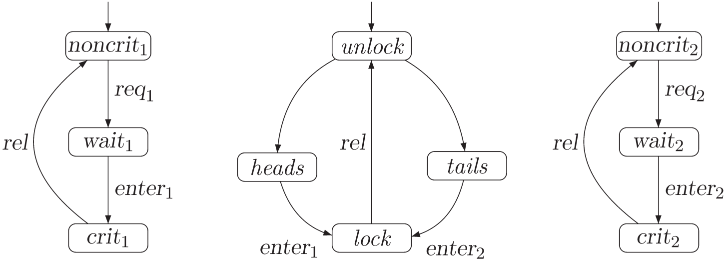
# Example: randomized arbiter for mutex




*Mutual exclusion with a randomized arbiter*

The arbiter chooses who gets access by tossing a coin: probabilities are abstracted by non-determinism.

Can process 1 access the section infinitely often?

# Example: randomized arbiter

$\hookrightarrow$ **No**, $\mathcal{T}_1 \mid\mid\mid \mathit{Arbiter} \mid\mid\mid \mathcal{T}_2 \not\models \Box\Diamond \mathit{req}_1 \rightarrow \Box\Diamond \mathit{crit}_1$ because the arbiter can always choose *tails*.

Intuitively, this is *unfair*: a real coin would lead to this with probability zero.

$\Longrightarrow$ LTL fairness assumption: $\Box\Diamond \mathit{heads} \wedge \Box\Diamond \mathit{tails}$.

$\hookrightarrow$ **The property is verified on fair executions**, i.e.,
$\mathcal{T}_1 \mid\mid\mid \mathit{Arbiter} \mid\mid\mid \mathcal{T}_2 \models_{\mathit{fair}} \bigwedge_{i \in \{1,2\}} (\Box\Diamond \mathit{req}_i \rightarrow \Box\Diamond \mathit{crit}_i)$.

# Handling fairness assumptions

Given a formula $\varphi$ and a fairness assumption *fair*, we can reduce $\models_{fair}$ to the classical satisfaction $\models$.

**From $\models_{\mathbf{fair}}$ to $\models$**

$$\mathcal{T} \models_{fair} \varphi \iff \mathcal{T} \models (fair \rightarrow \varphi).$$

# Summary and conclusions

## Linear temporal logic

Three very important things learned today:

- We now have a formal language to specify properties about systems: LTL
- We know how LTL formulas are interpreted over words
- We know how LTL formulas are interpreted over transition systems

## Moving forward

- Given a transition system and a formula, are there algorithms to determine whether the system satisfies the formula?